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# LIQUID BORNE NOISE REDUCTION
## Volume I
## MATHEMATICAL ANALYSIS
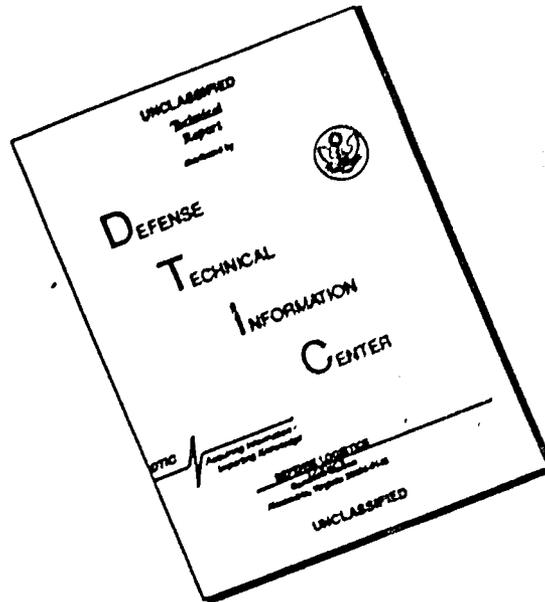
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# NOMENCLATURE

The definitions of the following symbols are used throughout the report except where otherwise stated. Some symbols defined in the text of the report are not included.

| | |
|---|---|
| $a$ . . . . . | Velocity of wave propagation in a liquid-filled pipe. |
| $A$ . . . . . | Cross sectional area of pipe. |
| $b$ . . . . . | Thickness of pipe wall. |
| $B$ . . . . . | Constant used in determining the frictional resistance to flow. |
| $B_i(j\omega)$ . . . | System function for the ith pipe at point $x_i$ . |
| $B_i'(j\omega)$ . . . | System function for the ith pipe at $x_i = \ell_i$ . |
| $c$ . . . . . | Velocity of wave propagation in an infinite medium. |
| $c_n$ . . . . . | Complex Fourier coefficient. |
| $C$ . . . . . | Coefficient of capacitance. |
| $D$ . . . . . | Inside diameter of pipe. |
| $e$ . . . . . | Base of natural logarithms $= 2.718\cdots$ . |
| $E$ . . . . . | Modulus of elasticity of pipe material. |
| $E_{pk}(x_i, j\omega)$. | Transfer function relating pressure at point $x_i$ to pressure at point $k$ . |
| $E_{qk}(x_i, j\omega)$. | Transfer function relating pressure at point $x_i$ to flow rate at point $k$ . |
| $f$ . . . . . | $\begin{cases} \text{Darcy-Weisbach friction factor.} \\ \text{Cyclic frequency.} \end{cases}$ |
| $f_1$ . . . . . | Fundamental frequency. |
| $f_c$ . . . . . | Cut-off frequency. |

$f_n$ . . . . . Discrete frequency.

$F_{pk}(x_i, j\omega)$. Transfer function relating power at point $x_i$ to pressure at point $k$ .

$F_{qk}(x_i, j\omega)$. Transfer function relating power at point $x_i$ to flow rate at point $k$ .

$G_i(j\omega)$ . . . Transfer function relating pressure at point $x_i$ to flow rate at terminus.

$G_i'(j\omega)$ . . . Transfer function relating pressure at point $\ell_i$ to flow rate at terminus.

$\overline{G(f)}$ . . . . Spectral density as defined by American Standard.

$h(\tau)$ . . . . . System weighting function.

$H_i(j\omega)$ . . . Transfer function relating flow rate at point $x_i$ to flow rate at terminus.

$H_i'(j\omega)$ . . . Transfer function relating flow rate at point $\ell_i$ to flow rate at terminus.

$i$ . . . . . . $\begin{cases} \text{Refers to ith segment of pipe.} \\ \text{General subscript.} \end{cases}$

$j$ . . . . . . $\sqrt{-1}$

$J_0$ and $J_1$ . Bessel functions of the first kind.

$k$ . . . . . General subscript.

$K$ . . . . . Bulk modulus of a liquid.

$K_{pi}$ . . . . Gain function relating pressure at point $x_i$ to flow rate at point $x_i = 0$ .

$K_{qi}$ . . . . Gain function relating flow rate at point $x_i$ to flow rate at point $x_i = 0$ .

$K_{qi}'$ . . . . Gain function relating flow rate at point $\ell_i$ to flow rate at point $x_i = 0$ .

$\overline{K}_{pi}$ . . . . Gain function relating pressure at point $x_i$ to flow rate at terminus.

$\overline{K}_{qi}$ . . . . Gain function relating flow rate at point $x_i$ to flow rate at terminus.

$\ell$ . . . . . Length.

$\ell_i$ . . . . . . Length of ith pipe segment.

$L$ . . . . . Coefficient of inertia.

$\mathcal{L}$ . . . . . One-sided Laplace transform.

$\mathcal{L}_T$ . . . . Two-sided Laplace transform.

$m$ . . . . . $\begin{cases}\text{Number of pipes in series.} \\ \text{Number of pipes in parallel.} \\ \text{General subscript.}\end{cases}$

$M_i(j\omega)$ . . . System function for the ith pipe at point $x_i$ .

$M_i'(j\omega)$ . . . System function for the ith pipe at $x_i = \ell_i$ .

$n$ . . . . : $\begin{cases}\text{Exponent of } \bar{q} \text{ for mean flow.} \\ \text{Refers to source end.} \\ \text{General subscript.}\end{cases}$

$p$ . . . . . Pressure variation as a function of distance and time.

$\bar{p}$ . . . . . Mean pressure as a function of distance.

$p_t$ . . . . . Instantaneous pressure as a function of distance and time $= \bar{p} + p$ .

$P(x_i, j\omega)$ . . Pressure variation at point $x_i$ transformed with respect to time with $s = j\omega$ .

$P_i(j\omega)$ . . . Transformed pressure variation at $x_i = 0$ with $s = j\omega$ .

$\hat{P}(\lambda, s)$ . . . Pressure variation transformed with respect to time and distance.

$q$ . . . . . Volume flow rate variation as a function of distance and time.

$\bar{q}$ . . . . . Constant mean volume flow rate.

$q_t$ . . . . . Instantaneous volume flow rate as a function of distance and time $= \bar{q} + q$ .

$Q(x_i, j\omega)$ . . Volume flow rate variation at point $x_i$ transformed with respect to time with $s = j\omega$ .

$Q_i(j\omega)$ . . . Transformed flow rate variation at $x_i = 0$ with $s = j\omega$ .

$\hat{Q}(\lambda, s)$ . . . Flow rate variation transformed with respect to time and distance.

$r$ . . . . . Refers to terminus.

$R$ . . . . . Coefficient of resistance.

$s$ . . . . . Laplace transform with respect to time operator.

$t$ . . . . . Time.

$T$ . . . . . $\begin{cases}\text{Limit of a period.}\\\text{Denotes truncation.}\end{cases}$

$T_i(j\omega)$ . . . System function for the ith pipe at point $x_i$ .

$T_i'(j\omega)$ . . . System function for the ith pipe at $x_i = \ell_i$ .

$u_t$ . . . . . Instantaneous fluid velocity as a function of distance and time.

$w$ . . . . . Power.

$W(x_i, j\omega)$ . . Transformed power at point $x_i$ with $s = j\omega$ .

$W_{ave}$ . . . Average power.

$x$ . . . . . . Axial distance.

$x_i$ . . . . . Axial distance measured from the terminating end of the ith segment.

$x(t)$ . . . . Function of time.

$X(s)$ . . . . Laplace transform of $x(t)$ .

$Y_{pk}(x_i, j\omega)$. Transfer function relating flow rate at point $x_i$ to pressure at point $k$ .

$z$ . . . . . Characteristic value.

$Z(x_i, j\omega)$ . . Ratio of transformed pressure to flow rate at point $x_i$ .

$Z_i(j\omega)$ . . . Ratio of transformed pressure to flow rate at $x_i = 0$ .

$Z_c$ . . . . . Characteristic impedance.

$Z_{ci}$ . . . . Characteristic impedance of the ith segment.

$Z_r$ . . . . . Ratio of transformed pressure to flow rate at terminus.

$Z_{qk}(x_i, j\omega)$. Transfer function relating pressure at point $x_i$ to flow rate at point $k$ .

$\alpha$ . . . . . Attenuation constant.

$\alpha_i$ . . . . . Attenuation constant of the ith segment.

$\beta$ . . . . . Phase constant.

$\beta_i$ . . . . . Phase constant of the ith segment.

$\gamma$ . . . . . Propagation coefficient.

$\gamma_i$ . . . . . Propagation coefficient of the ith segment.

$\Gamma$ . . . . . Reflection coefficient.

$\Gamma_i$ . . . . . Reflection coefficient of the ith segment.

$\theta$ . . . . . Phase angle of reflection coefficient.

$\lambda$ . . . . . Laplace transform with respect to distance operator.

$\rho$ . . . . . Mass density.

$\tau$ . . . . . $\begin{cases} \text{Dummy time variable.} \\ \text{Displacement in time.} \end{cases}$

$\phi_k$ . . . . . Phase angle of the ratio of pressure to flow at a point $k$ .

$\phi_{xx}(\tau)$ . . . Autocorrelation function of $x(t)$ .

$\phi_{xy}(\tau)$ . . . Cross-correlation function of $x(t)$ and $y(t)$ .

$\phi_{pq}(\tau)$ . . . Sound power.

$\Phi_{xx}(j\omega)$ . . Spectral density of $x(t)$ .

$\Phi_{xy}(j\omega)$ . . Cross-spectral density of $x(t)$ and $y(t)$ .

$\Phi_w(j\omega)$ . . Power spectral density.

$\omega$ . . . . . Radial frequency.

$\omega_1$ . . . . . Fundamental frequency.

$\omega_c$ . . . . . Cut-off frequency.

$\omega_n$ . . . . . Discrete frequency.

# CHAPTER I

## INTRODUCTION

### 1.1. Objectives.

*The*

~~This~~ report presents the results of a mathematical analysis of pressure, flow rate, and acoustic power levels at various points in an idealized piping system which is subjected to statistical inputs. Successful adaptation and use of the results will aid in the prevention of excessive liquidborne noise in shipboard piping systems.

### 1.2. Method of Attack.

After careful review of existing literature and technical reports, an analysis was begun to determine the feasibility of predicting the response of piping systems to liquidborne noise. The analysis was analytical in nature and used the results of previous scientific investigations. At various points in the analysis certain assumptions are made on the basis of engineering judgment. These points are clearly stated and discussed in this report.

It is shown that the pressure, flow rate, and power at any point in a complex piping system can be found in terms of the same quantities at some more convenient location in the piping system. A digital computer is needed for the complex systems. The results are expressible in terms of continuous spectrums (i.e. spectral densities) or in terms of discrete spectrums and bandwidths as are obtained from wave analyzers.

1

## 1.3.   Content of This Report.

This report contains background information about the physical system with the idealizing assumptions and the resulting equations which describe the response.  Further, background information on mathematical statistics is included to clarify terminology.  The mathematical symbols and terminology used is that acceptable as "American Standards"[1]* when applicable.

The analysis follows in which the physical system is analyzed using the mathematical techniques.  The results will be discussed elsewhere in this report.

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*Numbers in parentheses refer to references in the Bibliography.

# CHAPTER II

## BACKGROUND MATERIAL

### Mathematical Tools

### 2.1. Transform Methods.

The transform methods needed are the Fourier and the Two-Sided Laplace which become identical when $s = j\omega$ . Thus,

$$X(s) = \lim_{A \to \infty} \int_{-A}^{A} e^{-st} x(t)\, dt \ , \tag{2.1}$$

and

$$x(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X(s)\, e^{st}\, ds \tag{2.2}$$

are the definitions to be used in this analysis.

### 2.2. Fourier Series.

When a function $x(t)$ is periodic in time $2T$ , the Fourier series representation is

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_1 t + b_n \sin n\omega_1 t) \tag{2.3}$$

where

$$\omega_1 = \frac{\pi}{T} \ . \tag{2.4}$$

The Fourier coefficients are defined as

$$a_0 = \frac{1}{2T} \int_{t_0-T}^{t_0+T} x(t)\, dt \ ,$$

$$a_n = \frac{1}{T} \int_{t_0-T}^{t_0+T} x(t)\cos n\omega_1 t\, dt \ , \qquad \Bigg\} \qquad (2.5)$$

and

$$b_n = \frac{1}{T} \int_{t_0-T}^{t_0+T} x(t)\sin n\omega_1 t\, dt \ .$$

By further manipulation [2] the equations (2.3) and (2.5) can be placed in "exponential form" as follows:

$$x(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{jn\omega_1 t} \qquad (2.6)$$

where

$$c_n = \frac{1}{2}(a_n - jb_n) = \frac{1}{2T} \int_{t_0-T}^{t_0+T} x(t)\, e^{-jn\omega_1 t}\, dt \ , \qquad (2.7)$$

or [6]

$$c_n = \frac{1}{2T}\, \mathcal{L}\Big[x_T(t)\Big]_{s=jn\omega_1} \ , \qquad (2.8)$$

and

$$c_0 = a_0 \ . \qquad (2.9)$$

The function $x_T(t)$ is defined as

$$
\left.
\begin{aligned}
x_T(t) &= x(t) \ , \quad t_0-T \le t \le t_0+T \ ; \\
x_T(t) &= 0 \ \text{at all other points.}
\end{aligned}
\right\} \qquad (2.10)
$$

The Laplace operation in Equation (2.8) is the "one-sided" Laplace operation.

### 2.3. Transfer Functions.

A system represented by a linear differential equation

$$L\left[y(t)\right] = x(t) \tag{2.11}$$

can be transformed and written as

$$Y(s) = H(s) X(s) . \tag{2.12}$$

The solution of (2.12) is

$$y(t) = \mathcal{L}^{-1}\left[Y(s)\right] = \int_{-\infty}^{\infty} h(\tau) x(t - \tau) \, d\tau . \tag{2.13}$$

Thus, if the input $x(t)$ is known, the output $y(t)$ is found by an integral operation on the input using the proper weighting system function $h(\tau)$.

The function $H(s)$ is the transfer function and in some cases may be an impedence.

### 2.4. Output Responses.

When $x(t)$ is a prescribed input, the various output responses are obtained.

(a) Impulse response; $x(t) = \delta(t)$

$$\left.\begin{aligned} y(t) &= h(t) \\ Y(s) &= H(s) \end{aligned}\right\} \tag{2.14}$$

(b) Step response; $x(t) = u(t)$

$$\left.\begin{aligned} y(t) &= \int_0^t h(\tau) \, d\tau \\ Y(s) &= \frac{1}{s} H(s) \end{aligned}\right\} \tag{2.15}$$

(c) Frequency response; $x(t) = \sin \omega t$

$$\left.\begin{array}{l} y(t) = \displaystyle\int_0^t h(\tau) \sin \omega(t - \tau)\, d\tau \\[2em] Y(s) = \dfrac{\omega}{s^2 + \omega^2}\, H(s) \end{array}\right\} \tag{2.16}$$

where by common acceptance

$$H(j\omega) = \text{frequency response.} \tag{2.17}$$

It is assumed that the effect on $y(t)$ due to the poles of $H(s)$ has decayed as a transient and that only $s = \pm j\omega$ will be effective.

## 2.5. Statistical Concepts and Correlation.

The correlation between two continuous processes or motions is[3, 4, 5, 6]

$$\phi_{12}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x_1(t)\, x_2(t + \tau)\, dt = \overline{x_1(t)\, x_2(t + \tau)} \tag{2.18}$$

in which the process is stationary and ergodic. The above operation shown is denoted by $\overline{x_1(t)\, x_2(t + \tau)}$ and is called the "time" average. The time average is the scan of $x_1(t)$ by another function $x_2(t)$ by shifting an amount $\tau$ and averaging over a sufficiently large period $2T$. $\phi_{12}(\tau)$ is the cross-correlation between $x_1(t)$ and $x_2(t)$.

When $x_1 \equiv x_2$, Equation (2.18) yields the autocorrelation function

$$\phi_{11}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x_1(t)\, x_1(t + \tau)\, dt \ . \tag{2.19}$$

The mean square value is when $\tau = 0$, or

$$\phi_{11}(0) = \overline{x_1(t)^2} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left[x_1(t)\right]^2 dt \ . \tag{2.20}$$

The autocorrelation is an even function in $\tau$ .

Correlation functions are useful in any general approach to analysis of aperiodic processes and are instrumental in the frequency description of a random process.

## 2.6. Spectra and Spectral Densities.

The spectrum of a function is a plot of the Fourier coefficients $c_n$ versus the frequency $\omega_n$ . For periodic functions the spectrum is a series of discrete points at specific frequencies, and for aperiodic functions the spectrum becomes a continuous plot as the period becomes infinite[2, 4, 5]. This latter statement is equivalent to saying that the increment of frequency approaches zero.

The spectrum mostly used is that historically associated with power calculations. Power calculation results are expressed as the product of two functions as in Equations (2.18) and (2.19) in which the average value is obtained. The spectrum of the power is expressed as a function of the square of the Fourier coefficients.

A random vibration can be considered to be the sum of a large number of harmonic vibrations of appropriate amplitude and phase. The total power is the sum of the power of each component of the vibration. This power is distributed as a function of the frequency; therefore, the amount of power associated with each frequency is of interest and is called the power spectral density. The accepted mathematical definition of power spectral density for an ergodic process $x(t)$ [1] is

$$\overline{G(f)} = \lim_{T \to \infty} \frac{1}{T} \overline{\left| \int_{-T}^{T} x(t) \, e^{st} \, dt \right|^2} , \qquad (2.21)$$

where

$$s = 2\pi jf, \quad \omega > 0 \text{ in } (2.21), \text{ and } f = \frac{\omega}{2\pi} \, .$$

Applying Equations (2.3) through (2.10) to Equation (2.21) with

$$\omega = n\omega_1 \, , \quad f = nf_1 \, , \quad \text{and } f_1 = \frac{1}{2T} \tag{2.22}$$

there is obtained

$$\overline{G(f)} = \lim_{T \to \infty} \frac{1}{T} \left| \int_{-T}^{T} x(t) \cos n\omega_1 t \, dt + j \int_{-T}^{T} x(t) \sin n\omega_1 t \, dt \right|^2$$

which becomes

$$\overline{G(f)} = \lim_{T \to \infty} \frac{1}{T} \left| T a_n + jT b_n \right|^2 = \lim_{T \to \infty} \frac{1}{f_1} \left| c_n \right|^2 . \tag{2.23}$$

Thus, as a function becomes random, the power spectral density will be given by Equation (2.23).

A useful relationship for statistical signals is obtained from the equality[4]

$$\int_{-T}^{T-\tau} x_1(t) x_2(t + \tau) \, dt = \int_{-\infty}^{\infty} x_{T1}(t) x_{T2}(t + \tau) \, dt \, . \tag{2.24}$$

After substitution of the two-sided Laplace integral for $x_{T2}(t + \tau)$ and the inverse Laplace integral $X_{T1}(s)$ in the right side, dividing both sides by $2T$, taking the time average of both sides, integrating the left side, and letting $T$ go to infinity, Equation (2.24) becomes

$$\mathcal{L}_T \left[ \phi_{12}(\tau) \right] = \Phi_{12}(s) = \lim_{T \to \infty} \frac{1}{2T} \, \overline{X_{T1}(-s) X_{T2}(s)} \, . \tag{2.25}$$

And if $x_1(t) \equiv x_2(t)$,

$$\mathcal{L}_T \left[ \phi_{11}(\tau) \right] = \Phi_{11}(s) = \lim_{T \to \infty} \frac{1}{2T} \, \overline{X_{T1}(-s) X_{T1}(s)} \, . \tag{2.26}$$

The spectral density $\Phi_{11}(j\omega)$ is found by replacing $s$ by $j\omega$ in

Equation (2.26).

The power spectral density as given by Equation (2.21) is equivalent to

$$\overline{G(f)} = \lim_{T\to\infty} \frac{1}{T} \overline{X_T(s)\,X_T(-s)} = \lim_{T\to\infty} \frac{1}{T} \overline{\left| X_T(s) \right|^2} \qquad (2.27)$$

where $s = 2\pi jf$. Comparison of (2.26) and (2.27) shows that the power spectral density is

$$\overline{G(f)} = 2\Phi_{11}(f) . \qquad (2.28)$$

Modern control systems textbooks[3, 4, 6, 7, 8, 9] define the spectral density* as $\Phi_{11}(j\omega)$ while the American Standards[1] use $\overline{G(f)}$. No particular difficulty occurs in this if one is consistent, because most results will be normalized in the same manner. The factor of 2 is accounted for in that $\Phi_{11}(f)$ allows negative $f$ while $G(f)$ does not. Thus, both quantities when integrated over the frequency spectrum must give the same average power.

Equation (2.23) gives the relation between discrete and continuous signals. Using Equation (2.28),

$$\Phi_{11}(f) = \lim_{T\to\infty} \frac{1}{2f_1} \left| c_n \right|^2 = \lim_{f_1\to 0} \frac{1}{2f_1} \left| c_n \right|^2 . \qquad (2.29)$$

Equations (2.23) and (2.29) are more symbolic than useful and show how the spectral density may be found as a limiting procedure when the frequency $f_1 \to 0$.

2.7. Output Response for Statistical Inputs.

A system with multiple random inputs is shown in Fig. 1. The

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*Since any function can be resolved into a spectrum, the historic power connotation has been dropped and the term spectral density is used for any function. Thus, force may have a spectral density.
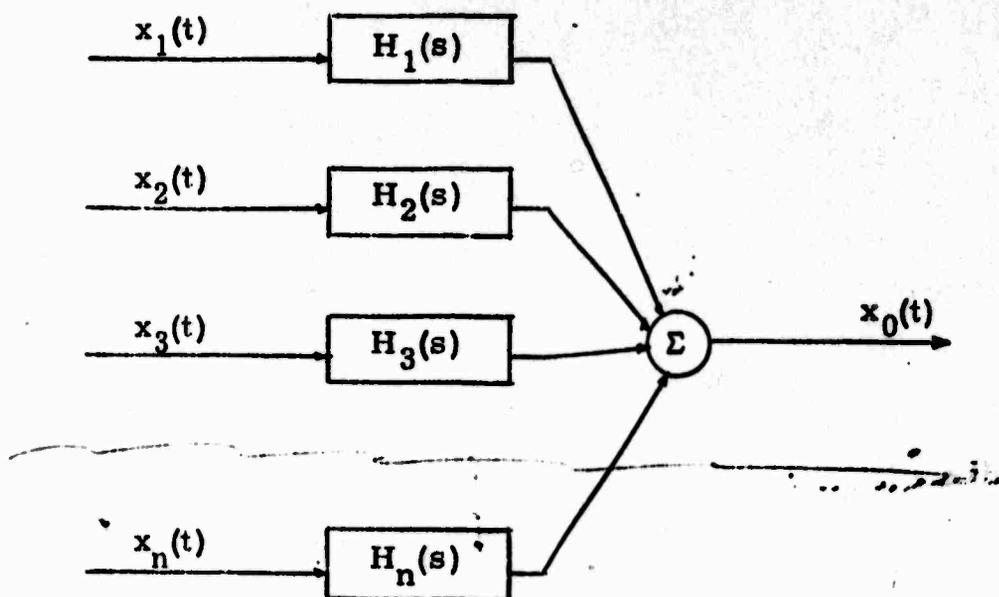
Fig. 1. System with Multiple Random Inputs

inputs are denoted by $x_i(t)$ and the respective system transfer functions by $H_i(s)$, $i = 1, 2, 3, \cdots, n$. After truncating the inputs as in the previous section, the following equations can be written:

$$X_{T0}(s) = \sum_{i=1}^{n} H_i(s) X_{Ti}(s) ; \qquad (2.30)$$

and

$$X_{T0}(-s) = \sum_{k=1}^{n} H_k(-s) X_{Tk}(-s) . \qquad (2.31)$$

Multiplying Equations (2.30) and (2.31) and dividing the results by $2T$,

$$\frac{1}{2T} X_{T0}(s) X_{T0}(-s) = \frac{1}{2T} \sum_{i=1}^{n} \sum_{k=1}^{n} H_i(s) H_k(-s) X_{Ti}(s) X_{Tk}(-s).$$
$$\qquad (2.32)$$

If the time average is taken over the above equation and $T$ is allowed

to approach infinity,

$$\Phi_{00}(j\omega) = \sum_{i=1}^{n} \sum_{k=1}^{n} H_i(j\omega) H_k(-j\omega) \Phi_{ki}(j\omega) \qquad (2.33)$$

where s has been replaced by $j\omega$.

Equation (2.33) can be used to obtain the spectral density of the output in terms of the spectral and cross-spectral densities of the inputs and the system functions. As an illustrative example, a single input system is analyzed.

Single input:

$$\Phi_{00}(j\omega) = \sum_{i=1}^{1} \sum_{k=1}^{1} H_i(j\omega) H_k(-j\omega) \Phi_{ki}(j\omega)$$

$$= \sum_{i=1}^{1} H_i(j\omega) H_1(-j\omega) \Phi_{1i}(j\omega)$$

$$= H_1(j\omega) H_1(-j\omega) \Phi_{11}(j\omega)$$

$$= \left| H_1(j\omega) \right|^2 \Phi_{11}(j\omega) . \qquad (2.34)$$

## 2.8.  Spectral Density and Amplitude-Frequency.

An amplitude-frequency plot of a process as obtained from a wave analyzer gives the absolute value of the Fourier coefficients $c_n$ for a bandwidth $\Delta\omega$ with center frequency $\omega_n$.

From Equation (2.26), for $\tau = 0$ and $s = j\omega$,

$$\phi_{11}(0) = \overline{x_1(t)^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_{11}(j\omega) \, d\omega . \qquad (2.35)$$

If $x_1(t)$ is represented by the complex Fourier series,

$$\overline{x_1(t)^2} = \frac{|c_n|^2}{2} \quad . \tag{2.36}$$

Then, for the bandwidth $\Delta\omega$ with center frequency $\omega_n$, Equation (2.35) becomes

$$\frac{|c_n|^2}{2} = \frac{1}{2\pi} \int_{\omega_n - \frac{\Delta\omega}{2}}^{\omega_n + \frac{\Delta\omega}{2}} \Phi_{11}(j\omega_n) \, d\omega \quad . \tag{2.37}$$

Carrying out the integration,

$$\Phi_{11}(j\omega_n) = \frac{\pi |c_n|^2}{\Delta\omega} \quad . \tag{2.38}$$

Thus, Equation (2.34) of the preceding section can be written

$$|c_n|_0 = |H_1(j\omega_n)| \, |c_n|_1 \tag{2.39}$$

where $|c_n|_1$ is the amplitude of the input $x_1(t)$ for some bandwidth $\Delta\omega$ with center frequency $\omega_n$; $|H_1(j\omega_n)|$ is the absolute value of the transfer function evaluated at $\omega = \omega_n$; and $|c_n|_0$ is the amplitude of the output $x_0(t)$ for the bandwidth $\Delta\omega$ with center frequency $\omega_n$.

## The Physical System

### 2.9. Introduction.

The following material contains the derivation of the differential equations describing compressible, turbulent flow in a non-rigid cylinder. The derivation is based on certain idealizing conditions that are pointed out and later discussed. Using Laplace transform methods, the frequency response solutions of the describing equations are obtained. The results of this investigation are applied in the next chapter to determine the response of a complex piping system, such as

shown in Fig. 2, to random inputs.

## 2.10. Derivation of Describing Equations.

Fig. 3 depicts a liquid filled cylinder with non-rigid walls containing compressible, turbulent flow. The necessary idealizing conditions follow and hereafter shall be referred to by number:

1. The flow is one-dimensional.

2. The frictional resistance is a function of some nonlinear operation on the fluid velocity and system parameters. It can be expressed as the product of a constant and the instantaneous flow rate variation.

3. The energy dissipation is due to heat transfer and is accounted for by Condition 2.

4. The rate of change in fluid velocity with respect to length is negligible as compared to its rate of change with respect to time.

5. The instantaneous pressure is constant over the cross section.

6. The rate of change in mass density with respect to length is negligible as compared to its rate of change with respect to time.

7. The velocity of wave propagation in the liquid cylinder is constant.

Using Conditions 1, 2, 3, 4, and 6, the fluid dynamical relationships completely describing the system are[10]:

Equation of Motion

$$\frac{\partial u_t}{\partial t} - \frac{1}{\rho} \frac{\partial p_t}{\partial x} + G u_t^n = 0 \; ; \tag{2.39}$$
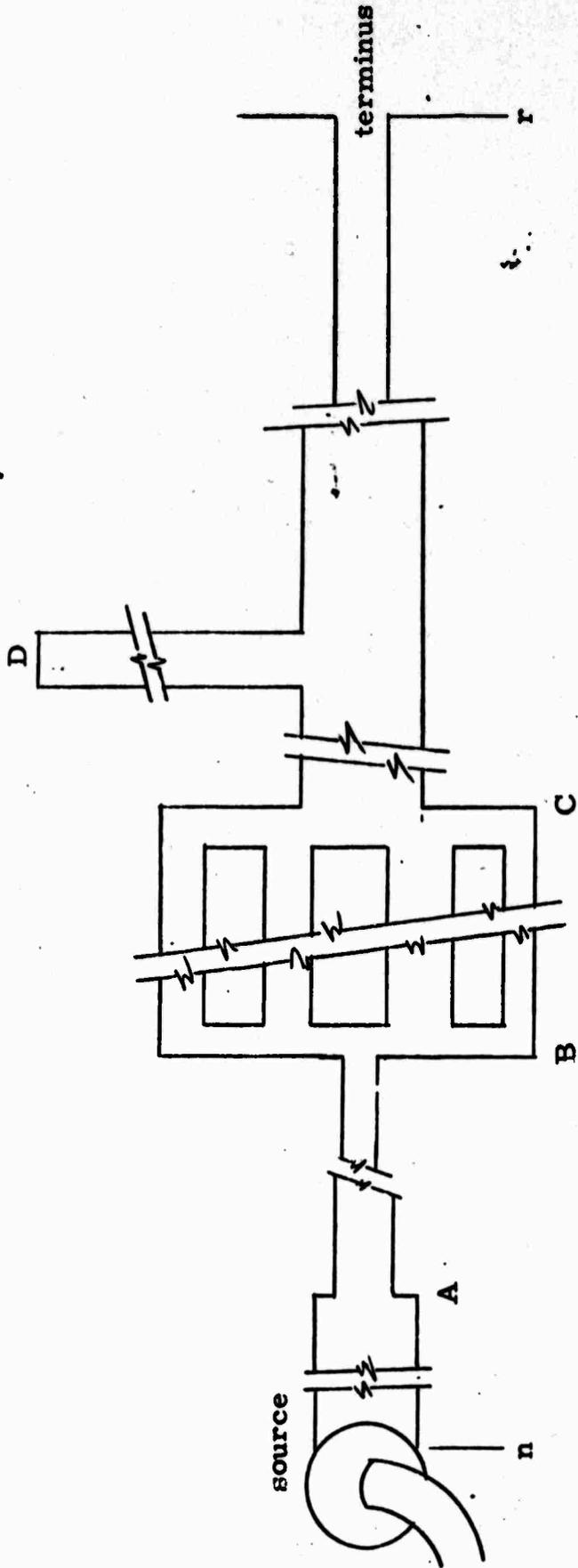
Fig. 2. A Complex Piping System
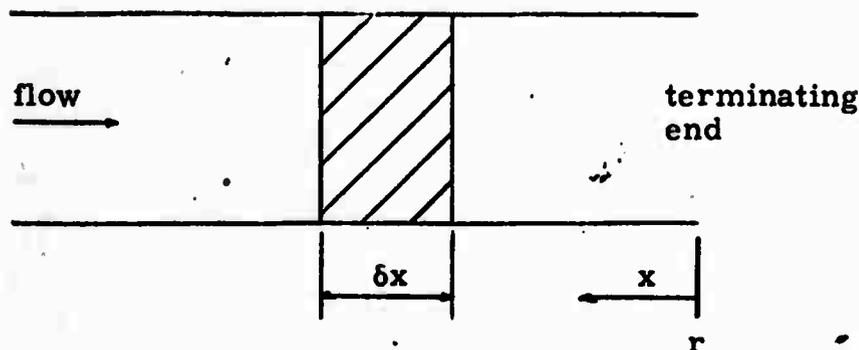
Fig. 3. The Physical System

Equation of Continuity

$$- \rho \frac{\partial u_t}{\partial x} + \frac{\partial \rho}{\partial t} = 0 \; ; \qquad (2.40)$$

Equation of State

$$\partial p_t = K' \frac{\partial \rho}{\rho} \qquad (2.41)$$

where K' is the bulk modulus of the system, i.e., the fluid and cylinder walls combined.

Equations (2.40) and (2.41) are combined to give

$$- \frac{\partial u_t}{\partial x} + \frac{1}{K'} \frac{\partial p_t}{\partial t} = 0 \; . \qquad (2.42)$$

Integrating Equations (2.39) and (2.42) over a control volume $A\delta x$,

$$- \frac{\partial p_t}{\partial x} + \frac{\rho}{A} \frac{\partial q_t}{\partial t} + B q_t^n = 0 \; , \qquad (2.43)$$

and

$$- \frac{\partial q_t}{\partial x} + \frac{A}{K'} \frac{\partial p_t}{\partial t} = 0 \; . \qquad (2.44)$$

When the volume flow rate is constant at a cross section, $q_t = \bar{q}$, Equation (2.43) reduces to the slope of the pressure grade

line, or

$$\frac{\partial \bar{p}}{\partial x} = B\bar{q}^n = \frac{f\rho \bar{q}^2}{2DA^2} . \tag{2.45}$$

Then,

$$B = \frac{f\rho \bar{q}^{2-n}}{2DA^2} = \frac{\bar{p}_f}{\ell \bar{q}^n} \tag{2.46}$$

where $\bar{p}_f$ is the pressure needed to overcome the frictional resistance of the pipe as determined from elementary fluid mechanics and $\bar{q}$ is the mean flow. The value of $n$ is estimated from experience or experiment and should range from 1.65 to 2.05.

The nonlinear term of Equation (2.43) is rewritten and expanded as follows:

$$q_t^n = (\bar{q} + q)^n$$

$$= \bar{q}^n (1 + \frac{q}{\bar{q}})^n$$

$$= \bar{q}^n \left[ 1 + n(\frac{q}{\bar{q}}) + \frac{n(n-1)}{2!} (\frac{q}{\bar{q}})^2 + \cdots \right] . \tag{2.47}$$

The ratio $\frac{q}{\bar{q}}$ is small, and the series in Equation (2.47) is sufficiently approximated by its first two terms. Then, Equation (2.43) becomes

$$-\frac{\partial p_t}{\partial x} + \frac{\rho}{A} \frac{\partial q_t}{\partial t} + \frac{\bar{p}_f}{\ell} + \frac{n\bar{p}_f}{\ell \bar{q}} q = 0 . \tag{2.48}$$

Since $q_t = \bar{q} + q$ and $p_t = \bar{p} + p$, Equations (2.44) and (2.48) can be reduced to the describing differential equations for the variations in flow rate and pressure. They are:

$$-\frac{\partial p}{\partial x} + \frac{\rho}{A} \frac{\partial q}{\partial t} + \frac{n\bar{p}_f}{\ell \bar{q}} q = 0 ; \tag{2.49}$$

and

$$-\frac{\partial q}{\partial x} + \frac{A}{K'}\frac{\partial p}{\partial t} = 0 \; . \tag{2.50}$$

It is noted that the above equations can be combined to form the classical acoustical wave equation including a dissipation term.

## 2.11. Idealizing Conditions.

The idealizing conditions listed in the previous section are discussed and validated to varying degrees as follows:

1. In any real piping system, a propagated wave is never one-dimensional, because the elasticity of the pipe walls allows radial motion. But over a certain frequency range from 0 to $\omega_c$ radians per second there exists only one propagated wave, or one mode of propagation, which is the axial, or plane wave, mode. This mode is denoted the $(0, 0)$ mode. The cut-off frequency $\omega_c$ at which the first radial mode of propagation occurs $\left[\text{the }(0, 1)\text{ mode}\right]$ for non-dissipative conditions can be determined by the following equations given by Jacobi[11]:

$$\frac{J_0(z)}{zJ_1(z)} = \frac{2\rho_2 b}{\rho_1 D} - \frac{2Eb}{\rho_1 D c^2 z^2} \; ; \tag{2.51}$$

and

$$\omega_c = \frac{2c}{D} z_{01} \tag{2.52}$$

where $z$ is some characteristic value, $J_0$ and $J_1$ are Bessel functions of the first kind, and $\rho_1$ and $\rho_2$ are the mass densities of the fluid and pipe material, respectively. The first real characteristic value $z_{01}$ that satisfies Equation (2.51) is substituted into Equation (2.52) to obtain $\omega_c$. For 8" standard steel pipe, $f_c = 6900$ cycles per second. For smaller sizes of pipe the value of $\omega_c$ increases substantially. To the best knowledge of the author, the subject of cut-off

frequencies for dissipative conditions is nonexistent in the literature. The cut-off frequency should be somewhat higher for dissipative conditions than that given by Equations (2.51) and (2.52). It is concluded that for a relatively rigid pipe, the one-dimensional condition is valid for frequencies up to the cut-off frequency.

2. The perturbation process used in the derivation of the describing equation to linearize the frictional dissipation term was proposed and verified experimentally for low frequencies by Waller[12]. There seems to be no good reason for discounting its validity in the frequency range of one-dimensional propagation.

3. The effect of heat transfer on a propagated wave could be considered resulting in a third describing differential equation as done by Brown[13], but it is believed that the rather empirical fricitional dissipation term used in the describing equations will account for this effect.

4. The condition that the rate of change in fluid velocity with respect to length is negligible has been used for many years in describing wave propagation in liquid cylinders. Phillips[14] in showing the acoustic wave equation adequate for describing turbulent flow in liquid-filled steel pipes has shown this term to be negligible.

5. If the one-dimensional wave condition is valid, it follows that the instantaneous pressure is essentially uniform over the cross section.

6. As in Condition 4, the rate of change in mass density with respect to length has classically been considered negligible.

7. Strictly speaking, the velocity of wave propagation in a liquid cylinder with non-rigid walls is not constant with frequency. With

increasing frequency, the flexural vibrations of the walls tend to impede the propagation causing a decrease in the wave velocity. Mathematical expressions to determine the velocity of wave propagation as a function of frequency are given in the paper by Jacobi[11]. For a relatively rigid pipe, such as a standard steel pipe, the velocity of wave propagation in the pipe, a, is essentially constant, and

$$a = \sqrt{\frac{\rho}{K'}} \qquad (2.53)$$

where[15]

$$K' = \frac{KbE}{KD + bE} . \qquad (2.54)$$

Rather than calculating the velocity of propagation by Equations (2.53) and (2.54), it can be determined for different pipe sizes and fluids from nomographs available in many papers and books[16].

## 2.12. Solution of the Differential Equations.

The describing differential Equations (2.49) and (2.50) for the variations in pressure and volume flow rate are:

$$-\frac{\partial p(x, t)}{\partial x} + L \frac{\partial q(x, t)}{\partial t} + R q(x, t) = 0 ; \qquad (2.55)$$

and

$$-\frac{\partial q(x, t)}{\partial x} + C \frac{\partial p(x, t)}{\partial t} = 0 \qquad (2.56)$$

where

$$L = \frac{\rho}{A} ; \qquad (2.57)$$

$$R = \frac{n \bar{p}_f}{\ell \bar{q}} ; \qquad (2.58)$$

and

$$C = \frac{A}{K'} = \frac{1}{La^2} . \qquad (2.59)$$

Equations (2.55) and (2.56) can be solved by Laplace transform methods for solutions in the frequency domain.

Transforming Equations (2.55) and (2.56) with the Laplace integral

$$F(s) = \int_0^\infty f(t) e^{-st} dt ,$$

$$-\frac{\partial P(x,s)}{\partial x} + (R + sL) Q(x,s) = 0 , \tag{2.60}$$

and

$$-\frac{\partial Q(x,s)}{\partial x} + s CP(x,s) = 0 \tag{2.61}$$

where $p(x,0)$ and $q(x,0)$ are zero. Transforming the above equations with the Laplace integral

$$\hat{F}(\lambda) = \int_0^\infty f(x) e^{-\lambda x} dx ,$$

$$-\lambda \hat{P}(\lambda,s) + (R + sL) \hat{Q}(\lambda,s) = -P(0,s) , \tag{2.62}$$

and

$$-\lambda \hat{Q}(\lambda,s) + sC \hat{P}(\lambda,s) = -Q(0,s) . \tag{2.63}$$

Solving Equations (2.62) and (2.63) simultaneously for $\hat{P}(\lambda,s)$ and $\hat{Q}(\lambda,s)$ ,

$$\hat{P}(\lambda,s) = Z_c Q(r,s)\left(\frac{\gamma}{\lambda^2 - \gamma^2}\right) + P(r,s)\left(\frac{\lambda}{\lambda^2 - \gamma^2}\right) , \tag{2.64}$$

and

$$\hat{Q}(\lambda,s) = \frac{P(r,s)}{Z_c}\left(\frac{\gamma}{\lambda^2 - \gamma^2}\right) + Q(r,s)\left(\frac{\lambda}{\lambda^2 - \gamma^2}\right) \tag{2.65}$$

where

$$\gamma^2 = sC(R + sL) , \tag{2.66}$$

and

$$Z_c^2 = \frac{R + sL}{sC} \, .$$ (2.67)

Evaluating Equations (2.64) and (2.65) with the inversion integral

$$f(x) = \frac{1}{2\pi j} \int_{\epsilon-j\infty}^{\epsilon+j\infty} \hat{F}(\lambda) \, e^{\lambda t} \, d\lambda \, ,$$

$$P(x,s) = P(r,s) \cosh \gamma x + Z_c \, Q(r,s) \sinh \lambda x \, ,$$ (2.68)

and

$$Q(x,s) = Q(r,s) \cosh \gamma x + \frac{P(r,s)}{Z_c} \sinh \gamma x$$ (2.69)

The frequency response of the system is obtained by replacing $s$ by $j\omega$ in Equations (2.66) through (2.69).

The propagation coefficient, $\gamma(j\omega)$, can be expressed in the form $\alpha + j\beta$ by finding the roots of Equation (2.66). The attenuation constant, $\alpha$, is given by

$$\alpha = \left[ \frac{\omega C}{2} ( \sqrt{R^2 + \omega^2 L^2} - \omega L) \right]^{\frac{1}{2}} \, .$$ (2.70)

The phase constant, $\beta$, is given by

$$\beta = \left[ \frac{\omega C}{2} ( \sqrt{R^2 + \omega^2 L^2} + \omega L) \right]^{\frac{1}{2}} \, .$$ (2.71)

The characteristic impedance, $Z_c(j\omega)$, can also be expressed in complex form by finding the roots of Equation (2.67). It is given by

$$Z_c = \frac{1}{\omega C} (\beta - j\alpha) \, .$$ (2.72)

By the same investigations as for $\gamma(j\omega)$ and $Z_c(j\omega)$, it can be shown that

$$\gamma(-j\omega) = \alpha - j\beta \, ,$$ (2.73)

and

$$Z_c(-j\omega) = \frac{1}{\omega C} (\beta + j\alpha) \, .$$ (2.74)

# CHAPTER III

## TRANSFER FUNCTION ANALYSIS

### 3.1. Mathematical Statement of the Problem.

The system which is symbolically depicted in Fig. 2 encompasses most any situation expected in shipboard piping systems. The noise source at n is a pump (or valve) which discharges into a complex system with typical changes in size as indicated by A, with looped lines (B to C), with stub lines (D), and a termination at r.

The fundamental problem is to specify the allowable noise levels of pressure, flow, and power at n or at some intermediate point which will not exceed allowable levels at the terminus r for a multitude of piping configurations n to r. It is expected that in the overall analysis it will be advantageous to express the noise levels in terms of known levels of pressure and flow at either end n or r. This requires determination of transfer relationships for the various cases.

The transformed pressure, flow, and power at a point $x_i$ in the pipe i are defined as follows:

$$P(x_i, s) = E_{pk}(x_i, s) P_k(s) , \tag{3.1}$$

and

$$P(x_i, s) = Z_{qk}(x_i, s) Q_k(s) ; \tag{3.2}$$

$$Q(x_i, s) = E_{qk}(x_i, s) Q_k(s) , \tag{3.3}$$

and

$$Q(x_i, s) = Y_{pk}(x_i, s) P_k(s) ; \tag{3.4}$$

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$$W(x_i, s) = F_{pk}(x_i, s) \left| P_k(s) \right|^2 \, , . \tag{3.5}$$

and

$$W(x_i, s) = F_{qk}(x_i, s) \left| Q_k(s) \right|^2 \, ; \tag{3.6}$$

where $k$ can be either the source $n$ or the terminus $r$ .

From the discussion preceding Equation (2.38) it is evident that the transfer functions $E$ , $Y$ , $Z$ , and $F$ are needed to relate the various spectra. Thus, the primary objective of this chapter is to determine these transfer functions and to place them in the best form for analysis.

Before proceeding to a discussion of transfer functions, the method of determination of power will be indicated.

The sound power at any point $x$ is found[7] as

$$\phi_{pq}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} p(x, t) \, q(x, t+\tau) \, dt \tag{3.7}$$

where the average power

$$W_{ave} = \phi_{pq}(0) = \frac{1}{2\pi j} \int_{-\infty}^{\infty} \Phi_{pq}(j\omega) \, d(j\omega) \, . \tag{3.8}$$

The power associated with any given frequency is found by observation of the spectrum of the power either in the form of a continuous plot,

$$\Phi_w(j\omega) = \lim_{T \to \infty} \frac{1}{2T} \overline{Q(x, j\omega) \, P(x, -j\omega)} \, , \tag{3.9}$$

or in terms of the discrete Fourier coefficients,

$$\Phi_w(j\omega_n) = \left[ \frac{\pi \left| c_n \right|^2 (power)}{\Delta \omega} \right]_{\Delta \omega \to 0} \, . . \tag{3.10}$$

From the preceding discussion, it is evident that once $P(x,s)$ and $Q(x,s)$ are known the power is obtained directly.

## 3.2. Transfer Function Determination.

To determine the overall transfer functions it is first necessary to work with a specific pipe in which the fundamental parameters $R$, $L$, and $C$ are constant with respect to that pipe. The relationships between the boundary conditions and pressure and flow at any point in the subsystem $i$, as shown in Fig. 4, are given by Equations (3.11) and (3.12)[18].

$$P(x_i, s) = P_i(s) \cosh \gamma_i x_i + Z_{ci} Q_i(s) \sinh \gamma_i x_i , \qquad (3.11)$$

and

$$Q(x_i, s) = Q_i(s) \cosh \gamma_i x_i + \frac{P_i(s)}{Z_{ci}} \sinh \gamma_i x_i . \qquad (3.12)$$
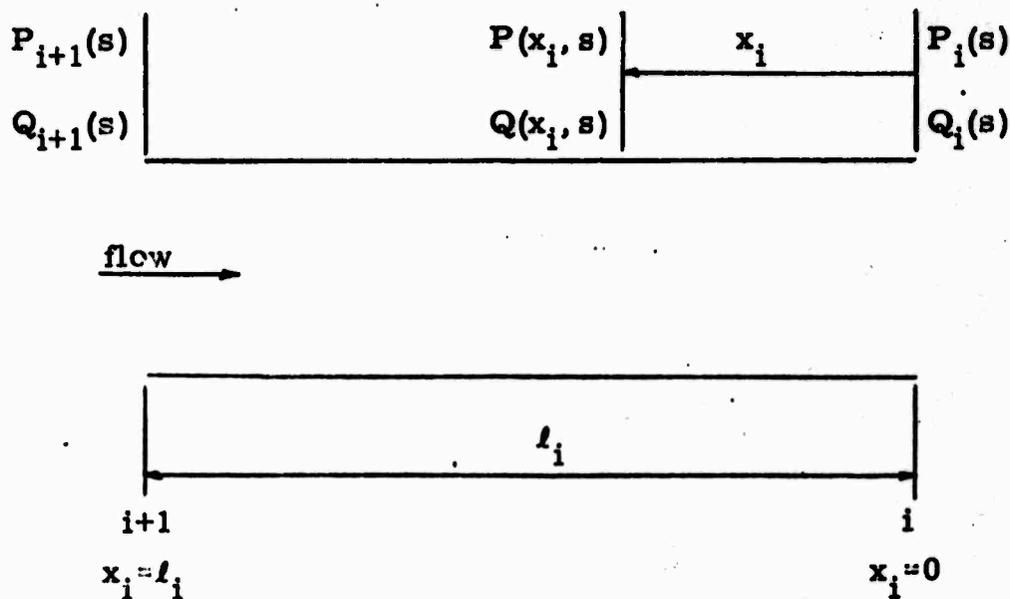


Fig. 4. Notation Sketch for the Pipe $i$

The calculations proceed from some known termination by successive application of Equations (3.11) and (3.12) with the notation

$$\left. \begin{array}{l} P_i(0,s) = P_i(s) \\ Q_i(0,s) = Q_i(s) \end{array} \right\} \text{ for } x_i = 0$$

$$\left. \begin{array}{l} P_i(\ell_i,s) = P_{i+1}(0,s) = P_{i+1}(s) \\ Q_i(\ell_i,s) = Q_{i+1}(0,s) = P_{i+1}(s) \end{array} \right\} \text{ for } x_i = \ell_i \quad (3.13)$$

Further,

$$\frac{P(x_i,s)}{Q(x_i,s)} = Z(x_i,s) \tag{3.14}$$

from which

$$\frac{P_i(0,s)}{Q_i(0,s)} = Z_i(0,s) = Z_i(s) . \tag{3.15}$$

Another term which will be useful in transfer function determination is the reflection coefficient

$$\Gamma_i = \frac{Z_i - Z_{ci}}{Z_i + Z_{ci}} = \left|\Gamma_i\right| e^{j\theta_i} \tag{3.16}$$

A further consideration is that the frequency response is all that is needed for spectral analysis; therefore, in all that is to follow

$$s = j\omega . \tag{3.17}$$

For the section $i$, the transfer relations are:

$$P(x_i,j\omega) = Q_i(j\omega)\left[Z_i \cosh \gamma_i x_i + Z_{ci} \sinh \gamma_i x_i\right] , \tag{3.18}$$

and

$$Q(x_i,j\omega) = Q_i(j\omega)\left[\cosh \gamma_i x_i + \frac{Z_i}{Z_{ci}} \sinh \gamma_i x_i\right] . \tag{3.19}$$

In an alternate form[19,20],

$$P(x_i, j\omega) = Q_i(j\omega) \, Z_{ci} \, e^{\gamma_i x_i} \left[ \frac{1 + \Gamma_i \, e^{-2\gamma_i x_i}}{1 - \Gamma_i} \right] , \qquad (3.20)$$

and

$$Q(x_i, j\omega) = Q_i(j\omega) \, e^{\gamma_i x_i} \left[ \frac{1 - \Gamma_i \, e^{-2\gamma_i x_i}}{1 - \Gamma_i} \right] . \qquad (3.21)$$

Proper application of Equations (3.15), (3.18), and (3.19) yields one method of calculating transfer functions. This will be termed Method A. A second method uses Equations (3.15), (3.20), and (3.21). This is Method B. Both methods have merit, and either would be sufficient provided adequate computer facilities are available.

### 3.3. Method A - Transfer Function Analysis[17].

This method uses Equations (3.15), (3.18), and (3.19) successively in a complex system. The system is shown in Fig. 5, in which a typical section is shown in Fig. 4.
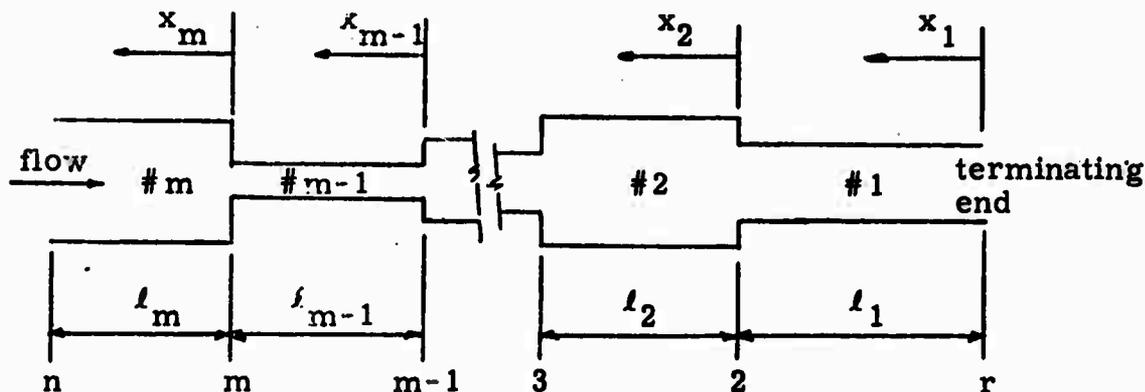


Fig. 5. A Series Piping System

The following notation is employed:

$$B_i(j\omega) = \cosh \gamma_i x_i \; ; \tag{3.22}$$

$$B_i'(j\omega) = \cosh \gamma_i \ell_i \; ;$$

$$T_i(j\omega) = Z_{ci} \sinh \gamma_i x_i \; ; \tag{3.23}$$

$$T_i'(j\omega) = Z_{ci} \sinh \gamma_i \ell_i \; ;$$

$$M_i(j\omega) = \frac{1}{Z_{ci}} \sinh \gamma_i x_i \; ; \tag{3.24}$$

$$M_i'(j\omega) = \frac{1}{Z_{ci}} \sinh \gamma_i \ell_i \; ;$$

where $i$ refers to the pipe number, $i = 1, 2, 3, \cdots, m$. Then, from Equations (3.18) and (3.19),

$$P(x_i, j\omega) = B_i(j\omega) P_i(j\omega) + T_i(j\omega) Q_i(j\omega) \; , \tag{3.25}$$

and

$$Q(x_i, j\omega) = B_i(j\omega) Q_i(j\omega) + M_i(j\omega) P_i(j\omega) \; . \tag{3.26}$$

Since

$$P_k(j\omega) = B_{k-1}'(j\omega) P_{k-1}(j\omega) + T_{k-1}'(j\omega) Q_{k-1}(j\omega) \tag{3.27}$$

and

$$Q_k(j\omega) = B_{k-1}'(j\omega) Q_{k-1}(j\omega) + M_{k-1}'(j\omega) P_{k-1}(j\omega) \; , \tag{3.28}$$

successive substitution for $P_k(j\omega)$ and $Q_k(j\omega)$, $k = i, i-1, \cdots, 2$, in Equations (3.25) and (3.26) will yield $P(x_i, j\omega)$ and $Q(x_i, j\omega)$ in the form

$$P(x_i, j\omega) = G_i(j\omega) Q_r(j\omega) \; , \tag{3.29}$$

$$P(x_i, j\omega) = \frac{G_i(j\omega)}{Z_r} P_r(j\omega) \; ,$$

$$Q(x_i, j\omega) = H_i(j\omega) \, Q_r(j\omega) \, , \qquad \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} \quad (3.30)$$

$$Q(x_i, j\omega) = \frac{H_i(j\omega)}{Z_r} \, P_r(j\omega) \, .$$

The transfer functions $G_i(j\omega)$ and $H_i(j\omega)$, $i = 2, 3, \cdots, m$, are given by

$$G_i = B_i \, G_{i-1}' + T_i \, H_{i-1}' \, ,$$

and

$$H_i = B_i \, H_{i-1}' + M_i \, G_{i-1}' \, , \qquad \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} \quad (3.31)$$

where

$$G_{i-1}' = B_{i-1}' \, G_{i-2}' + T_{i-1}' \, H_{i-2}' \, ,$$

and

$$H_{i-1}' = B_{i-1}' \, H_{i-2}' + M_{i-1}' \, G_{i-2}' \, , \qquad \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} \quad (3.32)$$

etc. The transfer functions for $i = 1$ at some point $x_1$ are

$$G_1 = \frac{B_1}{Z_r} + T_1 \, ,$$

and

$$H_1 = B_1 + \frac{M_1}{Z_r} \, . \qquad \left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} \quad (3.33)$$

For $x_1 = \ell_1$,

$$G_1' = \frac{B_1'}{Z_r} + T_1' \, ,$$

and

$$H_1' = B_1' + \frac{M_1'}{Z_r} \, . \qquad \left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} \quad (3.34)$$

Then, using the results of Equations (3.34) in Equations (3.31), the transfer functions for $i = 2$ at some point $x_2$ are

$$G_2 = B_2 \left[ \frac{B_1'}{Z_r} + T_1' \right] + T_2 \left[ B_1' + \frac{M_1'}{Z_r} \right] ,$$

and

$$H_2 = B_2 \left[ B_1' + \frac{M_1'}{Z_r} \right] + M_2 \left[ \frac{B_1'}{Z_r} + T_1' \right] . \qquad (3.35)$$

For $x_2 = l_2$ ,

$$G_2' = B_2' \left[ \frac{B_1'}{Z_r} + T_1' \right] + T_2' \left[ B_1' + \frac{M_1'}{Z_r} \right] ,$$

and

$$H_2' = B_2' \left[ B_1' + \frac{M_1'}{Z_r} \right] + M_2' \left[ \frac{B_1'}{Z_r} + T_1' \right] . \qquad (3.36)$$

This theory can be extended to any number of series components, but the transfer function expressions become increasingly complicated as the number of components increases.

The analysis of a complex piping system using Method A would proceed back from some termination with successive calculation of the transfer functions at all points and frequencies of interest including the junctions. Thus, Equations (3.31) would be used in the calculation of the transfer functions rather than expressions such as Equations (3.35). Because of this, Method A provides an easy way of computing the many transfer functions needed to completely analyze a series piping system. For any preliminary analysis, such as determining maximums and minimums, or for an insight into the overall performance of the system, Method A gives undesirable results. These things are more easily obtained from the results of Method B.

### 3.4.  Method B - Transfer Function Analysis.

This method makes use of Equations (3.15), (3.20), and (3.21) which are applied to one pipe to obtain the results in a form where the maxima and minima are more easily identified.  It may also reduce the computation time and make the analysis more amenable.

Taking the absolute values of Equations (3.20) and (3.21),

$$\left| P(x_i, j\omega) \right| \;=\; \left| Q_i(j\omega) \right| \left| Z_{ci} \right| \, e^{\alpha_i x_i} \left\{ \frac{1 + A_i^2 + 2A_i \cos \psi_i}{1 + \left|\Gamma_i\right|^2 - 2\left|\Gamma_i\right| \cos \theta_i} \right\}^{\frac{1}{2}} , \tag{3.37}$$

and

$$\left| Q(x_i, j\omega) \right| \;=\; \left| Q_i(j\omega) \right| \, e^{\alpha_i x_i} \left\{ \frac{1 + A_i^2 - 2 A_i \cos \psi_i}{1 + \left|\Gamma_i\right|^2 - 2\left|\Gamma_i\right| \cos \theta_i} \right\}^{\frac{1}{2}} \tag{3.38}$$

where

$$A_i \;=\; \left|\Gamma_i\right| e^{-2\alpha_i x_i} , \tag{3.39}$$

$\theta_i$ is defined in Equation (3.16), and

$$\psi_i \;=\; -2\beta_i x_i + \theta_i . \tag{3.40}$$

For $x_k = \ell_k$ , Equations (3.37) and (3.38) become

$$\left| P_{k+1}(j\omega) \right| \;=\; \left| Q_k(j\omega) \right| \left| Z_{ck} \right| e^{\alpha_k \ell_k} \left\{ \frac{1 + A_k'^2 + 2A_k' \cos \psi_k'}{1 + \left|\Gamma_k\right|^2 - 2\left|\Gamma_k\right| \cos \theta_k} \right\}^{\frac{1}{2}} , \tag{3.41}$$

and

$$\left| Q_{k+1}(j\omega) \right| \;=\; \left| Q_k(j\omega) \right| e^{\alpha_k \ell_k} \left\{ \frac{1 + A_k'^2 - 2A_k' \cos \psi_k'}{1 + \left|\Gamma_k\right|^2 - 2\left|\Gamma_k\right| \cos \theta_k} \right\}^{\frac{1}{2}} \tag{3.42}$$

where $A_k'$ and $\psi_k'$ are defined by Equations (3.39) and (3.40) with $i = k$ and $x_k = \ell_k$ .

Then,

$$\left|P(x_i, j\omega)\right| = K_{pi} \left|Q_i(j\omega)\right| , \tag{3.43}$$

$$\left|Q(x_i, j\omega)\right| = K_{qi} \left|Q_i(j\omega)\right| , \tag{3.44}$$

$$\left|P_{k+1}(j\omega)\right| = K'_{pk} \left|Q_k(j\omega)\right| , \tag{3.45}$$

and

$$\left|Q_{k+1}(j\omega)\right| = K'_{qk} \left|Q_k(j\omega)\right| \tag{3.46}$$

where $K_{pi}$, $K_{qi}$, $K'_{pk}$, and $K'_{qk}$ are defined by Equations (3.37), (3.38), (3.41), and (3.42), respectively. Successive substitution of Equation (3.46) for $Q_k(j\omega)$, $k = i, i-1, \cdots, 1$, in Equations (3.43) and (3.44) yields

$$\left|P(x_i, j\omega)\right| = \left|Q_r(j\omega)\right| K_{pi} \prod_{k=i-1}^{1} K'_{qk} = \left|Q_r(j\omega)\right| \overline{K}_{pi} \tag{3.47}$$

and

$$\left|Q(x_i, j\omega)\right| = \left|Q_r(j\omega)\right| K_{qi} \prod_{k=i-1}^{1} K'_{qk} = \left|Q_r(j\omega)\right| \overline{K}_{qi} \tag{3.48}$$

where $\overline{K}_{pi}$ and $\overline{K}_{qi}$ are identical to $\left|G_i(j\omega)\right|$ and $\left|H_i(j\omega)\right|$, respectively.

To calculate the various $\Gamma_k$'s, the point impedances,

$$Z_k(j\omega) = \left|\frac{P_k(j\omega)}{Q_k(j\omega)}\right| e^{j\phi_k} , \tag{3.49}$$

must be determined. Equations such as (3.41) and (3.42) give the absolute value of $Z_k$. Equations such as (3.20) and (3.21) give the phase angle as

$$\phi_k = \phi_{c(k-1)} + \tan^{-1} \frac{A'_{k-1} \sin \psi'_{k-1}}{1 + A'_{k-1} \cos \psi'_{k-1}}$$

$$- \tan^{-1} \frac{-A'_{k-1} \sin \psi'_{k-1}}{1 - A'_{k-1} \cos \psi'_{k-1}} \qquad (3.50)$$

where $\phi_{c(k-1)}$ is the phase angle of $Z_{c(k-1)}$. Reducing Equation (3.50),

$$\phi_k = \phi_{c(k-1)} + \tan^{-1} \frac{2A'_{k-1} \sin \psi'_{k-1}}{1 - A'^2_{k-1}} . \qquad (3.51)$$

Calculations start at the terminus $r$ with the necessary values known. The relationships derived in this section are used to compute the $K$ functions for any point in the system. With these, the desired spectrums can be determined.

### 3.5: Piping Systems with Parallel Components.

In Fig. 6 a piping system is shown containing pipes in parallel. The parallel components are not necessarily dimensionally or materially identical. The system to the right and left of the parallel pipes consists of either a single pipe or a series of pipes.
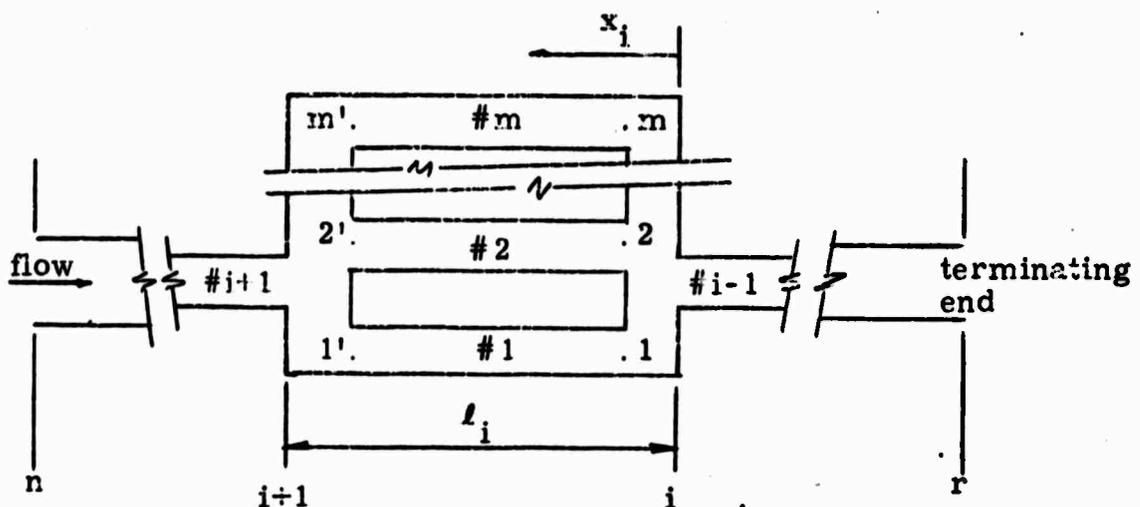


Fig. 6. Piping System with Parallel Components

From Section 3.3,

$$P_i(j\omega) = G'_{i-1}(j\omega) \, Q_r(j\omega) \tag{3.52}$$

and

$$Q_i(j\omega) = H'_{i-1}(j\omega) \, Q_r(j\omega) \,. \tag{3.53}$$

To analyze the complete system, $P_{i+1}(j\omega)$ and $Q_{i+1}(j\omega)$ must be found in terms of a transfer function and $Q_r(j\omega)$.

The following notation is employed:

$$B'_k(j\omega) = \cosh \gamma_k \ell_k \,; \tag{3.54}$$

$$T'_k(j\omega) = Z_{ck} \sinh \gamma_k \ell_k \,; \tag{3.55}$$

and

$$M'_k(j\omega) = \frac{1}{Z_{ck}} \sinh \gamma_k \ell_k \tag{3.56}$$

where $k$ refers to the parallel pipe number, $k = 1, 2, \cdots, m$. The relationships

$$P_i(j\omega) = P_1(j\omega) = P_2(j\omega) = \cdots = P_m(j\omega) \,, \tag{3.57}$$

$$Q_i(j\omega) = Q_1(j\omega) + Q_2(j\omega) + \cdots + Q_m(j\omega) \,, \tag{3.58}$$

$$P_{i+1}(j\omega) = P_{1'}(j\omega) = P_{2'}(j\omega) = \cdots = P_{m'}(j\omega) \,, \tag{3.59}$$

and

$$Q_{i+1}(j\omega) = Q_{1'}(j\omega) + Q_{2'}(j\omega) + \cdots + Q_{m'}(j\omega) \tag{3.60}$$

are valid where the subscripts $1, 2, \cdots, m$ and $1', 2', \cdots, m'$ refer to the points shown in Fig. 6. Using Equations (3.57) and (3.59),

$$\left.\begin{aligned}
P_{i+1}(j\omega) &= B'_1(j\omega) \, P_i(j\omega) + T'_1(j\omega) \, Q_1(j\omega) \,; \\
P_{i+1}(j\omega) &= B'_2(j\omega) \, P_i(j\omega) + T'_2(j\omega) \, Q_2(j\omega) \,; \\
&\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
P_{i+1}(j\omega) &= B'_m(j\omega) \, P_i(j\omega) + T'_m(j\omega) \, Q_m(j\omega) \,.
\end{aligned}\right\} \tag{3.61}$$

Also,

$$Q_{1'}(j\omega) = B_1^!(j\omega)\, Q_1(j\omega) + M_1^!(j\omega)\, P_i(j\omega) \;;$$

$$Q_{2'}(j\omega) = B_2^!(j\omega)\, Q_2(j\omega) + M_2^!(j\omega)\, P_i(j\omega) \;;$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$Q_{m'}(j\omega) = B_m^!(j\omega)\, Q_m(j\omega) + M_m^!(j\omega)\, P_i(j\omega) \;.$$

$$(3.62)$$

Equations (3.58), (3.60), (3.61), and (3.62) can be manipulated as follows to form $m+1$ independent equations. Substitute Equations (3.52) and (3.53) for all $P_i(j\omega)$'s and $Q_i(j\omega)$'s . In Equations (3.61) subtract each equation from the first equation. Add Equations (3.62) and substitute Equation (3.60). The results are:

$$T_1^! Q_1(j\omega) - T_2^! Q_2(j\omega) = G_{i-1}^!(B_2^! - B_1^!)\, Q_r(j\omega) \;;$$

$$T_1^! Q_1(j\omega) - T_3^! Q_3(j\omega) = G_{i-1}^!(B_3^! - B_1^!)\, Q_r(j\omega) \;;$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$T_1^! Q_1(j\omega) - T_k^! Q_k(j\omega) = G_{i-1}^!(B_k^! - B_1^!)\, Q_r(j\omega) \;;$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$T_1^! Q_1(j\omega) - T_m^! Q_m(j\omega) = G_{i-1}^!(B_m^! - B_1^!)\, Q_r(j\omega) \;; \quad (3.63)$$

$$Q_{i+1}(j\omega) - B_1^! Q_1(j\omega) - B_2^! Q_2(j\omega) - B_3^! Q_3(j\omega) - \cdots$$

$$- B_k^! Q_k(j\omega) - \cdots - B_m^! Q_m(j\omega) = G_{i-1}^!(M_1^!$$

$$+ M_2^! + M_3^! + \cdots + M_k^! + \cdots + M_m^!)Q_r(j\omega);$$

$$Q_1(j\omega) + Q_2(j\omega) + Q_3(j\omega) + \cdots + Q_k(j\omega) + \cdots$$

$$+ Q_m(j\omega) = H_{i-1}^! Q_r(j\omega) \;.$$

In matrix form,

$$
\begin{bmatrix}
0 & T'_1 & -T'_2 & 0 & \cdots & 0 & \cdots & 0 \\
0 & T'_1 & 0 & -T'_3 & \cdots & 0 & \cdots & 0 \\
\cdot & \cdot & \cdot & & \cdots & \cdot & \cdots & \cdot \\
0 & T'_1 & 0 & 0 & \cdots & -T'_k & \cdots & 0 \\
\cdot & \cdot & \cdot & & \cdots & \cdot & \cdots & \cdot \\
0 & T'_1 & 0 & 0 & \cdots & 0 & \cdots & -T'_m \\
1 & -B'_1 & -B'_2 & -B'_3 & \cdots & -B'_k & \cdots & -B'_m \\
0 & 1 & 1 & 1 & \cdots & 1 & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
Q_{i+1}(j\omega) \\
Q_1(j\omega) \\
Q_2(j\omega) \\
Q_3(j\omega) \\
\cdot \\
\cdot \\
Q_k(j\omega) \\
\cdot \\
\cdot \\
Q_m(j\omega)
\end{bmatrix}
= Q_r(j\omega)
\begin{bmatrix}
G'_{i-1}(B'_2-B'_1) \\
G'_{i-1}(B'_3-B'_1) \\
\cdot \\
G'_{i-1}(B'_k-B'_1) \\
\cdot \\
G'_{i-1}(B'_m-B'_1) \\
G'_{i-1}\sum_{k=1}^{m} M'_k \\
H'_{i-1}
\end{bmatrix}
$$

$$(3.64)$$

Equation (3.64) is solved for $Q_{i+1}(j\omega)$ and any one of the $Q_k(j\omega)$ s. Substitution of $Q_k(j\omega)$ and Equation (3.52) into the kth equation of Equations (3.61) will give $P_{i+1}(j\omega)$. The results in the desired form are:

$$P_{i+1}(j\omega) = G'_i(j\omega) Q_r(j\omega) ; \qquad (3.65)$$

and

$$Q_{i+1}(j\omega) = H'_i(j\omega) Q_r(j\omega) . \qquad (3.66)$$

For $m = 2$,

$$H'_i(j\omega) = \frac{H'_{i-1}(T'_1 B'_2 + T'_2 B'_1) + G'_{i-1}\left[B_1'^2 - B_2'^2 + (M'_1 + M'_2)(T'_1 + T'_2)\right]}{T'_1 + T'_2},$$

$$(3.67)$$

and

$$G_i^!(j\omega) = B_1^! G_{i-1}^! + T_1^! \left[ \frac{G_{i-1}^!(B_1^! - B_2^!) - T_2^! H_{i-1}^!}{T_1^! + T_2^!} \right] . \tag{3.68}$$

If the parallel components are dimensionally and materially identical,

$$G_i^!(j\omega) = B_1^! G_{i-1}^! + \frac{1}{m} T_1^! H_{i-1}^! \tag{3.69}$$

and

$$H_i^!(j\omega) = B_1^! H_{i-1}^! + m M_1^! G_{i-1}^! \tag{3.70}$$

for any number of pipes.

The transfer functions for parallel pipes will have to be determined as just shown for both Methods A and B.

### 3.6. Summary of Methods.

The results of the three previous sections provide adequate means by which the transfer functions of Equations (3.1) through (3.6) can be determined. For a series piping system, the results of Methods A and B are related to these transfer functions as follows:

For pressure,

$$\left| E_{pr}(x_i, j\omega) \right| = \left| \frac{P(x_i, j\omega)}{P_r(j\omega)} \right| = \left| \frac{G_i(j\omega)}{Z_r(j\omega)} \right| = \frac{\overline{K}_{pi}}{\left| Z_r(j\omega) \right|} ; \tag{3.71}$$

$$\left| E_{pn}(x_i, j\omega) \right| = \left| \frac{P(x_i, j\omega)}{P_n(j\omega)} \right| = \left| \frac{G_i(j\omega)}{G_n(j\omega)} \right| = \frac{K_{pi}}{K_{pn}} ; \tag{3.72}$$

$$\left| Z_{qr}(x_i, j\omega) \right| = \left| \frac{P(x_i, j\omega)}{Q_i(j\omega)} \right| = \left| G_i(j\omega) \right| = \overline{K}_{pi} ; \tag{3.73}$$

and

$$\left| Z_{qn}(x_i, j\omega) \right| = \left| \frac{P(x_i, j\omega)}{Q_n(j\omega)} \right| = \left| \frac{G_i(j\omega)}{H_n(j\omega)} \right| = \frac{K_{pi}}{K_{qn}} . \tag{3.74}$$

For flow,

$$\left| Y_{pr}(x_i, j\omega) \right| = \left| \frac{Q(x_i, j\omega)}{P_r(j\omega)} \right| = \left| \frac{H_i(j\omega)}{Z_r(j\omega)} \right| = \frac{K_{qi}}{\left| Z_r(j\omega) \right|} \quad ; \quad (3.75)$$

$$\left| Y_{pn}(x_i, j\omega) \right| = \left| \frac{Q(x_i, j\omega)}{P_n(j\omega)} \right| = \left| \frac{H_i(j\omega)}{G_n(j\omega)} \right| = \frac{K_{qi}}{K_{pn}} \quad ; \quad (3.76)$$

$$\left| E_{qr}(x_i, j\omega) \right| = \left| \frac{Q(x_i, j\omega)}{Q_r(j\omega)} \right| = \left| H_i(j\omega) \right| = K_{qi} \quad ; \quad (3.77)$$

and

$$\left| E_{qn}(x_i, j\omega) \right| = \left| \frac{Q(x_i, j\omega)}{Q_n(j\omega)} \right| = \left| \frac{H_i(j\omega)}{H_n(j\omega)} \right| = \frac{K_{qi}}{K_{qn}} \quad . \quad (3.78)$$

For power,

$$\left| F_{pr}(x_i, j\omega) \right| = \left| E_{pr}(x_i, j\omega) \right| \left| Y_{pr}(x_i, j\omega) \right| \quad ; \quad (3.79)$$

$$\left| F_{pn}(x_i, j\omega) \right| = \left| E_{pn}(x_i, j\omega) \right| \left| Y_{pn}(x_i, j\omega) \right| \quad ; \quad (3.80)$$

$$\left| F_{qr}(x_i, j\omega) \right| = \left| Z_{qr}(x_i, j\omega) \right| \left| E_{qr}(x_i, j\omega) \right| \quad ; \quad (3.81)$$

and

$$\left| F_{qn}(x_i, j\omega) \right| = \left| Z_{qn}(x_i, j\omega) \right| \left| F_{qn}(x_i, j\omega) \right| \quad . \quad (3.82)$$

# CHAPTER IV

## RESUME

### 4.1. Background Material.

The background material contained in Chapter II comprises much of the literature survey undertaken before the analysis began. Continuous evaluation of existing literature as concerned with the analysis is in progress, but the results of Chapter II are considered reasonably valid and sufficient to begin the analysis.

### 4.2. Analysis.

The results of Chapter II indicate that the analysis of a piping system with statistical inputs will best proceed from a "transfer function" type of investigation. In Chapter III, expressions for determining the transfer functions of series and parallel systems are obtained by manipulation of the system describing equations. The transfer functions given by Equations (3.71) through (3.82) are used in a relationship such as Equation (2.39) to relate pressure, flow rate, or power at some point in a series system to pressure, flow rate, or power at another point. The method of analysis for parallel components gives transfer functions which are easily employed in the overall analysis.

The analysis of a piping system will require electronic computation. For a complete investigation, these computations will be extensive, and Method A will probably be more appropriate due to its

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relative simplicity and possible shorter computing time. Method B provides relationships that will allow the maximum or minimum response of a system to be more clearly defined before numerical investigation. It also puts the transfer functions in a form that separates the effect on the response of each component.

### 4.3. Future Effort.

The analysis of a piping system by Method A using a digital computer is now in its latter stages. The same system will also be completely analyzed by Method B.

Preliminary investigations into the possibility of predicting upper and lower bounds of system response from the results of Method B have given favorable results. It is foreseen that an analysis such as this may make a complete analysis unnecessary. The results of this type of investigation will be compared with a complete solution for validation.

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