THE STATIONARITY OF AN ESTIMATED AUTOREGRESSIVE PROCESS

BY

T. W. ANDERSON

TECHNICAL REPORT NO. 7

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THEODORE W. ANDERSON, PROJECT DIRECTOR

DEPARTMENT OF STATISTICS

STANFORD UNIVERSITY

STANFORD, CALIFORNIA

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A stationary stochastic process $\{y_t\}$ with mean $\mathcal{E} y_t = 0$ satisfies a stochastic difference equation if there exist constants $\beta_0=1, \beta_1, \ldots , \beta_r$ such that $\{u_t\}$ defined by

$$\sum_{r=0}^{k} \beta_r y_{t-r} = u_t, \qquad t = \ldots , -1, 0, 1, \ldots , \tag{1}$$

consists of independently identically distributed random variables. The process $\{y_t\}$ is stationary and y_t is independent of u_{t+1}, $u_{t+2}, \ldots$ if and only if the roots of the associated polynomial equation

$$\sum_{r=0}^{p} \beta_r x^{p-r} = 0 \tag{2}$$

are less than 1 in absolute value. The process is autoregressive of order p. We assume $\mathcal{E} u_t = 0$ and $\mathcal{E} u_t^2 = \sigma^2$ with $0 < \sigma^2 < \infty$.

Let $y_1, \ldots , y_T$ be T successive observations on the process. To estimate the coefficients $\beta_1, \ldots , \beta_p$ one can solve the linear equations

$$\sum_{j=1}^{p} c_{i-j} b_j = -c_i, \qquad i=1, \ldots , p, \tag{3}$$

where

$$c_i = c_{-i} = \frac{1}{T} \sum_{t=1}^{T-i} y_{t+i} y_t, \qquad i=0, 1, \ldots , p. \tag{4}$$

See, for example, Section 5.4 of T. W. Anderson (1971). We assume that there are at least p different nonzero values of t observed. The purpose of this note is to show that the solution of (3) yields coefficients corresponding to a stationary process; that is, the roots of

(5) $$\sum_{r=0}^{p} b_r x^{p-r} = 0$$

are less than 1 in absolute value. Pagano (1971) has shown this result by a different method.

Let $y_{-p+1} = y_{-p+2} = \dots = y_0 = 0$ and $y_{T+1} = y_{T+2} = \dots = y_{T+p} = 0$. Define the vectors

(6) $$\tilde{y}_t = \begin{pmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t-p+1} \end{pmatrix}, \qquad t=0, 1, \dots, T+p.$$

The equations (3) can be written

(7) $$b' \sum_{t=1}^{T+p} \tilde{y}_t \tilde{y}_t' = - \sum_{t=1}^{T+p} \tilde{y}_t \tilde{y}_{t-1}'.$$

The equation (7) is the first row of

(8) $$B \sum_{t=1}^{T+p} \tilde{y}_t \tilde{y}_t' = - \sum_{t=1}^{T+p} \tilde{y}_t \tilde{y}_{t-1}'$$

and b' is the first row of B. The other $p-1$ rows of $\tilde{B}$ constitute the matrix $(-I \; 0)$.

Theorem 1. The matrix $\tilde{B}$ defined by (8) has characteristic roots less than 1 in absolute value.

Proof. If u is a characteristic vector of $\tilde{B}$ corresponding to a characteristic root λ

$$(9) \qquad \underset{\sim}{u}' \sum_{t=1}^{T+p} \tilde{\underset{\sim}{y}}_t \tilde{\underset{\sim}{y}}_t' = -\lambda \underset{\sim}{u}' \sum_{t=1}^{T+p} \tilde{\underset{\sim}{y}}_t \tilde{\underset{\sim}{y}}_{t-1}'.$$

Normalize $\underset{\sim}{u}$ so that

$$(10) \qquad 1 = \underset{\sim}{u}' \sum_{t=1}^{T+p} \tilde{\underset{\sim}{y}}_t \tilde{\underset{\sim}{y}}_t' \overline{\underset{\sim}{u}} = \sum_{t=1}^{T+p} (\underset{\sim}{u}' \tilde{\underset{\sim}{y}}_t) \overline{(\underset{\sim}{u}' \underset{\sim}{y}_t)}$$

$$= \underset{\sim}{u}' \sum_{t=1}^{T+p} \tilde{\underset{\sim}{y}}_{t-1} \tilde{\underset{\sim}{y}}_{t-1}' \overline{\underset{\sim}{u}} = \sum_{t=1}^{T+p} (\underset{\sim}{u}' \tilde{\underset{\sim}{y}}_{t-1}) \overline{(\underset{\sim}{u}' \underset{\sim}{y}_{t-1})} ,$$

where $\overline{\underset{\sim}{u}}$ is the complex conjugate of $\underset{\sim}{u}$. Then multiplication of (9) on the right by $\overline{\underset{\sim}{u}}$ gives

$$(11) \qquad \lambda = \underset{\sim}{u}' \sum_{t=1}^{T+p} \tilde{\underset{\sim}{y}}_t \tilde{\underset{\sim}{y}}_{t-1}' \overline{\underset{\sim}{u}} = - \sum_{t=1}^{T+p} (\underset{\sim}{u}' \tilde{\underset{\sim}{y}}_t) (\overline{\underset{\sim}{u}}' \tilde{\underset{\sim}{y}}_{t-1}).$$

By the Cauchy-Schwarz Inequality $|\lambda| \leq 1$. We can have equality only if $\underset{\sim}{u}' \tilde{\underset{\sim}{y}}_t = \underset{\sim}{u}' \tilde{\underset{\sim}{y}}_{t-1}$, $t=1, \ldots , T+p$, which is impossible. Q.E.D.

Since the characteristic roots of $\underset{\sim}{B}$ are the roots of (5), the desired result has been proved. [See Section 5.4 of T. W. Anderson (1971).]

Theorem 2. The roots of (5), where $b_1, \ldots, b_p$ are the solution to (3), are less than 1 in absolute value.

The result can be extended to the vector-valued autoregressive process $\{\underset{\sim}{y}_t\}$ satisfying

$$(12) \qquad \sum_{r=0}^{p} \underset{\sim}{B}_r \underset{\sim}{y}_{t-r} = \underset{\sim}{u}_t, \qquad t = \ldots , -1, 0, 1, \ldots ,$$

where $\underset{\sim}{y}_t$ and $\underset{\sim}{u}_t$ are q-component vectors and $\underset{\sim}{B}_0 = \underset{\sim}{I}, \underset{\sim}{B}_1, \ldots , \underset{\sim}{B}_p$ are q x q matrices, $\mathcal{E}\underset{\sim}{u}_t = \underset{\sim}{0}$, and $\mathcal{E}\underset{\sim}{u}_t \underset{\sim}{u}_t' = \underset{\sim}{\Sigma}$, positive definite and finite. The analogue of (2) is

(13) $$\left| \sum_{r=0}^{p} \underset{\sim}{B}_r \lambda^{p-r} \right| = 0.$$

We observe $\underset{\sim}{y}_1, \ldots, \underset{\sim}{y}_T$, and define

(14) $$\underset{\sim}{c}_i = \underset{\sim}{c}'_{-i} = \frac{1}{T} \sum_{t=1}^{T-i} \underset{\sim}{y}_{t+i} \underset{\sim}{y}'_t, \qquad i=0, 1, \ldots, p.$$

Then the estimates $\underset{\sim}{B}_1, \ldots, \underset{\sim}{B}_p$ are the solution to

(15) $$\sum_{j=1}^{p} \underset{\sim}{B}_j \underset{\sim}{c}_{i-j} = -\underset{\sim}{c}_i, \qquad i=1, \ldots, p.$$

The roots of (13) with $\underset{\sim}{B}_r$ replaced by $\underset{\sim}{B}_r$, r=1, ... , p, have roots less than 1 in absolute value.

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13. ABSTRACT

The estimated coefficients of an autoregressive process define a stationary process if they are computed from the (Toeplitz) matrix of sample moments computed from all available observations, using the same divisor.

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