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MAXIMUM LIKELIHOOD ESTIMATION AND HYPOTHESIS TESTING IN THE BIVARIATE EXPONENTIAL MODEL OF MARSHALL AND OLKIN

By

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1. INTRODUCTION AND SUMMARY

From reliability considerations, Marshall and Olkin [6] formulated a multivariate analog of the exponential distribution as a realistic model for a system where the component life times may be dependent due to shocks affecting two or more components simultaneously. In their bivariate model, two components A and B in a system are subject to three types of shocks which occur independently according to Poisson processes with intensity parameters λ_1, λ_2 and δ respectively. The first (second) type of shock affects only the component A (B) while the third type of shock causes the failure of both A and B so that their life times Y_1 and Y_2 will be dependent when $\delta>0$. It is shown in [6] that the joint distribution of (Y_1,Y_2) has the right hand cdf

$$\bar{F}(y_1,y_2) = P(Y_1>y_1,Y_2>y_2) = \exp[-\lambda_1 y_1 - \lambda_2 y_2 - \delta\max(y_1,y_2)]$$

$$\text{for } y_1,y_2 > 0. \tag{1.1}$$

Some properties of this distribution including the moment generating function, the distribution of $\min(Y_1,Y_2)$, etc. are studied in [6] and a natural extension to higher dimensions is also presented.

Although developed from a Poisson shock model analogously to the univariate exponential distribution, an analytical treatment of this bivariate exponential distribution is made difficult by the existence of a component which is singular with respect to the two-dimensional Lebesgue measure. Several authors [1], [2], [4] have mentioned this difficulty particularly in the context of maximum likelihood estimation. Problems of parameter estimation and testing certain hypotheses, based on a random sample $\underset{\sim}{Y}_i = (Y_{1i}, Y_{2i})$, $i=1,\ldots,n$ from (1.1), have been considered by Arnold [1] and George [4]. Due to the difficulty of explicitly writing out the likelihood function, both authors start with an initial reduction of the data to

$$
\begin{aligned}
T_1 &= \sum_{i=1}^{n} \min(Y_{1i}, Y_{2i}), \quad N_1 = \#(Y_{1i} > Y_{2i}) \ , \\
N_2 &= \#(Y_{1i} > Y_{2i}), \quad N_0 = \#(Y_{1i} = Y_{2i})
\end{aligned}
\tag{1.2}
$$

where the symbol # ("•") denotes the number of vectors $\underset{\sim}{Y}_i$ satisfying the statement "•". The counts (N_0, N_1, N_2) have a multinomial distribution and are independent of T_1 which has a gamma distribution. Based on this fact, unbiased estimators of λ_1, λ_2 and δ are obtained in [1] and likelihood ratio tests are formulated in [4]. However, aside from having a convenient distribution, (N_0, N_1, N_2, T_1) does not constitute a sufficient for (1.1) and therefore this reduction involves some loss of information. In this paper, we study procedures based on the complete random sample and its reduction to sufficient statistics.

A mixture of one- and two-dimensional Lebesgue measures is used in Section 2 as a dominating measure which leads to a joint density function and minimal sufficient statistics. The distributional properties of the sufficient statistics are studied and the lack of completeness is demonstrated for the case of identical marginals. Maximum likelihood estimators (MLE) are investigated in Section 3. It is shown that the MLE is realized as the unique root of the likelihood equation except in a subset of the sample space where it does not exist or is not unique. The probability of this set however tends to zero with increasing sample size. The structure of the MLE is compared with the simple unbiased estimators proposed by Arnold [1] and the asymptotic distribution of both estimators are obtained. Using the criterion of generalized variance, the asymptotic efficiency of the unbiased estimators relative to the MLE is derived and bounds of this efficiency are studied.

A case of particular interest in the model (1.1) is the one having identical marginal distributions, that is $\lambda_1=\lambda_2$. This model fits real life situations where the components which are connected in parallel in a system are similar in nature and are likely to experience the same sort of shocks component-wise in addition to occasionally being simultaneously affected by some catastrophic shocks. Since independence of life times introduces substantial simplification in system reliability studies, in Section 4 we consider the problem of testing for independence ($\delta=0$) in this model. Using the concept of a least favorable distribution, the uniformly most powerful (UMP) test for independence is derived in a convenient form. The proof indicates a strong optimality of some other tests in reliability studies. For instance, when testing for the equality of scale parameters in two exponential distributions against one sided alternatives, the usual F test is UMP rather than just UMP unbiased and the same property holds even for censored samples.

2. LIKELIHOOD, SUFFICIENCY AND COMPLETENESS

Let (Y_1,Y_2) have the bivariate exponential cdf given by (1.1) and denote this distribution by BVE $(\lambda_1,\lambda_2,\delta)$ where the parameter space is $\Omega = \{(\lambda_1,\lambda_2,\delta): 0<\lambda_i<\infty,\ i=1,2;\ 0\le\delta<\infty\}$. In order to obtain its probability density function (pdf), we consider the Lebesgue measure μ_2 on $(R_2^+,\ \ )$ where R_2^+ is the positive orthant of the (y_1,y_2) plane and is the corresponding Borel σ-field. In order to handle the singular component, we define another measure ν on $(R_2^+, \quad)$ as follows: let $C_0=\{(x,x): 0<x<\infty\}$ be the diagonal line in R_2^+ and for Borel sets B $\quad R_2^+$, set $\nu(B) = \mu_1(\{x: (x,x)\varepsilon B \quad C_0\})$ where μ_1 is the Lebesgue measure on the real line. ν is a σ-finite measure on $(R_2^+, \quad)$ and is singular with respect to μ_2. Finally, we let $\mu=\mu_2+\nu$ on $(R_2^+, \quad)$ and note that the probability measure determined by (1.1) is absolutely continuous with respect to the measure μ.

Let $C_1 = \{(y_1,y_2): 0<y_1<y_2<\infty\}$ and $C_2 = \{(y_1,y_2): 0<y_2<y_1<\infty\}$ be the subsets of R_2^+ which are above and below the diagonal respectively so that $R_2^+ = \bigcup_{\alpha=0}^{2} C_\alpha$. Then, from the properties of the distribution (1.1) discussed in [6], we observe that a determination of the pdf of (Y_1,Y_2), with respect to μ, is given by

$$f(y_1,y_2) = \Sigma_{\alpha=0}^{2} f_\alpha(y_1,y_2) I_{C_\alpha}(y_1,y_2) \tag{2.1}$$

where

$$\begin{aligned} f_0(y_1,y_2) &= \delta \exp[-(\lambda_1 + \lambda_2 + \delta)y_1] \\ f_1(y_1,y_2) &= \lambda_1(\lambda_2+\delta)\exp[-\lambda_1 y_1 - \lambda_2 y_2 - \delta y_2] \\ f_2(y_1,y_2) &= \lambda_2(\lambda_1+\delta)\exp[-\lambda_1 y_1 - \lambda_2 y_2 - \delta y_1] \end{aligned} \tag{2.2}$$

and I is the indicator of the set appearing in its suffix. The joint pdf of the random sample $\underset{\sim}{Y}_i = (Y_{1i}, Y_{2i})$, $i=1,\ldots,n$ is then the product $\prod_{i=1}^{n} f(y_{1i}, y_{2i})$ where f is defined by (2.1) and (2.2). To simplify the expression, we introduce the following notations:

$$n_\alpha = \Sigma_{i=1}^{n} I_{C_\alpha}(y_{1i}, y_{2i}),\ \alpha=0,1,2,\quad s_\alpha = \Sigma_{i=1}^{n} y_{\alpha i},\ \alpha=1,2$$

$$w_{1i} = \min(y_{1i}, y_{2i}),\quad w_{2i} = \max(y_{1i}, y_{2i}),\quad i=1,\ldots,n \tag{2.3}$$

$$t_1 = \sum_{i=1}^{n} w_{1i},\quad t_2 = \sum_{i=1}^{n} w_{2i},\quad v = t_2 - t_1.$$

Thus n_0, n_1 and n_2 are, respectively, the number of points which are on, above and below the diagonal line so that $n_0+n_1+n_2 = n$. Also we have $s_1+s_2 = t_1+t_2$. Noting that, for every point (y_1, y_2) in R_2^+, exactly one term in the r.h.s. of (2.1) is non-zero, the likelihood function simplifies to the form

$$\begin{aligned} \ell(\lambda_1, \lambda_2, \delta) &= \prod_{i=1}^{n} f(y_{1i}, y_{2i}) = \prod_{\alpha=0}^{2} \prod_{\underset{\sim}{y}_i \varepsilon C_\alpha} f_\alpha(y_{1i}, y_{2i}) \\ &= [\lambda_1(\lambda_2+\delta)]^{n_1} [\lambda_2(\lambda_1+\delta)]^{n_2} \delta^{n_0} \exp[-\lambda_1 s_1 - \lambda_2 s_2 - \delta t_2] \\ &\qquad \text{for } (\lambda_1, \lambda_2, \delta)\ \varepsilon\ \Omega. \end{aligned} \tag{2.4}$$

For the case $\delta=0$, the above functional form holds provided we interprete $0^0=1$. The likelihood in this case is then 0 if $n_0>0$ and $(\lambda_1\lambda_2)^n\exp[-\lambda_1 s_1 - \lambda_2 s_2]$ if $n_0=0$.

From the factorization criterion, it follows that a set of sufficient statistics is given by (N_1,N_2,S_1,T_1,T_2) or, equivalently, by (N_1,N_2,S_1,T_1,V) where the components are defined in (2.3) using small letters. The minimality of this sufficient statistics follows from the usual partitioning operation of the sample space (c.f. [8], p. 50). For the subfamily of (1.1) with $\delta=0$, a minimal sufficient statistic is (S_1,S_2) since $N_2=0$ with probability 1. This is also clear from the fact that, in this subfamily, Y_1 and Y_2 are independent and exponentially distributed.

In the special subfamily of (1.1) having identical marginals, we denote the common parameter $\lambda_1=\lambda_2$ by β and the parameter space by $\Omega_1 = \{(\beta,\delta)\colon 0<\beta<\infty,\ 0\leq\delta<\infty\}$. The likelihood function is then given by

$$\ell^*(\beta,\delta) = [\beta(\beta+\delta)]^{n-n_0}\delta^{n_0}\exp[-\beta(t_1+t_2) - \delta t_2],$$

$$(\beta,\delta)\in\Omega_1 \tag{2.5}$$

and (N_0,T_1,V) constitutes a set of minimal sufficient statistics.

We now list some distributional properties of this sufficient statistics for future reference, particularly for

Section 4 where we consider hypothesis testing in this subfamily. For abbreviation we shall write "X is $G(n,\theta)$" to mean that X has a gamma distribution with p.d.f. $\propto \exp[-\theta x]x^{n-1}$, $0<x<\infty$ and the corresponding cdf will be denoted by $G(x;\ n,\theta)$.

Theorem 2.1. *Let $\underset{\sim}{Y}_i$, $i=1,\ldots,n$ be a random sample from* $BVE(\beta,\beta,\delta)$ *and let* N_0, T_1, V *be defined as in* (2.3), *then the following hold*:

(a) T_1 *is* $G(n,\ 2\beta+\delta)$ *and is independent of* (N_0,V)

(b) N_0 *and* V *jointly have the mixed distribution given by*

$$\begin{aligned} P(N_0=k,\ V<v) &= \binom{n}{k}p^k(1-p)^{n-k}G(v;\ n-k,\beta+\delta) && \text{if } k=0,\ldots,n-1 \\ &= p^n && \text{if } k=n \end{aligned} \tag{2.6}$$

where $0<v<\infty$, *and* $p=\delta/(2\beta+\delta)$ *is the probability mass on the diagonal for a single observation*.

(c) *Conditionally given* $n_0=0$, V *is* $G(n,\beta+\delta)$.

The property (a) holds even in the general case of non-identical marginals in which case $2\beta+\delta$ is to be replaced by $\lambda_1+\lambda_2+\delta$. The distribution of T_1 has already been noted in [6]. Independence of T_1 and (N_0,V) can be verified by using (2.1) and (2.2) to write out the joint pdf of W_{1i} and $V_i=W_{2i}-W_{1i}$ and then factoring the pdf of W_{1i} and V_i. Since

$T_1 = \Sigma W_{1i}$, $V = \Sigma V_i$ and $N_0 = \#(V_i = 0)$, the result follows. To establish (b), one needs to write $P(N_0=k, V<v) = P(N_0=k)P(V<v|N_0=k)$ and note that when $k=n$, V has the constant value 0 and for any $k<n$, V is the sum of $(n-k)$ terms $\sum_{j=1}^{n-k} V_{i_j}$ where $V_1,\ldots,V_n$ are i.i.d. $G(1,\beta+\delta)$ and $(i_1,\ldots,i_n)$ is a permutation of the integers $(1,\ldots,n)$. (c) follows immediately from (b).

The following moments are obtained from the distributions stated in Theorem 2.1 using the properties of gamma distribution.

$$
\begin{aligned}
E(T_1) &= n(2\beta+\delta)^{-1} \\
E(N_0 T_1) &= n^2\delta(2\beta+\delta)^{-2} \\
E(V) &= E[E(V|N_0)] = 2n\beta(\beta+\delta)^{-1}(2\beta+\delta)^{-1} \\
E(VN_0) &= 2n(n-1)\beta\delta(\beta+\delta)^{-1}(2\beta+\delta)^{-2}.
\end{aligned}
\tag{2.7}
$$

For the family $BVE(\delta,\delta,\beta)$, we note that although the parameter space is two-dimensional, the minimal sufficient statistic obtained above has three components. To prove that the sufficient statistic is not complete we consider the statistic

$$T^* = V(n+N_0-1)[4n(n-1)]^{-1} - T_1(n-N_0)[2n^2]^{-1}. \tag{2.8}$$

By using the moments in (2.7) it is easy to verify that each of the two terms in the r.h.s. of (2.8) is an unbiased estimator of $\beta(2\beta+\delta)^{-2}$ so that T^* is an unbiased estimator of 0. Since T^* is not identically 0, the statistics (N_0, T_1, V) is not complete. Although the unbiased estimators constructed by Arnold [1] are functions of N_0 and T_1, the lack of completeness prevents one from concluding that these have minimum variance. In the general model $BVE(\lambda_1, \lambda_2, \delta)$ where the parameter space is three dimensional, the minimal sufficient statistics have five components. It is unlikely that these sufficient statistics are complete but we have not been able to prove it.

3. MAXIMUM LIKELIHOOD ESTIMATION

This section is devoted to the derivation and study of the asymptotic properties of the MLE for the parameters of the general model $BVE(\lambda_1, \lambda_2, \delta)$ as well as for the special model having identical marginals. The use of the dominating measure μ, introduced in Section 2, permits an explicit functional representation of the likelihood whose properties are readily studied. The investigation brings out some interesting facts about the model with regard to the existence and uniqueness of the MLE. We give the details for the general model and only summarize the results for the case of identical marginals.

Let $B=B_1 \quad B_2$ denote the boundary of the parameter space Ω of the general model BVE$(\lambda_1,\lambda_2,\delta)$ where $B_1 = [\delta=0,\ \lambda_1>0,\ \lambda_2>0]$ and $B_2 = [\lambda_1=0] \quad [\lambda_2=0]$. Note that $B_1 \quad \Omega$ whereas B_2 is disjoint from Ω, although it is in the closure of Ω. Using (2.4), the likelihood function is given by

$$\ell(\lambda_1,\lambda_2,\delta) = [\lambda_1(\lambda_2+\delta)]^{n_1}[\lambda_2(\lambda_1+\delta)]^{n_2}\delta^{n_0}\exp[-\lambda_1 s_1 - \lambda_2 s_2 - \delta t_2],$$

$$\text{on } \Omega-B_1$$

$$= [\lambda_1\lambda_2]^n \exp[-\lambda_1 s_1 - \lambda_2 s_2] I_{[n_0=0]},\ \text{on } B_1. \tag{3.1}$$

Equating the first partial derivatives of $\log\ell(\lambda_1,\lambda_2,\delta)$ on $\Omega-B_1$ to zero, we obtain the likelihood equations

$$\begin{aligned} n_1\lambda_1^{-1} + n_2(\lambda_1+\delta)^{-1} &= s_1 \\ n_1(\lambda_2+\delta)^{-1} + n_2\lambda_2^{-1} &= s_2 \\ n_1(\lambda_2+\delta)^{-1} + n_2(\lambda_1+\delta)^{-1} + n_0\delta^{-1} &= t_2, \end{aligned} \tag{3.2}$$

and the matrix Q of the second partial derivatives is given by

$$-Q = \begin{pmatrix} \frac{n_1}{\lambda_1^2} + \frac{n_2}{(\lambda_1+\delta)^2} & 0 & \frac{n_2}{(\lambda_1+\delta)^2} \\ 0 & \frac{n_2}{\lambda_2^2} + \frac{n_1}{(\lambda_2+\delta)^2} & \frac{n_1}{(\lambda_2+\delta)^2} \\ \frac{n_2}{(\lambda_1+\delta)^2} & \frac{n_1}{(\lambda_2+\delta)^2} & \frac{n_1}{(\lambda_2+\delta)^2} + \frac{n_2}{(\lambda_1+\delta)^2} + \frac{n_0}{\delta^2} \end{pmatrix} \quad (3.3)$$

The existence and uniqueness properties of MLE are given in the following theorem.

Theorem 3.1.

(i) *If n_0, n_1, n_2 are all non-zero, the MLE of $(\lambda_1, \lambda_2, \delta)$ is unique and is the unique root, belonging to $\Omega - B_1$, of the set of equations* (3.2).

(ii) *If $n_0=0$, $n_1>0$ and $n_2>0$, the unique MLE is given by* $\hat{\delta}=0$, $\hat{\lambda}_1 = n/s_1$, $\hat{\lambda}_2 = n/s_2$.

(iii) *If $n_0=0$ and either $n_1=0$ or $n_2=0$, the MLE exists but is not unique.*

(iv) *If $n_0>0$ and one or both of n_1 and n_2 are 0, the MLE does not exist in the sense that the supremum of the likelihood is not attained within the parameter space* Ω.

Proof. (i) We first note that when $n_i>0$, $i=0,1,2$, the diagonal matrix $D = \mathrm{diag}[n_1\lambda_1^{-2}, n_2\lambda_2^{-2}, n_0\delta^{-2}]$ is positive definite for all $(\lambda_1,\lambda_2,\delta) \in \Omega-B_1$ and also it is easy to verify that the matrix $-(Q+D)$ is positive semi-definite. It follows that when $(\lambda_1,\lambda_2,\delta) \in \Omega-B_1$ and all $n_i>0$, the matrix Q is negative definite. Thus log ℓ is a strictly concave function on $\Omega-B_1$. Also, $\ell=0$ on B_1 and $\ell\to 0$ as the argument $(\lambda_1,\lambda_2,\delta)$ approaches any point on the boundary B or tends to infinity in any component. Hence ℓ has a unique maximum within $\Omega-B_1$ and the maximum is attained at the root of (3.2).

(ii) When $n_0=0$, $n_1>0$, $n_2>0$ and $(\lambda_1,\lambda_2,\delta) \in \Omega-B_1$, Q is again negative definite so that log ℓ is strictly concave on $\Omega-B_1$. Since ℓ is continuous on B, it has a unique maximum either at an interior point of Ω or on the boundary B. If possible, suppose the maximum occurs at an interior point $(\tilde\lambda_1,\tilde\lambda_2,\tilde\delta)$. Then it must be a root of the equations (3.2). Substituting these values in (3.2) and subtracting the first two equations from the third, we obtain

$$n_1(\tilde\lambda_2+\tilde\delta)^{-1} - n_1\tilde\lambda_1^{-1} = t_2-s_1$$

$$n_2(\tilde\lambda_1+\tilde\delta)^{-1} - n_2\tilde\lambda_2^{-1} = t_2-s_2.$$

Since $t_2-s_1 \geq 0$ and $t_2-s_2 \geq 0$, we have $(\tilde\lambda_2+\tilde\delta)^{-1} \geq (\tilde\lambda_1+\tilde\delta)^{-1} \geq \tilde\lambda_2^{-1}$. But $(\tilde\lambda_1,\tilde\lambda_2,\tilde\delta)$, being an interior point of Ω, satisfies $0<\tilde\lambda_2<\tilde\lambda_2+\tilde\delta$; and we reach a contradiction. Therefore $\sup_{\Omega}\ell = \sup_{B_1}\ell$

and it follows from (3.1) that the sup on B_1 is attained at $\hat{\lambda}_1=n/s_1$, $\hat{\lambda}_2=n/s_2$.

(iii) Consider the case $n_0=n_1=0$, $n_2=n$ so that $s_1=t_2$ and $s_2=t_1$. Then we have

$$\ell(\lambda_1,\lambda_2,\delta) = [\lambda_2(\lambda_1+\delta)]^n \exp[-(\lambda_1+\delta)t_2 - \lambda_2 t_1] \text{ on } \Omega.$$

It is easy to see that ℓ is maximized at every point $(\hat{\lambda}_1,\hat{\lambda}_2,\hat{\delta})$ in Ω satisfying $\hat{\lambda}_2=n/t_1$ and $\hat{\lambda}_1+\hat{\delta}=n/t_2$. The MLE exists but it is not unique as far as λ_1 and δ are concerned. The case $n_0=n_2=0$, $n_1=n$ is entirely symmetric.

(iv) Let $n_0>0$, $n_2>0$ and $n_1=0$, $\log\ell$ is again strictly concave on $\Omega-B_1$. If (3.2) has a solution $(\tilde{\lambda}_1,\tilde{\lambda}_2,\tilde{\delta}) \in \Omega-B_1$, we have $n_0/\tilde{\delta}=0$ as can be seen from the first and third equations after noting that $s_1=t_2$ in this case. This is a contradiction. Hence $\sup\ell$ is not attained at any interior point of Ω, and on the boundary B_1 we have $\ell=0$. The MLE does not exist. However, if Ω is extended to include B_2, then $\sup\ell$ is attained on $\tilde{\Omega} = \Omega \quad B_2$ at the point $\hat{\lambda}_1=0$, $\hat{\lambda}_2=n_2/t_1$, $\hat{\delta}=n/t_2$ and, by convention, we may take this to be the MLE.

The other situation to be considered is when $n_0=n$ which implies $s_1=s_2=t_1=t_2$ and hence

$$\ell(\lambda_1,\lambda_2,\delta) = \delta^n \exp[-(\lambda_1+\lambda_2+\delta)t_1] \quad \text{on } \Omega-B_1$$

$$= 0 \qquad\qquad \text{otherwise.}$$

Supℓ is clearly not attained at any point in Ω although it is attained on the extended set $\tilde{\Omega}$ at the point $\hat{\lambda}_1=0$, $\hat{\lambda}_2=0$, $\hat{\delta}=n/t_1$. This may be taken to be the MLE, by convention. This concludes the proof of the theorem.

With increasing sample size, the probability that $N_1=0$ or $N_2=0$ approaches 0 exponentially and therefore the cases which are important in large samples are (i) and (ii) of the above theorem. For (i), a closed form expression of the MLE could not be obtained due to the non-linear form of the likelihood equations. In application, the estimates must be computed by an iterative procedure. The unbiased estimators of $\lambda_1,\lambda_2,\delta$ proposed by Arnold [1] are based on (N_0,N_1,N_2,T_1) and do not use all the components of the minimal sufficient statistic. In some parts of the sample space, these estimates are close to the MLE while in others they are quite different. The two sets of estimators are presented in Table 1 for comparison. In the first three columns, +(0) means that the corresponding N_i is greater than (equal to) zero. The cases (N_0,N_1,N_2) = (0,0,n) and (+,+,0) are not included in the table because they follow by symmetry from (0,n,0) and (+,0,+) respectively.

TABLE 3. COMPARISON OF MLE AND UNBIASED ESTIMATORS

$c_n = (n-1)/n$

N_0 N_1 N_2	Estimators of $(\lambda_1, \lambda_2, \delta)$	
	MLE	Unbiased
0 + +	$(n/S_1,\ n/S_2,\ 0)$	$c_n(N_1/T_1,\ N_2/T_1,\ 0)$
+ + +	Unique root of (3.2)	$c_n(N_1/T_1,\ N_2/T_1,\ N_0/T_1)$
0 n 0	$(n/T_1,\ n/T_2,\ 0)^*$	$c_n(N_1/T_1,\ 0,\ 0)$
n 0 0	$(0,\ 0,\ n/T_1)$	$c_n(0,\ 0,\ n/T_1)$
+ 0 +	$(0,\ N_2/T_1,\ n/T_2)$	$c_n(0,\ N_2/T_1,\ N_0/T_1)$

*a member in the class of MLE.

In the model BVE (β,β,δ) having identical marginals, the derivation of the MLE is essentially similar and a closed form expression can be obtained. However, even in this simplified model, there is a part of the sample space where the MLE does not exist. We state the findings without proof since the derivations are along the same lines as in Theorem 3.1.

Theorem 3.2. *If* $n_0<n$, *the* MLE of (β,δ) *in the model* $BVE(\beta,\beta,\delta)$ *exists, is unique and is given by* $\hat{\delta}=0$, $\hat{\beta}=2n/(t_1+t_2)$ *if* $n_0=0$ *and by*

$$\hat{\delta} = (2t_1t_2)^{-1}[\{n^2(t_2-t_1)^2+ 4n_0(2n-n_0)t_1t_2\}^{\frac{1}{2}} - n(t_2-t_1)]$$

$$\hat{\beta} = (n-n_0)\hat{\delta}[n_0+\hat{\delta}t_1]^{-1}$$

if $n_0>0$. *If* $n_0=n$, *the MLE does not exist.*

In order to investigate the asymptotic properties of the MLE in the model BVE $(\lambda_1,\lambda_2,\delta)$, we write $\hat{\underset{\sim}{\theta}}_n = (\hat{\lambda}_{1n},\hat{\lambda}_{2n},\hat{\delta}_n)$ for the MLE of $\underset{\sim}{\theta} = (\lambda_1,\lambda_2,\delta)$ where the suffix n indicates the sample size. Also let $\underset{\sim}{N}_n = (N_{0n},N_{1n},N_{2n})$ where we write N_{in} for N_i, i=0,1,2 of the previous sections. We consider first the case when the true $\underset{\sim}{\theta}$ is an interior point of Ω, that is, $\underset{\sim}{\theta}>\underset{\sim}{0}$. For every n, $\underset{\sim}{N}_n$ has the trinomial distribution $TN(n;\underset{\sim}{p})$ where $\underset{\sim}{p} = (p_0,p_1,p_2) = \lambda^{-1}(\delta,\lambda_1,\lambda_2)$ and $\lambda = \lambda_1+\lambda_2+\delta$. By the Borel-Cantelli lemma, almost surely $\underset{\sim}{N}_n>0$ for all but a finite number of n as $n\to\infty$. For all sufficiently large n, $\hat{\underset{\sim}{\theta}}_n$ is the unique root of the likelihood equation (3.2) and hence $\hat{\underset{\sim}{\theta}}_n\to\underset{\sim}{\theta}$ with probability 1 by the strong consistency property of MLE (c.f. Rao [7], p. 300). In the multiparameter case, it is necessary to use the fact that the log likelihood is dominated by an integrable function in some small neighborhood of $\underset{\sim}{\theta}$ in order to use the uniform strong law. Further, the likelihood function (3.1), restricted to $\Omega - B_1$, satisfies the Cramér conditions (c.f. Rao [7], p. 299) for asymptotic normality. The boundedness condition for the third partial derivatives of $\log \ell$ in a neighborhood of θ easily follows as these can be bounded by constants. Hence $n^{\frac{1}{2}}(\hat{\theta}_n - \theta)$ has asymptotically the trivariate normal distribution ${}_3(0,\Sigma)$ where $\Sigma^{-1} = E(-n^{-1}Q)$ is the information matrix and Q is given in (3.3). Letting

$$a = \lambda_2(\lambda_1+\delta)^{-2},\ b = \lambda_1(\lambda_2+\delta)^{-2}$$

$$\Sigma_1 = \lambda \text{ diag } (\lambda_1,\lambda_2,\delta)$$

$$C = \frac{1}{\lambda}\begin{pmatrix} a & 0 & a \\ 0 & b & b \\ a & b & a+b \end{pmatrix} \tag{3.4}$$

and computing $E(Q)$, we obtain

$$\Sigma^{-1} = \Sigma_1^{-1} + C. \tag{3.5}$$

The following lemma provides the limit distribution of the unbiased estimators.

Lemma 3.1. If $\theta \in \Omega - B_1$, the limiting distribution of $n^{\frac{1}{2}}(\theta_n^* - \theta)$, where $\theta_n^* = (n-1)(nT_{1n})^{-1}(N_{1n},N_{2n},N_{0n})$ is trivariate normal ${}_3(0,\Sigma_1)$ with Σ_1 given by (3.4).

Proof. From the properties of $BVE(\lambda_1,\lambda_2,\delta)$, we note that T_{1n} has the distribution $G(n,\lambda)$ and it is independent of $\underset{\sim}{N}_n$ which has a trinomial distribution. Letting $\underset{\sim}{Z}_n = (Z_{1n},Z_{2n},Z_{3n})$ where

$$Z_{in} = (\frac{N_{in}}{n} - p_i),\ i=1,2\ ;\ Z_{3n} = (\frac{n}{T_1} - \lambda), \qquad (3.6)$$

we see that $n^{\frac{1}{2}}\underset{\sim}{Z}_n$ is asymptotically $\quad_3(\underset{\sim}{0},\Gamma)$ where

$$\Gamma = \begin{pmatrix} p_1(1-p_1) & -p_1p_2 & 0 \\ -p_1p_2 & p_2(1-p_2) & 0 \\ 0 & 0 & \lambda^2 \end{pmatrix}. \qquad (3.7)$$

Employing the linear transformation $\underset{\sim}{U}_n = H\underset{\sim}{Z}_n$, where

$$H = \begin{pmatrix} \lambda & 0 & p_1 \\ 0 & \lambda & p_2 \\ -\lambda & -\lambda & p_3 \end{pmatrix} ,$$

it follows that $n^{\frac{1}{2}}\underset{\sim}{U}_n \sim \quad_3(\underset{\sim}{0},H\Gamma H')$. It is easy to check that $\underset{\sim}{V}_n = n^{\frac{1}{2}}(\underset{\sim}{\theta}^*_n - \underset{\sim}{\theta} - \underset{\sim}{U}_n) \xrightarrow{P} \underset{\sim}{0}$. For example, $V_{1n} = n^{\frac{1}{2}}(N_{1n}n^{-1} - p_1)(nT^{-1}_{1n} - \lambda) \xrightarrow{P} 0$ since $n^{\frac{1}{2}}(N_{1n}n^{-1} - p_1)$

has a limiting normal distribution and $nT_{1n}^{-1} \xrightarrow{P} \lambda$. Therefore $n^{\frac{1}{2}}(\boldsymbol{\theta}_n^* - \boldsymbol{\theta}) \sim {}_3(\mathbf{0}, H\Gamma H')$ and the lemma follows by checking that $H\Gamma H' = \Sigma_1$.

From (3.4) and (3.5) it is clear that $\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}_1^{-1}$ is non-negative definite so that $\boldsymbol{\Sigma}_1 - \boldsymbol{\Sigma}$ is non-negative definite. This has several implications on the concentration of the two asymptotic distributions. However, as a measure of the asymptotic efficiency of the vector estimator $\boldsymbol{\theta}_n^*$ relative to the MLE $\hat{\boldsymbol{\theta}}_n$, we only employ the criterion of the inverse ratio of their asymptotic generalized variances. From (3.5) and Lemma 3.1, the ARE of $\boldsymbol{\theta}_n^*$ relative to $\hat{\boldsymbol{\theta}}_n$ is given by

$$
\begin{aligned}
e(\lambda_1, \lambda_2, \delta) &= |\boldsymbol{\Sigma}|/|\boldsymbol{\Sigma}_1| = (|\boldsymbol{\Sigma}_1 \boldsymbol{\Sigma}^{-1}|)^{-1} \\
&= (|I + \boldsymbol{\Sigma}_1 \mathbf{C}|)^{-1} \\
&= [1 + \frac{\lambda_2}{\lambda_1+\delta} + \frac{\lambda_1}{\lambda_2+\delta} + \frac{\lambda_1\lambda_2}{(\lambda_1+\delta)^2(\lambda_2+\delta)^2}(\delta\lambda_1+\delta\lambda_2+\lambda_1\lambda_2)]^{-1}.
\end{aligned}
\tag{3.8}
$$

As for the bounds of the ARE, we note that $e<1$ for all $(\lambda_1, \lambda_2, \delta) > \mathbf{0}$. $e \to 0$ if λ_1 and δ are fixed and $\lambda_2 \to \infty$ or if λ_2 and δ are fixed and $\lambda_1 \to \infty$. Secondly, by keeping δ fixed and letting $\lambda_1 \to 0$, $\lambda_2 \to 0$, we have $e \to 1$. Summarizing these, we have

$$\inf_{\Omega-B_1} e(\lambda_1,\lambda_2,\delta) = 0 \ , \quad \sup_{\Omega-B_1} e(\lambda_1,\lambda_2,\delta) = 1. \tag{3.9}$$

When $\underset{\sim}{\theta}\varepsilon B_1$, we have $\delta=0$. In this case the MLE is given by $\hat{\lambda}_{in}=n/S_i$, $i=1,2$, $\hat{\delta}=0$, while the unbiased estimator is $\lambda^*_{in} = (n-1)N_{in}/(nT_{1n})$, $i=1,2$, $\delta^*=0$. The asymptotic normality of $n^{\frac{1}{2}}(\hat{\lambda}_{1n}-\lambda_1, \hat{\lambda}_{2n}-\lambda_2)$ and of $n^{\frac{1}{2}}(\lambda^*_{1n}-\lambda_1, \lambda^*_{2n}-\lambda_2)$ can be established using the above method. The ARE of the unbiased estimator relative to the MLE and its bounds are given by

$$e(\lambda_1,\lambda_2,0) = \lambda_1\lambda_2(\lambda_1+\lambda_2)^{-2} \tag{3.10}$$

$$\inf_{B_1} e(\lambda_1,\lambda_2,0) = 0 \ , \quad \sup_{B_1} e(\lambda_1,\lambda_2,0) = \frac{1}{4}. \tag{3.11}$$

The maximum efficiency occurs when the marginals have the same scale parameter and the minimum occurs when $\lambda_1/\lambda_2 \to 0$ or ∞.

(3.9) and (3.11) show that the unbiased estimators proposed in [1] ar~ asymptotically less efficient than the MLE and the loss in efficiency is serious in certain parts of the parameter space, particularly when δ is close to 0. However, it should be remarked that the unbiased estimators have a simple form even in the multidimensional case while the derivation of the MLE in higher dimensions is rather tedious.

4. TEST OF INDEPENDENCE IN BVE(δ,δ,β)

In this section, we restrict ourselves to the bivariate exponential distribution with identical marginals which, as noted earlier, is a plausible model in many practical contexts where identical components are connected in parallel. We proceed to derive an optimal test for the null hypothesis that the component life times are independent which is equivalent to testing H_0: $\delta=0$ against H_1: $\delta>0$. Without loss of generality, we can restrict attention to tests which are functions of the sufficient statistics (N_0,T_1,V). Their joint distribution, however, does not constitute an exponential family and therefore the standard procedure for deriving an optimal test in an exponential family does not apply.

Let $\Omega_1 = \{(\beta,\delta): 0<\beta<\infty,\ 0\leqq\delta<\infty\}$ denote the parameter space and $\omega = \{(\beta,0): 0<\beta<\infty\}$ denote the subset specified by the composite null hypothesis. We shall write $\theta=(\beta,\delta)$ to denote a point in Ω_1 and $\phi = \phi(N_0,T_1,V)$ a test function. The following lemma provides an essentially complete class of level α tests.

<u>Lemma 4.1.</u> <u>Let</u> $\mathcal{C} = \{\phi: E_\theta\phi\leqq\alpha,\ \theta\varepsilon\omega\}$ <u>be the class of level</u> α <u>tests for</u> H_0 vs. H_1 <u>and define</u> <u>a class of tests</u> $\mathcal{C}^*$ <u>as</u> <u>follows</u>:

$$\mathcal{C}^* = \{\phi^*: \phi^*=1 \text{ if } N_0>0,\ \phi^*=\phi\varepsilon\mathcal{C} \text{ if } N_0=0\}. \tag{4.1}$$

Then $\mathcal{C}^*\subset\mathcal{C}$ and $\mathcal{C}^*$ is an essentially complete class of level α tests.

Proof. Under H_0, the probability of the event $[N_0>0]$ is zero. Thus, for any test $\phi^*\varepsilon\,\mathcal{C}^*$ and any $\theta\varepsilon\Omega_1$ we have

$$E_\theta\phi^* = P_\theta(N_0>0) + P_\theta(N_0=0)E_\theta(\phi|N_0=0). \tag{4.2}$$

and $E_\theta\phi^* = E_\theta\phi$ if $\theta\varepsilon\omega$. Hence $\mathcal{C}^*\subset\mathcal{C}$. Also, since $0\leqq\phi\leqq1$, we have $P_\theta(N_0>0) \geqq E_\theta(\phi|N_0=0)P_\theta(N_0>0)$ and hence (4.2) yields $E_\theta\phi^* \geqq E_\theta\phi$ for all $\theta\ \varepsilon\ \Omega_1-\omega$. Consequently, every test $\phi\varepsilon\,\mathcal{C}$ has a power function which is dominated by a test in $\mathcal{C}^*$ and thus, by definition, $\mathcal{C}^*$ is essentially complete.

The lemma implies that we only need to look for a UMP test within the class $\mathcal{C}^*$. A UMP level α test ϕ_o^*, if one exists, will maximize $E_\theta(\phi_o^*|N_0=0)$ uniformly in $\theta\varepsilon\Omega_1-\omega$ subject to $E_\theta\phi_o^* = E_\theta(\phi_o^*|N_2=0) \leqq \alpha$, $\theta\varepsilon\omega$. This reduces the problem to finding a UMP test in the conditional problem given that $N_2=0$ is observed.

From Theorem 2.1, we note that, conditionally given $N_0=0$, T_1 and V have the joint pdf given by

$$g(t_1,v) = \frac{(2\beta+\delta)^n(\beta+\delta)^n}{[\Gamma(n)]^2} \exp[-(2\beta+\delta)t_1 - (\beta+\delta)v](t_1v)^{n-1},$$

$$0 < t_1,\quad v<\infty,\quad (\beta,\delta) \in \Omega_1. \tag{4.3}$$

<u>Theorem 4.1</u>. <u>The UMP level</u> α <u>test of</u> H_o: $\delta=0$ <u>vs.</u> H_1: $\delta>0$ <u>for the family of distributions</u> (4.3) <u>exists and is given by</u>

$$\begin{aligned}\phi_o(t_1,v) &= 1 \quad \text{if} \quad 2t_1/v > F_\alpha \\ &= 0 \quad \text{otherwise,}\end{aligned} \tag{4.4}$$

<u>where</u> F_α <u>is the upper</u> α <u>point of the central</u> F <u>distribution</u> <u>with</u> (2n,2n) <u>degrees of freedom</u>.

<u>Proof</u>. For convenience, we make the following transformations:

$$\begin{aligned}t_1 = u_1 \quad &, \quad v = 2u_2 \\ 2\beta+\delta = \xi \quad &, \quad 2(\beta+\delta) = \eta.\end{aligned} \tag{4.5}$$

The joint pdf of U_1, U_2 is given by

$$g(u_1,u_2;\xi,\eta) = c(\xi)c(\eta) \exp[-\xi u_1 - \eta u_2](u_1u_2)^{n-1},$$

$$0<u_1,\quad u_2<\infty, \tag{4.6}$$

where $c(\xi) = \xi^n/\Gamma(n)$. The transformed parameter space is $\Omega^* = \{(\xi,\eta): 0 < \eta/2 < \xi \leqq \eta < \infty\}$ and the hypotheses are equivalent to H_o: $\xi=\eta$ vs. H_1: $\xi<\eta$. Let $\omega^* = \{(\xi,\xi): 0<\xi<\infty\}$.

To derive the UMP test, we shall use the notion of least favorable distributions. Consider a fixed simple alternative $(\xi_1,\eta_1) \varepsilon \Omega^*-\omega^*$ and let ξ_o be a suitable constant, to be selected later, which satisfies $\xi_1<\xi_o<\eta_1$. Let $\lambda(\xi_o)$ be a prior probability distribution on Ω^* which concentrates mass 1 on the single point $\theta_o^* = (\xi_c,\xi_o)\varepsilon\omega^*$. Letting $g_\lambda(u_1,u_2) = \int g(u_1,u_2 \,;\, \xi,\eta)d\lambda(\xi_o)$, we have

$$g_\lambda(u_1,u_2) = c^2(\xi_o)\exp[-\xi_o(u_1+u_2)](u_1u_2)^{n-1}.$$

Consider the test ψ defined by

$$\psi(u_1,u_2) = 1, \text{ if } \frac{g(u_1,u_2 \,;\, \xi_1,\eta_1)}{g_\lambda(u_1,u_2)} > \frac{c(\xi_1)c(\eta_1)}{c^2(\xi_o)}$$
$$= 0 \quad \text{otherwise.} \tag{4.7}$$

After some simplification of the inequality, ψ is equivalently given by $\psi=1$ if $u_1/u_2 > (\eta_1-\xi_o)/(\xi_o-\xi_1)$ and $\psi=0$ otherwise. Now we choose ξ_o such that $(\eta_1-\xi_o)/(\xi_o-\xi_1) = F_\alpha$. Such a choice is always possible because the ratio tends to 0 and ∞ as ξ_o tends to η_1 and ξ_1 respectively. With this choice of ξ_o and hence of the prior distribution $\lambda(\xi_o)$, we have $\psi=1$ if $u_1/u_2 > F_\alpha$ and $\psi=0$ otherwise. Since under any $\theta^* \varepsilon \omega^*$, U_1/U_2 has a central $F(2n,2n)$ distribution, we have $E_{\theta_o^*}\psi(U_1,U_2) = E_{\theta^*}\psi(U_1,U_2) = \alpha$ for all $\theta^* \varepsilon \omega^*$. The conditions for Corollary 5 in p. 92 of Lehmann [5] are satisfied and therefore the test ψ is the most powerful level α test for H_o against the simple alternative (ξ_1,η_1). Transforming back to the variables t_1 and v, we recognize that ψ is identical with the test ϕ_o given by (4.4). As the test does not depend on the particular alternative (ξ_1,η_1), it is also the UMP test. This concludes the proof.

Combining Lemma 4.1 and Theorem 4.1, we have

Corollary 4.1. For testing H_o: $\delta=0$ vs. H_1: $\delta>0$ in the distribution $BVE(\beta,\beta,\delta)$, a UMP level α test exists and is given by

$$\phi(N_0,T_1,V) = 1 \text{ if } N_0>0 \text{ or } 2T_1/V > F_\alpha$$
$$= 0 \text{ otherwise.} \tag{4.8}$$

Incidentally, we note that the test (4.8) can also be derived from a natural invariance consideration. The problem is invariant under a common scale change in the two coordinates, that is, transformations of the form $y'_{1i}=cy_{1i}$,

$y'_{2i}=cy_{2i}$, $i=1,\ldots,n$, $c>0$. On the set of sufficient statistics, the induced transformation is $N'_0=N_0$, $T'_1=cT_1$, $V'=cV$ and a maximal invariant is given by (N_0,R) where $R=2T_1/V$. Conditionally, given $N_0=0$, R is distributed as kF where $k = (2\beta+\delta)/(2\beta+2\delta)$ and F has a central $F(2n,2n)$ distribution. For $0<k\leq 1$, the family has monotone likelihood ratio in R and hence the conditional UMP invariant test is the same as ϕ_o. Using Lemma 4.1, it then follows that the UMP invariant test is the one given by Corollary 4.1. Although it is easier to derive the test through this invariance argument, the use of least favorable distribution provides a stronger optimality property of the test.

The test (4.8), being invariant under common scale change, has a power function which depends only on $\rho=\delta/\beta$ which is the maximal invariant in the parameter space. Using (4.2) and the distributional properties mentioned above, the power function $\gamma(\rho)=E_\rho\phi$ is given by

$$\gamma(\rho) = 1 - \left(\frac{2\beta}{2\beta+\delta}\right)^n + \left(\frac{2\beta}{2\beta+\delta}\right)^n P\left(\frac{(2\beta+\delta)F}{2(\beta+\delta)} > F_\alpha\right)$$

$$= 1 - [2/(2+\rho)]^n H(2F_\alpha(1+\rho)/(2+\rho)) \qquad (4.9)$$

where $H(\cdot)$ is the cdf of a central $F(2n,2n)$. For given n and ρ, the power $\gamma(\rho)$ can be easily computed with the help of an incomplete beta function table. The power is strictly increasing in ρ for every n. To see this, let $q(\rho)= 2F_{\alpha}(1+\rho)/(2+\rho)$ and note that the derivative of $\gamma(\rho)$ is proportional to $J(\rho) = \{nH[q(\rho)] - q(\rho)h[q(\rho)]\}$ where $h(\cdot)$ is the pdf of $F(2n,2n)$. For $n\geq 2$, the pdf $h(x)$ is strictly concave over 0 to the mode $(n-1)/(n+1)$, so that $2H(x) > xh(x)$ for all $x>0$ and hence $J(\rho)>0$. The case $n=1$ is immediate since $h(x)$ is monotone decreasing.

Remark. It is apparent from the proof of Theorem 4.1 that a UMP test for testing H_o: $\xi=\eta$ vs. H_1: $\xi<\eta$ can be constructed in the same way even when U_1 and U_2 are independent $G(n_1,\xi)$ and $g(n_2,\eta)$ respectively, and the parameter space is $\Omega^* = \{(\xi,\eta): 0<\xi\leq\eta<\infty\}$, and n_1 and n_2 may be different. For an application in life testing, let $X_1,\ldots,X_{n_1}$ and $Y_1,\ldots,Y_{n_2}$ be two independent random samples from $f(x) = \eta\exp(-\eta x)$ and $g(y) = \xi\exp(-\xi y)$ respectively. Epstein and Tsao [3] considered the problem of testing H_o: $\xi=\eta$ against two-sided alternatives and showed that the UMP unbiased test rejects for large and small values of $\bar{X}/\bar{Y}$. The above theorem shows that for one-sided alternatives H_1: $\xi<\eta$, the test which rejects for large values of $\bar{X}/\bar{Y}$ is UMP rather than just UMP unbiased. Same property holds with the usual modification of the test statistic when the samples are censored at fixed numbers of order statistics.

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