Naval Warfare Research Center
Research Memorandum RM-64

January 1972

# SEMI-MARKOV MODELS OF SEARCH IN THE PRESENCE OF DECOYS

By: JAMES M. MOORE



*Prepared for:*

NAVAL ANALYSIS PROGRAMS
OFFICE OF NAVAL RESEARCH
ARLINGTON, VIRGINIA 22217

CONTRACT N00014-71-C-0119
Task No. NR 274-008-34



STANFORD RESEARCH INSTITUTE
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**13. ABSTRACT**

A semi-Markov model representation of a submarine searching for a high value target in a field of decoys has been developed. The model was used to assess the potentials of tactical deception techniques in antisubmarine warfare. This research memorandum describes the details of the model structure. The assessment results are published in a separate, classified, final project report. In its present form, the model represents the first-generation of a promising approach that can address a large class of tactical deception assessment problems. Possible extensions and uses of the present model together with some areas potentially requiring new formulation of the semi-Markov approach are suggested in the research memorandum.

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## PREFACE

The work reported in this research memorandum was conducted as a subtask within a larger project directed toward the assessment of tactical deception techniques in antisubmarine warfare. The project was sponsored by the Naval Analysis Programs Group, Mr. R. J. Miller, Director, in the Office of Naval Research. Mr. J. G. Smith was the ONR Project Scientific Officer.

The research effort was performed by the Naval Warfare Research Center, Mr. L. J. Low, Director, of Stanford Research Institute. Mr. A. Bien of NWRC was the project leader.

The author wishes to acknowledge other members of NWRC, in particular Mr. G. W. Black and Mr. M. W. Zumwalt, for their help in discussions which led to formulation of the problem, resolution of some perplexing difficulties, and correction of some technical errors. Mrs. Terry Silvia did an excellent job in typing a difficult manuscript.

## SUMMARY

### Model Framework

The semi-Markov process is an effective framework within which to study questions regarding the capability of decoys to delay a searcher looking for a particular type of target.

The problem of search in the presence of decoys naturally lends itself to representation by a semi-Markov process because the state of the searcher and the transitions between states are well defined. Although the necessary assumptions for a continuous time Markov chain can be intuitively justified in some cases, placing the problem in the format of a more general semi-Markov process provides a framework which can accept experimental data that might not be well adapted to the more specialized Markov chain.

A primary feature of the work reported on herein is that the solutions to the various models considered are in an easily calculable closed form.

The specific situation represented is as follows. A number of high value targets (HVTs) are operating within a certain specified area. Operating within the same area are a number of low value targets called decoys. A searcher enters the area at time zero and begins looking for the high value targets. (We assume the searcher wishes to destroy the HVTs, but this assumption is not crucial to model development.) In the search process the searcher encounters the decoys, which have characteristics similar to the HVTs. Because destruction of a target involves expenditure of a limited resource, as well as possible compromise of searcher concealment, the searcher does not wish to destroy decoys; he must therefore spend time to classify a target as high value or decoy. The searcher wants to minimize search time; the targets wish to maximize it. The targets are not able to observe the searcher, and the searcher must devote his attention to a single target if he encounters several at one time or over a short interval of time.

Model Application

A three-state semi-Markov model was implemented that incorporates the events: (1) start search, (2) start classifying a decoy, and (3) start classifying an HVT. Event (3) is defined as the absorbing state. The following assumptions governed the structure of the model:

- At any fixed instant in time, positions of HVTs and decoys are distributed over the operating area according to a uniform probability distribution
- Over short intervals of time, everything moves in straight lines with uniformly random headings
- Detection is represented by a definite range law
- All detection circles lie inside the operating area (i.e., the searcher as well as the HVT knows what the operating area is)
- When a decoy is classified, information obtained by the searcher is dissipated rapidly enough so that the classification has no operational effect on the density of decoys
- When presented with an array of targets (both decoys and HVTs) from which one is to be selected for classification, the targets are equally likely to be chosen (i.e., the decoys are identical and are indistinguishable from the identical HVTs until a classification is made)

- Stochastic independence of motion is assumed among decoys as a group, among HVTs as a group, and between the two groups except in the case of reduced overlap when HVTs avoid decoys.

In addition, it was assumed that search and classification times are exponentially distributed (reducing the semi-Markov model to a Markov model). Simulation studies, conducted as an adjunct to the semi-Markov model formulation, demonstrated that the assumed exponential search time distribution and the adopted model for determining mean time to target detection are in fact valid with respect to the assumptions made concerning the search process (primarily the definite range detection law). The validity of assuming exponential distribution for classification time remains to be demonstrated. Mean time to classification is handled as an input variable in the current study. The three-state model was used to address the following questions:

1. How long does it take the searcher to find a high value target (i.e., what is the mean time)?
2. How many decoys are encountered before an HVT is found (i.e., mean number)?
3. What are the relative effects of decoy characteristics such as number, speed, detectability, and realism (classification time) on the quantities above?
4. How many decoys with the given characteristics are required to provide a certain level of safety for the high value targets (e.g., provide a certain minimum level of probability of an HVT being detected over a specified time interval)?

Answers to the above questions for a realistic operational situation are presented in a separate, classified, final project report.*

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*A. Bien; "Evaluation of Tactical Deception Techniques in Carrier Task Force Defense" (U), Final Report; SRI Project 1016-245; Contract N00014-71-C-0119; Stanford Research Institute, Menlo Park, California, December 1971 (SECRET)

## Model Extension

The following are specific recommendations directed toward the potential extension of the semi-Markov model approach:

1. Derive output expressions for a searcher starting in the classify decoy state.

2. Given the assumption of a constant range detection law, the three-state model is a suitable representation of the search problem, within the limits of the random motion/position assumption, the restrictions due to overlap, and a large (relative to detection radii) operating area. In order to extend these limits it is recommended that the effects of geometry and time on the state transition mechanism in the three-state model be studied in detail and the three-state model be expanded to four states by including a secondary search state*.

3. A ubiquitous assumption in this study is the definite range detection law. The realized detection range in any real encounter is a random variable that is represented in the models discussed here by a fixed range R (which depends on the type of target and its speed as well as searcher and environmental characteristics). If the target comes within R of the searcher, the target is assumed detected; no detections occur at ranges greater than R. The value assigned to R is usually the median value of a distribution generated by models outside the scope of this study. The search models presented in this document permit sensitivity studies on R but they do not take into account the inherent variability in realized detection ranges. Hence, it is recommended that a study be conducted to determine the effect of this inherent variability on the results produced by the current search models. Such a study could be conducted with Markov process models and could examine fade zone effects as well as variable detection range.

4. An area for analytic extension is the situation of stationary decoys where the searcher can plot the position of classified decoys and thus render them relatively ineffective. The three-state model is not applicable in this situation because the transition probabilities change with each decoy that is

classified. What is needed is a general n-state formulation.

5. Several interesting questions involving constrained optimization arise. Loosely speaking, it is clear that the more decoys that are available the better for the HVT. However, it is also clear that the choice of decoy configuration and number of decoys is a constrained problem. This constrained choice problem will assume characteristics dependent on the circumstance within which it arises. Some possibilities are:

   - Given specified limited funds, how many decoys of what configuration should be built to optimize some operational variable, such as probability of HVT detection?
   - In initial planning stages, how do the optimum number and configuration in the preceding question vary as the amount of available funds varies?
   - Funds are "limited" but not specified. Therefore, it is desired to meet some operational performance threshold (such as a minimum acceptable HVT detection probability) with minimum cost. How many decoys of what configuration should be built and what is the cost?

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## 1. INTRODUCTION

This report discusses semi-Markov models of a random search process with decoys. The primary situation envisaged is as follows. A number of high value targets (HVTs) are operating within a certain specified area. Operating within the same area are a number of low value targets called decoys. A searcher enters the area at time zero and begins looking for the high value targets. We assume the searcher wishes to destroy the HVTs, but this assumption is not crucial to model development. In the search process the searcher encounters the decoys. The decoys have characteristics similar to the HVTs. Because destruction of a target involves expenditure of a limited resource, as well as possible compromise of searcher concealment, the searcher does not wish to destroy decoys; therefore he must spend time to classify a target as high value or decoy. The searcher wants to minimize search time while the targets wish to maximize it. The targets are not able to observe the searcher, and the searcher must devote his attention to a single target if he encounters several at one time or over a short interval of time. Targets and searcher are assumed to be moving randomly in the following sense

i) At any fixed instant of time positions are uniformly random over the operating area

ii) Over short intervals of time everything moves in straight lines with uniformly random headings.

These two assumptions, plus later assumptions regarding detection radius, raise some difficult questions of validity near the boundaries of a finite operating region. These questions do not lend themselves to easy analytical treatment. There are two methods of treating them. The first is to assume the errors introduced are negligible or that there are compensating errors. The errors will certainly be negligible if the operating area is large enough. An example of compensating errors is

i) Target density within detection range of the searcher will be less on the boundary of operating area than within the area (due to the fact that part of the area swept by the searcher has zero target density),

ii) But the searcher will know this and concentrate his efforts accordingly.

The second method of treatment is to drop explicit consideration of area and numbers of targets and searchers and consider only densities instead. This approach requires only minor adjustment of details and will be discussed later.

Certain questions regarding the above situation are of interest

i) How long does it take the searcher to find a high value target (i.e., what is the mean time)?

ii) How many decoys are encountered before an HVT is found (i.e., mean number)?

iii) What are the relative effects of decoy characteristics such as number, speed, detectability, and realism (classification time) on the quantities in (i) and (ii) above?

iv) How many decoys with the given characteristics are required to provide a certain level of safety for the high value targets (e.g., provide a certain minimum level of probability of an HVT being detected over a specified time interval)?

The above notions are random in nature and will be discussed in terms of semi-Markov random processes.

The problem of search in the presence of decoys naturally lends itself to representation by a semi-Markov process because the state of the searcher and the transitions between states are well defined. Although the necessary assumptions for a continuous time Markov chain can be intuitively justified in some cases, placing the problem in the format of a more general semi-Markov process provides a framework which can accept experimental data that might not be well adapted to the more specialized Markov chain. Moreover, the semi-Markov process allows us to look into some of those cases that definitely do not meet the assumptions of the Markov process. Analysis of "sprint-drift" is a case in point. In sprint-drift holding times in certain states are bounded and the ensuing results are of a form that is quite different from the results obtained if one assumes all holding times are (unbounded) exponentially distributed.

A primary feature of the work reported on herein is that the solutions to the various models considered are in an easily calculable closed form. For example, all matrix inversions required for model solutions have been carried out in symbolic manipulations. Parametric studies are therefore inexpensive to conduct. Extension of the work reported here to more complicated state spaces or distributions may or may not require more complicated numerical methods.

Analysis begins in Section 2 with a review of Koopman's formula for detection rate. A general discussion of semi-Markov processes follows in Section 3. A three-state search model with constant speeds is developed in Section 4, and some sample results are given in Section 5. A four-state model for a variable speed searcher is developed in Section 6. Finally, conclusions and recommendations are given in Section 7, where several model extensions and optimization studies using the models are suggested.

## 2. DETECTION RATE AND TIME

From Ref. 3 we adopt a model for determining rate of detection. The following assumptions are made

i) The searcher is progressing on its course at constant velocity u

ii) The searcher is moving among a uniform random distribution of targets with uniformly randomly distributed headings

iii) The targets are progressing at constant velocity v

iv) The density of targets is $\delta$ (targets per square nautical mile, say)

v) The searcher immediately detects all targets which come within a range R (i.e., definite range rule).

Then the rate $\rho$ (i.e., number per unit time) at which targets are detected is given by

$$(2.1) \qquad \rho = \frac{4R\delta}{\pi}(u+v)\int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2\varphi}\, d\varphi$$

where $$k = \frac{2\sqrt{uv}}{u+v}$$

and the integral is known as the complete elliptic integral of the second kind. Polynomial approximations for this integral may be found in Ref. 6 (p 592).

Mean time to detection is given by the inverse rate, i.e., $1/\rho$.

Some example mean times are given in Table 2-1 for various ranges and speeds. The target density is equivalent to one target per $\pi(200)^2$ square nautical miles. Ranges are nautical miles and speeds in knots. For example, a searcher speed of 20 kt, a target speed of 15 kt, and a



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Table 2-1

MEAN DETECTION TIME (HOURS)

| Searcher Speed (u) - Knots | Target Speed (v) - Knots: 5 | 10 | 15 | 20 | 25 | 30 |
|---|---|---|---|---|---|---|
| 5 | 328.99 | 196.92 | 135.82 | 103.10 | 82.95 | 69.33 |
| 10 | 196.92 | 164.49 | 125.27 | 98.46 | 80.52 | 67.91 |
| 15 | 135.82 | 125.27 | 109.66 | 91.34 | 76.70 | 65.64 |
| 20 | 103.10 | 98.46 | 91.34 | 82.25 | 71.74 | 62.64 |
| 25 | 82.95 | 80.52 | 76.70 | 71.74 | 65.80 | 59.02 |
| 30 | 69.33 | 67.91 | 65.64 | 62.64 | 59.02 | 54.83 |
| | | | R = 30.0 nmi | | | |
| 5 | 164.49 | 98.46 | 67.91 | 51.55 | 41.47 | 34.67 |
| 10 | 98.46 | 82.25 | 62.64 | 49.23 | 40.26 | 33.96 |
| 15 | 67.91 | 62.64 | 54.83 | 45.67 | 38.35 | 32.82 |
| 20 | 51.55 | 49.23 | 45.67 | 41.12 | 35.87 | 31.32 |
| 25 | 41.47 | 40.26 | 38.35 | 35.87 | 32.90 | 29.51 |
| 30 | 34.67 | 33.96 | 32.82 | 31.32 | 29.51 | 27.42 |
| | | | R = 60.0 nmi | | | |
| 5 | 109.66 | 65.64 | 45.27 | 34.37 | 27.65 | 23.11 |
| 10 | 65.64 | 54.83 | 41.76 | 32.82 | 26.84 | 22.64 |
| 15 | 45.27 | 41.76 | 36.55 | 30.45 | 25.57 | 21.88 |
| 20 | 34.37 | 32.82 | 30.45 | 27.42 | 23.91 | 20.88 |
| 25 | 27.65 | 26.84 | 25.57 | 23.91 | 21.93 | 19.67 |
| 30 | 23.11 | 22.64 | 21.88 | 20.88 | 19.67 | 18.28 |
| | | | R = 90.0 nmi | | | |

$\left(\delta = \frac{1}{\pi(200)^2} \text{ targets per square nmi}\right)$

detection range of 30 nmi yield a mean detection time of 91.34 hours. For interpolation and extrapolation, inverse mean time (rate) is linear in range and target density.

Simulation studies* have shown that formula (2.1) works quite well for a 200-nmi radius circular area under the following condition. A given vehicle (target or searcher) pursues a randomly selected heading until the area boundary is reached, where a new random heading within the area is selected and followed until the boundary is again reached, and so on.

Finally it is noted that no distinction is made here between the state of nature and the searcher's state of knowledge of the state of nature, so that detection is synonymous with encounter.

---

*E. L. Wong; "Simulation Model of Search in the Presence of Decoys," NWRC TN-33; SRI Project 1016-245, Contract N00014-71-C-0119; Stanford Research Institute, Menlo Park, California, July 1971

# 3. THE SEMI-MARKOV PROCESS

## 3.1 Definition

The semi-Markov process (SMP) is a stochastic process in which time is the independent variable, and the dependent variable can assume only a denumerable number of discrete values. A given value of the dependent variable is called a state, and the collection of all possible values is the state space. Transition from a present state to a future state depends only on the present state, while the time required for such a transition may, in general, depend on both the present state and the future state. Consider a space of functions $X = \{x_\omega(t);\ t \geq 0,\ \omega\epsilon\Omega\}$. This space is the sample space and consists of functions of time, with the generic functional form parameterized by $\omega\epsilon\Omega$. Thus, a sample "point" consists of the set of points $\{x_\omega(t);\ t \geq 0\}$; such a sample point is called a "realization" of the process and will be denoted simply $x_\omega(\cdot)$. We will consider only a finite number of states, with a state denoted by one of the numbers 1, 2, ..., N. Thus, for a fixed value of t, $x_\omega(t)\epsilon\ \{1,2,\ldots,N\}$. Hence, it is characteristic of the process that each realization $x_\omega(\cdot)$ is a staircase function; we will assume each realization continuous from the right. This assumption is not universal. For example, Cox and Miller (Ref. 1) analyze SMPs which are continuous from the left, although they also consider separately in some detail the special case of Markov processes (MP) which are continuous from the right. A typical realization is plotted in Fig. 3-1 for a four-state SMP. The dots in the figure represent "events," that is, points where a transition into a state occurs. Transitions from a state into the same state may occur. If a transition from state i to state j ($i \neq j$) occurs at time $\tau$, then $x_\omega(\cdot)$ will be continuous from the right but discontinuous from the left



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at $t = \tau$. Formally, we say the process is in state j at time t if the last event to occur was of type j, where $j \in \{1,\ldots,N\}$.

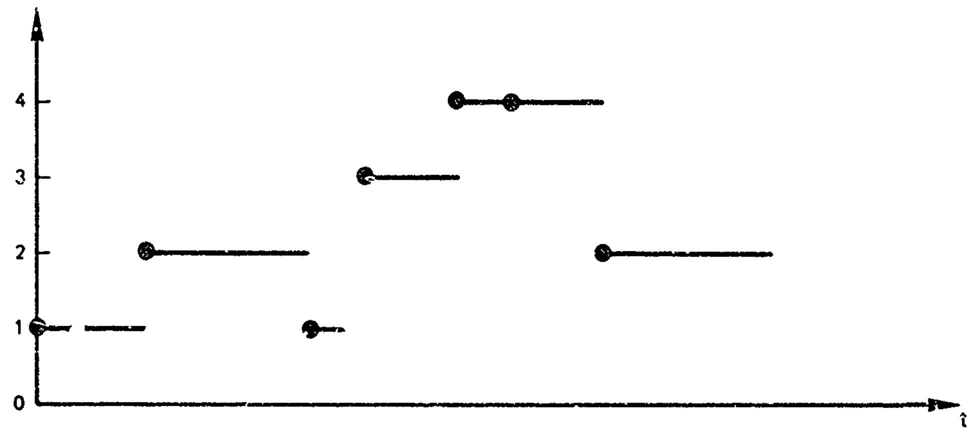


FIGURE 3-1 A TYPICAL REALIZATION OF A SMP

A SMP is often described in terms of what is called a semi-Markov matrix $F = [F_{ij}(t)]$, where $F_{ij}(t)$ is a distribution function (or sub-distribution function) and

$$F_{ij}(t-t_n) = Pr\{x(t_{n+1}) = j \quad , \quad t_{n+1} \le t \mid x(t_n) = i\}$$

where $t_n$ is the random variable denoting the time the process makes the $n^{th}$ transition (Refs. 8 and 9).

However, an alternative method (Refs. 1 and 10) more appropriate for our purposes is to describe the process in terms of a transition probability (stochastic) matrix $A = (\alpha_{ij})$ and a probability density function (p.d.f.) matrix $f = [f_{ij}(t)]$.

We have

$$\alpha_{ij} = \Pr\{\text{next event is of type } j \mid \text{last event of type } i\}$$

and where

$f_{ij}(t)$ = probability density function of transition time given that last event was type i and next event is type j.

The special case where $f_{ij}(t)$ is an exponential distribution, depending only on i, is a continuous time Markov process. This formulation of the continuous time Markov processes differs somewhat from the so-called "minimal process" formulation given in Refs. 1 and 2. A limited discussion of the minimal process approach is given in Appendix D.

### 3.2 Characteristics of the SMP

We now proceed to discuss and develop some interesting characteristics of semi-Markov processes. Cox and Miller (Ref. 1) show the following. If we let

$$h_{ij}(t) = \lim_{\Delta t \to 0} \frac{\Pr\{\text{event of type } j \text{ in } (t, t+\Delta t) \mid \text{event of type } i \text{ at } 0\}}{\Delta t}$$

then the conditional (on event type i at 0) expected number $H_{ij}(t)$ of type j events in (0, t) is given by

$$H_{ij}(t) = \int_0^t h_{ij}(\tau)d\tau \quad . \tag{3.1}$$

If we denote the Laplace transform of a general function m(x) by

$m^*(s) = \int_0^\infty e^{-sx} m(x)dx$ and introduce matrix notation, then

$$(3.2) \qquad h^*(s) = [h^*_{ij}(s)]$$
$$= g^*(s) [I - g^*(s)]^{-1}$$

where

$$g^*_{ij}(s) = \alpha_{ij} f^*_{ij}(s) \quad .$$

Furthermore, from the well-known properties of the Laplace transform we have

$$(3.3) \qquad H^*_{ij}(s) = \frac{1}{s} h^*_{ij}(s)$$

and H(t) may be obtained by inverting the transforms. Cox and Miller derived expression (3.2) in the context of a SMP continuous from the left, whereas ours are continuous from the right. However, inspection of their derivation shows that the direction of continuity is irrelevant.

We say the system is in state j at time t if the last event to occur before (or at) t was of type j. Let

$$p_{ij}(t) = Pr\{\text{in state } j \text{ at time } t | \text{event of type } i \text{ at time } 0\}$$

and

$$P(t) = [p_{ij}(t)].$$

Cox and Miller also develop an expression for P(t). However, this derivation, in contrast to that for $h^*(s)$, is heavily dependent on the direction of continuity. We proceed now to develop an expression for $sP^*(s)$ via integral equations in a manner similar to Cox and Miller.

We start with a two state process. The integral equations are of the form

(3.4) $$p_{11}(t) = \alpha_{11}F_{11}(t) + \alpha_{12}F_{12}(t) + \alpha_{11}\int_0^t h_{11}(t-u)F_{11}(u)du + \alpha_{12}\int_0^t h_{11}(t-u)F_{12}(u)du$$

(3.5) $$p_{12}(t) = \alpha_{22}\int_0^t h_{12}(t-u)F_{22}(u)du + \alpha_{21}\int_0^t h_{12}(t-u)F_{21}(u)du$$

where

(3.6) $$F_{ij}(t) = \int_t^\infty f_{ij}(u)du \quad .$$

One obtains $p_{22}(t)$ and $p_{21}(t)$ in an obviously similar manner.

To understand the derivation of (3.4) and (3.5), one should study Figs. 3-2 through 3-4. Figure 3-2 represents the Cox and Miller derivation of $p_{11}(t)$, while Figs. 3-3 and 3-4 correspond to our derivation of $p_{11}(t)$ and $p_{12}(t)$, respectively. The multiple components of each of the figures represent an exhaustive classification of the ways in which the desired state can be reached at time t. The expressions on the right of the figures represent corresponding components in integral equations.

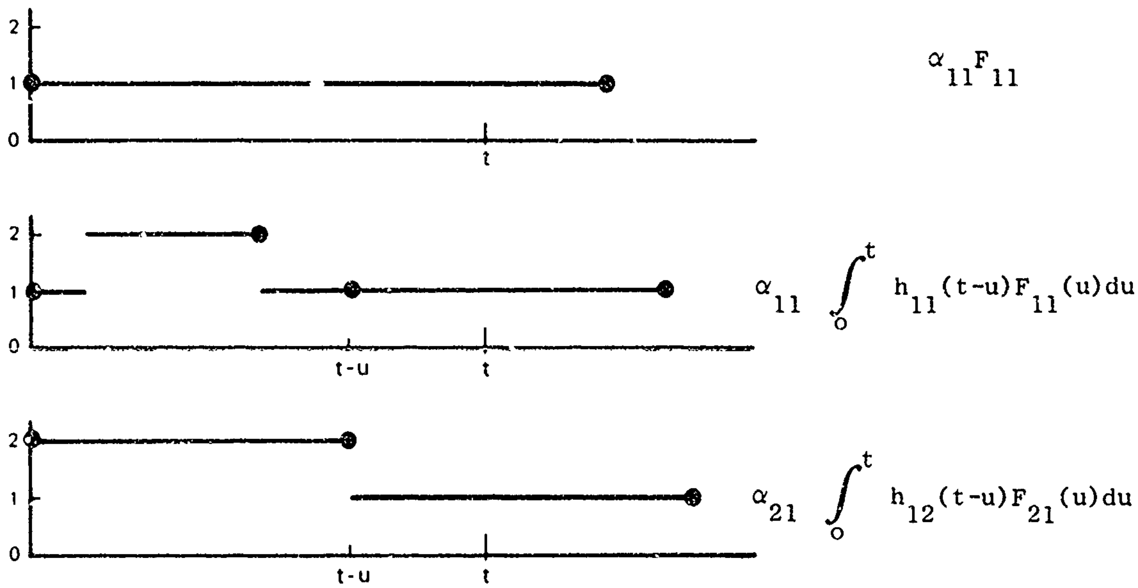


FIGURE 3-2 $p_{11}(t)$ CONTINUOUS FROM LEFT (Cox & Miller)

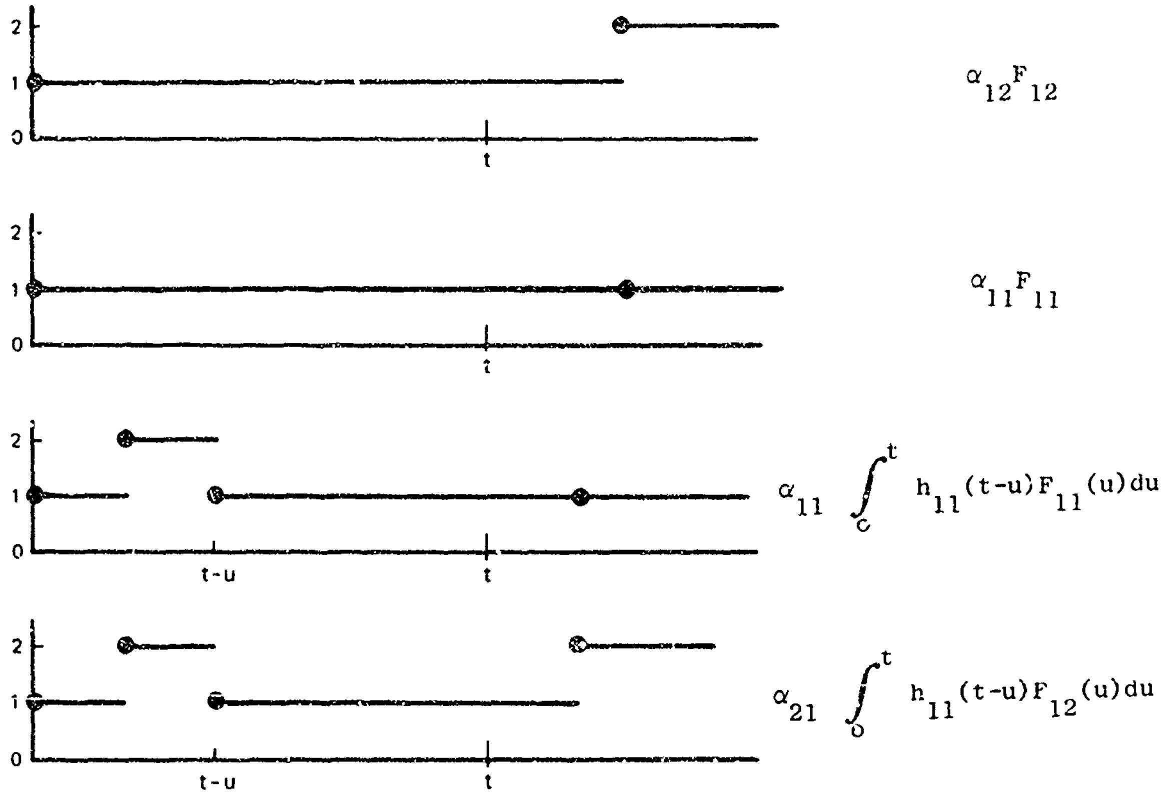


FIGURE 3-3 $p_{11}(t)$ CONTINUOUS FROM RIGHT

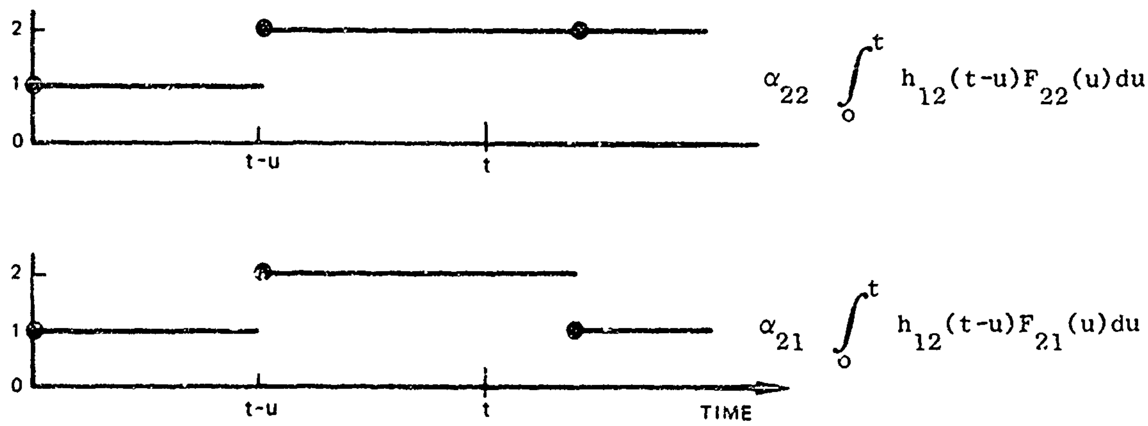


FIGURE 3-4 $p_{12}(t)$ CONTINUOUS FROM RIGHT

Note that

$$F_{ij}(t) \;=\; \Pr\{\text{the transition from } i \text{ to } j \text{ takes } t \text{ or longer}\} \quad .$$

Thus in (3.4) $p_{11}(t)$ consists of the components

$$\alpha_{11}F_{11}(t) \;=\; \Pr\{\text{1st transition is to state 1 and takes } t \text{ or longer}\}$$

$$\alpha_{12}F_{12}(t) \;=\; \Pr\{\text{1st transition is to state 2 and takes } t \text{ or longer}\}$$

$$\int_0^t h_{11}(t-u)\alpha_{11}F_{11}(u)du$$

$$= \int_0^t \Pr\{\text{1st transition leads to a sequence of events yielding an event 1 at } t-u \text{ followed by a transition to state 1 taking } u \text{ or longer}\}du$$

$$\int_0^t h_{11}(t-u)\alpha_{12}F_{12}(u)du$$

$$= \int_0^t \Pr\{\text{1st transition leads to a sequence of events yielding an event 1 at } t-u \text{ followed by a transition to state 2 taking } u \text{ or longer}\}du.$$

Note the general classification according to whether the first transition occurs before or after time t. In determining $p_{ij}(t)$, $i \neq j$, the first transition must occur before t.

Thus we can write, dropping the "s" arguments for clarity,

$$(3.7)\qquad \begin{cases} p^*_{11} = \alpha_{11}F^*_{11} + \alpha_{12}F^*_{12} + \alpha_{11}h^*_{11}F^*_{11} + \alpha_{12}h^*_{11}F^*_{12} \\ p^*_{12} = \alpha_{21}h^*_{12}F^*_{12} + \alpha_{22}h^*_{12}F^*_{22} \\ p^*_{22} = \alpha_{21}F^*_{21} + \alpha_{22}F^*_{22} + \alpha_{21}h^*_{22}F^*_{21} + \alpha_{22}h^*_{22}F^*_{22} \\ p^*_{21} = \alpha_{11}h^*_{21}F^*_{11} + \alpha_{12}h^*_{21}F^*_{12} \end{cases}$$

or in matrix form

$$\begin{bmatrix} p^*_{11} & p^*_{12} \\ p^*_{21} & p^*_{22} \end{bmatrix} = \begin{bmatrix} \alpha_{11}F^*_{11} + \alpha_{12}F^*_{12} & 0 \\ 0 & \alpha_{21}F^*_{21} + \alpha_{22}F^*_{22} \end{bmatrix}$$

$$(3.8)\qquad + \begin{bmatrix} h^*_{11} & h^*_{12} \\ h^*_{21} & h^*_{22} \end{bmatrix} \begin{bmatrix} \alpha_{11}F^*_{11} + \alpha_{12}F^*_{12} & 0 \\ 0 & \alpha_{21}F^*_{21} + \alpha_{22}F^*_{22} \end{bmatrix}$$

$$= [I + h^*] \begin{bmatrix} \alpha_{11}F^*_{11} + \alpha_{12}F^*_{12} & 0 \\ 0 & \alpha_{21}F^*_{21} + \alpha_{22}F^*_{22} \end{bmatrix} .$$

Noting that $sF^*_{ij} = 1 - f^*_{ij}$, we have

$$(3.9)\qquad sP^*(s) = [I + h^*] \begin{bmatrix} \alpha_{11}(1-f^*_{11}) + \alpha_{12}(1-f^*_{12}) & 0 \\ 0 & \alpha_{21}(1-f^*_{21}) + \alpha_{22}(1-f^*_{22}) \end{bmatrix} .$$

In a three state process we have

$$
(3.10)\quad \begin{cases} p_{11}^*(s) = \alpha_{11}F_{11}^* + \alpha_{12}F_{12}^* + \alpha_{13}F_{13}^* + \alpha_{11}h_{11}^*F_{11}^* + \alpha_{12}h_{11}^*F_{12}^* + \alpha_{13}h_{11}^*F_{13}^* \\ p_{12}^*(s) = \alpha_{21}h_{12}^*F_{21}^* + \alpha_{22}h_{12}^*F_{22}^* + \alpha_{23}h_{12}^*F_{23}^* \\ \ldots \end{cases}
$$

So it is clear that (3.9) generalizes to the n-dimensional case

$$
(3.11)\qquad sP^*(s) = [I + h^*(s)]W(s)
$$

where

$$
W(s) = [W_{ij}(s)]
$$

$$
W_{ii}(s) = \sum_{k=1}^{n} \alpha_{ik}[1-f_{ik}^*(s)] \quad , \quad i = j
$$

$$
W_{ij}(s) = 0 \quad , \quad i \neq j \quad .
$$

Note in the special case where $f_{ij} = f_i$ for all (i, j), we have $W_{ii} = 1 - f_i^*$. Recall $h^*(s)$ is given by (3.2).

Now, suppose an event of type i occurs at time t = 0. Let $T_{ij}$ be the (random) time the system first reaches state j; $T_{ij}$ is thus referred to as the conditional (on i) first passage time. We would like to develop an expression for the expected value $E(T_{ij})$. Adapting a method used by Cox and Miller (Ref. 1, p 196) for Markov processes, we proceed as follows. Replace the matrix $A = (\alpha_{\ell k})$ by $A_o = (\alpha^o_{\ell k})$ with $\alpha^o_{\ell k} = \alpha_{\ell k}$ except $\alpha^o_{jk} = 0$, $k \neq j$, $k = 1, \ldots, n$ and $\alpha^o_{jj} = 1$. Thus j becomes an "absorbing" state--once the process reaches state j, it stays there, although the event j may reoccur. Let the new sample space be $Y = \{y_\omega(t);\ t \geq 0,\ \omega\epsilon\Omega\}$. Let $r_{ij}(t)$ be the probability in the new process that state j is occupied at time t if started with event i at time 0. Thus $r_{ij}(t)$ in the new process corresponds to $p_{ij}(t)$ in the old process and $r_{jj}(0) = 1$. Then

$$(3.12) \qquad P(T_{ij} \le t) = r_{ij}(t) \quad .$$

Note that

$$(3.13) \quad P(T_{ij} \le t) = P(\omega\epsilon\Omega | x_\omega(0) = i, \ x_\omega(T) = j, \ x_\omega(\tau) \ne j, \text{ all } \tau < T, \text{ some } T < t)$$

and

$$(3.14) \qquad r_{ij}(t) = P(\omega\epsilon\Omega | y_\omega(0) = i, \ y_\omega(t) = j) \quad .$$

By construction we have $y_\omega(0) = i \Leftrightarrow x_\omega(0) = i$ and $y_\omega(t) = j \Leftrightarrow (x_\omega(T) = j, \ x_\omega(\tau) \ne j, \text{ all } \tau < T, \text{ some } T < t)$. So the set of $\omega\epsilon\Omega$ in (3.13) and the set of $\omega\epsilon\Omega$ in (3.14) are the same, and hence the probabilities are the same. (We have assumed the probability measure on $\Omega$ is the same in the new prosess as in the old.)

Now we have

$$(3.15) \qquad E\left(e^{-sT_{ij}}\right) = \int_0^\infty e^{-st} P'(T_{ij} \le t)dt$$

$$= \int_0^\infty e^{-st} r'_{ij}(t)dt$$

$$= sr^*_{ij}(s) + r_{ij}(+0)$$

$$= sr^*_{ij}(s)$$

where the last equality holds because we assume no instantaneous transitions at $t = 0$ can occur, that is, $f_{ij}(t)$ has no probability mass concentrated at $t = 0$.

To obtain $E(T_{ij})$ we note that

$$(3.16) \qquad -\frac{d}{ds} E\left(e^{sT_{ij}}\right)\Bigg|_{s=0} = E(T_{ij}) \quad .$$

One is now tempted to write

$$(3.17) \qquad E(T_{ij}) = -\frac{d}{ds}\Big[sr^*_{ij}(s)\Big]_{s=0} = -\Big[sr'^*_{ij}(s) + r^*_{ij}(s)\Big]_{s=0}$$

but $sr'^*_{ij}(s)\Big|_{s=0}$ may be indeterminate, i.e., $r'^*_{ij}(0) = \infty$, so the entire function $sr^*_{ij}(s)$ must be developed.

Thus we have developed formulas for

$p_{ij}(t)$ = time dependent transition probability

$H_{ij}(t)$ = conditional expected number of events

$E(T_{ij})$ = conditional expected first passage time.

## 3.3 Initial State Probabilities

So far the discussion of the SMP has centered on the transition mechanism and the associated conditional probabilities, conditional times, and conditional events, where the conditioning has been on the initial state of the process. If the probability distribution of the initial state is specified, then these conditional quantities can be combined in weighted averages to obtain the corresponding unconditional quantities. Suppose we have the state space $S = \{1, 2, \ldots, N\}$. Let $p_i(t)$ denote the probability of being in state $i \in S$ at time $t \geq 0$. The probability distribution of the initial state assigns a probability $p_i(0) = p_i$ such that $\sum_{i=1}^{N} p_i = 1$. The time dependent state probability $p_i(t)$ is distinct from the time dependent transition probability $p_{ij}(t)$; the row vector of $p_i(t)$ is denoted by $p(t)$, and the matrix of $p_{ij}(t)$ by $P(t)$. Note that for the "well-behaved" processes under consideration $P(0) = I$. We now have

$$p_j(t) = \sum_{i=1}^{N} p_i(0) p_{ij}(t) \qquad \text{for } t \geq 0 \tag{3.18}$$

or

$$p(t) = p(0)P(t) \qquad \text{for } t \geq 0 \tag{3.19}$$

Similarly, if we define

$H_j(t)$ = unconditional expected number of type $j$ events in $(0, t)$

$T_j$ = the time the system first reaches state $j$

then

$$\begin{aligned} H_j(t) &= \sum_{i=1}^{N} p_i(0) H_{ij}(t) \\ E(T_j) &= \sum_{i=1}^{N} p_i(0) E(T_{ij}) \quad . \end{aligned} \tag{3.20}$$

## 4. A THREE-STATE SEMI-MARKOV SEARCH MODEL

### 4.1 Definition

Referring to the operational situation described in Section 1 and the semi-Markov process described in Section 3, the following identifications yield a model of the search process with decoys present.

Let the events be

1. Start search
2. Start classifying a decoy
3. Start classifying an HVT.

The times of interest are

$T_{13}$ = time to start of HVT classification given searcher starts in search state

$T_{23}$ = time to start of HVT classification given searcher starts in decoy classification state

$T_3$ = time to start of HVT classification (unconditional).

The state generated by event (3) is considered absorbing because

a. Event (3) results in the destruction (or probable destruction) of an HVT

b. Hence (3) is a highly significant event in itself

c. In the cases of interest in the present study, loss of one of the small number of HVTs would cause significant changes in HVT density, which in turn would require considerable extension of the model.

Note, however, that (3) being an absorbing state has no direct consequences on the interesting output except

a. On the form of $H_{12}(t)$ and $H_{22}(t)$

b. On the interpretation of $T_{13}$ and $T_{23}$ as absorbtion times.



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It would be a very straightforward modification to consider (3) to be other than absorbing. A situation of interest in this case would be where the searcher is interested only in surveying the field of targets to determine, for example, the number or density of HVTs.*

In the absorbing case, we postulate the transition matrix

$$(4.1) \qquad A = \begin{pmatrix} 0 & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

and the probability density function

$$(4.2) \qquad f = \begin{pmatrix} f_1 & f_1 & f_1 \\ f_2 & f_2 & f_2 \\ f_3 & f_3 & f_3 \end{pmatrix} .$$

Here the densities are

$f_1(t)$ = p.d.f. of the time $\tau_1$ to detect a target

$f_2(t)$ = p.d.f. of the time $\tau_2$ to classify a decoy

$f_3(t)$ = p.d.f. of the time $\tau_3$ to classify a HVT

and the conditional probabilities are

$\alpha_{12}$ = probability the detected target is a decoy

$\alpha_{13}$ = probability the detected target is a HVT

---

* An interesting problem is: given such a model and a search history, what are good estimates of density or number. Assume speeds and detection ranges are measured with error.

$\alpha_{21}$ = probability return to search after classify a decoy (i.e., no HVT or other decoy present)

$\alpha_{22}$ = probability that, after classify a decoy, another decoy is present and selected, from the field of visible targets, to be classified

$\alpha_{23}$ = probability that, after classify a decoy, an HVT is present and selected, from the field of visible targets, to be classified.

Note that since the columns of the matrix f are identical we have a Markov process if the $f_i$'s are exponential. The characteristics of $f_3(t)$ other than finite expected value turn out to be irrelevant to any of the subsequent analysis due to the fact that nothing of interest (in the present model) happens after state 3 is reached; let $\mu^{-1}$ be the expected value. If the model were modified to a nonabsorbing situation (for example, to include HVT mission consideration), then $f_3(t)$ would acquire more importance.

We assume that all decoys are identical and all HVTs are identical in their operating characteristics. A decoy may have characteristics different from an HVT. If more than one type of decoy or HVT is required, modification of the subsequent analysis is required. Specifically, additional states and/or modified transition probabilities would be required, depending on the type of information required. Many types of information can probably be obtained through modification of the transition probabilities.

## 4.2 Conditional Expected First Passage Time

Referring to equations (3.2) and (3.11) we have

$$sP^*(s) = [I + h^*] W(s) \tag{4.3}$$

where

$$
(4.4) \qquad W(s) = \begin{bmatrix} 1-f_1^* & 0 & 0 \\ 0 & 1-f_2^* & 0 \\ 0 & 0 & 1-f_3^* \end{bmatrix}
$$

$$
(4.5) \qquad h^*(s) = g^*(s)[I - g^*(s)]^{-1}
$$

and

$$
(4.6) \qquad g^*(s) = \begin{bmatrix} 0 & \alpha_{12}f_1^* & \alpha_{13}f_1^* \\ \alpha_{21}f_2^* & \alpha_{22}f_2^* & \alpha_{23}f_3^* \\ 0 & 0 & f_3^* \end{bmatrix} .
$$

Thus

$$
(4.7) \qquad I - g^*(s) = \begin{bmatrix} 1 & -\alpha_{12}f_1^* & -\alpha_{13}f_1^* \\ -\alpha_{21}f_2^* & 1-\alpha_{22}f_2^* & -\alpha_{23}f_2^* \\ 0 & 0 & 1-f_3^* \end{bmatrix}
$$

$$
\triangleq \begin{bmatrix} 1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & c_3 \end{bmatrix} .
$$

So we can write

$$(4.8)\qquad [I - g^*(s)]^{-1} \triangleq \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ 0 & 0 & g_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{b_2}{b_2 - a_2 b_1} & \dfrac{-a_2}{b_2 - a_2 b_1} & \dfrac{a_2 b_3 - a_3 b_2}{c_3(b_2 - a_2 b_1)} \\ \dfrac{-b_1}{b_2 - a_2 b_1} & \dfrac{1}{b_2 - a_2 b_1} & \dfrac{a_3 b_1 - b_3}{c_3(b_2 - a_2 b_1)} \\ 0 & 0 & \dfrac{1}{c_3} \end{bmatrix} .$$

Noting that in the terminology of Section 3 we now have $p_{13}(t) \equiv r_{13}(t)$ and $p_{23}(t) \equiv r_{23}(t)$, we proceed to get $E(T_{13})$ and $E(T_{23})$ by using equation (3.16). We have*

$$(4.9)\qquad \begin{aligned} sP^*_{13}(s) &= [I + h^*]_{1\cdot} W_{\cdot 3} \\ &= [I + h^*]_{13} W_{33} \qquad \text{since } W_{13} = W_{23} = 0 \\ &= h^*_{13} W_{33} = W_{33} g^*_{1\cdot}(s)\,[I - g^*(s)]^{-1}_{\cdot 3} \\ &= W_{33}[0,\ \alpha_{12} f^*_1,\ \alpha_{13} f^*_1] \begin{bmatrix} g_{13} \\ g_{23} \\ g_{33} \end{bmatrix} \\ &= (1-f^*_3) f^*_1 (\alpha_{12} g_{23} + \alpha_{13} g_{33}) \end{aligned}$$

---

*Notation: $A_{\cdot j}$ and $A_{i\cdot}$ refer to $j^{th}$ column and the $i^{th}$ row, respectively, of the matrix A.

where

$$(4.10)\quad \begin{cases} g_{23} = \dfrac{\alpha_{23} f_2 + \alpha_{21}\alpha_{13} f_1 f_2}{(1-f_3^*)(1-\alpha_{22} f_2^* - \alpha_{12}\alpha_{21} f_1^* f_2^*)} \,. \\[2ex] g_{33} = \dfrac{1}{1 - f_3^*} \end{cases}$$

So

$$(4.11)\qquad sP_{13}^*(s) = \left[\frac{\alpha_{13} f_1^* + (\alpha_{12}\alpha_{23} - \alpha_{13}\alpha_{22}) f_1^* f_2^*}{1 - \alpha_{22} f_2^* - \alpha_{12}\alpha_{21} f_1^* f_2^*}\right] \,.$$

Differentiating with respect to s, negating, setting $s = 0$, and using $f_i^*(0) = 1$ and $f_i'^*(0) = -E(\tau_i)$, we find

$$(4.12)\qquad E(T_{13}) = \frac{(\alpha_{21} + \alpha_{23})E(\tau_1) + \alpha_{12}E(\tau_2)}{\alpha_{23} + \alpha_{21}\alpha_{13}} \,.$$

In working with the $\alpha_{ij}$'s, the following identities are useful:

$$(4.13)\quad \begin{cases} 1 - \alpha_{22} - \alpha_{12}\alpha_{21} = \alpha_{13} + \alpha_{12}\alpha_{23} - \alpha_{13}\alpha_{22} \\ \qquad\qquad = \alpha_{23} + \alpha_{21}\alpha_{13} \\ \text{and} \\ \alpha_{22} + \alpha_{12}\alpha_{21} + \alpha_{12}\alpha_{23} - \alpha_{13}\alpha_{22} = \alpha_{12} \,. \end{cases}$$

Similarly, we find that

$$(4.14a) \qquad sP_{23}^*(s) = \frac{\alpha_{23}f_2^* + \alpha_{13}\alpha_{21}f_1^*f_2^*}{1 - \alpha_{22}f_2^* - \alpha_{12}\alpha_{21}f_1^*f_2^*}$$

and

$$(4.14b) \qquad E(T_{23}) = \frac{\alpha_{21}E(\tau_1) + E(\tau_2)}{\alpha_{23} + \alpha_{21}\alpha_{13}} \quad .$$

Of course,

$$(4.14c) \qquad E(T_{33}) \equiv 0 \quad .$$

## 4.3 The Transition Matrix

The matrix

$$A = \begin{pmatrix} 0 & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

is comprised of the transition probabilities.

Note that

$$(4.15) \qquad \begin{aligned} \alpha_{12} + \alpha_{13} &= 1 \\ \alpha_{21} + \alpha_{22} + \alpha_{23} &= 1 \quad . \end{aligned}$$

The absorbing nature of state 3 is reflected in $\alpha_{33} = 1$. The value of $\alpha_{11} = 0$ is implied by the fact that the end of a search leg is marked by the detection of a target which must be classified.

From (4.15) we see that once $\alpha_{12}$ is determined, $\alpha_{13}$ is also determined. Also, $\alpha_{21}$ is determined by $\alpha_{22}$ and $\alpha_{23}$. We proceed to discuss these independent quantities.

### 4.3.1 Determination of $\alpha_{12}$

If we let

$n_c$ = number of HVTs

$n_d$ = number of decoys

it seems natural at first to define

$$\alpha_{12} = \frac{n_d}{n_c + n_d} \quad . \tag{4.16}$$

This procedure, however, neglects the effects of relative detectability of the two types of targets. We have seen in Section 2 that "detectability" is a function of speed and detection range as well as number or density.

An alternative procedure, which yields a result equally intuitive for simple assumptions, is the following. Define the random variables

Y = time to detect a decoy

X = time to detect a high value target.

Then

$$\alpha_{12} = P(Y < X) \tag{4.17}$$

$$= \int_0^\infty P(Y < X | X = x) f_X(x) dx$$

where $f_X(x)$ is the p.d.f. of X, and in the following, $f_Y(y)$ is the p.d.f.

of Y. The value of $\alpha_{12}$ thus depends on the forms that are assumed for the p.d.f.'s. We consider two cases: exponential and Erlangian with parameter k = 2.

- <u>Exponential Case</u>

In this case

$$f_Y(y) = \begin{cases} \lambda e^{-\lambda y} & y > 0 \\ 0 & \text{otherwise} \end{cases}$$

(4.17a)

$$f_X(x) = \begin{cases} \beta e^{-\beta x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

so

$$P(Y < X | X = x) = P(Y < x) = \int_0^x \lambda e^{-\lambda y} dy \tag{4.18}$$

$$= 1 - e^{-\lambda x} \quad .$$

Hence

$$\alpha_{12} = \int_0^\infty (1-e^{-\lambda x})\beta e^{-\beta x} dx \tag{4.19}$$

$$= \frac{\lambda}{\lambda+\beta} \quad .$$

Since $\lambda$ is rate at which decoys are detected and $\beta$ is rate at which HVTs are detected, (4.19) is a highly intuitive result.

It is of interest to no

(4.20)
$$\begin{cases} E(Y) = \frac{1}{\lambda} , & E(X) = \frac{1}{\beta} \\ Var(Y) = \frac{1}{\lambda^2} , & Var(X) = \frac{1}{\beta^2} \end{cases} .$$

- Erlangian Case

The Erlang distribution is discussed in Ref. 5 (Hillier and Lieberman, pp 303-304). It should be noted that the Erlang distribution is similar to the gamma distribution but they are not different names for the same thing. Although they have the same "shape," custom has the parameters arranged with slight differences. If Z is a random variable of Erlang distribution with parameters $m > 0$ and $k > 0$, then the p.d.f. of Z is

(4.21)
$$f_Z(z) = \frac{(mk)^k}{\Gamma(k)} z^{k-1} e^{-kmz} dz \qquad \text{for } z \geq 0.$$

The mean and variance are $\frac{1}{m}$ and $\frac{1}{km^2}$. Note that $k = 1$ yields the exponential distribution. Figure 4-1 exhibits how k changes the shape of the distribution for a given m.

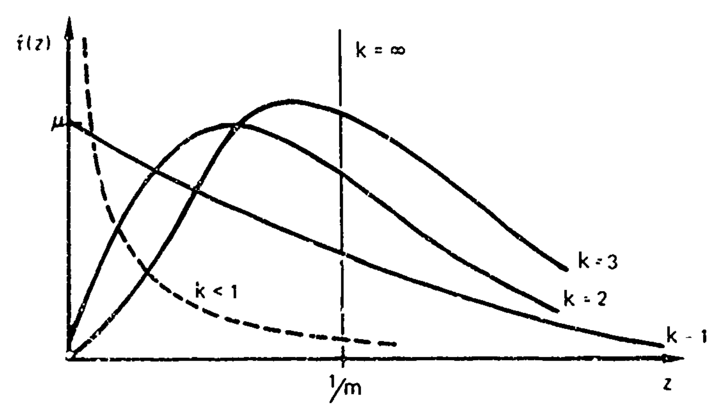


FIGURE 4-1 ERLANG DISTRIBUTION FOR VARIOUS VALUES OF k

We consider the case k = 2. Thus

$$f_Y(y) = 4\lambda^2 y e^{-2\lambda y} \qquad y \geq 0$$

(4.21a)

$$f_X(x) = 4\beta^2 x e^{-2\beta x} \qquad x \geq 0 \quad .$$

So

(4.22) $$P(Y < x) = 1 - (2\lambda x+1)e^{-2\lambda x}$$

and

$$\alpha_{12} = \int_0^\infty [1-(2\lambda x+1)e^{-2\lambda x}]4\beta^2 x e^{-2\beta x} dx$$

(4.23)

$$= \frac{\lambda^2(\lambda+3\beta)}{(\lambda+\beta)^3} \quad .$$

We note that if $\lambda = \beta$ we have $\alpha_{12} = \frac{1}{2}$, which is intuitively reassuring.

It is useful to note that if experimental data are available and appear to fit the family of Erlang distributions, the parameters may be estimated. It seems likely that the only other type of experimentally derived distribution (in the context of what we are trying to model) would be a bimodal (or multimodal) distribution. In that case it may be feasible to fit the data with a mixture of Erlang distributions, that is, a distribution with p.d.f. given by

$$p_1 f_{Z_1}(z_1) + p_2 f_{Z_2}(z_2)$$

where $Z_1$ and $Z_2$ represent two different Erlang distributions and $p_1 + p_2 = 1$, $0 \leq p_1 \leq 1$, $0 \leq p_2 \leq 1$. Some interesting technical questions are what conditions (if any) on the means and variances of

$Z_1$ and $Z_2$ and on $p_1$ and $p_2$ yield a bimodal distribution? and what are good estimators of the six parameters? (It appears that maximum likelihood estimators may be very nonlinear.)

#### 4.3.2 Determination of Detection Rates

The computation of $\alpha_{12}$ requires determination of the detection rates $\lambda$ and $\beta$. In the absence of experimental data or for ease of parametric analysis, reliance must be placed on equation (2.1). Specifically

$$(4.24)\quad \begin{cases} \lambda = \dfrac{4R_d\delta_d}{\pi}(V_d+V_s)\displaystyle\int_0^{\frac{\pi}{2}} \sqrt{1 - k_1^2\sin^2\varphi}\,d\varphi \;, & k_1 = \dfrac{2\sqrt{V_dV_s}}{V_d + V_s} \\[2ex] \beta = \dfrac{4R_c\delta_c}{\pi}(V_c+V_s)\displaystyle\int_0^{\frac{\pi}{2}} \sqrt{1 - k_2^2\sin^2\varphi}\,d\varphi \;, & k_2 = \dfrac{2\sqrt{V_cV_s}}{V_c + V_s} \end{cases}$$

where
$V_s$ = speed of searcher
$V_d$ = speed of decoys
$V_c$ = speed of HVTs
$R_d$ = detection range of decoy
$R_c$ = detection range of HVTs
$\delta_d$ = density of decoys
$\delta_c$ = density of HVTs
$\beta$ = rate at which searcher detects HVTs
$\lambda$ = rate at which searcher detects decoy.

The values of $\lambda$ and $\beta$ are expressed in inverse time units, and may intuitively be considered detections per unit time.

Two approaches may be taken regarding the densities $\delta_c$ and $\delta_d$. First of all, the situation to be modelled may be considered purely in terms of density and all questions of specific area size and number of targets ignored. On the other hand, we may compute

$$(4.25) \qquad \delta_c = \frac{n_c}{A_o}, \qquad \delta_d = \frac{n_d}{A_o}$$

where

$n_c$ = number of high value targets

$n_d$ = number of decoys

$A_o$ = area of operating region containing the targets.

The former approach may be more theoretically aesthetic, while the latter has more practical application. The choice of alternatives here should be consistent with the choice made in the next subsection for $\alpha_{22}$ and $\alpha_{23}$.

### 4.3.3 Determination of $\alpha_{22}$ and $\alpha_{23}$

The probabilities $\alpha_{22}$ and $\alpha_{23}$ are defined by

$$(4.26) \quad \begin{cases} \alpha_{22} = \Pr\{\text{next event is "classify decoy"} \mid \text{last event was "classify decoy"}\} \\ \alpha_{23} = \Pr\{\text{next event is "classify HVT"} \mid \text{last event was "classify decoy"}\}. \end{cases}$$

In other words, $\alpha_{22}$ represents the probability of going from a decoy to another decoy and $\alpha_{23}$ the probability of going from a decoy to a high value target. By the law of total probability $\alpha_{21}$, representing the probability of going from a decoy to search, is given by $1 - \alpha_{22} - \alpha_{23}$.

It turns out that certain results are highly sensitive to $\alpha_{22}$ and $\alpha_{23}$. Also, certain unexpected and unusual, but explicable, effects are due to $\alpha_{23}$. Hence a high degree of care must be exercised in specifying $\alpha_{22}$ and $\alpha_{23}$.

As with $\alpha_{12}$, there are two approaches to considering $\alpha_{22}$ and $\alpha_{23}$: density or area. In addition, there is the question of adapting the assumption of random movement to movement which is random with a condition of reduced interaction or "overlap" on the targets.

- Area Approach

Consider an area of $A_o$ square nautical miles containing $n_d$ decoys, $n_c$ high value targets, and one searcher. Assume that at any given instant the decoys and HVTs are uniformly distributed at random over the area.

The complete classification of a decoy anticipates an event. The next event to occur can be either start of search, start of classification of (another) decoy, or start classification of an HVT.

In order for the next event to be start of classification of another decoy, at least one other decoy must be present and, if one or more HVTs are also present, a decoy must be chosen from the array of targets.

So

$$(4.27)\quad \begin{cases} \alpha_{22} = P\{\text{at least one new decoy present and pick decoy} \mid \text{just finished a decoy}\} \\ \quad = \sum_{k=1}^{n_d-1} P\{k \text{ new decoys present and pick decoy} \mid \text{just finished decoy}\} \\ \quad = \sum_{k=1}^{n_d-1} \sum_{\ell=0}^{n_c} P\{k \text{ new decoys present, } \ell \text{ HVTs present, pick decoy} \mid \text{just finished decoy}\} \end{cases}$$

i.e.,

$$(4.28) \qquad \alpha_{22} = \sum_{k=1}^{n_d-1} \sum_{\ell=0}^{n_c} \binom{n_d-1}{k} P_d^k (1-P_d)^{n_d-1-k} \binom{n_c}{\ell} P_c^{\ell} (1-P_c)^{n_c-\ell} \frac{k}{\ell+k} .$$

This last statement follows from

$$(4.29) \qquad P\{k \text{ new decoys present}, \ell \text{ HVTs present, pick decoy}\}$$

$$= P\{k \text{ new decoys present}\} \cdot P\{\ell \text{ HVTs present}\}$$

$$\cdot P\{\text{pick decoy} | k \text{ new decoys and } \ell \text{ HVTs present}\}$$

plus the assumption that the motions of decoys and HVTs are stochastically independent and the fact that

$$P(A \cap B \cap C) = P(A \cap B)P(C|A \cap B) = P(A)P(B)P(C|A \cap B)$$

if A and B are independent events. Also

$$P\{k \text{ new decoys present}\} = \binom{n_d-1}{k} P_d^k (1-P_d)^{n_d-1-k}$$

$$(4.30) \qquad P\{\ell \text{ HVTs present}\} = \binom{n_c}{\ell} P_c^{\ell} (1-P_c)^{n_c-\ell}$$

$$P\{\text{pick decoy} | k \text{ new decoys and } \ell \text{ HVTs}\} = \frac{k}{k+\ell}$$

and

$$P_c = \pi R_c^2 / A_o$$

$$P_d = \pi R_d^2 / A_o .$$

$P_c$ is the probability any one HVT is present (within range of the searcher). $P_d$ is analogous.

In a similar manner we write

$$(4.31) \qquad \alpha_{23} = \sum_{k=0}^{n_d-1} \sum_{\ell=1}^{n_c} \binom{n_d-1}{k} P_d^k (1-P_d)^{n_d-1-k} \binom{n_c}{\ell} P_c^{\ell} (1-P_c)^{n_c-\ell} \frac{\ell}{\ell+k} \quad .$$

Note the difference between $\alpha_{22}$ and $\alpha_{23}$ is in the limits of summation and the probability of picking the appropriate target.

Let's consider the special case where $n_c = 1$. Then (4.31) becomes

$$(4.32) \qquad \alpha_{23} = P_c \sum_{k=0}^{n_d-1} \frac{1}{k+1} \binom{n_d-1}{k} P_d^k (1-P_d)^{n_d-1-k}$$

$$= \frac{P_c}{P_d} \frac{1}{n_d} \sum_{k=1}^{n_d} \binom{n_d}{k} P_d^k (1-P_d)^{n_d-k}$$

$$= \frac{P_c}{P_d n_d} \left[ 1 - (1-P_d)^{n_d} \right] \quad .$$

Similarly, (4.28) becomes

$$(4.33) \qquad \alpha_{22} = (1-P_c) \sum_{k=1}^{n_d-1} \binom{n_d-1}{k} P_d^k (1-P_d)^{n_d-1-k}$$

$$+ P_c \sum_{k=1}^{n_d-1} \frac{k}{1+k} \binom{n_d-1}{k} P_d^k (1-P_d)^{n_d-1-k}$$

$$= (1-P_c)\left[ 1 - (1-P_d)^{n_d-1} \right] + \frac{P_c}{P_d n_d} \left[ n_d P_d + (1-P_d)^{n_d} - 1 \right] \quad .$$

It is also clear that in general

(4.34) $$\alpha_{21} \triangleq \Pr\{\text{no HVTs or new decoys present} \mid \text{just finished a decoy}\}$$

$$= (1-P_c)^{n_c}(1-P_d)^{n_d-1} \quad .$$

The case $n_c = 1$ now makes a good check case for $\alpha_{21} + \alpha_{22} + \alpha_{23} = 1$.

- Density Approach

It is possible to drop explicit consideration of area and number of targets. Equations (4.27) and (4.28) still hold, but the binomial distribution is replaced by the Poisson distribution. The results are

(4.35)
$$\alpha_{21} = m\,y$$
$$\alpha_{23} = m\,x \sum_{k=1}^{\infty} \frac{(1+\rho)^k y^k - x^k}{k(k!)}$$
$$\alpha_{22} = 1 - \alpha_{21} - \alpha_{23}$$

where

$$x = \delta_c \pi R_c^2$$
$$y = \delta_d \pi R_d^2$$
$$\rho = x/y$$
$$m = \frac{e^{-x-y}}{1-e^{-y}} \quad .$$

Details are given in Appendix A.

- Effects of Overlap

One of the main assumptions has been random motion of the targets and uniform random distribution of the targets at any given instant in time. One suspects that the evader might improve his situation by not allowing targets to congregate, as will happen occasionally under purely random motion. It happens that this suspicion is supported by analysis to be given shortly. On the other hand, one also suspects that, as the target density increases, it becomes increasingly difficult to prevent congregation. This matter will be discussed also.

Suppose that it can be guaranteed that HVTs are kept a sufficient distance from decoys to cause $\alpha_{23} = 0$. Compare this with the situation where $\alpha_{23} \neq 0$, i.e., normal overlap. Consider the case where the $f_i$ are exponential distributions (the Markov process) for one HVT. We then have

$$(4.36) \qquad E(T_{13} | n_d = 1, \alpha_{23} = 0) > E(T_{13} | n_d = 1, \alpha_{23} \neq 0)$$

from (4.12) with $n_d = 1$. That is, for $\alpha_{23} \neq 0$

$$(4.37) \qquad \begin{cases} \alpha_{12} = \dfrac{\lambda}{\lambda+\beta}, & \alpha_{13} = \dfrac{\beta}{\lambda+\beta} \\ \alpha_{21} = 1 - P_c, & \alpha_{22} = 0, \quad \alpha_{23} = P_c \end{cases}$$

so

$$(4.38) \qquad E(T_{13} | n_d = 1, \alpha_{23} \neq 0) = \frac{\mu+\lambda}{(\beta+P_c\lambda)\mu}$$

where $\mu$ is the mean of $f_3(t)$.

For $\alpha_{23} = 0$

(4.39)
$$\begin{cases} \alpha_{12} = \dfrac{\lambda}{\lambda+\beta}, & \alpha_{13} = \dfrac{\beta}{\lambda+\beta} \\ \alpha_{21} = 1, & \alpha_{22} = \alpha_{23} = 0 \end{cases}$$

so

(4.40)
$$E(T_{13} | n_d = 1, \alpha_{23} = 0) = \frac{\mu+\lambda}{\beta\mu} .$$

The inequality is now clear. Thus the evader <u>does wish</u> to separate an HVT and a decoy, at least after the searcher has deployed and is searching.

- <u>Model Restrictions Due to Overlap</u>

Turning to another question, one might ask whether it is always of advantage to the evader to use decoys. If the decoys and HVTs are independent, the intuitive answer is yes. However, the model sometimes answers "No!" There are reasonable cases where $E(T_{13} | n_d = n)$ and $E(T_3 | n_d = n)$ will at first decrease as n is increased from zero. This phenomenon is an artifact due to failure of the random motion/position assumption. To gain at least a partial understanding of why it happens, let us examine the case where the $f_i$ are exponential distributions (the Markov process) for one HVT with first zero and then one decoy. Considering first $T_{13}$ we have

(4.41)
$$E(T_{13} | n_d = 0) = \frac{1}{\beta}$$

and

(4.42)
$$E(T_{13} | n_d = 1) = \frac{\mu+\lambda}{(\beta+P_c\lambda)} .$$

since with $n_d = 1$ we have

$$(4.43) \quad \begin{cases} \alpha_{12} = \dfrac{\lambda}{\lambda+\beta} , & \alpha_{13} = \dfrac{\beta}{\lambda+\beta} \\ \alpha_{21} = 1 - P_c , & \alpha_{22} = 0 , \end{cases}$$

The desirable situation for the HVT is $E(T_{13}|n_d = 1) > E(T_{13}|n_d = 0)$. We thus require

$$(4.44) \qquad \frac{\mu+\lambda}{(\beta+P_c\lambda)\mu} > \frac{1}{\beta}$$

or

$$(4.45) \qquad \beta > \mu P_c .$$

The relation (4.45) can be interpreted as follows. At the beginning of decoy classification, the HVT is known not to be within detection range. As long as the mean classification time is long enough (i.e., (4.45) satisfied), then this initial condition will dissipate, and the HVT position will be uniformly random over the entire area at the end of decoy classification. Failure of (4.45) to hold means the HVT speed (or the classification time) is not high enough to dissipate this initial condition. Notice that if overlap is prohibited so that $\alpha_{23} = 0$, then (4.44) becomes $\frac{\lambda}{\mu} > 0$, which always holds (for all practical situations).

Now consider $T_3$ in the same manner we have just examined $T_{13}$. Then

$$(4.46) \qquad E(T_3|n_d = 0) = p_1 E(T_{13}|n_d = 0) = (1-F_c)\frac{1}{\beta}$$

since $p_1 = 1 - P_c$ when $n_d = 0$, and from (4.38)

$$(4.47) \qquad E(T_3 | n_d = 1) \; = \; p_1 E(T_{13} | n_d = 1) + p_2 E(T_{23} | n_d = 1)$$

where $E(T_{13} | n_d = 1)$ is given above by (4.42). Substituting the values of (4.43) into (4.14b) we get

$$(4.48) \qquad E(T_{23} | n_d = 1) \; = \; \frac{\mu(1-P_c) + \lambda+\beta}{\mu(P_c\lambda+\beta)} \quad .$$

So

$$(4.49) \qquad E(T_3 | n_d = 1) \; = \; \frac{p_1(\mu+\lambda) + p_2\mu(1-P_c) + p_2(\lambda+\beta)}{\mu(\beta+P_c\lambda)} \quad .$$

Thus in order to have $E(T_3 | n_d = 1) > E(T_3 | n_d = 0)$ we require

$$(4.50) \qquad p_2\beta^2 + \beta\left\{p_1(\mu+\lambda) + p_2[\mu(1-P_c) + \lambda] - (1-P_c)\mu\right\} - (1-P_c)P_c\mu\lambda \;\geq\; 0$$

or, substituting

$$a \; = \; p_2$$

$$(4.51) \qquad b \; = \; p_1(\mu+\lambda) + p_2[\mu(1-P_c) + \lambda] - (1-P_c)\mu$$

$$c \; = \; (1-P_c)P_c\mu\lambda \quad ,$$

we require

$$(4.52) \qquad a\beta^2 + b\beta - c \;\geq\; 0$$

where

$$(4.53) \qquad a > 0 \quad , \quad c > 0 \quad .$$

Completing the square, (4.52) can be written as

$$(4.54) \qquad p(\beta) = \left[\beta + \frac{b}{2a}\right]^2 - \frac{c}{a} - \frac{b^2}{4a^2} \geq 0 \quad .$$

Now $p(\beta)$ is a quadraic form in $\beta$ with two real roots $-\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 + 4ac}$ and a minimum of $-\frac{1}{4a^2}(b^2 + 4ac)$ at $\beta = -\frac{b}{2a}$. Further

$$b^2 + 4ac > b^2$$

since $a > 0$ and $c > 0$ so

$$\frac{1}{2a}\sqrt{b^2 + 4ac} > \frac{b}{2a}$$

and hence

$$(4.55) \qquad \begin{aligned} p(\beta) &< 0 \quad \text{for} \quad 0 < \beta < \frac{1}{2a}\left\{\sqrt{b^2 + 4ac} - b\right\} \\ p(\beta) &\geq 0 \quad \text{for} \quad \beta \geq \frac{1}{2a}\left\{\sqrt{b^2 + 4ac} - b\right\} \quad . \end{aligned}$$

Since $E(T_3|n_d = 1) > E(T_3|n_d = 0)$ if and only if $p(\beta) \geq 0$, it is also possible for the deployment of a decoy to be disadvantageous to the HVT when using $E(T_3)$ as a measure. Unfortunately, this threshold on $\beta$ is not as easily interpreted as when using $E(T_{13})$. However, if $b > 0$ we note that $b + \sqrt{4ac} > \sqrt{b^2 + 4ac}$ and hence $p(\beta) > 0$ if $\beta > \sqrt{\frac{c}{a}}$. In particular, if $\mu < 1$ (i.e., mean decoy classification time is greater than 1 hour) and if $\sqrt{\frac{\lambda}{2p_2}} < P_c$ then $\mu P_c > \sqrt{\frac{c}{a}}$ since $\sqrt{\mu} > \mu$; moreover,

$\beta > P_c\mu$ implies $\beta > \sqrt{\frac{c}{a}}$. Hence in this case, any $\beta$ good for $E(T_{13})$ is good for $E(T_3)$, and there may be a range of $\beta$ good for $E(T_3)$ which is not good for $E(T_{13})$.*

If overlap is not permitted so that $\alpha_{23} = 0$, then since in this case $p_3 = P_c$ instead of (4.50), we have merely $(p_1 + p_2)\lambda + p_2\beta \geq 0$ which always holds.

To summarize thus far, if overlap is permitted we must check that $\beta \geq \mu P_c$ and $p(\beta) \geq 0$ to guard against the failure of the random position assumption. It is useful to note, however, that as the number of decoys increases this problem will diminish because eventually the target selection problem will overshadow the error in the probability of the HVT being present. Possibilities for removing these restrictions are discussed in Section 7.

Unfortunately, these conditions alone are neither necessary nor sufficient to guarantee the accuracy of the model, even though the preceding discussion may seem to imply them. The reason is that a further problem remains in the fact that after having classified a decoy and returned to search, the search rate is not $\lambda+\beta$ immediately. In the case of a single decoy the search rate is simply $\beta$ until the effect of the information obtained in the last decoy classification has dissipated, at which time the search rate is restored to $\lambda+\beta$. In the

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* Sample case: anticipating Section 4.7 we have for $n_d = 1$ that $p_2 = P_d + P_c[1-2P_d]/2$; using formulas for $P_c$ and $P_d$ given with (4.30) and letting $V_s = 15$, $V_c = V_d = 20$, $R_c = R_d = 60$, we get $\lambda < .002$ or $1/\lambda > 500$; from Table 2-1 we see this is <u>not</u> satisfied since $1/\lambda = 45.67$; since this is a typical case, the bound above doesn't say too much.

case of multiple decoys, the search rate immediately after classification is something less than $\lambda+\beta$ until an appropriate time for restoration to $\lambda+\beta$. The transition probabilities are similarly affected. In the case where the obtained information does indeed dissipate, this problem is solvable by adding another state to the model. This additional state would be a secondary search state that accounts for the information dependency; a transition to the primary search state would be made after a suitable length of time if the HVT is not found in the interim. The case where the information does not dissipate (e.g., position plotting of stationary decoys) must be treated in a different manner. A possible approach exploiting special matrix structure is discussed in Section 7.

- Reduced Overlap

Earlier, under the subheading "Effects of Overlap" it was shown that it may be advantageous to eliminate, or at lease reduce, overlap between HVTs and decoys. We now consider some ways of incorporating this reduced overlap into the model.

Suppose the decoys are not stationary (i.e., are mobile). Then it may be feasible for them to know or recognize where the HVTs are going to be and simply stay a suitable distance, call it $R_s$, away from them. Let $R_s = R_c + \rho_d$ where $\rho_d$ = decoy classification range. In this case, if $\rho_d$ is small, a simple approximation is to set $\alpha_{22} = 0$. (See the end of Appendix D for further discussion of this approach.)

Suppose now that the decoys are stationary or that it is not feasible for the decoys to know or recognize where the HVTs will be. Two basic methods of approaching this problem have been considered. Both require that an HVT have the capability to recognize and avoid a decoy. The first method is a queuing model which requires only specification of an operational parameter reflecting the objectives of the HVT activities.

The second method involves a class of models which basically requires estimation of a parameter describing fleet operations or fitting a curve to observed data. Since the first method seems more suitable to the course of the current analysis, it is presented here and the second method is discussed in Appendix B.

Let $\lambda_c$, the rate at which HVTs encounter decoys, be defined as

$$\lambda_c = \frac{4R_s n_c n_d}{\pi A_o} (V_d + V_c) \, E\left(\frac{2\sqrt{V_d V_c}}{V_c + V_d}\right) \qquad (4.56)$$

where E(k) is the elliptic integral discussed in Section 2. Also define

(4.57) $\lambda_o$ = maximum acceptable rate of HVT course change for purpose of decoy avoidance.

For example, an HVT may be subject to constraints such that it would be willing to change course to avoid a decoy on the average only once in every 5 hours, in which case $\lambda_o = 0.2 \text{ hr}^{-1}$. We can now model the situation of a single HVT as a single server queue with customer balking, that is, when the customer arrives if the server is busy the arriving customer leaves. In this case the HVT is the server and the customer is a decoy. The customer arrival rate is $\lambda_c$ and the service rate is $\lambda_o$. Customer arrival corresponds to HVT contact of a decoy. Customer service corresponds to the length of time after a decoy avoidance maneuver before the HVT is willing to make another such maneuver. Such a queuing system can be represented by a two-state Markov process, where the states 0 or 1 correspond to the number of customers being served or, equivalently, whether or not the server is busy. The process can be represented by the generator matrix

(4.58) $$Q = \begin{pmatrix} -\lambda_c & \lambda_c \\ \lambda_o & -\lambda_o \end{pmatrix} .$$

Let B denote "system in state 1" and $\bar{B}$ "system in state 0." Assume the system is in steady state. Let P(B) = Pr(B) and P($\bar{B}$) = Pr($\bar{B}$) in steady state. Then from Ref. 2 (p 194) we have

(4.59)
$$P(B) = \frac{\lambda_c}{\lambda_o + \lambda_c}$$

$$P(\bar{B}) = \frac{\lambda_o}{\lambda_o + \lambda_c} .$$

We refer now to Fig. 4-2. $A_s$ represents the area $\pi(R_c + \rho_d)^2$ and $A^*$ the area $\pi R_c^2$. The decoy is centered in $A_s$, the searcher in $A^*$. Denote "HVT in area A" by "H$\epsilon$A." We assume $H\epsilon A_s \Rightarrow B$ and hence $P(H\epsilon A^* \& \bar{B}|H\epsilon A_s) = 0$ and $P(B|H\epsilon A_s) = 1$. Then

(4.60)
$$\begin{aligned} P(H\epsilon A^*|H\epsilon A_s) &= P(H\epsilon A^* \& B|H\epsilon A_s) + P(H\epsilon A^* \& \bar{B}|H\epsilon A_s) \\ &= P(H\epsilon A^* \& B|H\epsilon A_s) \\ &= P(H\epsilon A^*|B \& H\epsilon A_s)P(B|H\epsilon A_s) \\ &= P(H\epsilon A^*|H\epsilon A_s) \\ &= A^*/A_s \\ &= (R_c/R_s)^2 . \end{aligned}$$

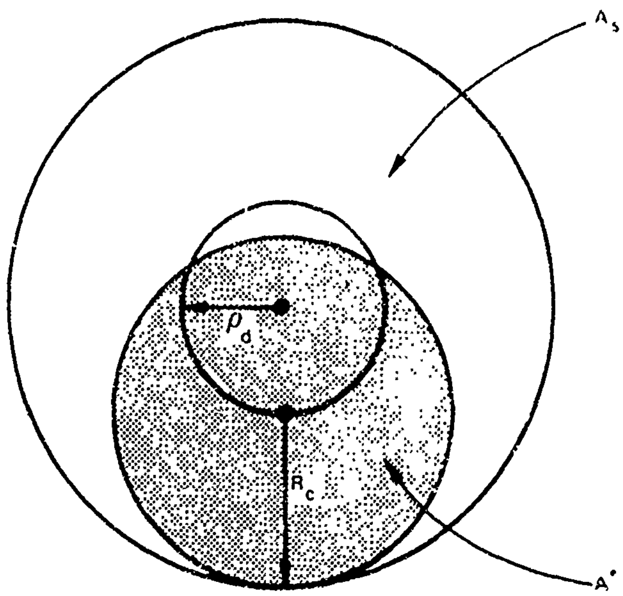


FIGURE 4-2 AREAS IN QUEUE MODEL OF OVERLAP

Now

(4.61) $$P(H \epsilon A_s | B) = \pi (R_c + \rho_d)^2 / A_o$$

so

(4.62) $$\begin{aligned} P(H \epsilon A_s) &= P(H \epsilon A_s \;\&\; \bar{B}) + P(H \epsilon A_s \;\&\; B) \\ &= P(H \epsilon A_s | B) P(B) \\ &= [\pi (R_c + \rho_d)^2 / A_o] \frac{\lambda_c}{\lambda_o + \lambda_c} \quad . \end{aligned}$$

We interpret $P_c$ as $P_c = P(H\epsilon A^*)$. Furthermore, $P(H\epsilon A^* | H \notin A_s) = 0$. Hence

$$(4.63) \qquad P(H\epsilon A^*) = P(H\epsilon A^* | H\epsilon A_s)P(H\epsilon A_s) + P(H\epsilon A^* | H\notin A_s)P(H\notin A_s)$$

so

$$(4.64) \qquad P_c(\lambda_o) = \frac{\pi(R_c + \rho_d)^2}{A_o}\left(\frac{R_c}{R_s}\right)^2 \frac{\lambda_c}{\lambda_o + \lambda_c}$$

$$= \frac{\pi R_c^2}{A_o} \frac{\lambda_c}{\lambda_o + \lambda_c} .$$

Note that $\rho_d$ enters the equation through $\lambda_c$. We can now substitute $P_c(\lambda_o)$ for $P_c$ in the computation of $\alpha_{22}$ and $\alpha_{23}$. This approach could perhaps be generalized to priority queuing system with balking and two types of customers, decoys and HVT activities, where the HVT activities have priority over decoy avoidance. This generalization is not developed here for lack of time. A discussion of possible error due to a failure of the implicit assumption that $A^*/A_s = (R_c/R_s)^2$ is given in Appendix C.

## 4.4 The P.D.F. Matrix

The form of $f_1(t)$ is dependent on the way in which $\alpha_{12}$ and $\alpha_{13}$ are derived. The forms of $f_2(t)$ and $f_3(t)$ may be specified freely, and, as noted previously, the form of $f_3(t)$ is irrelevant.

Referring to the discussion surrounding (4.17) we see that we should write*

$$(4.65) \qquad P(\tau_1 \le t) = P(\tau_1 \le t \;\&\; Y \le X) + P(\tau_1 \le t \;\&\; X < Y)$$

$$= P(Y \le X \;\&\; Y \le t) + P(X < Y \;\&\; X \le t)$$

*Consider $\tau_1 = \min(X,Y)$.

where, recall,

$$Y = \text{time to detect decoy}$$

$$X = \text{time to detect an HVT} \quad .$$

Now

$$(4.66) \qquad P(Y \le X \;\&\; Y \le t) = \int_0^\infty P(Y \le \tau \;\&\; Y \le t) f_X(\tau) d\tau$$

$$= \int_0^t P(Y \le \tau) f_X(\tau) d\tau + P(Y \le t) \int_t^\infty f_X(\tau) d\tau \quad .$$

Similarly

$$P(X < Y \;\&\; X \le t) = \int_0^t P(X \le t) f_Y(\tau) d\tau + P(X \le t) \int_t^\infty f_Y(\tau) d\tau \quad .$$

We now examine the exponential and Erlangian cases.

### 4.4.1 Exponential Case

Recall that in this case

$$f_X(\tau) = \beta e^{-\beta\tau}, \ \tau > 0 \quad \text{and} \quad f_Y(\tau) = \lambda e^{-\lambda\tau}, \ \tau > 0 \quad .$$

Thus, using (4.66) we have

$$(4.67) \qquad P(Y \le X \;\&\; Y \le t) = \int_0^t (1 - e^{-\lambda\tau}) \beta e^{-\beta\tau} d\tau + (1 - e^{-\lambda t}) e^{-\beta t}$$

$$= 1 - \frac{\beta}{\lambda + \beta} + \frac{\beta}{\lambda + \beta} e^{-(\lambda+\beta)t} - e^{-(\lambda+\beta)t} \quad .$$

Similarly

$$(4.68) \qquad P(X < Y \;\&\; X \le t) = 1 - \frac{\lambda}{\lambda+\beta} + \frac{\lambda}{\lambda+\beta} e^{-(\lambda+\beta)t} - e^{-(\lambda+\beta)t} \; .$$

Hence (4.65) gives us

$$(4.69) \qquad P(T_1 \le t) = 1 - e^{-(\lambda+\beta)t}$$

or

$$(4.70) \qquad f_1(t) = (\lambda + \beta)\, e^{-(\lambda+\beta)t} \; .$$

4.4.2 Erlangian Case

Refer to the Erlangian part of Section 4.3.1. Using (4.21a) and (4.22) we have

$$(4.71) \qquad P(Y \le X \;\&\; Y \le t) = \int_0^t [1 - (2\lambda\tau+1)e^{-2\lambda\tau}]4\beta^2\tau e^{-2\beta\tau} d\tau$$

$$+ [1 - (2\lambda t+1)e^{-2\lambda t}][2\beta t+1]e^{-2\beta t}$$

$$= 1 - \left[\frac{\beta^3+3\beta^2\lambda}{(\lambda+\beta)^3}\right] + \left[\frac{\beta^3+3\beta^2\lambda}{(\lambda+\beta)^3}\right] [2(\lambda+\beta)t + 1]e^{-2(\lambda+\beta)t}$$

$$+ \frac{4\lambda\beta^2 t^2 e^{-2(\lambda+\beta)t}}{\lambda+\beta} - (2\lambda t+1)(2\beta t+1)e^{-2(\lambda+\beta)t} \; .$$

A similar expression holds for $P(X \le Y \;\&\; X \le t)$. Add these expressions to get

$$(4.72) \qquad P(T_1 \le t) = 1 + [2(\lambda+\beta)t+1 - 2(2\lambda t+1)(2\beta t+1) + 8\lambda\beta t^2]e^{-2(\lambda+\beta)t}$$

$$= 1 - [2(\lambda+\beta)t+1]e^{-2(\lambda+\beta)t}$$

which is Erlang with parameters 2 and $\lambda+\beta$. Hence

$$f_1(t) = 4(\lambda+\beta)^2 te^{-2(\lambda+\beta)t} \qquad t > 0 \quad . \tag{4.73}$$

4.5 Renewal

As mentioned in Section 1, a quantity of interest is the number of decoys encountered by the searcher. Here the renewal density $h_{ij}(t)$ and its integral $H_{ij}(t)$, as discussed in Section 3, come into play. Recall that $H_{ij}(t)$ is the expected number of events of type j occurring in the time interval (0,t) given that an event of type i occurred at time zero. (Recall also that an event occurs when the process first enters or renews a state.) We are interested in events of type j=2, first encounters of decoys. From (3.2) and (3.3) we have

$$H^*(s) = \frac{1}{s} h^*(s)$$

where

$$h^*(s) = g^*(s) \, [I - g^*(s)]^{-1}$$

with $g^*(s)$ and $[I - g^*(s)]^{-1}$ given by (4.6) and (4.8). Thus

$$\begin{aligned} h^*_{12}(s) &= g^*_{1\cdot}(s) \, [I - g^*(s)]^{-1}_{\cdot 2} \\ &= [g^*_{11}(s), \; g^*_{12}(s), \; g^*_{13}(s)] \begin{bmatrix} g_{12} \\ g_{22} \\ g_{32} \end{bmatrix} \\ &= g_{22} \cdot g^*_{12}(s) = \alpha_{12} f^*_1 g_{22} \\ &= \frac{\alpha_{12} f^*_1}{1 - \alpha_{22} f^*_2 - \alpha_{12}\alpha_{21} f^*_1 f^*_2} \quad . \end{aligned} \tag{4.74}$$

Similarly

$$(4.75)\qquad h_{22}^*(s) = \frac{\alpha_{21}^2 f_2^{*2} + \alpha_{22} f_2^*}{1 - \alpha_{22} f_2 - \alpha_{12}\alpha_{21} f_1^* f_2^*} \quad .$$

We will now consider computation of $H_{12}(t)$ for some special cases of $f_1$ and $f_2$.

### 4.5.1 Exponential Case

In this case we have, using (4.70),

$$(4.76)\qquad f_1^*(s) = \frac{\lambda+\beta}{\lambda+\beta+s} \quad \text{and} \quad f_2^*(s) = \frac{\mu}{\mu+s}$$

and also

$$\alpha_{12} = \frac{\lambda}{\lambda+\beta} \quad \text{and} \quad \alpha_{13} = \frac{\beta}{\lambda+\beta} \quad .$$

So

$$(4.77)\qquad h_{12}^*(s) = \alpha_{12} \frac{\frac{\lambda+\beta}{\lambda+\beta+s}}{1 - \alpha_{22}\frac{\mu}{\mu+s} - \alpha_{12}\alpha_{21}\frac{\lambda+\beta}{\lambda+\beta+s}\cdot\frac{\mu}{\mu+s}}$$

$$= \alpha_{12}(\lambda+\beta)\frac{s+\mu}{s^2 + c_1 s + c_2}$$

where

$$c_1 = (1-\alpha_{22})\mu + \lambda + \beta$$

$$c_2 = (\alpha_{23}+\alpha_{21}\alpha_{13})\mu(\lambda+\beta) \quad .$$

Hence

$$(4.78)\qquad H_{12}^*(s) = \frac{1}{s} h_{12}^*(s)$$

$$= \alpha_{12}(\lambda+\beta)[m_1^*(s) + \mu m_2^*(s)]$$

where

$$m_1^*(s) = \frac{1}{s^2 + c_1 s + c_2}$$

$$m_2^*(s) = \frac{1}{s(s^2 + c_1 s + c_2)} = \frac{1}{s} m_1^*(s) \quad .$$

By completing the square in the denominator we have

(4.79)
$$m_1^*(s) = \frac{1}{\left[s + \frac{c_1}{2}\right]^2 - \left[\frac{c_1^2}{4} - c_2\right]}$$

where it can be shown that $\frac{c_1^2}{4} - c_2 \geq 0$.

Let

(4.80)
$$m_3^*(s) = \frac{1}{s^2 - p^2}$$

where
$$p = \frac{1}{2}\sqrt{c_1^2 - 4c_2}$$

then
$$m_1^*(s) = m_3^*(s-a) \qquad \text{where } a = -\frac{c_1}{2} \ .$$

Now

$$m_3^*(s) \longleftrightarrow F_3(t) = \frac{1}{p} \sinh pt$$

and

$$m_3^*(s-a) \longleftrightarrow e^{at} F_3(t)$$

so

(4.81)
$$m_1^*(s) \longleftrightarrow F_1(t) = e^{at} F_3(t) = \frac{e^{at}}{p} \sinh pt$$

$$= \frac{1}{2p} \left[e^{(a+p)t} - e^{(a-p)t}\right] \quad .$$

By noting that $\frac{1}{s} m^*(s) \longleftrightarrow \int_0^t F(\tau)d\tau$ we obtain $m_2^*(s) = \frac{1}{s} m_1^*(s)$ by integrating $F_1(t)$. Thus

$$(4.82) \qquad F_2(t) = \int_0^t F_1(\tau)d\tau = \frac{1}{c_2} + \frac{1}{2p}\left[\frac{e^{(a+p)t}}{a+p} - \frac{e^{(a-p)t}}{a-p}\right] .$$

From (4.78) it follows that

$$(4.83) \qquad H_{12}(t) = \alpha_{12}(\lambda+\beta)\left[F_1(t) + \mu F_2(t)\right]$$

i.e.,

$$(4.84) \qquad H_{12}(t) = \frac{\alpha_{12}}{\alpha_{23}+\alpha_{21}\alpha_{13}} + \frac{\alpha_{12}(\lambda+\beta)}{2p}\left[\left(1+\frac{\mu}{a+p}\right)e^{(a+p)t} - \left(1+\frac{\mu}{a-p}\right)e^{(a-p)t}\right]$$

where

$$c_1 = (1-\alpha_{22})\mu + \lambda + \beta$$

$$c_2 = (\alpha_{23} + \alpha_{21}\alpha_{13})\mu(\lambda+\beta)$$

$$a = -c_1/2$$

$$p = \frac{1}{2}\sqrt{c_1^2 - 4c_2} .$$

Typically $a + p$ is negative and very close to zero, so the right-hand term in (4.84) decays to zero as $t \to \infty$ and $H_{12}(t) \nearrow \frac{\alpha_{12}}{\alpha_{23} + \alpha_{21}\alpha_{13}}$ as $t \to \infty$. Note $H_{12}(0) = 0$.

By a similar approach $H_{22}(t)$ could also be found. Then, given $p_1$ and $p_2$, the unconditional expected number of decoys encountered is given by $p_1H_{12}(t) + p_2H_{22}(t)$.

### 4.5.2 Erlangian (k=2) Case

Suppose, as in (4.73), that

$$f_1(t) = 4(\lambda+\beta)^2 t\, e^{-2(\lambda+\beta)t} \qquad t > 0$$

and

$$f_2(t) = 4\mu^2 t\, e^{-2\mu t} \qquad t > 0 \quad .$$

Then

$$(4.85) \qquad f_1^*(s) = \left[\frac{2(\lambda+\beta)}{2(\lambda+\beta) + s}\right]^2 \quad \text{and} \quad f_2^*(s) = \left[\frac{2\mu}{2\mu + s}\right]^2$$

since

$$\int_0^\infty 4a^2 t\, e^{-(2a+s)t} = 4a^2 \frac{e^{-(2a+s)t}}{(2a+s)^2}\left[-(2a+s) - 1\right]\Bigg|_{t=0}^{\infty} = \left[\frac{2a}{2a + s}\right]^2 \quad .$$

Thus

$$(4.86) \qquad h_{12}^*(s) = \alpha_{12} \frac{\left[\frac{2(\lambda+\beta)}{2(\lambda+\beta)+s}\right]^2}{1 - \alpha_{22}\left[\frac{2\mu}{2\mu+s}\right]^2 - \alpha_{12}\alpha_{21}\left[\frac{2\mu}{2\mu+s}\right]^2\left[\frac{2(\lambda+\beta)}{2(\lambda+\beta)+s}\right]^2}$$

$$= \alpha_{12} \frac{[2(\lambda+\beta)]^2[2\mu+s]^2}{[2(\lambda+\beta)+s]^2[2\mu+s]^2 - \alpha_{22}[2\mu]^2[2(\lambda+\beta)+s]^2 - \alpha_{12}\alpha_{21}[4\mu(\lambda+\beta)]^2}$$

$$= 4\alpha_{12}(\lambda+\beta)^2 \frac{p(s)}{q(s)}$$

where

$$p(s) = s^2 + 4\mu s + 4\mu^2$$

$$q(s) = s^4 + 4(\lambda+\beta+\mu)s^3 + 4[(\lambda+\beta+\mu)^2 + 2\mu(\lambda+\beta) - \alpha_{22}\mu^2]s^2$$

$$+ 16\mu(\lambda+\beta)[\lambda+\beta + (1 - \alpha_{22})\mu]s - [4\mu(\lambda+\beta)]^2(\alpha_{22}+\alpha_{12}\alpha_{21}) \quad .$$

Moreover

$$H^*_{12}(s) = 4\alpha_{12}(\lambda+\beta)^2 \frac{sp(s)}{q(s)} \quad . \tag{4.87}$$

Referring to the Heaviside expansion theorem (e.g., Ref. 6, 1021), we can write

$$H_{12}(t) = \sum_{k=1}^{r} \frac{a_k p(a_k)}{q'(a_k)} e^{a_k t} \tag{4.88}$$

where $r = 4$ is the number of roots of $q(s) = 0$ and $a_k$, $k=1, \ldots, r$ are the roots. Reference 4 (p 66 ff) discusses methods for obtaining roots of polynomials numerically; such methods are commonly available as computer programs.

### 4.6 Distribution Function of First Passage Time

First passage time, $T_{ij}$, was defined in general in Section 3.2. Expected first passage times for the search model were considered in Section 4.2. In this section we derive the probability distribution function $G_{13}(t)$ for the conditional first passage (or absorption) time $T_{13}$ in the case of exponential $f_i(t)$. The algebra here is quite similar to that of the preceding section.

From (3.12) we have $G_{13}(t) = P(T_{13} \leq t) = r_{13}(t)$. Recalling that in the terminology of Section 3, $p_{13}(t) \equiv r_{13}(t)$, and using (4.11), we have that

$$(4.89) \qquad sG^*_{13}(s) = sr^*_{13}(s) = sp^*_{13}(s)$$

$$\equiv \frac{\left[\alpha_{13}f^*_1(s) + (\alpha_{12}\alpha_{23} - \alpha_{13}\alpha_{22})f^*_1(s)f^*_2(s)\right]}{1 - \alpha_{22}f^*_2(s) - \alpha_{12}\alpha_{21}f^*_1(s)f^*_2(s)} .$$

Using (4.76), we obtain after simplification

$$(4.90) \qquad sG^*_{13}(s) = (\lambda+\beta)\frac{\alpha_{13}s + c_o}{s^2 + c_1 s + c_2}$$

where

$$(4.91) \qquad c_o = \mu(1 - \alpha_{22} - \alpha_{12}\alpha_{21})$$

$$c_1 = (1 - \alpha_{22})\mu + \lambda + \beta$$

$$c_2 = (\alpha_{23} + \alpha_{21}\alpha_{13})\mu(\lambda+\beta) = (\lambda+\beta)c_o .$$

Hence

$$(4.92) \qquad G^*_{13}(s) = (\lambda+\beta)\alpha_{13}m^*_1(s) + (\lambda+\beta)c_o m^*_2(s)$$

where

$$m^*_1(s) = \frac{1}{s^2 + c_1 s + c_2}$$

$$m^*_2(s) = \frac{1}{s(s^2 + c_1 s + c_2)} .$$

Using the methods of Section 4.5 we arrive at

$$(4.93) \qquad G_{13}(t) = 1 + \frac{1}{2p}\left\{\left[(\lambda+\beta)\alpha_{13} + \frac{c_2}{a+p}\right]e^{(a+p)t} - \left[(\lambda+\beta)\alpha_{13} + \frac{c_2}{a-p}\right]e^{(a-p)t}\right\}$$

where

$$a = -c_1/2$$

$$p = \frac{1}{2}\sqrt{c_1^2 - 4c_2} \quad .$$

Note

$$a + p < 0$$

$$a - p < 0 \quad .$$

Hence $F(0) = 0$ and $F(+\infty) = 1$. For a check on the result we can use the fact

$$(4.94) \qquad E(T_{13}) = \int_0^{\infty} [1 - G_{13}(t)]dt \quad .$$

A simple check is obtainable in the case $n_d = 1$, where we can use equations (4.43) with (4.12) to obtain $E(T_{13}|n_d = 1) = \frac{\mu + \lambda}{(\beta + P_c\lambda)\mu}$; the same result is obtained using (4.94).

By a similar approach, $G_{23}(t)$ could also be found. Then, given $p_1$ and $p_2$, the distribution function $G_3(t)$ of the unconditional time $T_3$ is given by

$$(4.95) \qquad G_3(t) = p_1G_{13}(t) + p_2G_{23}(t) \quad .$$

## 4.7 Initial State Probability Vector

The concept of initial state probability was introduced in Section 3.3. In the present section we discuss a simple submodel for computing these initial probabilities for the three-state search model under consideration.

Assume that the searcher enter the operating area $A_o$ in a blind condition (at high speed, say) and remains blind until both the HVT and decoy detection circles are inside $A_o$. Some time after the circles are inside $A_o$(immediately perhaps), the search process begins when the searcher instantaneously scans his search field. This initial scan results in an event which places the searcher in one of the three states of the model

1. The searcher sees no targets, and starts search
2. The searcher sees at least one decoy, selects a decoy for classification, and starts classifying
3. The searcher sees at least one HVT, selects an HVT for classification, and starts classifying.

These events are mutually exclusive and exhaustive. If we label them with i, i = 1, 2, 3, then i occurs with probability $p_i$ and $p_1 + p_3 + p_3 = 1$. We have

$$p_1 = P\{\text{no targets present}\}$$

(4.96) $$p_2 = P\{\text{at least one decoy present and pick decoy}\}$$

$$p_3 = P\{\text{at least one HVT present and pick HVT}\} \quad .$$

Determination of the $p_i$ is very similar to the determination of the $\alpha_{2i}$ in Section 4.3.3. The reader is referred there for details.

As before

$$P_c = \pi R_c^2/A_o$$

$$P_d = \pi R_d^2/A_o$$

unless we are using the reduced overlap submodel, in which case we use $P_c(\lambda_o)$ instead of $P_c$. Then

$$(4.97) \qquad p_1 = (1-P_c)^{n_c}(1-P_d)^{n_d}$$

$$p_2 = \sum_{k=1}^{n_d} P\{k \text{ decoys present and pick decoy}\}$$

$$= \sum_{k=1}^{n_d} \sum_{\ell=0}^{n_c} \binom{n_d}{k} P_d^k (1-P_d)^{n_d-k} \binom{n_c}{\ell} P_c^{\ell}(1-P_c)^{n_c-\ell} \frac{k}{\ell+k}$$

$$p_3 = \sum_{\ell=0}^{n_c} P\{\ell \text{ HVTs present and pick HVT}\}$$

$$= \sum_{k=1}^{n_d} \sum_{\ell=0}^{n_c} \binom{n_d}{k} P_d^k (1-P_d)^{n_d-k} \binom{n_c}{\ell} P_c^{\ell}(1-P_c)^{n_c-\ell} \frac{\ell}{\ell+k} \quad .$$

Of special interest is $n_c = 1$, in which the above expressions reduce to

$$(4.98) \qquad p_1 = (1-P_c)(1-P_d)^{n_d}$$

$$p_2 = P_c(1-P_d)^{n_d} + \left[1 - (1-P_d)^{n_d}\right]\left[P_d(n_d+1) - P_c\right]/\left[P_d(n_d+1)\right]$$

$$p_3 = P_c\left[1 - (1-P_d)^{n_d+1}\right]/\left[P_d(n_d+1)\right] \quad .$$

An alternative approach is to assume the searcher makes his initial scan at some point in time prior to entry of the complete detection circles into $A_o$. In this case appropriate area ratios other than $P_c$ and $P_d$ would be used. Note that the forward half of the circumferences of the detection circles must be in $A_o$ in order for the previously discussed detection rates to be valid.

### 4.8 Summary of Three-State Model

A summary of the three-state model is given in Fig. 4-3 in the form of a flow graph. For purposes of this flow graph the area approach is used for calculating transition probabilities, and it is assumed that search and classification times are exponentially distributed.

Assumptions that have been made, explicitly or implicitly, thus far are

- At any fixed instant in time, positions of HVTs and decoys are distributed over the operating area according to a uniform probability distribution
- Over short intervals of time, everything moves in straight lines with uniformly random headings
- Definite range law for detection
- All detection circles lie inside the operating area (i.e., the searcher as well as the HVT knows what the operating area is; see Appendix C for an analysis of one place where this assumption may fail)
- When a decoy is classified, information obtained by the searcher is dissipated rapidly enough so that the classification has no operational effect on the density of decoys
- When presented with an array of targets (both decoys and HVTs) from which one is to be selected for classification, the targets are equally likely to be chosen (i.e., the decoys are identical and are indistinguishable from the identical HVTs until a classification is made)
- Stochastic independence of motion is assumed among decoys as a group, among HVTs as a group, and between the two groups except in the case of reduced overlap when HVTs avoid decoys.

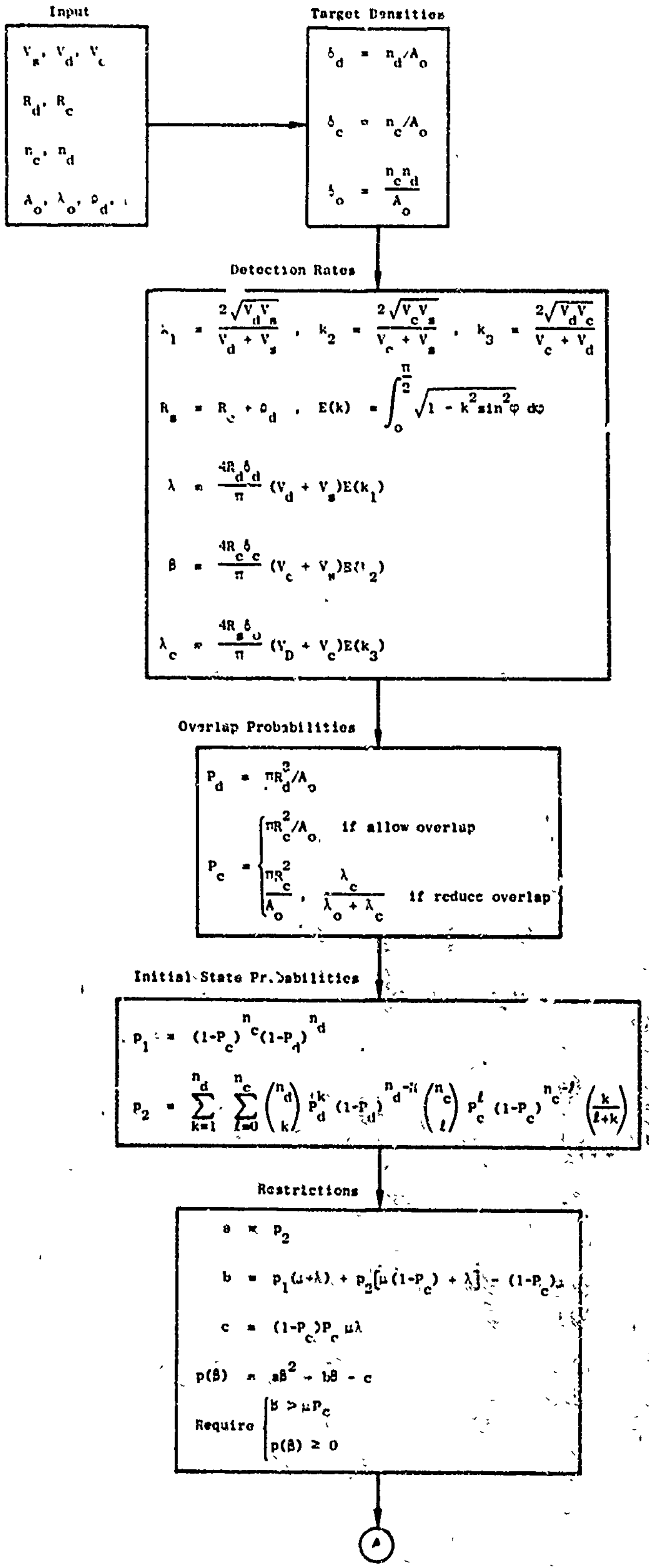


FIGURE 4-3 LOGICAL FLOW OF THREE-STATE MODEL WITH EXPONENTIAL DISTRIBUTIONS

(A)

**Transition Probabilities**

$$\alpha_{12} = \frac{\lambda}{\lambda+\beta} \qquad \alpha_{13} = 1 - \alpha_{12}$$

$$\alpha_{21} = (1-P_c)^{n_c}(1-P_d)^{n_d-1}$$

$$\alpha_{23} = \sum_{k=0}^{n_d-1} \sum_{\ell=1}^{n_c} \binom{n_d-1}{k} P_d^k (1-P_d)^{n_d-1-k} \binom{n_c}{\ell} P_c^\ell (1-P_c)^{n_c-\ell} \frac{\ell}{\ell+k}$$

$$\alpha_{22} = 1 - \alpha_{21} - \alpha_{23}$$

**Intermediate Variables**

$$c_1 = (1-\alpha_{22})\mu + \lambda + \beta$$

$$c_2 = (\alpha_{23}+\alpha_{21}\alpha_{13})\mu(\lambda+\beta)$$

$$a = -c_1/2$$

$$p = \frac{1}{2}\sqrt{c_1^2 - 4c_2}$$

**Secondary Output**

$$E(T_{13}) = \frac{(\alpha_{21}+\alpha_{23})E(\tau_1) + \alpha_{12}E(\tau_2)}{\alpha_{23} + \alpha_{21}\alpha_{13}}$$

$$E(T_{23}) = \frac{\alpha_{21}E(\tau_1) + E(\tau_2)}{\alpha_{23} + \alpha_{21}\alpha_{13}}$$

$$H_{12}(t) = \frac{\alpha_{12}}{\alpha_{23} + \alpha_{21}\alpha_{13}} + \frac{\alpha_{12}(\lambda+\beta)}{2p}\left[\left(1 + \frac{\mu}{a+p}\right)e^{(a+p)t} - \left(1 + \frac{\mu}{a-p}\right)e^{(a-p)t}\right]$$

$H_{22}(t)$ = remains to be derived

$$G_{13}(t) = 1 + \frac{1}{2p}\left\{\left[(\lambda+\beta)\alpha_{13} + \frac{c_2}{a+p}\right]e^{(a+p)t} - \left[(\lambda+\beta)\alpha_{13} + \frac{c_2}{a-p}\right]e^{(a-p)t}\right\}$$

$G_{23}(t)$ = remains to be derived

**Primary Output**

$$E(T_3) = p_1E(T_{13}) + p_2E(T_{13})$$

$$H_2(t) = p_1H_{12}(t) + p_2H_{22}(t)$$

$$G_3(t) = p_1G_{13}(t) + p_2G_{23}(t)$$

FIGURE 4-3 LOGICAL FLOW OF THREE-STATE MODEL WITH EXPONENTIAL DISTRIBUTIONS (Concluded)

## 5. EXAMPLE RESULTS

Eight cases are defined for the purpose of numerical illustration of the three-state search model. The area approach to computing transition probabilities is used with a circular area of 200-nautical mile radius. A single HVT is assumed throughout ($n_c = 1$); a decoy classification time of 5 hours is used in all cases ($E(\tau_2) \equiv \mu^{-1} = 5$).

The cases are defined in Table 5-1. In cases 1-4 the decoys operate like the HVT, while in cases 5-8 the decoys operate in a different manner than the HVT. In particular, in cases 7-8 the decoys are stationary. With one exception, to be discussed later, all search times are taken to be exponentially distributed as in (4.70).

Table 5-1

CASES FOR NUMERICAL ILLUSTRATION

| Case | $V_s$ (kts) | $V_c$ (kts) | $V_d$ (kts) | $R_c$ (nmi) | $R_d$ (nmi) | |
|---|---|---|---|---|---|---|
| 1 | 5 | 10 | 10 | 25 | 25 | HVT and decoy same |
| 2 | 5 | 20 | 20 | 80 | 80 | |
| 3 | 10 | 10 | 10 | 15 | 15 | |
| 4 | 10 | 20 | 20 | 50 | 50 | |
| 5 | 5 | 10 | 20 | 25 | 80 | HVT and decoy different |
| 6 | 10 | 10 | 20 | 15 | 50 | |
| 7 | 5 | 10 | 0 | 25 | 80 | |
| 8 | 10 | 10 | 0 | 15 | 50 | |

Figure 5-1 displays conditional expected time (in hours) to HVT detection $E(T_{13}|n_d)$ versus number of decoys $n_d$. The broken versus solid line distinguishes searcher speed (broken- $V_s = 5$ kts; solid- $V_s = 10$ kts). The striking feature of the data in this figure is the



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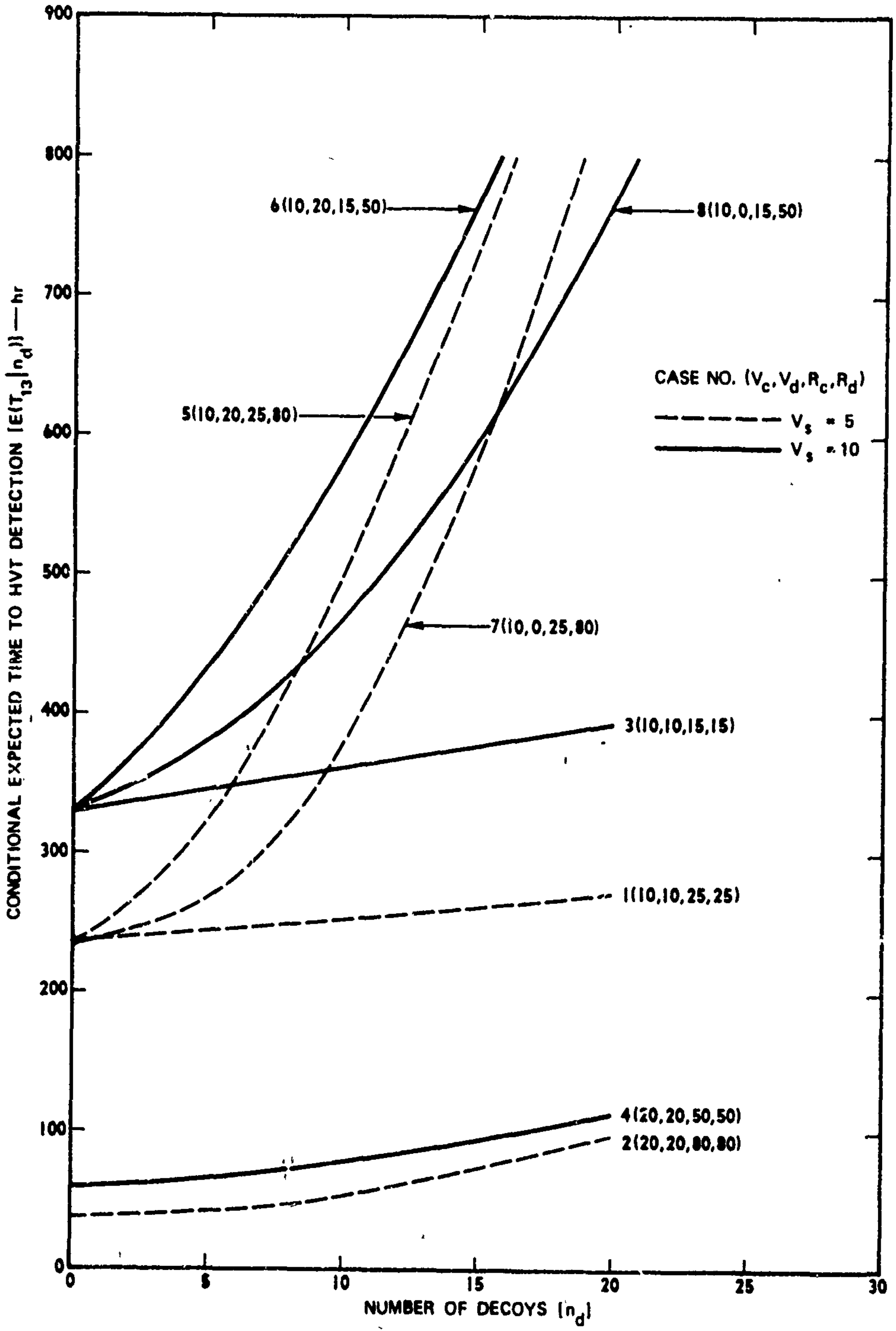


FIGURE 5-1 NUMBER OF DECOYS vs CONDITIONAL EXPECTED TIME TO HVT DETECTION

effect of nonsimilarity between decoy and HVT. In all the sharply rising curves the decoys are different in detectability and speed than the HVT.

Figure 5-2 illustrates the effect on $E(T_{13}|n_d)$ of reduced overlap between decoys and the HVT. Cases 4 and 7 are considered. A notable feature in these data is the significant interaction between $\lambda_o$ and case considered. Recall that $\lambda_o$ is the maximum acceptable rate of HVT course change for purpose of decoy avoidance, i.e., $\lambda_o^{-1}$ is the mean time between such course changes. The principal expression for the queue model of reduced overlap is given by (4.64).

Figure 5-3 shows, for Case 5, the conditional (on starting in state 1) expected number of decoys encountered prior to HVT detection $H_{13}(t|n_d)$ as a function of time in days and number of decoys deployed. The broken line is $E(T_{13}|n_d)$ in units of days. The limiting values of $H_{13}(t|n_d)$ as $t \to \infty$ are shown on the right.

Figure 5-4 shows, for Case 5, the conditional cumulative probability $G_{13}(t|n_d)$ of HVT detection versus time and number of decoys. Recall that $G_{13}(t|n_d)$ is the probability the HVT is detected at or before t given there are $n_d$ decoys deployed. The curves for $n_d = 0$ and $n_d = 1$ are practically coincident and are represented by a single line. Thus, for example, in a 9-day operation, deployment of 20 decoys of Case 5 type reduces the probability of HVT detection down to .2 from the .6 obtained with no decoys.

Figure 5-5 is the exception to the exponential distribution mentioned earlier. In this figure the solid lines represent search times with Erlang distribution as given by (4.73); the broken lines are exponential data shown earlier in Fig. 5-1 and repeated here for comparison. The peculiar shape of the curves resulting from the Erlang distribution is probably due to a failure of the random motion/position

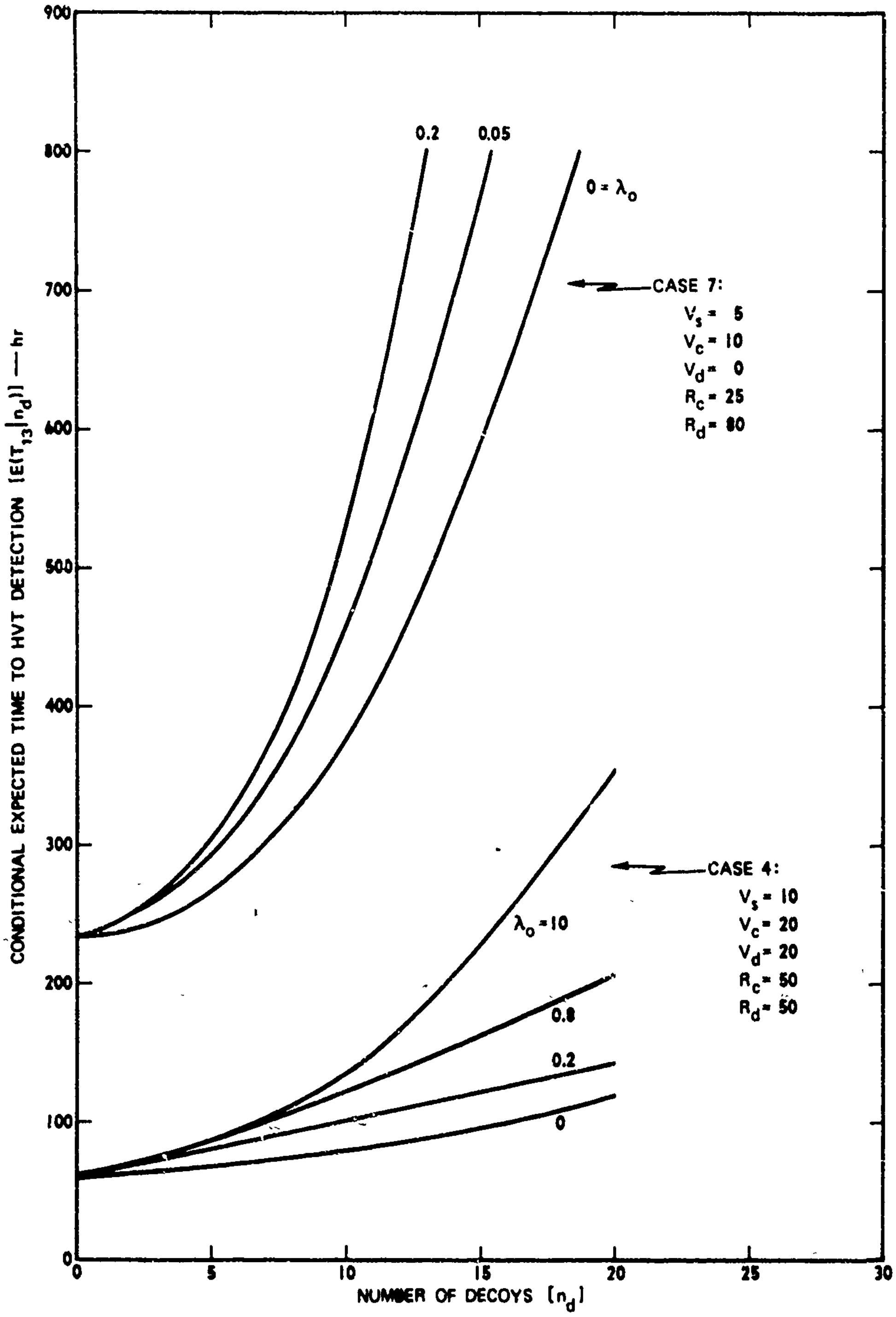


FIGURE 5-2 EFFECT OF REDUCED OVERLAP (CASES 4 AND 7)

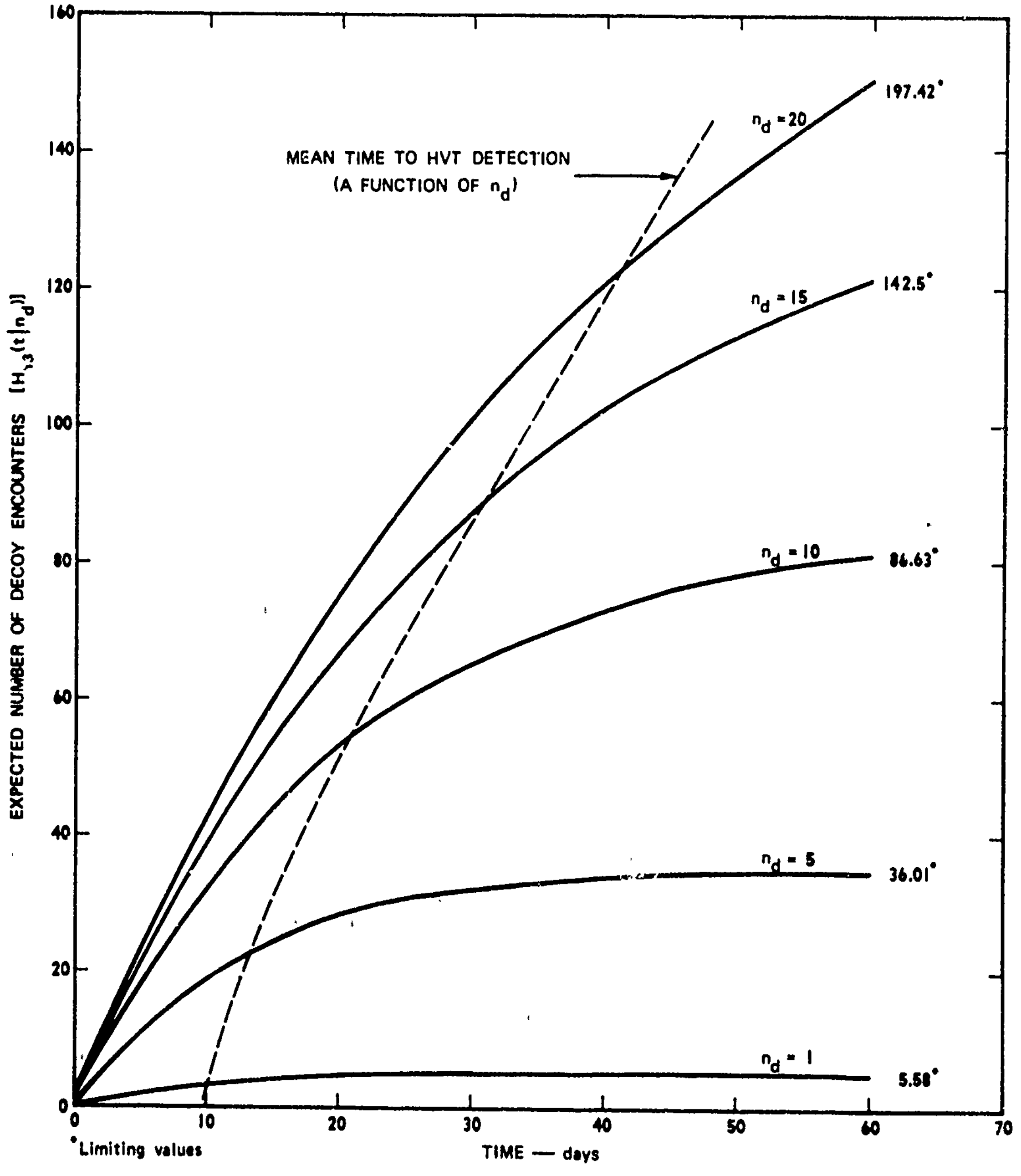


FIGURE 5-3 EXPECTED NUMBER OF DECOYS ENCOUNTERED vs TIME (CASE 5)

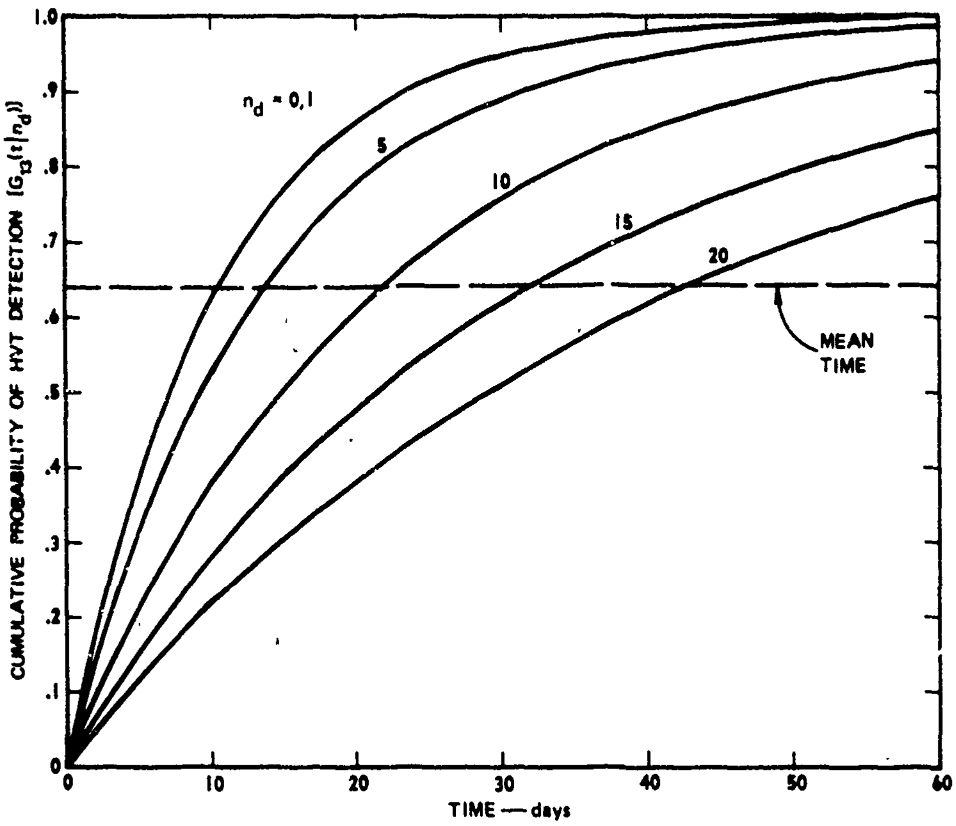


FIGURE 5-4 CUMULATIVE PROBABILITY OF HVT DETECTION vs TIME (CASE 5)

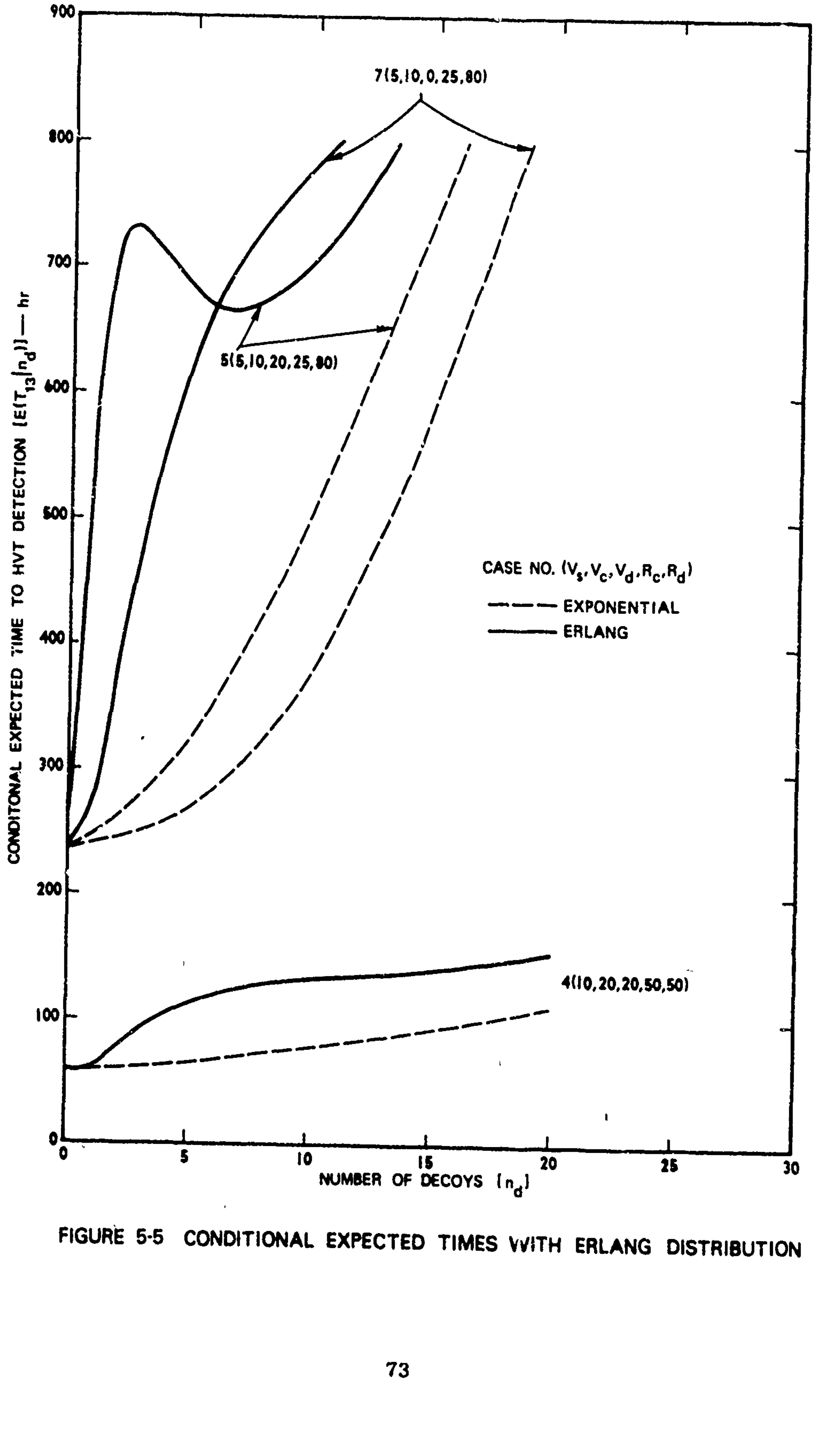


FIGURE 5-5 CONDITIONAL EXPECTED TIMES WITH ERLANG DISTRIBUTION

assumption and other considerations discussed for the exponential distribution in Section 4.3.3 under "Model Restrictions Due to Overlap." Other data not included here indicate the Erlang results approach the exponential results as $n_d$ becomes very large. However, for small numbers of decoys there is a significant difference in the implications of the two distributions. Specifically, decoys are much more effective with Erlang detection times than with exponential detection times. This is especially true since the errors in the Erlang curves presented here are most likely in the direction of decreased effectiveness of the decoys; the exponential curves can be considered relatively free of error

## 6. A FOUR-STATE MODEL

This section demonstrates how the three-state model can be extended to a four-state model to analyze the particular search procedure of "sprint-drift" motion. It is important to emphasize that this is a demonstration and not a final model in any sense. In particular, not included in the model is a representation of the capability for the searcher to look behind himself as he changes from sprint to drift. Such a capability is analogous to the instantaneous scan discussed in Section 4.7 with respect to initial state probabilities, except that the scan is now periodic instead of occurring only at time zero. This capability could be incorporated into a model but it is not done here because of lack of time.

In using the "sprint-drift" method, the searcher alternates between high ($V_{s1}$) and low ($V_{s2}$) speed, with corresponding detection ranges $R_{c_1}$, $R_{c_2}$, $R_{d_1}$, $R_{d_2}$, and encounter rates $\lambda_1$, $\lambda_2$, $\beta_1$, and $\beta_2$. Typically, $V_{s1} \gg V_{s2}$ and $V_{s2}$ will be close to zero. The speed $V_{s_i}$ is maintained for a time $t_i$ after which speed is changed, unless a detection occurs before $t_i$, in which case classification commences.

Let the events be

1. Start search ($V_{s1}$)
2. Start search ($V_{s2}$)
3. Start decoy classification
4. Start HVT classification

As before, we assume state 4 to be absorbing. The transition matrix is then

$$(6.1)\qquad A = \begin{bmatrix} 0 & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & 0 & \alpha_{23} & \alpha_{24} \\ \alpha_{31} & 0 & \alpha_{33} & \alpha_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

Note we are assuming that after classifying a decoy, if no other targets are immediately visible, the searcher makes a high speed sprint away from the decoy. Also, we allow a limited detection capability at sprint speed. The f matrix is

$$(6.2)\qquad f = \begin{bmatrix} f_1 & f_1 & f_1 & f_1 \\ f_2 & f_2 & f_2 & f_2 \\ f_3 & f_3 & f_3 & f_3 \\ f_4 & f_4 & f_4 & f_4 \end{bmatrix} .$$

## 6.1 The Transition Matrix

Consider first $\alpha_{13}$, which is actually $\alpha_{13}(t_1)$ a function of search (sprint) time $t_1$. Thus

$$\alpha_{13}(t_1) = \Pr\{\text{Contact a decoy before an HVT and before } t_1 | \text{ started search at speed } V_{s1} \text{ at time zero}\} .$$

As before, let

$$X_1 = \text{time to detect an HVT at } V_{s1}$$

$$Y_1 = \text{time to detect a decoy at } V_{s1} ,$$

Then

$$\alpha_{13}(t_1) = P\{Y_1 \le X_1 \text{ and } Y_1 \le t_1\}$$

$$= P\{Y_1 \le X_1 | Y_1 \le t_1\} P\{Y_1 \le t_1\}$$

$$= P\{Y_1 \le t_1\} \int_0^\infty P\{Y_1 \le t | Y_1 \le t_1\} f_{X_1}(t) dt \quad .$$

Writing

$$F_{X_1}(t) = \int_0^t f_{X_1}(\tau) d\tau \quad \text{and} \quad F_{Y_1}(t) = \int_0^t f_{Y_1}(\tau) d\tau$$

$$= P\{X_1 \le t\} \qquad\qquad = P\{Y_1 \le t\}$$

we have

$$P\{Y_1 \le t | Y_1 \le t_1\} = \begin{cases} \dfrac{F_{Y_1}(t)}{F_{Y_1}(t_1)} & \text{if } t \le t_1 \\ 1 & \text{if } t > t_1 \end{cases}$$

so that

$$\alpha_{13}(t_1) = \int_0^{t_1} F_{Y_1}(t) f_{X_1}(t) dt + F_{Y_1}(t_1) \left[1 - F_{X_1}(t_1)\right] \quad .$$

Analogously,

$$\alpha_{14}(t_1) = \int_0^{t_1} F_{X_1}(t) f_{Y_1}(t) dt + F_{X_1}(t_1) \left[1 - F_2(t_1)\right]$$

and $$\alpha_{12} = 1 - \alpha_{13} - \alpha_{14} \quad .$$

There are also obvious analogies for $\alpha_{21}(t_2)$, $\alpha_{23}(t_2)$, and $\alpha_{24}(t_2)$. One may compute $\alpha_{31}$, $\alpha_{33}$, $\alpha_{34}$ in the same manner as $\alpha_{21}$, $\alpha_{22}$, $\alpha_{23}$ were computed in the three-state model. We consider now the case where $f_{X_i}(t)$ and $f_{Y_i}(t)$ are exponential as in (4.17a). Then for i = 1, 2

$$\alpha_{13}(t_i) = 1 - \frac{\beta_i}{\lambda_i + \beta_i}\left[1 - e^{-(\lambda_i+\beta_i)t_i}\right] - e^{-(\lambda_i+\beta_i)t_i}$$

$$\alpha_{14}(t_i) = 1 - \frac{\lambda_i}{\lambda_i + \beta_i}\left[1 - e^{-(\lambda_i+\beta_i)t_i}\right] - e^{-(\lambda_i+\beta_i)t_i}$$

$$\alpha_{12}(t_1) = e^{-(\lambda_1+\beta_1)t_1} \quad .$$

$$\alpha_{21}(t_2) = e^{-(\lambda_2+\beta_2)t_2} \quad .$$

6.2 The $f^*$ Matrix and Expected Values

In this model we don't have a "p.d.f" matrix because $\tau_1$ and $\tau_2$ (the length of time in states 1 and 2, respectively) are not continuous random variables. Rather they are mixed random variables with a continuous and a discrete component each. This is all right, though, because all we need is the $f^*$ matrix (which still exists) and expected values of the $\tau_i$. As in the three-state model, the forms of $f_3$ and $f_4$ are arbitrary, while the forms of $f_1$ and $f_2$ must remain consistent with the $\alpha_{ij}$'s.

We proceed now to examine the probability distribution of $\tau_1$. The analysis for $\tau_2$ is identical.

We have

$$P(\tau_1 \le t) = \begin{cases} 1 & \text{if } t \ge t_1 \\ P(Y_1 \le X_1 \;\&\; Y_1 \le t) + P(X_1 \le Y_1 \;\&\; X_1 \le t) & \text{if } 0 < t < t_1 \end{cases}$$

$$= \begin{cases} 1 & t \ge t_1 \\ \alpha_{13}(t) + \alpha_{14}(t) & 0 < t < t_1 \end{cases} \quad .$$

Hence, in general, $P(\tau_1 \le t)$ is discontinuous at $t = t_1$. We have*

$$P(\tau_1 = t) = \begin{cases} 0 & t \ne t_1 \\ 1 - \alpha_{13}(t) - \alpha_{14}(t) = \alpha_{12}(t) & \text{if } t = t_1 \end{cases}$$

$$f_1(t|0 < t < t_1) = \alpha'_{13}(t) + \alpha'_{14}(t) = \alpha'_{12}(t) \qquad \text{for } 0 < t < t_1 \quad .$$

Thus

$$E(\tau_1) = t_1 \cdot \alpha_{12}(t_1) + \int_0^{t_1} t\alpha'_{12}(t)dt \quad .$$

In the case where $f_{X_1}(t)$ and $f_{X_2}(t)$ are exponential we have

$$E(\tau_1) = t_1 e^{-(\lambda_1+\beta_1)t_1} + \int_0^{t_1} t(\lambda_1+\beta_1)e^{-(\lambda_1+\beta_1)t} dt$$

$$= t_1 e^{-(\lambda_1+\beta_1)t_1} - t_1 e^{-(\lambda_1+\beta_1)t_1} - \frac{e^{-(\lambda+\beta_1)t_1} - 1}{\lambda + \beta}$$

$$= \frac{1 - e^{-(\lambda_1+\beta_1)t_1}}{\lambda_1 + \beta_1} \quad .$$

An analogous equation holds for $E(\tau_2)$. These terms are required in the following subsection for computing $E(T_{14})$.

---

*Where the prime indicates derivative with respect to t.

## 6.3 Conditional Expected First Passage Time

Referring to (3.2), (3.11), and (3.16) we proceed to derive $E(T_{14})$. Dropping the s arguments of the $f_i^*(s)$ for sake of clarity, we have

$$g^*(s) = \begin{bmatrix} 0 & \alpha_{12}f_1^* & \alpha_{13}f_1^* & \alpha_{14}f_1^* \\ \alpha_{21}f_2^* & 0 & \alpha_{23}f_2^* & \alpha_{24}f_2^* \\ \alpha_{31}f_3^* & 0 & \alpha_{33}f_3^* & \alpha_{34}f_3^* \\ 0 & 0 & 0 & f_4^* \end{bmatrix} .$$

So

$$I - g^*(s) = \begin{bmatrix} 1 & -\alpha_{12}f_1^* & -\alpha_{13}f_1^* & -\alpha_{14}f_1^* \\ -\alpha_{21}f_2^* & 1 & -\alpha_{23}f_2^* & -\alpha_{24}f_2^* \\ -\alpha_{31}f_3^* & 0 & 1-\alpha_{33}f_3^* & -\alpha_{34}f_3^* \\ 0 & 0 & 0 & 1-f_4^* \end{bmatrix}$$

$$\triangleq \begin{bmatrix} 1 & a_2 & a_3 & a_4 \\ b_1 & 1 & b_3 & b_4 \\ c_1 & 0 & c_3 & c_4 \\ 0 & 0 & 0 & d_4 \end{bmatrix} .$$

Thus

$$[I - g^*(s)]^{-1} \triangleq \begin{bmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ 0 & 0 & 0 & g_{44} \end{bmatrix}$$

$$= \frac{1}{k} \begin{bmatrix} c_3 d_4 & -a_2 c_3 d_4 & d_4(a_2 b_3 - a_3) & -a_2(b_3 c_4 - b_4 c_3) + (a_3 c_4 - a_4 c_3) \\ -d_4(b_1 c_3 - c_1 b_3) & d_4(c_3 - a_3 c_1) & -d_4(b_3 - a_3 b_1) & (b_3 c_4 - b_4 c_3) - b_1(a_3 c_4 - a_4 c_3) + c_1(a_3 b_4 - a_4 b_3) \\ -d_4 c_1 & d_4 a_2 c_1 & d_4(1 - a_2 b_1) & -c_1(a_2 b_4 - a_4) - c_4(1 - a_2 b_1) \\ 0 & 0 & 0 & c_1(a_2 b_3 - a_3) + c_3(1 - a_2 b_1) \end{bmatrix}$$

where $k = c_1 d_4(a_2 b_3 - a_3) + c_3 d_4(1 - a_2 b_1)$ .

Now

$$sP^*_{14}(s) = [I+h^*]_{1\cdot} \cdot W_{\cdot 4}(s) = (1-f^*_4)[I+h^*]_{14}$$

$$= (1-f^*_4)h^*_{14}$$

$$= (1-f^*_4)g^*_{1\cdot}(s)[I-g^*(s)]^{-1}_{\cdot 4}$$

$$= (1-f^*_4)f^*_1\left[\alpha_{12}g_{24}+\alpha_{13}g_{34}+\alpha_{14}g_{44}\right]$$

where

$$g_{24} = \frac{1}{k} \cdot \alpha_{24}f^*_2 + (\alpha_{23}\alpha_{34}-\alpha_{24}\alpha_{33})f^*_2f^*_3 + \alpha_{21}\alpha_{14}f^*_1f^*_2$$
$$+ (\alpha_{21}\alpha_{13}\alpha_{34}-\alpha_{31}\alpha_{13}\alpha_{24}+\alpha_{31}\alpha_{14}\alpha_{23}-\alpha_{21}\alpha_{14}\alpha_{33})f^*_1f^*_2f^*_3$$

$$g_{34} = \frac{1}{k} \cdot \alpha_{34}f^*_3 + \alpha_{14}\alpha_{31}f^*_1f^*_3 + (\alpha_{12}\alpha_{24}\alpha_{31}-\alpha_{12}\alpha_{21}\alpha_{34})f^*_1f^*_2f^*_3$$

$$g_{44} = \frac{1}{k} \cdot 1 - \alpha_{33}f^*_3 - \alpha_{13}\alpha_{31}f^*_1f^*_3 - \alpha_{12}\alpha_{21}f^*_1f^*_2$$
$$+ (\alpha_{12}\alpha_{21}\alpha_{33}-\alpha_{31}\alpha_{12}\alpha_{23})f^*_1f^*_2f^*_3$$

and

$$k = -\alpha_{31}f_3^*(1-f_4^*)\left[\alpha_{12}\alpha_{23}f_1^*f_2^*+\alpha_{13}f_1^*\right] + (1-f_4^*)(1-\alpha_{33}f_3^*)(1-\alpha_{12}\alpha_{21}f_1^*f_2^*)$$

$$= (1-f_4^*)\left[1-\alpha_{33}f_3^*-\alpha_{13}\alpha_{31}f_1^*f_3^*-\alpha_{12}\alpha_{21}f_1^*f_2^*+(\alpha_{12}\alpha_{21}\alpha_{33}-\alpha_{12}\alpha_{23}\alpha_{31})f_1^*f_2^*f_3^*\right] .$$

So

$$sp_{14}^*(s) = \frac{\alpha_{14}f_1^* + \alpha_{12}\alpha_{24}f_1^*f_2^* + (\alpha_{13}\alpha_{34}-\alpha_{14}\alpha_{33})f_1^*f_3^* + \alpha_{12}(\alpha_{23}\alpha_{34}-\alpha_{24}\alpha_{33})f_1^*f_2^*f_3^*}{1 - \alpha_{33}f_3^* - \alpha_{13}\alpha_{31}f_1^*f_3^* - \alpha_{12}\alpha_{21}f_1^*f_2^* + \alpha_{12}(\alpha_{21}\alpha_{33}-\alpha_{23}\alpha_{31})f_1^*f_2^*f_3^*}$$

$$= \frac{u(s)}{v(s)}$$

$$E(T_{14}) = -\frac{d}{ds}\left[sP_{14}^*(s)\right]\Bigg|_{s=0} = \frac{u(s)v'(s) - v(s)u'(s)}{v^2(s)}\Bigg|_{s=0} .$$

But we find that $u(0) = v(0)$ since

$$\alpha_{14} + \alpha_{12}\alpha_{24} + \alpha_{13}\alpha_{34} - \alpha_{14}\alpha_{33} + \alpha_{12}(\alpha_{23}\alpha_{34}-\alpha_{24}\alpha_{33})$$

$$= 1 - \alpha_{33} - \alpha_{13}\alpha_{31} - \alpha_{12}\alpha_{21} + \alpha_{12}(\alpha_{21}\alpha_{33}-\alpha_{23}\alpha_{31})$$

where

$$\alpha_{12} + \alpha_{13} + \alpha_{14} = 1,\ \alpha_{21} + \alpha_{23} + \alpha_{24} = 1,\ \alpha_{31} + \alpha_{33} + \alpha_{34} = 1.$$

Hence

$$E(T_{14}) = \frac{v'(0) - u'(0)}{v(0)}$$

$$= \frac{\begin{array}{l}(1-\alpha_{12})(1-\alpha_{33})E(\tau_1) + \alpha_{12}(\alpha_{21}+\alpha_{24})\left[E(\tau_1) + E(\tau_2)\right] \\ + \left[\alpha_{33}(1-\alpha_{14}) + \alpha_{13}(\alpha_{31}+\alpha_{34})\right]E(\tau_3) \\ + \alpha_{12}\left[\alpha_{23}(\alpha_{31}+\alpha_{34}) - \alpha_{33}(\alpha_{21}+\alpha_{24})\right]\left[E(\tau_1) + E(\tau_2) + E(\tau_3)\right]\end{array}}{1 - \alpha_{33} - \alpha_{13}\alpha_{31} - \alpha_{12}\alpha_{21} + \alpha_{12}(\alpha_{21}\alpha_{33}-\alpha_{23}\alpha_{31})}$$

where $E(\tau_i) = -\frac{d}{ds} f_i^*(s)\Big|_{s=0}$ .

Note that if we set $\alpha_{12} = 0$, $\alpha_{21} = 1$, $E(\tau_2) = 0$ and make the proper identifications, we get (4.12).

## 7. CONCLUSIONS AND RECOMMENDATIONS

This study has shown that the semi-Markov process is an effective framework within which to study questions regarding the effectiveness of decoys in delaying a searcher looking for a particular type of target. In particular, the four questions raised in the Introduction have been answered in detail for a constant speed searcher and constant speed targets. However, output expressions $H_{22}(t)$ and $G_{23}(t)$, respectively the number of decoys encountered and the cumulative distribution of first passage time for a searcher starting in the classify decoy state, remain to be derived. The question of mean time has been answered to a degree for a variable speed searcher. Applications to other search and related problems are implicit in the type of analysis that has been conducted. For example, optimization of the "sprint-drift" search tactic is a latent application of the four-state model discussed. A slightly upgraded four-state model could provide much insight into "sprint-drift."

Given the assumption of a constant range detection law, the three-state model is a suitable representation of the search problem, within the limits of the random motion/position assumption, the restrictions due to overlap, and a large (relative to detection radii) operating area. In order to extend these limits it is recommended that the effects of geometry and time on the state transition mechanism in the three-state model be studied in detail and the three-state model be expanded to four states by including a secondary search state as discussed in Section 4.3.3. The geometry/time study would likely include numerical calculations to provide functional relations for generalized use within the context of the three-state model and extensions. Such a study is necessary if the model is to be used for optimization (where it is likely to be pushed to extremes) or for study of relatively large decoys (where decoy coverage approaches the size of the operating area).

Several interesting questions involving constrained optimization arise. Loosely speaking, it is clear that the more decoys that are available the better for the HVT. However, it is also clear that the choice of decoy configuration and number of decoys is a constrained problem. This constrained choice problem will assume characteristics dependent on the circumstance within which it arises. Some possibilities are

1. Given specified limited funds how many decoys of what configuration should be built to optimize some operational variable, such as probability of HVT detection?
2. Funds are "limited" but not specified. Therefore, it is desired to meet some operational performance threshold (such as a minimum acceptable HVT detection probability) with minimum cost. How many decoys of what configuration should be built and what is the cost?

A third question, of some interest in initial planning stages, is how the optimum number and configuration in (1) above vary as the amount of available funds varies. For example, it may be of considerable interest to know whether or not only the number and not the configuration varies with a change in the budget constraint. Of course, examination of any of these questions requires development of the relation between per unit decoy cost and decoy detection range, speed, and holding (classification) time. In optimization studies such as suggested here, scenario dependencies (such as size of operating area and operation duration) can be removed to some extent by considering various scenarios and applying probabilistic weighting factors at the appropriate places in the analysis.

Another area for analytic extension is the situation of stationary decoys where the searcher can plot the position of classified decoys and thus render them relatively ineffective. The three-state model is not applicable in this situation because the transition probabilities change with each decoy that is classified. What is needed is a general n-state formulation. Such a formulation may be obtainable by exploiting and generalizing the special structure of the following six-state transition matrix (where the stars denote allowable transitions and the state definitions follow)

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 0 | * | 0 | 0 | 0 | * |
| 2 | 0 | 0 | * | 0 | 0 | * |
| 3 | 0 | 0 | 0 | * | 0 | * |
| 4 | 0 | 0 | 0 | 0 | * | * |
| 5 | 0 | 0 | 0 | 0 | 0 | * |
| 6 | 0 | 0 | 0 | 0 | 0 | 1 |

where

state 1 = search with 2 decoys unclassified

state 3 = search with 1 decoy unclassified

state 5 = search with 0 decoys unclassified

state 2 = classify decoy with 2 decoys unclassified

state 4 = classify decoy with 1 decoy unclassified

state 6 = classify HVT.

Results analogous to those for birth-death processes might be obtainable.

A ubiquitous assumption in this study is the definite range detection law. The realized detection range in any real encounter is a random variable that is represented in the models discussed here by a fixed range R (which depends on the type of target and its speed as well as searcher and environmental characteristics). If the target comes within R of the searcher, the target is assumed detected; no detections occur at ranges greater than R. The value assigned to R is usually the median value of a distribution generated by models outside the scope of this study. The search models presented in this document permit sensitivity studies on R but they do not take into account the inherent variability in realized detection ranges. Hence, it is recommended that a study be conducted to determine the effect of this inherent variability on the results produced by the current search models. Such a study could be conducted with Markov process models and could examine fade zone effects as well as variable detection range.

Recommendations for future work may be summarized as follows:

- Derive output expressions for a searcher starting in the classify decoy state
- Incorporate geometry/time interactions into the transition mechanism of the three-state model
- Expand the three-state model to four states by including a secondary search state
- Optimize decoy configuration using the three-state model to gain insight into dependency of optimum configuration on budget constraints
- Incorporate a "clearing-turn" maneuver into the four-state model and subsequent optimization study of "sprint-drift" (e.g., find the optimum durations for sprinting and drifting)
- Develop a model to examine the situation where decoy classification information obtained by the searcher does not dissipate with time
- Study the effects of detection range variation and fade zone phenomena.

## Appendix A: ALTERNATE COMPUTATION OF $\alpha_{22}$ AND $\alpha_{23}$

In Section 4.3.3 we determined $\alpha_{22}$ and $\alpha_{23}$ on the basis of a specified number of targets and finite specified area. We now drop explicit consideration of area and number and develop an approach based on target density. Equations (4.27) and (4.28) still hold, but we replace the binomial distribution by the Poisson distribution. That is, we assume that as an element of area $\Delta A \to 0$ we have

(A.1)
$$\begin{aligned} Pr(0 \text{ decoys in } \Delta A) &= 1 - \lambda \Delta A + o(\Delta A) \\ Pr(1 \text{ decoy in } \Delta A) &= \lambda \Delta A + o(\Delta A) \\ Pr(0 \text{ HVTs in } \Delta A) &= 1 - \beta \Delta A + o(\Delta A) \\ Pr(1 \text{ decoy in } \Delta A) &= \beta \Delta A + o(\Delta A) \quad . \end{aligned}$$

Consult Refs. 1 or 2 on the Poisson process. We further assume that the number of decoys (or HVTs) in nonoverlapping areas are mutually independent. Let K = number of decoys present and L = number of HVTs present. Then

(A.2)
$$P(\ell \text{ HVTs present}) = P(L = \ell) = e^{-x} \frac{x^{\ell}}{\ell!} \qquad \ell = 0, 1, \ldots$$

where $x = \delta_c \pi R_c^2$

(A.3)
$$\begin{aligned} P(k \text{ new decoys present}) &= P(k{+}1 \text{ total decoys present} \mid \text{at least one decoy present}) \\ &= P(K = k{+}1 \mid K \geq 1) \\ &= \frac{P(K = k{+}1 \ \& \ K \geq 1)}{P(K \geq 1)} \qquad k = 0, 1, \ldots \\ &= P(K = k{+}1)/P(K \geq 1) \qquad k = 0, 1, \ldots \\ &= \frac{e^{-y}}{1 - e^{-y}} \frac{y^{k+1}}{(k+1)!} \qquad k = 0, 1, \ldots \end{aligned}$$

where $y = \delta_d \pi R_d^2$ .



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It is clear that

$$\text{(A.4)} \qquad \alpha_{21} = P(K = 1|K \geq 1) \cdot P(L = 0) = \frac{y\,e^{-x-y}}{1 - e^{-y}}$$

and that

$$\text{(A.5)} \qquad \alpha_{23} = \sum_{k=0}^{\infty} \sum_{\ell=1}^{\infty} P(K = k+1|K \geq 1)P(L = \ell)\,\frac{\ell}{\ell+k}$$

$$= \frac{e^{-x-y}}{1 - e^{-y}} \sum_{k=0}^{\infty} \sum_{\ell=1}^{\infty} \frac{\ell}{\ell+k}\,\frac{x^{\ell}}{\ell!} \cdot \frac{y^{k+1}}{(k+1)!} .$$

Also

$$\text{(A.6)} \qquad \alpha_{22} = \sum_{k=1}^{\infty} \sum_{\ell=0}^{\infty} \frac{k}{\ell+k}\,P(K = k+1|K \geq 1)P(L = \ell)$$

$$= \frac{e^{-x-y}}{1 - e^{-y}} \sum_{k=1}^{\infty} \sum_{\ell=0}^{\infty} \frac{k}{\ell+k}\,\frac{x^{\ell}}{\ell!}\,\frac{y^{k+1}}{(k+1)!} .$$

It happens that (A.6) is the more complicated expression and is best obtained as $\alpha_{22} = 1 - \alpha_{21} - \alpha_{23}$. So we examine (A.5). Let

$$\text{(A.7)} \qquad m = \frac{e^{-x-y}}{1 - e^{-y}} \quad \text{and} \quad \rho = \frac{x}{y} .$$

Then we can write

(A.8)
$$\alpha_{23} = m\,x \sum_{k=0}^{\infty} \sum_{\ell=1}^{\infty} \frac{1}{\ell+k} \frac{x^{\ell-1}\, y^{k+1}}{(\ell-1)!\,(k+1)!}$$

$$= m\,x \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{1}{\ell+k+1} \frac{x^{\ell}}{\ell!} \frac{y^{k+1}}{(k+1)!}$$

$$= m\,x \sum_{k=1}^{\infty} \sum_{\ell=0}^{\infty} \frac{1}{\ell+k} \frac{x^{\ell}}{\ell!} \frac{y^{k}}{k!}$$

$$= m\,x \sum_{k=1}^{\infty} \sum_{\ell=0}^{\infty} \frac{\rho^{\ell}}{\rho!\,k!} \int y^{\ell+k-1}\,dy$$

$$= m\,x \int \frac{1}{y} \sum_{k=1}^{\infty} \sum_{\ell=0}^{\infty} \frac{\rho^{\ell} y^{\ell}}{\ell!} \frac{y^{k}}{k!}\,dy$$

$$= m\,x \int \frac{1}{y} \left[e^{(1+\rho)y} - e^{\rho y}\right] dy$$

$$= m\,x \left[\sum_{k=1}^{\infty} \frac{(1+\rho)^{k} y^{k}}{k \cdot k!} - \sum_{k=1}^{\infty} \frac{\rho^{k} y^{k}}{k \cdot k!}\right]$$

$$= m\,x \sum_{k=1}^{\infty} \frac{\left[(1+\rho)^{k} - \rho^{k}\right] y^{k}}{k \cdot k!}$$

$$= m\,x \sum_{k=1}^{\infty} \frac{(1+\rho)^{k} y^{k} - x^{k}}{k \cdot k!} .$$

We have thus replaced a double sum by a single sum, but the summation operation is still infinite.

## Appendix B: ALTERNATE REDUCED OVERLAP MODELS

In Section 4.3.3 under the heading "Reduced Overlap" we considered a queuing model of HVT/decoy interaction due to overlap. This appendix presents an alternative class of models for representing the interaction. This alternative class requires specialized data and has no special decision variable for controlling overlap per se. The queuing model is therefore viewed as a more fruitful approach and the alternative class is presented basically for completeness.

We first consider $P_c$ simply as a function of number of decoys or decoy density. Define

(B.1) $$P_{co} = \pi R_c^2 / A_o$$

and redefine $P_c$ a function $P_c(n_d)$ or $P_c(\delta_d)$ as $P_{co}$ multiplied by weighting factor. For example, we could write

(B.2) $$P_c(n_d) = P_{co}\left[1 - e^{-an_d}\right] .$$

If we could establish, for example, that $P_c(10) = \frac{1}{2} P_{co}$, then we would have a = 0.069315. This function is depicted in Fig. B-1.

More generally, we can consider the contact rate $\lambda_c$ as defined in Section 4.3.3. We could then write

(B.3) $$P_c(\lambda_c) = P_{co}\left[1 - e^{-a\lambda_c}\right]$$

or alternatively

(B.4) $$P_c(\lambda_c) = P_{co}\left[1 - (2a\lambda_c + 1)\, e^{-2a\lambda_c}\right] .$$



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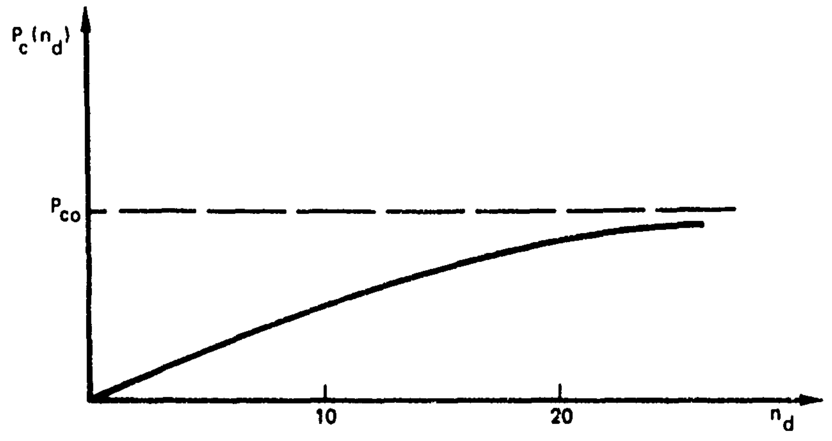


FIGURE B-1 $P_c$ AS A FUNCTION OF NUMBER OF DECOYS

Using (B.4), suppose we knew $P_c(1) = \frac{1}{2} P_{co}$. Then a = 0.85 and we have Fig. B-2.

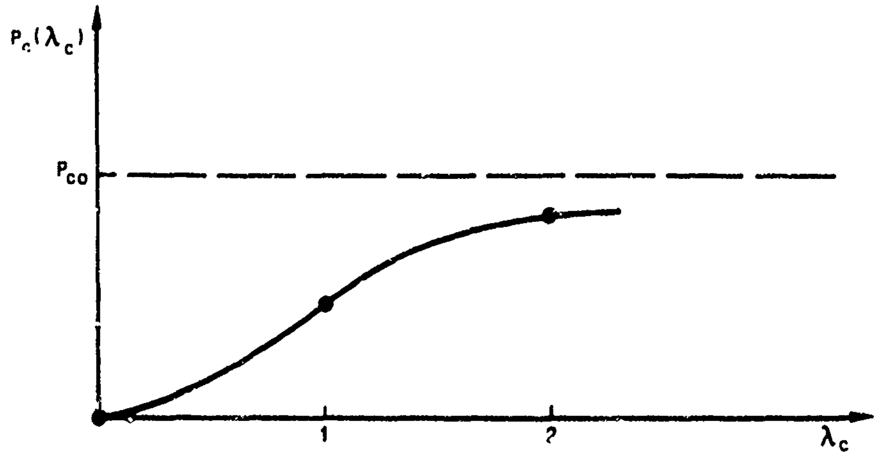


FIGURE B-2 $P_c$ AS A FUNCTION OF ENCOUNTER RATE

Extending this approach, suppose it could be determined that the situation looks like Fig. B-3. That is, things can be kept pretty well

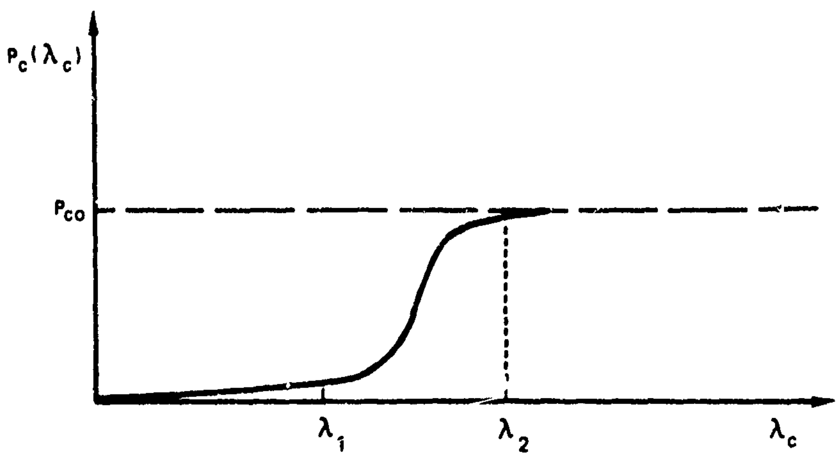


FIGURE B-3 EXTENSION OF $P_c$ AS A FUNCTION OF ENCOUNTER RATE

under control until the encounter rate $\lambda_c$ reaches some critical level, after which the ability to separate deteriorates rapidly. Such a curve can be obtained by fitting data to the equation

$$P_c(\lambda_c) = P_{co} \frac{a^k k^k}{\Gamma(k)} \int_0^{\lambda_c} u^{k-1} e^{-kau} du \quad . \tag{B.5}$$

In the absence of data, (B.5) could be established by specifying two points on the curve such as $(\lambda_1, P_c(\lambda_1))$ and $(\lambda_2, P_c(\lambda_2))$ shown in Fig. B-3.

For example, we could establish "critical points" $\lambda_1$ and $\lambda_2$

$$\lambda_1 : \frac{P_c(\lambda_1)}{P_{co}} = .05$$

$$\lambda_2 : \frac{P_c(\lambda_2)}{P_{co}} = .95$$

where, respectively, "things start to deteriorate" and "chaos begins to reign."

## Appendix C: OVERLAP MODEL ERROR STUDY

In the discussion of the reduced overlap model in Section 4.3.3 it was recognized that the assumption $A^*/A_s = (R_c/R_s)^2$ could fail. Indeed it will fail if the decoy being classified is within $R_c + \rho_d$ of the perimeter of the operating area. In order to get a rough assessment of the error introduced by this failure we examine a particular case with $R_c = 70$ nmi and $\rho_d = 25$ nmi. Let "r" denote the event "decoy being classified is located r nmi from the center of the operating area." Assume r is uniformly distributed between 0 and $R_o$, the radius of the operating area. Assuming that "r" is independent of "$H \epsilon A_s$" we can write

$$\text{(C.1)} \qquad P(H \epsilon A^* | H \epsilon A_s) = \sum_r P(H \epsilon A^* | H \epsilon A_s \text{ \& } r) P(r)$$

where we are implicitly quantizing r and P(r) is the probability of a specific value of r. Actually, we consider 11 equally likely values of r,

$$r_i = 20_i \qquad i = 0, \ldots, 10$$

$$P(r_i) = 1/11 \quad .$$

Hence,

$$\text{(C.2)} \qquad P(H \epsilon A^* | H \epsilon A_s) = (1/11) \sum_{i=0}^{10} P(H \epsilon A^* | H \epsilon A_s \text{ \& } r_i).$$

$P(H \epsilon A^* | H \epsilon A_s \text{ \& } r_i)$ is further decomposed by conditioning on the location of the searcher on the perimeter of the classification circle of radius $\rho_d$ about the decoy. We assume the searcher is uniformly distributed along that portion of the classification perimeter that is located <u>inside</u> the operating area. This assumption is questionable for those values of r where the classification perimeter is entirely inside the operating



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area but is "close" to the perimeter of the operating area; for now we ignore this possible source of error. This searcher distribution is also quantized into discrete points (6 to 10 points depending on the value of r and the resulting geometry). We then have

$$(C.3) \qquad P(H\epsilon A^*|H\epsilon A_s \ \& \ r_i) = (1/n_r) \sum_{j=1}^{n_r} P(H\epsilon A^*|H\epsilon A^*|H\epsilon A_s \ \& \ r_i \ \& \ s_j)$$

where "$s_j$" denotes searcher is in position j and $n_r$ is the number of searcher positions for distance r.

The area $P(H\epsilon A^*|H\epsilon A_s \ \& \ r_j \ \& \ s_j)$ can now be measured. A planimeter was used for the small study but the procedure could easily be programmed for a digital computer. The situation is depicted in Fig. C-1 for a given value of r and $s_j$. The ratio of the shaded area to the crosshatched area is the probability to be measured. The reader can compare Fig. C-1 with Fig. 4-2. The results obtained for $P(H\epsilon A^*|H\epsilon A_s \ \& \ r_i)$ are shown in Fig. C-2 with a smooth curve fitted to the data.

$P(H\epsilon A^*|H\epsilon A_s)$ is then calculated to be 0.583 via the formula (C.3). This is to be compared with the approximation $(R_c/R_s)^2 = 0.544$. Thus, the error appears to be small. An appropriate modification of the reduced overlap model including the calculated value 0.583 was exercised and the resulting change from using $(R_c/R_s)^2$ in the operational measures $E(T_{13})$, $H_{13}(t)$, and $G_{13}(t)$ was found to be negligible in the case examined ($R_c = 70$, $\rho_d = 25$, $R_o = 200$).

Thus, although it is by no means a definitive error study, this set of calculations indicates $P(H\epsilon A^*|H\epsilon A_s) = A^*/A_s$ is a reasonable approximation. The approximation can be expected to break down as $R_c + \rho_d$ approaches or exceeds $R_o$.

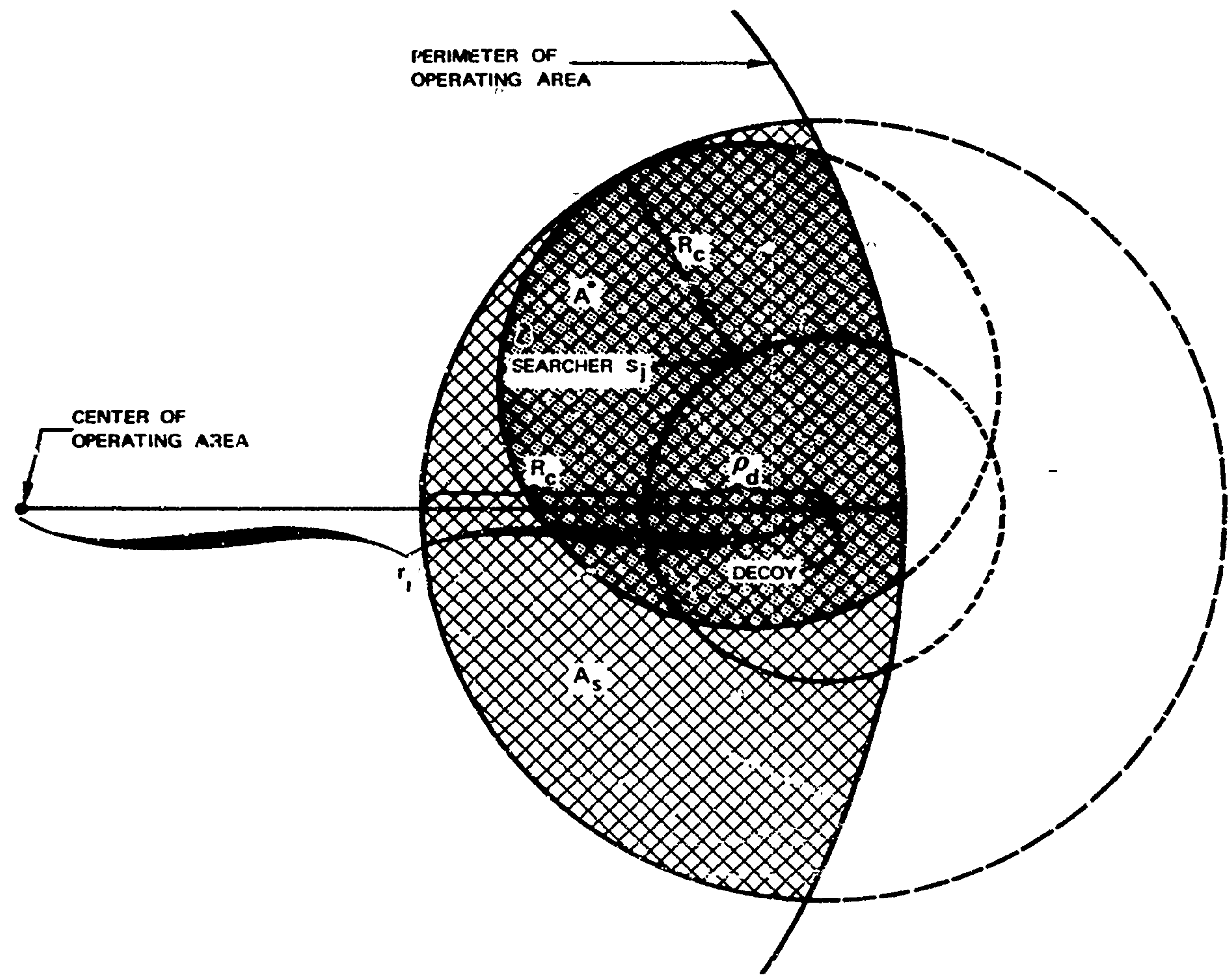


FIGURE C-1 GEOMETRY FOR MEASURING $P(H \in A^* | H \in A_s \;\&\; r_1 \;\&\; s_j)$

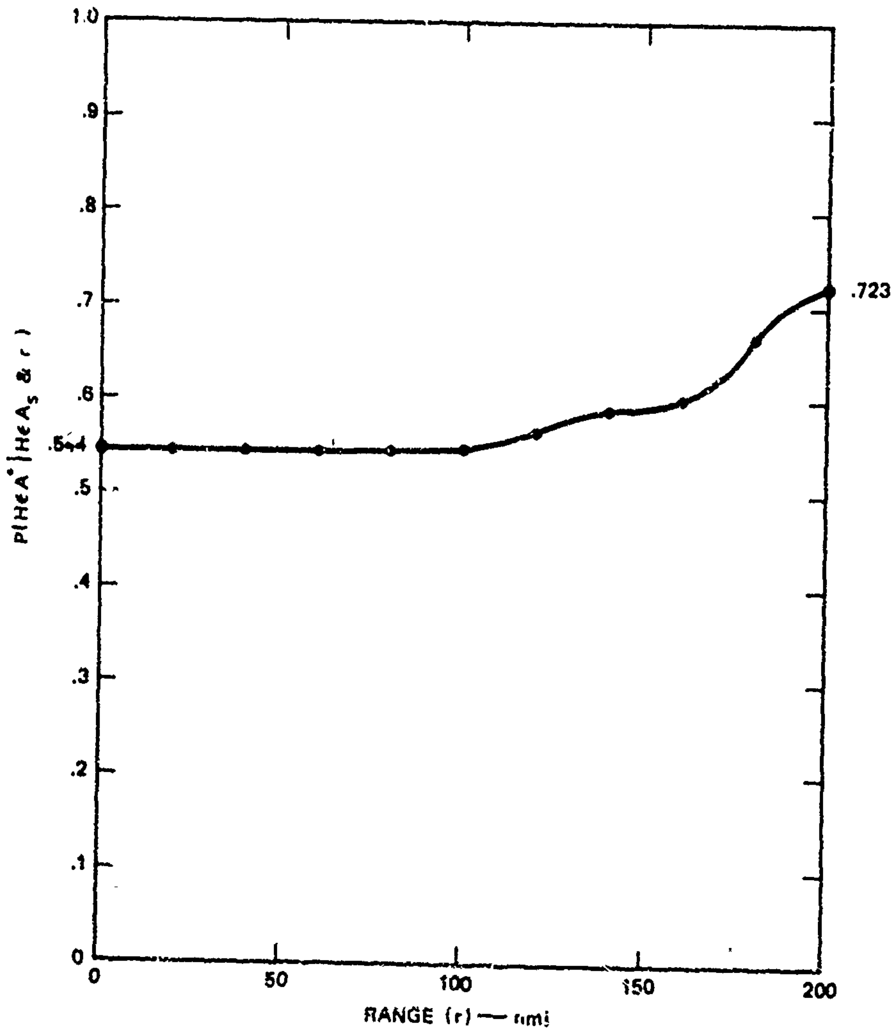


FIGURE C-2 INTERMEDIATE CONDITIONAL PROBABILITIES FOR COMPUTING $P(H \epsilon A^* | H \epsilon A_s) = 0.583$

## Appendix D: A MINIMAL MARKOV PROCESS FORMULATION

As mentioned in Section 3 the special case where $f_{ij}(t) = f_i(t)$ is an exponential distribution is a Markov process. Markov processes are discussed extensively in Refs. 1 and 2 where they are given another characterization, that of the minimal process, which arises from particular assumptions. The concept of minimal process, with its "infinitesimal generator matrix," and its relation to our present characterization of the SMP is discussed by Karlin (Ref. 2, Chs. 7 and 8). A brief discussion is included here for completeness. The more classical minimal process approach was not used in the main body of the study because the alternative was more suited to the events occurring in the real phenomena under consideration.

We assume a stationary continuous (in t) transition probability matrix $P(t) = [p_{ij}(t)]$ defined by $p_{ij}(t) = \Pr\{X(t+s) = j | X(s) = i\}$ where, using our previous terminology from Section 3

$$\Pr\{X(t+s) = j | X(s) = i\} = \Pr\{A_j(t+s) | A_i(s)\}$$

with

$$\Omega \supset A_j(t+s) = \{\omega \epsilon \Omega : x_\omega(t+s) = j\}$$

$$\Omega \supset A_i(s) = \{\omega \epsilon \Omega : x_\omega(s) = i\} \quad .$$

The relationship to our SMP terminology is given by

$$q_i = \lim_{h \to 0+} \frac{1 - p_{ii}(h)}{h}$$

$$q_{ij} = \lim_{h \to 0+} \frac{p_{ij}(h)}{h}$$



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$$\alpha_{ij} = \begin{cases} q_{ij}/q_i & i \neq j \\ 0 & i = j \end{cases}$$

and

$$\lambda_{ij}(t) = f_i(t) \triangleq \frac{1}{q_i} e^{-\frac{1}{q_i}t} \qquad i, j=1, \ldots, n$$

The question can be raised as to how to formulate our search problem so as to have a minimal Markov process. Let us assume that for small $\Delta t$

$$p_{12}(\Delta t) = \lambda \cdot \Delta t + o(\Delta t)$$

$$p_{13}(\Delta t) = \beta \cdot \Delta t + o(\Delta t)$$

$$p_{21}(\Delta t) = [\mu \cdot \Delta t + o(\Delta t)]\alpha_1$$

$$p_{23}(\Delta t) = [\mu \cdot \Delta t + o(\Delta t)]\alpha_3$$

$$p_{31}(\Delta t) = p_{32}(\Delta t) = 0 \quad \text{for all } \Delta t$$

where

$\alpha_1$ = P{no new targets in view when finished classifying decoy}

$\alpha_3$ = P{at least one HVT in view and chosen from field of targets when finished classifying decoy}.

Note $\alpha_3$ is given by (4.31), or (4.32) if $n_c = 1$, or (A.5). We have $\alpha_1$ given by $\alpha_1 = (1-P_c)^{n_c}(1-P_d)^{n_d-1}$ where $P_c$ and $P_d$ are as before; or $\alpha_1$ can be given by (A.4).

We require

$$\text{(i)} \quad \sum_{j=1}^{N} p_{ij}(t) = 1 \qquad \text{all } t \geq 0, \text{ all } i=1, \ldots, N$$

$$\text{(ii)} \quad p_{ij}(t) \geq 0 \qquad \text{all } t \geq 0, \text{ all } i, j=1, \ldots, N$$

$$\text{(iii)} \quad P(t+s) = P(t)P(s)$$

$$\text{(iv)} \quad \lim_{t \to 0+} p_{ij}(t) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} .$$

From the continuity properties of P(t) it can be shown that $p'_{ij}(0)$ exists for all i and j.

We define the state transition rate matrix (or infinitesimal generator matrix) $Q = (q_{ij})$ as follows

$$q_{ij} \triangleq p'_{ij}(0) = \lim_{h \to 0} \frac{p_{ij}(h)}{h} < \infty \qquad \text{all } i \neq j$$

$$q_{ii} \triangleq -q_i \text{ where } q_i \triangleq -p'_{ii}(0) = \lim_{h \to 0} \frac{1 - p_{ii}(h)}{h} .$$

In our case of a finite state space, $q_{ij} < \infty$ all i, j and $\sum_{j=1}^{N} q_{ij} = 0$. Q is nonpositive on the main diagonal, non-negative off the main diagonal, and its row sums are zero. It can be shown that $P(t) = e^{Qt}$ (matrix exponential). Then by definition we have

$$q_{12} = \lambda, \; q_{13} = \beta$$

$$q_{21} = \alpha_1\mu, \; q_{22} = \alpha_3\mu$$

$$q_{31} = 0 = q_{32}$$

and since the row sums are zero, we complete with

$$q_{11} = -(\lambda+\beta), \; q_{22} = -(\alpha_1+\alpha_3)\mu, \; q_{33} = 0.$$

Thus

$$Q = \begin{bmatrix} -(\lambda+\beta) & \lambda & \beta \\ \alpha_1\mu & -(\alpha_1+\alpha_3)\mu & \alpha_3\mu \\ 0 & 0 & 0 \end{bmatrix} .$$

Using the method of Cox and Miller (Ref. 1, p 196 ff) we obtain

$$E(T_{13}) = \frac{(\alpha_1+\alpha_3)\mu + \lambda}{[\beta(\alpha_1+\alpha_3) + \alpha_3\lambda]\mu} .$$

This is identical with (4.12) when the following identifications are made

$$\alpha_1 \longleftrightarrow \alpha_{21} , \qquad \alpha_3 \longleftrightarrow \alpha_{23}$$

$$\alpha_{13} \longleftrightarrow \frac{\lambda}{\lambda+\beta} , \qquad \alpha_{14} \longleftrightarrow \frac{\beta}{\lambda+\beta}$$

$$E(\tau_1) \longleftrightarrow \frac{1}{\lambda+\beta} , \qquad E(\tau_2) \longleftrightarrow \frac{1}{\mu} .$$

If we use the area approach in computing $\alpha_1$ and $\alpha_3$ and if we consider the special case $n_c = 1$, we can use (4.32) to compute $\alpha_3$ and $\alpha_1 = (1-P_c)(1-P_d)^{n_d-1}$, so that

$$\alpha_1 + \alpha_3 = \frac{P_d n_d - (1-P_c)\left[1-(1-P_d)^{n_d-1}\right] + P_c\left[P_d n_d + (1-P_d)^{n_d} - 1\right]}{P_d n_d}$$

and

$$E(T_{13}|n_c=1) =$$

$$\frac{\left\{P_d n_d+(1-P_c)\left[1-(1-P_d)^{n_d-1}\right] + P_c\left[P_d n_d+(1-P_d)^{n_d}-1\right]\right\}\mu + P_d n_d\lambda}{\left(\left\{P_d n_d+(1-P_c)\left[1-(1-P_d)^{n_d-1}\right] + P_c\left[P_d n_d+(1-P_d)^{n_d}-1\right]\right\}\beta + P_c\left[1-(1-P_d)^{n_d}\right]\lambda\right)\mu} .$$

On the other hand, consider the case of no overlap, so that $\alpha_1 = 1$, $\alpha_3 = 0$. Then

$$E(T_{13} | \alpha_1 = 1) = \frac{1}{\beta} + \frac{\lambda}{\beta} \cdot \frac{1}{\mu} .$$

So once all the variables determining $\lambda$ and $\beta$ are fixed $E(T_{13}|\alpha_1 = 1)$ is a linear increasing function of $\mu^{-1}$, the mean classification time.

## REFERENCES AND ANNOTATED BIBLIOGRAPHY

The following books and papers provided useful background as well as explicitly usable material.


1. D. Cox and H. Miller; "The Theory of Stochastic Processes"; Wiley; 1965 (Note especially: pp 351-356 on Semi-Markov Processes, pp 178-185 on Markov Processes, pp 146 ff on the Poisson Processes)

2. S. Karlin; "A First Course in Stochastic Processes"; Academic Press; 1968 (Note especially: pp 206-208, Ch. 8, Markov Processes; Karlin does not discuss Semi-Markov Processes)

3. "A Theoretical Basis for Methods of Search and Screening," Summary Technical Report of Division 6; NDRC; Washington, D.C.; 1946 (Note especially: pp 7-8 on encounter ate. This material also in: B. O. Koopman's; "The Theory of Search: Part I, Kinematic Bases"; O.R. (US) Vol 4; 1956; pp 324-346

4. DiStefano, et al; "Feedback and Control Systems," Schaum Outline Series; McGraw-Hill; 1967 (Note especially: pp 64-69 regarding inversion of rational fraction Laplace transforms)

5. Hillier and Lieberman; "Introduction to Operations Research"; Holden-Day; 1967

6. M. Abramowitz and I. Stegun, editors; "Handbook of Mathematical Functions," National Bureau of Standards Applied Mathematics Series, No. 55; U.S. Government Printing Office; Washington, D.C.; 1965

7. Luce, et al; "Handbook of Mathematical Psychology," Vol. III; Wiley; 1965 (Note: pp 467-473 and 484-485 on continuous time Markov chains with denumerable state space)

8. Lars Kristiansson; "On the Semi-Markov Process and Its Block Diagram Representation with Applications to Aircraft Mission Analysis," Technical Note 55; Svenska Aeroplan Aktiebolaget (Saab Aircraft Co.); Linköping, Sweden; March 1964

9. D. L. Iglehart; "Applied Probability," lectures delivered at Stanford University; Stanford, California; Autumn 1970

10. Ronald A. Howard; "Dynamic Probabilistic Systems," Volumes I and II; Wiley; 1971 (Note: a significant reference for those interested in applying the concepts of Markov and semi-Markov processes)




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