MAXIMUM DIAMETER OF ABSTRACT POLYTOPES

BY

ILAN ADLER and GEORGE B. DANTZIG

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13. ABSTRACT

Walkup and Klee studied the diameter of ordinary convex polytopes which is defined as the smallest integer k such that all pairs of vertices can be joined by a path of k or less neighboring vertices. The well known d-step (or Hirsch) conjecture for d dimensional polytopes with n facets states that the maximum diameter is $n - d$. Walkup and Klee showed the conjecture as correct for all $n - d \leq 5$. In an effort to extend the results, the authors of ~~this~~ the paper abstract (in the form of three simple axioms) some of the combinatorial properties of ordinary polytopes and show that these are sufficient to establish the maximum diameter is $n - d$ for all $n - d \leq 5$ and a variety of bounds for other values of n and d. Abstract polytopes are a special class of pseudo manifolds. The results provide insight into the number of iterations required to solve a linear program using the simplex method.

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1. Abstract polytope--definition and notation.

Given a finite set T of symbols, a family P of subsets of T (called vertices) forms a d-*dimensional abstract polytope* if the following three axioms are satisfied:

(i) Every vertex of P has cardinality d.

(ii) Any subset of $d-1$ symbols of T is either contained in no vertices of P or in exactly two (called *neighbors* or *adjacent*).

(iii) Given any pair of vertices $v, \bar{v} \in P$, there exists a sequence of vertices $v = v_0, \ldots, v_k = \bar{v}$ such that

(a) v_i, v_{i+1} are neighbors $(i = 0, \ldots, k-1)$

(b) $\{v \cap \bar{v}\} \subset v_i \quad (i = 0, \ldots, k)$.

It is convenient to delete from T all symbols that are not used to define vertices. Hence we denote $\cup P = \{\cup v | v \in P\}$.

Let u be a subset of P such that $|u| = k$ ($|u|$ denotes the cardinality of u). If $P' = \{v \in P | v \supset u\}$ is nonempty we say that P' is the *face* of P which is *generated by* u and denote it by $F_P(u)$ or simply $F(u)$ if the abstract polytope P is clear. It is not difficult to verify that the family $\{v-u | v \in F_P(u)\}$ of subsets obtained by deleting u from each vertex of such a face is a $(d-k)$-dimensional abstract polytope. In the sequel we shall use this property of faces extensively. Whenever we mention a face as an abstract polytope, it is to be understood that the deleting of common symbols has been performed. Since $F_P(u)$ corresponds to a $(d-k)$-dimensional abstract polytope we say that it is a $(d-k)$-dimensional face of P. Zero, one and d-1 dimensional faces are called, respectively, vertices, edges, and facets.

A d-dimensional abstract polytope with n facets is called an (n,d)-abstract polytope. (Note that $n = |\cup P|$.) We denote by $\mathcal{P}(n,d)$ the class of all (n,d)-abstract polytopes.

The graph $G(P)$ of an abstract polytope P is the graph whose vertices and edges correspond 1-1 to the vertices and edges of P, respectively.

Note that axiom (iii) is satisfied by P if, and only if, the graph of every face of P is connected, and that if we augment P by including all subsets of the vertices of P, then axioms (i)-(iii)(a) define a $(d-1)$-dimensional pseudo-manifold (with no boundary).

2. Relation between abstract and simple polytopes.

Abstract polytopes are (combinatorially) closely related to simple polytopes. A simple polytope can be expressed as the set of solutions of a bounded and non-degenerate linear program [4]. Suppose the latter consists of m equations, in n non-negative variables whose coefficient matrix is of rank m. One can associate n symbols with the index set of the n columns of the coefficient matrix. Then the family of subsets of symbols which correspond to the non-basic columns of all the basic feasible solutions (i.e., vertices) of the linear program forms an (n,d)-abstract polytope where $d = n-m$.

This is true because any feasible solution is defined uniquely by the subset of $d = n-m$ non-basic variables set to zero (axiom (i)). Given a basic feasible solution, a new basic solution can be obtained by dropping any one of the d non-basic variables. Exactly one of the basic variables can be set equal to zero in its place (under non-degeneracy and boundness). This generates a neighboring vertex (axiom (ii)). Given any two vertices v and $\bar{v}$, then by restricting ourselves to the lowest dimensional face common to v and $\bar{v}$ (i.e., holding at zero value the subset of non-basic variables common to the two vertices), a path of neighboring vertices from v to $\bar{v}$ can be found (e.g., by using the simplex method and a suitably chosen objective function) (axiom (iii).

Although the class of abstract polytopes includes (combinatorially) that of simple polytopes the converse is not true. Indeed by a theorem of Steinitz (see [2]) the graph of every 3-dimensional abstract polytope is planar. However, the graph of the 3-dimensional abstract polytope displayed in Figure 2 is easily shown to be non-planar. Hence no simple polytope can have the graph structure of this particular abstract polytope.

3. Paths and diameters.

Let P be an abstract polytope and let $v, \bar{v} \in P$. A path of length k from v to $\bar{v}$ in P is a sequence of vertices $v = v_0, \ldots, v_k = \bar{v}$ such that v_i, v_{i+1} are neighbors $(i = 0, \ldots, k-1)$. (Note that vertices of the path are not required to be in $F_P(v \cap \bar{v})$.) The diameter $\delta(P)$ of P is the smallest integer k such that any two vertices of P can be joined by a path of length less than or equal to k. We denote by $\Delta_a(n,d)$ the maximum of $\delta(P)$ over all (n,d)-abstract polytopes. This corresponds to Klee and Walkup's $\Delta_b(n,d)$ for ordinary simple polytopes [1]. In general, of course, $\Delta_a(n,d) \geq \Delta_b(n,d)$.

Our main objective is to establish values and bounds for $\Delta_a(n,d)$. We shall show in particular that the unsolved d-step (or Hirsch) conjecture (that $\Delta_a(n,d) \leq n-d$) holds for $n-d \leq 5$ thus parelleling results of Klee and Walkup [1] for ordinary polytopes. Our arguments, however, are based on fewer axioms and imply theirs as a special case.

4. Some preliminary results.

We shall make frequent use of the following theorem:

Theorem 1. (Adler, Dantzig, Murty [3]) Given an abstract polytope P, if two vertices v, $\bar{v}$ in P do not have a symbol (say A) in common then there exists an "A—avoiding path" joining them; i.e., there exists a path from v to $\bar{v}$ such that no vertex along the path contains A.

The next Theorem is the analog of a result of Klee and Walkup in [1]. The proof here is similar.

Theorem 2. For $k = 0, 1, 2, \ldots$

(i) $\Delta_a(n,d) \leq \Delta_a(n+k, d+k)$

(ii) $\Delta_a(n,d) \leq \Delta_a(n+k,d)$

(iii) $\Delta_a(n,d) \leq \Delta_a(n+2k, d+k) - k$.

(iv) $\Delta_a(2d,d) = \Delta_a(d+k, k)$, $k \geq d$.

Proof. We shall prove (i)-(iii) for $k = 1$, the extension to $k > 1$ is trivial.

Let P be an (n,d)-abstract polytope such that $\delta(P) = \Delta_a(n,d)$.

(i) Let $A \in \cup P$ and let $A' \notin \cup P$ be a new symbol, define P' as an abstract polytope identical with P except the symbol A' replaces A. Define $\tilde{P}$ as a new abstract polytope with vertices $v \cup A'$ and $v' \cup A$ for all $v \in P$ and all $v' \in P'$.

It is easy to verify that $\tilde{P}$ is an $(n+1,d+1)$-abstract polytope with a diameter at least as big as $\delta(P)$, thus

$$\Delta_a(n,d) = \delta(P) \leq \delta(\tilde{P}) \leq \Delta_a(n+1,\ d+1)$$

(ii) Let $A' \notin \cup P$ be a new symbol and $v' \in P$. Let $v_1, \ldots, v_d$ be the d subsets of v' with cardinality $d-1$. Define $\tilde{P} = (P-v') \cup \bigcup_{i=1}^{d} \{v_i \cup A'\}$. It is obvious that $\tilde{P} \in \mathcal{P}(n+1,d)$ (i.e., $\tilde{P}$ is an $(n+1,d)$-abstract polytope) and that $\delta(\tilde{P}) \geq \delta(P)$, hence

$$\Delta_a(n+1,\ d) \geq \delta(\tilde{P}) \geq \delta(P) = \Delta_a(n,d).$$

(iii) Let $A_i' \notin \cup P$ $(i = 1,2)$ be distinct new symbols. Define $P_i = \{(v \cup A_i')\,|\,v \in P\}$, $i = 1,2$. Then $P_1 \cup P_2 \in \mathcal{P}(n+2,\ d+1)$ and $\delta(P_1 \cup P_2) = \delta(P) + 1$. So

$$\Delta_a(n+2,\ d+1) - 1 \geq \delta(P_1 \cup P_2) - 1 = \delta(P) = \Delta_a(n,d)\ .$$

(iv) Let $P \in \mathcal{P}(d+k, k)$ $(k \geq d)$ where $\delta(P) = \Delta_a(d+k, k)$. Choose $v, \bar{v} \in P$ so that the shortest path from v to $\bar{v}$ has length $\delta(P)$. Consider the face $P' = F_P(v \cap \bar{v})$ of P which corresponds 1-1 to a $(d+k', k')$-abstract polytope (where $k'' = k - |v \cap \bar{v}| \leq d$). Since $P' \subset P$, the length of the shortest path from v to $\bar{v}$ in P' is at least as large as $\delta(P)$. Hence

$$\Delta_a(d+k'', k') \geq \delta(P') \geq \delta(P) = \Delta_a(d+k, k) .$$

However by (i) (since $d+k \geq 2d \geq d+k'$)

$$\Delta_a(d+k, k) \geq \Delta_a(2d, d) \geq \Delta_a(d+k', k')$$

Hence

$$\Delta_a(d+k, k) = \Delta_a(2d, d) \qquad \text{for } k \geq d. \qquad \square$$

We shall make frequent use of the notion of a "shell" bordering a set of vertices of an abstract polytope. Let P be an abstract polytope and let $Z \subset P$. A vertex v of P belongs to the <u>i-th shell</u> $N_P^i(Z)$ of Z in P if, and only if i is the minimum length of all of the paths in P joining v to the various vertices of Z. The o-shell of Z is Z itself. The 1-shell of Z is the set of vertices which are adjacent to but not in Z. In general

$$N_P^i(Z) = N_P^1\left(\bigcup_{j=1}^{i-1} N_P^j(Z)\right) .$$

For simplicity, the 1-shell of Z in P will also be denoted by $N_P(Z)$ or simply $N(Z)$ if P is clear.

Theorem 3 below will be used (in section 3) to establish the values of $\Delta_a(n,2)$ and the values of $\Delta_a(n,d)$ for all n, d such that $n-d \leq 5$.

Theorem 3. Given $P \in \mathcal{P}(2d,d)$ (i.e., P is a $(2d,d)$-abstract polytope) and $v_0, \bar{v}_0 \in P$ such that $v_0, \bar{v}_0$ partition $\cup P$. Let $(v_0, v_1, \ldots, v_k)$ and $(\bar{v}_0, \bar{v}_1, \ldots, \bar{v}_{\bar{k}})$ be two paths in P with the property $|v_i \cap \bar{v}_j| = i+j$, then such paths exist for

(i) $d \geq 1$, $k = 0$, $\bar{k} = 1$ where $\bar{v}_1$ is any given vertex in $N_P(\bar{v}_0)$.

(ii) $d \geq 2$, $k = 1$, $\bar{k} = 1$ where $\bar{v}_1$ is any given vertex in $N_P(\bar{v}_0)$.

(iii) $d \geq 3$, $k = 2$, $\bar{k} = 1$ where $\bar{v}_1$ is any given vertex in $N_P(\bar{v}_0)$.

(iv) $d \geq 4$, $k = 2$, $\bar{k} = 2$.

Proof. (Except part (iv) for $d \geq 5$.)

Let $v_0 = \{A_1, \ldots, A_d\}$, $\bar{v}_0 = \{\bar{A}_1, \ldots, \bar{A}_d\}$ partition P. The symbols $v_i, \bar{v}_j$ where used below satisfy $|v_i \cap \bar{v}_j| = i+j$ or will be shown to do so.

(i) Obvious by the second axiom of abstract polytopes.

(ii) By (i), $\bar{v}_1 \in N(\bar{v}_0)$ implies that $|v_0 \cap \bar{v}_1| = 1$. Relable so that $\bar{v}_1 = \{A_1, \bar{A}_1, \ldots, \bar{A}_{d-1}\}$. Note that $\bar{A}_d \not\subset \{v_0 \cup \bar{v}_1\}$. By Theorem 1 there exists an $\bar{A}_d$-avoiding path between v_0 and $\bar{v}_1$ in $F(v_0 \cap \bar{v}_1)$ (i.e., all the vertices of that path contain $v_0 \cap \bar{v}_1 = A_1$ but do not contain $\bar{A}_d$). Let v_1 be the neighbor of v_0 in this path. Since $\bar{A}_d \not\subset v_1$, v_1 must contain, for $d \geq 2$, one symbol different from those in $v_0 \cup \bar{A}_d$. Hence, since $A_1 \subset v_1$, $|v_1 \cap \bar{v}_1| = 2$.

(iii) Let $\bar{v}_1 \in N(\bar{v}_0)$ then by (ii) there exists a vertex $v_1 \in N(v_0)$ such that $|v_1 \cap \bar{v}_1| = 2$. By relabeling let $v_1 = \{A_1, \ldots, A_{d-1}, \bar{A}_1\}$, $\bar{v}_1 = \{A_1, \bar{A}_1, \ldots, \bar{A}_{d-1}\}$. Define $P' = F(v_1 \cap \bar{v}_1)$ and $W = N(v_0) \cap P'$. Note that W is the set of all vertices of $N(v_0)$ which contain both A_1 and $\bar{A}_1$.

By Theorem 1 there exists an $\bar{A}_d$– avoiding path from v_1 to $\bar{v}_1$ in P'. Let v_2 be a vertex of this path which belongs to $N_{P'}(W)$ (such vertex exists because $v_1 \in W$ while $\bar{v}_1 \notin W$ for $d \geq 3$). But v_2 contains $\{A_1, \bar{A}_1\}$ and one symbol out of $\bar{A}_2, \ldots, \bar{A}_{d-1}$, hence $|v_2 \cap \bar{v}_1| = 3$.

(iv) $(d = 4)$ By (iii) there exists $\bar{v}_1 \in N(\bar{v}_0)$ and $v_2 \in N^2(v_0)$ such that $|v_2 \cap \bar{v}_1| = 3$. Since $d = 4$, the second axiom of abstract polytopes implies that v_2 is a neighbor of $\bar{v}_1$. Thus letting $\bar{v}_2 = v_2$ completes the proof for this case. $\square$

Part (iv) for $d = 5$ will be established in stages via Theorems 4 and 5 and for $d \geq 6$ via Theorems 6 and 7.

<u>Theorem 4</u>. Given $P \in \mathcal{P}(2d,d)$ (where $d \geq 4$) and vertices $v_0, v_1, \bar{v}_0 \in P$ satisfying $|v_i \cap \bar{v}_j| = i+j$ and $\bar{v}_2', \bar{v}_2'' \in N^2(\bar{v}_0)$ satisfying $|v_1 \cap \bar{v}_2' \cap \bar{v}_2''| = 3$ then there exist $v_2 \in N^2(v_0)$ such that $|v_2 \cap \bar{v}_2| = 4$ where either $\bar{v}_2 = \bar{v}_2'$ or $\bar{v}_2 = \bar{v}_2''$.

Proof: By relabeling for $d \geq 4$ we are assuming

$$v_0 = \{A_1, \dots, A_d\};\ \bar{v}_0 = \{\bar{A}_1, \dots, \bar{A}_d\};\ v_1 = \{A_1, \dots, A_{d-1}, \bar{A}_1\};$$
$$\bar{v}_2' = \{A_1, A_2, \bar{A}_1, \dots, \bar{A}_d\} - \{\bar{A}_i, \bar{A}_j\};$$
$$\bar{v}_2'' = \{A_1, A_2, \bar{A}_1, \dots, \bar{A}_d\} - \{\bar{A}_k, \bar{A}_\ell\} \qquad (2 \leq i, j, k, \ell \leq d)$$

where i, j, k, ℓ are all distinct or $i = k$ and i, j, ℓ are distinct. Note that $v_1 \cap \bar{v}_2' \cap \bar{v}_2'' = \{A_1, A_2, \bar{A}_1\}$.

By Theorem 1 there exists an $\bar{A}_i$—avoiding path from v_1 to $\bar{v}_2'$ in $P' = F(v_1 \cap \bar{v}_2')$. Let $Z = N(v_0) \cap P'$. Since $v_1 \in Z$, while $\bar{v}_2' \notin Z$, this path intersects $N_{P'}(Z)$, say at v_2. Note that $v_2 \in N_{P'}(Z) \subset N_P^2(v_0)$.

By definition, all the vertices of $P' = F(v_1 \cap \bar{v}_2')$ contain the symbols A_1, A_2 and $\bar{A}_1$ and v_2 contains also some $\bar{A}_s \in \{\bar{A}_t | t \in [2, \dots, d], t \neq i\}$. But, either $\bar{v}_2'$ or $\bar{v}_2''$ must contain $\bar{A}_s$. Hence either $\bar{v}_2 = \bar{v}_2'$ or $\bar{v}_2 = \bar{v}_2''$ satisfies $|v_2 \cap \bar{v}_2| = 4$. $\square$

Theorem 5. Let $P \in \mathcal{P}(10,5)$ and let (v_0, v_1, v_2); $(\bar{v}_0, \bar{v}_1)$ be paths in P satisfying $|v_i \cap \bar{v}_j| = i+j$. Define $P' = F(v_1 \cap \bar{v}_1)$, $W = N(v_0) \cap P'$ and $\bar{W} = N(\bar{v}_0) \cap P'$. Then either there exists a path of length 3 connecting a vertex in W to a vertex in $\bar{W}$ or:

(a) $F(v_2 \cap \bar{v}_1)$ is (by relabeling) the 7-vertex 2-dimensional abstract polytope given in Figure 1.

$$\begin{tikzcd}[column sep=small, row sep=small]
v_2 = \{A_1,A_2,A_3,\bar{A}_1,\bar{A}_2\} \arrow[r, dash] \arrow[dd, dash] & \{A_1,A_3,A_5,\bar{A}_1,\bar{A}_2\} \arrow[r, dash] & \{A_1,A_5,\bar{A}_1,\bar{A}_2,\bar{A}_4\} \arrow[dr, dash] & \\
 & & & \{A_1,\bar{A}_1,\bar{A}_2,\bar{A}_3,\bar{A}_4\} = \bar{v}_1 \\
\{A_1,A_2,\bar{A}_1,\bar{A}_2,\bar{A}_5\} \arrow[r, dash] & \{A_1,A_4,\bar{A}_1,\bar{A}_2,\bar{A}_5\} \arrow[r, dash] & \{A_1,A_4,\bar{A}_1,\bar{A}_2,\bar{A}_3\} \arrow[ur, dash] &
\end{tikzcd}$$

Figure 1

(b) $F(v_1 \cap \bar{v}_1)$ is (by relabeling) the 3-dimensional abstract polytope given in Figure 2 (note that the graph of $F(v_1 \cap \bar{v}_1)$ is non-planar).

<u>Proof</u>: Assume (by relabeling if necessary) that

$v_0 = \{A_1, \ldots, A_d\}$, $v_1 = \{A_1, \ldots, A_{d-1}, \bar{A}_1\}$, $v_2 = \{A_1, \ldots, A_{d-2}, \bar{A}_1, \bar{A}_2\}$,
$\bar{v}_0 = \{\bar{A}_1, \ldots, \bar{A}_d\}$, $\bar{v}_1 = \{A_1, \bar{A}_1, \ldots, \bar{A}_{d-1}\}$.

(a) Since $|v_2 \cap \bar{v}_1| = 3$, $P'' = F(v_2 \cap \bar{v}_1)$ corresponds 1-1 to an $(n,2)$-abstract polytope with $3 \leq n \leq 7$. It is easy to show that every $(n,2)$-abstract polytope Q has exactly n vertices and that every symbol of $\cup Q$ is contained by two adjacent vertices of Q. Furthermore, the graph of Q forms a simple cycle. We shall consider three cases.

(a1) $|P''| \leq 5$. Obviously there exists a path of length less than or equal to 3 joining v_1 in W to $\bar{v}_1$ in $\bar{W}$.

(a2) $|P''| = 6$. In this case P'' has the form:

$$\begin{tikzcd}[column sep=small, row sep=small]
 & v_3' \arrow[r, dash] & v_4' \arrow[dr, dash] & \\
v_2 = \{A_1,A_2,A_3,\bar{A}_1,\bar{A}_2\} \arrow[ur, dash] \arrow[dr, dash] & & & \{A_1,\bar{A}_1,\bar{A}_2,\bar{A}_3,\bar{A}_4\} = \bar{v}_1 \\
 & v_3'' \arrow[r, dash] & v_4'' \arrow[ur, dash] &
\end{tikzcd}$$

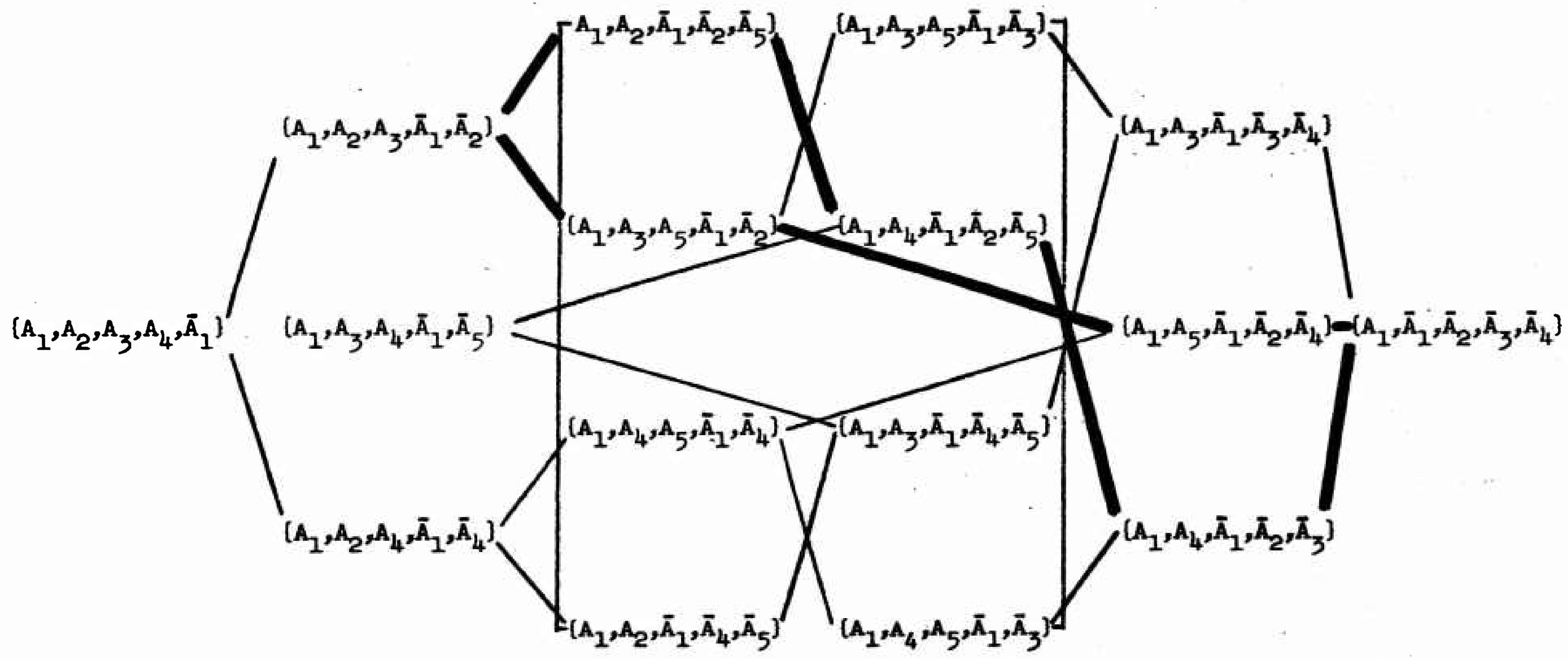
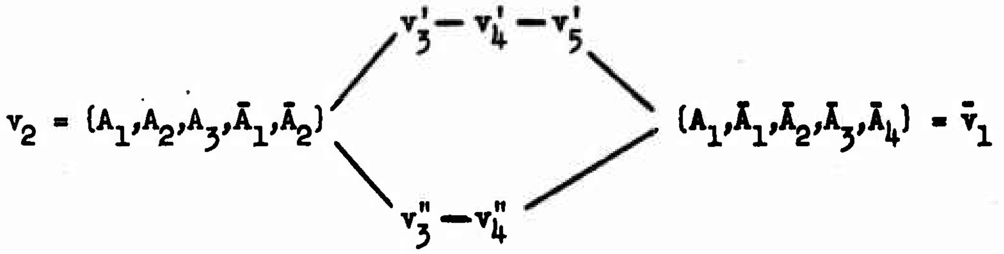

{$A_1, A_2, A_3, A_4, \bar{A}_1$}
{$A_1, A_2, A_3, \bar{A}_1, \bar{A}_2$}
{$A_1, A_3, A_4, \bar{A}_1, \bar{A}_5$}
{$A_1, A_2, A_4, \bar{A}_1, \bar{A}_4$}
{$A_1, A_2, \bar{A}_1, \bar{A}_2, \bar{A}_5$}
{$A_1, A_3, A_5, \bar{A}_1, \bar{A}_2$}
{$A_1, A_4, A_5, \bar{A}_1, \bar{A}_4$}
{$A_1, A_2, \bar{A}_1, \bar{A}_4, \bar{A}_5$}
{$A_1, A_3, A_5, \bar{A}_1, \bar{A}_3$}
{$A_1, A_4, \bar{A}_1, \bar{A}_2, \bar{A}_5$}
{$A_1, A_3, \bar{A}_1, \bar{A}_4, \bar{A}_5$}
{$A_1, A_4, A_5, \bar{A}_1, \bar{A}_3$}
{$A_1, A_3, \bar{A}_1, \bar{A}_3, \bar{A}_4$}
{$A_1, A_5, \bar{A}_1, \bar{A}_2, \bar{A}_4$}
{$A_1, A_4, \bar{A}_1, \bar{A}_2, \bar{A}_3$}
{$A_1, \bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4$}

Figure 2

Since A_1, $\bar{A}_1$ and $\bar{A}_2$ are contained by all the vertices of P'', and $v_2 \cup \bar{v}_1 = \{A_1,A_2,A_3,\bar{A}_1,\bar{A}_2,\bar{A}_3,\bar{A}_4\}$, one of the remaining symbols A_4, A_5 or $\bar{A}_5$ is contained by the two adjacent vertices v_3', v_4' and another by v_3'', v_4''. But if A_4 is contained by v_3', v_4' (or v_3'', v_4'') then v_3' (or v_3'') is a neighbor of v_1. If $\bar{A}_5$ is contained by v_3', v_4' (or v_3'', v_4'') then v_4' (or v_4'') is a member of $\bar{W}$. In both cases there exists a path of length 3 from a member of W to a member of $\bar{W}$.

(a3) $|P''| = 7$. Here P'' has the form:



Using the same arguments as in (a2) we see that if every path from a member of W to a member of $\bar{W}$ has a length of at least 4 then v_3', v_4' must contain $\bar{A}_5$, v_4', v_5' must contain A_4 and v_3'', v_4'' must contain A_5. Thus P'' has the form of Figure 1 except for possible interchange of symbols A_2 with A_3 and $\bar{A}_3$ with $\bar{A}_4$.

(b) Suppose very path in P' joining a member of W to a member of $\bar{W}$ has a length larger than 3. So P'' has the form given in Figure 1. Let us denote $\bar{v}_2 = \{A_1,A_4,\bar{A}_1,\bar{A}_2,\bar{A}_3\}$ we can apply now the above analysis to the face $F(v_1 \cap \bar{v}_2)$ where we replace the symbols

$\{A_1,A_2,A_3,A_4,A_5\}$ above by $\{\bar{A}_1,\bar{A}_3,\bar{A}_2,\bar{A}_4,\bar{A}_5\}$, and $\{\bar{A}_1,\bar{A}_2,\bar{A}_3,\bar{A}_4,\bar{A}_5\}$ by $\{A_1,A_4,A_2,A_3,A_5\}$ respectively. Thus $F(v_1 \cap \bar{v}_2)$ has the following form (with the possible interchanging of A_2, A_3 and $\bar{A}_2$, $\bar{A}_3$):

$$\begin{tikzcd}[column sep=small]
 & \{A_1,A_3,A_4,\bar{A}_1,\bar{A}_5\} \arrow[r, dash] & \{A_1,A_4,\bar{A}_1,\bar{A}_2,\bar{A}_5\} \arrow[r, dash] & \{A_1,A_4,\bar{A}_1,\bar{A}_2,\bar{A}_3\} = \bar{v}_2 \arrow[dd, dash] \\
v_1 = \{A_1,A_2,A_3,A_4,\bar{A}_1\} \arrow[ur, dash] \arrow[dr, dash] & & & \\
 & \{A_1,A_2,A_4,\bar{A}_1,\bar{A}_4\} \arrow[r, dash] & \{A_1,A_4,A_5,\bar{A}_1,\bar{A}_4\} \arrow[r, dash] & \{A_1,A_4,A_5,\bar{A}_1,\bar{A}_3\}
\end{tikzcd}$$

Note that interchanging $\bar{A}_2$, $\bar{A}_3$ is not possible since $\{A_1,A_4,\bar{A}_1,\bar{A}_2,\bar{A}_5\}$ is forced as a neighbor of $\bar{v}_2$ because it already exists as a vertex of $F(v_2 \cap \bar{v}_1)$

We now let $v_2' = \{A_1,A_2,A_4,\bar{A}_1,\bar{A}_4\}$ and apply the argument of (a) on $F(v_2' \cap \bar{v}_1)$. Since $\{A_1,A_5,\bar{A}_1,\bar{A}_2,\bar{A}_4\} \in F(v_2 \cap \bar{v}_1)$ and $\{A_1,A_4,A_5,\bar{A}_1,\bar{A}_4\} \in F(v_1 \cap \bar{v}_2)$ we get a unique form for $F(v_2' \cap \bar{v}_1)$ as follows:

$$\begin{tikzcd}[column sep=small]
v_2' = \{A_1,A_2,A_4,\bar{A}_1,\bar{A}_4\} \arrow[r, dash] \arrow[dd, dash] & \{A_1,A_4,A_5,\bar{A}_1,\bar{A}_4\} \arrow[r, dash] & \{A_1,A_5,\bar{A}_1,\bar{A}_2,\bar{A}_4\} \arrow[dr, dash] & \\
 & & & \{A_1,\bar{A}_1,\bar{A}_2,\bar{A}_3,\bar{A}_4\} = \bar{v}_1 \\
\{A_1,A_2,\bar{A}_1,\bar{A}_4,\bar{A}_5\} \arrow[r, dash] & \{A_1,A_3,\bar{A}_1,\bar{A}_4,\bar{A}_5\} \arrow[r, dash] & \{A_1,A_3,\bar{A}_1,\bar{A}_3,\bar{A}_4\} \arrow[ur, dash] &
\end{tikzcd}$$

If we would have interchange A_2 with A_3 in $F(v_1 \cap \bar{v}_2)$ then we would have $\{A_1,A_2,\bar{A}_1,\bar{A}_4,\bar{A}_5\} \in F(v_2' \cap \bar{v}_1)$ and $\{A_1,A_2,A_4,\bar{A}_1,\bar{A}_5\} \in F(v_1 \cap \bar{v}_2) \subset P$ but $\{A_1,A_2,\bar{A}_1,\bar{A}_2,\bar{A}_5\} \in F(v_2 \cap \bar{v}_1) \subset P$. So axiom (ii) of abstract polytopes would have been violated since $\{A_1,A_2,\bar{A}_1,\bar{A}_5\}$ is contained by 3 vertices of P.

Hence, we can conclude that A_2 is not interchangable with A_3 in $F(v_1 \cap \bar{v}_2)$ (i.e., the structure of this face which is given above is the only structure which is compatible with $F(v_2 \cap \bar{v}_1)$ given in (a) and the assumption that no path of length less than 4 joined a member of W to a member of $\bar{W}$).

Finally, letting $\bar{v}_2' = \{A_1,A_3,\bar{A}_1,\bar{A}_3,\bar{A}_4\}$, considering $F(v_1 \cap \bar{v}_2')$ and applying (a) (notice that $\{A_1,A_3,A_5,\bar{A}_1,\bar{A}_2\}$; $\{A_1,A_2,A_3,\bar{A}_1,\bar{A}_2\}$; $\{A_1,A_3,A_4,\bar{A}_1,\bar{A}_5\}$ and $\{A_1,A_3,\bar{A}_1,\bar{A}_4,\bar{A}_5\}$ belong to $F(v_1 \cap \bar{v}_2')$) we get that the form of $F(v_1 \cap \bar{v}_2')$ is necessarily as follows:

$$
v_1 = \{A_1,A_2,A_3,A_4,\bar{A}_1\} \begin{array}{l} \nearrow \{A_1,A_3,A_4,\bar{A}_1,\bar{A}_5\}-\{A_1,A_3,\bar{A}_1,\bar{A}_4,\bar{A}_5\}-\{A_1,A_3,\bar{A}_1,\bar{A}_3,\bar{A}_4\} = \bar{v}_2' \\ \searrow \{A_1,A_2,A_3,\bar{A}_1,\bar{A}_2\}-\{A_1,A_3,A_5,\bar{A}_1,\bar{A}_2\}-\{A_1,A_3,A_5,\bar{A}_1,\bar{A}_3\} - \bar{v}_2' \end{array}
$$

Collecting all the vertices of the four 2-dimensional faces considered above it is not difficult to verify that they form a 3-dimensional abstract polytope which has the structure described in Figure 2. $\square$

<u>Remark</u>. Klee and Walkup [1] name the property that every path from a member of W to a member of $\bar{W}$ has a length of at most 3, as "Property A". They show that every 3-dimensional simple polytope with 8 facets satisfies property A. We have shown that all 3-dimensional abstract polytopes with 8 facets also satisfy property A except one, namely the one with structure given in Figure 2.

<u>Proof of Theorem 3, Part (iv) for $d = 5$</u>: It is obvious that (iv) holds if and only if there exists a path of length 5 from v_0 to $\bar{v}_0$, i.e., if, and only if, there exists a path of length 3 from a neighbor of v_0 to a neighbor of $\bar{v}_0$.

Suppose (iv) does not hold, then by Theorem 5-b every 3-dimensional face of P which is generated by a member of $N(v_0)$ and a member of $N(\bar{v}_0)$ has the structure of Figure 2 after relabeling. In particular let $v_1 = \{A_1,A_2,A_3,A_4,\bar{A}_1\}$, $v_2 = \{A_1,A_2,A_3,\bar{A}_1,\bar{A}_2\}$, $\bar{v}_1 = \{A_1,\bar{A}_1,\bar{A}_2,\bar{A}_3,\bar{A}_4\}$ and $P' = F(v_1 \cap \bar{v}_1)$ has the form of Figure 2. Consider $v_0 = \{A_1,A_2,A_3,A_4,A_5\}$ and its incident edge generated by $\{A_1,A_3,A_4,A_5\}$. The other vertex incident to this edge cannot be $\{A_1,A_3,A_4,A_5,\bar{A}_i\}$ where $i = 1,2,3,4$ since this would imply that there is a path of length 5 from v_0 to $\bar{v}_0$ via that edge and one of the following four vertices of P':

$\{A_1,A_3,A_5,\bar{A}_1,\bar{A}_3\}$; $\{A_1,A_4,A_5,\bar{A}_1,\bar{A}_3\}$; $\{A_1,A_3,A_5,\bar{A}_1,\bar{A}_2\}$ and $\{A_1,A_4,A_5,\bar{A}_1,\bar{A}_4\}$

Hence $\{A_1,A_3,A_4,A_5,\bar{A}_5\}$ is a vertex adjacent to v_0.

Similar arguments lead to the conclusion that either $\{A_1,A_2,A_4,A_5,\bar{A}_5\}$ or $\{A_1,A_2,A_4,A_5,\bar{A}_2\}$ is the other vertex (beside v_0) which is incident to the edge generated by $\{A_1,A_2,A_4,A_5\}$.

The same argument with respect to $\bar{v}_0$, the edge generated by $\{\bar{A}_1,\bar{A}_2,\bar{A}_4,A_5\}$ and vertices $\{A_1,A_4,\bar{A}_1,\bar{A}_2,\bar{A}_5\}$; $\{A_1,A_3,\bar{A}_1,\bar{A}_4,\bar{A}_5\}$; $\{A_1,A_2,\bar{A}_1,\bar{A}_2,\bar{A}_5\}$; $\{A_1,A_2,\bar{A}_1,\bar{A}_4,\bar{A}_5\}$ in P' implies that $\{A_5,\bar{A}_1,\bar{A}_2,\bar{A}_4,\bar{A}_5\} \in N_P(\bar{v}_0)$. Consider now vertices $\{A_1,A_3,A_4,A_5,\bar{A}_5\}$ and $\{A_5,\bar{A}_1,\bar{A}_2,\bar{A}_4,\bar{A}_5\}$, by Theorem 5 in order not to have a path of length 3 joining these two vertices the face of their intersection must have the structure of Figure 2. But note in Figure 2 that the three neighbors of v_1 have the property that no two are neighbors. This rules out the possibility that $\{A_1,A_2,A_4,A_5,\bar{A}_5\}$ is a vertex as it would lie in the face and would be a neighbor of $\{A_1,A_3,A_4,A_5,\bar{A}_5\}$. Hence if (iv) does not hold, $\{A_1,A_2,A_4,A_5,\bar{A}_2\} \in N(v_0)$.

Let us now consider the face $F(\{A_1,A_2,A_4,A_5,\bar{A}_2\} \cap \{A_1,\bar{A}_1,\bar{A}_2,\bar{A}_3,\bar{A}_4\}) = F(\{A_1,\bar{A}_2\})$. It contains the non-empty 3-dimensional face $F(\{A_1,A_2,A_3,\bar{A}_1,\bar{A}_2\} \cap \{A_1,\bar{A}_1,\bar{A}_2,\bar{A}_3,\bar{A}_4\}) = F(\{A_1,\bar{A}_1,\bar{A}_2\})$. By Theorem 5, under the assumption that (iv) does not hold, $F(\{A_1,\bar{A}_2\})$ must have the structure of Figure 2. Note that $F(\{A_1,\bar{A}_1,\bar{A}_2\})$ also lies in $F(v_1 \cap \bar{v}_1)$ and has 7 vertices (shown connected by heavy arcs in Figure 2). But all two dimensional faces with seven vertices of the abstract polytope given in Figure 2 have the property that one of its vertices is adjacent to v_1 in P' and thus analogously to $\{A_1,A_2,A_4,A_5,\bar{A}_2\}$ in $F(\{A_1,\bar{A}_2\})$, but in fact none are, a contradiction. So (iv) must hold. $\square$

The last part of Theorem 3-(iv), for $d \geq 6$, will be proved via Theorems 6 and 7.

<u>Theorem 6</u>. Let $P \in \mathcal{P}(2d,d)$ and let $\{v_0,v_1,v_2\}$, $\{\bar{v}_0,\bar{v}_1\}$ be two paths in P such that $|v_i \cap \bar{v}_j| = i+j$. Let $W = N(v_0) \cap F(v_1 \cap \bar{v}_1)$; then if $d \geq 6$ and $W \geq 2$ there exists $v_2' \in N^2(v_0)$ and $\bar{v}_2' \in N^2(\bar{v}_0)$ such that $|v_2' \cap \bar{v}_2'| = 4$.

<u>Proof</u>. By relabeling let $v_0 = \{A_1,\ldots,A_d\}$; $v_1 = \{A_1,\ldots,A_{d-1},\bar{A}_1\}$; $v_2 = \{A_1,\ldots,A_{d-2},\bar{A}_1,\bar{A}_2\}$; $\bar{v}_0 = \{\bar{A}_1,\ldots,\bar{A}_d\}$, and $\bar{v}_1 = \{A_1,\bar{A}_1,\ldots,\bar{A}_{d-1}\}$. Define $P' = F(v_1 \cap \bar{v}_1)$. (Thus $W = N(v_0) \cap P'$); $P'' = F(v_2 \cap \bar{v}_1)$; $\bar{Z} = N(\bar{v}_0) \cap P''$ and $\bar{U}_i = \{v \in N_{P''}(\bar{Z}) | A_i \subset v\}$ $(i = 2,\ldots,d)$. The proof of Theorem 6 obviously is a result of the following Lemma [part (b4)].

<u>Lemma</u>.

(a) $\bar{U}_2, \ldots ; \bar{U}_d$ partitions $N_{P''}(\bar{Z})$.

(b) Either there exist $v_2' \in N^2(v_0)$ and $\bar{v}_2' \in N^2(\bar{v}_0)$ such that $|v_2' \cap \bar{v}_2'| = 4$ or:

(b1) $|\bar{U}_i| = 0$ for $i = 2,\ldots,d-2$

(b2) $|\bar{U}_{d-1}| = 1$

(b3) $|\bar{U}_d| \geq d-4$

(b4) $|W| = 1$ for $d \geq 6$.

<u>Proof of the Lemma</u>.

(a) Since every vertex of P'' contains A_1 and since every vertex of $N_{P''}(\bar{Z})$ contains exactly two non-barred symbols, obviously:

$$N_{P''}(\bar{Z}) = \bigcup_{i=2}^{d} \bar{U}_i \quad \text{and} \quad \bar{U}_i \cap \bar{U}_j = \emptyset \qquad \text{for } i,j = 2, \ldots, d,\ i \neq j.$$

(b1) Assume $\bar{U}_{i_0} \neq \emptyset$ and $\bar{v}_2' \in \bar{U}_{i_0}$ for some i_0, $2 \leq i_0 \leq d-2$, then $\{A_1,A_{i_0},\bar{A}_1,\bar{A}_2\} \subset \bar{v}_2'$. Hence $|v_2 \cap \bar{v}_2'| = 4$. Moreover, $\bar{v}_2' \in N_{P''}(\bar{Z}) \subset N^2(\bar{v}_0)$.

(b2) By Theorem 1 there exists an A_d—avoiding path joining v_2 to $\bar{v}_1$ in P''. Since $\bar{v}_1 \in \bar{Z}$ and $v_2 \notin \bar{Z}$ this path must intersect $N_{P''}(\bar{Z})$. Thus, there exists a vertex $\bar{v}_2' \in N_{P''}(\bar{Z}) \subset N^2(\bar{v}_0)$ which does not contain A_d implying at least one $|\bar{U}_i| \neq 0$ for $i = 2, \ldots, d-1$. So by (b1) either there exists $\bar{v}_2' \in N_p^2(\bar{v}_0)$ such that $|v_2 \cap \bar{v}_2'| = 4$ or $|\bar{U}_{d-1}| > 0$.

Assume now that $|\bar{U}_{d-1}| \geq 2$ and let $\bar{v}_2', \bar{v}_2'' \in \bar{U}_{d-1}$. Since $\bar{v}_2'$ and $\bar{v}_2''$ both contain A_1, A_{d-1} and $\bar{A}_1$ we have $|\bar{v}_1 \cap \bar{v}_2 \cap \bar{v}_2'| = 3$. Furthermore $\bar{v}_2', \bar{v}_2'' \in N_{P''}(\bar{Z}) \subset N^2(\bar{v}_0)$, so by Theorem 4 there exists $v_2' \in N^2(v_0)$ such that either $|v_2' \cap \bar{v}_2'| = 4$ or $|v_2' \cap \bar{v}_2''| = 4$.

Thus we conclude that either $|\bar{U}_{d-1}| = 1$ or there exists $v_2' \in N^2(v_0)$ and $\bar{v}_2' \in N^2(\bar{v}_0)$ such that $|v_2' \cap \bar{v}_2'| = 4$.

(b3) Suppose $|\bar{Z}| = k$. Note that $k \geq 1$ because $\bar{v}_1 \in \bar{Z}$. The vertices of $\bar{Z}$ have the form $\{A_1, \bar{A}_1, \ldots, \bar{A}_d\} - \{\bar{A}_i\}$, $i \in R$, where R is a subset of k indices of $\{3, \ldots, d\}$.

By the second axiom of abstract polytopes the subset $\{A_1, \bar{A}_1, \ldots, \bar{A}_d\} - \{\bar{A}_i, \bar{A}_j\}$, $(i \in R,\ j \in \{3, \ldots, d\},\ j \notin R)$ is contained by two vertices of P''. Thus every vertex of $\bar{Z}$ gives rise to $d-2-k$ distinct vertices in $N_{P''}(\bar{Z})$. Therefore $|N_{P''}(\bar{Z})| = k(d-2-k)$. Hence by (a), (b1) and (b2), either there exists $v_2' \in N^2(v_0)$, $\bar{v}_2' \in N^2(\bar{v}_0)$ such that $|v_2' \cap \bar{v}_2'| = 4$ or

$$|N_{P''}(\bar{Z})| = |\bar{U}_d| + 1 = k(d-2-k)\ .$$

The last expression implies that

$$0 < k < d-2 \qquad \text{and} \qquad |\bar{U}_d| \geq d-4 .$$

(b4) Finally let us assume that $d \geq 6$ and $|W| \geq 2$ and let $v_1' \in W$ be distinct from v_1. Thus $\{A_1, A_d, \bar{A}_1\} \subset v_1'$. By (b3) either there exists $v_2' \in N^2(v_0)$ and $\bar{v}_2' \in N^2(\bar{v}_0)$ such that $|v_2' \cap \bar{v}_2'| = 4$ or $|\bar{U}_d| \geq d-4 \geq 2$ for $d \geq 6$. Accordingly let $\bar{v}_2', \bar{v}_2'' \in \bar{U}_d$. Since both $\bar{v}_2'$ and $\bar{v}_2''$ contain $\{A_1, A_d, \bar{A}_1\}$ we have $|v_1' \cap \bar{v}_2' \cap \bar{v}_2''| = 3$ which by Theorem 4 implies either $|v_2 \cap v_2'| = 4$ or $|v_2 \cap \bar{v}_2''| = 4$, which completes the proof of the Lemma.

□

<u>Theorem 7</u>: $d \geq 6$, $k = 2$, $\bar{k} = 2$ holds for Theorem 3, if there exist $v_1', v_1'' \subset N(v_0)$ and $\bar{v}_1 \subset N(\bar{v}_0)$ such that $|v_1' \cap v_1'' \cap \bar{v}_1| = 2$.

<u>Proof</u>. Without loss of generality we can assume that $v_0 = \{A_1, \ldots, A_d\}$; $v_1 = \{A_1, \ldots, A_{d-1}, \bar{A}_1\}$; $v_1' = \{A_1, \ldots, A_{d-2}, A_d, \bar{A}_1\}$; $\bar{v}_1 = \{A_1, \bar{A}_1, \ldots, \bar{A}_{d-1}\}$; $\bar{v}_0 = \{\bar{A}_1, \ldots, \bar{A}_d\}$.

We wish to show that if $d \geq 6$ then there exist $v_2' \in N_P^2(v_0)$ and $\bar{v}_2' \in N_P^2(\bar{v}_0)$ such that $|v_2' \cap \bar{v}_2'| = 4$.

By Theorem 1 there exists an $\bar{A}_d$— avoiding path from v_1 to $\bar{v}_1$ in $P' = F(v_1 \cap \bar{v}_1)$. This path intersects $N_{P'}(W)$ at, say, v_2. In this case $v_0, v_1, v_2, \bar{v}_0, \bar{v}_1$ satisfy the conditions of Theorem 6. Moreover, $|W| \geq 2$ since $v_1, v_1' \subset W$ and $d \geq 6$ so that by Theorem 6 there exist $v_2' \in N_P^2(v_0)$ and $\bar{v}_2' \in N_P^2(\bar{v}_0)$ such that $|v_2' \cap \bar{v}_2'| = 4$.

□

Proof of Theorem 3 Part (iv) for $d \geq 6$: By Theorem 3 (iii) we can assume the existence of paths (v_0, v_1, v_2); $(\bar{v}_0, \bar{v}_1)$ such that $|v_i \cap \bar{v}_j| = i+j$. Without loss of generality we can assume that

$$v_0 = \{A_1, \ldots, A_d\}; \quad v_1 = \{A_1, \ldots, A_{d-1}, \bar{A}_1\}; \quad v_2 = \{A_1, \ldots, A_{d-2}, \bar{A}_1, \bar{A}_2\} \text{ and}$$

$$\bar{v}_1 = \{A_1, \bar{A}_1, \ldots, \bar{A}_{d-1}\}; \quad \bar{v}_0 = \{\bar{A}_1, \ldots, \bar{A}_d\}.$$

Let us define P', P'', W, $\bar{Z}$ and $\bar{U}_i$ $(i = 2, \ldots, d)$ as in the preceding Lemma . Since we assume that $d \geq 6$ we have, by the Lemma that $|\bar{U}_d| \geq 2$. Let $\bar{v}_2, \bar{v}_2' \in \bar{U}_d$.

If $|\bar{Z}| \geq 2$ then (considering the two vertices in $\bar{Z}$ and v_1) (iv) holds by Theorem 7. If $|\bar{Z}| = 1$, then, necessarily $\bar{v}_2, \bar{v}_2'$ have the form:

$$\bar{v}_2 = \{A_1, A_d, \bar{A}_1, \ldots, \bar{A}_{d-1}\} - \{\bar{A}_i\}; \quad \bar{v}_2' = \{A_1, A_d, \bar{A}_1, \ldots, \bar{A}_{d-1}\} - \{\bar{A}_j\}$$

$$\text{for some } i, j : 3 \leq i, j \leq d-1 \text{ and } i \neq j .$$

Thus by Theorem 4 either (iv) holds or every vertex of $W' = N_P(v_0) \cap F(v_0 \cap \bar{v}_1)$ (except v_1) contains one symbol out of $\bar{A}_i$, $\bar{A}_j$ and $\bar{A}_d$. But since $d \geq 6$, $|W'| = d-1 \geq 5$. Thus, at least two vertices of W', say v_1', v_1'' are adjacent. But then $|v_1' \cap v_1'' \cap \bar{v}_1| = 2$ and (iv) holds by Theorem 7.

□

Corollary 1.

(i) $\Delta_a(2d+k, d) \le \Delta_a(2d+k-1, d-1) + 1$, $k = 0, 1$

(ii) $\Delta_a(2d,d) \le \Delta_a(2d-k, d-k) + k$, $k = 1, 2, 3, 4$.

Proof:

(i) follows from Theorem 3-(i), (ii).

(ii) follows from Theorem 3. $\square$

Note that since every simple polytope satisfies the axioms of an abstract polytope, Theorem 3 holds for simple polytopes and Corollary 1 holds if one replaces $\Delta_a(n,d)$ by $\Delta_b(n,d)$ (the maximum diameter of ordinary polytopes over all d-dimensional polytopes with n facets).

5. Maximum diameters of abstract polytopes and the Hirsch conjecture.

Hirsch conjecture. Corresponding to the Hirsch conjecture of polytopes, Dantzig [4], is the conjecture for abstract polytopes that

$$\Delta_a(n,d) \le n-d \qquad (d \ge 1,\ n \ge d+1)\ .$$

Theorem 8 below is the analog of the results of Klee and Walkup [1] for abstract polytopes (except for $\Delta_a(n,3)$, $n \ge 9$) and is mainly based on Theorem 3.

Theorem 8. The values of $\Delta_a(n,d)$ for $d \leq 2$ and for $n-d \leq 5$ are given in the following table:

d \ $n-d$	1	2	3	4	5
2	"	2	2	3	3
3	"	"	3	3	4
4	"	"	"	4	5
$d \geq 5$	"	"	"	"	5

... $\Delta_a(n,2) = [n/2]$

Table 1: Values of $\Delta_a(n,d)$

(The double quote mark indicates that each column is constant from the main diagonal downwards.)

Proof: As was pointed out in Section 3, $\Delta_a(n,d) \geq \Delta_b(n,d)$. Thus since Table 1 holds for $\Delta_b(n,d)$ (Klee and Walkup [1]), it is sufficient to show that the values in Table 1 are upper bounds for $\Delta_a(n,d)$.

Let $P \in \mathcal{P}(n,d)$.

(a) $2d > n$: By Theorem 2-(iv) each column of Table 1 is constant from the main diagonal downwards.

(b) $d = 2$, $n \geq 4$: Since P is a 2-dimensional abstract polytope, the number of vertices of P is equal to the number of its edges, therefore the graph of P forms a simple cycle with n vertices. Hence $\Delta_a(n,2) = [n/2]$.

(c) $n-d = 3$: By (b) and Corollary 1, $\Delta_a(6,3) = 3$.

(d) $n-d = 4$:

(d1) $n = 7$: If $v \cap \bar{v} \neq \emptyset$ for every pair of vertices $v, \bar{v} \in P$ then $F(v \cap \bar{v})$ is an (n',d')-abstract polytope with $n' \leq 6$ and $d' \leq 2$. Thus, by (b) and (c), $\delta(P) \leq 3$.

Suppose now, that there exist $v, \bar{v} \in P$ such that $v \cap \bar{v} = \emptyset$. Let $UP - \{v \cup \bar{v}\} = A$ then by Theorem 1 there exists an A avoiding path between v and $\bar{v}$. This path intersects $N^2(v)$, say at v_2. Since every vertex in $N^2(v)$ contains two symbols of $UP - v$, v_2 is necessarily adjacent to $\bar{v}$. Hence $\Delta(7,3) \leq 3$.

(d2) $n = 8$: By (d1) and Corollary 1, $\Delta_a(8,4) \leq 4$.

(e) $n-d = 5$:

(e1) $n = 8$: If $v \cap \bar{v} \neq \emptyset$ for all $v, \bar{v} \in P$ then $P' = F(v \cap \bar{v})$ is d'-dimensional face of P where $d' \leq 2$ and $|UP - \{v \cap \bar{v}\}| \leq 7$, thus by (a) and (b), $\delta(P) \leq 3$.

Suppose there exists $v, \bar{v} \in P$ such that $v \cap \bar{v} = \emptyset$. Let $UP = \{A_1,A_2,A_3,A_4,A_5,A_6,A_7,A_8\}$, $v = \{A_1,A_2,A_3\}$ and $\bar{v} = \{A_4,A_5,A_6\}$.

If $\delta(P) > 4$ then every vertex in $N(v)$ and $N(\bar{v})$ contains either A_7 or A_8 (otherwise a vertex in $N(v)$ (or in $N(\bar{v})$) and $\bar{v}$ (or v), both contain the same symbol which, by (iii) implies that $\delta(P) \leq 4$).

Without loss of generality we can assume that

$$N(v) = \{\{A_1,A_2,A_7\};\ \{A_1,A_3,A_7\};\ \{A_2,A_3,A_8\}\}$$

and either

$$N(\bar{v}) = \{\{A_4,A_5,A_7\};\ \{A_4,A_6,A_7\};\ \{A_5,A_6,A_8\}\}$$

or

$$N(\bar{v}) = \{\{A_4,A_5,A_7\};\ \{A_4,A_6,A_8\};\ \{A_5,A_6,A_8\}\}.$$

In the first case the graph of $F(\{A_7\})$ forms a simple cycle with at most 7 vertices while in the second case it forms a simple cycle with at most 6 vertices. (Since in the second case $F(\{A_6\}) = \{\{A_4,A_5,A_6\},\ \{A_4,A_6,A_8\},\ \{A_5,A_6,A_8\}\}$ which implies that $F(\{A_7\}) \cap F(\{A_6\}) = \emptyset$.) In both cases $\delta(P) \leq 4$. Hence, $\Delta_a(8,3) \leq 4$.

(e2) $n = 9$: By (e1) and Corollary 1, $\Delta_a(9,4) \leq 5$.

(e3) $n = 10$: If $v \cap \bar{v} \neq \emptyset$ for all $v, \bar{v} \in P$ then $F(v \cap \bar{v})$ is an (n',d')-abstract polytope with $n' \leq 9$ and $d' \leq 4$. Therefore, by (e2), $\delta(P) \leq 5$. Suppose, now, that there exist $v_0, \bar{v}_0$ in P such that $v_0 \cap \bar{v}_0 = \emptyset$. Without loss of generality we can assume that $v_0 = \{A_1,A_2,A_3,A_4,A_5\}$, $\bar{v}_0 = \{A_1,A_2,A_3,A_4,A_5\}$. Then by Theorem 3-(iv) $\delta(P) = 5$.

$\square$

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