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CENTRAL LIMIT THEOREMS FOR CONDITIONALLY LINEAR RANDOM PROCESSES

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PREFACE

The work reported in this paper is part of Rand's broad studies of radar signal-processing methods. Models of radar clutter interference are developed which are both physically reasonable and more general than the usual Gaussian model or linear-process model. These are called conditionally linear processes. The major contribution of this paper is the development of simple conditions under which linear functionals of data generated by the model will have a Gaussian distribution. The result enables the computation of detection probabilities and false alarms for linear signal detectors.

The proof of the main result described herein makes use of some advanced probabilistic concepts; however, understanding of the other results and the applications described requires only limited knowledge of probability theory.

In an earlier form this report appeared as *Central Limit Theorems for Conditionally Linear Random Processes with Applications to Models of Radar Clutter*, by Percy A. Pierre, April 1969. The more mathematical portions have been revised, and some of the proofs have been strengthened and simplified. Also, the discussion of applications and some of

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the straightforward calculations appearing in the earlier versions have been omitted in this revision.

SUMMARY

A random process is called a linear process if it is an infinite sum of statistically independent component random processes. A particular example of a linear process is the output of a filter driven by a sequence of impulses whose times of occurrence are the times of a Poisson process. If the filter responses are also random and the responses to impulses applied at different times are statistically independent, the filter output is still a linear process. If, however, either the impulses do not occur according to a Poisson process or the filter responses are not independent, the process is called conditionally linear. The latter situation can be used to describe the radar echoes from randomly dispersed scatterers which, however, exhibit some phase coupling.

It has been shown in many special cases that when a linear process is passed through a low-pass filter, the output is approximately Gaussian. These are the well-known central limit theorems for linear processes. This paper presents a very general and widely applicable technique for proving such theorems. More importantly, however, central limit theorems are obtained for the conditionally linear processes described above.

CONTENTS

I. INTRODUCTION

Increasingly, clutter and other reverberation noises are being modeled by a class of random processes derived from a Poisson process, e.g., Middleton.[1] Typically,

$$x(t) = \sum_{i=-\infty}^{\infty} \phi_i(t - t_i), \qquad t \in (-\infty,\infty), \tag{1}$$

where $\{t_i\}$ are the times of a Poisson process and $\{\phi_i(t)\}$ are independent, identically distributed random functions. We call such processes linear processes. Different versions of this process have been extensively studied.[2,11] Unfortunately, this process sometimes proves physically unreasonable. For example, the times $\{t_i\}$ may not be Poisson. On the other hand, if the average time interval $t_i - t_{i-1}$ is small, the *assumption that* $\phi_i(t)$ *and* $\phi_j(t)$ *are independent whenever* $i \neq j$ may not hold.

In this paper we shall obtain central limit theorems for sequences of linear functionals of $x(t)$,

$$\ell_k = \int_{-\infty}^{\infty} x(t)dL_k(t), \qquad k = 0, 1, \ldots, \tag{2}$$

where *either one of the two* "independence" assumptions in Eq. (1) is dropped. In this case, $x(t)$ is called conditionally linear. Not surprisingly, we will find that the asymptotic distribution of ℓ_k is not always normal.

To this end, we shall prove two central limit theorems for conditionally linear random processes--one involving continuous-parameter random processes and the other involving discrete-parameter random processes.

A simplified version of a discrete-parameter conditionally linear process is

$$Z_i = x_i y_i, \qquad i = 0, \pm 1, \pm 2, \ldots,$$

where $\{y_i\}$ are independent random variables, while the processes $\{x_i\}$ and $\{y_i\}$ are independent of each other. We show that if $Ey_i = 0$, $Ey_i^2 = 1$, $Ey_i^4 < M$, and

$$\lim_{k\to\infty} k^{-1} \sum_{i=1}^{k} x_i^2 = \sigma^2 \qquad (\text{w.p.1}),^* \tag{3}$$

then

$$\ell_k = k^{-1/2} \sum_{i=1}^{k} Z_i \tag{4}$$

is asymptotically normal with variance σ^2.

Two things are worth noting. First, the sequence $\{Z_i\}$ need not obey any of the more common criteria for central limit theorems, such as independence, M-dependence, or strong mixing. Second, the independent structure of the y_i's seems to be more important in determining the shape of the asymptotic distribution of ℓ_k than the dependent structure of the x_i's. It is sufficient that the sequence $\{x_i\}$ obey Eq. (3). Translated to Eq. (1), this hints at the conclusion that when $\{t_i\}$ are Poisson, the ϕ_i's need not be independent. If the ϕ_i's

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*With probability one.

are independent, then the t_i's need not be Poisson. Examples illustrating each of these situations are presented in Section IV.

The second major result concerns the conditionally linear process*

$$x(t) = \int_{-\infty}^{\infty} h(t,\lambda)dZ(\lambda) \tag{5}$$

and the linear functionals

$$\ell_k = \int_{-\infty}^{\infty} \Phi_k(\lambda)dZ(\lambda), \tag{6}$$

where

$$\Phi_k(\lambda) = \int_{-\infty}^{\infty} h(t,\lambda)dL_k(t),$$

$Z(\lambda)$ is a process with stationary independent increments, and $\Phi_k(\lambda)$ is a random process independent of $Z(\lambda)$. Here $dZ(\lambda)$ plays the role of the independent random variables $\{y_i\}$. We show that if $EZ(\lambda) = 0$; if

$$\sigma_k^2 = \int_{-\infty}^{\infty} \Phi_k^2(\lambda)d\lambda \tag{7}$$

converges w.p.1 as $k \to \infty$ to a random variable with distribution function $S(x)$; and if

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} \Phi_k^4(\lambda)d\lambda \to 0 \qquad (\text{w.p.1}), \tag{8}$$

*In Pierre[11] this process is called linear.

then the asymptotic characteristic function of ℓ_k is a weighted normal-- in fact, weighted by the distribution function S(x) (Theorem 3.1).

The integrals in Eqs. (7) and (8) are assumed to exist by at least one of the conventional modes of convergence. We also prove a central limit theorem for a discrete-parameter version of Eq. (5).

It will almost always be assumed that the "white-noise" process (the one having the independent structure) has zero mean. This is a nontrivial assumption, since without it the nature of the possible conclusions changes.

The mathematical problem we consider was perhaps most recently investigated by Lugannani and Thomas[(12)] and Mallows,[(13)] who considered continuous and discrete-parameter linear processes, respectively. Our results for the special case of linear processes are somewhat weaker than theirs (e.g., we require finite fourth moments). However, in those cases where fourth moments do exist, our condition for the central limit theorem is much simpler and, in fact, is of such a form that it can be meaningfully applied to conditionally linear processes.

II. THE CONDITIONING TECHNIQUE FOR DEPENDENT SEQUENCES

Before considering general conditionally linear processes, we consider a special case involving finite sums, which illustrates the basic idea but avoids many of the technical points to be raised later.

We consider the discrete-parameter random process Z_i defined on the probability space $(\Omega,\mathfrak{F},P)$, $\omega \in \Omega$, such that

$$Z_i = g_i(x_i,y_i), \qquad i = 0, \pm 1, \pm 2, \ldots,$$

where $\{y_i\}$ are independent random variables, the sets $\{y_i\}$ and $\{x_i\}$ are independent of each other, and $\{g_i\}$ are known functions. Ordinarily we will suppress the variable ω. Occasionally, we will exhibit the ω-dependence to emphasize that a quantity is a random variable.

Let

$$\ell_n = n^{-1/2} \sum_{i=1}^{n} Z_i.$$

We will show the following.

<u>Theorem 2.1</u>. *If*

$$\lim_{n\to\infty} n^{-1/2} \sum_{i=1}^{n} E\{Z_i|x_i\} \to \mu(\omega) \qquad \text{(w.p.1)},$$

$$\lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} \text{var}\ \{Z_i|x_i\} \to \sigma^2(\omega) \qquad \text{(w.p.1)},$$

where $\sigma^2(\omega)$ *and* $\mu(\omega)$ *are random variables with joint distribution function* $F(\mu,\sigma^2)$; *and if*

$$n^{-2} \sum_{i=1}^{n} E\{(Z_i - E[Z_i|x_i])^4|x_i\} \to 0 \qquad \text{(w.p.1)},$$

then the asymptotic characteristic function of ℓ_n *is*

$$C(u) = \int_{-\infty}^{\infty} \exp\,[i\mu u - \tfrac{1}{2}\sigma^2 u^2]dF(\mu,\sigma^2).$$

Proof. Let $C_n(u) = E\{\exp\,[iu\ell_n] = E\{E[\exp\, iu\ell_n|x_1, \ldots, x_n]\} = E\{C_n^*(u)\}$, where $C_n^*(u)$ is the conditional characteristic function of ℓ_n. Since $|C_n^*(u)| \leq 1$, the dominated convergence theorem implies that $\lim C_n(u) = E \lim C_n^*(u)$. We shall show that for almost all ω, $\lim C_n^*(u)$ is the characteristic function of a normally distributed random variable with mean $\mu(\omega)$ and variance $\sigma^2(\omega)$.

For almost all realizations of $\{x_i\}$, the following application of Lyapunov's condition for normal convergence holds. First, ℓ_n is the sum of a finite number of conditionally independent random variables whose conditional moments are easy to calculate. Let $\underline{x} = (x_1, \ldots, x_n)$. Then var $[\ell_n|\underline{x}] \to \sigma^2(\omega)$. The fourth-moment condition of the theorem is just a Lyapunov condition for the conditionally centered variates $Z_i - E[Z_i|x_i]$. Thus the conditional distributions of the centered variates converge, and since the conditional means converge, the conditional distribution functions of the uncentered variates converge for almost all ω.

An important and simple special case is the following.

Corollary 2.1. *Let* $g_i(x_i,y_i) = x_i y_i$, $Ey_i = 0$, var $y_i = 1$, *and* $Ey_i^4 < M$. *If for some constant* σ^2,

$$\lim n^{-1} \sum_{i=1}^{n} x_i^2 = \sigma^2 \qquad \text{(w.p.1)},$$

then Y_n *is asymptotically normal,* $N(0,\sigma^2)$.

The process $Z_i = x_i y_i$ can be thought of as resulting from passing white noise, $\{y_i\}$, through a particularly simple random filter. An interesting application is the case where y_i is multiplicative noise on a random signal x_i.

Proof. One can show (Pierre[14]) that if

$$n^{-1} \sum_{i=1}^{n} x_i^2 \to \sigma^2 \qquad \text{(w.p.1)},$$

then

$$\max n^{-1} x_i^2 \to 0 \qquad \text{(w.p.1)}.$$

The fourth-moment condition of Theorem 2.1 takes the form

$$n^{-2} \sum_{i=1}^{n} x_i^4 E y_i^4 \leq M \left(\max_{1 \leq i \leq n} n^{-1} x_i^2 \right) \sum_i n^{-1} x_i^2 .$$

We see that the right-hand side converges to zero. We apply Theorem 2.1 to complete the proof.

The extension of these results to situations involving infinite sums runs into two problems: The first is whether or not conditional expectations exist when the number of conditioning random variables is infinite. The second problem is more difficult. We were able to compute conditional moments by taking conditional expectations under summation signs, because the sums were always finite. Justifying this for infinite sums is considerably more difficult. This problem is considered in Section III.

Note that in Theorem 2.1 neither of the processes $\{x_i\}$ or $\{y_i\}$ was required to have a zero mean.

III. CENTRAL LIMIT THEOREMS FOR STOCHASTIC SUMS AND INTEGRALS

First we consider sequences

$$\ell_k = \int_{-\infty}^{\infty} \phi_k(\lambda) dZ(\lambda), \qquad k = 0, 1, 2, \ldots,$$

where $Z(\lambda)$ is a process with *stationary independent increments* with $EZ(\lambda) = 0$, $EZ^4(\lambda) < \infty$. Here, $\{\phi_k(\lambda)\}$ are random processes independent of $Z(\lambda)$, and the integral is defined as the *limit in the fourth mean* of the Riemann-Stieltjes partial sums. (See Pierre[11] for a discussion of this integral and a computation of its moments; see also Pierre[14].) We recall from Ref. 11 that $E\ell_k = 0$,

$$\mathrm{var}\ \ell_k = \int E\phi_k^2(\lambda) d\lambda$$

and

$$E\ell_k^4 - 3\{E\ell_k^2\}^2 = C\int E\phi_k^4(\lambda) d\lambda$$

for some constant C. If $Z(\lambda)$ is a Poisson process, $C\lambda$ is its variance; if $Z(\lambda)$ is Brownian motion, $C = 0$; in general, $E|dZ(\lambda)|^4 = Cd\lambda$.

We will also need the following known results.

<u>Definition</u>. A distribution function $F(x)$ with characteristic function $C(u)$ is *infinitely divisible* (I.D.) if for every positive integer n, there exists a characteristic function $C_n(u)$ such that $C(u) = [C_n(u)]^n$.

<u>Lemma 3.1 (Doob[15])</u>. *A distribution function of a random variable X is infinitely divisible if and only if for every* $\varepsilon > 0$ *there exist independent random variables* $Y_1, \ldots, Y_n$ *such that*

$$P\{|Y_i| > \varepsilon\} < \varepsilon, \qquad i \leq n,$$

and $Y_1 + Y_2 + \ldots + Y_n$ *has the same distribution function as* X.

Lemma 3.2 (Pierre[16]). *If* $\{S_n\}$ *is a sequence of I.D. random variables such that* $ES_n = 0$, var $S_n \to \sigma^2$, *and*

$$k_4(S_n) = ES_n^4 - 3\{ES_n^2\}^2$$
$$\to 0 \quad \text{as} \quad n \to \infty,$$

then S_n *is asymptotically normal.*

For the case in which $\Phi_k(\lambda)$ is nonrandom, it is clear that ℓ_k is infinitely divisible. Thus the condition that $k_4(\ell_k) \to 0$ provides an easily checked sufficient condition for asymptotic normality.

Lemma 3.3. *If* $P_n \to P$ *in the* γ*-th mean (i.e.,* $E|P_n - P|^\gamma \to 0$ *as* $n \to \infty$*), then there is a subsequence* $P_{n'}$ *such that* $P_{n'} \to P'$ w.p.1, *and* P *and* P′ *differ only on a set of measure* 0. *Also,* $E|P|^\beta = \lim E|P_n|^\beta$ *for* $\beta < \gamma$.

Lemma 3.3 is known and can be proved using the Borel-Cantelli lemma and the Markov inequality.

Theorem 3.1. *If for* $k = 1, 2, \ldots,$

$$\sigma_k^2(\omega) = \int_{-\infty}^{\infty} \Phi_k^2(\lambda)d\lambda$$

converges as $k \to \infty$ w.p.1 *to a random variable* $\sigma^2(\omega)$ *with distribution function* S(x) *and*

$$\int_{-\infty}^{\infty} \Phi_k^4(\lambda)d\lambda \to 0 \qquad \text{(w.p.1)},$$

then the asymptotic characteristic function of ℓ_k *is*

$$C(u) = \int_{-\infty}^{\infty} \exp\,[-\tfrac{1}{2}xu^2]\,dS(x).$$

The integrals of $\phi_k^2(\lambda)$ *and* $\phi_k^4(\lambda)$ *are assumed to exist as limits of the Riemann partial sums in mean square, in probability, or* w.p.1.

Proof. First we note that ℓ_k can be defined using only a denumerable set of the random variables $\{\phi_k(\lambda)\}$ and $\{Z(\lambda)\}$ corresponding to a denumerable parameter set D, and D is independent of k. Our basic probability space will be the one generated by these random variables. More explicitly, the space is $(\Omega_1 \times \Omega_2, \mathfrak{F}_1 \times \mathfrak{F}_2, P_1 \times P_2)$, where Ω_1 and Ω_2 are the spaces on which the independent random processes $\{\phi_k(\lambda):\lambda \in D, k = 1, 2, \ldots\}$ and $Z(\lambda)$, $\lambda \in D$, respectively, are defined. The elements of Ω_1 and Ω_2 are ω_1 and ω_2, respectively.

Let $\mathfrak{B}$ be the σ-field generated by the set $\{\phi_k(\lambda):k = 1, 2, \ldots; \lambda \in D\}$; i.e., $\mathfrak{B} = \mathfrak{F}_1$. Then the characteristic function of ℓ_k is given by

$$\begin{aligned} C_k(u) &= E \exp\,[iu\ell_k] \\ &= E\{E^{\mathfrak{B}} \exp\,[iu\ell_k]\}, \end{aligned}$$

where $E^{\mathfrak{B}}$ stands for conditional expectation. Let $C_k^{\mathfrak{B}}(u;\omega) = E^{\mathfrak{B}} \exp\,[iu\ell_k]$, where the ω-dependence of the conditional characteristic function is made explicit. Actually, $C_k^{\mathfrak{B}}$ is measureable on $C_1^{\mathfrak{B}}$ and is thus a function of ω_1 only. Thus we will often write $C_k^{\mathfrak{B}}(u;\omega_1)$.

Since $|C_k^{\mathfrak{B}}| \leq 1$, by the dominated convergence theorem, $\lim_k C_k(u) = E \lim_k C_k^{\mathfrak{B}}(u)$. It will be sufficient to show that for almost all ω_1, $C_k^{\mathfrak{B}}(u)$ is asymptotically normal with variance $\sigma^2(\omega_1)$.

A regular conditional distribution, $F_k^{\mathfrak{B}}(y;\omega)$, of ℓ_k, given $\mathfrak{B}$, exists for all k as a consequence of the probability space we have chosen (see Loève[17]); i.e., for all ω, $F_k^{\mathfrak{B}}(y;\omega) = F_k^{\mathfrak{B}}(y;\omega_1)$ is a distribution function.

In order to apply Lemma 3.2, we need to show that under $\mathfrak{B}$, ℓ_k is I.D., that $\mathrm{var}^{\mathfrak{B}}\,\ell_k = \sigma_k^2(\omega_1)$, and that

$$k_4^{\mathfrak{B}}(\ell_k) = E^{\mathfrak{B}}\ell_k^4 - 3\{E^{\mathfrak{B}}\ell_k^2\}^2 = \int_{-\infty}^{\infty} \phi_k^4(\lambda)d\lambda$$

except on a null set N_k.

Consider the points $\{\lambda_i\}_n \subset D \cap [a,b]$ such that

$$\ell_k = \underset{\substack{b\to\infty \\ a\to-\infty}}{\mathrm{l.i.m.}}\ \underset{n\to\infty}{\mathrm{l.i.m.}} \sum_i \phi_k(\lambda_i)(Z(\lambda_i) - Z(\lambda_{i-1})),$$

where l.i.m. stands for limit in the fourth mean (see Lemma 3.3), and $\{\lambda_i\}_n$ increases monotonically to $D \cap [a,b]$. Since ℓ_k is a limit in the fourth mean, it represents a class of random variables, any two members of which differ on at most a null set. We will show that for each k, at least one member of the class satisfies the conditions of the previous paragraph. Consider the inner limit for a given [a,b] and call the partial sums P_n. By assumption, ℓ_k is convergent in the fourth mean: i.e.,

$$E\left|\int_a^b \phi_k(\lambda)dZ(\lambda) - P_n\right|^4 \to 0$$

as $n \to \infty$. Then, clearly,

$$M_n^{\mathfrak{B}} = E^{\mathfrak{B}}\left|\int_a^b \Phi_k(\lambda)\,dZ(\lambda) - P_n\right|^4 \to 0$$

in the first absolute mean. Using the results of Lemma 3.3, a subsequence $P_{n'}$ can be chosen so that $M_{n'}^{\mathfrak{B}} \to 0$ except on a null set $N_k(a,b)$: i.e., we get conditional convergence of P_n in the fourth mean for almost all ω. Let

$$\left[\int_a^b \Phi_k(\lambda)\,dZ(\lambda)\right]^*$$

be the version of the truncated integral obtained from the subsequence $P_{n'}$. For this version, the conditional moments for $\omega \notin N_k(a,b)$ are obtained as the limit of the conditional moments of $P_{n'}$. In a similar way, we can let $b \to \infty$ and $a \to -\infty$ through a denumerable set of values so that the union of the null sets $N_k(a,b)$ is still null and fast enough so that the limit exists except on the null set N_k'.

For the version of ℓ_k obtained as described above,

$$\mathrm{var}^{\mathfrak{B}}\,\ell_k = \left[\int \Phi_k^2(\lambda)\,d\lambda\right]^* \quad \text{and} \quad k_4^{\mathfrak{B}}(\ell_k) = \left[\int \Phi_k^4(\lambda)\,d\lambda\right]^*.$$

By assumption, the integrals without asterisks exist in at least one of the conventional senses. They can differ from the integrals above on at most a null set; thus

$$\mathrm{var}^{\mathfrak{B}}\,\ell_k = \int \Phi_k^2(\lambda)\,d\lambda \quad \text{and} \quad k_4^{\mathfrak{B}}(\ell_k) = \int \Phi_k^4(\lambda)\,d\lambda$$

except on the null set N_k'.

The random variables ℓ_k are conditionally I.D., since under the conditioning $\mathfrak{B}$, they are fourth-mean limits of I.D. partial sums.

If N is the union of the null sets N_k', it is also a null set. Fix $\omega \notin N$; then the sequence of distribution functions $F_k^{\mathfrak{B}}(y;\omega)$ is asymptotically normal, and the proof is complete.

It should be pointed out here that the main difficulty in the proof of the theorem is obtaining an expression for the conditional fourth moments. We showed that these conditional moments can always be expressed in terms of a special integral--an integral whose Riemann partial sums converge w.p.1 when the sequence of partitions is especially chosen. The theorem could have been stated in terms of this special integral. However, we decided to assume the existence of the relevant integrals in some conventional sense: i.e., for *arbitrary* sequences of partitions becoming dense in the interval concerned, the Riemann partial sums converge in some conventional sense. Under this latter assumption, it is clear that our special integral and the conventional integral are equal w.p.1.

STOCHASTIC SUMS

Next we consider sequences of sums

$$\ell_k = \sum_{j=-\infty}^{\infty} \phi_{jk}\varepsilon_j, \qquad k = 0, \pm 1, \pm 2, \ldots,$$

where $\{\varepsilon_j : j = 0, \pm 1, \pm 2, \ldots\}$ is a sequence of independent random variables. Let the random matrix $\{\phi_{jk}\}$ be independent of $\{\varepsilon_j\}$ and let the sums exist as *limits in the fourth mean*. Thus ℓ_k for random sums is a natural generalization of ℓ_k for random integrals.

Finally, we will need a modification of a result from Pierre[(16)*].

Lemma 3.4. *Let*

$$S_k = \sum_{j=0}^{j_k} X_{kj},$$

where $k = 0, 1, \ldots,$ *and for each* k, $\{X_{kj}\}$ *are independent random variables. If* $EX_{kj} = 0$, $\sum_j \operatorname{var} X_{kj} \to \sigma^2 < \infty$, $\max_j \operatorname{var} X_{kj} \to 0$, *and*

$$ES_k^4 - 3[ES_k^2]^2 = \sum_j \{EX_{kj}^4 - 3[EX_{kj}^2]^2\} \to 0,$$

then S_k *is asymptotically normal.*

Proof. Since

$$\sum_j [EX_{kj}^2]^2 \leqq \max_j EX_{kj}^2 \sum_j EX_{kj}^2 \to 0$$

as $k \to \infty$, the Lyapunov condition for $\delta = 2$ is satisfied (see Loève[(17)]), and S_k is asymptotically normal.

Theorem 3.2. *If* $E\varepsilon_j = 0$, $E\varepsilon_j^2 = 1$ *and*

$$\sigma_k^2(\omega) = \sum_j \phi_{jk}^2$$

converges as $k \to \infty$ w.p.1 *to a random variable* $\sigma^2(\omega)$ *with distribution function* $S(x)$; *and if either*

(a) $\lim_{k\to\infty} \sum_j |\phi_{jk}|^4 E|\varepsilon_j|^4 = 0$ (w.p.1) *or*

(b) $\{\varepsilon_j\}$ *are infinitely divisible and* $\lim_{k\to\infty} \sum_j \phi_{jk}^4 (E\varepsilon_j^4 - 3[E\varepsilon_j^2]^2) = 0$ (w.p.1), *then the asymptotic characteristic function of* ℓ_k *is*

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*This modification was brought to the author's attention by C. N. Morris of The Rand Corporation.

$$C(y) = \int_{-\infty}^{\infty} \exp\,[-\tfrac{1}{2}xy^2]\,dS(x).$$

The sums above are assumed to exist as limits either in mean-square in probability, or w.p.1.

Proof. The proof of (b) follows from the fact that the sum of independent I.D. random variables is I.D. and that if $\mathfrak{B}$ is the σ-field generated by $\{\phi_{jk}\}$, the same arguments as used in Theorem 3.1 indicate that, w.p.1,

$$E^{\mathfrak{B}}\ell_k^4 - 3[E^{\mathfrak{B}}\ell_k^2]^2 = \sum_j \phi_{jk}^4 (E\varepsilon_j^4 - 3[E\varepsilon_j^2]^2).$$

The rest follows just as in Theorem 3.1.

Lemma 3.4 together with condition (a) and the conditioning argument is used to prove the other result. Condition (a) implies that for almost all ω,

$$\max_j |\phi_{jk}|^4 E|\varepsilon_j|^4 \to 0$$

as $k \to \infty$. Since $E\varepsilon_j^4 \geq (E\varepsilon_j^2)^2 = 1$,

$$\max_j |\phi_{jk}|^2 = \max_j \operatorname{var}^{\mathfrak{B}}(\phi_{jk}\varepsilon_j) \to 0$$

as $k \to \infty$ w.p.1. Computing conditional moments as we did in Theorem 3.1, we see that

$$E^{\mathfrak{B}}\ell_k^4 - 3[E^{\mathfrak{B}}\ell_k^2]^2 = \sum_j \phi_{jk}^4 (E\varepsilon_j^4 - 3[E\varepsilon_j^2]^2).$$

Each of the two sums on the right-hand side converges to zero--the first by condition (a) and the second because it is less than three times the first. All that remains is to express ℓ_k conditioned by $\mathscr{B}$ as *finite* sums satisfying the conditions of Lemma 3.4. Let

$$X_{k0} = \phi_{k0}\varepsilon_0,$$

$$X_{kj} = \phi_{kj}\varepsilon_j + \phi_{k,-j}\varepsilon_{-j}$$

for $1 \leqq j \leqq j_k - 1$ (j_k is defined below), and

$$X_{kj_k} = \sum_{|j| \geqq j_k} \phi_{jk}\varepsilon_j,$$

where $j_k = j_k(\omega)$ is a random positive integer chosen large enough so that for almost all ω,

$$\operatorname{var}^{\mathscr{B}} X_{kj_k} \leqq \max_j |\phi_{jk}|^2.$$

This can be done, since X_{kj_k} is the tail sum of ℓ_k and

$$\operatorname{var}^{\mathscr{B}} \ell_k = \sum_j \phi_{jk}^2 < \infty \qquad \text{(w.p.1)}.$$

Then

$$\ell_k = \sum_{k=0}^{j_k(\omega)} X_{kj}$$

and all the conditions of Lemma 3.4 are conditionally satisfied. This completes the proof.

If $Z(\cdot)$ has nonstationary, infinitely divisible, independent increments, it is handled in the same way as in Theorem 3.1 except that now

$$\text{var } \ell_k = \int E\Phi_k^2(\lambda)dV_2(\lambda)$$

and

$$E\ell_k^4 - 3\{E\ell_k^2\}^2 = C\int E\Phi_k^4(\lambda)dV_4(\lambda)$$

for some monotone increasing functions $V_2(\lambda)$ and $V_4(\lambda)$. (This situation was considered in Appendix B of Ref. 11. Then (b) of Theorem 3.2 would be a special case of this more general theorem.

We note that $\{\varepsilon_j\}$ being infinitely divisible in Theorem 3.2 (b) is a special case of Theorem 3.2 (a). However, the condition (b) is weaker than condition (a).

Even in the case in which Φ_{jk} is nonrandom, the results of Theorem 3.2 are useful. For example, most central limit theorems for discrete-parameter linear processes require that the ε_j be identically distributed. This is not required in Theorem 3.2.

IV. APPLICATIONS

Perhaps the most important special cases of the theorems of Section III occur when $\sigma^2(\omega)$ is a constant w.p.1. A closer look at $\sigma_k^2(\omega)$ will show that this occurs surprisingly often.

NORMAL AND WEIGHTED NORMAL CONVERGENCE FOR RANDOM INTEGRALS

Consider the random process

$$x(t) = \int f(t,\lambda)dZ(\lambda)$$

and the linear functionals

$$\ell_k = \int x(t)dL_k(t),$$

where

$$dL_k(t) = k^{-1/2} \cos 2\pi f_0 t \; dt$$

or

$$dL_k(t) = k^{-1/2} \sin 2\pi f_0 t \; dt$$

for $0 \leqq t \leqq k$, and 0 elsewhere. If $f(t,\lambda)$ is mean-square continuous and

$$I = \int Ef(t,\lambda)f(s,\lambda)d[L_k(t)L_k(s)\lambda] \tag{9}$$

exists, then the integrals may be interchanged (Pierre[14]) to obtain

$$\ell_k = \int_{-\infty}^{\infty} \Phi_k(\lambda)dZ(\lambda),$$

where

$$\Phi_k(\lambda) = \int_0^k f(t,\lambda)dL_k(t).$$

Finally, we require that

$$\sigma_k^2(\omega) = \int_{-\infty}^{\infty} \left(\int_0^k f(t,\lambda)dL_k(t)\right)^2 d\lambda$$

$$= \int_0^k \int_0^k \int_{-\infty}^{\infty} f(t,\lambda)f(s,\lambda)d\lambda dL_k(t)dL_k(s) \qquad (\text{w.p.1}). \tag{10}$$

This will be the case under conditions analogous to Eq. (9). As before, we assume $EZ(\lambda) = 0$.

<u>Example</u>. Let $f(t,\lambda) = h(t - \lambda)$; let

$$\int |h(t)|dt < \infty \qquad (\text{w.p.1});$$

let

$$H(f) = \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}\, dt;$$

and let $\frac{1}{2}|H(f_0)|^2$ have distribution function $S(x)$. Then if $f_0 \neq 0$, ℓ_k has the asymptotic characteristic function

$$C(y) = \int_{-\infty}^{\infty} \exp\, [-\tfrac{1}{2}xy^2]dS(x).$$

If $f_0 = 0$, $S(x)$ is the distribution function of $|H(0)|^2$.

<u>Proof</u>. The function

$$r(t - s) = \int_{-\infty}^{\infty} h(t - \lambda)h(s - \lambda)d\lambda$$

is a stationary covariance function w.p.1. From Davenport and Root,[18] we have for $f_0 \neq 0$,

$$\sigma_k^2(\omega) = \frac{1}{k}\int_0^k\int_0^k r(t - s) \cos (2\pi f_0 s) \cos (2\pi f_0 t) ds\ dt$$

$$\rightarrow \tfrac{1}{2}|H(f_0)|^2 \tag{11}$$

and this holds for almost all h(t). In Eq. (11) the integral is interpreted as an integral along the sample paths of r(t - s), while in Eq. (10) the integral is a mean-square integral. However, since they both exist, they are equal. Equation (11) is also true when cosine is replaced by sine. If $f_0 = 0$, we get $|H(0)|^2$.

Finally, using the sample path interpretation of the following integral we have

$$\frac{1}{k^2}\int_{-\infty}^{\infty}\left(\int_0^k h(t - \lambda) \cos (2\pi f_0 t)dt\right)^4 d\lambda \leq \frac{1}{k^2}\int_{-\infty}^{\infty}\left(\int_0^k |h(t - \lambda)|dt\right)^4 d\lambda$$

$$\leq \frac{1}{k}\left[\max_{\lambda}\int_0^k |h(t - \lambda)|dt\right]^2\sigma_k^2(\omega)$$

$$\leq \frac{\sigma_k^2}{k}\left[\int|h(t)|dt\right]^2$$

$$\rightarrow 0 \qquad (\text{w.p.1}).$$

Thus the conditions of Theorem 3.1 are satisfied and the proof is complete.

Two things worth noting are: First, the asymptotic distribution of ℓ_k does not depend on the phase of h(t). Thus if h(t) is random, but $|H(f)|^2$ is nonrandom, ℓ_k is asymptotically normal. Second, if h(t) is a Gaussian random process, the weight function S(x) is a noncentral chi-square distribution with one degree of freedom.

<u>CONDITIONALLY LINEAR PROCESSES DERIVED FROM STATIONARY POINT PROCESSES</u>

Let

$$x(t) = \sum_i A_i e^{-(t-t_i)} u(t - t_i),$$

where $\{A_i : i = 0, \pm 1, \pm 2, \ldots\}$ are independent random variables with $EA_i = 0$, $EA_i^2 = 1$, and EA_i^4 uniformly bounded; $u(t) = 1$ for $t > 0$ and zero otherwise; and $\{t_i\}$ form a point process with the following properties:

(i) If N_k is the number of points t_i occurring in the interval $[0,k]$, then $\lim k^{-1} N_k = N$ (w.p.1) for some number $N < \infty$.

(ii) For any number $a < k$, $N_k = N_a + N_{k,a}$, where $N_{k,a}$ is the number of points t_i occurring in the interval $[a,k]$.

We consider the linear functionals

$$\ell_k = k^{-1/2} \int_0^k x(t)dt.$$

Assume that x(t) may be integrated termwise, and further assume that the resulting sum converges in the fourth mean. Then

$$\ell_k = \sum_i A_i k^{-1/2} \int_0^k e^{-(t-t_i)} u(t - t_i)dt$$

$$= \sum_{t_i<0} A_i k^{-1/2} e^{t_i}(e^{-k} - 1)$$

$$+ \sum_{0\leq t_i\leq k} A_i k^{-1/2} e^{t_i} \left(e^{-k} - e^{-t_i}\right).$$

The conditional variance of ℓ_k given $\{t_i\}$ is

$$\mathrm{var}^{\mathcal{B}}\, \ell_k = k^{-1}(e^{-k} - 1)^2 \sum_{t_i<0} e^{2t_i}$$

$$+ \sum_{0\leq t_i\leq k} k^{-1}\left(1 - 2e^{-(k-t_i)} + e^{-2(k-t_i)}\right).$$

The first summation converges to zero as $k \to \infty$ w.p.1. The first term of the second summation contributes $k^{-1}N_k \to N$. Consider

$$p_k = k^{-1}e^{-k} \sum_{0\leq t_i\leq k} e^{t_i}$$

$$= k^{-1}e^{-k}\left(\sum_{0\leq t_i\leq(1-\varepsilon)k} e^{t_i} + \sum_{(1-\varepsilon)k\leq t_i\leq k} e^{t_i}\right)$$

for some $\varepsilon > 0$. If in each of the sums above we substitute the largest value of t_i, then

$$p_k \leq k^{-1}N_{(1-\varepsilon)k}e^{-\varepsilon k} + k^{-1}N_{k,(1-\varepsilon)k}$$

$$\to 0 + \varepsilon N.$$

Since ε is arbitrary, $p_k \to 0$ w.p.1. Thus $\mathrm{var}^{\mathcal{B}}\, \ell_k \to N$. A similar analysis will show that

$$\sum_i k^{-2}\left(\int_0^k e^{-(t-t_i)} u(t - t_i)dt\right)^4 \to 0 \qquad \text{(w.p.1)}.$$

Thus the conditions of Theorem 3.2 (a) are satisfied and $\{\ell_k\}$ is asymptotically normal.

Results completely analogous to those above can be stated for discrete-parameter processes.

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