$AD\ 735349$

A MINIMAX LATTICE POINT LOCATION PROBLEM

Technical Report No. 57

SUBMITTED BY:

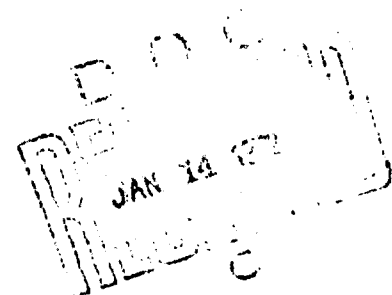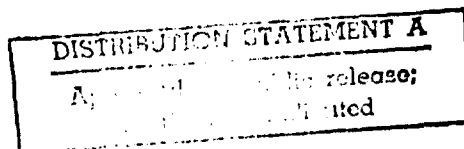APPROVED BY:

R. L. Francis

R. L. Francis
Principal Investigator

B. D. Sivazlian

B. D. Sivazlian
Editor, Project THEMIS

Department of Industrial and Systems Engineering
The University of Florida
Gainesville, Florida 32601

November 1971

DISCLAIMER NOTICE

THIS DOCUMENT IS BEST QUALITY AVAILABLE. THE COPY FURNISHED TO DTIC CONTAINED A SIGNIFICANT NUMBER OF PAGES WHICH DO NOT REPRODUCE LEGIBLY.

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)	2a. REPORT SECURITY CLASSIFICATION
University of Florida	Unclassified
	2b. GROUP
	Unclassified

3. REPORT TITLE

A MINIMAX LATTICE POINT LOCATION PROBLEM

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

5. AUTHOR(S) *(Last name, first name, initial)*

FRANCIS, R. L.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
November 1971	19	28

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
DAH C04 68C0002	Technical Report No. 57
b. PROJECT NO.	
2TO 14501E81B	
c.	9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*
d.	

10. AVAILABILITY/LIMITATION NOTICES

This document has been approved for public release and sale; its distribution is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	ARO-Durham

13. ABSTRACT

The problem is considered of assigning n facilities, or departments, to locations so that the maximum of the rectilinear distances between departments is minimized. The locations are considered to be points in a lattice. A simple expression is developed for the minimum value of the objective function for all values of n; the expression has as a corollary a necessary and sufficient condition for an assignment, or a configuration, to be minimax; also a simple geometrical procedure is developed for constructing minimax configurations. A closed form solution is obtained for an analogous continuous problem.

DD FORM 1473

A MINIMAX LATTICE POINT LOCATION PROBLEM

Technical Report No. 57

SUBMITTED BY:

R. L. Francis

R. L. Francis
Principal Investigator

APPROVED BY:

B. D. Sivazlian
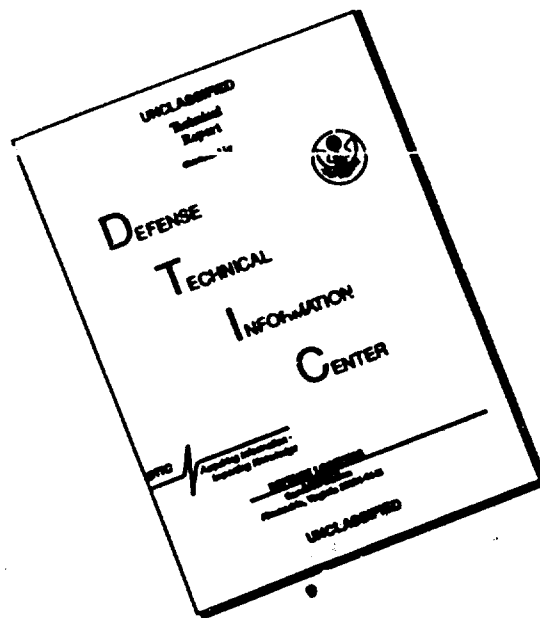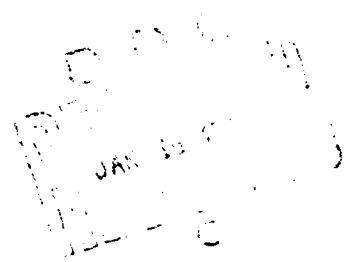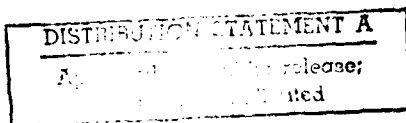
B. D. Sivazlian
Editor, Project THEMIS

Department of Industrial and Systems Engineering
The University of Florida
Gainesville, Florida 32601

November 1971

ABSTRACT

The problem is considered of assigning n facilities, or departments, to locations so that the maximum of the rectilinear distances between departments is minimized. The locations are considered to be points in a lattice. A simple expression is developed for the minimum value of the objective function for all values of n; the expression has as a corollary a necessary and sufficient condition for an assign t, or a configuration, to be minimax; also a simple geometrical procedur s developed for constructing minimax configurations. A closed form solution is obtained for an analogous continuous problem.

TABLE OF CONTENTS

LIST OF FIGURES

INTRODUCTION

Define a lattice point in the plane to be any point in the plane such that each entry in the point is an integer. Note that, with reference to Figure 1, each lattice point is the center of a square of unit dimensions, so that n distinct lattice points may be considered to represent n nonoverlapping unit squares. Each unit square may be considered, for example, to be a department of a plant, and the problem of interest would be to find a best configuration of n departments, where "best" will be defined subsequently. Alternatively, a "facility" may be considered to consist of n unit squares each having a lattice point as a center, with the individual squares having no particular interpretation; the problem of interest would then be to find a "best" facility configuration. Note that, by choosing dimensions appropriately, the length of each side of each square may be assumed to be one without any loss of generality.

Some notation and definitions will now be useful. Given any two points $X_1 = (x_1, y_1)$ and $X_2 = (x_2, y_2)$ in the plane, the rectilinear distance between the two points will be denoted by $r(X_1, X_2)$, where, by definition, $r(X_1, X_2) = |x_1 - x_2| + |y_1 - y_2|$. It should be noted that the use of a rectilinear distance, rather than a Euclidean distance, is more appropriate in commonly occurring industrial situations where travel is carried out on a set of rectilinear aisles, each of which is orthogonal to either the x or the y axis; thus the rectilinear distance between two lattice points in a configuration would be an approximation to the distance items would travel between the two departments, unit squares, having lattice points as their centers.

Denote the set of all lattice points in the plane by L. Let n be a given positive integer, at least two, and denote by S_n a set consisting of n distinct lattice points. The collection of all sets of n distinct lattice

points will be denoted by $H_n(L)$. A set S_n will be called a <u>configuration of</u> <u>size n</u> if and only if $S_n \in H_n(L)$. For any configuration S_n of size n, the diameter of S_n, denoted by $d(S_n)$, will be defined to be the maximum of the rectilinear distances among all distinct pairs of lattice points in S_n. Note that the rectilinear distance between any two distinct lattice points will be a positive integer, so that $d(S_n)$ will be a positive integer. Intuitively, the diameter of a configuration of size n is a measure of the closeness of the lattice points in the configuration.

The concern in the sequel will be with finding configurations of size n of minimum diameter, that is, with finding minimax configurations. Note that the set $\{d(S_n): S_n \in H_n(L)\}$ is a collection of positive integers, so that the principle of the smallest integer [3] guarantees that the function f(n), where

$$f(n) = \min \{d(S_n): S_n \in H_n(L)\} \qquad (1)$$

is well defined; f(n) is the minimum diameter of all configurations of size n. Define the integer valued strictly increasing function g(i) as follows:

$$g(i) = (i^2 + 2i + 2)/2 \text{ if i is an even, nonnegative integer}$$
$$g(i) = (i^2 + 2i + 1)/2 \text{ if i is an odd, positive integer.} \qquad (2)$$

Denote by I^+ the collection of all positive integers. Then the main result of this paper is as follows: for any positive integer i, f(n) = i for $n \in [g(i - 1) + 1, g(i)] \cap I^+$. A proof of the result will be given subsequently. As an illustration of the result, f(2) = 1; f(n) = 2 for n = 3,4,5; f(n) = 3 for n = 6, 7, 8. Thus, for example, the minimum diameter of all configurations of size 6 is 3, and so any configuration with a diameter of 3 will be a minimax configuration if it is of size 6. In the sequel a systematic method will be given for constructing minimax configurations.

As concerns related literature in location theory, the area of literature most closely related to the problem being considered appears to be that

involving the quadratic assignment problem, first formulated by Koopmans and Beckmann [15]. A special case of the quadratic assignment problem represents the case where n lattice points are to be chosen from a finite number of lattice points in such a way that a total cost consisting of terms directly proportional to distances between lattice points is minimized. An extensive discussion of, and list of references on, the quadratic assignment problem has been given recently by Pierce and Crowston [24]. However, as pointed out by Nugent, Vollmann, and Ruml [23], the quadratic assignment problem is still largely computationally intractable, so that heuristic approaches to solving the problem have been of considerable interest. Hillier, [12], [13], has considered a heuristic approach which is equivalent to choosing from among a finite number of lattice points; he also considered distances between lattice points to be rectilinear. The location problem considered herein, due to its objective function, is simpler than the quadratic assignment problem, and permits the ready obtainment of optimum solutions. It is hoped that the study of location problems with special structure will eventually result in a fruitful approach to the study of more general problems.

It may be worth noting that in at least one sense the problem being considered here is quite general, since all lattice points in the plane are considered as candidates for locations, rather than a finite number of lattice points; interestingly enough, considering all lattice points substantially simplifies the analysis. Further, there are certainly instances in which a minimax criterion is at least as acceptable as other criteria; the minimax approach requires less data than does a total cost approach, for instance. Particularly if a new facility is being built, the cost data required for the quadratic assignment problem formulation, for example, may be unavailable. Alternatively, the facility may be intended to

3

serve a number of different uses at different times, so that total cost data, if available, might change in an unknown manner with time. Also, the minimax criterion has sometimes been suggested as being appropriate for emergency situations; there may be instances where it may not be as important to minimize long term total costs of movement as it is to be able to travel between any two points in the facility as quickly as is possible.

There exists other literature on minimax location problems, for example, [7] and [8]; however, other than a statement of a minimax problem by Newman [18], there appears to be no literature on minimax location problems involving distances between points which cannot share common locations. There is some literature on somewhat related max-min location problems [10], [28]; these problems appear to be extremely difficult to solve. It should also be pointed out that while there is quite a large literature on lattices (see, for example, [1] and [4]) only the very simplest properties of lattices, employed in the next section, are used in this paper. There does exist some literature involving packing problems [6], [26] which has location theory aspects, and in which lattices are sometimes employed, but this literature, with the exception of the max-min location problems referred to earlier, where lattices are not employed, does not appear to be directly relevant to the problem being studied here, due to differences in the objective functions being considered. The material in some of the foregoing references is discussed by Saaty [27], as well as a number of related geometric optimization topics; Saaty gives an extensive list of references.

SOME GEOMETRIC PRELIMINARIES

Crucial to the development to follow are properties of the rectilinear distance. In this section the properties needed subsequently are developed.

Given a point P in the plane and a nonnegative number t, the set $D(P, t) = \{X \in E_2: r(P, X) \leq t\}$ will be called a diamond with center P and

radius t; the diameter of $D(P, t)$ will be defined to be twice the radius. A graph of $D(P, t)$ shows that the diamond is just a square which has been rotated 45 degrees; graphs of the boundaries of diamonds of diameter 4 appear in Figures 2 and 3. It is readily verified that the rectilinear distance between any two points on opposite edges of a diamond is equal to the diameter of the diamond, and that the rectilinear distance between any two points in a diamond is equal to or less than the diameter of the diamond.

Given any nonempty closed and bounded set S in the plane, a diamond $D(P, t)$ will be called a smallest diamond containing S if $D(P, t)$ contains S and the radius of any other diamond containing S is at least as large as t. In [9], Francis establishes the existence of a smallest diamond containing S by giving an explicit procedure for constructing the diamond; his work is a sequel to earlier work by Elzinga and Hearn [5], who established the existence of a smallest containing diamond for the case where S consists of a finite number of points, and also developed a solution procedure. For any points X and Y in S, since X and Y are in $D(P, t)$, it follows that $r(X, Y) \leq 2t$. Further, an examination of the procedure for constructing a smallest containing diamond $D(P, t)$ shows that at least two opposite edges of $D(P, t)$ have nonempty intersections with S, so that there exist points, say X* and Y*, in S and in $D(P, t)$ such that $r(X*, Y*) = 2t$. Thus the maximum of the rectilinear distances between all pairs of points in S is equal to the diameter of any smallest diamond containing S. Hence it makes sense to define the rectilinear diameter of S, denoted by $d(S)$, as follows:

$$d(S) = \max \{r(X, Y):X, Y \in S\}. \tag{3}$$

Note that when S is a collection of n distinct lattice points, say S_n, the previous definition of the diameter of S_n agrees with the definition (3), so that the diameter of S_n is equal to the diameter of any smallest diamond containing S_n.

5

The foregoing discussion of the question of the smallest diamond containing a set is directly analogous to the question of the smallest disk containing a set. For this latter question there is quite a substantial literature, beginning with the work of Jung, in 1901 and 1909, who developed an inequality relating the Euclidean diameter of a set to the diameter of the smallest disk (or, more generally, hypersphere) containing the set. A discussion of Jung's work, as well as extensions of it, is given by Blumenthal and Whalin [2]. Klee [14], discusses, and gives references to, subsequent generalizations of Jung's inequality, as well as necessary and sufficient conditions for the inequality to hold as an equality; equality does not always hold when the Euclidean distance is used. However, the rectilinear diameter of a compact set in the plane is always equal to the diameter of any smallest diamond containing the set; extensive use is made of this fact in the sequel.

At this point it is convenient to give names to several types of diamonds which will be employed in the subsequent analysis. A Type I diamond will be a diamond having exactly two opposite vertices coincident with lattice points and having as its diameter an odd positive integer. A Type II-a diamond will be a diamond having all four vertices coincident with lattice points and having as its diameter an even positive integer. A Type II-b diamond will be a diamond having as its center the center of a unit square which has a lattice point at each corner and having as its diameter an even positive integer. Illustrations of Type II-a and Type II-b diamonds are given in Figures 2 and 3 respectively.

It is easy to verify, simply by counting, that the number of lattice points in a Type I diamond is $(i^2 + 2i + 1)/2$ if i is the diameter of the diamond. Likewise, the number of lattice points in a Type II-a diamond of diameter i is $(i^2 + 2i + 2)/2$, and the number of lattice points in a Type II-b diamond of diameter i is $(i^2 + 2i)/2$.

Given any diamond D in the plane of diameter i, if i is an odd positive integer it can be shown, using simple geometrical arguments, that there exists a Type I diamond of the same diameter such that every lattice point in D is also in the Type I diamond. Likewise, given any diamond in the plane of diameter i, if i is an even positive integer, it can be shown that there exists a Type II-a or a Type II-b diamond such that every lattice point in the given diamond is in the Type II-a or Type II-b diamond. Thus it follows that the maximum number of lattice points which any diamond of diameter i, where i is a positive integer, can contain is $g(i)$, where $g(i)$ is defined by (2). A number of geometric problems in number theory, some quite famous [17], have to do with finding the least or greatest number of lattice points in some type of planar set, [16], [20], [21], [22], [25]. The problem of finding the maximum number of lattice points a diamond of integral diameter can contain, while related to these problems, especially to the one considered by Newman [19], is a conceptually much simpler problem, and does not appear to be a special case of any of the problems in the literature.

MAIN RESULTS

The main results of the paper will now be developed. It will be useful in the sequel to make use of the easily established fact that the sequence $\{f(n)\}_{n=2}^{\infty}$, where $f(n)$ is defined by (1), is integer valued and nondecreasing.

The result stated in the first section will be proven first.

Lemma 1 For any positive integer i,

$$f(n) = i \quad \text{for all} \quad n \varepsilon [g(i - 1) + 1, g(i)] \cap I^+ .$$

Proof The proof will be by induction on i. Let i = 1; then $g(i - 1) + 1 = 2 = g(i)$ and it is geometrically evident that $f(2) = 1$, since any two distinct lattice points must be a distance apart of at least 1.

Now assume the lemma is true for i = t. By assumption, $t = f[g(t)]$, and so $t \leq f[g(t) + 1]$. Let $p = g(t) + 1$ and suppose $t = f(p)$. Then there

7

exists at least one $S_p \in H_p(L)$ such that $d(S_p) = t$. Thus if D_p is a smallest

diamond containing S_p, then the diameter of D_p is also t. However, D_p contains

$p = g(t) + 1$ lattice points, and the maximum number of lattice points that

any diamond of diameter t can contain is g(t). Thus $t < f[g(t) + 1]$, and so,

since f is integer valued, $t + 1 \leq f[g(t) + 1]$. Thus $t + 1 \leq f(n)$ for all

$n \in [g(t) + 1, g(t + 1)] \cap I^+$. Now since a diamond of diameter $t + 1$ can

contain $g(t + 1)$ lattice points, let $p = g(t + 1)$, let D_p be a diamond

of diameter $t + 1$ containing p lattice points, and let S_p be the collection

of all distinct lattice points in D_p, so that $S_p \in H_p(L)$. Then $d(S_p) \leq t + 1$,

and so $f[g(t + 1)] \leq d(S_p) \leq t + 1$. Thus $t + 1 \leq f(n) \leq t + 1$ for all

$n \in [g(t) + 1, g(t + 1)] \cap I^+$, and so the proof is complete.

The lemma may be readily used to obtain necessary and sufficient conditions for a configuration of size n to be a minimax configuration.

Corollary Let $S_n \in H_n(L)$. Then S_n is a minimax configuration of size n if and only if

$$d(S_n) = i \ ,$$

where i is the unique positive integer such that $n \in [g(i - 1) + 1, g(i)] \cap I^+$.

Proof If $d(S_n) = i$, where i is the unique positive integer such that

$n \in [g(i - 1) + 1, g(i)] \cap I^+$, then Lemma 1 implies $d(S_n) = f(n)$, and so S_n is

a minimax configuration.

Suppose S_n is a minimax configuration in $H_n(L)$, so that $d(S_n) = f(n)$.

Clearly it is possible to choose an integer $i \in I^+$ such that $n \in [g(i - 1) + 1,$

$g(i)] \cap I^+$; then, by Lemma 1, $f(n) = i$ and so $d(S_n) = i$. Further the

choice of i is unique, since the intervals $[g(i - 1) + 1, g(i)]$ and $[g(j-1)+1,$

$g(j)]$ are disjoint for distinct positive integers i and j.

8

CONSTRUCTING MINIMAX CONFIGURATIONS

It is easy to use Lemma 1 to determine the minimum diameter for configurations of size n. Table 1 shows the minimum diameter for n as large as 221; the computation of the table entries is facilitated by using the readily established fact that $g(i) = g(i - 1) + i$ if i is an odd positive integer, and that $g(i) = g(i-1) + i + 1$ if i is an even positive integer. Table 1 may be readily used in conjunction with Lemma 1 as follows. The minimum diameter of configurations of size 2 is 1; of configurations of size 3 through 5 is 2; of configurations of size 6 through 8 is 3; etc.; the minimum diameter of configurations of size 201 through 221 is 20.

As well as finding the minimum diameter for configurations of a given size, it is also of interest, of course, to be able to construct configurations which have a minimum diameter. The following propositions address the question of constructing minimum diameter configurations.

TABLE 1

i	g(i)	i	g(i)
1	2	11	72
2	5	12	85
3	8	13	98
4	13	14	113
5	18	15	128
6	25	16	145
7	32	17	162
8	41	18	181
9	50	19	200
10	61	20	221

Proposition 1 Let n be a given positive integer, at least 2, suppose S_n^* is a minimax configuration in $H_n(L)$, and let $i = d(S_n^*)$. If i is an odd integer, then S_n^* is contained in a Type I diamond of diameter i, say $D(i)$. Further, for any configuration S_n such that $S_n \subset D(i)$ and $S_n \in H_n(L)$, $d(S_n) = i$, and there are $\binom{g(i)}{n}$ such configurations.

Proof Let $D'(i)$ be a smallest diamond containing S_n^*. Then $D'(i)$ has a diameter of i, and the same approach as in the preliminaries section can be used to show that there exists a Type I diamond, say $D(i)$, also of diameter i, such that $S_n^* \subset D(i)$. Now let $S_n \in H_n(L)$ be such that $S_n \subset D(i)$; then certainly $d(S_n) \le i$. Suppose that $d(S_n) \le i - 1$; then there exists a diamond of diameter no greater than $i - 1$, say $D(i - 1)$, containing S_n. By the corollary, i is the unique integer such that $n \in [g(i - 1) + 1, g(i)] \cap I^+$, and thus $g(i - 1) + 1 \le n$. However, $D(i - 1)$ can contain at most $g(i - 1)$ distinct lattice points, and $S_n \subset D(i - 1)$ implies $n \le g(i - 1)$, giving a contradiction. Thus $d(S_n) = i$. Since $D(i)$ contains $g(i)$ lattice points, there will be $\binom{g(i)}{n}$ different choices of n distinct lattice points from among the lattice points in $D(i)$.

Notice that there is no physical reason to distinguish between two different configurations of the same size if one configuration can be obtained from the other by a sequence of translations or rotations. Thus for a given n such that $i = f(n)$ is an odd integer it suffices to consider only configurations of size n in a single Type I diamond of diameter i when constructing minimax configurations of size n. Even then, it may be possible to obtain some configurations with the given diamond from others in the diamond by means of rotations.

When $f(n) = i$ and i is an even positive integer, the procedure for constructing minimax configurations is given below in Proposition 2. The proof of Proposition 2 will not be given, as it is similar to the proof of

Proposition 1; the main point of difference in the proofs is that use is made in the proof of Proposition 2 of the fact that all of the lattice points in S_n^* are in a Type II-a diamond or a Type II-b diamond.

<u>Proposition 2</u>. Let n be a given positive integer, at least 2. Suppose S_n^* is a minimax configuration in $H_n(L)$ and let $i = d(S_n^*)$. If i is an even positive integer then S_n^* is contained in a Type II-a diamond of diameter i, say $D^*(i)$, or in a Type II-b diamond of diameter i, say $D^{**}(i)$. Further, for any configuration such that $S_n \subset D^*(i)$ and $S_n \in H_n(L)$, $d(S_n) = i$ and there are $\binom{g(i)}{n}$ such configurations. For any configuration S_n such that $S_n \subset D^{**}(i)$ and $S_n \in H_n(L)$, $d(S_n) = i$ and there are $\binom{g(i)-1}{n}$ such configurations.

Due to Proposition 2, when $f(n) = i$ and i is an even positive integer, it is only necessary to consider a single Type II-a diamond and a single Type II-b diamond when constructing minimax configurations of size n. Notice that if $n = g(i)$ then there is a unique configuration of size n, which must be contained in a Type II-a diamond. Define two configurations of the same size to be <u>equivalent</u> if one configuration can be obtained from the other by a sequence of translations or rotations. Then when $g(i - 1) + 1 \leq n < g(i)$ there may be a number of configurations which are not equivalent. For example, consider the Type II-a and II-b diamonds shown in Figures 2 and 3 respectively. Each diamond has a diameter of 4; minimax configurations of size 9 through 13 have a diameter of 4. By an inspection of the figures it is easy to see that there are, for example, a number of minimax configurations of size 9 in the Type II-a diamond which are not equivalent to any minimax configuration of size 9 in the Type II-b diamond. Thus it is not enough to consider only Type II-a diamonds or Type II-b diamonds when constructing minimax configurations; both types must be considered in order to find all possible minimax configurations of a given size which are not equivalent. On the other hand, there are cases when Type II-a and Type II-b diamonds of

11

the same diameter contain equivalent minimax configurations, as can be easily seen by constructing Type II-a and II-b diamonds of diameter 2 and considering minimax configurations of size 3.

Figures 2 and 3 may serve to illustrate several other points as well. There exist minimax configurations which are perhaps what might be expected intuitively, and there also exist minimax configurations which perhaps might not be expected intuitively, as can be seen by examining non-equivalent configurations of size 9 contained in the diamond of Figure 2; note also that there are a number of such non-equivalent minimax configurations of size 9. One other point can be illustrated by means of the figures; the largest size minimax configuration which the Type II-a diamond can contain is 13, while the largest size minimax configuration which the Type II-b diamond can contain is 12.

A CONTINUOUS PROBLEM

As indicated previously, a facility could be considered to consist of the union of n unit squares with centers at lattice points, so that the facility could be thought of as a set in the plane of area n. If the lattice point assumption is no longer made, so that no distinctions are made between departments, then an analogous "continuous" problem could be considered of finding a compact set in the plane of a given area, say A, having the smallest possible rectilinear diameter. The lemma below established that a compact set of area A has a minimum rectilinear diameter if and only if it is a diamond.

Lemma 2   If S is any compact set in the plane of area A, the rectilinear diameter of S is equal to or greater than $(2A)^{1/2}$, the rectilinear diameter of a diamond, say D*, of area A. Conversely, any compact set of area A having a minimum rectilinear diameter is a diamond.

<u>Proof</u> A direct computation establishes that the diameter of any diamond of area A is $(2A)^{1/2}$. Now let S be any compact set of area A, and let D be a smallest diamond containing S, so that $d(D) = d(S)$. Since S is contained in D, the area of D will be at least A; denote the area of D by A'. Since A' $\geq$ A, $d(S) = d(D) = (2A')^{1/2} \geq (2A)^{1/2} = d(D^*)$, which completes the first half of the proof.

To establish the converse, let S be a compact set of area A of minimum rectilinear diameter and let D be a smallest diamond containing S, so that $d(D) = d(S) = (2A)^{1/2}$. As above, S contained in D implies the area of D is at least A, and if the area of D is greater than A then $d(D) > (2A)^{1/2} = d(S)$. Thus the area of D is A, so it follows that the area of the set D-S is zero. Now let Y ϵ D; we shall show Y ϵ S, so that D is contained in S and the conclusion will then follow. Since Y ϵ D and D is a closed set, every epsilon neighborhood of Y, $N(Y, \epsilon)$, where $\epsilon > 0$, contains a point of D. Suppose Y is not in S; then, since S is a closed set, there exists $\epsilon > 0$ such that $N(Y, \epsilon) \cap S$ is empty. Now geometrical considerations evidently imply there exists Z ϵ $N(Y, \epsilon)$ and $\epsilon' > 0$ with $\epsilon' \leq \epsilon$ such that $N(Z, \epsilon') \subset N(Y,\epsilon)$, and $N(Z, \epsilon') \subset D$. But since $N(Z, \epsilon') \cap S$ is empty, $N(Z, \epsilon') \subset D-S$, implying the area of D-S is positive, which is a contradiction. Thus Y ϵ S and the proof is complete.

By way of comparison, the rectilinear diameter of a square of area A, which has the smallest rectilinear diameter of all rectangles of area A, is $2(A)^{1/2}$. The rectilinear diameter of a disk of area A is $(2/\pi^{1/2})(2A)^{1/2} \doteq (1.13)(2A)^{1/2}$.

When a diamond of area A is rotated 45 degrees, a square of area A is obtained, of course; however, since the axes are also rotated, each aisle in the system of rectilinear aisles now makes an angle of 45 degrees or -45

degrees with some edge of the square. In other words, a set of the smallest rectilinear diameter may still be considered to be a square of area A; it simply has an aisle structure rotated 45 degrees from the usual aisle structure. It is not difficult to imagine arranging desks in an office, for example, so that such a rotated aisle structure could be realized. Even if the answer provided by Lemma 2 cannot be utilized directly, it may still be useful as a design benchmark. Further, Lemma 2 provides additional insight into the minimax solutions obtained to the lattice point problem. Roughly speaking, minimax configurations of size n are obtained by choosing a smallest diamond of integer diameter which will contain n lattice points. For the continuous case, a minimax solution is obtained by choosing a smallest diamond which will contain a set of area A; namely, a diamond of area A. It should be evident, however, that the lattice point "discrete" problem and the continuous problem each has its own special points of interest.
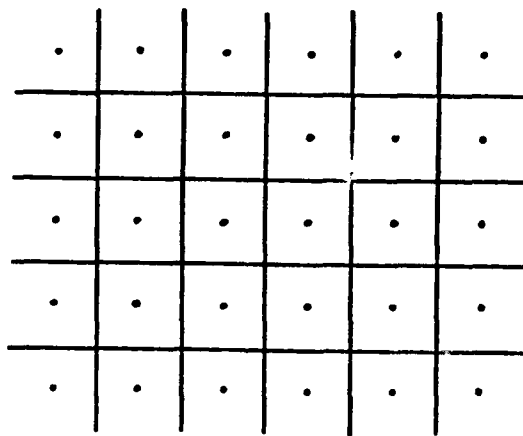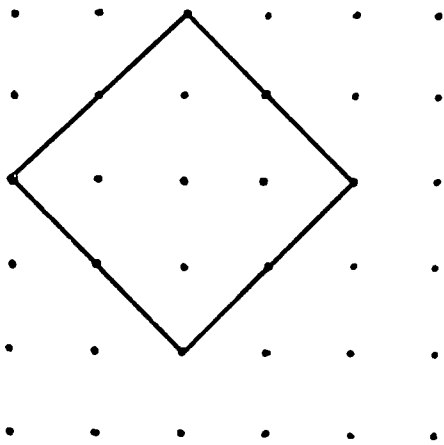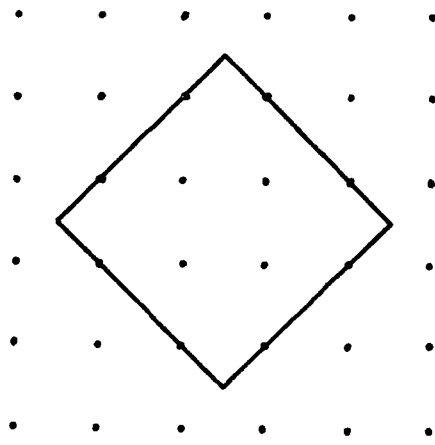
Figure 1

Figure 2

Figure 3

REFERENCES

1. G. Birkhoff, <u>Lattice Theory</u>, American Mathematical Society Colloquium
    Publications, Vol. XXV, American Mathematical Society, Providence,
    Rhode Island (1967).

2. L. M. Blumenthal and G. E. Whalin, "On the Spherical Surface of a
    Smallest Radius Enclosing a Bounded Subset of n-Dimensional Euclidean
    Space," <u>Amer. Math. Soc. Bull.</u>, 47, 771-777 (1941).

3. R. Courant and H. Robbins, <u>What is Mathematics?</u>, p. 18, Oxford University
    Press, Oxford (1941).

4. T. Donnellan, <u>Lattice Theory</u>, Pergamon Press, Oxford (1966).

5. J. Elzinga and D. W. Hearn, "Geometrical Solutions for some Minimax Location
    Problems," submitted for publication (1971).

6. L. Fejes Tóth, <u>Lagerungen in der Ebene auf der Kugel und in Raum</u>, Springer-
    Verlag, Berlin (1953).

7. R. L. Francis, "Sufficient Conditions for Some Optimum-Property Facility
    Designs," <u>Opns. Res.</u>, 15, 448-456 (1967).

8. R. L. Francis, "Some Aspects of a Minimax Location Problem," <u>Opns. Res.</u>,
    15, 1163-1168 (1967).

9. R. L. Francis, "A Geometrical Solution Procedure for a Rectilinear Distance
    Minimax Location Problem," submitted for publication (1971).

10. M. Goldberg, "The Packing of Equal Circles in a Square," <u>Math. Magazine</u>,
    43, 24-30 (1970).

11. H. Hancock, <u>Development of the Minkowski Geometry of Numbers</u>, The Macmillan
    Co., New York (1939).

12. F. S. Hillier, "Quantitative Tools for Plant Layout Analysis," <u>J. of Ind. Engr.</u>,
    14, 33-40 (1963).

13. F. S. Hillier and M. M. Connors, "Quadratic Assignment Problem Algorithms and
    the Location of Indivisible Facilities," <u>Management Science</u>, 13, 42-57 (1966).

14. V. Klee, "What is a Convex Set?," <u>Math. Assoc. Monthly</u>, 616-631 (1971).

15. J. C. Koopmans and M. Beckmann, "Assignment Problems and the Location of
    Economic Activities," <u>Econometrica</u>, 25, 53-76 (1957).

16. E. A. Maier, "On the Minimal Rectangular Region which has the Lattice Point
    Covering Property," <u>Math. Magazine</u>, 42, 84-85 (1969).

17. H. Minkowski, <u>Geometrie der Zahlen</u>, Chelsea Publishing Co., New York (1953).

18. D. J. Newman, "A Parking Lot Design," <u>SIAM Review</u>, 6, 62-66 (1964).

19. D. J. Newman, "A Maximum Covering of Lattice Points by a Square," *Amer. Math. Monthly*, 75 (1968).

20. I. Niven, *Diophantine Approximations*, p. 54, Interscience Publishers, New York (1963).

21. I. Niven and H. S. Zukerman, "Lattice Point Covering by Plane Figures," *Amer. Math. Monthly*, 74, 353-362 (1967).

22. I. Niven and H. S. Zuckerman, "The Lattice Point Covering Theorem for Rectangles," *Math. Magazine*, 42, 85-86 (1969).

23. C. E. Nugent, T. E. Vollmann, and J. Ruml, "An Experimental Comparison of Techniques for the Assignment of Facilities to Locations," *Opns. Res.*, 16, 150-173 (1968).

24. J. F. Pierce and W. B. Crowston, "Tree-Search Algorithms for Quadratic Assignment Problems," *Nav. Res. Logis. Quart.*, 18, 1-36 (1971).

25. S. Reich, "Two-dimensional Lattices and Convex Domains," *Math. Magazine*, 43, 219-220 (1970).

26. C. A. Rogers, *Packing and Covering*, Cambridge University Press, Cambridge (1964).

27. T. L. Saaty, *Optimization in Integers and Related Extremal Problems*, Chapter 2, McGraw-Hill Book Co., New York (1970).

28. J. Schaer, "The Densest Packing of 9 Circles in a Square," *Canad. Math Bull.*, 8, 273-277 (1965).