United States Naval Postgraduate School

THESIS

A FINITE MARKOV CHAIN
MODEL OF THE COMBAT PROCESS

by

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13. ABSTRACT

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A Finite Markov Chain

Model of the Combat Process

by

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I. INTRODUCTION

The analysis of a military combat operation can be divided into two general areas of interest. Emphasis can be placed either on the decision making process or on the physical process. This thesis describes an application of the mathematical theory of finite Markov chains to the study of both of these aspects of a military combat operation. However, the greatest portion of the emphasis of this study will be placed on the physical aspects of the combat process. Prior to formulating the mathematical model, the rationale behind the approach used will be explained and the purpose of the study will be established.

A. A MILITARY COMBAT OPERATION AND ITS GENERAL ENVIRONMENT

Before beginning a study of the physical and decision making aspects of a combat operation, it is appropriate to identify and briefly explain the structure or framework within which a military weapons system operates. Conceptually, any military weapons system can be thought of as a man-machine system which operates in a specified environment or space of operations. The term weapons system as used in this thesis refers to a friendly weapons system. The term target system as used in this thesis refers to an enemy weapons system. This definition of the term target system does not mean to imply that an enemy weapons system is the only possible type of target system. Target and weapons systems can be categorized by type according to their differing specialized functions or roles and the general type of activities which they conduct. The geographic region where weapons and target systems operate is called a space of operations. A space of operations in its most general form is a finite three dimensional, complex combination of weather, terrain, vegetation, and bodies of water. The modern weapons system can be positioned in a variety of spaces of operation separated by relatively large distances within short time spans.

Individual weapons and target systems can be organized into either larger systems or organizations of essentially homogeneous weapons and target systems or into larger systems or organizations of predominately heterogeneous weapons and target systems in order to accomplish assigned tasks or missions. Between these two extremes there can exist a continum of weapons system organizations. The modern military commander allocates his weapons system organizations to specific spaces of operations based on a prediction or forecast of the number of target systems anticipated to be located in a given space at a given time and on the number of weapons systems that he has available in his weapons systems inventory at a given time. The commander is continually evaluating the target system forecasts and reallocating and reorganizing his weapons systems from his inventory of reserves to his spaces of operations.

A weapons system which has been positioned in a given space of operations requires continual inputs from support systems. A support system is defined as any type of system which provides input resources to a weapons system organization. Weapons systems use input resources during the execution of a combat operation. Support systems are external to the weapons system organization and can be in the form of additional weapons systems which have varying types of control, maneuver, target acquisition, and target destruction sub-systems, or they can be in the form of personnel and logistical systems. These systems provide inputs that are used to augment and to sustain the activities of the weapons systems. The inputs provided by personnel and logistical systems consist of people, supplies, and equipment needed to replace resources which have been expended during the conduct of a combat operation. Personnel and logistical resources are obtained from a relatively secure base space. These replacement items then proceed along some route or system of routes and eventually arrive in the space of operations.

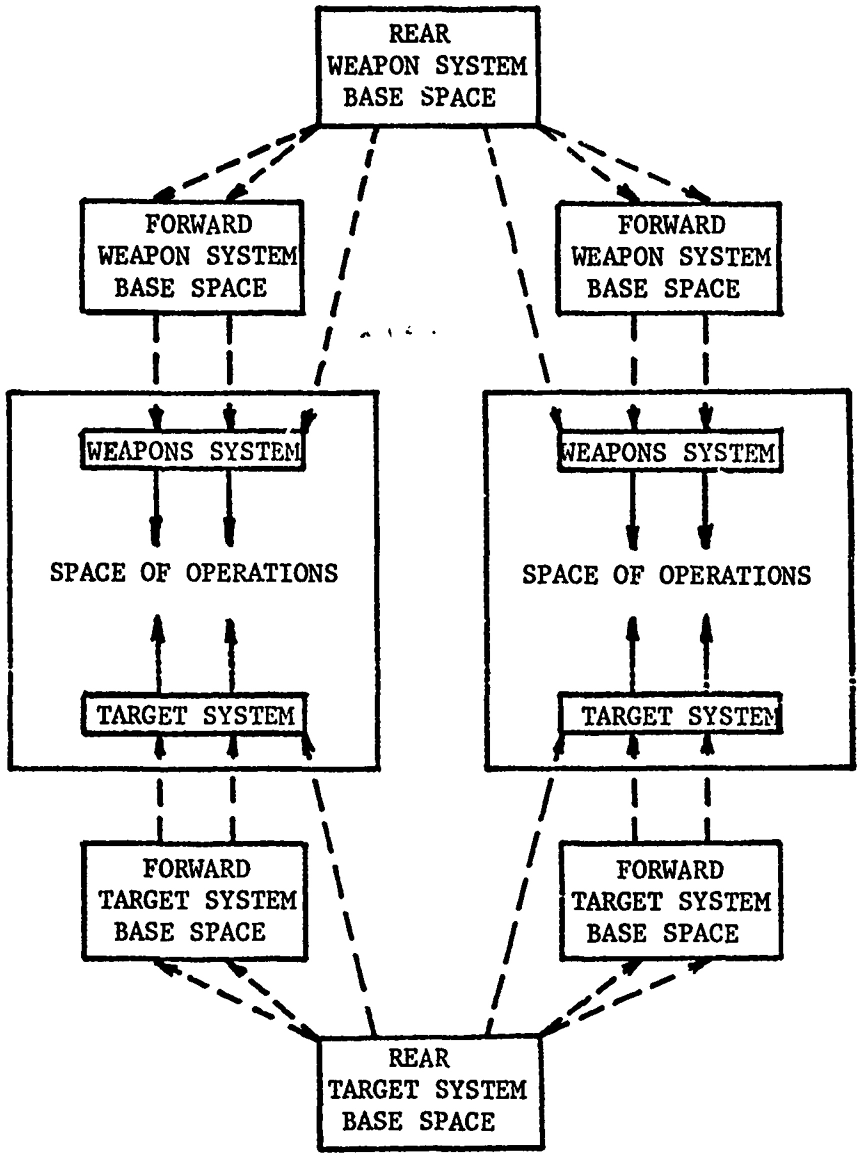
Weapons systems which have been placed in a space of operations can function in either specialized or generalized roles. For example, a tank can function in an anti-aircraft role or in an anti-armor role or in both roles. The manner in which a system operates in any one of its roles is established by some type of standard operating procedures document. The number of roles which a weapons system can execute is determined by the current state of technology.

However, regardless of the type of roles in which the weapons and target systems are employed they generally are able to perform four primary activities. These four primary activities are control, maneuver, target acquisition, and target destruction. The definitions of these activities will be given in the next chapter. An operation or a sequence of these activities is not as definable as the activities which comprise it. The sequence and number of activities which occur during an operation will vary depending on the mission statement, the type of weapons system, and the type of target system. In order to provide further clarification of the concepts which have been discussed, a graphic depiction of a possible structure or frame work for the combat process is given in Figure 1.

The preceeding discussion of a combat process and its environment establishes a basis for a more detailed qualitative and quantitative analysis of the combat process. Prior to continuing the analysis, the purpose and scope of this thesis will be established.

B. OBJECTIVE AND SCOPE

The purpose of this thesis is to present a mathematical model which can be used to describe the activities comprising a military combat operation. The model which will be used to describe a combat operation should facilitate a clearer understanding of the combat process and permit the development of measures of the effectiveness of the weapons systems involved in this combat process. In the next chapter the elements of a combat operation will be defined and explained in detail. The mathematical model of the combat process will be formulated as a finite Markov chain. The properties of the Markov chain model will be presented and a method of estimating the transition probabilities will be given. The properties of the model will then be used to develop measures of the effectiveness of the systems involved in the combat process. Having introduced the idea of a combat process in this chapter, the next chapter will provide a more definitive discussion of the combat process.



Dotted lines represent routes of personnel and logistical inputs.
Solid lines represent control, maneuver, target acquisition and target destruction outputs.
Arrows represent the direction in which inputs and outputs move.

Figure 1: Scheme of the interactions between weapons and target systems.

II. THE COMBAT PROCESS AND ITS STATES

The combat process, as previously described, involves weapons systems, target systems, their assigned mission statements, and a space of operations. As previously mentioned, the weapons and target systems conduct four primary activities in the space of operations. They are control, maneuver, target acquisition, and target destruction. This chapter will place most of its emphasis on analyzing these four system outputs or activities. These system activities are being emphasized because they represent the means by which a system accomplishes its objectives or goals as established by assigned mission statements. Having an understanding of these system outputs will provide a rational basis for future studies on system inputs such as personnel and logistical replacement items.

An important first step is to explicitly define and explain these four output activities.

A. DEFINITIONS

The four primary activities of a weapons system are defined as follows:

The control activity consists of the set of all actions which cause the weapons system to conduct all of its other activities in a purposeful, coordinated, and procedural manner.

The maneuver activity consists of the following controlled actions: preparation and occupation of a specified position, movement to or from a specified position, and navigation of the weapons system as it moves.

The target acquisition activity consists of the following controlled actions: searching for a specified target system, evaluating a detected target system, and maintaining surveillance of a specified target system.

The target destruction activity consists of the following controlled actions: firing at a specified target, adjusting the firing action, and assessing the amount of target destruction caused by firing at a specified target.

Each activity of a weapons or target system has a time associated with it. This activity time is defined as the amount of time that elapses during the conduct of any one of the four primary activities of a weapons or target system.

The definitions of a weapons system and a target system are as follows:

A weapons system is an organized, friendly man-machine system which has sub-systems that conduct the following activities: control, maneuver, target acquisition, and target destruction.

A target system is defined in this thesis as an enemy weapons system. As previously mentioned weapons and target systems can be organized or designed so as to accomplish either generalized or specialized roles or functions with their sub-systems. The sub-systems of weapons and target systems can conduct one or more of the four primary activities.

B. EXPLANATION OF THE COMBAT PROCESS

These definitions and activities of a weapons system and a target system require further elaboration. Weapons and target systems conduct operations in order to attain objectives. However, the primary objectives of both systems require that each system attempt to destroy the other system while at the same time attempting to preserve itself.

During the conduct of an operation the activity time or the amount of time required by a weapons or target system to accomplish an activity is not fixed. The amount of time required to conduct the activities of the weapons system and target system will continually vary depending on the space of operations. The space of operations contains topographical, geological, hydrographical, botanical, and meteorological factors which cause the activity times of the weapons systems and target systems to vary. The actions and counteractions of both systems also modify the activity times of each system.

As a weapons system conducts operations, it uses logistical and personnel inputs. Since weapons and target systems often have limited internal personnel and logistical storage capacities, these two types of inputs are either used immediately upon being issued to the weapons and target systems or limited amounts are stored by the weapons and target systems for future use. The amount of personnel and logistical inputs required will depend on the effects of the mission statement, the space of operations, and the target system.

In summary, the activity times or output times of both the weapons and target systems depend on the interactions between the two systems, the interactions between each of the systems and the space of operations, and the interactions among each of the systems and their personnel and logistical inputs.

In order for a weapons system to survive and to destroy the target system in the space of operations, it executes all of its four primary activities in some sequence. It is assumed that the definitions of the four primary activities are such that no two activities can occur simultaneously. These definitions of the four primary activities are mutually exclusive definitions and they also collectively exhaust the set of possible output actions of a weapons or a target system. The four primary activities are conducted in accordance with the standard operating procedures which apply to a particular weapons system and the mission statement which is in effect. The mission statement specifies the tasks to be accomplished by the weapons system, the initial positions or initial space of responsibility in the space of operations, and the objectives of the operation. These documents are flexible enough to permit the weapons system to adapt to unexpected changes in its own activity times, in the activity times of the target system, in the space of operations, and in the assigned missions.

C. ACTIVITIES AND STATES OF THE COMBAT PROCESS

The control activity will be the first activity to be discussed. The mission statement assigns the weapons system a set of tasks to be accomplished, a space of operations, initial positions or an initial space of responsibility in the space of operations, and specifies the objectives of the operation. The sub-system of the weapons system that conducts the control activity then requires that the other sub-systems of the weapons system communicate information concerning the detection of a target system, the maneuvering of the weapons system, the firing activities of the weapons system, an estimate of the amount of damage to the target system, and an assessment of the status of the weapons system's personnel and logistical needs. This information concerning the activities of the weapons system is used by the control sub-system in a decision making process. The output from this decision making process is a set of instructions called a mission statement. Each sub-system of the weapons system receives a mission statement. If each sub-system of a weapons system executes its activities in accordance with its mission statement, then it is considered to be operating in a controlled and coordinated manner. This information which is transmitted to the control sub-system is also used as a basis for the allocation of adequate personnel and logistical inputs to the weapons system. If the assigned tasks are not being accomplished as scheduled, then the control sub-system must either ensure that there is a decrease in the amount of time which elapses during the conduct of the activities of the committed weapons systems or it must ask for either a change in its mission statement or for additional weapons systems.

In other words, the weapons system is given a task and it commences execution of this task. The control sub-system receives progress information. This information is compared with stored information which the control sub-system has previously received concerning what tasks the weapons system is capable of accomplishing and the approximate amount of time required to complete an assigned task. The control sub-system also has a knowledge of the procedures which will be used by the weapons system to accomplish its assigned tasks. If the weapons system is not operating as desired, then the control sub-system transmits the appropriate mission statement changes to the weapons system. These changes should cause the weapons system to operate in the desired manner. Without proper control the weapons system can not use its other activities to produce an accomplished mission.

Another important aspect of the control activity is adaptation. As the weapons system conducts its control actions. it must properly adapt its control actions to its environment in the space of operations and to the actions of the target system.

The actions which comprise the maneuver activity will be defined prior to discussing the target acquisition and destruction activities. The ultimate purposes of maneuver are to position the weapons systems so that they can maintain continuous surveillance of the target systems and to do so in such a manner that the target systems become relatively isolated from their support systems. The weapons systems should then be maneuvered against the target systems in such a manner that the weapons systems achieve a decisive advantage over the target systems in terms of the relative activity times of the opposing systems. This advantage can be attained either by maneuvering a large number of weapons systems with high activity times or by maneuvering a small number of weapons systems with low activity times against the target systems. The achievement of target acquisition and destruction is related to the degree of activity time advantage that the weapons systems have over the target systems and to the length of time that this advantage can be sustained. Upon completion of the target acquisition and destruction phases, the weapons systems will normally continue maneuvering in order to survive and to perform other missions.

Having established the purpose of maneuver, it is now appropriate to describe the conduct of this activity. If the weapons system is in an adequate position, then it is able to attempt target acquisition or destruction and it also has ensured itself some degree of survivability. If the weapons system determines that it is in a position that does not permit commencement of target acquisition and destruction and which does not provide some degree of survivability, then it will move from its current position to another position that does meet these requirements. The position that it moves to will be selected by the weapons system and approved by control. As the weapons system moves, it either maintains surveillance over the target or it continues to search for the target as it moves. As the weapons system moves, it also conducts a navigation action. This action is executed by the weapons system in order to ensure that its movement actions are being made in the proper direction. When the system determines that it is in its selected position, some type of preparation is normally required in order to ensure that target acquisition or destruction can commence and that survivability is enhanced. An important aspect of this maneuver activity is the proper adaptation of the maneuver actions of the weapons system to its environment in the space of operations and to the actions of the target system. This adaptation is essential if the weapons system is to acquire and destroy the target and preclude its own destruction. Adaptation and the ability to survive are functions not only of the design of the system but of the manner in which the system is employed in relation to the space of operations and the target system.

The next activity to be discussed is target acquisition. If a weapons system is not able to acquire targets, then it will have difficulty destroying targets. The weapons system is assigned a search space and is then positioned so that it can search this space. It either determines that it has detected or it determines that it has not detected a target system in its assigned space. This information is reported to the control sub-system. If it has not detected a target system, it is then either assigned a new search space or it remains in its present search space and continues to search for a target. If it has detected a target, then it should maintain surveillance over the target in order to evaluate the target system. The information obtained from target surveillance is also reported to the control sub-system which can then either direct that the target destruction phase of the operation begin or direct the weapons system to maintain further surveillance over the target. An important aspect of target acquisition is also adaptation. As the weapons system conducts its target acquisition activity, it must properly adapt its target acquisition actions to its environment in the space of operations and to the actions of the target system.

The target destruction phase commences when the control sub-system has adequate information on the target system and when all necessary maneuver activities have been completed. A weapons system which is located in an acceptable firing position is ready to begin the execution of its target destruction activity. Since the target has been under surveillance, sufficient information should be available to commence firing on the target. This firing action includes such sub-actions as loading and aiming of the weapon and time of flight of the projectile. The projectile or round may then be assessed by the weapons system as having either hit or missed the target. If a target miss is assessed, then some type of adjustment in the impact point of the projectile may be made and the firing action undertaken again. It should be realized that once a target is fired upon, it most probably will either commence evasive actions or begin to fire at the weapons system. If a miss occurs, the target can begin to maneuver and this can result in the weapons system losing surveillance of the target. This result may require the combat process to return to its target acquisition phase. If the weapons system fires and assesses that it has hit the target, then further assessment may be conducted to determine if a target kill has occurred. The protection afforded the target by either its design or by available terrain cover can preclude its total destruction. If a target kill is not assessed, then the firing action may be repeated. Again depending on the extent of the damage the target may take some additional type of evasive or firing action and the possibility exists for the loss of surveillance over the target. If continual surveillance is maintained over the target, then sufficient time should be available for the weapons system to fire, to determine if a hit has occurred, and to determine if the target has been destroyed. Upon completing the destruction of the target, the control sub-system is informed and the process normally will return to the target acquisition phase. Another important aspect of target destruction is adaptation. As the weapons system conducts its target destruction activity, it must properly adapt its target destruction actions to its environment in the space of operations and to the actions of the target system. It should also be understood that target destruction in its most general connotation means that a weapons system has performed some actions which have resulted in the target system being unable to perform one or all of its activities.

The activities of the combat process are similar to the activities which occur during the conduct of a competitive game. In competitive and conflict processes the players are trying to defeat or destroy their opponents while simultaneously attempting to avoid their own defeat or destruction. In competitive and conflict processes varying degrees of information are available to the players concerning the results of operations.

It should be noted that throughout the previously mentioned operation the target system was also conducting the same activities with its objective being to destroy the weapons system. These activities of both systems can have either a positive or negative effect on all the activity times of each system. For example, consider the effect of suppressive fire. If the target system's suppressive fire is effective, it may increase all the activity times of the weapons system. If it is not effective, then it may decrease all the activity times of the weapons system. This interaction between the target and weapons systems is one of the factors which affect the states of the combat process.

A typical combat operation scenario might be described by the following sequence of activities:

Control

Maneuver

Control

Target Acquisition

Control

Maneuver

Control

Target Destruction

Control

Maneuver

Control

The combat operation that was just described is not necessarily unique in the following sense: The activities could and probably will occur in a differing sequence the next time an independent operation of the same type is conducted. These differing sequences of activities are due to the complexity of the combat process and to the numerous factors involved. An attempt to attribute the outcome of the combat process to any one factor or set of factors appears to be questionable with regards to the validity of the results which follow from this type of analysis. A better method of analysis might be to carefully define the activities which comprise an operation and then to observe the sequence of these activities over the time period of a specified type of operation. This experiment would be repeated for a spacified number of replications so that the required data sample sizes can be obtained. The results of this type of experimentation can then be used to obtain the distribution of the amount of time required to conduct a given activity and thus the amount of time required to conduct the operation. This resulting distribution can be considered as a measure of the effectiveness of a weapons system. A similar distribution could be obtained for the target system under the same experimental conditions.

The combat operation which has been described has well defined states with estimable probability distributions associated with the transition from one state to another state. The transition from one state to another state depends only on the present or current state of the system. Also an initial state distribution of the system can be easily obtained. Thus, this combat operation satisfies the general prerequisites or assumptions which are required if a finite Markov chain stochastic model is to be used to describe a combat operation.

Having defined and explained the combat operation qualitatively, it is now appropriate to begin the development of a mathematical model which will enable a combat operation to be described in an analytical manner.

III. FINITE MARKOV CHAIN MODEL OF THE COMBAT PROCESS

This chapter will present a finite Markov chain model which can be used to describe a military combat operation. First, the definition of a finite Markov chain will be given. The states of the finite Markov chain model of the combat process will then be defined. A sequence of activities for a typical combat operation will also be given. Using this information it will then be possible to construct the one-step transition probability matrix or the finite Markov chain model of the combat process. This model which will be called Model I will be constructed on the basis of the additional assumption that each of the activities which comprise a combat operation will have equal activity times. The effect of relaxing this assumption will then be investigated. A new procedure for constructing the one-step transition probability matrix will then be presented. This procedure will permit the construction of a one-step transition probability matrix when the assumption of equal activity times does not hold. The one-step transition probability matrix which can be constructed by the use of this procedure will then be presented. This model will be called Model II. It will then be shown that some pertinent results from the theory of finite Markov chains can be used to compute the mean and the variance of the first passage time from an initial state s_i to a state s_j. The specific first passage time which is of interest will be defined as the first passage time from the control state s_o to the damage assessment state s_{12}. This first passage time can be interpreted as being a measure of the effectiveness of a weapons system. The reason for selecting the first passage time from the control state s_o to the damage assessment state s_{12} will also be explained. It will then be shown that some pertinent results from the theory of finite Markov chains can be used to compute a probability statement concerning the number of times in the first n steps that a finite Markov chain process is in a specified state s_j. The reason for interpreting this probability statement as being another measure of the effectiveness of a weapons system will then be explained. This presentation of two measures of effectiveness for a weapons system will be followed by a short discussion of some ways in which they can be used to aid in an analysis of the combat process. Having outlined the content of this chapter which describes the finite Markov chain model of the combat process, it is appropriate to begin the presentation of this model by stating the basic concepts upon which this model is based.

A. BASIC CONCEPTS OF MARKOV CHAINS

If a finite Markov chain model is to be used to describe a combat operation or a combat process, then a combat operation must satisfy the following definitions: If the stochastic process $\left\{f_n,\ n=0, 1, 2, \ldots\right\}$ where f_n is a sequence of outcome functions with state space $\left\{s_o, s_1, \ldots, s_n\right\}$ is given, then this stochastic process is defined by Kemeny and Snell [7] as a finite Markov process if the probability statement $P\left[f_{n+1}=s_{n+1} \,\middle|\, f_o=s_o, \ldots, f_n=s_n\right] = P\left[f_{n+1}=s_{n+1} \,\middle|\, f_n=s_n\right]$ is true for all $s_o, s_1, \ldots, s_n$ elements of the state space and for all $n=0, 1, 2, \ldots$ elements of the index space. A Markov process is, therefore, a process in which a knowledge of the present outcome f_n is all that is needed in order to probabilistically predict the future outcome

f_{n+1}. All information concerning the past, i. e., the outcomes $f_o=s_o, \ldots, f_{n-1} = s_{n-1}$, may be ignored.

The one-step transition probabilities for a Markov process denoted by $p_{ij}(n)$ are defined as $p_{ij}(n)=P\left[f_{n+1}=s_j \mid f_n=s_i\right]$ for all s_i and s_j elements of the state space [7]. A finite Markov chain will be defined as a finite Markov process such that the one-step transition probabilities $p_{ij}(n)$ do not depend on n, i. e., $p_{ij}(n)=p_{ij}$ where n is defined as a general element of the index space for the Markov chain [7]. A finite Markov chain which satisfies this definition is often called a homogeneous finite Markov chain with stationary one-step transition probabilities. The index space for the finite Markov chain is the set of all non-negative integers. If n=2, for example, then the process has taken its second step or second transition. The amount of time that elapses when the process is in any state is called the time unit of the process.

The matrix of transition probabilities p_{ij} will be denoted as $\underline{P}$. The special type of Markov chain that is to be used to model the combat process is a regular finite Markov chain. A regular finite Markov chain is a finite Markov chain that consists of a set of states and a set of one-step transition probabilities. However, in a regular finite Markov chain once the process has moved from state s_i, it will eventually return to state s_i with probability one. This statement is true for all of the states of the state space. It is also possible for the process to return to state s_i in n steps where n can be 1 or 2 or 3 or ... steps [3]. This type of Markov chain was selected because it was felt that it was the most appropriate type of Markov chain model which could be used to describe the combat process as it was described in Chapters I and II of this thesis.

B. FINITE MARKOV CHAIN MODEL OF THE COMBAT PROCESS

It will be assumed that the combat process, which was described in the Chapters I and II, satisfies the definitions of a regular finite Markov chain, i.e., the combat process can be described by a finite number of states and a set of one-step transitions which connect these states. Also, a one-step transition probability p_{ij}, which is not a function of n, can be associated with each possible transition. The states of the combat process are defined as follows in terms of a general weapons or target system:

The system is said to be in

<u>state s_o</u> when it is executing a control action.
This single state comprises the complete control activity of a weapons or target system.

The system is said to be in

<u>state s_1</u> when it is executing a maneuver control action.

<u>state s_2</u> when it is executing any type of movement action

<u>state s_3</u> when it is executing a movement evaluation or navigation action.

<u>state s_4</u> when it is executing any type of preparation and occupation of a position action.

These four states comprise the maneuver activity of a weapons or target system.

The system is said to be in

<u>state s_5</u> when it is executing a target acquisition control action.

<u>state s_6</u> when it is executing a target search action.

state s_7 when it is executing a target detection evaluation action.

state s_8 when it is executing a target surveillance action.

These four states comprise the target acquisition activity of a weapons or target system.

The system is said to be in

state s_9 when it is executing a fire control action.

state s_{10} when it is executing a firing action.

state s_{11} when it is executing a fire adjustment action.

state s_{12} when it is executing a damage assessment action.

These four states comprise the target destruction activity of a weapons or target system.

The combat operation which will be described by a finite Markov chain model will consist of the following sequence of four activities: control, maneuver, target acquisition, and target destruction. In order to clarify the discussion that follows, an example of a typical weapons system which could be conducting this type of combat operation will be presented. For example, suppose that the weapons system being modeled is a tank. The control sub-system of the tank might consist of those components of the tank's radio communications equipment which are capable of communicating with a control headquarters external to the tank. The maneuver sub-system of the tank might consist of the tank commander, the propulsion system, and the tank commander's compass. The target acquisition sub-system of the tank might consist of the eyes of the tank commander. The fire control sub-system might consist of the tank commander and the main tank gun. The description of the tank sub-systems which has just been given is not meant to be a unique description of the tank sub-systems. It merely represents a possible description of the tank sub-systems.

These tank sub-systems execute the set of all actions which comprise the combat operation to be modeled. Figure 2 gives a pictorial representation of the states of the combat operation being modeled and a typical set of one-step transitions connecting these states. The arrows in Figure 2 represent the directions of possible one-step transitions connecting the states. A one-step transition probability is associated with each arrow. A time unit is associated with each state. This time unit represents the amount of time that elapses between the transitions of the combat process. The sequence of transitions between the states given in Figure 2 is not meant to be a unique description of all the possible sequences of transitions. It merely represents a possible description of the sequence of transitions between the states of the combat process. The states and sequence of transitions given in Figure 2 are based on the discussions of the combat process which are included in Chapters I and II.

Using the information which has just been given concerning a typical combat operation, it is now possible to construct the one-step transition probability matrix or the finite Markov chain model of this combat process. A typical transition matrix for a regular finite Markov chain model of the combat process is given in Figure 3. The elements of this matrix are the one-step transition probabilities. The one-step transition probability p_{ij} denotes the probability of passing from state s_i to s_j in one time step. One time step or time unit for the combat process will be assumed to represent the passage of ten

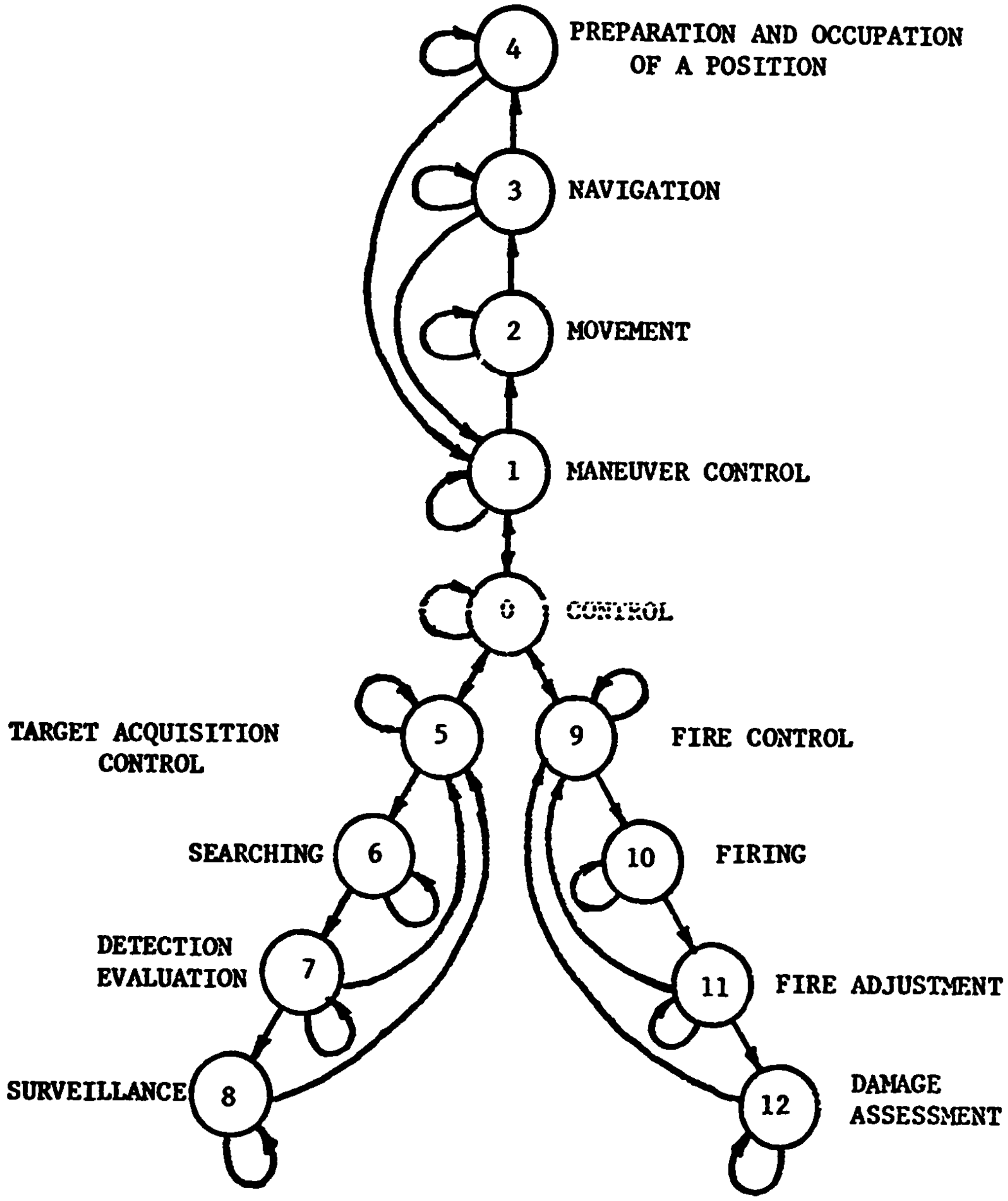


Figure 2: States of the combat process and typical one-step transitions between these states.

S T A T E S

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\text{STATES}\quad
\begin{array}{c}
\\ s_0 \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \\ s_7 \\ s_8 \\ s_9 \\ s_{10} \\ s_{11} \\ s_{12}
\end{array}
\begin{array}{c}
\begin{array}{ccccccccccccc}
s_o & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 & s_{10} & s_{11} & s_{12}
\end{array} \\
\begin{bmatrix}
0 & p_{01} & 0 & 0 & 0 & p_{05} & 0 & 0 & 0 & p_{09} & 0 & 0 & 0 \\
p_{10} & 0 & p_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & p_{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & p_{31} & 0 & 0 & p_{34} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & p_{41} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p_{50} & 0 & 0 & 0 & 0 & 0 & p_{56} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & p_{67} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & p_{75} & 0 & 0 & p_{78} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & p_{85} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p_{90} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_{910} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_{1011} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_{119} & 0 & 0 & p_{1112} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_{129} & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$

Figure 3: One-step transition probability matrix for a regular finite Markov chain model of the combat process

minutes of actual time. The sum of the one-step transition probabilities in each row of the matrix is one. The one-step transition probability matrix given in Figure 3 was constructed on the basis of the information which was given in this chapter concerning finite Markov chain models. The size of this matrix is thirteen by thirteen. This size is determined by the number of states which comprise the combat process. The one-step transition probabilities p_{ij} do not depend on n where n is a general element of the index space of the Markov chain. This model will be called Model I. One of the assumptions which had to be satisfied in order to construct Model I was the assumption that in the combat process the one-step transition probabilities $p_{ij}(n)$ can be defined as p_{ij}.

Suppose that the assumption that $p_{ij}(n)=p_{ij}$ does not hold. For example, an actual military combat operation might be conducted in such a manner that each of the activity times is not of equal duration. If this is the case, and a Markov chain model is used to describe this operation, then the one-step transition probabilities given in Model I will become $p_{ij}(n)$. In other words, the amount of time which elapses between the nth and the n+1st step is not a constant. The time unit is, thus, a different size for each state and, therefore, the one-step transition probabilities will depend on n. If this situation occurs, then it will not be possible to model the combat process with a regular finite Markov chain model. The one-step transition probability matrix given in Figure 3 needs to be modified so that the one-step transition probabilities $p_{ij}(n)$ can again be assumed to be one-step transition probabilities p_{ij} which are not a function of n. A method will now be given which will permit the construction of a one-step transition probability matrix such that the one-step transition probabilities are not a function of n, i.e., so that different states may have different action times.

It will be assumed that the combat operation which is to be modeled is identical to the one that was described by Model I. The only exception to this assumption is that the activity times will no longer be assumed to be equal. It will also be assumed that each of the activities has a finite determinate activity time. An algorithm will now be presented which will permit the construction of a one-step transition probability matrix such that the one-step transition probabilities are no longer a function of n. In order to provide a clearer understanding of the algorithm, the one-step transition probability matrix for only the movement state s_2, the navigation state s_3, and the preparation and occupation of a position state s_4 will be constructed. The steps of this algorithm can be repeated for each of the remaining states of the combat process and a one-step transition probability matrix $\underline{P}$ will result such that the one-step transition probabilities p_{ij} can be assumed to be independent of n. Figure 4 pictorially represents the one-step transition probability matrix which can be constructed by the use of this algorithm.

1. The first step in the construction of the transition matrix is to select an appropriate time unit for the transition matrix. Suppose that the action time or the amount of time which elapses during the conduct of the actions that comprise each of the states is as follows:

(a) for state s_2 it is given as ten minutes,

(b) for state s_3 it is given as eight minutes,

(c) for state s_4 it is given as two minutes.

$$
\begin{array}{cc}
 & \underline{\text{S T A T E S}} \\
 & \begin{array}{cccccccccccccc} s_o & \cdot\ \cdot & s_{21} & s_{22} & s_{23} & s_{24} & s_{25} & s_{31} & s_{32} & s_{33} & s_{34} & s_{41} & \cdot\ \cdot & s_{12} \end{array} \\
\begin{array}{c} \text{S} \\ \text{T} \\ \text{A} \\ \text{T} \\ \text{E} \\ \text{S} \end{array}
\begin{array}{c} s_o \\ \cdot \\ \cdot \\ s_{21} \\ s_{22} \\ s_{23} \\ s_{24} \\ s_{25} \\ s_{31} \\ s_{32} \\ s_{33} \\ s_{34} \\ s_{41} \\ \cdot \\ \cdot \\ s_{12} \end{array}
&
\left[\begin{array}{cccccccccccccc}
 & & & & & & & & & & & & & \\
 & & & & & & & & & & & & & \\
 & & & & & \cdot & \cdot & \cdot & \cdot & & & & & \\
 & & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
 & & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
 & & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
 & \cdot & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdot & \\
 & \cdot & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdot & \\
 & \cdot & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdot & \\
 & \cdot & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdot & \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & & \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & P_{34} & & \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
 & & & & & & & & & & & & & \\
 & & & & & \cdot & \cdot & \cdot & \cdot & & & & & \\
 & & & & & & & & & & & & &
\end{array}\right]
\end{array}
$$

Figure 4: Modified one-step transition probability matrix for a regular finite Markov chain model of the combat process with determinate action times

The time unit which should be used is that number which is the greatest common divisor of all of the given action times. For this example the time unit is two minutes.

2. The action time for each of the states given in this example will now be divided by the time unit of two minutes. The quotients which result represent the number of sub-states that will comprise one original state of the combat process. In other words, the original state s_2 has now been divided into the following set of sub-states: $\left\{s_{21}, s_{22}, s_{23}, s_{24}, s_{25}\right\}$. A similar set of sub-states results for state s_3 and state s_4. The resulting one-step transition probability matrix is given in Figure 4.

Another modification of the one-step transition probability matrix given for Model I can result when the activity times are no longer assumed to be equal. This modification assumes that each of the activities has a continuous activity time probability distribution associated with it. An algorithm will be presented which will permit the construction of a one-step transition probability matrix. In order to provide a clearer understanding of the algorithm, the one-step transition probability matrix for only the navigation state s_3 will be constructed. The steps of this algorithm can be repeated for each of the remaining states of the combat process and a one-step transition probability matrix $\underline{P}$ will result such that the one-step transition probabilities p_{ij} can be assumed to be independent of n. Figure 5 pictorially represents the one-step transition probability matrix which can be constructed by the use of this algorithm.

1. The first step in the construction of the transition matrix is to select an appropriate time unit for the transition matrix. Suppose

$$
\begin{array}{ccc}
 & & \underline{\text{S T A T E S}} \\
 & & \begin{array}{ccccccccccccc} s_o & s_1 & \cdots & s_{25} & s_{31} & s_{32} & s_{33} & s_{34} & s_{35} & s_{41} & \cdots & s_{11} & s_{12} \end{array} \\
\text{STATES} & \begin{array}{c} s_o \\ s_1 \\ \cdot \\ \cdot \\ \cdot \\ s_{25} \\ s_{31} \\ s_{32} \\ s_{33} \\ s_{34} \\ s_{35} \\ s_{41} \\ \cdot \\ \cdot \\ \cdot \\ s_{11} \\ s_{12} \end{array} & \left[\begin{array}{ccccccccccccc}
 & & & & & & & & & & & & \\
 & & & & \cdot & \cdot & \cdot & \cdot & \cdot & & & & \\
 & & & & & & & & & & & & \\
 & & & & & & & & & & & & \\
 & & & & & & & & & & & & \\
 & & & 0 & P_{2531} & 0 & 0 & 0 & 0 & 0 & & & \\
 & \cdot & & 0 & 0 & P_{3132} & 0 & 0 & 0 & 0 & \cdot & & \\
 & \cdot & & 0 & 0 & 0 & P_{3233} & 0 & 0 & P_{3241} & \cdot & & \\
 & \cdot & & 0 & 0 & 0 & 0 & P_{3534} & 0 & P_{3341} & \cdot & & \\
 & \cdot & & 0 & 0 & 0 & 0 & 0 & P_{3435} & P_{3441} & \cdot & & \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & P_{3541} & & & \\
 & & & & & & & & & & & & \\
 & & & & \cdot & \cdot & \cdot & \cdot & \cdot & & & & \\
 & & & & & & & & & & & & \\
 & & & & & & & & & & & & \\
 & & & & & & & & & & & & \\
 & & & & & & & & & & & &
\end{array} \right]
\end{array}
$$

Figure 5: Modified one-step transition probability matrix for a regular finite Markov chain model of the combat process with stochastic action times

that the action time or the amount of time which elapses during the conduct of the actions that comprise state s_3 is known to be ten minutes. A time unit should be selected which gives an adequate amount of information concerning the amount of time that the process remains in state s_3 and a time unit which does not cause the transition matrix to become so large that it exceeds the storage capacity of the computer being used. Suppose a time unit of two minutes is selected. This time unit represents the amount of time which elapses between each iteration or transition of the process. The selection of the time unit in this algorithm represents an approximation of the continuous probability distribution of the amount of time that the process remains in a state by a discrete probability distribution.

2. The action time of ten minutes will now be divided by the time unit of two minutes. The resulting number is five. This number represents the number of sub-states that will comprise one original state of the combat process. In other words, the original state s_3 has now been divided into the following set of sub-states: $\left\{s_{31}, s_{32}, s_{33}, s_{34}, s_{35}\right\}$. The resulting one-step transition probability matrix is given in Figure 5. These extensions of basic Markov chain theory have modified the one-step transition probabilities in such a manner that the one-step transition probabilities are again independent of n. As previously mentioned, the steps of these algorithms can be repeated for each of the remaining states of the combat process and a one-step transition probability matrix $\underline{P}$ will result. The matrix $\underline{P}$ which is given in Figure 5 will be called Model II. The one-step transition probabilities which could be contained in Model II are not meant to be a unique description of all the

possible sequences of transitions. They merely represent a possible description of the sequences of transitions between the states of the combat process. The states and sequence of transitions which could be described by Model II are based on the discussions of the combat process which were given in Chapters I and II.

C. MEASURE OF EFFECTIVENESS

This section will present two useful measures of the effectiveness of a weapons system. Also, some pertinent results from the theory of finite Markov chains will be presented. These theoretical results will permit the computation of the measures of weapons systems' effectiveness. Prior to presenting the formulas which can be used to compute these measures of effectiveness, it is appropriate to briefly discuss and define a measure of effectiveness. A measure of the effectiveness of a weapons system will be defined as a mathematical function which is a quantitative measure of how effectively a weapons system operates as it attempts to attain its primary objective, which is to destroy the target system.

The first measures of effectiveness which can be obtained from the finite Markov chain models of the combat process are the matrices of the mean and variance of the first passage times. The first passage time which is of primary interest is the first passage time from the control state s_o to the damage assessment state s_{12}. If the combat process begins in the control state s_o and then passes to the damage assessment state s_{12} according to the one-step transition probabilities given in Model I, it will have conducted a set of actions that should result in some type of damage being inflicted on the target system. When the weapons system has inflicted this damage, it then attempts to assess this damage. The assessment of target damage can be considered as being the final action which must be accomplished by a weapons system. The completion of this action implies that the weapons system has accomplished its objective of destroying the target system. Therefore, the mean first passage time from the control state s_o to the damage assessment state s_{12} is a measure of the average amount of time that elapses as a weapons system attains its primary objective. This measure can be interpreted as being a measure of how effectively a weapons system operates as it attains its primary objective. Prior to presenting the formulas which will permit computation of the mean and variance of the first passage time from state s_o to state s_{12}, a brief discussion of some pertinent results from the theory of finite Markov chains will be given. The use of this theory can enable the computation of the mean and the variance of the first passage time matrices.

It may be shown that for a regular finite Markov chain the first passage time f_j is a function whose value is the number of steps before entering state s_j for the first time after departing state s_i. This statement is true for all i, j indexes of the state space. The mean first passage time matrix denoted by $\underline{M}$ is the matrix with entries m_{ij} defined to be equivalent to the expected value of the first passage time function f_j. For any i, an index of the state space, the expected value of the function f_k is finite. If $\underline{a}$ is the limiting probability vector for the transition matrix $\underline{P}$, then $m_{ii}=1/a_i$ where a_i is an arbitrary element of the limiting probability vector $\underline{a}$. If $\underline{P}$ is a regular transition matrix, then $\underline{P}^n$ will approach a limiting probability matrix $\underline{A}$

as n approaches infinity. Each row of the matrix $\underline{A}$ is the same limiting probability vector $\underline{a}$. In order to compute the mean first passage matrix $\underline{M}$, the fundamental matrix of a regular Markov chain must be known. If $\underline{P}$ is the transition matrix for a regular Markov chain and $\underline{A}$ is the limiting matrix, then it can be shown that the fundamental matrix $\underline{Z}$ for a regular finite Markov chain can be defined as $(\underline{I} - (\underline{P} - \underline{A}))^{-1}$. This fundamental matrix Z can also be defined as $I + \sum_{n=1}^{\infty} (\underline{P}^n - \underline{A})$ where $\underline{I}$ is the identity matrix. These results have been shown by Kemeny and Snell [7]. These results may be used to compute the mean first passage matrix $\underline{M}$. The matrix $\underline{M}$ is defined as being equal to the quantity $(\underline{I} - \underline{Z} + \underline{EZ}_{dg})\underline{D}$ where $\underline{D}$ is the diagonal matrix with diagonal elements $d_{ii} = 1/a_i$, $\underline{E}$ is a matrix of unit elements, and the matrix $\underline{Z}_{dg}$ results from the matrix $\underline{Z}$ by setting off-diagonal entries equal to zero.

In order to compute the variance of the first passage times, the matrix $\underline{Z}$ must again be used. It will also be necessary to make use of the fact that the variance of the first passage time f_j for any state s_i where i is an index of the state space is given by the following equation:

$$\mathrm{Var}_i(f_j) = E_i(f_j^2) - (E_i(f_j))^2 .$$

Since the value of $E_i(f_j)$ or the expected value of the first passage time f_j is known, it is only necessary to find $E_i(f_j^2)$ or the expected value of the first passage time f_j^2. The matrix of all the expected values of the first passage times f_j^2 will be denoted as $\left\{E_i(f_j^2)\right\} = \underline{W}$. It can be shown that the matrix $\underline{W} = \underline{M}(2\,\underline{Z}_{dg}\,\underline{D} - \underline{I}) + 2(\underline{ZM} - \underline{E}(\underline{ZM})_{dg})$ where all of the matrices in this equation are as previously defined. This statement implies that the variance matrix which will be defined as $\underline{V}$

is equivalent to $\left\{\mathrm{Var}_i(f_j)\right\}$ and is equal to $\underline{W} - \underline{M}_{sq}$ where $\underline{M}_{sq}$ results by squaring each entry of $\underline{M}$. These results also have been shown by Kemeny and Snell [7].

By using the results which have just been presented, it is possible to compute the mean and the variance of the first passage time from the control state s_o to the damage assessment state s_{12} for the one-step transition probability matrix $\underline{P}$.

A second measure of the effectiveness of a weapons system which can be computed by using a set of probability statements concerning the number of times in the first n-steps that the process is in an arbitrary state s_j will be presented in this section. If a probability statement is made concerning the number of times in the first n steps that the combat process is in an arbitrary state s_j, then it can also be computed for all states. If the time unit of the process is also known, then it will be possible to determine the amount of time that the combat process remains in each one of the states. These probability statements will, therefore, aid in determining which set of the actions of a weapons system require the greatest amount of time to accomplish. If the judgment and experience of the analyst indicate that the amount of time which is spent in a state is too long, then a need for further analysis might be required. Either the weapons system is improperly organized, i.e., it is the wrong type of weapons system, or a sufficient number of the right type of weapons systems are not available to accomplish the assigned tasks. Also, the level of the intensity of combat might be such that it is physically impossible to acquire more targets. Since these probability statements provide a quantitative measure of how effectively a weapons system operates as it attempts to attain its primary objective, they can be interpreted as being a useful measure of the effectiveness of a weapons system. In order to compute these probability statements, a brief discussion of some pertinent results from the theory of finite Markov chains will be given. The use of this theory can enable the computation of these probability statements.

It may be shown for a regular finite Markov chain model that if there exists a regular finite Markov chain with limiting probability vector $\underline{a}$, then for any initial probability vector $\underline{b}$ the mean fraction of times in the first n steps that the process moves to state s_j approaches a_j as n approaches infinity. The limiting probability vector $\underline{a}$ has been previously defined; however, it should be noted that it is equivalent to the following notation: $\left\{a_1, a_2, \ldots, a_j, \ldots, a_{n-1}, a_n\right\}$ where a_j is an arbitrary element of the n-dimensional vector. The initial probability vector $\underline{b}$ is a probability vector which defines the probabilities of the process being in each one of the states of the combat process at the start of the process.

This limiting result and the following limiting results are being presented so that the analyst will have a better understanding of the central limit theorem of Markov chains. This central limit theorem permits the computation of limiting probability statements concerning the number of times that the combat process is in the state s_j in the first n steps.

Let f be a function defined on the states of a regular chain and let this function be denoted as $f(s_i) = f_i$. Also let $f^{(n)}$ be the value of this function on the n-th step. Then the limiting variance of f can be denoted as the $\lim_{n \to \infty} \frac{1}{n} \text{Var}_{\underline{b}} \left[\sum_{k=1}^{n} f^{(k)} \right]$. This variance can be shown to be defined as being equivalent to $\sum_{i=1}^{r} \sum_{i=1}^{r} f_i c_{ij} f_j + \sum_{i=1}^{r} a_i f_i (1 - f_i)$ where f is a function that takes on the value 1 with probability f_i in state s_i and is 0 otherwise. In order to compute the limiting variances, the variance-covariance matrix $\underline{C}$ must be known. This matrix can be obtained from previous results and is given by the following formula:

$$\underline{C} = \underline{A}_{dg}\underline{Z} + \left[\underline{A}_{dg}\underline{Z}\right]' - \underline{A}_{dg}\underline{I} - \underline{A}'\underline{A}_{dg}$$

All of the elements of this matrix have been previously defined with the following exception: $\underline{A}'$ denotes the transpose of the matrix $\underline{A}$. An arbitrary element of the matrix $\underline{C}$ is the element c_{ij}. The limiting variances are the diagonal elements c_{jj} of this matrix $\underline{C}$.

It is now possible to state the central limit theorem of regular finite Markov chains. For an ergodic chain, that is, a chain in which it is possible to go from every state to every other state, let $y^{(n)}_j$ be the number of times that the process is in state s_j in the first n steps. Also, let $\underline{a} = \{a_j\}$ be the fixed limiting probability vector and let $\underline{c} = \{c_{jj}\}$ be the vector of limiting variances. Then if c_{jj} is not equal to zero for any numbers $r < s$, the probability statement which is denoted as $P_k \left[r < \frac{y^{(n)}_j - na_j}{\sqrt{nc_{jj}}} < s \right]$ approaches a standard normal cumulative distribution function denoted as $\frac{1}{\sqrt{2\pi}} \int_r^s e^{-x^2/2} dx$ as n approaches infinity for any choice of starting state k [7].

By using these results, it is possible to compute a set of probability statements which state that in the first n steps the occurrence of a particular state s_j will with a certain probability not deviate from a certain mean number of steps by more than a certain number of deviation steps. This type of probability statement can be computed for each of the states of the combat process.

The next section of this chapter will present some examples of how these two measures of effectiveness can be used to aid in the analysis of a combat process.

D. APPLICATIONS OF MEASURES OF EFFECTIVENESS

If two weapons systems are to be compared, the analyst normally desires to know if one weapons system is better than another weapons system. These weapons systems can be compared by using a measure of the weapons systems' effectiveness. The specific measures of weapons systems' effectiveness which can be used to compare two weapons systems are the mean and the variance of the first passage times from the control state to the damage assessment state for both weapons systems. The mean and variance matrices $\underline{M}$ and $\underline{V}$ of the first passage times will probably depend on the following factors for each of the weapons systems: the type of target system, the weapons system's mission, the target system's mission, the type of space of operations, and the initial positions of both weapons in the space of operation. These relationships follow from the fact that the transition matrix $\underline{P}$ was a function of the above factors. If all of the above factors which determine the mean and variance of the first passage time are fixed with the exception of the type of weapons systems, then two weapons systems can be compared in terms of the mean and variance of the first passage times. This use of the mean and variance of the first passage time measures of effectiveness is one example of an application of these measures of effectiveness. An example of an application of the limiting probability statement measure of effectiveness will be given next.

The limiting probability statements enable the analyst to make a quantitative statement concerning the number of times in the first n steps that the combat process is in an arbitrary state s_j. A limiting probability statement can be computed for all of the states of the combat process. As previously mentioned, if the time unit of the process is known, then it will be possible to determine the amount of time that the combat process remains in each one of the states. These probability statements will, therefore, aid in determining which set of the actions of a weapons system require the greatest amount of time to accomplish. For example, it should be expected that a considerable amount of time will be spent in the control state s_o. This does not necessarily represent an inefficiency on the part of the weapons system. However, if the judgment and experience of the analyst indicate that the amount of time which is spent in the target acquisition states as opposed to the target destruction states is too long, then a need for further analysis might be required. Either the weapons system is improperly organized, i.e., it is the wrong type of weapons system or a sufficient number of the right type of weapons systems are not available, or the level of intensity of combat is such that it is physically impossible to acquire more targets. The probable answer to a problem such as this lies somewhere between the organization or design of the weapons system and the number of the correct types of weapons systems. This use of the limiting probability statements is merely one example of their application in the field of weapons systems analysis.

These two measures of effectiveness can also be used to aid in the analysis of target systems. This type of analysis can be quite useful because it not only indicates the ability of the weapons system to survive, but it also indicates some possible strengths and weaknesses that a target system might have in comparison to a given weapons system.

The versatility of this regular finite Markov chain model of the combat process is a direct consequence of the ability of the one-step transition probabilities to realistically describe the combat process. These one-step transition probabilities can be obtained from data which is obtained from a variety of sources. The interpretations and conclusions which can be inferred from the model follow directly from the estimated values of the one-step transition probabilities. Consequently, the subject of estimation of the one-step transition probabilities is important for without good estimates of the one-step transition probabilities, useful results probably will not be obtained from the model.

IV. DATA SOURCES AND ONE-STEP TRANSITION PROBABILITY ESTIMATION

The finite Markov chain model as described in Chapter III consisted of a transition matrix $\underline{P}$. This chapter will present some statistical procedures which could be used to estimate the one-step transition probabilities for a typical weapons system and some possible sources from which data may be obtained. This chapter will also present a method for constructing a transition matrix $\underline{P}$.

A. SOURCES OF DATA

In order to estimate the one-step transition probabilities of the finite Markov chain model, the weapons systems analyst would like to obtain data under controlled experimental conditions; however, this is not always possible, and data from other types of data sources may have to be used. It is the purpose of this section to briefly mention some possible sources of data on a tank weapons system whose actions can be modeled by the use of a regular finite Markov chain.

Data on tank and armor operations in general can be obtained from the records of the official history of the United States Army. Private historical records of armor operations are also available. The works of Fuller [4], Hart [6], Rommel [11], and Guderian [5] are some possible sources of data. Another means of obtaining data on armor combat operations is to conduct combat interviews of the types which have been conducted by S. L. A. Marshall [9]. These interviews can be conducted with individuals or with groups of individuals. The quality of the data obtained from a combat interview is quite variable due to the fact that the individuals being interviewed are or have been under the stresses of combat. Data which has been collected from the sources that were just mentioned can be interpreted as being uncontrolled data. In other words, no experimental controls were in effect.

The next data sources which will be mentioned represent data sources which provide data that has been collected under conditions where some degree of control may possibly have been imposed. Data can be obtained from military organizations which are not involved in the actual conduct of combat operations. These organizations are normally conducting some type of training for possible combat operations. United States Army armor units in Europe and the continental United States are typical examples of this type of organization. They could be the source of extensive amounts of data which can be collected during the conduct of training exercises in which these units participate.

The next two sources of data that will be mentioned can provide data which has been obtained under experimental conditions. These two sources can be separated into two broad categories. They are research and development organizations and test and evaluation organizations. Typical examples of these types of organizations are the United States Army Materiel Command and the Human Resources Research Organization.

Other possible sources of data that has been obtained under experimental conditions are management science, operations research, and systems analysis organizations. These organizations obtain most of the data which they use from the data sources that have been previously mentioned. However, they also generate data by the use of mathematical models and computer simulations. Typical examples of this type of organization are the United States Army's Combat Developn s Command and organizations such as the Research Analysis Corporation.

These sources of data are representative of the type of organizations which can provide information on a tank weapons system. This list could be made more extensive in nature if a specific problem were being addressed. These sources of data have been discussed in very general terms so that the weapons systems analyst with a more specific problem might benefit from a broad overview of some of the sources of data for a tank weapons system.

Having briefly discussed data sources, one important use of this data will be presented in the next section.

B. ESTIMATION OF ONE-STEP TRANSITION PROBABILITIES

The models of the combat process that were presented in Chapter III describe the actions of a weapons system which is conducting a combat operation. The weapons and target systems are both assumed to operate in such a manner that their actions can be described by the previously mentioned thirteen states of a regular finite Markov Chain. The subject of estimation of the one-step transiton probabilities can be divided into two parts. The first part is concerned with the estimation of the amount of time that the process remains in a given state once it arrives in the given state. The second part is concerned with the estimation of the probability of transitioning from a given state to another state or set of states.

The amount of time that the actions of a system are in a given state s_j between transitions is a random variable which shall be designated as T. A random sample $T_1, T_2, \ldots, T_n$ can be obtained from data. The sample mean and the sample variance can be computed. The distribution of the random variable T can be estimated by the use of standard statistical techniques and available data. The next task is to develop a method for estimating the one-step transition probabilities.

The estimation of the one-step transition probabilities for a regular Markov chain transition matrix can be accomplished by the use of the statistical procedures which follow and available data. The state space for the combat process is denoted as S where S is defined as the set $\left\{s_o, s_1, s_2, \ldots, s_{12}\right\}$. The random variable X_i will be defined as a function X_i which maps its domain S onto euclidian one-space as follows: $X_i(s)$ will be defined as 1 if $s = s_i$. It will be defined as 0 if $s \neq s_i$ where $i = 0, 1, \ldots, 12$. The random vector $\underline{X}$ will be defined as a thirteen dimensional random vector $\left\{x_o, x_1, x_2, \ldots, x_{12}\right\}$. The probability mass function associated with the random vector $\underline{X}$ is defined as

$$p_{\underline{X}}(x_o, x_1, \ldots, x_{12}) = p_o^{x_o} p_1^{x_1} p_2^{x_2} \ldots p_{12}^{x_{12}}$$

where $\sum_{i=0}^{12} x_i = 1$. The random vector $\underline{X}$ is often called a n-dimensional Bernoulli random vector. A realization of $\underline{X}$ or a multinomial trial results in the process passing from one initial state to one and only one final state in one time step of the process. The likelihood function for the random vectors $\underline{X}_1, \underline{X}_2, \ldots \underline{X}_n$ where $\underline{X}_i$'s are independent and identically distributed random vectors is

$$L(p_o, p_1, p_2, \ldots, p_{12}) = p_o^{y_o} p_1^{y_1} p_2^{y_2} \ldots p_{12}^{y_{12}}, \text{ i.e., } y_j = \sum_{i=1}^{n} x_{ij}$$

where y_j is defined as the number of times that the process is in the state s_j in the first n steps and $j = 0, 1, \ldots, 12$. Standard techniques of the calculus are used to maximize this function. The vector of transition probabilities $\underline{p}$ which is defined as $(p_o, p_1, p_2, \ldots p_{12})$ will be estimated by the vector $\hat{\underline{p}} = (y_o/n. y_1/n, \ldots, y_{12}/n)$ where n is the sample size. If the estimation of p is to be in accordance with the idea of conditional transition probabilities, then the following sampling procedure must be used: An independent and identically distributed observation for a given initial state s_i can only be obtained when the combat process can be described as being in the initial state s_i. This statement implies that a family of thirteen probability mass functions can be obtained. Only thirteen probability mass functions are required because it will be assumed that the process passes to the same state or set of states regardless of how long it remains in a given initial state. This assumption can be relaxed if a more sophisticated sampling plan is used.

The estimation procedures which have been discussed are sufficient to permit the estimation of all of the one-step transition probabilities of the regular finite Markov chain transition matrix. A method will now be given which can be used to construct the transition matrix $\underline{P}$ when stochastic activity times are being modeled.

C. CONSTRUCTION OF A TRANSITION MATRIX

In this section a method will be presented for constructing a one-step transition probability matrix. For illustrative purposes the steps of this method will only be used to construct the transition matrix for the navigation state s_3. The steps of this method can be repeated for each of the other states of the combat process in order to construct the complete transition matrix $\underline{P}$. The action time T for the state s_3 will be assumed to be a random variable with an exponential distribution. The probability density function of the exponential distribution is given by the equation $f(t) = re^{-rt}$ where r is the rate at which transitions occur. The mean of this distribution is $1/r$. Since the transitions between the states of a finite Markov chain occur at discrete points in time, this continuous distribution of the action time may be approximated by a geometric distribution with a probability distribution given by the equation $p(t) = p(1-p)^{t-2}$ for $t = 2, 3, \ldots$. The mean of this distribution is $1 + \frac{1}{p}$. The geometric distribution will be fitted to the exponential distribution by equating the means of the two distributions permitting the value of p to be computed in terms of r. The value for the parameter p of the geometric distribution is defined as $r/(1-r)$. The transition matrix for the navigation state s_3 can be constructed as depicted in Figure 6. Assuming that the mean action time $1/r$ is known, then this value for p represents the probability of transitioning from state s_{31} to state s_{32}.

This chapter has presented a discussion on some possible sources from which data can be obtained. Some statistical procedures were also presented which can be used in conjunction with data in order to estimate the one-step transition probabilities of the regular finite Markov chain model. This chapter also presented a method for constructing a transition matrix $\underline{P}$. The next chapter will compare the finite Markov chain model to another probability model which can be used to describe the combat process and it will also discuss some possible extensions of the finite Markov chain model presented in this thesis.

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 & \underline{\text{S T A T E S}} \\
 & \begin{array}{ccccccccccc} s_o & s_1 & s_2 & \cdots & s_{31} & s_{32} & s_{41} & \cdots & s_{10} & s_{11} & s_{12} \end{array} \\
\begin{array}{c} \\ \\ \\ \\ \\ \\ \text{S} \\ \text{T} \\ \text{A} \\ \text{T} \\ \text{E} \\ \text{S} \\ \\ \\ \\ \end{array}
\begin{array}{c} s_o \\ s_1 \\ s_2 \\ \cdot \\ \cdot \\ \cdot \\ s_{31} \\ \\ s_{32} \\ \\ s_{41} \\ \cdot \\ \cdot \\ \cdot \\ s_{10} \\ s_{11} \\ s_{12} \end{array}
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Figure 6: A portion of the transition probability matrix for the Markov chain model where state s_3 has an exponentially distributed action time

V. CONCLUSIONS AND RECOMMENDATIONS

The purpose of this thesis as outlined in the introductory remarks was to develop a method for analyzing the combat stochastic process. The regular Markov chain model which was presented should aid in the analysis of the combat process and provide measures of effectiveness of the weapons systems involved in the combat process. In this chapter the degree to which the original purpose of the thesis was accomplished will be examined. It will also be shown that the reciprocal of the mean first passage time from the control state s_o to the target destruction state s_{12} can be defined as the Lanchester attrition coefficient for a square law combat process. The manner in which the model can be extended to other situations will be mentioned and some possible areas of future work will be suggested.

A. ACCOMPLISHMENTS

This section will briefly review and discuss the two quantitative measures of effectiveness of a weapons system which were presented in Chapter III. These two measures of effectiveness can be obtained from the mathematical model which was used to describe the combat process.

From the regular Markov chain model of the combat process, it is possible to obtain an estimate of the mean and the variance of the first passage time from the control state s_o to the damage assessment state s_{12}. This mean first passage time can be considered to be a measure of the effectiveness of a weapons system for the following reason: Consider a weapons system which was initially in the control state s_o. This weapons system then performs a set of actions that result in some type of damage being inflicted on the target. When the weapons system has inflicted this damage, it then attempts to assess this damage. The assessment of target damage can be considered as being the final action which must be accomplished by a weapons system. The completion of this action implies that the weapons system has accomplished its objective of destroying the target system. In other words, the mean first passage time is a measure of the average amount of time that elapses as a weapons system attains its primary objective. This measure can, therefore, be interpreted as being a measure of how effectively a weapons system operates as it attains its primary objective.

Another measure of the effectiveness of a weapons system which can be obtained from the model is a set of probability statements concerning the number of times in the first n steps that the process is in an arbitrary state s_j. These probability statements can be obtained for each one of the states of the combat process by applying the Central Limit Theorem of Markov chains to the combat process. These probability statements provide a means by which the activity times of a weapons system can be quantitatively analyzed.

Both of these measures of the effectiveness should not be used without a clear understanding of the experimental context from which the transition probabilities of the model were estimated. If the analyst requires additional measures of the effectiveness in order to adequately analyze a weapons system, then extensions of the model might be helpful. Prior to considering extensions of the finite Markov chain model, it will be shown that the reciprocal of the mean first passage time from the control state to the damage assessment state can be defined as being the Lanchester attrition coefficient for a square law combat process. The significance of this contribution to the Lanchester theory of combat will also be discussed.

B. CONTRIBUTIONS TO THE LANCHESTER THEORY OF COMBAT

The Lanchester theory of combat proposes that combat between two homogeneous forces can be modeled using the following set of ordinary differential equations:

$$\frac{dx(t)}{dt} = -\,ay(t) \quad (0) \quad \text{and} \quad \frac{dy(t)}{dt} = -\,bx(t) \qquad (1)$$

These equations model what is known as a deterministic Lanchester square law process with parameters a and b. These parameters are often called the Lanchester square law attrition coefficients. The attrition coefficient a is defined as the average rate at which a single unit of the y force destroys units of the x force where x(t) is often defined as the expected number of x force units at time t after the beginning of the engagement. Suppose T, a random variable, is the amount of time that elapses as one unit of the y force destroys units of the x force. Barfoot [2] proposed that the Lanchester square law attrition coefficient may be defined as the reciprocal of the expected value of the random variable T.

For Markov dependent fire it has been shown by Taylor [12] that the expected value of the random variable T which will be denoted as E [T] can be defined as indicated in the following equation:

$$E[T] = t_a + t_1 - t_h + \frac{(t_h + t_f)}{P(K|H)} + \frac{(t_m + t_f)}{P(h|m)}\left[\frac{(1-P(h|h)}{P(K|H)} + P(h|h) - p\right] \quad (2)$$

where t_a = time to acquire a target

t_1 = time to fire the first round after the target is acquired

t_h = time to fire a round after sensing a hit on the previous round

t_m = time to fire a round after sensing a miss on the previous round

t_f = time of flight of the projectile

$P(K|H)$ = the probability of a kill given that the target was hit on the previous round

$P(h|m)$ = the probability of a hit given that the target was missed on the previous round

$P(h|h)$ = the probability of a hit given that the target was hit on the previous round

p = the probability of a hit on the first round

The mean of the first passage time from the control state s_o to the damage assessment state s_{12} can be interpreted as being the average amount of time that elapses as one weapons system destroys a target system. Suppose that a weapons system conducts the following sequence of activities: control, maneuver, control, target acquisition, control, maneuver, control, and target destruction. If a finite Markov chain model of this combat operation is constructed using the methods developed in Chapter III, it would then be possible to compute the mean of the first passage time from the control state s_o to the damage assessment state s_{12}. This mean first passage time can be considered to represent the average amount of time that elapses as a weapons system accomplishes its objective of destroying the target system. The reciprocal of the mean first passage time from the control state s_o to the damage assessment state s_{12} can be defined as the Lanchester square law attrition coefficient for a Markov combat process. The value of the Lanchester square law attrition coefficient that is obtained by using a finite Markov chain model of a combat operation should be more realistic than a value obtained by using equation (2), because the finite Markov chain model contains transition probabilities which describe not only the target acquisition and target destruction actions of the weapons system but also the maneuver and control actions of the weapons system. In contrast, equation (2) does not contain information on the maneuver and control actions of a weapons system. If the value of the Lanchester attrition coefficient that is used in equation (0) is computed by the use of a finite Markov chain model, then this equation should also model combat in a more realistic manner. Since equation (0) is one of the equations that is used to describe combat in the Lanchester theory of combat, an improvement in the Lanchester theory of combat should also result.

C. EXTENSIONS

The regular Markov chain model which was developed in this thesis describes in a single model the control, maneuver, target acquisition, and target destruction activities of a weapons system. These activities were defined so that they represent all of the output actions of a weapons system. This section will cover three possible extensions of the original model.

The first extension concerns the effects of increasing and decreasing the number of weapons systems in a weapons system organization. In an abstract sense this allocation of additional weapons systems merely represents a change in the type of weapons system which is being analyzed, and the model should reflect the effect of these allocations by indicating a change in the amount of time needed to conduct a given activity. This change in the type of weapons system organization should not necessitate a change in the state space of the model. It should merely require that the one-step probabilities which are contained in the transition matrix $\underline{P}$ be estimated again.

Another extension of the model which would aid in the analysis of a weapons system would be an explicit treatment of the interactions between the weapons and target systems. The model developed in this thesis focuses only on the actions of a weapons system or only on the actions of a target system. It does not explicitly address the interactions between a weapons and a target system. An extension of the model to cover explicitly the actions of both the target and weapons system would involve the development of a model which would have a state space consisting of three hundred and twenty-five different states. The method for estimating the parameters that was given in Chapter IV would also have to be modified in order to estimate the parameters of this interaction model. These extensions and modifications would greatly increase the complexity of the model.

As a final extension to the model presented in this thesis, it is recommended that the personnel and logistical activities of a weapons system also be modeled. Personnel and logistical activities are input activities in contrast to control, maneuver, target acquisition, and target destruction which are output activities. Since ideally input activities occur simultaneously with the control, maneuver, target acquisition, and target destruction activities, it appears that they should be modeled separately. A regular Markov chain model could also be used to model these input activities. The estimation of the one-step transition probabilities contained in such a model would be conducted in a manner similar to that which was discussed in Chapter IV.

D. CONCLUDING COMMENTS

This thesis has attempted to carefully define the activities of a military weapons system involved in a combat process. A deliberate effort has been made to explain all of the actions which comprise these activities. A regular Markov chain model was used to aid in the analysis of a combat process. It is hoped that the analysis of the combat process contained in this thesis will be of assistance to weapons system analysts and that it will generate an interest in additional investigations of the combat process.

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