RADC-TR-71-225
Technical Report
November 1971

NARROW-BAND SYSTEMS AND GAUSSIANITY

Polytechnic Institute of Brooklyn

**Approved for public release;
distribution unlimited.**

DI-69-11

Rome Air Development Center
Air Force Systems Command
Griffiss Air Force Base, New York

UNCLASSIFIED

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)

Polytechnic Institute of Brooklyn
Long Island Graduate Center
Route 110, Farmingdale, NY 11735

2a. REPORT SECURITY CLASSIFICATION

UNCLASSIFIED

2b. GROUP

N/A

3. REPORT TITLE

NARROW-BAND SYSTEMS AND GAUSSIANITY

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Technical Report

5. AUTHOR(S) (First name, middle initial, last name)

A. Papoulis

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
November 1971	30	15

8a. CONTRACT OR GRANT NO.

F30602-69-C-0331

Job Order No. 01720000

DI-69-11

9a. ORIGINATOR'S REPORT NUMBER(S)

9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

RADC-TR-71-225

10. DISTRIBUTION STATEMENT

Approved for public release; distribution unlimited.

11. SUPPLEMENTARY NOTES

12. SPONSORING MILITARY ACTIVITY

Rome Air Development Center (ISCP)
Griffiss Air Force Base, New York 13440

13. ABSTRACT

The approach to Gaussianity of the output $y(t)$ of a narrowband system $h(t)$ is investigated. It is assumed that the input $x(t)$ is an a-dependent process in the sense that the random variables $x(t)$ and $x(t+u)$ are independent for $u > a$. With $F(y)$ and $G(y)$ the distribution function of $y(t)$ and of a suitable normal process, a realistic bound B of the difference $F(y) - G(y)$ is determined and it is shown that $B \to 0$ as the bandwidth ω_0 of the system tends to zero.

DD FORM 1 NOV 65 1473

UNCLASSIFIED

Security Classification

UNCLASSIFIED
Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Narrow Band Systems						
Gaussianity						
Random Processes						

UNCLASSIFIED
Security Classification

TABLE OF CONTENTS

NARROW-BAND SYSTEMS AND GAUSSIANITY

A. Papoulis

Polytechnic Institute of Brooklyn

DI-69-11

FOREWORD

This Phase Report was submitted by Polytechnic Institute of Brooklyn, Long Island Graduate Center, Route 110, Farmingdale, New York, under Contract F30602-69-C-0331, Job Order Number 01720000, with Rome Air Development Center, Griffiss Air Force Base, New York. Haywood E. Webb, Jr., was the RADC Project Engineer.

Research results reported herein were funded by the Laboratory Director's Fund, DI-69-11.

This report has been reviewed by the Information Office, OI, and is releasable to the National Technical Information Service, NTIS.

This technical report has been reviewed and is approved.

Haywood E. Webb, Jr.

Approved: HAYWOOD E. WEBB, JR
Project Engineer

Daniel R. Loreto

Approved: DANIEL R. LORETO, Chief
Computer Technology Branch
Information Sciences Division

ABSTRACT

The approach to Gaussianity of the output $\underset{\sim}{y}(t)$ of a narrow-band system $h(t)$ is investigated. It is assumed that the input $\underset{\sim}{x}(t)$ is an <u>a-dependent</u> process in the sense that the random variables $\underset{\sim}{x}(t)$ and $\underset{\sim}{x}(t+u)$ are independent for $u > a$. With $F(y)$ and $G(y)$ the distribution function of $\underset{\sim}{y}(t)$ and of a suitable normal process, a realistic bound B of the difference $F(y) - G(y)$ is determined and it is shown that $B \to 0$ as the bandwidth ω_o of the system tends to zero. In the special case of the shot noise process

$$\underset{\sim}{y}(t) = \sum_i h(t-\underset{\sim}{t}_i)$$

it is shown that

$$\left| F(y) - G(y) \right| < \sqrt{\frac{\omega_o}{\lambda}}$$

where λ is the average density of the Poisson points $\underset{\sim}{t}_i$.

1. Introduction

INTRODUCTION

In the engineering applications of random signals it is often assumed that if a stationary process $\underset{\sim}{x}(t)$ is the input to a linear system, then the resulting response

$$\underset{\sim}{y}(t) = \int_{-\infty}^{\infty} \underset{\sim}{x}(t-\alpha)\, h(\alpha)\, d\alpha \tag{1}$$

tends to a normal process as the bandwidth ω_o of the system tends to zero.

This theorem is not always true as one can show with a trivial counter example. However, it holds under fairly general conditions. To apply it meaningfully, we need to establish not only conditions for its asymptotic validity, but also realistic bounds for the deviation of $\underset{\sim}{y}(t)$ from Gaussianity for a given $\omega_o \neq 0$.

As one might expect from the central limit theorem, $\underset{\sim}{y}(t)$ will approach Gaussianity if the past $\underset{\sim}{x}(u)$, $u \leq t$ of the process $\underset{\sim}{x}(t)$ is "almost" independent of its future $\underset{\sim}{x}(u)$, $u \geq t + \tau$ for sufficiently large τ. This loose requirement is precisely formulated in Rosenblatt's classic paper [1] as follows:

Let B_t and F_τ be the Borel fields generated by the random variables $\underset{\sim}{x}(u)$ for $u \leq t$ and $u \geq t + \tau$ respectively. We say that the process $\underset{\sim}{x}(t)$ satisfies the strong mixing condition if there is a function $g(\alpha)$ with

$$0 \leq g(\alpha) \downarrow 0 \quad \text{as} \quad \alpha \to \infty \tag{2}$$

such that for any pair of events $B \in B_t$, $F \in F_\tau$

$$| P(BF) - P(B)\, P(F) | < g(\tau) \tag{3}$$

Assuming further that the moments of $\underset{\sim}{x}(t)$ of order up to four exist and satisfy certain conditions and that its power spectrum $S(\omega)$ is such that

$$S(\omega) > 0 \tag{4}$$

he shows that the output $\underset{\sim}{y}(t)$ of a certain class of filters tends to Gaussianity.

In applying Rosenblatt's result, we are faced with the problem of testing the mixing condition (3). Furthermore, the problem of establishing realistic bounds for the distance of $\underset{\sim}{y}(t)$ from Gaussianity remains. In this paper we shall overcome these difficulties but only at the sacrifice of generality. We shall base our analysis on the assumption that the process $\underset{\sim}{x}(t)$ is a-dependent, i.e., that the events B and F are independent for $\tau > a$. This assumption is equivalent to the condition

$$g(\alpha) = 0 \qquad \text{for} \qquad \alpha > a \tag{5}$$

The above is, of course, more restrictive than (2), however, it holds in many applications and can be often readily established. Suppose, for example, that $\underset{\sim}{x}(t)$ is the output of a memory-less non-linear system with input a normal process $\underset{\sim}{s}(t)$. In this case, condition (5) is equivalent to the assumption that the autocovariance $C(\tau)$ of $\underset{\sim}{s}(t)$ vanishes for $\tau > a$.

2. The Berry-Esseén Theorem

The output $\underset{\sim}{y}(t)$ of our system is a linear combination of a non-countable infinity of dependent random variables. As we show in section 5, if the input $\underset{\sim}{x}(t)$ is a-dependent, then the principal part of $\underset{\sim}{y}(t)$ can be expressed as a sum of independent random variables. To solve our problem we shall need, therefore, a bound of the deviation of such a sum from Gaussianity. Such a bound is given in the following important theorem due to A. C. Berry and G. Esseén [2, 3]. We present it here for easy reference and also because in the proof we make use of a useful lemma (Appendix A) which is an improvement of the corresponding lemma 1 in Feller [2, p. 510]. With its help, we obtain the constant 4 in (8), whereas, the corresponding constant in Feller is 33/4. However, as it is stated in [2], unpublished calculations mention the constant 2.9 (Esseén, 1956) and 2.05 (Wallace,1958).

Consider a sequence $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \cdots$ of independent r.v. (random variables) such that

$$E\{\underset{\sim}{x}_i\} = 0\,, \qquad E\{\underset{\sim}{x}_i^2\} = \sigma_i^2, \quad E\{|\underset{\sim}{x}_i|^3\} = c_i$$

We form the sum

$$\underset{\sim}{x} = \frac{1}{\sigma}\sum_i \underset{\sim}{x}_i \qquad \text{where} \qquad \sigma^2 = \sum_i \sigma_i^2 < \infty \tag{6}$$

Clearly, $E\{\underset{\sim}{x}^2\} = 1$. With $F(x)$ the distribution function of $\underset{\sim}{x}$ and

$$G(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\,dy \tag{7}$$

that of a normal r.v. with the same mean and variance, we wish to bound the "distance" $|F(x) - G(x)|$ of $\underset{\sim}{x}$ from Gaussianity.

<u>Theorem</u>. If

$$\frac{c_i}{\sigma_i^2} \le \lambda$$

then

$$|F(x) - G(x)| < 4\frac{\lambda}{\sigma} \tag{8}$$

<u>Proof</u>. With $\Phi_i(\omega)$ and $\Phi(\omega)$ the characteristic functions of the r.v. $\underset{\sim}{x}_i$ and $\underset{\sim}{x}$ respectively, we conclude from (6) and the independence of the $\underset{\sim}{x}_i$'s that

$$\Phi(\omega) = \Phi_1\left(\frac{\omega}{\sigma}\right)\cdot \Phi_2\left(\frac{\omega}{\sigma}\right)\cdots \tag{9}$$

As we see from (B-6)

$$\Phi_i(\omega) = e^{\frac{\sigma_i^2\omega^2}{2} + \theta_i \frac{5c_i}{21}|\omega|^3}\,, \qquad |\omega| \le \frac{1}{2\lambda} \le \frac{\sigma_i^2}{2c_i} \tag{10}$$

(throughout the paper the letter θ will always be such that $|\theta| \le 1$),

hence,

$$\Phi(\omega) = e^{-\frac{\omega^2}{2} + \theta_o \frac{5}{21} \frac{|\omega|^3}{\sigma^3} \sum_i c_i} = e^{-\frac{\omega^2}{2} + \theta \frac{5\lambda}{21\sigma} |\omega|^3}, \quad |\omega| \le \frac{\sigma}{2\lambda} \tag{11}$$

because

$$\frac{\sum_i c_i}{\sum_i \sigma_i^2} < \text{Max} \frac{c_i}{\sigma_i^2} \le \lambda$$

But for any z

$$\left|\frac{e^z - 1}{z}\right| = \left|1 + \frac{z}{2} + \frac{z^2}{3!} + \cdots\right| \le 1 + |z| + \frac{|z|^2}{2} + \cdots = e^{|z|} \tag{12}$$

hence,

$$\left|\Phi(\omega) - e^{-\frac{\omega^2}{2}}\right| \le e^{-\frac{\omega^2}{2}} \left| e^{\frac{5\theta\lambda}{21\sigma}|\omega|^3} - 1 \right| \le$$

$$|\omega| \le \frac{\sigma}{2\lambda}$$

$$\frac{5\lambda}{21\sigma} |\omega|^3 e^{-\frac{\omega^2}{2}\left[1 - \frac{10\lambda}{21\sigma}|\omega|\right]} < \frac{5\lambda}{21\sigma} |\omega|^3 e^{-\frac{\omega^2}{4}} \tag{13}$$

We now introduce a function $r(x)$ as in (A-1) with Fourier transform $R(\omega)$ such that

$$R(\omega) = 0 \qquad \text{for} \qquad |\omega| > \omega_1 \equiv \frac{\sigma}{2\lambda} \tag{14}$$

As we see from (A-1)

$$|R(\omega)| \le R(0) = 1 \tag{15}$$

The transform of the convolution

$$g(x) \equiv [\,F(x) - G(x)\,] * r(x) \tag{16}$$

is given by [4]

$$\frac{\Phi(\omega) - e^{-\omega^2/2}}{j\,\omega} \; R(\omega)$$

hence,

$$2\pi \; |g(x)| = \left| \int_{-\omega_1}^{\omega_1} \frac{\Phi(\omega) - e^{-\omega^2/2}}{\omega} \; R(\omega)\, d\omega \right| \le \frac{5\lambda}{21\sigma} \int_{-\omega_1}^{\omega_1} \omega^2 \, e^{-\omega^2/4} \, d\omega$$

$$< \frac{5\lambda}{21\sigma} \int_{-\infty}^{\infty} \omega^2 \, e^{-\omega^2/4} \, d\omega = \frac{5\lambda}{21\sigma} \, 4\sqrt{\pi}$$

Thus

$$|g(x)| \le \frac{10\,\lambda}{21\,\sigma\sqrt{\pi}} \tag{17}$$

Since

$$G'(x) = \frac{1}{\sqrt{2\pi}} \, e^{-x^2/2} \le \frac{1}{\sqrt{2\pi}}$$

we conclude from (17) and (A-15) that

$$|F(x) - G(x)| \le \gamma \sqrt{\frac{2}{\pi}} \tag{18}$$

where the constant γ is such that

$$\int_0^{\gamma} r(x)\, dx = \frac{1}{3} + \frac{5\sqrt{2}\,\lambda}{63\,\sigma\,\gamma} \tag{19}$$

We now choose for $r(x)$ the Fejèr kernel [4]

$$r(x) = \frac{\sin^2(\omega_1 x/2)}{\pi\, \omega_1\, x^2/2} \tag{20}$$

whose transform is a triangle satisfying (14). And with

$$w = \frac{\omega_1 \gamma}{2} = \frac{\sigma \gamma}{4\lambda} \tag{21}$$

we conclude that

$$| F(x) - G(x) | \leq \frac{4\lambda}{\sigma} \sqrt{\frac{2}{\pi}}\, w \tag{22}$$

where w is such that

$$\frac{1}{\pi} \int_0^w \frac{\sin^2 x}{x^2}\, dx = \frac{1}{3} + \frac{5}{126\sqrt{2}\, w} \tag{23}$$

Solving we find

$$w \simeq 1.26$$

Inserting into (22) we obtain (8).

Corollary. If the r.v. $\underset{\sim}{z}_i$ are independent with

$$E\{\underset{\sim}{z}_i\} = 0 \quad , \quad E\{\underset{\sim}{z}_i^2\} = \sigma^2, \quad E\{|z_i|^3\} \leq c \tag{24}$$

and

$$\underset{\sim}{z} = \sum_i \alpha_i \underset{\sim}{z}_i \quad , \quad |\alpha_i| \leq A \quad , \quad \sum \alpha_i^2 = \alpha^2$$

then

$$\left| F(z) - G\left(\frac{z}{\alpha\sigma}\right) \right| \leq 4 \frac{Ac}{\alpha\sigma^3} \tag{25}$$

The above follows from (8) with $\underset{\sim}{x}_i = \alpha_i \underset{\sim}{z}_i$.

3. Systems driven by an impulse train

Consider a band-limited system with energy E

$$E = \int_{-\infty}^{\infty} h^2(t)\, dt \qquad H(\omega) = 0 \quad \text{for} \quad |\omega| \geq \omega_0 \tag{26}$$

As it is known [5] ,

$$|h(t)| \le \sqrt{\frac{E\omega_o}{\pi}} , \quad \sum_{n=-\infty}^{\infty} h^2(t-nT) = \frac{E}{T} \quad \text{for} \quad T < \frac{\pi}{\omega_o} \tag{27}$$

With $\underset{\sim}{z}_n$ a sequence of r. v. as in (24) we form the impulse train

$$\underset{\sim}{x}(t) = \sum_m \underset{\sim}{z}_n \, \delta(t-nT) \qquad T < \frac{\pi}{\omega_o} \tag{28}$$

If $\underset{\sim}{x}(t)$ is the input to our system, then the resulting output is given by

$$y(t) = \sum_n \underset{\sim}{z}_n \, h(t-nT) \tag{29}$$

Clearly,

$$E\{\underset{\sim}{y}(t) = 0 , \qquad E\{\underset{\sim}{y}^2(t)\} \equiv \sigma_y^{\,2} = \frac{E\sigma^2}{T}$$

With $F(y)$ the distribution function of $\underset{\sim}{y}(t)$, it follows from (25) and (27) that

$$F(y) - G\left(\frac{y}{\sigma_y}\right) \le \frac{4c}{\sigma^3} \sqrt{\frac{T\omega_o}{\pi}} \tag{30}$$

From the above it follows that $\underset{\sim}{y}(t)$ tends to Gaussianity with $\omega_o \to 0$.

4. Shot noise

Consider the random points $\underset{\sim}{t}_n$ of a Poisson process with average density λ . If the sequence of impulses

$$\underset{\sim}{x}(t) = \sum_n \delta(t-\underset{\sim}{t}_n) \tag{31}$$

is the input to our system, then the resulting output

$$\underset{\sim}{y}(t) = \sum_n h(t - \underset{\sim}{t}_n) \tag{32}$$

is the familiar shot noise process [6, 7]. As it is known, the characteristic function $\Phi_o(\omega)$ of $\underset{\sim}{y}(t)$ is given by [8, p. 567]

$$\Phi_o(\omega) = e^{\lambda \int_{-\infty}^{\infty} [e^{j\omega h(t)} - 1]\, dt} \tag{33}$$

and its mean η and variance σ^2 by (Campbell's theorem [8, p. 569])

$$\eta = \lambda \int_{-\infty}^{\infty} h(t)\, dt \quad , \qquad \sigma^2 = \lambda \int_{-\infty}^{\infty} h^2(t)\, dt = \lambda\ E \tag{34}$$

To simplify notations we shall consider the normalized process

$$\underset{\sim}{s}(t) = \frac{\underset{\sim}{y}(t) - \eta}{\sigma} \tag{35}$$

whose mean is zero, variance one, and characteristic function

$$\Phi(\omega) = e^{-j\frac{\eta\omega}{\sigma}}\, \Phi_o\left(\frac{\omega}{\sigma}\right) \tag{36}$$

With $F(s)$ the distribution function of $\underset{\sim}{s}(t)$ we shall show that

$$|\, F(s) - G(s)\, | < \frac{2.6}{\pi} \sqrt{\frac{\omega_o}{\lambda}} \tag{37}$$

<u>Proof</u>. Since [see (B-6)]

$$e^{j\omega h(t)} = 1 - j\,\omega h(t) - \frac{\omega^2 h^2(t)}{2} + \theta(t)\frac{\omega^3 h^3(t)}{6}, \quad |\,\theta(t)\,| \le 1 \tag{38}$$

we conclude from (33) and (36) that all ω

$$\Phi(\omega) = e^{-\frac{\omega^2}{2} + \theta\beta\omega^3} \qquad\qquad |\,\theta\,| \le 1 \tag{39}$$

where

$$\beta \equiv \frac{\lambda \int_{-\infty}^{\infty} |h(t)|^3 \, dt}{6\sqrt{\lambda^3 E^3}} \leq \frac{1}{6} \sqrt{\frac{\omega_o}{\pi \lambda}} \tag{40}$$

because [see (27)]

$$\int_{-\infty}^{\infty} |h(t)|^3 \, dt < \sqrt{\frac{E\omega_o}{\pi}} \int_{-\infty}^{\infty} h^2(t) \, dt = E\sqrt{\frac{E\omega_o}{\pi}} \tag{41}$$

From (39) and (12) we obtain

$$\left| \Phi(\omega) - e^{-\omega^2/2} \right| = e^{-\frac{\omega^2}{2}} \left| e^{\theta\beta\omega^3} - 1 \right| \leq \beta \, |\omega|^3 \, e^{-\frac{\omega^2}{2} + \beta|\omega|^3} \tag{42}$$

valid for all ω. With $\epsilon < 1$ a constant to be soon determined, we observe that

$$-\frac{\omega^2}{2} + \beta \, |\omega|^3 < -\frac{\omega^2}{2\epsilon^2} \quad \text{for} \quad |\omega| \leq \frac{1}{2\beta} \left(1 - \frac{1}{\epsilon^2}\right) \equiv \omega_1 \tag{43}$$

hence,

$$\left| \Phi(\omega) - e^{-\omega^2/2} \right| \leq \beta \, |\omega|^3 \, e^{\frac{\omega^2}{2\epsilon^2}} \, , \qquad |\omega| \leq \omega_1 \tag{44}$$

We now proceed as in section 2: with $r(x)$ and $g(x)$ as in (16), we have

$$2\pi \, |g(x)| = \left| \int_{-\omega_1}^{\omega_1} \frac{\Phi(\omega) - e^{-\omega^2/2}}{\omega} R(\omega) \, d\omega \right| \leq \beta \int_{-\omega_1}^{\omega_1} \omega^2 e^{-\omega^2/2 \, \epsilon^2} \, d\omega < \beta \, \epsilon^2 \sqrt{2\pi}$$

Therefore, [see (17) to (23)]

$$|g(x)| \leq \frac{\beta \epsilon^3}{\sqrt{2\pi}} \tag{45}$$

and

$$|F(x) - G(x)| \leq \frac{2w}{\omega_1}\sqrt{\frac{2}{\pi}} = \frac{8\beta}{\sqrt{2\pi}} \frac{\epsilon^2}{\epsilon^2 - 1} w \tag{46}$$

where w is such that

$$\frac{1}{\pi}\int_0^w \frac{\sin^2 x}{x^2} dx = \frac{1}{3} + \frac{\epsilon(\epsilon^2-1)}{24w} \tag{47}$$

It remains to solve (47) for w as a function of ϵ, and find ϵ such as to minimize (46). We find that the optimum ϵ and the corresponding w are given by

$$\epsilon^2 \simeq 3 \qquad\qquad w \simeq 1.9$$

Inserting into (46) we obtain (37). The bound (44) yields

$$\left| \Phi(\omega) - e^{-\omega^2/2} \right| \leq \beta \, |\omega|^3 e^{-\omega^2/6}, \qquad |\omega| < \frac{1}{3\beta}$$

The value of ϵ is not critical.

5. Band-limited systems with a-dependent inputs

Consider a stationary process with zero mean, autocorrelation $R(\tau)$, and power spectrum $S(\omega)$

$$E\{\underset{\sim}{x}(t)\} = 0 \ , \quad R(\tau) = E\{\underset{\sim}{x}(t+\tau)\underset{\sim}{x}(t)\} \longleftrightarrow S(\omega) \tag{48}$$

We shall assume that $\underset{\sim}{x}(t)$ is a-dependent [10] as defined in section 1. From the definition it follows that if the instances t_r and t_s are such that

$$\max_{r}\ t_r + a < \min_{s}\ t_s \tag{49}$$

then the r. v. $\underset{\sim}{x}(t_r)$ and $\underset{\sim}{x}(t_s)$ are independent. Hence,

$$R(\tau) = 0 \qquad \text{for} \quad |\tau| > a\,, \qquad S(0) = \int_{-a}^{a} R(\tau)\, d\tau \tag{50}$$

We define the constant α by

$$\frac{1}{a}\int_{-a}^{a} |\tau|\, R(\tau)\, d\tau = \alpha S(0) \tag{51}$$

It is easy to see that $\alpha < 1$.

In this section we shall bound the deviation of the output $\underset{\sim}{y}(t)$ of our system (26) in terms of the above quantities and the third absolute moment

$$E\{\, |\underset{\sim}{x}(t)|^3 \,\} = c \tag{52}$$

of the input $\underset{\sim}{x}(t)$.

For this purpose, we express the impulse response $h(t)$ of our system as a sum (Fig. 1)

$$h(t) = \bar{h}(t) + \epsilon(t) \tag{53}$$

of a staircase function

$$\bar{h}(t) = h(nT) \qquad (n - \tfrac{1}{2})\, T \le t < (n + \tfrac{1}{2})\, T \tag{54}$$

and an error term $\epsilon(t)$, where T is such that

$$2a < T < \frac{\pi}{\omega_o} \qquad T \equiv a + b \tag{55}$$

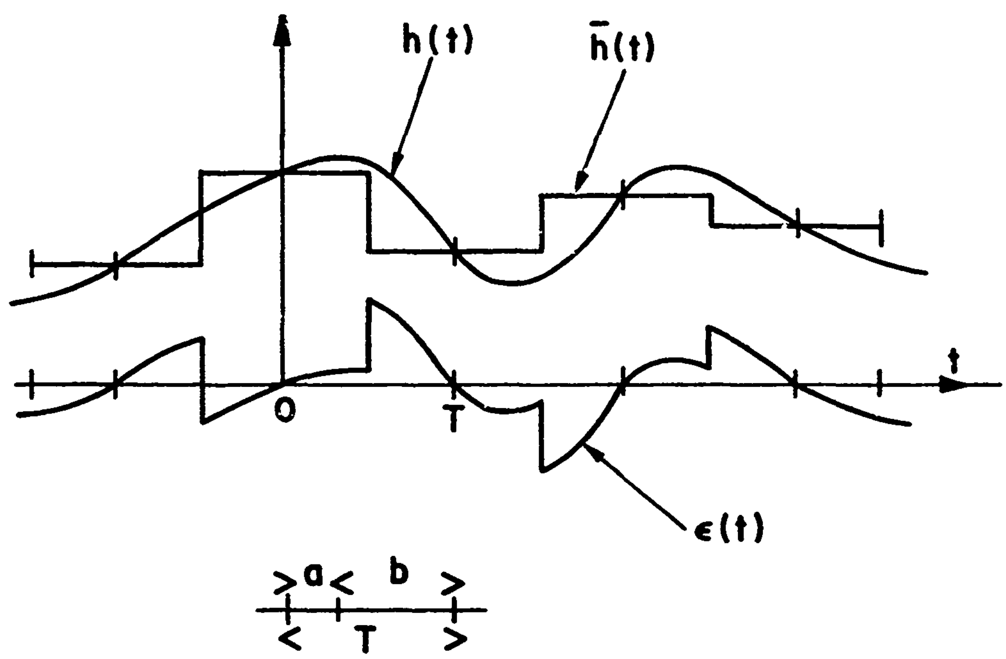


Fig. 1

We next form the r.v.

$$\underset{\sim}{z}_n = \int_{(n-\frac{1}{2})T}^{(n-\frac{1}{2})T+B} \underset{\sim}{x}(t-\alpha)\, d\alpha \qquad \underset{\sim}{w}_n = \int_{(n-\frac{1}{2})T+b}^{(n+\frac{1}{2})T} \underset{\sim}{x}(t-\alpha)\, d\alpha \tag{56}$$

and

$$\underset{\sim}{z} = \sum_n h(nT)\, \underset{\sim}{z}_n \qquad \underset{\sim}{w} = \sum_n h(nT)\, \underset{\sim}{w}_n \tag{57}$$

As we see from (53), the output $\underset{\sim}{y}(t)$ of the system is given by

$$\underset{\sim}{y}(t) = \int_{-\infty}^{\infty} \overline{h}(\alpha)\, \underset{\sim}{x}(t-\alpha)\, d\alpha + \underset{\sim}{e} \tag{58}$$

where

$$\underset{\sim}{e} = \int_{-\infty}^{\infty} \epsilon(\alpha)\, \underset{\sim}{x}(t-\alpha)\, d\alpha \tag{59}$$

From the above it follows that

$$\underset{\sim}{y}(t) = \underset{\sim}{z} + \underset{\sim}{w} + \underset{\sim}{\epsilon} \tag{60}$$

As we shall presently see, if

$$a << T << \pi/\omega_o \tag{61}$$

then the dominant term in (60) is the r.v. $\underset{\sim}{z}$.

From (56) and (57) and the stationarity of $\underset{\sim}{x}(t)$ it follows that [9 p. 346]

$$E\{\underset{\sim}{z}_n\} = 0 \quad , \qquad E\{\underset{\sim}{w}_n\} = 0$$

$$E\{\underset{\sim}{z}_n^2\} = b\int_{-a}^{a} R(\tau)\left[1 - \frac{|\tau|}{b}\right] d\tau = S(0)\left(1 - \alpha\frac{a}{b}\right) b \tag{62}$$

$$E\{\underset{\sim}{w}_n^2\} = a\int_{-a}^{a} R(\tau)\left[1 - \frac{|\tau|}{a}\right] d\tau = S(0)(1 - \alpha)\, a \tag{63}$$

$$E\{|\underset{\sim}{z}_n|^3\} \le \int_0^b\int_0^b\int_0^b E\{|\underset{\sim}{x}(\alpha_1)\,\underset{\sim}{x}(\alpha_2)\,\underset{\sim}{x}(\alpha_3)|\}\, d\alpha_1\, d\alpha_2\, d\alpha_3 \tag{64}$$

But

$$E^3\{|\underset{\sim}{x}\underset{\sim}{y}\underset{\sim}{z}|\} \le E\{|\underset{\sim}{x}|^3\}\; E\{|\underset{\sim}{y}|^3\}\; E\{|\underset{\sim}{z}|^3\} \tag{65}$$

as one can show from Hölder's inequality, hence, [see (52)]

$$E\{|\underset{\sim}{z}_n|^3\} \le b^3 c \tag{66}$$

From the a-dependence of $\underset{\sim}{x}(t)$ it follows that the r.v. $\underset{\sim}{z}_n$ are independent, hence, [see (27)]

$$\sigma_o^2 = E\{\underset{\sim}{z}^2\} = \sum_n h^2(nT)\, E\{\underset{\sim}{z}_n^2\} = \frac{E}{T} S(0)\left(1 - \alpha\frac{a}{b}\right) b \tag{67}$$

But $|h(nT)| \le \sqrt{\frac{E\omega_o}{\pi}}$, therefore, [see (25) and (68)]

$$\left| F(z) - G\left(\frac{z}{\sigma_z}\right)\right| \le A_1 T^2 \sqrt{\omega_o} \tag{68}$$

where

$$A_1 = \frac{4\,c^3}{\sqrt{\pi S^3(0)\,(1 - |\alpha|\,a/b)^3}} \tag{69}$$

The r. v. $\underset{\sim}{w}_n$ are also independent, hence,

$$\sigma_w^{\,2} = E\{\underset{\sim}{w}^2\} = \sum_n h^2\,(nT)\,E\{\underset{\sim}{w}_n^{\,2}\} = E\,A_2^{\,2}\frac{a}{T} \tag{70}$$

where

$$A_2^{\,2} = S(0)\,(1-\alpha) \tag{71}$$

From (59) it follows that [9, p. 346)]

$$\sigma_e^{\,2} < \rho\,(0)\int_{-a}^{a} R\,|\alpha|\,d\,\alpha < E_\epsilon\,2aR(0) \le E\,A_3^{\,2}\frac{a}{T} \tag{72}$$

where

$$A_3^{\,2} = \frac{1}{6}\,R(0)\,T^3\,\omega_o^{\,2} \tag{73}$$

From the above it follows that

$$E\{(\underset{\sim}{w} + \underset{\sim}{e})^2\} \le (\sigma_w + \sigma_e)^2 = E(A_2 + A_3)^2\,\frac{a}{T} \tag{74}$$

The constant T is still to be determined. If it is small, then the bound in (68) is small, however, the variance of the term $\underset{\sim}{w} + \underset{\sim}{e}$ is large. To account for its effect on the distribution $F_y(y)$ of $\underset{\sim}{y}(t)$ [see (62)] we shall use the bound (D-3) in Appendix D. As we see from (74) and (67)

$$\frac{E\{(\underset{\sim}{w}+\underset{\sim}{e})^2\}}{E\{\underset{\sim}{z}^2\}} \leq \frac{E\,(A_2+A_3)^2\frac{a}{T}}{E\,S(0)(1-\alpha a/b)} = A_4\frac{a}{T} \tag{75}$$

where

$$A_4 = \frac{(A_2+A_3)^2\,(b+a)}{S_o\,(b-\alpha a)} \tag{76}$$

We note that the quantities A_1, A_2 and A_4 are essentially independent of the system. Furthermore, the final result is useful only if ω_o is so small that $a << T << \pi/\omega_o$. In this case, $A_3 << A_2$, $b \simeq T$ and A_4 is close to unity.

From (75), (68), (60), and (D-4) it follows that

$$\left|F_y(y) - G(\frac{y}{\sigma_o})\right| < A_1\,T^2\sqrt{\omega_o} + 1.03\sqrt[3]{A_4\frac{a}{T}} \tag{77}$$

Choosing T so as to minimize the above, we finally obtain

$$\left|F_y(y) - G\,(\frac{y}{\sigma_o})\right| < 2\sqrt[7]{A_1\,A_4{}^2 a^2\sqrt{\omega_o}} \tag{78}$$

From the preceding discussion it follows that if the input to a band-limited system is an a-dependent process such that

$$E\{|\underset{\sim}{x}|^3\} < \infty \text{ and } S(0) \neq 0$$

then the resulting output tends to Gaussianity with $\omega_o \to 0$.

Appendix A

A useful lemma. Given an even non-negative function r(x) of unit area

$$\int_{-\infty}^{\infty} r(x)\, dx = 1 , \qquad r(-x) = r(x) , \qquad r(x) \geq 0 \tag{A-1}$$

and a differentiable function $\varphi(x)$ such that

$$\varphi(\pm\infty) = 0 , \qquad \varphi'(x) \geq -A < 0 \tag{A-2}$$

we form the convolution integral

$$g(x) = \int_{-\infty}^{\infty} \varphi(\xi)\, r(x-\xi)\, d\xi = \varphi(x) * r(x) \tag{A-3}$$

We maintain that if

$$|g(x)| \leq B \tag{A-4}$$

then [9, 11, 12]

$$|\varphi(x)| \leq 2A\gamma \tag{A-5}$$

where γ is such that

$$\int_0^{\gamma} r(x)\, dx = \frac{1}{3} + \frac{B}{6A\gamma} \tag{A-6}$$

Proof. Suppose that the maximum of $\varphi(x)$ equals C. Without loss of generality we can assume that this maximum is reached at $x = 0$. Thus

$$|\varphi(x)| \leq |\varphi(0)| = C \tag{A-7}$$

Consider first the case $\varphi(0) > 0$. Clearly,

$$\varphi(x) - C = \int_0^{x} \varphi'(\xi)\, d\xi$$

Hence,

$$\varphi(x) \geq C - Ax \tag{A-8}$$

Furthermore,

$$g(x) = \int_{-\infty}^{0} + \int_{0}^{2x} + \int_{2x}^{\infty} \varphi(\xi)\, r(x-\xi)\, d\xi$$

From (A-8) and the evenness of $r(x)$ it follows with $x - \xi = y$ that

$$\int_{0}^{2x} \varphi(\xi)\, r(x-\xi)\, d\xi \geq \int_{0}^{2x} (C - A\xi)\, r(x-\xi)\, d\xi = 2\,[C - Ax] \int_{0}^{x} r(y)\, dy$$

But

$$\int_{-\infty}^{0} + \int_{2x}^{\infty} \varphi(\xi)\, r(x-\xi)\, d\xi \leq C \left[\int_{-\infty}^{0} + \int_{2x}^{\infty} r(x-\xi)\, d\xi \right] = C \left[1 - 2 \int_{0}^{x} r(y)\, dy \right]$$

Hence,

$$g(x) > [\,4C - 2Ax\,] \int_{0}^{x} r(y)\, dy - C$$

Letting

$$x = \frac{C}{2A} \tag{A-9}$$

in (A-9) we conclude with (A-4) that

$$B \geq 3C \int_{0}^{C/2A} r(y)\, dy - C \tag{A-10}$$

If our assertion (A-5) is not true then

$$\varphi(0) = C > 2\, A\, \gamma \tag{A-11}$$

But this leads to a contradiction because then (A-10) would yield [see (A-6)]

$$B \geq 3\,C \int_0^{\gamma} r(y)\,dy = \frac{CB}{2A\gamma} > B$$

hence, (A-4) is true.

The case $\varphi(0) < 0$ can be reduced to the above. Indeed, with

$$\varphi_1(x) = -\varphi(-x), \quad g_1(x) = \varphi_1(x) * r(x)$$

we have

$$\varphi_1'(x) = \varphi'(-x) \quad g_1(x) = g(-x)$$

hence,

$$|\varphi_1(x)| \leq \varphi_1(0) > 0, \quad |g_1(x)| \leq B$$

From the above it follows that $|\varphi_1(x)| \leq 2A\gamma$ and the proof of the theorem is thus complete.

Corollary. Consider two distribution functions $F_1(x)$, $F_2(x)$ and a function $r(x)$ satisfying (A-1). With

$$g(x) = [F_1(x) - F_2(x)] * r(x) \tag{A-12}$$

we maintain that if

$$F_2'(x) \leq A \tag{A-13}$$

and

$$|g(x)| \leq B \tag{A-14}$$

then

$$|F_1(x) - F_2(x)| \leq 2A\gamma \tag{A-15}$$

where γ is as in (A-6).

Proof. With

$$\varphi(x) = F_1(x) - F_2(x)$$

we have $\varphi(\pm\infty) = 0$. Furthermore,

$$\varphi(x) \geq C - Ax$$

as we see from (A-13) and the monotonicity of $F_1(x)$. The desired bound (A-15) follows as in the lemma.

Appendix B

Consider a random variable $\underset{\sim}{x}$ such that

$$E\{\underset{\sim}{x}\} = 0 \ , \qquad E\{\underset{\sim}{x}^2\} = \sigma^2, \qquad E\{|x|^3\} = c \tag{B-1}$$

Fith $f(x)$ its density function and

$$\Phi(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{j\omega x}\, dx \tag{B-2}$$

the corresponding characteristic function, we maintain that for all ω

$$|\Phi(\omega) - 1| \leq \frac{\omega^2\sigma^2}{2} \tag{B-3}$$

$$\left|\Phi(\omega) - 1 - \frac{\omega^2\sigma^2}{2}\right| \leq \frac{\omega^3 c}{6} \tag{B-4}$$

and

$$\Phi(\omega) = e^{-\frac{\sigma^2\omega^2}{2} + \theta\frac{5c}{21}|\omega|^3} \qquad \text{for} \quad |\omega| < \frac{\sigma^2}{2c} \tag{B-5}$$

Proof

As it is known [2, 13]

$$e^{j\omega x} = 1 + j\,\omega x + \theta_1 \frac{\omega^2 x^2}{2}, \qquad |\theta_1| \le 1$$

$$e^{j\omega x} = 1 + j\,\omega x - \frac{\omega^2 x^2}{2} + \theta_2 \frac{\omega^3 x^3}{6}, \quad |\theta_2| \le 1 \tag{B-6}$$

Inserting into (B-2) we obtain (B-3) and (B-4).

From the expansion

$$-\log(1-z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots \qquad |z| < 1 \tag{B-7}$$

it follows with $z = 1 - \Phi(\omega)$ that if $|1 - \Phi(\omega)| < 1$, then

$$-\log\{1 - [1-\Phi(\omega)]\} = [1 - \Phi(\omega)] + \sum_{n=2}^{\infty} \frac{1}{n}[1 - \Phi(\omega)]^n \tag{B-8}$$

hence,

$$\left|\log \Phi(\omega) + \frac{1}{2}\omega^2\sigma^2\right| \le \left|\Phi(\omega) - 1 - \frac{\omega^2\sigma^2}{2}\right| + \sum_{n=2}^{\infty} \frac{1}{n}\left|1 - \Phi(\omega)\right|^n \tag{B-9}$$

As it is known, $\sigma^3 \le c$; hence, for

$$|\omega| < \frac{\sigma^2}{2c} \le \frac{1}{2\sigma} \tag{B-10}$$

$$|1 - \Phi(\omega)| \le \frac{\omega^2\sigma^2}{2} \equiv r < \frac{1}{8}$$

Therefore,

$$\sum_{n=2}^{\infty} \frac{1}{n} \left| 1 - \Phi(\omega) \right|^n \leq \frac{r^2}{2} \left[1 + \frac{2}{3} r + \frac{2}{4} r^2 + \dots \right] < \frac{r^2}{2} \frac{1}{1-r} < \frac{4r^2}{7}$$

and [see (B-6), (B-9) and (B-4)]

$$\left| \log \Phi(\omega) + \frac{\omega^2 \sigma^2}{2} \right| \leq \frac{c|\omega|^3}{6} + \frac{\sigma^4 \omega^4}{7} < \frac{5c}{21} |\omega|^3 \tag{B-11}$$

from which (B-5) follows.

Appendix C

In the decomposition

$$h(t) = \overline{h}(t) + \epsilon(t) \tag{C-1}$$

of Fig. 1, the transform $\overline{H}(\omega)$ of $\overline{h}(t)$ is given by [14]

$$\overline{H}(\omega) = \frac{2 \sin(\omega T/2)}{T \omega} \sum_n H(\omega + \frac{n\pi}{T}) \tag{C-2}$$

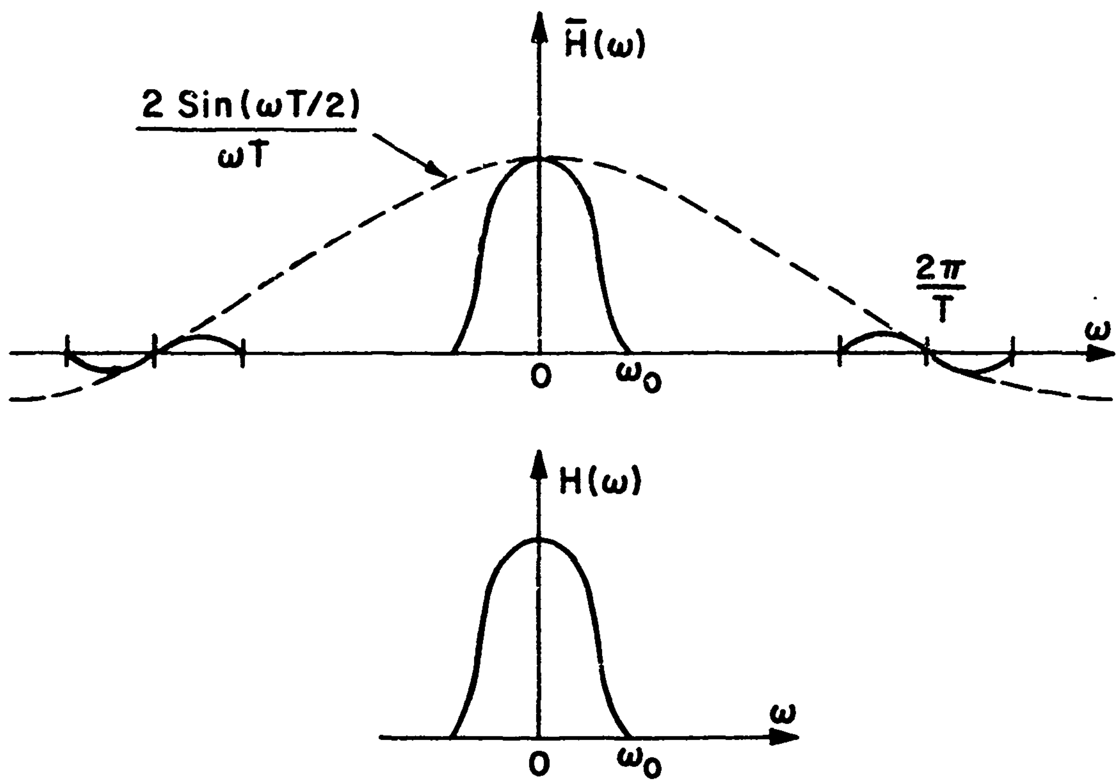
(Fig. 2) and its energy [15, p. 122] by

$$\overline{E} = \int_{-\infty}^{\infty} \overline{h}^2(t) dt = T \sum_n h^2(nT) = E \tag{C-3}$$

We shall show that

$$E_\epsilon = \int_{-\infty}^{\infty} \epsilon^2(t)\, dt < \frac{E}{12} T^2 \omega_o^2 \tag{C-4}$$

Proof. Since $H(\omega) = 0$ for $|\omega| \geq \omega_o$ we conclude from Parseval's formula that



$\bar{H}(\omega)$
$\frac{2 \sin(\omega T/2)}{\omega T}$
$\frac{2\pi}{T}$
ω
0
ω_0
$H(\omega)$
ω
0
ω_0

Fig. 2

$$\int_{-\infty}^{\infty} h(t)\,\bar{h}(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)\,\overline{H}^{*}(\omega)\,d\omega = \frac{1}{2\pi}\int_{-\omega_o}^{\omega_o} |H(\omega)|^2\,\frac{2\sin(\omega T/2)}{T\,\omega}\,d\omega \qquad \text{(C-5)}$$

hence,

$$E_\epsilon = \int_{-\infty}^{\infty} [h(t)-\bar{h}(t)]^2\,dt = \frac{1}{\pi}\int_{\omega_o}^{\omega_o} |H(\omega)|^2\left[1 - \frac{2\sin(\omega T/2)}{\omega T}\right] d\omega$$

But

$$|\sin x - x| \le \frac{x^3}{6}$$

hence

$$E_\epsilon \le \frac{T^2}{24\pi}\int_{-\omega_o}^{\omega_o} \omega^2\,|H(\omega)|^2\,d\omega \le \frac{E}{12}\,T^2\,\omega_o^{\,2}$$

Appendix D

The r.v. $\underset{\sim}{x}$ is such that $E\{\underset{\sim}{x}\} = 0$, $E\{\underset{\sim}{x}^2\} = 1$. With $F(x)$ its distribution function, w assume that [see (7)]

$$|F(x) - G(x)| < \delta \qquad \text{for all } x \qquad \text{(D-1)}$$

The r.v. $\underset{\sim}{y}$ is such that

$$E\{\underset{\sim}{y}\} = 0 \qquad E\{|\underset{\sim}{y}|\} = M_1 \qquad E\{\underset{\sim}{y}^2\} = \sigma^2 \qquad \text{(D-2)}$$

We form the sum

$$\underset{\sim}{z} = \underset{\sim}{x} + \underset{\sim}{y}$$

With $F_z(z)$ the distribution function of $\underset{\sim}{z}$ we shall show that:

a) If the r.v. $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are independent, then

$$|F_z(z) - G(z)| < \delta + \frac{\sigma}{\sqrt{2\pi}} \tag{D-3}$$

b) In any case,

$$|F_z(z) - G(z)| < \delta + 1.03\sqrt[3]{\sigma^2} \tag{D-4}$$

Proof. a) With $f_y(y)$ the density of $\underset{\sim}{y}$, it follows from the independence of $\underset{\sim}{x}$ and $\underset{\sim}{y}$ that

$$F_z(z) = \int_{-\infty}^{\infty} F(z-\xi)\, f_y(\xi)\, d\xi \tag{D-5}$$

hence,

$$F_z(z) - G(z) = \int_{-\infty}^{\infty} [\, F(z-\xi) - G(z-\xi) + G(z-\xi) - G(z)\,]\; f_y(\xi)\, d\xi \tag{D-6}$$

But

$$|G(z-\xi) - G(z)| = |G'(z-\theta\xi)\,\xi| \leq \frac{|\xi|}{\sqrt{2\pi}} \tag{D-7}$$

therefore,

$$|F_z(z) - G(z)| < \delta + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |\xi|\, f_y(\xi)\, d\xi = \delta + \frac{M_1}{\sqrt{2\pi}}$$

and (D-3) follows because $M_1 < \sigma$.

b) $F_z(z)$ equals the probability masses in the region $x + y \leq z$ of the x-y plane. With ε a constant to be soon determined, we see that

$$F(z-\varepsilon) - P\{\underset{\sim}{y} > \varepsilon\} < F_z(z) < F(z+\varepsilon) + P\{\underset{\sim}{y} < -\varepsilon\} \tag{D-8}$$

hence, [see (D-1)]

$$G(z-\epsilon) - \delta - P\{\underset{\sim}{y}>\epsilon\} - G(z) < F_z(z) - G(z) < G(z+\epsilon) + \delta + P\{\underset{\sim}{y}<-\epsilon\} - G(z)$$

and [see (D-7)]

$$|F_z(z) - G(z)| < \delta + \frac{\epsilon}{\sqrt{2\pi}} + P\{|y| > \epsilon\} \qquad \text{(D-9)}$$

It remains to select ϵ so as to minimize the above bound. From Tchebycheff's inequality we have [9, p. 150]

$$P\{|\underset{\sim}{y}| \geq \epsilon\} \leq \frac{\sigma^2}{\epsilon^2}$$

hence,

$$|F_z(z) - G(z)| < \delta + \frac{\epsilon}{\sqrt{2\pi}} + \frac{\sigma^2}{\epsilon^2} \qquad \text{(D-10)}$$

This quantity is minimum for

$$\epsilon = \sqrt[3]{2\sigma^2\sqrt{2\pi}}$$

Inserting into (D-10) we obtain (D-4).

If the variance of $\underset{\sim}{x}$ is not one but σ_x^2, then

$$\left|F(x) - G\left(\frac{x}{\sigma_x}\right)\right| < \delta + \sqrt[3]{\frac{27\sigma^2}{8\pi\sigma_x^2}} \qquad \text{(D-11)}$$

This follows readily from (D-4) by a simple scaling.

References

[1] M. Rosenblatt, Some Comments on Band-Pass Filters, Quart. Appl. Math. 18, pp. 387-393 (1961).

[2] W. Feller, "An Introduction to Probability Theory and Its Applications," Vol. II, John Wiley and Sons, N. Y., 1966, pp. 510-524.

[3] M. Loève, "Probability Theory," Van Nostrand Co., N. Y., 1954, pp. 282-288.

[4] A. Papoulis, "The Fourier Integrals and its Applications," McGraw-Hill, 1962.

[5] A. Papoulis, Limits on Band-Limited Signals, Proc. IEEE, Vol. 55, pp. 1677-1686, October 1967.

[6] A. Blanc-Lapierre and R. Fortet, "Theorie des fonctions aléatoires," Masson et Cie, Paris, 1953.

[7] J. L. Doob, "Stochastic Processes," John Wiley and Sons, Inc., New York.

[8] A. Papoulis, "Probability, Random Variables, and Stochastic Processes," McGraw-Hill Book Co., New York, 1965.

[9] A. Papoulis, High Density Shot Noise and Gaussianity, J. Appl. Prob., March 1971, Vol. 8, No. 1, pp. 118-127.

[10] W. Hoeffding and H. Robbins, The Central Limit Theorem for Dependent Random variables, Duke Math. J., 15, 1948, pp. 773-780.

[11] H. Bergstrom, On the Central Limit Theorem in the Case of Not Equally Distributed Random Variables, Skandinarisk Aktuarietidskrift, 32, pp. 37-62.

[12] N. A. Sapogov, The Independent Components of the Sum of Random Variables Distributed Approximately Normally, Vestnik Lenningrad un-ta, 19, 78-105 (1959).

[13] A. Papoulis, IEEE Trans. Inform. Theory, Vol. IT-11, pp. 593-594, October 1965.

[14] A. Papoulis, Error Analysis in Sampling Theory, Proc.IEEE, 54, 1966, pp. 947-955.

[15] A. Papoulis, "Systems and Transforms with Applications in Optics," McGraw-Hill, N. Y., 1968.