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Response of Structural Systems to Nonstationary Random Excitation

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RESPONSE OF STRUCTURAL SYSTEMS
TO NONSTATIONARY RANDOM EXCITATION*

ABSTRACT

In this work, the response of lumped parameter, second order systems to nonstationary random excitations is examined. Included is a brief introduction to the probabilistic theory of structural dynamics and various basic concepts required for subsequent work.

More specifically, the second order central moment (covariance) response of structural systems to random excitations is studied. In the course of the analysis, an approximate method for the calculation of system response to a class of nonstationary excitation processes is constructed. This class of excitations we have called "slowly varying" nonstationary random processes. By this is meant that the nonstationary variation of the correlation functions of the process is small compared with the time variation of the impulse response functions of the system considered.

It is shown how this approximation technique may be applied to the estimation of inertial loads in the structural members of a payload during the launch phase of flight. Employing previous rocket engine test data, the excitations to the payload are idealized as a "slowly varying" nonstationary random excitation. An approximation procedure is then developed for the calculation of the second-order central moments of the payload response.

Accepted for the Air Force
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Chief, Lincoln Laboratory Project Office

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CHAPTER I

INTRODUCTION

1.1 General Nature of the Problem

This work is concerned with the estimation of inertial loads in a payload structure during the launch phase of flight. It has been motivated by research on this problem with regard to the Lincoln Experimental Satellite (LES) series for which the Titan III launch vehicles have been used. Thus, it is toward the LES program that this thesis is generally directed.

For LES-5, structural qualification was based on a dynamic analysis of the payload system which consisted of the dispenser truss and all satellites. To accomplish this analysis the satellite contractors supplied Martin-Marietta Corporation with the relevant characteristics of the satellites. Martin-Marietta then determined the system response to excitations obtained from 3 of 27 booster engine test firings. The analysis was carried out only for the core engine cut-off phase of flight since it is during this period (approximately 2 seconds) that the payload is subjected to the most severe dynamic loading.

It was subsequently assumed that:

A. The dynamic excitations are such that the maximum response of the payload system[1] occurs in the low frequency range (less than 50 Hertz).

B. The six components of rigid body acceleration of the booster/payload interface are virtually independent of the payload impedance for payloads in a given weight category.(1) Thus, in estimating the low frequency response of a satellite, the acceleration time histories at the booster/payload interface were assumed to be the basic dynamic inputs. We shall adhere to this assumption in the following.

In this work, we shall be dealing with the low frequency response of a payload to excitations obtained from the acceleration time histories mentioned above.

1.2 Interface Accelerations Considered as Random Processes

Typical plots of the booster/interface accelerations are shown on page 12. The non-reproducibility of these data are to be noted. Indeed, from the structural designers point of view, these time histories can only be described as random. The apparently random nature of the data suggest the following approach to the estimation of structural member loads:

A. The 27 time histories of the six acceleration components are considered to be ensemble members of a nonstationary random process.

B. Statistical parameters of the stochastic response of the payload are to be calculated with the interface accelerations considered as the stochastic inputs.

(a)

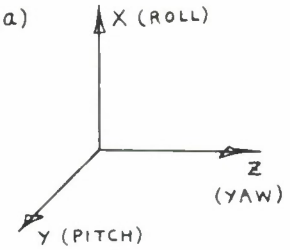
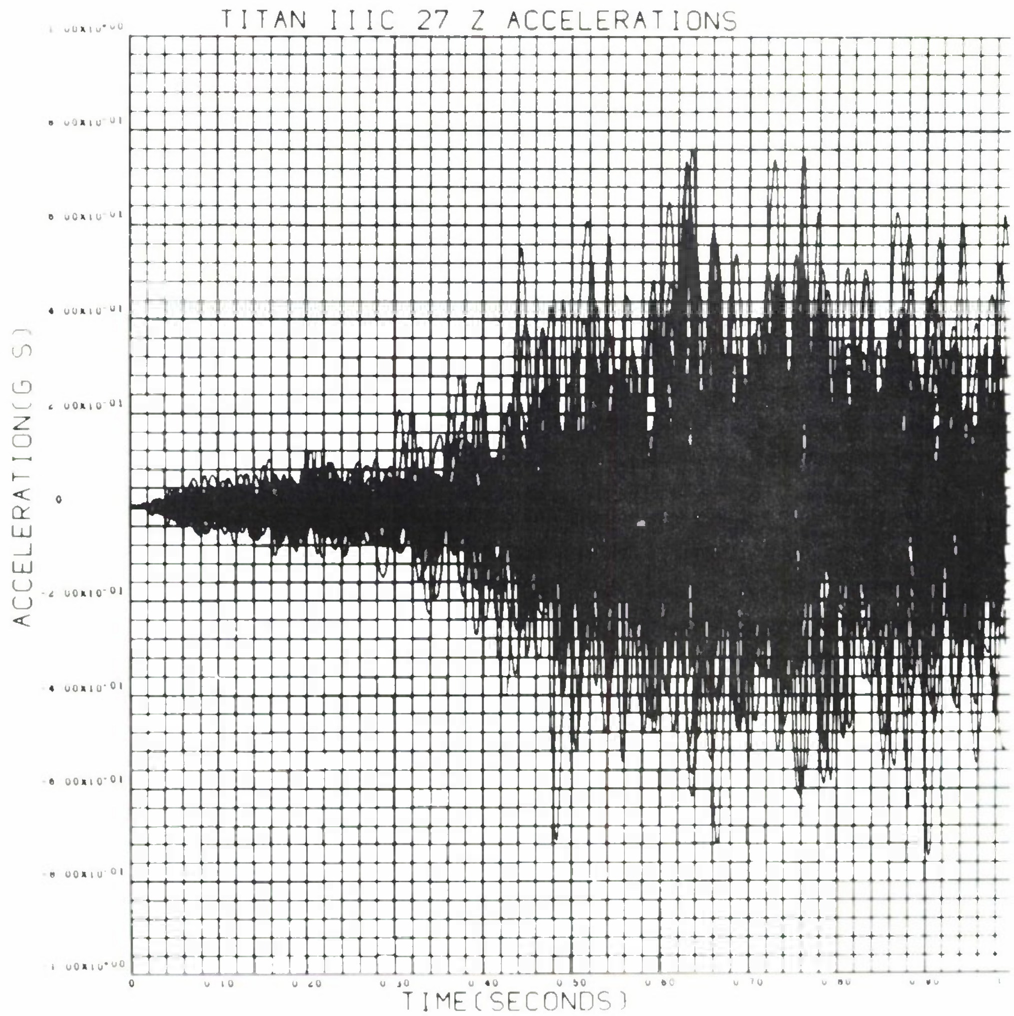


Fig. 1 The coordinate system for the interface accelerations is shown at left; (a). A composite plot of all the z acceleration time histories from the 27 booster engine test firings are shown below; (b).

(b)



The most important response parameters to be obtained are the means and mean squares of the loads on the payload structural members. For if x is one of the stresses in some member of the structure, μ_x the expectation value of x, and Ψ_x^2 the expectation of x^2 then in the customary notation[(2)]:

$$P\{|x-\mu_x| \geq k\sigma\} \leq \frac{1}{k^2} \tag{1.1}$$

$$\sigma_x^2 = \Psi_x^2 - \mu_x^2 \tag{1.2}$$

where $P\{A\}$ is the probability of event A, and σ_x^2 is called the variance or second central moment of x. We thus see that a knowledge of μ_x and Ψ_x allows us to determine an upper bound to the probability that the member stresses will exceed the values for which failure occurs. Since the evaluation of the means is relatively simple, we shall be mainly concerned with the determination of the variance of the member loads. To obtain these it is necessary to determine the complete set of second-order central moments of the payload displacements.*

In summary, the approach to be presented involves the construction of a "stochastic equivalent" to the interface accelerations and the subsequent evaluation of the second

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* See reference (2), p. 24 or Chapter 2 for definition of second central moments. See also Section 2.8 for a justification of this assertion.

moment response of a satellite. In general, the object of this thesis is the development of a plan of analysis whereby the approach may be conveniently applied. The two major aspects of the problem are discussed in more detail below.

1.3 Statistics of the Interface Accelerations*

In the analysis, three basic assumptions have been made concerning the general nature of the interface accelerations.

A. The interface accelerations are the response of a lumped parameter, second order, linear system (i.e. the Titan III launch vehicles) to a nonstationary vector shot noise excitation.

B. The intensity functions of the components of the vector shot noise are the same to within multiplicative constants.

C. The excitations to different normal modes of the booster are statistically uncorrelated.

In chapter VI these assumptions are discussed. Furthermore it is shown that under the above assumptions, the generalized spectral density matrix of the six booster-payload interface accelerations can be determined from the general properties of the time histories and appropriate averages of the Fourier transforms of these time histories.

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* For definitions of statistical terms, see references (3), (2) or (4).

1.4 Evaluation of Second Moment Response

The techniques for the calculation of the response of a system to both stationary and non-stationary random excitation are well established. However, the evaluation of response to nonstationary processes is considerably more taxing than in the stationary case. Indeed, for the simplest systems, the analytic expression for the second moment response may be quite unwieldy.[5,6]

From Section 1.3, the accelerations are assumed to represent the output of a nonstationary random vector process. Furthermore, they are the inputs to a complicated structural system. Therefore, the need for reliable approximation procedures is evident.

One such procedure, developed by Bucciarelli and Kuo,[7] deals with excitations of the form

$$x(t) = g(t)n(t) \tag{1.3}$$

where $n(t)$ is a stationary random process and $g(t)$ is a deterministic function of time. When $g(t)$ satisfies certain restrictions, Reimann's lemma[8] may be used to eliminate the high frequency terms in the expression for the second moment response. This analysis has been applied successfully to single degree of freedom systems.[9]

The present work is largely concerned with the improvement and generalization of the approach mentioned above. A procedure is developed for the approximate calculation of the second moment response of a payload system to excitations of the form presented in Section 1.3. An analysis of the statistical properties of the interface accelerations is also included but no attempt will be made at a detailed calculation of the confidence intervals of the estimated statistical parameters.

1.5 Summary of Contents

In Chapter II we review the aspects of the probabilistic theory of structural dynamics relevant to this study. Since the proposed approximation technique involves extensive use of the Fourier integral, a discussion of its elementary properties is included in Chapter III. At the conclusion of Chapter III we present two related theorems which form the basis of the approximation procedure. The general response formulation of Chapter IV leads directly to the application of the theorems developed in Chapter III to the approximate evaluation of the second moment response in Chapter V. Finally, in Chapter VI we discuss the statistical analysis of the interface accelerations and outline a general procedure for the calculation of the variances of the member loads in an actual payload structure.

CHAPTER II

PROBABILISTIC THEORY OF STRUCTURAL DYNAMICS

2.1 Introduction

Here we review certain aspects of the theory of structural dynamics and of the theory of random processes needed to formulate the response of a structural system to random excitations. In Sections 2.2, 2.3 and 2.4 are presented the deterministic equations of response. Sections 2.5 through 2.8 include a discussion of certain results in random theory. The chapter concludes with the formulation of the second moment displacement response of a structure to random excitations and a consideration of two special classes of random processes.

2.2 The Idealization of Elastic Structures

a. The system has a finite number, N, of degrees of freedom. Thus the state of the system can be specified by a vector, $\{x\}$, whose components are the generalized coordinates of the system.*

b. The inertial properties can be represented by a symmetric matrix of constants, $\underset{(N\times N)}{[M]}$, called the mass matrix.

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* For an explanation of the matrix notation see reference (10).

The total kinetic energy of the system during its motion is $\frac{1}{2}\ \dot{x}\ [M]\ \{\dot{x}\}$.

c. Each component of the structure is assumed to be linearly elastic. Hence, the total potential energy due to elastic deformations is $\frac{1}{2}\ \lfloor x \rfloor\ [K]\ \{x\}$ where $[K]$ is a symmetric matrix of constants, called the stiffness matrix.

d. Because only small motions are considered, the potential energy due to the action of externally applied forces can be written as $-\lfloor F \rfloor\ \{x\}$.[11] The components of $\{F\}$ are the components of the generalized forces corresponding to each generalized coordinate.

e. Forces which retard the motion (damping forces) are assumed to be linearly related to the generalized velocities. Hence the column vector of damping forces equals $-[C]\ \{\dot{x}\}$ where $[C]$ is a symmetric matrix of constants.

Under the above assumptions, the equations of motion in terms of the generalized coordinates are[12]

$$[M]\ \{\ddot{x}\} + [C]\ \{\dot{x}\} + [K]\ \{x\} = \{F\} \qquad (2.1)$$

For a linear elastic structure, we have

$$\{S\} = [J]\ \{x\} \qquad (2.2)$$

where the elements of $\{S\}$ are the components of the internal forces within each member of the structure. $[J]$ is a matrix of constants depending on the elastic properties of the structural members.

Once equations (2.1) have been written for a given structure, they may be greatly simplified according to the specific nature of the problem. We present the following example which is of utmost importance to the present study.

Example 2.1 Consider the case in which the structural system is not subjected to external forces but $L (6 \leq L < N)$ of its generalized coordinates are prescribed functions of time.

We can partition the matrices of Eq. (2.1) to obtain

$$\left[\begin{array}{c|c} M_{rr} & M_{rf} \\ \hline M_{fr} & M_{ff} \end{array}\right] \left\{\begin{array}{c} \ddot{\Delta}_R \\ \hline \ddot{\Delta}_f \end{array}\right\} + \left[\begin{array}{c|c} C_{rr} & C_{rf} \\ \hline C_{fr} & C_{ff} \end{array}\right] \left\{\begin{array}{c} \dot{\Delta}_R \\ \hline \dot{\Delta}_f \end{array}\right\} + \left[\begin{array}{c|c} K_{rr} & K_{rf} \\ \hline K_{fr} & K_{ff} \end{array}\right] \left\{\begin{array}{c} \Delta_R \\ \hline \Delta_f \end{array}\right\}$$

$$= \left\{\begin{array}{c} F_r \\ \hline 0 \end{array}\right\} \qquad (2.3)$$

where $\{\Delta_R\}$ are those generalized displacements which are given functions of time, and $\{F_R\}$ are the reaction forces at the locations where displacements are prescribed.

In expanded form, Eq. (2.3) can be written

$$[M_{rr}]\{\ddot{\Delta}_R\} + [M_{rf}]\{\ddot{\Delta}_f\} + [C_{rr}]\{\dot{\Delta}_R\} + [C_{rf}]\{\dot{\Delta}_f\}$$

$$+ [K_{rr}]\{\Delta_R\} + [K_{rf}]\{\Delta_f\} = \{F_r\} \qquad (2.4)$$

$$[M_{fr}]\{\ddot{\Delta}_R\} + [M_{ff}]\{\ddot{\Delta}_f\} + [C_{fr}]\{\dot{\Delta}_R\} + [C_{ff}]\{\dot{\Delta}_f\}$$

$$+ [K_{fr}]\{\Delta_R\} + [K_{ff}]\{\Delta_f\} = \{0\} \qquad (2.5)$$

Now, we define the <u>relative displacement vector</u>, $\{u\}$;

$$\{\Delta_f\} = -[K_{ff}]^{-1}[K_{fr}]\{\Delta_R\} + \{u\} \qquad (2.6)$$

Substitution of Eq. (2.6) into Eq. (2.5) yields

$$[M_{ff}]\{\ddot{u}\} + [C_{ff}]\{\dot{u}\} + [K_{ff}]\{u\} = [P]\{\ddot{\Delta}_R\} + [R]\{\dot{\Delta}_R\} \qquad (2.7)$$

where
$$[P] = [M_{ff}][K_{ff}]^{-1}[K_{fr}] - [M_{fr}] \qquad (2.8)$$

$$[R] = [C_{ff}][K_{ff}]^{-1}[K_{fr}] - [C_{fr}] \qquad (2.9)$$

Substitution of Eq. (2.6) into Eq. (2.4) gives expressions for the reaction forces in terms of $\{u\}$, $\{\Delta_R\}$ and their derivatives. Thus, in this case, (2.7) is the matrix equation of motion.

In the case for which the damping forces are small, the last term on the right-hand side of Eq. (2.7) may be neglected.[12] Hence, for small damping, the approximate equations of motion are

$$\underset{(N-L)\times(N-L)}{[M_{ff}]}\{\ddot{u}\} + [C_{ff}]\{\dot{u}\} + [K_{ff}]\{u\} = [P]\{\ddot{\Delta}_R\} \qquad (2.10)$$

where $[P]$ is again given by Eq. (2.8).

2.3 <u>Normal Equations of Motion</u>

For the moment, we shall consider Eq. (2.1) without the damping term and with the applied forces set equal to zero.

$$[M]\{\ddot{x}\} + [K]\{x\} = \{0\} \qquad (2.11)$$

That is, we shall consider the free vibration of an undamped structure.

Since (2.11) is linear with constant coefficients, we may try a solution of the form

$$\{x\} = \{a\} \exp(i\omega t) \tag{2.12}$$

where $\{a\}$ is a constant vector.

Substitution of (2.12) into (2.11) yields

$$-[M]\omega^2\{a\} + [K]\{a\} = \{0\} \tag{2.13}$$

after the complex exponential factor has been divided out. The condition to be satisfied for (2.13) to yield a non-trivial solution for the components of $\{a\}$ is

$$|[K] - \omega^2[M]| = 0 \tag{2.14}$$

This is an algebraic equation of order N in ω^2, called the <u>characteristic equation</u>.[12] Let us denote the solutions by ω_i^2 $(i = 1,2,\ldots,N)$. To each separate root of (2.14), ω_i^2, there corresponds a solution to (2.13) which we denote by $\{a_i\}$.

Now, let us define the matrix, $\underset{(N\times N)}{[\varphi]}$, whose i^{th} column is composed of the elements of $\{a_i\}$ and make the transformation

$$\{x\} = [\varphi]\{\xi\} \tag{2.15}$$

It can be shown that the application of this transformation to Eq. (2.13) with the term $\{F\}$ produces uncoupled equations of motion in terms of the components of $\{\xi\}$:[11,12]

$$\lceil m \rfloor\{\ddot{\xi}\} + \lceil k \rfloor\{\xi\} = \{0\}$$

(equation continued)

where

$$\lceil m \rfloor = [\varphi]^T[M][\varphi]$$
$$\lceil k \rfloor = [\varphi]^T[K][\varphi] = \lceil \omega_i^2 \rfloor \lceil m \rfloor \qquad (2.16)$$
$$\{Q\} = [\varphi]^T\{F\}$$

To extend (2.16) to the case wherein damping is present, we assume that [C] is proportional to either [K] or [M]. Under this assumption the transformation (2.15) also diagonalizes [C]. It can then be shown that the full equations of motion are[12]

$$\{\ddot{\xi}\} + 2\lceil \xi_i \omega_i \rfloor \{\dot{\xi}\} + \lceil \omega_i^2 \rfloor \{\xi\} = \lceil m \rfloor^{-1}\{Q\} \qquad (2.17)$$

where $\lceil m \rfloor = [\varphi]^T[M][\varphi]$

$$\{Q\} = [\varphi]^T[F], \quad 2\lceil \xi_i \omega_i \rfloor = \lceil m \rfloor^{-1}[\varphi]^T[C][\varphi]$$

The ω_i are called the natural frequencies of the structural system each of which is associated with a particular mode of free vibration.[12] The ξ_i are called the damping ratios. Eq. (2.17) are the normal equations of motion and the ξ_i are the normal coordinates.[11]

We shall assume hereafter that ξ_i (i=1,2,...,N) << 1.0 since this is true for most engineering structures and particularly for payload structures. When this holds, the free motion of the system takes the form of lightly damped oscillations.[11]

Finally, from (2.15), it can be seen that the member stresses, {S} are related to the normal coordinates by

$$\{S\} = [J][\varphi]\{\xi\} \qquad (2.18)$$

<u>Example 2.2:</u> Derive the normal equations corresponding to Example 2.1.

We write the equations of undamped, free vibration:

$$[M_{ff}]\{\ddot{u}\} + [K_{ff}]\{u\} = 0$$

From (2.13) and (2.14) the natural frequencies and the matrix $[\varphi]$ are obtained from

$$([K_{ff}] - \omega^2[M_{ff}])\{a\} = 0$$

and $$|[K_{ff}] - \omega^2[M_{ff}]| = 0$$

Assuming that $[C_{ff}]$ is proportional to $[M_{ff}]$ or $[K_{ff}]$ we can write

$$\{\ddot{\eta}\} + 2[\ulcorner\xi_i\omega_i\lrcorner]\{\dot{\eta}\} + [\ulcorner\omega_i^2\lrcorner]\{\eta\} = [\ulcorner m\lrcorner]^{-1}[\varphi]^T[P]\{\ddot{\Delta}_r\}$$

where $$\{u\} = [\varphi]\{\eta\} \tag{2.19}$$

$$2[\ulcorner\xi_i\omega_i\lrcorner] = [\ulcorner m\lrcorner]^{-1}[\varphi]^T[C_{ff}][\varphi]$$

$$[P] = [M_{ff}][K_{ff}]^{-1}[K_{fr}] - [M_{fr}]$$

Since equations (2.17) and (2.19) will form the basis of much later work, we consider their solution in the next section.

2.4 <u>Response Formulation to Deterministic Forces</u>

Consider the k^{th} normal equation from (2.17);

$$\ddot{\xi}_k + 2\zeta_k\omega_k\dot{\xi}_k + \omega_k^2\xi_k = \frac{Q_k(t)}{m_k} \tag{2.20}$$

and suppose that $Q_k(t)$ is a unit delta function;

$$Q_k(t) = \delta(t-\tau) \tag{2.21}$$

We shall denote the resulting $\xi_k(t)$ by $h_k(t,\tau)$. $h_k(t,\tau)$ is the <u>impulse response function</u> of the k^{th} mode.[12]

The solution to Eq. (2.20) is then easily obtained:[2]

$$h_k(t-\tau) = \begin{cases} \dfrac{1}{\omega_{kd} m_k} e^{-\zeta_k \omega_k (t-\tau)} \sin \omega_{kd}(t-\tau) & t-\tau \geq 0 \\ 0 & t-\tau < 0 \end{cases} \tag{2.22}$$

where $\omega_{kd} = \omega_k \sqrt{1-\zeta_k^2}$.

Now, any arbitrary $Q_k(t)$ may be constructed from a series of impulses:

$$Q_k(t) = \int_0^{\infty} Q_k(\tau)\delta(t-\tau)d\tau \tag{2.23}$$

assuming that $Q_k(t) = 0$ for $t < 0$.

Hence, when the structure starts its motion from rest at $t = 0$, the response of the k^{th} normal displacement is[2]

$$\xi_k(t) = \int_0^{t} h_k(t-\tau)Q_k(\tau)d\tau \tag{2.24}$$

with $h_k(t-\tau)$ given by Eq. (2.22).

Since this equation holds for all the normal coordinates of the system, it represents a complete response formulation to deterministic excitations which commence at t=0.

<u>Example 2.3</u>: Obtain the response of the system considered in Example 2.1, assuming that $\{\ddot{\Delta}_r\} = 0$ for $t < 0$.

We can write (2.24) in the matrix form:

$$\{\xi\} = \int_0^t \lceil h_k(t-\tau) \rfloor \{Q(\tau\} \, d\tau$$

Now, $\{\xi\}$ is replaced by $\{\eta\}$ and $\{Q(\tau)\}$ is replaced by $\lceil m \rfloor^{-1}[\varphi]^T[P]\{\ddot{\Delta}_r(\tau)\}$. Hence, we have

$$\{\eta\} = \int_0^t \lceil h_k(t-\tau) \rfloor [T]\{\ddot{\Delta}_r(\tau)\} d\tau \tag{2.25}$$

where

$$[T] = \lceil m \rfloor^{-1}[Q]^T([M_{ff}][K_{ff}]^{-1}[K_{fr}] - [M_{fr}]) \tag{2.26}$$

and the $h_k(t-\tau)$ $(k=1,2,\ldots,N-L)$ are given by Eq. (2.22).

In the next several sections we consider the response of a structure to random excitations.

2.5 Brief Discussion of Stochastic Processes

Probability theory will not be discussed here but we shall give a heuristic description of a random process.

Suppose we have a family of random variables, $X_i(t_i)$ $(i=1,2,\ldots,n)$ which are distinguished by the value of some parameter, t, within a range of values, (t_0,t_n). That is, there is a one-to-one correspondence between each random variable, X_i, and a distinct value of a parameter t, $t=t_i$. If we conduct successive experiments whose outcomes correspond to the vector $\{X_1,X_2,X_3,\ldots,X_n\}$ and plot each of the components of the vector against the corresponding values of t, we obtain for each experiment a function defined only at the points $t=t_i$ $(i=1,\ldots,n)$. Figure 2 shows the outcomes of two such experiments.

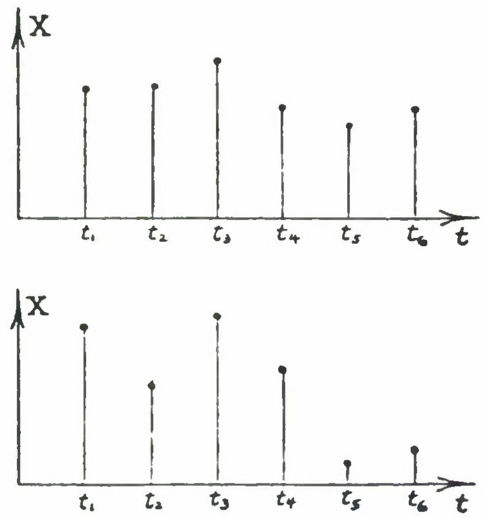

Fig. 2

When the number, n, of random variables, and of corresponding values of the parameter, t, increases without limit within the t interval under consideration, we have a continuously parametered <u>stochastic process</u> in the parameter t, which we denote by X(t).[4] For our work, the parameter t will henceforth be associated with time.

To completely describe a random process, we must specify something like

$$f_x(x_1(t_1), x_2(t_2), \ldots, x_n(t_n))$$

where $f_x(\ldots)$ is the n^{th} order joint probability density of the random variables $X_1(t_1), \ldots, X_n(t_n)$.*[2] Since this is usually impractical, we must be content with more limited information such as a determination of

$f_x(x(t))$, and $f_x(x(t_1), x(t_2))$ for all t, t_1, and t_2.

or of other first and second order statistics.

---

* We denote a random variable by a capital letter whereas we denote the specific values it may assume by the small letter.

In the next sections, the ensemble averaging operation is defined, and its elementary properties are considered.

2.6 The Ensemble Average

The ensemble average of a function of several random variables is defined as the integral of the function multiplied by the joint probability density of the random variables taken over the entire state space. For example, if $G(X_1, X_2, \ldots X_n)$ is a (Borel) function of the n random variables X_1, X_2, and X_n then its ensemble average, denoted by $E[G(X_1, X_2, \ldots X_n)]$ is (Ref. 2, p. 23, Eq. (2-58))

$$E[G] = \int_{x_1=-\infty}^{\infty} \int_{x_2=-\infty}^{\infty} \cdots \int_{x_n=-\infty}^{\infty} G(x_1, x_2, \ldots x_n) F(x_1, x_2, \ldots x_n) dx_1 \ldots dx_n \tag{2.27}$$

provided that the integral remains finite if G is replaced by $|G|$. Here, $f(x_1, x_2, \ldots, x_n)$ is the joint probability density function of the random variables X_1, $X_2, \ldots, X_n$.

Using the ensemble averaging operation, we may define certain useful statistics of random processes. It is to be noted that for a specified value of t, the stochastic process X(t) is a random variable and for a different value of t the process is, in general, a different random variable. Thus, for the stochastic process, X(t), we may define the following statistical averages[2,3,4]

a. The mean of $X(t)$ is $E\{X(t)\}$ and is usually denoted by $\mu_x(t)$

b. The mean square of X is $E\{X^2(t)\}$.

c. The variance of $X(t)$ is $E\{X(t) - \mu_x(t))^2\} = \sigma^2$

d. For n processes, $X_1(t) \ldots X_n(t)$, the average

$$\kappa_{ij} = E[(X_i(t) - \mu_{xi}(t))(X_j(t) - \mu_{xj}(t))]$$

is the second order central moment of $X_i(t)$ and $X_j(t)$. When $\mu_{xi}(t) = \mu_{xj}(t) = 0$ it is simply the second order moment of $X_i(t)$ and $X_j(t)$. In any case, we shall most frequently use the latter designation.

e. The autocorrelation of $X(t)$ is

$$R_x(t_1,t_2) \equiv E[X(t_1)X(t_2)]$$

f. The autocovariance of $X(t)$ is

$$C_x(t_1,t_2) \equiv E[(X(t_1) - \mu_x(t_1))(X(t_2) - \mu_x(t_2))]$$
$$= R_x(t_1,t_2) - \mu_x(t_1)\mu_x(t_2)$$

g. The cross-correlation between two random processes $X(t)$ and $Y(t)$ is

$$R_{xy}(t_1,t_2) = E\{X(t_1)Y(t_2)\}$$

h. The cross-covariance between $X(t)$ and $Y(t)$ is defined as

$$C_{xy}(t_1,t_2) = E[(X(t_1) - \mu_x(t_1))(Y(t_2) - \mu_y(t_2))]$$
$$= R_{xy}(t_1,t_2) - \mu_x(t_1)\mu_2(t_2)$$

A most important result associated with definitions a. and c. is the Tchebycheff inequality:[2,4]

$$P\{|X(t) - \mu_x(t)| \geq k\sigma_x(t)\} \leq \frac{1}{k^2} \tag{2.28}$$

where P{A} is the probability of event A. The existence of this inequality makes the evaluation of the mean and variance of a random process of the utmost importance.

One more important ensemble average will now be introduced. If we take the Fourier transform*(13) of each of the ensemble members of a stochastic process X(t), we obtain under suitable conditions another random process, $\hat{X}(\omega)$, defined in the frequency domain. The <u>generalized spectral density</u> of X(t) can be defined as**

$$\hat{\Phi}_x(\omega_1,\omega_2) = E[\hat{X}(\omega_1)\hat{X}^*(\omega_2)] - E[\hat{X}(\omega_1)]E[\hat{X}^*(\omega_2)] \tag{2.29}$$

This quantity is related to the autocovariance function by the formulae

$$\hat{\Phi}_x(\omega_1,\omega_2) = \int\int_{-\infty}^{\infty} C_{xx}(t_1,t_2)\exp(-i(\omega_1 t_1-\omega_2 t_2))dt_1 dt_2 \tag{2.30}$$

$$C_{xx}(t_1,t_2) = \frac{1}{(2\pi)^2}\int\int_{-\infty}^{\infty} \hat{\Phi}_x(\omega_1,\omega_2)\exp(i(\omega_1 t_1-\omega_2 t_2))d\omega_1 d\omega_2 \tag{2.31}$$

Similarly, for two random processes, X(t) and Y(t), we have the generalized cross-spectral density:

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* See also Chapter III for the definition.

** For a full discussion of this material see Ref. (2), section 3.8.

$$\hat{\Phi}_{xy}(\omega_1,\omega_2) = E[\hat{X}(\omega_1)\hat{Y}^*(\omega_2)] - E[\hat{X}(\omega_1)]E[\hat{Y}^*(\omega_2)] \tag{2.32}$$

which is related to the cross-covariance function by

$$\hat{\Phi}_{xy}(\omega_1,\omega_2) = \int_{-\infty}^{\infty}\int C_{xy}(t_1,t_2)\exp(-i(\omega_1 t_1-\omega_2 t_2))dt_1 dt_2 \tag{2.33}$$

$$C_{xy}(t_1,t_2) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int \Phi_{xy}(\omega_1,\omega_2)\exp(i(\omega_1 t_1-\omega_2 t_2))d\omega_1 d\omega_2 \tag{2.34}$$

2.7 Theorems on Linear, Time Invariant Systems

Now we shall consider two theorems concerning linear time invariant systems, excited by stochastic time functions. Let the input process, $X(t)$, be related to the response of the system, $Y(t)$ by

$$Y(t) = L[X(t)] \tag{2.35}$$

where L is an operator in t which may be algebraic, differential, integral, or any combination of these. The system is linear when

$$L[a_1X_1(t) + a_2X_2(t)] = a_1L[X_1(t)] + a_2L[X_2(t)] \tag{2.36}$$

where $X_1(t)$ and $Y_2(t)$ are two different input processes and a_1 and a_2 are constants. The system is time-invariant when

$$Y(t + \epsilon) = L[X(t+\epsilon)] \tag{2.37}$$

That is, when the input is translated in time by ϵ, the output is translated in the same way.

For a system satisfying Eqs. (2.36) and (2.37) the following result, within certain restrictions,(4) holds true:

$$E[Y(t)] = L[E[X(t)]] \tag{2.38}$$

i.e., the operators E[ ] and L[ ] commute.

As a result of (2.38):

$$C_{xy}(t_1,t_2) = L_{t_2}[C_{xx}(t_1,t_2)] \tag{2.39}$$

$$C_{yy}(t_1,t_2) = L_{t_1}[C_{xy}(t_1,t_2)] \tag{2.40}$$

and so, $$C_{yy}(t_1,t_2) = L_{t_1}[C_{xx}(t_1,t_2)] \tag{2.41}$$

where L_{t_1} and L_{t_2} are the operator L expressed in terms of t_1 and t_2 respectively.

Thus, from (2.41) the autocovariance of the output can be expressed in terms of the autocovariance of the input.

2.8 Autocovariance Response of the Normal Coordinates

The form of Equation (2.24) shows that the structural system, as we have idealized it, satisfies (2.36) and (2.37) and is therefore a linear, time invariant system. If $Q_k(t)$ is a stochastic process then it follows that $\xi_k(t)$ is stochastic. The response equations for $\xi_k(t_1)$ and $\xi_j(t_2)$ are

$$\xi_k(t_1) = \int_0^{t_1} h_k(t_1-\tau_1)Q_k(\tau_1)d\tau_1 \tag{2.42}$$

$$\xi_j(t_2) = \int_0^{t_2} h_j(t_2-\tau_2)Q_j(\tau_2)d\tau_2 \tag{2.43}$$

Taking ensemble averages of both sides of the above equations we obtain:

$$\mu_k(t_1) = \int_0^{t_1} h_k(t_1-\tau_1)E[Q_k(\tau_1)]d\tau_1 \tag{2.44}$$

$$\mu_j(t_2) = \int_0^{t_2} h_j(t_2-\tau_2)E[Q_j(\tau_2)]d\tau_2 \tag{2.45}$$

Multiplying $\xi_k(t_1)$ by $\xi_j(t_2)$ and taking the ensemble average and using (2.38), (2.42) and (2.43):

$$E[\xi_k(t_1)\xi_j(t_2)] = R_{kj}(t_1,t_2) = \int_0^{t_1}\int_0^{t_2} h_k(t_1-\tau_1) \cdot h_j(t_2-\tau_2)E[Q_k(\tau_1)Q_j(\tau_2)]d\tau_1 d\tau_2 \tag{2.46}$$

Multiplying $\mu_k(t_1)$ by $\mu_j(t_2)$ and employing (2.44) and (2.45):

$$\mu_k(t_1)\mu_j(t_2) = \int_0^{t_1}\int_0^{t_2} h_k(t_1-\tau_1)h_j(t_2-\tau_2) \cdot E[Q_k(\tau_1)]E[Q_j(\tau_2)]d\tau_1 d\tau_2 \tag{2.47}$$

Now, if we subtract Eq. (2.47) from (2.46) and employ definitions g. and h. of section 2.6 we see that

$$C_{kj}(t_1,t_2) = \int_0^{t_1}\int_0^{t_2} h_k(t_1-\tau_1)h_j(t_2-\tau_2)C_{Q_{kj}}(\tau_1,\tau_2)d\tau_1 d\tau_2 \tag{2.48}$$

where $C_{kj}(t_1,t_2)$ is the cross covariance of the k^{th} and j^{th} normal coordinates and $C_{Q_{kj}}(\tau_1,\tau_2)$ is the cross-covariance of the k^{th} and j^{th} components of the excitation.

Furthermore from definition d. section 2.6 it is apparent that

$$[C_{kj}(t_1,t_2)]_{t_1=t_2=t} = \kappa_{ij}(t) \tag{2.49}$$

Therefore, from Eq. (2.48):

$$\kappa_{kj}(t) = \int_0^t \int_0^t h_k(t-\tau_1)h_j(t-\tau_2)C_{Q_{kj}}(\tau_1,\tau_2)d\tau_1 d\tau_2 \tag{2.50}$$

Now, from (2.18) and (2.38):

$$E[\{S\}] = [J][\varphi]E[\{\xi\}]$$

and
$$E[\{S\}\lfloor S\rfloor] = [J][\varphi]E[\{\xi\}\lfloor\xi\rfloor][\varphi]^T[J]^T$$

so that from definition d., section 2.6:

$$[\kappa_{S_{kj}}] = [J][\varphi][\kappa_{kj}(t)][\varphi]^T[J]^T \tag{2.51}$$

where $[\kappa_{S_{kj}}]$ is the matrix of second order moments of the components of the stress vector. The diagonal terms of $[\kappa_{S_{kj}}(t)]$ are the variances of the member loads. These can be determined, according to Eq. (2.51), only when $[\kappa_{kj}(t)]$ is first known. Thus, in later sections we shall employ Eq. (2.50) repeatedly.

In the final section of this chapter two important classes of stochastic processes which will be useful in later discussions, are introduced.

2.9 Two Special Classes of Stochastic Input Processes

Here we consider weakly stationary and shot noise processes.

A stochastic process is said to be weakly stationary when its mean is a constant and its autocorrelation can be expressed as a function of $|t_1-t_2|$, i.e., $X(t)$ is weakly stationary when[2]

$$\mu_x(t) = \text{constant} \tag{2.52}$$

and
$$R_x(t_1,t_2) = R_x(t_2-t_1) \tag{2.53}$$

where $R_x(t_2-t_1)$ is an even function.

It can be shown that if the input to a linear time invariant system is weakly stationary, so is the response of the system.[3,4]

Furthermore, it can be shown that the generalized spectral density of a stationary process is of the form[4]

$$\hat{\Phi}_x(\omega_1,\omega_2) = S_x(\omega_1)\delta(\omega_2-\omega_1)2\pi \tag{2.54}$$

and
$$S_x(\omega_1) = \int_{-\infty}^{\infty} R_x(\tau)e^{-i\omega_1\tau}\, d\tau \tag{2.55}$$

where $\tau = |t_2-t_1|$

Finally, for a stationary excitation, the second order moment between the k^{th} and j^{th} normal displacements is given by[3]

$$\kappa_{kj} = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_k(\omega)H_j^*(\omega)S_x(\omega)d\omega \tag{2.56}$$

where $H_k(\omega)$ and $H_j(\omega)$ are the Fourier transforms of $h_k(t)$ and $h_j(t)$ respectively and are called the <u>complex frequency response functions</u> of the k^{th} and j^{th} modes.

Finally, we say that a random process, S(t), is a short noise when its mean and covariance functions are given by[2]

$$\mu_s(t) = 0 \tag{2.57}$$

$$C_{ss}(t_1,t_2) = I(t_1)\delta(t_2-t_1) \tag{2.58}$$

where $I(t)$ is called the intensity function of the shot noise.

When each of the components of a vector $\{F\}$ is a shot noise process, we call $\{F\}$ a <u>vector shot noise</u>.[2] In this case:

$$[C_{s_{ij}}(t_1,t_2)] = [I_{ij}(t_1)]\delta(t_2-t_1) \qquad (2.59)$$

This concludes the discussion of the probabilistic theory of structural dynamics. It is hoped that this brief introduction will suffice in providing the foundations of our later work.

CHAPTER III

ELEMENTARY DISCUSSION OF THE FOURIER INTEGRAL

3.1 Introduction

In this chapter, we define the Fourier integral and describe its elementary properties. The applications of the Fourier integral to analysis of system response to both deterministic and random excitations is also developed. The chapter concludes with the derivation of an approximate inversion formula which is shown to be closely related to Reimann's lemma.

3.2 Definitions and Simple Theorems*

We shall define the Fourier integral, $F(\omega)$, of a function of time, $f(t)$, as

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t} dt \tag{3.1}$$

where the parameter ω is called the frequency and is always a real quantity. Note that from (3.1), $F(\omega)$ is, in general, a complex quantity. $F(\omega)$ is also called the Fourier transform of $f(t)$. Henceforth, we shall adhere to this designation.

It is known that $f(t)$ may be represented in terms of its Fourier transform by the following equation, known as the inversion formula:

* The material in this section is taken from Ref. (13).

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t} d\omega \qquad (3.2)$$

The above representation is valid only at continuity points of f(t). However, the inversion formula (3.2) also holds at discontinuity points if it is assumed that

$$f(a) = \frac{1}{2} \left(f(a^+) + f(a^-)\right) \qquad (3.3)$$

where t=a is a point of discontinuity. $f(a^+)$ and $f(a^-)$ denote the value of the function as t approaches a from the right and from the left respectively.

Sufficient conditions for the validity of (3.2) are given in Ref. (13), p.9. The functions we shall deal with will be assumed to satisfy these conditions.

When a function of time, f(t), and a function, $F(\omega)$, are related as in Eqs. (3.1) and (3.2), we shall indicate the relationship by

$$f(t) \leftrightarrow F(\omega)$$

We shall now consider the following simple theorems:

<u>Theorem 1</u>: Linearity

It is apparent from the definitions that if

$$f_1(t) \leftrightarrow F_1(\omega) \text{ and } f_2(t) \leftrightarrow F_2(\omega)$$

then

$$\int_{-\infty}^{\infty} (a_1 f_1(t) + a_2 f_2(t)) e^{-i\omega t} dt = a_1 F_1(\omega) + a_2 F_2(\omega)$$

or, $$a_1 f_1(t) + a_2 f_2(t) \leftrightarrow a_1 F_1(\omega) + a_2 F_2(\omega) \tag{3.4}$$

The above can be extended to finite sums

$$a_1 f_1(t) + \ldots + a_n f_n(t) \leftrightarrow a_1 F_1(\omega) + \ldots + a_n F_n(\omega)$$

but the extension to infinite sums is not always valid.

Thus, as we see from (3.4), the Fourier integral operator is linear.

<u>Theorem 2</u>: Time Shifting

Let us replace t in Eq. (3.2) by $t-t_0$ where t_0 is a prescribed (real) constant. Then $d(t-t_0) = dt$ and

$$f(t-t_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega(t-t_0)} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} [F(\omega) e^{-i\omega t_0}] e^{i\omega t} d\omega$$

or

$$f(t-t_0) \leftrightarrow F(\omega) e^{-i\omega t_0} \tag{3.5}$$

where $F(\omega)$ is the transform of $f(t)$.

<u>Theorem 3</u>: Time Differentiation

Differentiating both sides of Eq. (3.2) with respect to time, we obtain:

$$\frac{df}{dt} = \frac{1}{2\pi} \int_{-\infty}^{\infty} [F(\omega) i\omega] e^{i\omega t} d\omega$$

or, $$\frac{df(t)}{dt} \leftrightarrow i\omega\, F(\omega)$$

Doing this n times, we have

$$\frac{d^n f(t)}{dt^n} \leftrightarrow (i\omega)^n F(\omega) \tag{3.6}$$

Theorem 4: Symmetry

If we rewrite (3.2) in the form

$$2\pi f(-t) = \int_{-\infty}^{\infty} F(\omega)e^{-i\omega t}d\omega$$

and interchange t and ω we obtain a statement of the symmetrical character of Eqs. (3.1) and (3.2):

$$F(t) \leftrightarrow 2\pi f(-\omega) \tag{3.7}$$

Thus, theorems obtained in the time domain may be easily extended to the frequency domain.

We shall now state without proof, two theorems fundamental to the application of the Fourier transform to the analysis of linear, time invariant systems.

Theorem 5: Time Convolution

Let g(t) and f(t) be two time functions for which the inversion formula is valid. The integral $\int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)dt$ is the convolution of f(t) and g(t) in the time domain. We shall use the notation

$$\int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau = f(t)*g(t) \tag{3.8}$$

The Fourier transform of f(t)*g(t) is given by

$$f(t)*g(t) \leftrightarrow F(\omega)G(\omega) \tag{3.9}$$

where $F(\omega)$ and $G(\omega)$ are the Fourier transforms of $f(t)$ and $g(t)$ respectively.

<u>Theorem 6</u>: Frequency Convolution

From Eqs. (3.7) and (3.9)

$$f(t)g(t) \leftrightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega-y)G(y)dy$$

or

$$f(t)g(t) \leftrightarrow \frac{1}{2\pi} F(\omega)*G(\omega) \tag{3.10}$$

where now, the convolution is taken in the frequency domain.

For much of our work, we shall need the <u>delta function</u>, $\delta(t-t_0)$. This is not a function in the usual sense, since it is not defined by its values for given values of the argument, but by the following integral properties

$$\int_{-\infty}^{\infty} \delta(t-t_0)\varphi(t)dt = \varphi(t_0) \tag{3.11}$$

where $\varphi(t)$ is an arbitrary function, continuous at the given point t_0. Similarly:

$$\int_{-\infty}^{\infty} \frac{d^n \delta(t-t_0)}{dt^n} \varphi(t)dt = (-1)^n \frac{d^n \varphi(t_0)}{dt^n} \tag{3.12}$$

The Fourier transform of the delta function is readily obtained from (3.2) and (3.11):

$$F(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} \delta(t)dt = 1$$

or

$$\delta(t) \leftrightarrow 1 \tag{3.13}$$

Applying Theorem 2 to (3.13) we see that

$$\delta(t-t_0) \leftrightarrow e^{-i\omega t_0} \tag{3.14}$$

From this result and Eq. (3.7) we have

$$e^{i\omega_0 t} \leftrightarrow 2\pi\ \delta(\omega-\omega_0) \tag{3.15}$$

Since the delta function appears often in later discussions we shall make frequent use of the above results.

In the next section, the Fourier transform is applied to the determination of the response of a linear system to deterministic excitations.

3.3 System Response to Deterministic Excitations

Consider the k^{th} normal coordinate of a structural system. Its response to a deterministic excitation, $Q_k(t)$, is given by Eq. (2.24):

$$\xi_k(t) = \int_0^t h_k(t-\tau)Q_k(\tau)d\tau \tag{3.16}$$

Since $Q_k(\tau)$ is zero for $\tau < 0$ and $h_k(t-\tau)$ is zero for $\tau > t$, the limits of integration in (3.16) can be replaced by $-\infty$ and ∞:

$$\xi_k(t) = \int_{-\infty}^{\infty} h_k(t-\tau)Q_k(\tau)d\tau \tag{3.17}$$

Taking the Fourier transform of both sides of (3.17) and applying Eq. (3.9) gives

$$\Xi_k(\omega) = H_k(\omega)\hat{Q}_k(\omega) \tag{3.18}$$

where $\Xi_k(\omega)$, $H_k(\omega)$ and $\hat{Q}_k(\omega)$ are the transforms of ξ_k, $h_k(t)$ and Q_k respectively. We can express (3.18) more conveniently in matrix notation as follows

$$\Xi_k(\omega) \quad [H_k]\{\hat{Q}_k(\omega)\} \tag{3.19}$$

For the impulse response function given by (2.22) the Fourier transform is[2]

$$H_k(\omega) = \frac{1}{m_k}\,\frac{1}{\omega_k^2-\omega^2+2i\zeta_k\omega_k\omega} \tag{3.20}$$

Eq. (3.18) shows that once the transform of the excitation is obtained, the transform of the response can be readily calculated and application of the inversion formula (3.2) gives the corresponding time function.

3.4 Formulation of Response To Random Excitations

Recall the definition of the generalized spectral density of a random process, X(t):

$$\hat{\Phi}_X(\omega_1,\omega_2) = E[\hat{X}(\omega_1)\hat{X}^*(\omega_2)] - E[\hat{X}(\omega_1)]E[\hat{X}^*(\omega_2)] \tag{3.21}$$

This can be immediately extended to a vector process, $\{X_i(t)\}$:

$$[\hat{\Phi}_{ij}(\omega_1,\omega_2)] = E[\{\hat{X}_i(\omega_1)\} \lfloor \hat{X}^*_i(\omega_2)\rfloor]$$

$$- E[\{\hat{X}_i(\omega_1)\}]E[\lfloor \hat{X}^*_i(\omega_2\rfloor] \tag{3.22}$$

where $[\hat{\Phi}_{ij}(\omega_1,\omega_2)]$ is the matrix of generalized spectral densities of the components of the vector $\{X_i(t)\}$.

$[\hat{\Phi}_{ij}(\omega_1,\omega_2)]$ is related to the matrix of covariance functions, $[C_{ij}(t_1,t_2)]$ by[2]

$$[\hat{\Phi}_{ij}(\omega_1,\omega_2)] = \iint_{-\infty}^{\infty} [C_{ij}(t_1,t_2)]\exp(-i(\omega_1 t_1 - \omega_2 t_2))dt_1 dt_2 \tag{3.23}$$

$$[C_{ij}(t_1,t_2)] = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty}[\hat{\Phi}_{ij}(\omega_1,\omega_2)]\exp(i(\omega_1 t_1 + \omega_2 t_2))d\omega_1 d\omega_2 \tag{3.24}$$

With the above definitions and the results of section 3.3, we can relate the spectral density of $\{\xi_i(t)\}$ to the spectral density of $\{Q_k(t)\}$.

Replacing ω in Eq. (3.18) by ω_1, we have

$$\{\Xi_k(\omega_1)\} = \lceil H_k(\omega_1)\rfloor \{\hat{Q}_k(\omega_1)\} \tag{3.25}$$

Now, replacing ω by ω_2, taking the transpose and then the conjugate of Eq. (3.18), we have

$$\Xi^*_k(\omega_2) = \lfloor \hat{Q}^*_k(\omega_2)\rfloor \lceil H^*_k(\omega_2)\rfloor \tag{3.26}$$

From (3.19) and (3.20), the quantitites $E[\{\Xi_k(\omega_1)\}\lfloor \Xi^*_k(\omega_2)\rfloor]$ and $E[\{\Xi_k(\omega_1)\}]E[\lfloor \Xi^*_k(\omega_2)\rfloor]$ are given by

$$E[\{\Xi_k(\omega_1)\}\lfloor\Xi_k^*(\omega_2)\rfloor] = [\diagdown H_k(\omega_1)\diagdown]E[\{\hat{Q}_k(\omega_1)\}\lfloor\hat{Q}_k^*(\omega_2)\rfloor]$$

$$\cdot[\diagdown H_k^*(\omega_2)\diagdown] \qquad (3.27)$$

$$E[\{\Xi_k(\omega_1)\}]\; E[\lfloor\Xi_k^*(\omega_2)\rfloor]$$

$$= [\diagdown H_k(\omega_1)\diagdown]E[\{\hat{Q}_k(\omega_1)\}]E[\lfloor\hat{Q}_k^*(\omega_2)\rfloor][\diagdown H_k^*(\omega_2)\diagdown] \qquad (3.28)$$

Subtracting (3.28) from (3.27) and employing (3.22), we obtain

$$[\hat{\Phi}_\xi(\omega_1,\omega_2)] = [\diagdown H_k(\omega_1)\diagdown][\hat{\Phi}_Q(\omega_1,\omega_2)][\diagdown H_k^*(\omega_2)\diagdown] \qquad (3.29)$$

where $[\hat{\Phi}_Q(\omega_1,\omega_2)]$ is the matrix of generalized spectral densities of the vector process $\{Q_k(t)\}$.

It can be seen that the important result (3.29) is the counterpart of Eq. (2.48) in the frequency domain. When $[\hat{\Phi}_Q(\omega_1,\omega_2)]$ is known, $[\hat{\Phi}_\xi(\omega_1,\omega_2)]$ may be readily obtained and the correlation matrix of the response calculated by a double application of the inversion integral, Eq. (3.2)

3.5 Evaluation of the Inverse Transform

It is clear from the preceeding sections that efficient application of the Fourier transform methods to both deterministic and random response of a system depends upon the application of the inversion formula, Eq. (3.2). Therefore, we shall consider the calculation of the integral $\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega$ when $F(\omega)$ is a given function of ω.

The integral may be conveniently calculated if we allow x to assume values in the complex plane. That is, we replace x by the complex number $z = x + iy$ and consider the integral

$$I = \frac{1}{2\pi} \oint_c F(z)e^{izt} dz \tag{3.30}$$

where c is a conveniently chosen closed curve in the complex plane. In this discussion we assume that F(z) is analytic everywhere in the finite z plane except at certain isolated points $a_1, a_2, \ldots, a_n$ where it has poles of various orders. When this is true we may use the calculus of residues to evaluate the inversion integral. To do this, however, we must introduce the following essential theorems, stated without proof.*

Theorem 1:

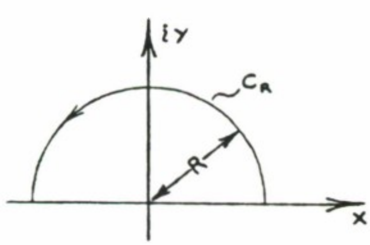

Fig. 3(a)

Suppose that, on a circular arc C_R with radius R and center at the origin, $F(z) \to 0$ uniformly as $R \to \infty$. Then:

$$\lim_{R\to\infty} \int_{C_R} e^{imz} F(z)dz = 0 \quad (m>0) \tag{3.31}$$

if C_R is in the first and/or second quadrants. This result is known as Jordan's lemma.

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* See reference (14), p. 556.

Theorem 2:

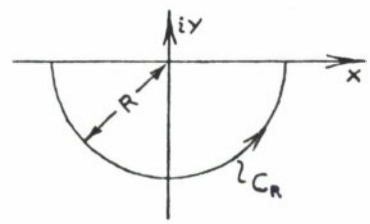


Fig. 3(b)

Now let C_R extend over the third and/or fourth quadrants. Then

$$\lim_{R\to\infty} \int_{C_R} e^{-imz} F(z)dz = 0 \quad (m>0) \qquad (3.32)$$

With these theorems a procedure for evaluating the inversion integral can be outlined. We shall consider separately the two cases $t > 0$ and $t < 0$.

$t > 0$: Let the path of integration in Eq. (3.30) be taken in the counterclockwise direction along the closed curve consisting of a line segment on the real axis and a circular arc, C_R, as shown.

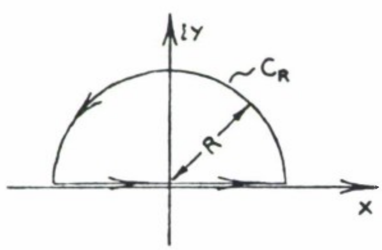


Fig. 4

Eq. (3.30) can then be written

$$I = \frac{1}{2\pi} \int_{-R}^{R} F(x)e^{ixt}dx + \frac{1}{2\pi} \int_{C_R} F(z)e^{izt}dz \qquad (3.33)$$

Taking the limit of I as $R \to \infty$ and applying The. 1 with m replaced by t, we see that

$$\lim_{R\to\infty} I = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(x)e^{ixt}dx \qquad (3.34)$$

which is the desired inversion integral. But, by the calculus of residues:[14]

$$\lim_{R\to\infty} I = i \sum_k \text{Res}(F(z)e^{izt};a_k) \tag{3.35}$$

where $\text{Res}(F(z)e^{izt};a_k)$ is the residue of $F(z)e^{izt}$ at the point of singularity a_k and the summation is taken over the singularities in the entire upper half z plane. Equating the left hand sides of Eqs. (3.34) and (3.35) we achieve the final result:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega = i \sum_k \text{Res}(F(z)e^{izt};a_k) \tag{3.36}$$

for $t > 0$ and for all a_k such that $\text{Im}(a_k) \geq 0$

t < 0: Now let the path of integration be as shown at left

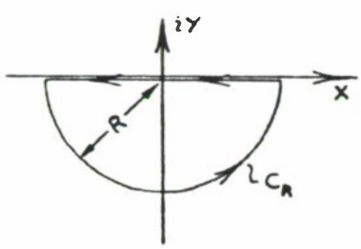


Fig. 5

and let $\tilde{t} = |t|$. Then

$$I = -\frac{1}{2\pi}\int_{-R}^{R} F(x)e^{-ix\tilde{t}}dx + \int_{C_R} F(z)e^{-iz\tilde{t}}dz$$

Again, taking the limit as $R \to \infty$, and applying The. 2, we have

$$\lim_{R\to\infty} I = -\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega\tilde{t}}d\omega \tag{3.37}$$

Employing the same reasoning as for $t > 0$ we conclude that

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega = -i \sum_k \text{Res}(F(z)e^{izt};a_k) \tag{3.38}$$

for $t < 0$ and for all a_k such that $\text{Im}(a_k) < 0$.

When $f(0^+) \neq f(0^-)$ the value of $[f(t)]_{t=0}$ is

$$[f(t)]_{t=0} = \frac{1}{2}\,(f(0^+) + f(0^-)) \tag{3.39}$$

as has been stated in section 3.2.

In our applications, F(z) will be a rational function; so that all its singularities will be poles. For this case, the residues are most simply calculated. If F(z) has a pole of order m at z=a, then[14]

$$\mathrm{Res}(F(z)e^{izt};a) = \frac{1}{(m-1)!}\;\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m\,F(z)e^{izt}\}_{z=a} \tag{3.40}$$

since e^{izt} has no singularities in the finite z-plane.

To illustrate the above methods, we present the following example.

<u>Example 3.1</u>

Consider a simple oscillator for which the displacement, $\xi(t)$, is given by

$$\ddot{\xi} + 2\zeta\Omega\dot{\xi} + \Omega^2\xi = \frac{Q(t)}{m} \tag{3.41}$$

where Q(t) is a shot noise with a constant intensity function. Calculate the correlation function and mean square of $\xi(t)$.

Let the constant intensity function be I; then from (2.58):

$$C_Q(t_1,t_2) = I\,\delta(t_2-t_1) \tag{3.42}$$

Thus Q is a stationary process. With this correlation function, we have from Eq. (3.23)

$$\hat{\Phi}_Q(\omega_1,\omega_2) = I \iint_{-\infty}^{\infty} \delta(t_2-t_1)\exp(-i(\omega_1 t_1-\omega_2 t_2))dt_1 dt_2$$

$$= 2\pi\ I\ \delta(\omega_1-\omega_2) \qquad (3.43)$$

by Eqs. (3.11) and (3.15).

Therefore, from (3.29):

$$\hat{\Phi}_\xi(\omega_1,\omega_2) = 2\pi\ I\ \delta(\omega_1-\omega_2)H(\omega_1)H^*(\omega_2) \qquad (3.44)$$

where $H(\omega)$ is the frequency response function associated with (3.41). Substituting the expression for $\hat{\Phi}_\xi(\omega_1,\omega_2)$ into (3.24) we have

$$C_\xi(t_1,t_2) = \frac{I}{2\pi} \iint_{-\infty}^{\infty} \delta(\omega_1-\omega_2)H(\omega_1)H^*(\omega_2)$$

$$\exp(i(\omega_1 t_1-\omega_2 t_2))d\omega_1 d\omega_2$$

or

$$C_\xi(t_1,t_2) = \frac{I}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 e^{i\omega(t_1-t_2)} d\omega \qquad (3.45)$$

by (3.11). Since the combination (t_1-t_2) appears in the exponent in Eq. (3.45), we see that C_ξ is a function of (t_1-t_2) only. Hence, the response is weakly stationary and we can write:

$$C_\xi(\tau) = \frac{I}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2\ e^{i\omega\tau} d\omega \qquad (3.46)$$

where $\tau \equiv t_1-t_2$

We can now apply the methods of this section to the evaluation of the integral in (3.46).

From (3.20), the frequency response function is $\frac{1}{m^2}(\Omega^2-\omega^2 + 2i\zeta\Omega\omega)^{-1}$ and so:

$$|H(\omega)|^2 = \frac{1/m^2}{(\Omega^2-\omega^2)^2 + 4\zeta^2\Omega^2\omega^2}$$

This can be written as

$$|H(\omega)|^2 = \frac{1/m^2}{(\omega-a_1)(\omega-a_2)(\omega+a_1)(\omega+a_2)} \qquad (3.47)$$

where $a_1 = i\zeta\Omega + \Omega_d$, $a_2 = i\zeta\Omega - \Omega_d$ and $\Omega_d = \Omega\sqrt{1-\zeta^2}$.

Substituting (3.47) into (3.46), we obtain

$$C_\xi(\tau) = \frac{I}{2\pi m^2} \int_{-\infty}^{\infty} \frac{e^{i\omega\tau} d\omega}{(\omega-a_1)(\omega-a_2)(\omega+a_1)(\omega+a_2)} \qquad (3.48)$$

Let us first consider the case $\tau > 0$. We first replace ω by $z = x + iy$ and adopt the contour of integration shown in Fig. 3(a). Employing (3.36) and (3.40), in the expression for $C_\xi(\tau)$, we have

$$C_\xi(\tau) = \frac{I\, e^{-\zeta\Omega\tau}}{4\zeta\Omega^3 m^2} \left[\cos \Omega_d\tau + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \Omega_d\tau\right]$$

for $\tau > 0$.

For the case $\tau < 0$, we adopt the contour of Fig. 3(b). Setting $\tilde{\tau} = |\tau|$ and employing (3.38) and (3.40):

$$C_\xi(\tau) = \frac{I\ e^{-\zeta\Omega\tilde{\tau}}}{4\zeta\Omega^3 m^2}\,[\cos\Omega_d\tilde{\tau} + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\Omega_d\tilde{\tau}]$$

Hence, for all τ, the correlation function of $\xi(t)$ is given by

$$C_\xi(\tau) = \frac{I\ e^{-\zeta\Omega|\tau|}}{4\zeta\Omega^3 m^2}\,(\cos\Omega_d\tau + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\Omega_d|\tau| \qquad (3.49)$$

Since the mean of $\xi(t)$ is zero, its mean square is

$$\kappa_\xi = E[\xi^2(t)] = \frac{I}{4\zeta\Omega^3 m^2} \qquad (3.50)$$

3.6 Approximate Inversion of the Fourier Transform

In most of our considerations, we shall be dealing with functions possessing Fourier transforms which are rational functions. Hence, it would appear that the evaluation of the inverse Fourier transforms in our applications is perfectly simple. However, when the Fourier transform to be inverted has many singularities in the complex plane, the resulting expression for the inverse transform may be quite complicated. In such cases, it is more desirable to obtain useful approximate forms of the transform such that its inverse is algebraically simplified. Here, we present one such approximation, valid for many of the cases to be considered. Essentially, the approximations involve ignoring the presence of singularities for which the residues of the transform are small. We shall consider only transforms which are rational functions and which possess no singularities in the lower half z-plane. As can be seen from Eq. (3.38), the latter restriction implies that the inverse transforms are zero for negative t. Such functions are called causal functions of time;[13] and our later discussions will invariably involve them.

With the above considerations in mind, consider the evaluation of the integral

$$I(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{J(\omega)\, e^{i\omega t}}{F(\omega)G(\omega)}\, d\omega \qquad (t > 0) \tag{3.51}$$

where $F(z)$ and $G(z)$ are polynomials and $J(z)$ is analytic in the upper ahlf z plane and possesses no zeros in common with $F(z)$ and $G(z)$. We shall assume the following:

a. $F(z)$ has zeros of order one at the distinct points $a_1, a_2, \ldots, a_n$ and

$$|a_i - a_j| \leq \delta_1 \tag{3.52}$$

b. $G(z)$ has zeros of order one at the distinct points $b_1, b_2, \ldots, b_m$ and

$$|b_i - b_j| \geq \epsilon_2 \tag{3.53}$$

c. Finally:

$$\gamma_1 \leq |a_i - b_j| \leq \gamma_2 \quad \text{where} \quad \gamma_1, \gamma_2 > 0 \tag{3.54}$$

With these assumptions, it can be shown* that

---

* See Appendix I.

$$I(t) = \left\{\frac{1}{2\pi G(a)} \int_{-\infty}^{\infty} \frac{J(\omega)}{F(\omega)} e^{i\omega t} d\omega\right\} (1+T_1)(1+T_2) \tag{3.55}$$

where

$$O\{T_1\} = O\left\{\frac{m\,\delta_1^{n-1}\,\gamma_2^{m}\ \mathrm{MAX}[|J(b_k)|]}{n\,\epsilon_2^{m-1}\,\gamma_1^{n}\ \mathrm{MIN}[|J(a_k)|]}\right\} \tag{3.56}$$

$$O\{T_2\} = O\left\{\ \mathrm{MAX}\left[\frac{G(a)}{g(a+\Delta_j)} - 1\right]\right\} \tag{3.57}$$

and where

$$a_j = a + \Delta_j\ ,\ \Delta_j < \delta_1 \qquad (j = 1,\ldots,n) \tag{3.58}$$

that is, a is some point in the vicinity of the a_j's.

If the Fourier transform can be factored into the form of the integrand in (3.51) in such a way that the quantities given by Eqs. (3.56) and (3.57) are small, then

$$I(t) \approx \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{J(\omega)}{G(a)F(\omega)} e^{i\omega t} d\omega \tag{3.59}$$

with a relative error of $(T_1 + T_2 + T_1T_2)$.

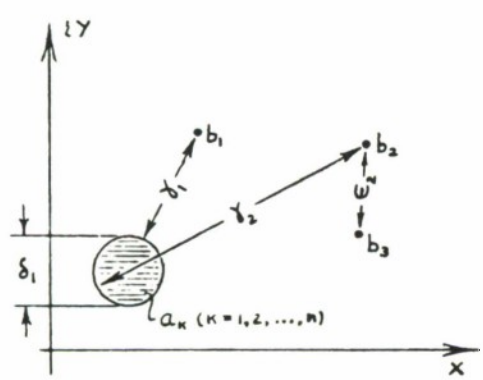


Fig. 6

A situation for which (3.59) is valid occurs when the zeros of F(z) are clustered within a small region of the z plane (Fig. 6) far removed from the zeros of G(z) which are themselves widely scattered. As an illustration of this situation, we present the following example.

Example 3.2:

a) Consider the integral:

$$K(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(\omega) \frac{\beta e^{i\omega t} d\omega}{(\alpha + i\omega)^2 + \beta^2} \tag{3.60}$$

where α and β are real and positive, and where $\Phi(\omega) = \frac{1}{F(\omega)}$

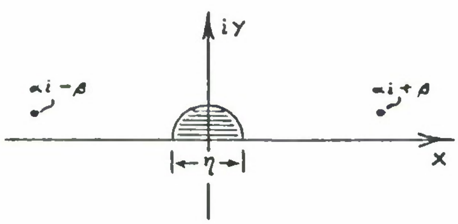


Fig. 7

and the n poles of $\frac{1}{F(\omega)}$ are confined between the real axis and a circular arc of radius $\eta/2$ with center at the origin (as shown in Fig. 7). Factoring the term $(\alpha + i\omega)^2 + \beta^2$, we may write

$$K(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(\omega) \frac{-\beta \ e^{i\omega t} d\omega}{(\omega - \alpha i + \beta)(\omega - \alpha i - \beta)}$$

Thus the poles of $\frac{1}{(\alpha + i\omega)^2 + \beta^2}$ occur at $\omega = \alpha i \pm \beta$. We shall assume that $|\alpha i \pm \beta| >> \eta$ so that we may apply Eqs. (3.55) through (3.57).

In this case:

$$m = 2, \quad \delta_1 = \eta, \quad \epsilon_2 = 2\beta$$

$$\gamma_1 \approx \sqrt{\alpha^2 + \beta^2} - \frac{\eta}{2}, \quad \gamma_2 \approx \sqrt{\alpha^2 + \beta^2} + \frac{\eta}{2}$$

Furthermore, since $J(\omega) = 1$, $\frac{\text{MAX}\,[|J(b_k)|]}{\text{MIN}\,[|J(a_k)|]} = 1$. The point $z = a$ may be conveniently chosen as the origin, i.e., $a = 0$. Hence, from (3.55), (3.56) and (3.57) we have

$$K(t) = \{\frac{1}{2\pi} \frac{\beta}{\alpha^2+\beta^2} \int_{-\infty}^{\infty} \Phi(\omega)\, e^{i\omega t} d\omega\}\ (1+T_1)(1+T_2) \qquad (3.61)$$

where

$$O\,\{T_1\} = O\,\{\frac{1}{n} \frac{\eta^{n-1}}{\beta} (\alpha^2 + \beta^2)^{1-\frac{\eta}{2}}\} \qquad (3.62)$$

since $\sqrt{\alpha^2+\beta^2} \pm \frac{\eta}{2} \approx \sqrt{\alpha^2 + \beta^2}$, and

$$O\,\{T_2\} = O\,\{\frac{2\eta}{\sqrt{\alpha^2 + \beta^2}}\} \qquad (3.63)$$

b) Now, let $\varphi(t) \leftrightarrow \Phi(\omega)$. Then by the convolution theorem and the fact that $e^{-\alpha t}\,\text{SIN}\,\beta t\; U(t) \leftrightarrow \frac{\beta}{(\alpha+i\omega)^2 + \beta^2}$:

$$K(t) = \int_0^t \varphi(t)\; e^{-\alpha(t-\tau)}\,\text{SIN}\,\beta(t-\tau)d\tau$$

Hence, from (3.61):

$$\int_0^t \varphi(t-\tau)\; e^{-\alpha\tau}\;\text{SIN}\,\beta\tau\; d\tau = \frac{\beta}{\alpha^2+\beta^2}\,\varphi(t)\;(1+T_1)(1+T_2) \qquad (3.64)$$

Using Eqs. (3.62) through (3.64) we can estimate the order of magnitude of $K(t)$. Consider the case $n=1$. We have for small α;

$$O\,\{T_1\} = O\,\{\frac{\sqrt{\alpha^2+\beta^2}}{\beta}\} \approx 1$$

$$O\{T_2\} = O\{\frac{2\pi}{\beta}\} << 1$$

Hence, if $O\{\varphi u\} \leq A$, then

$$O\{\int_0^t \varphi(t-\tau)\, e^{-\alpha\tau}\, \text{SIN}\, \beta\tau\, d\tau\} \leq O\{\frac{2A}{\beta}\} \qquad (3.65)$$

Now when $n \geq 2$, the order of T_1 is always << 1 from (3.75). Hence for $n \geq 2$:

$$O\{\int_0^t \varphi(t-\tau)\, e^{-\alpha\tau}\, \text{SIN}\, \beta\tau\, d\tau\} \approx O\{\frac{A}{\beta}\} \qquad (3.66)$$

Since β is large, we can write the general result:

$$O\{\int_0^t \varphi(t-\tau)\, e^{-\alpha\tau}\, \text{SIN}\, \beta\tau\, d\tau\} = O\{\frac{A}{\beta}\} \qquad (3.67)$$

In particular, for $\alpha = 0$;

$$O\{\int_0^t \varphi(t-\tau)\, \text{SIN}\, \beta\tau\, d\tau\} = O\{\frac{A}{\beta}\} \qquad (3.68)$$

3.7 Reimann's Lemma

The result (3.68) in the example of the last section is a special case of a result known as Reimann's lemma. This can be stated as follows:

If $\varphi(\tau)$ is of bounded variation in the range a to b, and has an upper bound of order one, then

$$O\{\int_a^b \varphi(\tau)\cos\beta\tau\, d\tau\} = O\{\frac{1}{\beta}\} \qquad (3.69)$$

$$O\{\int_a^b \varphi(\tau)\sin\beta\tau\, d\tau\} = O\{\frac{1}{\beta}\} \qquad (3.70)$$

when β is large.[8]

All functions under consideration will be assumed of bounded variation. Simply stated, this means that they can be represented by a curve of finite length in any finite interval of time.

An estimate of the variation of $\varphi(t)$ is given[13] by

$$|\varphi(t_2) - \varphi(t_1)| \leq M_1 |t_2 - t_1|$$
$$M_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\omega| |\Phi(\omega)| d\omega \tag{3.71}$$

where M_1 is called the first moment of the transform, $\Phi(\omega)$. From (3.83) we see that $\varphi(t)$ is of bounded variation when $\Phi(z)$ has no singularities at infinity. This is a necessary condition for the validity of (3.80), so that Reimann's lemma is seen to be closely related to the results of section (3.6).

To conclude, we now have, both in the time and frequency domains, techniques for the approximate analysis of system response to random excitation. In the following chapters, we examine the second moment response of various systems and apply the techniques of this chapter to their approximate evaluation.

CHAPTER IV

FORMULATION OF SECOND MOMENT RESPONSE

4.1 Introduction

In this chapter we consider the response of structural systems to random excitations as measured by the matrix of second order moments of the normal displacements. In section 4.2 the Fourier transform of the second moments is derived while in later sections the set of variances is given a special treatment. The chapter concludes with a consideration of response to shot noise excitations.

4.2 Second Moment Response

The responses of the j^{th} and k^{th} normal coordinates of a structure for which the equation of motion is (2.17) are given by

$$\xi_k(t) = \int_0^t h_k(t-\tau_1)Q_k(\tau_1)d\tau_1 \tag{4.1}$$

$$\xi_j(t) = \int_0^t h_j(t-\tau_2)Q_j(\tau_2)d\tau_2 \tag{4.2}$$

where we assume that $\{Q\}$ is a random vector process the components of which are zero for negative time.

From definition d. of section 2.6, the matrix of second order central moments of the normal coordinates is given by:

$$[\kappa_{ij}(t)] = E[\{\xi_k - \mu_{\xi_k}\} \lfloor \xi_j - \mu_{\xi_j} \rfloor]$$

$$= E[\{\xi_k\} \lfloor \xi_j \rfloor] - \{\mu_{\xi_k}\} \lfloor \mu_{\xi_j} \rfloor \tag{4.3}$$

From (4.3) and Eq. (2.50) we have

$$[\kappa_{kj}] = \int_0^t \int_0^t [\diagdown h_k(t-\tau_1) \diagdown][C_{Q_{kj}}(\tau_1,\tau_2)][\diagdown h_j(t-\tau_2) \diagdown] d\tau_1 d\tau_2 \tag{4.4}$$

where $[C_{Q_{kj}}(\tau_1,\tau_2)]$ is the correlation matrix of the excitation, $\{Q(t)\}$, and the impulse response functions are given by Eq. (2.22).

Now, we consider only the component $\kappa_{kj}(t)$ of the matrix of second moments and take the Fourier transform of it. Denoting the transform of $\kappa_{kj}(t)$ by $\hat{\kappa}_{kj}(\omega)$, we have from (4.4):

$$\hat{\kappa}_{kj}(\omega) = \int_{-\infty}^{\infty} \{\int_0^t \int_0^t h_k(t-\tau_1) h_j(t-\tau_2) C_{Q_{kj}}(\tau_1,\tau_2) d\tau_1 d\tau_2\} e^{-i\omega t} dt \tag{4.5}$$

$C_{Q_{kj}}(\tau_1,\tau_2)$ is zero for τ_1 or $\tau_2 < 0$. Also, $h_k(t-\tau_1)$ and $h_j(t-\tau_2)$ are zero for $\tau_1 > t$ and $\tau_2 > t$ respectively. Hence the limits of integration in the double integral within braces in (4.5) may be replaced by $-\infty$ and $+\infty$, so that

$$\hat{\kappa}_{kj}(\omega) = \int_{-\infty}^{\infty} \{\int_{-\infty}^{\infty}\int h_k(t-\tau_1) h_j(t-\tau_2) C_{Q_{kj}}(\tau_1,\tau_2) d\tau_1 d\tau_2\} e^{-i\omega t} dt$$

Assuming the order of integration may be reversed, we have

$$\hat{\kappa}_{kj}(\omega) = \int_{-\infty}^{\infty}\int \{\int_{-\infty}^{\infty} h_k(t-\tau_1) h_j(t-\tau_2) e^{-i\omega t} dt\} C_{Q_{kj}}(\tau_1,\tau_2) d\tau_1 d\tau_2 \tag{4.6}$$

Thus, we need only take the transform of the product $h_k(t-\tau_1)$ $h_j(t-\tau_2)$. From (3.5):

$$h_k(t-\tau_1) \leftrightarrow H_k(\omega)e^{-i\omega\tau_1}, \quad h_j(t-\tau_2) \leftrightarrow H_j(\omega)e^{-i\omega\tau_2}$$

and from the frequency convolution theorem:

$$h_k(t-\tau_1)h_j(t-\tau_2) \leftrightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} H_k(y)e^{-iy\tau_1} H_j(\omega-y)e^{-i\tau_2(\omega-y)} dy \tag{4.7}$$

Substitution of Eq. (4.7) into (4.6) yields

$$\hat{\kappa}_{kj}(\omega) = \frac{1}{2\pi} \iint_{-\infty}^{\infty} \{ \int_{-\infty}^{\infty} H_k(y)H_j(\omega-y)e^{-i(\tau_1 y-\tau_2(y-\omega))} dy\} \cdot C_{Q_{kj}}(\tau_1,\tau_2)d\tau_1 d\tau_2 \tag{4.8}$$

Again reversing the order of integration, we have

$$\hat{\kappa}_{kj}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H_k(y)H_j(\omega-y) \{\iint_{-\infty}^{\infty} C_{Q_{kj}}(\tau_1,\tau_2)e^{-i(\tau_1 y-\tau_2(y-\omega))} d\tau_1 d\tau_2\}dy \tag{4.9}$$

But, from (2.33), the term in braces is nothing but the generalized cross-spectral density of the k^{th} and j^{th} excitation components, $\hat{\Phi}_{Q_{kj}}$, expressed in terms of y and $y-\omega$. That is

$$\iint_{-\infty}^{\infty} C_{Q_{kj}}(\tau_1,\tau_2)e^{-i(\tau_1 y-\tau_2(y-\omega))} d\tau_1 d\tau_2 = \hat{\Phi}_{Q_{kj}}(y,y-\omega) \tag{4.10}$$

Hence:

$$\hat{\kappa}_{kj}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H_k(y)H_j(\omega-y)\, \hat{\Phi}_{Q_{kj}}(y,y-\omega)dy \tag{4.11}$$

or in matrix notation

$$[\hat{\kappa}_{kj}(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\ulcorner H_k(y) \lrcorner][\hat{\Phi}_Q(y, y-\omega)][\ulcorner H_j(\omega-y) \lrcorner] dy \tag{4.12}$$

With $[\hat{\Phi}_Q(\omega_1, \omega_2)]$ given, we can calculate $[\hat{\kappa}_{kj}(\omega)]$ and apply the approximation procedure of section 3.6 to obtain a simplified representation of $[\kappa_{kj}(t)]$. However, in the next section, we derive a special result for the variances (i.e. the diagonal components of $[\kappa_{kj}(t)]$) for which approximations may be obtained directly, without an integration of the type considered in (4.12).

4.3 Special Treatment of the Variances

The equation of motion for the k^{th} normal coordinate is, from Eq. (2.17):

$$\ddot{\xi}_k + 2\zeta_k \omega_k \dot{\xi}_k + \omega_k^2 \xi_k = \frac{1}{m_k} Q_k \tag{4.13}$$

where $\xi_k(0) = \dot{\xi}_k(0) = 0$.

Taking ensemble averages of both sides of (4.13) gives

$$\ddot{\mu}_k + 2\zeta_k \omega_k \dot{\mu}_k + \omega_k^2 \mu_k = \frac{1}{m_k} E[Q_k] \tag{4.14}$$

Subtracting (4.14) from (4.13) and letting $(\xi_k - \mu_k) = y$ and $(Q_k - E[Q_k]) = x$, we have

$$\frac{d^2 y}{dt^2} + 2\zeta_k \omega_k \frac{dy}{dt} + \omega_k^2 y = \frac{1}{m_k} x$$

This equation may also be written in the form

$$\frac{dy_1}{dt} = y_2 \;, \quad \frac{dy_2}{dt} = -2\zeta_k \omega_k y_2 - \omega_k^2 y_1 + \frac{1}{m_k} x \tag{4.15}$$

where $y_1 = y$ and $y_2 = \frac{dy}{dt}$.

If the equations of (4.15) are multiplied by y_1 and y_2 in turn, we obtain:

$$\frac{1}{2} \frac{dy_1^2}{dt} = y_1 y_2$$

$$\frac{1}{2} \frac{dy_2^2}{dt} = -2\zeta_k \omega_k y_2^2 - \omega_k^2 y_1 y_2 + \frac{1}{m_k} x y_2$$

$$y_2 \frac{dy_1}{dt} = y_2^2$$

$$y_1 \frac{dy_2}{dt} = -2\zeta_k \omega_k y_1 y_2 - \omega_k^2 y_1^2 + \frac{1}{m_k} x y_1$$

Adding the last two equations and taking ensemble averages of the resulting equations:

$$\frac{1}{2} \frac{dS_{11}}{dt} = S_{12} \tag{4.16}$$

$$\frac{1}{2} \frac{dS_{22}}{dt} = -2\zeta_k \omega_k S_{22} - \omega_k^2 S_{12} + \frac{1}{m_k} E[x\, y_2] \tag{4.17}$$

$$\frac{dS_{12}}{dt} = S_{22} - 2\zeta_k \omega_k S_{12} - \Omega^2 S_{11} + \frac{1}{m_k} E[x\, y_1] \tag{4.18}$$

where:

$$S_{11} = E[y_1^2]$$

$$S_{12} = E[y_1 y_2]$$

$$S_{22} = E[Y_2^2]$$

Substitution of S_{12} as given by (4.16) into (4.17) and (4.18) gives

$$\frac{1}{2}\frac{dS_{22}}{dt} = -2\zeta_k\omega_k S_{22} - \frac{\omega_k^2}{2}\frac{dS_{11}}{dt} + \frac{1}{m_k} E[x\, y_2] \tag{4.19}$$

$$\frac{1}{2}\frac{d^2 S_{11}}{dt^2} = S_{22} - \zeta_k\omega_k \frac{dS_{11}}{dt} - \omega_k^2 S_{11} + \frac{1}{m_k} E[x\, y_1] \tag{4.20}$$

From (4.19) and (4.20) we may solve for S_{22} and substitute this into the remaining equation to obtain a single equation in S_{11}. Recognizing that $S_{11} = \kappa_{kk}(t)$ we may write this as:

$$\frac{1}{4}\frac{d^3\kappa_{kk}}{dt^3} + \frac{3}{2}\zeta_k\omega_k\frac{d^2\kappa_{kk}}{dt^2} + (\omega_k^2 + 2\zeta_k^2\omega_k^2)\frac{d\kappa_{kk}}{dt} + 2\zeta_k\omega_k^3\kappa_{kk}$$

$$= \frac{1}{m_k}\left[\frac{1}{2}\dot{E}[xy_1] + 2\zeta_k\omega_k E[xy_1] + E[xy_1]\right] \tag{4.21}$$

which can be factored as follows

$$\left(\frac{d}{dt} + 2\zeta_k\omega_k\right)\left(\frac{1}{4}\frac{d^2}{dt^2} + \zeta_k\omega_k\frac{d}{dt} + \omega_k^2\right)\kappa_{kk}$$

$$= \frac{1}{m_k}\left(\frac{1}{2}\frac{d}{dt} + 2\zeta_k\omega_k\right)E[xy] + \frac{1}{m_k}E[xy] \tag{4.22}$$

Now, from (2.24) we have

$$y = (\xi_k - \mu_k) = \int_0^t h_k(t-\tau)x(\tau)d\tau = \int_{-\infty}^{\infty} h_k(t-\tau)x(\tau)d\tau \tag{4.23}$$

and

$$\dot{y} = \int_0^t [\frac{\partial}{\partial t} h_k(t-\tau)]x(\tau)d\tau = \int_{-\infty}^{\infty} [\frac{\partial}{\partial t} h_k(t-\tau)]x(\tau)d\tau \qquad (4.24)$$

Multiplying Eqs. (4.23) and (4.24) by x(t), taking ensemble averages; and noting that $E[x(t)x(\tau)]$ is $C_{Q_{kk}}(t,\tau)$:

$$E[xy] = \int_{-\infty}^{\infty} h_k(t-\tau)C_{Q_{kk}}(t,\tau)d\tau \qquad (4.25)$$

$$E[x\dot{y}] = \int_{-\infty}^{\infty} \dot{h}_k(t-\tau)C_{Q_{kk}}(t,\tau)d\tau \qquad (4.26)$$

Hence, Eq. (4.22) becomes

$$(\frac{d}{dt} + 2\zeta_k\omega_k)(\frac{1}{4}\frac{d^2}{dt^2} + \zeta_k\omega_k\frac{d}{dt} + \omega_k^2)\kappa_{kk}$$

$$= \frac{1}{m_k}(\frac{1}{2}\frac{d}{dt} + 2\zeta_k\omega_k)\int_{-\infty}^{\infty} h_k(t-\tau)C_{Q_{kk}}(t,\tau)d\tau + \frac{1}{m_k}\int_{-\infty}^{\infty}\dot{h}_k(t-\tau)C_{Q_{kk}}(t,\tau)d\tau \qquad (4.27)$$

We shall now take the Fourier Transform of both sides of Eq. (4.27).

From (3.6), we have

$$(\frac{d}{dt} + 2\zeta_k\omega_k)(\frac{1}{4}\frac{d^2}{dt^2} + \zeta_k\omega_k\frac{d}{dt} + \omega_k^2)\kappa_{kk}(t)$$

$$\leftrightarrow (i\omega + 2\zeta_k\omega_k)(-\frac{1}{4}\omega^2 + i\zeta_k\omega_k\omega + \omega_k^2)\hat{\kappa}_{kk}(\omega) \qquad (4.28)$$

Since $h_k(t-\tau) \leftrightarrow H_k(\omega)e^{-i\tau\omega}$,

$$E[xy] \leftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty}\{H_k(y)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\tau y}e^{-it(\omega-y)}C_{Q_{kk}}(t,\tau)dtd\tau\}\, dy$$

or

$$E[xy] \leftrightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} H_k(y)\, \hat{\Phi}_{Q_{kk}}(y, y-\omega)\,dy \tag{4.29}$$

by (4.25) and the frequency convolution theorem.

Employing (3.6), we have:

$$\dot{E}[xy] \leftrightarrow \frac{1}{2\pi}\, i\omega \int_{\infty}^{\infty} H_k(y)\, \hat{\Phi}_{Q_{kk}}(y, y-\omega)\,dy \tag{4.30}$$

and

$$E[x\dot{y}] \leftrightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} iy\, H_k(y)\, \hat{\Phi}_{Q_{kk}}(y, y-\omega)\,dy \tag{4.31}$$

Therefore, from (4.28) through (4.31) the Fourier transform of $\kappa_{kk}(t)$ is

$$\hat{\kappa}_{kk}(\omega) = \frac{\frac{1}{2\pi} \int_{-\infty}^{\infty} (\frac{\omega}{2} - 2\zeta_k \omega_k i + y) H_k(y) \hat{\Phi}_{Q_{kk}}(y, y-\omega)\,dy}{m_k(\omega - 2\zeta_k \omega_k i)(- \frac{\omega^2}{4} + i\zeta_k \omega_k \omega + \omega_k^2} \tag{4.32}$$

4.4 <u>Summary of the Response Formulation</u>

In general, the second moment response to random excitations is given by Eq. (4.12)

$$[\hat{\kappa}_{kj}(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\, H_k(y)\,][\hat{\Phi}_Q(y, y-\omega)][\, H_j(\omega - y)\,]dy \tag{4.12}$$

and an alternative form for the diagonal elements of $[\hat{\kappa}_{kj}]$ (i.e. the variances of the normal coordinates) is given by Eq. (4.32)

$$\hat{\kappa}_{kk}(\omega) = \frac{\frac{1}{2\pi}\int_{-\infty}^{\infty}(\frac{\omega}{2} - 2\zeta_k\omega_k i+y)H_k(y)\,\Phi_{Q_{kk}}(y,y-\omega)dy}{m_k(\omega - 2\zeta_k\omega_k i)(-\frac{\omega^2}{4} + i\zeta_k\omega_k\omega + \omega_k^2)} \tag{4.32}$$

These two equations summarize the response formulation of our work and will provide the basis of all subsequent considerations.

A cautionary note should be given on the evaluation of the integrals appearing in (4.12) and (4.32). Let us denote either of the integrands by $J(\omega,y)$ and the integrals themselves by $I(\omega)$. Then we may write

$$I(\omega) = [\frac{1}{2\pi}\int_{-\infty}^{\infty} J(\omega,y)e^{iyt}dy]_{t=0}$$

which, by (3.3) becomes

$$I(\omega) = \frac{1}{2}[\lim_{t\to 0^+}[\frac{1}{2\pi}\int_{-\infty}^{\infty} J(\omega,y)e^{iyt}dy] + \lim_{t\to 0^-}[\frac{1}{2\pi}\int_{-\infty}^{\infty} J(\omega,y)e^{iyt}dy]] \tag{4.33}$$

From the discussions of section 3.5 it is clear that the first integral within the brackets is to be evaluated from the residues of the integrand in the upper half z-plane. Similarly, the second integral is to be evaluated from the residues in the lower half plane. Hence, $I(\omega)$ is one half the sum of the results obtained when the integral is evaluated from the residues in the upper and the lower halves of the z plane. This we represent symbolically by

$$I(\omega) = \frac{1}{2}(I_u + I_\ell)$$

A consideration of Eq. (4.12) for the case j=k shows that it does not reduce to the corresponding result in Eq. (4.32) in any obvious way. Consequently, in Appendix II, we present a demonstration that the two expressions for the variances are, in fact, equivalent.

In the next sections, we consider the cases for which the excitation $\{Q_k\}$ is (i) a stationary vector process and (ii) a vector shot noise.

4.5 The Stationary Case

Analogous to Eq. (2.52) and (2.53) written for the case of a single random variable, we define a stationary vector process, $\{X_i(t)\}$ as one for which

$$[C_{x_{kj}}(t_1,t_2)] = E[\{X_k - \mu_{x_k}\}\lfloor X_j - \mu_{x_j}\rfloor] = [C_{x_{kj}}(\tau)] \qquad (4.34)$$

where $(t_2-t_1) = \tau$. The matrix of generalized spectral densities is given by

$$[\Phi_{x_{kj}}(\omega_1,\omega_2)] = 2\pi\,\delta(\omega_1-\omega_2)[S_{x_{kj}}(\omega_1)] \qquad (4.35)$$

where

$$[S_{x_{kj}}(\omega_1)] = \int_{-\infty}^{\infty} [C_{x_{kj}}(\tau)]e^{-i\omega_1\tau}\,d\tau \qquad (4.36)$$

If $\{Q_k(t)\}$ is such a process we conclude that, quite apart from any considerations involving Eqs. (4.12) and (4.32)

$$[C_{\xi_{kj}}(\tau)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\diagdown H_k(\omega) \diagdown][S_{a_{kj}}(\omega)][\diagdown H_j^*(\omega) \diagdown] e^{i\omega\tau} d\omega \qquad (4.37)$$

from Eq. (3.29) and an application of (3.15). Hence:

$$[\kappa_{kj}] = ([C_{\xi_{kj}}(\tau)])_{\tau=0} = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\diagdown H_k(\omega) \diagdown][S_{Q_{kj}}(\omega)][\diagdown H_j^*(\omega) \diagdown] d\omega \qquad (4.38)$$

Now, let us examine Eq. (4.12). From (4.35):

$$[\Phi_{Q_{kj}}(y, y-\omega)] = 2\pi\, \delta(\omega)[S_{Q_{kj}}(y)] \qquad (4.39)$$

Substituting this into (4.12) gives

$$[\hat{\kappa}_{kj}(\omega)] = \int_{-\infty}^{\infty} [\ H_k(y)\ ][S_{Q_{kj}}]\ \delta(\omega)[\ H_j(\omega-y)\ ]dy$$

so that

$$[\kappa_{kj}] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} [\diagdown H_k(y) \diagdown][S_{Q_{kj}}(y)][\diagdown H_j(\omega-y) \diagdown] \cdot \delta(\omega)\, e^{i\omega t}\, d\omega \right\} dy$$

Applying (3.15) and noting that $H_j(-y) = H_j^*(y)$, we have

$$[\kappa_{kj}] = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\diagdown H_k(y) \diagdown][S_{Q_{kj}}(y)][\diagdown H_j^*(y) \diagdown] dy$$

which is in complete agreement with (4.38)

Substituting (4.35) into (4.32):

$$\hat{\kappa}_{kk}(\omega) = \frac{\delta(\omega) \int_{-\infty}^{\infty} (\frac{\omega}{2} - 2\zeta_k \omega_k i + y) H_k(y) S_{Q_{kk}}(y) dy}{m_k(\omega - 2\zeta_k \omega_k i)(-\frac{\omega^2}{4} + i\zeta_k \omega_k \omega + \omega_k^2}$$

Taking the inverse transform and employing (3.15), this becomes

$$\kappa_{kk}(t) = \frac{1}{2\pi m_k} \int_{-\infty}^{\infty} \frac{2\zeta_k\omega_k i - y}{2\zeta_k\omega_k i} H_k(y) S_{Q_{kk}}(y) dy \tag{4.40}$$

If both numerator and denominator of the integrand is multiplied by $(\omega_k^2 - y^2 - 2i\zeta_k\omega_k y)$ and using Eq. (3.20) for $H_k(y)$:

$$\kappa_{kk} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\left(\frac{1}{\omega_k^2} + \frac{iy}{2\zeta_k\omega_k^2}\right)(\omega_k^2 - y^2 - 2i\zeta_k\omega_k y)}{m_k^2((\omega_k^2 - y^2)^2 + 4\zeta_k^2\omega_k^2 y^2)} S_{Q_{kk}}(y) dy$$

Noting that $m_k^2((\omega_k^2 - y^2)^2 + 4\zeta_k^2\omega_k^2 y^2)$ is $|H_k(y)|^2$ and expanding the numerator of the integrand, we have

$$\kappa_{kk} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[1 + \frac{iy}{2\zeta_k\omega_k} - \frac{2i\zeta_k y}{\omega_k} - \frac{iy^3}{2\zeta_k\omega_k^3}\right] |H_k(y)|^2 S_{Q_{kk}}(y) dy \tag{4.41}$$

Finally, since $S_{Q_{kk}}(y)|H_k(y)|^2$ is an even function of y, the terms involving y and y^3 in (4.41) vanish.

Hence, from (4.41):

$$\kappa_{kk} = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H_k(y)|^2 S_{Q_{kk}}(y) dy$$

which agrees with the results corresponding to the diagonal terms of both sides of Eq. (4.38).

In later sections, we shall repeatedly use the result (4.38) to check results obtained for non-stationary excitations via (4.12) or (4.32).

4.6 Shot Noise Excitations

If $\{Q_k(t)\}$ is a vector shot noise, then according to (2.59):

$$[C_{Q_{kj}}] = [I_{jk}(t_1)]\,\delta(t_2-t_1)$$

So that, from (3.23):

$$[\Phi_{Q_{kj}}(\omega_1,\omega_2)] = \iint_{-\infty}^{\infty}[I_{jk}(t_1)]\delta(t_2-t_1)\exp(-i(\omega_1 t_1-\omega_2 t_2))dt_1 dt_2$$

$$= \int_{-\infty}^{\infty} [I_{jk}(t)]\, e^{-it(\omega_1-\omega_2)} dt \tag{4.42}$$

Or

$$[\Phi_{Q_{kj}}(y,y-\omega)] = [\hat{I}_{jk}(\omega)] \tag{4.43}$$

where $\hat{I}_{jk}(\omega)$ is the Fourier transform of the intensity function, $I_{jk}(t)$.

Substitution of this into (4.32) gives

$$[\hat{\kappa}_{kj}(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\diagdown H_k(y)\diagdown][\hat{I}_{jk}(\omega)][\diagdown H_j(\omega-y)\diagdown]dy \tag{4.44}$$

Without further information about the intensity functions we cannot carry out the integration of (4.44). Thus, (4.44) represents the most general result that can be obtained via (4.12).

However, we can carry the determination of the variances of the response further than (4.44) by employing (4.32).

Substitution of (4.43) into (4.32) gives

$$\hat{\kappa}_{kk}(\omega) = \frac{\frac{1}{2\pi}\hat{I}_{kk}(\omega)\int_{-\infty}^{\infty}(\frac{\omega}{2} - 2\zeta_k\omega_k i+y)H_k(y)dy}{m_k(\omega-2\zeta_k\omega_k i)(-\frac{\omega^2}{4} + i\zeta_k\omega_k\omega+\omega_k^2)} \tag{4.45}$$

The integral on the left hand side is easily evaluated when we realize that

$$\int_{-\infty}^{\infty} H_k(y)dy = 2\pi \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} H_k(y)e^{+iyt}dy\right)_{t=0}$$

$$\int_{-\infty}^{\infty} yH_k(y)dy = -2\pi i \left\{\frac{d}{dt}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}H_k(y)e^{iyt}dy\right]\right\}_{t=0}$$

from the definitions of section 3.2. Employing (2.22), (3.20) and the above formulae, we have

$$\int_{-\infty}^{\infty} H_k(y)dy = 2\pi \left. \frac{1}{\omega_{kd}m_k} e^{-\zeta_k\omega_k t} \text{Sin } \omega_{kd}t \right|_{t\to0^+} = 0 \tag{4.46}$$

$$\begin{aligned}\int_{-\infty}^{\infty} yH_k(y)dy &= -2\pi i \left\{ \frac{1}{2}\left[\frac{-\zeta_k\omega_k}{\omega_{kd}m_k} e^{-\zeta_k\omega_k t} \text{Sin } \omega_{kd}t \right.\right. \\ &\quad \left.\left. + \frac{1}{m_k} e^{-\zeta_k\omega_k t} \cos \omega_{kd}t\right]_{t\to0^+} \right\} \\ &= \frac{\pi}{im_k}\end{aligned} \tag{4.47}$$

where we have noted that t=0 is a discontinuity point of the derivative of the impulse response function. Finally, substitution of (4.46) and (4.47) into (4.45) yields

$$\hat{\kappa}_{kk}(\omega) = \frac{\frac{i}{2}\,\hat{I}_{kk}(\omega)}{m_k^2(\omega-2\zeta_k\omega_k i)(\frac{\omega^2}{4} - i\zeta_k\omega_k\omega-\omega_k^2)} \tag{4.48}$$

We can now take the inverse Fourier transforms of both sides of (4.48) to obtain the variance of the k^{th} normal coordinate in terms of a convolution integral between $I_{kk}(t)$ and the inverse of $\dfrac{i}{2m_k^2(\omega-2\zeta_k\omega_k i)(\frac{\omega^2}{4} - i\zeta_k\omega_k\omega-\omega_k^2)}$. Hence the integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{i\,e^{i\omega t}d\omega}{2m_k^2(\omega-2\zeta_k\omega_k i)(\frac{\omega^2}{4} - i\zeta_k\omega_k\omega-\omega_k^2)}$$

must be evaluated. Upon replacing ω by $z = x+iy$ and employing Eqs. (3.36) and (3.40), the result is found to be

$$\frac{e^{-2\zeta_k\omega_k t}}{2m_k^2\,\omega_{kd}^2}(1 - \cos 2\omega_{kd}t)U(t) \leftrightarrow \frac{i}{2m_k^2(\omega-2\zeta_k\omega_k i)(\frac{\omega^2}{4} - 2\zeta_k\omega_k\omega-\omega_k^2)} \tag{4.49}$$

where $U(t)$ is the unit step function and $\omega_{kd} = \omega_k\sqrt{1-\zeta_k^2}$.

Thus, using the time convolution theorem and assuming that $I_{kk}(t)$ is zero for negative time, we have the result

$$\kappa_{kk}(t) = \frac{1}{2m_k^2\omega_{kd}^2}\int_0^t I_{kk}(t-\tau)\,e^{-2\zeta_k\omega_k\tau}(1-\cos 2\omega_{kd}\tau)d\tau \tag{4.50}$$

which is the general expression for the variances of the response to shot noise excitations. We conclude this chapter with an illustration of the application of Eq. (4.50).

Example 4.1:

Consider the variance of the response of a one dimensional oscillator to a shot noise excitation, for which the intensity function is a unit step. If its equation of motion in terms of the displacement, ξ, is

$$\ddot{\xi} + 2\zeta\Omega\dot{\xi} + \Omega^2\xi = \frac{1}{m} Q$$

then the mean of ξ is zero, and from (4.50):

$$\kappa(t) = E[\xi^2(t)] = \frac{1}{2m^2\Omega_d^2} \int_0^t I(\tau)e^{-2\zeta\Omega(t-\tau)}(1-\cos 2\Omega_d(t-\tau))d\tau$$

where $\Omega_d = \Omega\sqrt{1-\zeta^2}$. Since

$$I(\tau) = \begin{cases} I & \tau > 0 \\ 0 & \tau < 0 \end{cases}$$

where I is some constant, we have

$$\kappa(t) = \frac{I}{2m^2\Omega_d^2} e^{-2\zeta\Omega t} \int_0^t e^{2\zeta\Omega\tau}(1- \cos 2\Omega_d(t-\tau))d\tau$$

which, after a direct integration by elementary methods, becomes:

$$\kappa(t) = \frac{I}{4m^2\zeta\Omega^3} [1 - e^{-2\zeta\Omega t}(1+ \frac{\zeta\Omega}{\Omega_d} \sin 2\Omega_d t + 2(\frac{\zeta\Omega}{\Omega_d})^2 \sin^2\Omega_d t)]$$

This result is in complete agreement with that obtained in Ref. (6). When t tends to infinity, $\kappa(t)$ tends to the value obtained in the Example of section 3.5:

$$\lim_{t \to \infty} \kappa(t) = \frac{I}{4m^2 \zeta \Omega^3}$$

so that for large t, $\kappa(t)$ approaches its stationary value.

CHAPTER V

APPROXIMATE FORMULATION OF RESPONSE

5.1 Introduction

In this chapter, we use the results of sections 3.6 and 3.7 to obtain approximations to the second moment response of a structural system to random excitations. This is accomplished by a consideration of Eqs. (4.12) and (4.32) for a particular class of input processes, described in the next section. A special treatment of the variances is given in section 5.3 since Eq. (4.12) is amenable to direct approximation. In the remaining sections we treat cases involving both shot noise, and filtered shot noise excitations.

5.2 Slowly Varying Excitation Processes

Recall the expression for the generalized spectral density matrix of $\{Q\}$ in terms of the correlation matrix, $[C_{Q_{ij}}]$. Replacing ω_1 by y and ω_2 by $y-\omega$:

$$[\Phi_{Q_{ij}}(y,y-\omega)] = \int\int_{-\infty}^{\infty}[C_{Q_{ij}}(t_1,t_2)]\exp(-i(t_1y-(y-\omega)t_2))dt_1dt_2 \tag{5.1}$$

Now, consider the transformation

$$\tau = t_1-t_2 \quad \text{and} \quad t^* = t_2 \tag{5.2}$$

Since the Jacobian of the transformation is one, Eq. (5.1) becomes

$$[\Phi_{Q_{ij}}(y,y-\omega)] = \iint_{-\infty}^{\infty} [C_{Q_{ij}}(\tau,t^*)]\, e^{-iy\tau} e^{-i\omega t^*} dt^* d\tau \quad (5.3)$$

We can also write this as

$$[\Phi_{Q_{ij}}(y,y-\omega)] = \int_{-\infty}^{\infty} [\Gamma_{ij}(\tau,\omega)]\, e^{-iy\tau} d\tau \quad (5.4)$$

where

$$[\Gamma_{ij}(\tau,\omega)] = \int_{-\infty}^{\infty} [C_{Q_{ij}}(\tau,t^*)]\, e^{-i\omega t^*} dt^* \quad (5.5)$$

From Eq. (3.83), we see that an estimate of the variation of $C_{Q_{ij}}(\tau,t^*)$ with respect to t^* is given by

$$|C_{Q_{ij}}(\tau,t_2^*) - C_{Q_{ij}}(\tau,t_1^*)| \leq M_{ij}\, |t_2^* - t_1^*|$$

$$M_{ij} = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\omega|\, |\Gamma_{ij}(\tau,\omega)|\, d\omega \quad (5.6)$$

Hence, the variation of the correlation functions with respect to t^* depends on the behavior of the generalized spectral densities with respect to ω.

We designate as slowly varying those non-stationary processes, $\{Q_i(t)\}$, whose correlation functions vary slowly (in a sense to be specified later) with respect to t^* when the transformation (5.2) is effected. This restriction implies that the M_{ij} $(i,j = 1,N)$ of (5.6) are small for all τ. It can be seen that the M_{ij} are small when most of the singularities of $\Gamma_{ij}(\tau,z)$ are near the origin of the complex plane. For convenience in later discussions it will be assumed that all the singularities of $\Gamma_{ij}(\tau,z)$ lie within some small distance to the origin.

When the damping coefficients of the system are small, ($\zeta_i = O(10^{-2})$) then the methods of sections 3.6 and 3.7 may be used to approximate the second moments of the response. In general, this can be done through the use of Eq. (4.12) only when the integration in y is first performed. However, approximations to the variances can be obtained directly from Eq. (4.32) and in a general form. Thus, we give the set of variances a special treatment in the next sections.

5.3 Approximate Evaluation of the Variances

In the following discussion we assume that

a. The damping coefficients of the structure, ζ_k (k=1,N) are small (of the order of 10^{-2})

b. The natural frequencies, ω_k, are large ($> 10^1$).

c. The excitation, $\{Q_k(t)\}$, are slowly varying as described in the last section in the sense that all the singularities of $\Gamma_{kk}(\tau,z)$ lie within the shaded region shown in Fig. 8.

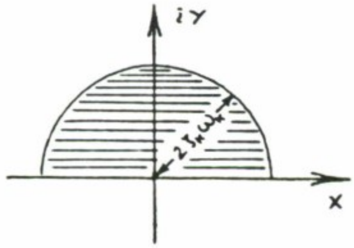


Fig. 8

d. The functions $\Gamma_{kk}(\tau,z)$ (k=1,...,N) can be expressed in the form

$$\Gamma_{kk}(\tau,z) = \frac{J_k(\tau,z)}{F_k(\tau,z)}$$

where J_k is analytic in the finite z plane and $\frac{1}{F_k}$ has p poles at $a_1, \ldots, a_p$ in the upper half z plane.

Having made these assumptions, we now consider Eq. (4.32). Recognizing the term $(\frac{\omega}{2} - 2\zeta_k\omega_k i + y)$ in the numerator is $\frac{1}{2}(\omega - 2\zeta_k\omega_k i) + (y - \zeta_k\omega_k i)$, (4.32) becomes

$$\hat{\kappa}_{kk}(\omega) = \frac{1}{2\pi m_k}\left[\int_{-\infty}^{\infty} H_k(y)\frac{\hat{\Phi}_{kk}(y,y-\omega)}{2D_1}dy + \int_{-\infty}^{\infty}(y-\zeta_k\omega_k i)\, H_k(y)\frac{\hat{\Phi}_{kk}(y,y-\omega)}{D_2}dy\right] \tag{5.7}$$

where $D_1 = (-\frac{\omega^2}{4} + i\zeta_k\omega_k\omega + \omega_k^2)$, $D_2 = (\omega - 2\zeta_k\omega_k i)D_1$

And, employing (5.4) and (5.5), we have

$$\hat{\kappa}_{kk}(\omega) = \frac{1}{2\pi m_k}\int_{-\infty}^{\infty}\Big\{\left[\int_{-\infty}^{\infty}H_k(y)e^{-iy\tau}dy\right]\frac{\Gamma_{kk}(\tau,\omega)}{2D_1(\omega)} + \left[\int_{-\infty}^{\infty}(y-\zeta_k\omega_k i)H_k(y)e^{-iy\tau}dy\right] * \frac{\Gamma_{kk}(\tau,\omega)}{D_2(\omega)}\Big\}\,d\tau \tag{5.8}$$

$$= \frac{1}{m_k}\int_{-\infty}^{0}\Big\{\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}H_k(y)e^{-iy\tau}dy\right]\frac{\Gamma_{kk}(\tau,\omega)}{2D_1(\omega)} + \left[\frac{1}{2\pi}\int_{-\infty}^{\infty}(iy + \zeta_k\omega_k)H_k(y)e^{-iy\tau}dy\right]\frac{\Gamma_{kk}(\tau,\omega)}{i\,D_2(\omega)}\Big\}\,d\tau \tag{5.9}$$

since $H_k(y)$ is the transform of a causal function.

We shall now estimate the relative order of magnitude of the two terms in (5.9).

First, notice that the integral $\frac{1}{2\pi}\int_{-\infty}^{\infty} H_k(y)e^{-iy\tau}dy \quad (\tau < 0)$ is $h_k(-\tau)$, so that by Eq. (3.6), $\frac{1}{2\pi}\int_{-\infty}^{\infty} H_k(y)\, iy\, e^{-iy\tau}dy$ is $-\frac{d}{d\tau}(h_k(-\tau))$. Hence, from (2.22) and assumptions a. and b.:

$$O\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}(iy - \zeta_k\omega_k)H_k(y)e^{-iy\tau}dy\right\}$$

$$= \omega_k\, O\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} H_k(y)e^{-iy\tau}dy\right\} \tag{5.10}$$

And since $h_k(-\tau)$ and $-\frac{d}{d\tau}(h_k(-\tau))$ are both exponentially decaying sinusoids with the same frequency and exponential factor:

$$O\left\{\frac{1}{2\pi}\int_{-\infty}^{0}\int_{-\infty}^{\infty}(iy - \zeta_k\omega_k)H_k(y)\Gamma_{kk}(\tau,y)e^{-iy\tau}dy\,d\tau\right\}$$

$$= \omega_k\, O\left\{\frac{1}{2\pi}\int_{-\infty}^{0}\int_{-\infty}^{\infty} H_k(y)\Gamma_{kk}(\tau,y)e^{-iy\tau}dy\,d\tau\right\} \tag{5.11}$$

Taking the inverse Fourier transform of (5.9), we see that $\kappa_{kk}(t)$ involves the two integrals

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Gamma_{kk}(\tau,\omega)}{2D_1(\omega)}e^{i\omega t}d\omega \qquad \text{and} \qquad \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Gamma_{kk}(\tau,\omega)}{D_2}e^{i\omega t}d\omega$$

Using (3.36) and (3.40) and noting assumptions c. and d. we have

$$O\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Gamma_{kk}(\tau,\omega)}{2D_1(\omega)}e^{i\omega t}d\omega\right\} \approx \zeta_k\, O\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Gamma_{kk}(\tau,\omega)}{D_2}e^{i\omega t}d\omega\right\} \tag{5.12}$$

Hence, combining Eqs. (5.11) and (5.12), we conclude that the first term in (5.9) gives a time function that is of the order of ζ_k/ω_k multiplied by the order of the time function resulting from the second term.

With this result, from (5.9), we have

$$\kappa_{kk}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{2\pi m_k}\int_{-\infty}^{0}\int_{-\infty}^{\infty}(y-i\zeta_k\omega_k)H_k(y)e^{-iy\tau}\frac{\Gamma_{kk}(\tau,\omega)}{D_2(\omega)}\,dy\,d\tau\right]$$

$$e^{i\omega t}\,d\omega\,(1+T_3)$$

$$= \frac{1}{2\pi m_k}\iint_{-\infty}^{\infty}(y-i\zeta_k\omega_k)H_k(y)e^{-iy\tau}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Gamma_{kk}(\tau,\omega)}{D_2(\omega)}e^{i\omega t}d\omega\right\}dy\,d\tau$$

$$(1 + T_3) \qquad (5.13)$$

where $O\{T_3\} = O(\frac{\zeta_k}{\omega_k})$

Consider the integral $\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Gamma_{kk}(\tau,\omega)}{D_2(\omega)}e^{i\omega t}d\omega$. This can be written

$$I = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{-4\,J_k(\tau,\omega)e^{i\omega t}d\omega}{F_k(\tau,\omega)(\omega-2\zeta_k\omega_k i)(\omega^2-4i\zeta_k\omega_k\omega-4\omega_k^2)} \qquad (5.14)$$

from assumption d. and (5.7).

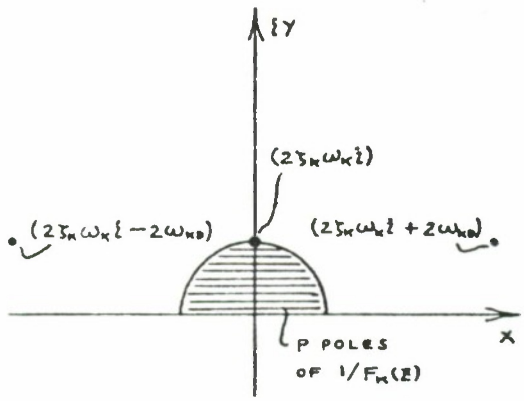


Fig. 9

The positions of the poles of the integrand are shown in Fig. 9. Since, by assumption a., the poles of the term $\frac{1}{(\omega^2-4i\zeta_k\omega_k\omega-4\omega_k^2)}$ are far distant from all the other poles, the treatment given in section 3.6 is certainly applicable. Referring to section 3.6, we can set

$$(\omega^2-4i\zeta_k\omega_k\omega - 4\omega_k^2) = G(\omega)$$

$$F_k(\tau,\omega)(\omega - 2\zeta_k\omega_k i) = F(\omega)$$

$$n = 1 + p$$

$$m = 2$$

$$\delta_1 = 4\zeta_k\omega_k$$

$$\epsilon_2 = 4\omega_{kD}$$

$$\gamma_1 \approx 2\omega_{kD} \approx \gamma_2$$

$$a = 0$$

$$b_1 = 2\zeta_k\omega_k i + 2\omega_{kD}\;,\quad b_2 = 2\zeta_k\omega_k i - 2\omega_{kD}$$

From (3.69) to (3.71), it can be seen that

$$I = \left[\frac{1}{\omega_k^2}\,\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Gamma_{kk}(\tau,\omega)e^{i\omega t}d\omega}{(\omega-2\zeta_k\omega_k i)}\right](1+T_1)(1+T_2) \tag{5.15}$$

where

$$O\{T_1\} = O\{\frac{2(4\zeta_k\omega_k)^p(2\omega_{kD})^2 \text{MAX}[|J_k(b_j)|]}{(1+p)(4\omega_{kD})(2\omega_{kD})^{p+1}\text{MIN}[|J_k(a_j)|]}\}$$

$$\approx O\{\frac{2^p\zeta_k^p}{(1+p)}\frac{\text{MAX}[|J_k(b_j)|]}{\text{MIN}[|J_k(a_j)|]}\} \tag{5.16}$$

$$O\{T_2) < O\{|(\frac{1}{G(a-\delta_1/2)} - \frac{1}{G(a)})G(a)|\}$$

$$\approx O\{\zeta_k^2\} \tag{5.17}$$

Finally a substitution of (5.15) into (5.13) results in the approximate Fourier integral

$$\hat{\kappa}_{kk}(\omega) \approx \frac{1}{2\pi m_k\omega_k^2}\int_{-\infty}^{\infty}\frac{(y-i\zeta_k\omega_k)H_k(y)}{\omega - 2\zeta_k\omega_k i}\hat{\Phi}_{kk}(y,y-\omega)dy \tag{5.18}$$

which, upon inversion gives $\kappa_{kk}(t)$ correct to within $O\{T_1 + T_2 + T_3\}$ (neglecting higher order terms in the expansion of $(1+T_1)(1+T_2)(1+T_3)$), where

$$O\{T_1\} = O\{\frac{2^p\gamma_k^p}{1+p}\frac{\text{MAX}[|J_k(b_j)|]}{\text{MIN}[|J_k(a_j)|]}\} \tag{5.19}$$

$$O\{T_2\} < O\{\zeta_k^2\} \tag{5.20}$$

$$O\{T_3\} < O\{\frac{\zeta_k}{\omega_k}\} \tag{5.21}$$

It should be noted that for shot noise excitations, the first term of (5.7) is

$$\frac{1}{2\pi m_k}\int_{-\infty}^{\infty} H_k(y)\ \frac{\hat{I}_{kk}(\omega)}{2D_1(\omega)}\ dy = \frac{\hat{I}_{kk}(\omega)}{4\pi m_k D_1(\omega)}\int_{-\infty}^{\infty} H_k(y)\ dy$$

and vanishes identically. Hence the error to which this term gives rise, T_3, is zero.

When the singularities of $\Gamma_{kk}(\tau,\omega)$ lie within a distance ϵ_k from the origin such that $\epsilon_k > 2\zeta_k\omega_k$, it is obvious from the preceding arguments that if $\epsilon_k << 2\omega_k$ then formulae (5.18) to (5.21) are still valid provided ζ_k is replaced by $\frac{\epsilon_k}{2\omega_k}$. Thus, it is now possible to clarify the sense in which the excitations considered here are "slowly varying". The excitations are slowly varying when the function $\Gamma_{kk}(\tau,\omega)$ (of Eq. 5.5) takes significant values only for $|\omega| < \epsilon$ where $\epsilon << 2\omega_{kD}$.

To illustrate the application of the above ideas we shall consider the following example.

<u>Example 5.1</u>

Let us consider once more the situation of Example 4.1. For this case, the spectral density is

$$\hat{\Phi}(y,y-\omega) = \hat{I}(\omega)\cdot I \qquad (5.22)$$

where $\hat{I}(\omega)$ is the Fourier transform of the unit step function, $U(t)$. Since $U(t) \leftrightarrow \pi\delta(\omega) + \frac{1}{i\omega}$, (5.22) becomes

$$\hat{\Phi}(y,y-\omega) = I(\pi\delta(\omega) + \frac{1}{i\omega}) \qquad (5.23)$$

Substituting the first term of (5.23) into (5.18) gives

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2\pi m_k\omega_k^2}\int_{-\infty}^{\infty}\frac{(y-i\zeta_k\omega_k)H_k(y)}{\omega-2\zeta_k\omega_k i}\, I\,\pi\delta(\omega)\,dy\,e^{i\omega t}d\omega$$

$$=\frac{I}{2}\frac{1}{2\pi m\Omega^2}\frac{1}{(2\zeta\Omega)}\int_{-\infty}^{\infty}(iy+\zeta\Omega)H(y)dy$$

$$=\frac{I}{8m^2\Omega^2\zeta} \tag{5.24}$$

by (3.11) and (4.47). It can be seen that the exact expression, Eq. (4.48) yields the same result. Thus, for the first term of (5.23), (5.24) involves no approximation.

Similarly, in general the expression (5.18) becomes

$$\hat{\kappa}(\omega)=\frac{1}{2m^2\Omega^2 i}\frac{\hat{I}(\omega)}{\omega-2\zeta\Omega i} \tag{5.25}$$

Substituting the second term of (5.23) for $\hat{I}(\omega)$ above and taking the inverse Fourier transform, we have

$$\frac{I}{2\pi(2m^2\Omega^2)}\int_{-\infty}^{\infty}\frac{i\,e^{i\omega t}d\omega}{i\omega(\omega-2\zeta\Omega i)}=\frac{iI}{2m^2\Omega^2}\left[\frac{1}{4\zeta\Omega i}-\frac{e^{-2\zeta\Omega t}}{2\zeta\Omega i}\right]$$

$$=\frac{I}{8m^2\zeta\Omega^3}\left[1-2e^{-2\zeta\Omega t}\right] \tag{5.26}$$

$\kappa_k(t)$ is just the sum of expressions (5.24) and (5.26) with suitable correction terms:

$$\kappa(t)=\frac{I}{4m^2\zeta\Omega^3}\left(\frac{1}{2}+\left(\frac{1}{2}-e^{-2\zeta\Omega t}\right)(1+T_1+T_2)\right)$$

and from (5.19) and (5.20):

$$O(T_1) = O\{\zeta\}$$

$$O(T_2) = O\{\zeta^2\}$$

since $J(\omega) = 1$, and $P=1$.

Thus:

$$\kappa(t) = \frac{I}{4m^2\zeta\Omega^3}\left[1 + \frac{T}{2} - e^{-2\zeta\Omega t}(1+T)\right] \tag{5.27}$$

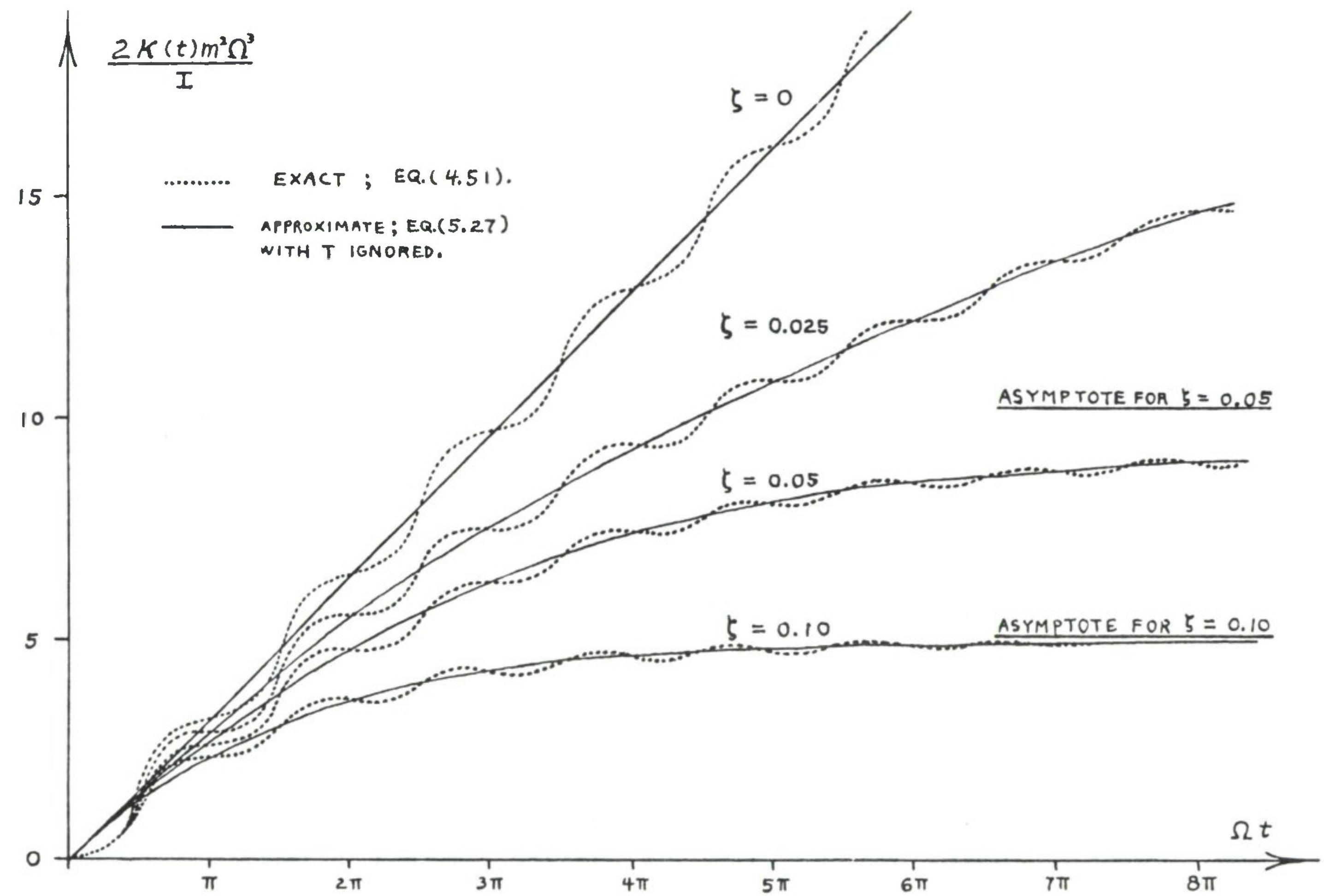
where T is of order ζ. This result can be directly verified by comparing it with the exact solution, (4.51). In Fig. 10, the two expressions are shown graphically. Here, the quantity $\frac{2\kappa(t)m^2\Omega^3}{I}$ is plotted as a function of Ωt. Now, when the unit step commences at $t=t_0$, by Eq. (3.5), $I(t-t_0) \leftrightarrow e^{-i\omega t_0}\hat{I}(\omega)$. And from (5.25) we see that $\kappa(t)$ is merely shifted in time by t_0. Hence, for a white noise excitation modulated by a step function beginning at t_0:

$$\kappa(t) = \begin{cases} \frac{I}{4m^2\zeta\Omega^3}\left(1 - e^{-2\zeta\Omega(t-t_0)}\right)(1+T) & , \ t > t_0 \\ 0 & , \ t < t_0 \end{cases} \tag{5.28}$$

where $O\{T\} = O\{\zeta + \zeta^2\}$.

Example 5.1 shows the validity of Eq. (5.18), when $C_{Q_{ij}}(\tau,t^*)$ has points of discontinuity but is otherwise slowly varying in t^*. For suppose that $C_{Q_{ij}}(\tau,t^*)$ has a finite jump of magnitude C at $t^* = t_0^*$. Then we may write

FIG. 10


$\frac{2K(t)m^2\Omega^3}{I}$
EXACT ; EQ.(4.51).
APPROXIMATE ; EQ.(5.27)
WITH T IGNORED.
$\zeta = 0$
$\zeta = 0.025$
$\zeta = 0.05$
$\zeta = 0.10$
ASYMPTOTE FOR $\zeta = 0.05$
ASYMPTOTE FOR $\zeta = 0.10$
0
5
10
15
π
2π
3π
4π
5π
6π
7π
8π
Ωt

$$C_{Q_{ij}}(\tau,t^*) = C^{(c)}_{Q_{ij}}(\tau,t^*) + CU(t^*-t_0) \tag{5.29}$$

where $C^{(c)}_{Q_{ij}}(\tau,t^*)$ is continuous in t^*. The generalized spectral density of $C_{Q_{ij}}$ is

$$\hat{\Phi}_{Q_{ij}}(y,y-\omega) = \hat{\Phi}^{(c)}_{Q_{ij}}(y,y-\omega) + C(\pi\delta(\omega) + \frac{1}{i\omega})e^{-i\omega t_0} \tag{5.30}$$

where $\Phi^{(c)}_{Q_{ij}}$ corresponds to $C^{(c)}_{Q_{ij}}$ in (5.29).

But the second term of (5.30) is the spectral density of Example 5.1 with $I = C$. Then from (5.18) and (5.28):

$$\kappa_{kk}(t) = \frac{1/2\pi}{2\pi m_k \omega_k^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{(y-i\zeta_k\omega_k)H_k(y)}{\omega - 2\zeta_k\omega_k i} \hat{\Phi}^{(c)}_{kk}(y,y-\omega)dy\, e^{i\omega t} d\omega$$

$$+ \frac{I\, U(t_0)}{4m^2\zeta_k\omega_k^3}(1 - \exp(-2\zeta_k\omega_k(t-t_0)))(1+T) \tag{5.31}$$

5.4 Approximate Variance Formulation in the Time Domain

In some cases, it may be advantageous to work with the slowly varying case in the time domain; that is to express the inverse transform of Eq. (5.18) in a general form. We can write (5.18) in the form

$$\hat{\kappa}_{kk}(\omega) = \frac{\Lambda(\omega)}{m_k\omega_k^2(\omega-2\zeta_k\omega_k i)i}$$

$$\Lambda(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(iy+\zeta_k\omega_k)H_k(y)\,\hat{\Phi}_{kk}(y,y-\omega)dy \tag{5.32}$$

From the frequency convolution theorem and the definition of the generalized power spectral density:

$$\int_0^t [\zeta_k \omega_k h_k(t-\tau) + \frac{d}{dt} h_k(t-\tau)] C_{Q_{kk}}(t,\tau)d\tau \leftrightarrow \Lambda(\omega) \qquad (5.33)$$

Substituting (5.33) into (5.32) and employing the time convolution theorem, we have

$$\kappa_{kk}(t) = \frac{1}{m_k \omega_k^2} \int_0^t \int_0^{t_1} e^{-\zeta_k \omega_k (t-t_1)} [\zeta_k \omega_k h_k(t_1-t_2) + \frac{d}{dt_1} h(t_1-t_2)] \, C_{Q_{kk}}(t_1,t_2)dt_1 dt_2 \qquad (5.34)$$

where the fact that $U(t)\, e^{-2\zeta_k \omega_k t} \leftrightarrow \frac{1}{(\omega - 2\zeta_k \omega_k i)i}$ was used.

From Eq. (2.22):

$$[\zeta_k \omega_k h_k(t_1-t_2) + \frac{d}{dt_1} h(t_1-t_2)]$$

$$= \frac{1}{\omega_{kD} m_k} [\zeta_k \omega_k \, e^{-\zeta_k \omega_k (t_1-t_2)} \operatorname{Sin} \omega_{kD}(t_1-t_2)$$

$$- \zeta_k \omega_k \, e^{-\zeta_k \omega_k (t_1-t_2)} \operatorname{Sin} \omega_{kD}(t_1-t_2)$$

$$+ \omega_{kD} \, e^{-\zeta_k \omega_k (t_1-t_2)} \cos \omega_{kD}(t_1-t_2)]$$

$$= \frac{1}{m_k} \, e^{-\zeta_k \omega_k (t_1-t_2)} \cos \omega_{kD}(t_1-t_2)$$

Substitution of this result into Eq. (5.34) gives

$$\kappa_{kk}(t) = \frac{1}{m_k^2 \omega_k^2} \int_0^t \int_0^{t_1} e^{-2\zeta_k \omega_k (t-t_1)} e^{-\zeta_k \omega_k (t_1-t_2)} \cos \omega_{kD}(t_1-t_2) \cdot C_{Q_{kk}}(t_1,t_2)dt_1, dt_2 \qquad (5.35)$$

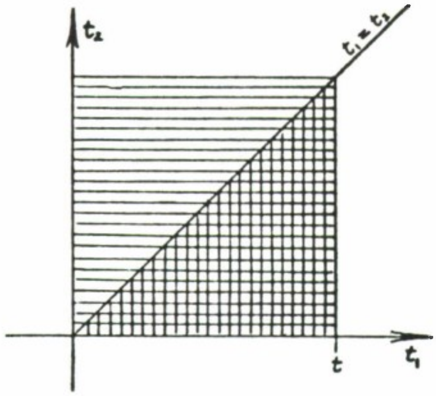


Fig. 11

The region of integration is shown as the cross-hatched area in Fig. 11. But from the definition of $C_{Q_{kk}}(t_1,t_2)$ it follows that it is symmetric in t_1 and t_2. Hence, the entire integrand in Eq. (5.35) is symmetric with respect to t_1 and t_2. Thus the integral of (5.35) is one half the integral taken over the entire shaded region of Fig. 11. That is:

$$\kappa_{kk}(t) = \frac{e^{-2\zeta_k \omega_k t}}{2m_k^2 \omega_k^2} \int_0^t \int_0^t e^{\zeta_k \omega_k (t_1+t_2)} \cos \omega_{kD}(t_1-t_2) C_{Q_{kk}}(t_1,t_2) \cdot dt_1 dt_2 \qquad (5.36)$$

where from (5.19) to (5.21), this is correct to within terms of order ζ_k. This is the general approximation to the variances of the response, and proves to be more convenient than (5.18) when the generalized spectral densities cannot be easily calculated. The following example illustrates this advantage.

Example 5.2

Suppose that the simple oscillator of Example 4.1 is excited by a random force, Q(t), for which the correlation function is an exponential cosine in τ enveloped by a slowly varying deterministic function, $g(t)$. That is:

$$C_Q(t_1,t_2) = C_0\, e^{-\alpha|\tau|} \cos \Omega_0\tau\; g(t_1)g(t_2) \tag{5.37}$$

where α is small and $\tau = t_2-t_1$. And, the equation of motion is

$$\ddot{\xi} + 2\zeta\Omega\dot{\xi} + \Omega^2\xi = \frac{Q(t)}{m} \tag{3.41}$$

It can be seen that the generalized spectral density of Q corresponding to (5.37) is quite complicated. Therefore, we use (5.36) in calculating the variance of the response. We shall consider only the case for which Ω_0 is in the vicinity of Ω.

Substitution of (5.37) into (5.36) yields

$$\kappa_{kk}(t) = C_0 \frac{e^{-2\zeta\Omega t}}{2m\ \Omega^2} \int_0^t\!\!\int_0^t e^{\zeta\Omega(t_1+t_2)} \cos \Omega_D(t_1-t_2)$$

$$\cdot\ e^{-\alpha|\tau|} \cos \Omega_0\tau\; g(t_1)g(t_2)\, dt_1dt_2$$

$$= \frac{C_0\, e^{-2\zeta\Omega\tau}}{2m^2\Omega^2} \int_0^t\!\!\int_0^t e^{\zeta\Omega(t_1+t_2)}\, e^{-\alpha|t_2-t_1|} g(t_1)g(t_2)$$

$$\cdot\ \frac{1}{2}\,[\cos\, (\Omega_D+\Omega_0)(t_2-t_1) + \cos(\Omega_D-\Omega_0)(t_2-t_1)]dt_1dt_2$$

But, by Reimann's lemma, when $|\Omega_D-\Omega_0| << (\Omega_D+\Omega_0) \approx 2\Omega_D$

then to order $\dfrac{|\Omega_D-\Omega_0|}{2\Omega_D}$,

$$\kappa_{kk}(t) = \frac{C_0 e^{-2\zeta\Omega t}}{4m^2\Omega^2} \int_0^t\int_0^t e^{\zeta\Omega(t_1+t_2)} e^{-\alpha|t_2-t_1|} \cos(\Omega_D-\Omega_0)(t_2-t_1)$$

$$\cdot\ g(t_1)g(t_2)dt_1dt_2 \qquad (5.38)$$

Or, if we let

$$\mu \equiv \frac{\alpha}{\epsilon\Omega_0}\ ,\ \epsilon \equiv \zeta\frac{\Omega}{\Omega_0}\ ,\ \tau_1 \equiv \epsilon\Omega_D t_1\ ,\ \tau_2 \equiv \epsilon\Omega_D t_2$$

$$\eta \equiv \epsilon\Omega_D t \quad \text{and } \lambda = \frac{\Omega_0}{\Omega_D} = 1 + \epsilon\beta$$

then Eq. (3.42) becomes

$$\kappa_{kk}(t) = \frac{C_0 e^{-2\zeta\Omega t}}{4m^2\Omega^2\Omega_D^2\epsilon^2} \int_0^\eta\int_0^\eta e^{(\tau_1+\tau_2)} e^{-\mu\lambda|\tau_2-\tau_1|} g(\tau_1)g(\tau_2)$$

$$\cdot\ \cos\beta(\tau_2-\tau_1)d\tau_1d\tau_2 \qquad (5.39)$$

to order $\frac{|\Omega_D-\Omega_0|}{2\Omega_D}$. This is in complete agreement with the result obtained in reference (7).

This concludes our special treatment of the variances. In the remaining sections of this chapter, we show how, for two important cases, methods analogous to those in preceeding pages may be applied to the approximations of the entire matrix of second moments.

5.5 Applications for Shot Noise Excitations

Suppose that a structural system governed by Eqs. (2.17) is driven by a vector shot noise excitation. Then according to (4.43) of section (4.6) the matrix of generalized spectral densities of the excitation is given by

$$[\Phi_{Q_{kj}}(y, y-\omega)] = [\hat{I}_{jk}(\omega)] \tag{4.43}$$

where $\hat{I}_{jk}(\omega)$ is the Fourier transform of the intensity function, $I_{jk}(t)$. Hence the components of the matrix $[\hat{\kappa}_{kj}(\omega)]$ are given by (4.12) in the form

$$\hat{\kappa}_{kj}(\omega) = \frac{\hat{I}_{kj}(\omega)}{2\pi} \int_{-\infty}^{\infty} H_k(y)\, H_j(\omega-y)\,dy \tag{5.40}$$

By Eq. (3.20) we can write (5.40) in the form

$$\left.\begin{aligned} \hat{\kappa}_{kj}(\omega) &= \frac{\hat{I}_{kj}(\omega)}{2\pi m_k m_j} \int_{-\infty}^{\infty} \frac{dy}{(y-a_1)(y-a_2)(\omega-y-a_3)(\omega-y-a_4)} \\ \text{where} \quad a_1 &= i\zeta_k\omega_k + \omega_{kD}\,, \quad a_2 = i\zeta_k\omega_k - \omega_{kD} \\ a_3 &= i\zeta_j\omega_j + \omega_{jD}\,, \quad a_4 = i\zeta_j\omega_j - \omega_{jD} \end{aligned}\right\} \tag{5.41}$$

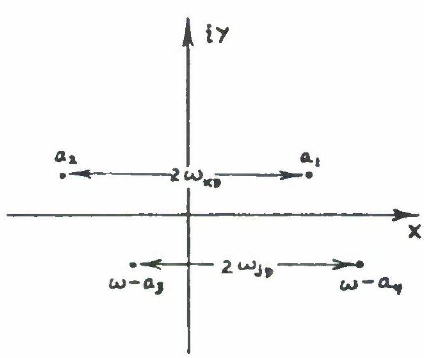


Fig. 12

When y is replaced by z the integrand has simple poles at the points shown in Fig. 12. Integrating over the upper and lower half planes along the contours of Figs. 3(a) and 3(b) we have

$$(\hat{\kappa}_{kj})_u = \frac{i\hat{I}_{kj}(\omega)}{2m_k m_j \omega_{kD}} \left[\frac{1}{(\omega - i\delta_{kj} - \mu_{kj})(\omega - i\delta_{kj} - \nu_{kj})} - \frac{1}{(\omega - i\delta_{kj} + \nu_{kj})(\omega - i\delta_{kj} + \mu_{kj})} \right] \tag{5.42}$$

$$(\hat{\kappa}_{kj})_\ell = \frac{i\hat{I}_{kj}(\omega)}{2m_k m_j \omega_{jD}} \left[\frac{1}{(\omega - i\delta_{kj} - \mu_{kj})(\omega - i\delta_{kj} + \nu_{kj})} - \frac{1}{(\omega - i\delta_{kj} - \nu_{kj})(\omega - i\delta_{kj} + \mu_{kj})} \right] \tag{5.43}$$

where

$$\left.\begin{aligned} \delta_{kj} &\equiv \zeta_k \omega_k + \zeta_j \omega_j \\ \mu_{kj} &\equiv (\omega_{kD} + \omega_{jD}) \\ \nu_{kj} &\equiv (\omega_{kD} - \omega_{jD}) \end{aligned}\right\} \tag{5.44}$$

and from (4.34), $\hat{\kappa}_{kj} = \frac{1}{2}((\hat{\kappa}_{kj})_u + (\hat{\kappa}_{kj})_\ell)$.

Now, we assume that the intensity functions, $I_{jk}(t)$, are slowly varying in the sense that the singularities of $\hat{I}_{jk}(\omega)$ lie within a distance, ϵ, from the origin where $\epsilon \approx \delta_{jk} << \omega_{kD}$ or ω_{jD}. We also assume that $\hat{I}_{jk}(\omega)$ can be expressed in the form

$$\hat{I}_{jk}(\omega) = \frac{J_{kj}(\omega)}{F_{kj}(\omega)} \tag{5.44}$$

where $J_{kj}(z)$ is analytic in the finite z plane and $F_{kj}(z)$ has p poles at $a_1, \ldots, a_p$ in the upper half z plane.

We shall first treat of the case for which $\nu_{kj} = (\omega_{kD} - \omega_{jD})$ is of order ϵ. Consider the first term within the brackets of Eq. (5.42); its inverse transform is

$$I(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\hat{I}_{kj}(\omega)e^{i\omega t}d\omega}{(\omega - i\delta_{kj} - \mu_{kj})(\omega - i\delta_{kj} - \nu_{kj})} \tag{5.45}$$

Referring to section 3.6, we can set

$$(\omega - i\delta_{kj} - \mu_{kj}) = G(\omega)$$

$$a = i\delta_{kj} + \nu_{kj}$$

so that $G(a) = -2\omega_{jD}$ (5.46)

Also:

$$\left.\begin{aligned} m &= 1 \\ n &= p+1 \\ \delta_1 &= \epsilon \\ \epsilon_2 &= 1 \\ \gamma_1 &\approx \mu_{kj} \approx \gamma_2 \end{aligned}\right\} \tag{5.47}$$

And substitution of (5.46) and (5.47) into Eqs. (3.69) to (3.70) gives

$$I(t) = -\frac{1}{2\pi}\frac{1}{2\omega_{jD}}\int_{-\infty}^{\infty}\frac{\hat{I}_{kj}(\omega)\,e^{i\omega t}d\omega}{\omega - i\delta_{kj} - \nu_{kj}}(1+T_1+T_2) \tag{5.48}$$

$$O\{T_1\} = O\{\frac{1}{p+1}(\frac{\varepsilon}{\mu_{kj}})^p \frac{|J(i\delta_{kj}+\nu_{kj})|}{\mathrm{MIN}\{|J(a_k)|\}}\}$$

$$O\{T_2\} < O\{\frac{\varepsilon}{\mu_{kj}}\} \tag{5.49}$$

We can get similar results for the other terms of the inverse transform of $(\hat{\kappa}_{ij}(\omega))_u$ and $(\hat{\kappa}_{ij}(\omega))_\ell$. The results are

$$(\kappa_{kj}(t))_u = \frac{-i/2\pi}{4m_k m_j \omega_{kD}\omega_{jD}} \int_{-\infty}^{\infty} e^{i\omega t}\, \hat{I}_{ij}(\omega)$$

$$\cdot\left[\frac{1}{\omega-i\delta_{kj}-\nu_{kj}} + \frac{1}{\omega-i\delta_{kj}+\nu_{kj}}\right] d\omega\ (1+T_1+T_2)$$

$$(\kappa_{kj}(t))_\ell = \frac{-i/2\pi}{4m_k m_j \omega_{kD}\omega_{jD}} \int_{-\infty}^{\infty} e^{i\omega t}\, \hat{I}_{ij}(\omega)$$

$$\cdot\left[\frac{1}{\omega-i\delta_{kj}+\nu_{kj}} + \frac{1}{\omega-i\delta_{kj}-\nu_{kj}}\right] d\omega\ (1+T_1+T_2)$$

Hence, the approximate Fourier transform of $\kappa_{ij}(t)$ is

$$\hat{\kappa}_{kj}(\omega) = \frac{-i\hat{I}_{jk}(\omega)}{4m_k m_j \omega_{kD}\omega_{jD}}\left[\frac{1}{\omega-i\delta_{kj}+\nu_{kj}} + \frac{1}{\omega-i\delta_{kj}-\nu_{kj}}\right] \tag{5.50}$$

This gives, upon inversion, the second moments correct to within terms of order $\{\frac{1}{p+1}(\frac{\varepsilon}{\mu_{kj}})^p \frac{|J(i\delta_{kj}+\nu_{kj})|}{\mathrm{MIN}\{|J(a_k)|\}} + \frac{\varepsilon}{\mu_{kj}}\}$.

We shall now express (5.50) as a convolution integral.

Since

$$2\, e^{-\delta_{kj} t} \cos \nu_{kj} t \leftrightarrow -i \left[\frac{1}{\omega - i\delta_{kj} + \nu_{kj}} + \frac{1}{\omega - i\delta_{kj} - \nu_{kj}} \right]$$

by the time convolution theorem, we have

$$\kappa_{kj}(t) = \left\{ \frac{1}{2m_k m_j \omega_{kD} \omega_{jD}} \int_0^t I_{jk}(t-\tau)\, e^{-\delta_{kj}\tau} \cos \nu_{kj}\tau \, d\tau \right\}$$

$$(1+T_1+T_2) \tag{5.51}$$

where

$$\left.\begin{aligned} O\{T_1\} &= O\left\{ \frac{1}{p+1} \left(\frac{\varepsilon}{\mu_{kj}}\right)^p \frac{|J(i\delta_{kj} + \nu_{kj})|}{\mathrm{MIN}\,\{|J(a_k)|\}} \right\} \\ O\{T_2\} &= O\left\{ \frac{\varepsilon}{\mu_{kj}} \right\} \end{aligned}\right\} \tag{5.52}$$

and for small ν_{kj}.

Now suppose that $O(\nu_{kj}) \approx O(\omega_{kD})$ or $O(\omega_{jD})$ and hence that $O(\nu_{kj}) >> \varepsilon$. Consider the first term of the inverse transform of Eq. (5.42); given by Eq. (5.45)

$$I(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\hat{I}_{kj}(\omega) e^{i\omega t} d\omega}{(\omega - i\delta_{kj} - \mu_{kj})(\omega - i\delta_{kj} - \nu_{kj})} \tag{5.45}$$

In this case, we can set

$$G(\omega) = (\omega - i\delta_{kj} - \mu_{kj})(\omega - i\delta_{kj} - \nu_{kj})$$

$$a = 0$$

so that

$$G(a) = (i\delta_{kj} + \mu_{kj})(i\delta_{kj} + \nu_{kj}) \tag{5.53}$$

Also as in the previous case:

$$\left.\begin{aligned} m &= 2 \\ n &= p \\ \delta_1 &= \varepsilon \\ \varepsilon_2 &= |\mu_{kj} - \nu_{kj}| = 2\omega_{jD} \geq 2\ \mathrm{MIN}(\omega_{kD}, \omega_{jD}) \\ \gamma_1 &\approx |\nu_{kj}| \\ \gamma_2 &\approx \mu_{kj} \\ b_1, b_2 &= i\delta_{kj} + \mu_{kj},\quad b_3, b_4 = i\delta_{kj} + \nu_{kj} \end{aligned}\right\} \tag{5.54}$$

And substitution of (5.53) and (5.54) into Eqs. (3.69) through (3.70) gives

$$I(t) = \frac{1}{2\pi}\frac{1}{G(a)} \int_{-\infty}^{\infty} \hat{I}_{kj}(\omega) e^{i\omega t} d\omega\ (1+T_1+T_2) \tag{5.55}$$

$$\left.\begin{aligned} O\{T_1\} &\leq O\left\{\frac{\varepsilon^{p-1}\ \mu_{kj}^2\ \mathrm{MAX}\,[|J(b_k)|]}{p\ \mathrm{MIN}(\omega_{kD}\omega_{jD})\ \nu_{kj}^p} \frac{1}{\mathrm{MIN}\,[|J(b_k)|]}\right\} \\ O\{T_2\} &< O\left\{\frac{2\varepsilon}{\nu_{kj}}\right\} \end{aligned}\right\} \tag{5.56}$$

Performing the same analysis for the other terms, we have:

$$\kappa_{kj}(t) = \frac{2\ I_{jk}(t)\ \delta_{kj}}{m_k m_j\ \mu_{kj}^2\ \nu_{kj}^2} (1+T_1+T_2+T_3) \tag{5.57}$$

where the orders of magnitude of T_1 and T_2 are given by (5.56) and

$$O\{T_3\} = O\left\{\left(\frac{2\varepsilon}{\mu_{kj}}\right)^2\right\} \tag{5.58}$$

T_3 arises from the approximations made of $\frac{1}{G(a)}$ for the terms in (5.42) and (5.43).

Equations (5.51) and (5.57) form the complete approximate response formulation to vector shot noise excitations. We consider one application in the following example.

Example 5.3

Suppose that we have obtained the second moments of the response of a system by experimental measurements, and that we know the excitation is a vector shot noise but the specific intensity functions are unknown. Eqs. (5.51) and (5.57) make it possible in certain cases to estimate quite easily these intensity functions from a knowledge of $\kappa_{kj}(t)$.

Let us first consider the case $\nu_{kj} \approx O\{\epsilon\}$, so that we employ Eq. (5.51):

$$\kappa_{kj}(t) \approx \frac{1}{2m_k m_j \omega_{kD} \omega_{jD}} \int_0^t I_{jk}(t-\tau)\, e^{-\delta_{kj}\tau} \cos \nu_{kj}\tau \, d\tau$$

$$\approx \frac{1}{2m_k m_j \omega_{kD} \omega_{jD}} \int_0^t I_{jk}(\tau)\, e^{-\delta_{kj}(t-\tau)} \, d\tau \qquad (5.59)$$

to order $\frac{\nu_{kj}}{\mu_{kj}}$.

Assuming that $I_{jk}(t=0^+) = 0$ and by repeatedly integrating (5.59) by parts, we obtain the series

$$\kappa_{kj}(t) \approx \frac{1}{2m_k m_j \omega_{kD} \omega_{jD} \delta_{jk}} [I_{jk}(t) - \frac{1}{\delta} [\dot{I}_{jk}(t) - e^{-\delta t} \dot{I}_{jk}(0)] + \ldots]$$

Now, that $I_{kj}(t)$ is slowly varying implies that its derivative is small. With $\alpha = O\{\frac{\dot{I}_{jk}}{I_{jk}}\}$ where α is small, we have

$$I_{jk}(t) \approx (2m_k m_j \omega_{kD} \omega_{jD} \delta_{jk} \kappa_{jk}(t)) \qquad (5.60)$$

correct to order α.

A consideration of (5.57) shows that for the case $\nu_{kj} >> \epsilon$, we have

$$I_{jk}(t) \approx \frac{m_k m_j \mu_{kj}^2 \nu_{kj}^2}{2\delta_{kj}} \kappa_{jk}(t) \qquad (5.61)$$

Thus, with $\kappa_{kj}(t) = 0$ at $t=0$, the intensities may be estimated from a direct examination of the second moments.

5.6 Applications for Filtered Shot Noise Excitations

Here we treat a case of the greatest importance to the estimation of Titan III-C flight loads.

Suppose that a shot noise with intensity function $I(t)$ is the excitation, $Q(t)$, to the system of (3.41):

$$\ddot{\xi} + 2\zeta\Omega\dot{\xi} + \Omega^2\xi = \frac{Q(t)}{m} \qquad (3.41)$$

The displacement, ξ, is called a filtered shot noise. Now let each normal coordinate of an N degree of freedom system be driven by the displacement of (3.41). That is:

$$\ddot{\xi}_k + 2\zeta_k\omega_k\dot{\xi}_k + \omega_k^2\xi_k = \frac{\xi(t)}{m_k} \quad (k=1,N) \qquad (5.62)$$

We shall apply Eq. (4.12) and the methods of the previous sections to the approximate calculation of the second moments of the ξ_k.

From (3.29), the generalized spectral density of ξ is

$$\hat{\Phi}_{\xi}(\omega_1,\omega_2) = H(\omega_1)\ \hat{\Phi}_s(\omega_1,\omega_2)H^*(\omega_2) \tag{5.63}$$

and by (4.43) the spectral density of the shot noise, $\hat{\Phi}_s(\omega_1,\omega_2)$ is given by

$$\hat{\Phi}_s(\omega_1,\omega_2) = \hat{I}(\omega_1-\omega_2) \tag{5.64}$$

Thus:

$$\hat{\Phi}_{\xi}(\omega_1,\omega_2) = H(\omega_1)\hat{I}(\omega_1-\omega_2)H^*(\omega_2)$$

or

$$\hat{\Phi}_{\xi}(y,y-\omega) = H(y)\hat{I}(\omega)H^*(y-\omega) \tag{5.65}$$

where

$$\left.\begin{aligned} H(\omega) &= \frac{1}{m}\,(\Omega^2 - \omega^2 + 2i\zeta\Omega\omega)^{-1} \\ \hat{I}(\omega) &= \int_{-\infty}^{\infty} I(t)\ e^{-i\omega t}\,dt \end{aligned}\right\} \tag{5.66}$$

Thus, for this case (4.12) becomes

$$\hat{\kappa}_{kj}(\omega) = \frac{1}{2\pi}\ \hat{I}(\omega)\int_{-\infty}^{\infty} H_k(y)H(y)H(\omega-y)H_j(\omega-y)dy \tag{5.67}$$

since $H^*(y-\omega) = H(\omega-y)$.

From Eq. (3.20) this can be written as

$$\hat{\kappa}_{kj}(\omega) = \Gamma(\omega)\hat{I}(\omega) \tag{5.68}$$

$$\Gamma(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_k(y)H(y)H(\omega-y)H_j(\omega-y)dy \tag{5.69}$$

In what follows, we shall attempt to evaluate approximately the inverse transform of $\Gamma(\omega)$ (denoted by $\gamma(t)$) so that

$$\kappa_{kj}(t) = \int_0^t I(t-\tau)\gamma(\tau)d\tau \tag{5.70}$$

As in the last chapter, we assume that ζ, ζ_k, and ζ_j are small (of the order of 10^{-2}) and that $I(t)$ is a slowly varying function.

From (3.20), $\Gamma(\omega)$ may be expressed as

$$\Gamma(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1/m_k}{(y-a_1)(y-a_2)} \cdot \frac{1/m_j}{(\omega-y-a_3)(\omega-y-a_4)} \cdot \frac{1/m}{(y-a_5)(y-a_6)} \cdot \frac{dy/m}{(\omega-y-a_5)(\omega-y-a_6)} \tag{5.71}$$

where

$$\left.\begin{aligned} a_1 &= i\zeta_k\omega_k + \omega_{kD} \quad , \quad a_2 = i\zeta_k\omega_k - \omega_{kD} \\ a_3 &= i\zeta_j\omega_j + \omega_{jD} \quad , \quad a_4 = i\zeta_j\omega_j - \omega_{jD} \\ a_5 &= i\zeta\Omega + \Omega_D \quad , \quad a_6 = i\zeta\Omega - \Omega_D \end{aligned}\right\} \tag{5.72}$$

We first assume that $|\omega_{kD}-\omega_{jD}|$, $|\omega_{kD}-\Omega|$, $|\omega_{jD}-\Omega| \ll \omega_{kD}$, ω_{jD}, or Ω_D. Upon replacing y by z and employing the calculus of residues the integral of (5.71) can be evaluated. Subsequently, the inverse transform of $\Gamma(\omega)$ can be calculated approximately, according to the methods of preceding sections. The result is*

---

* See Appendix III.

$$\gamma(t) \approx \frac{1}{m^2 m_k m_j} \frac{1}{8\omega_{kD}\omega_{jD}\Omega_D} [\mathrm{Re}(\frac{H(a_1)e^{i(a_1+a_4)t}}{a_4 - a_6})$$

$$+ \mathrm{Re}(\frac{H(a_3)e^{i(a_2+a_3)t}}{a_2 - a_6}) - \mathrm{Re}(\frac{H(a_1)e^{i(a_1+a_6)t}}{a_4 - a_6})$$

$$- \mathrm{Re}(\frac{H(a_3)e^{i(a_3+a_6)t}}{a_2 - a_6})]$$

$$+ \frac{1}{8\omega_{jD}\Omega_D^2} [\mathrm{Re}(\frac{H_k(a_5)e^{i(a_4+a_5)t}}{a_4 - a_6}) - \mathrm{Re}(\frac{H_k(a_5)e^{i(a_5+a_6)t}}{a_4 - a_6})]$$

$$+ \frac{1}{8\omega_{kD}\Omega_D^2} [\mathrm{Re}(\frac{H_j(a_5)e^{i(a_2+a_5)t}}{a_2 - a_6}) - \mathrm{Re}(\frac{H_j(a_5)e^{i(a_5+a_6)t}}{a_2 - a_6})]$$

(5.73)

correct to terms which we denote by T_1+T_2. According to Eq. (III.15):

$$O\{T_1\} <$$

$$O\{\frac{2[\mathrm{MAX}(|a_k^{(-)}|]^{p+1}[\mathrm{MAX}(|a_k^{(+)}-a_j^{(-)}|]^2 \mathrm{MAX}[|J(a_k^{(+)})|]}{(p+2)[\mathrm{MIN}(|a_k^{(+)}-a_j^{(+)})][\mathrm{MIN}(|a_k^{(+)}-a_j^{(-)}|)]^{p+z}\mathrm{MIN}[|J(a_k^{(-)})|]}\}$$

$$O\{T_2\} < \{\frac{2\ \mathrm{MAX}\ (|a_k^{(-)}|)}{\mathrm{MIN}\ (|a_k^{(+)}|)}\} \qquad \text{(III.15)}$$

where these are the orders of magnitude <u>relative to</u> $\gamma(t)$. This can be approximated still further when we note that

$$H(a_1) = [(\Omega^2 - \omega_k^2 + 2\zeta_k^2\omega_k^2 - 2\zeta\Omega\zeta_k\Omega_k) - 2i\Omega_D(\zeta\Omega - \zeta_k\omega_k)]^{-1}$$

$$= [(\Omega_D^2 - \omega_{kD}^2 + \zeta_k^2\omega_k^2 + \zeta^2\Omega^2 - 2\zeta\Omega\zeta_k\Omega_k) - 2i\Omega_D(\zeta\Omega - \zeta_k\omega_k)]^{-1}$$

$$\approx [(\Omega_D^2 - \omega_{kD}^2) - 2i\Omega_D(\zeta\Omega - \zeta_k\omega_k)]^{-1}$$

to order ζ^2. Now we expand this in a Taylor series in terms of the quantity $(\zeta\Omega - \zeta_k\omega_k)$:

$$H(a_1) = \frac{1}{\Omega_D^2 - \omega_{kD}^2} + \frac{1}{\Omega_D^2 - \omega_{kD}^2}\left[\frac{2i\Omega_D(\zeta\Omega - \zeta_k\omega_k)}{(\Omega_D^2 - \omega_{kD}^2)}\right] + \ldots$$

Since the ζ's are small, this series converges rapidly and we can say that

$$H(a_1) \approx \frac{1}{\Omega_D^2 - \omega_{kD}^2} \tag{5.75}$$

to order less than $\mathrm{MAX}(\zeta, \zeta_k)$ since the magnitude of the term

$$\frac{2i\Omega_D(\Omega - \frac{\zeta_k}{\zeta}\omega_k)}{\Omega_D^2 - \omega_{kD}^2}$$ is of order one.

Also if we expand

$$\frac{1}{a_4 - a_6} = \frac{1}{k(\zeta_j\omega_j - \zeta\Omega) + (\Omega_D - \omega_{jD})}$$

in terms of the quantity $\zeta_j\omega_j - \zeta\Omega$, we obtain

$$\frac{1}{(a_4 - a_6)} = \frac{1}{\Omega_D - \omega_{jD}} - \frac{1}{\Omega_D - \omega_{jD}}\left[\frac{i(\zeta_j\omega_j - \zeta\Omega)}{\Omega_D - \omega_{jD}}\right] + \ldots$$

Since the magnitude of $\dfrac{i(\zeta_j\omega_j-\zeta\Omega)}{\Omega_D-\omega_{jD}}$ is of order $\mathrm{MAX}(\zeta,\zeta_j)$:

$$\frac{1}{a_4-a_6} \approx \frac{1}{\Omega_D-\omega_{jD}} \tag{5.76}$$

to order MAX (ζ,ζ_j).

Combining (5.75) and (5.76) we have

$$\frac{H(a_1)}{a_4-a_6} \approx \frac{1}{(\Omega_D^2-\omega_k^2)(\Omega_D-\omega_{jD})} \tag{5.77}$$

to order less than $\mathrm{MAX}(\zeta,\zeta_j,\zeta_k)$.

By a similar procedure, we can obtain

$$\frac{H(a_3)}{a_2-a_6} \approx \frac{1}{(\Omega_D^2-\omega_{jD}^2)(\Omega_D-\omega_{kD})} \tag{5.78}$$

$$\frac{H_k(a_5)}{a_4-a_6} \approx \frac{1}{(\omega_{kD}^2-\Omega_D^2)(\Omega_D-\omega_{jD})} \tag{5.79}$$

$$\frac{H_j(a_5)}{a_2-a_6} \approx \frac{1}{(\omega_{jD}^2-\Omega_D^2)(\Omega_D-\omega_{kD})} \tag{5.80}$$

all to order less than MAX (ζ,ζ_j,ζ_k).

Finally, by virtue of Eqs. (5.72), Eq. (5.73) and Eqs. (5.77)-(5.80), we have:

$$
\left.
\begin{aligned}
\gamma_{kj}(t) \approx A[B_1 & e^{-(\zeta_k\omega_k+\zeta_j\omega_j)t} \cos(\omega_{kD}-\omega_{jD})t \\
-B_2 & e^{-(\zeta_k\omega_k+\zeta\Omega)t} \cos(\omega_{kD}-\Omega_D)t \\
-B_3 & e^{-(\zeta_j\omega_j+\zeta\Omega)t} \cos(\omega_{jD}-\Omega_D)t + B_4\, e^{-2\zeta\Omega t} \\
A &= \frac{1}{8\omega_{kD}\omega_{jD}\Omega_D(\Omega_D^2-\omega_{jD}^2)(\Omega_D^2-\omega_{kD}^2)} \\
B_1 &= 2\Omega_D + \omega_{kD} + \omega_{jD} \\
B_2 &= \frac{\omega_{jD}}{\Omega_D}(\Omega_D + \omega_{kD}) + (\Omega_D + \omega_{jD}) \\
B_3 &= \frac{\omega_{kD}}{\Omega_D}(\Omega_D + \omega_{jD}) + (\Omega_D + \omega_{kD}) \\
B_4 &= \frac{\omega_{jD}}{\Omega_D}(\omega_{kD} + \Omega_D) + \frac{\omega_{kD}}{\Omega_D}(\omega_{jD} + \Omega_D)
\end{aligned}
\right\} \qquad (5.81)
$$

which yields a result correct to $(T_1+T_2+T_3)$ where $O\{T_1\}$ and $O\{T_2\}$ are given by (III.15) and T_3 arises from the approximations of the frequency response functions so that $O\{T_3\} < O\{MAX(\zeta,\zeta_k,\zeta_j)\}$.

It is instructive to note the results for the following special cases.

A. j=k:

From (5.81):

$$
\gamma_{kk}(t) \approx \frac{1}{8\omega_{kD}^2\Omega_D(\Omega_D^2-\omega_{kD}^2)^2}\,[2(\Omega_D+\omega_{kD})e^{-2\zeta_k\omega_k t} - 2(\frac{\omega_{kD}}{\Omega_D} + 1)
$$

(equation continued)

$$(\Omega_D + \omega_{kD})\, e^{-(\zeta_k\omega_k+\zeta\Omega)t} \cos(\omega_{kD} - \Omega_D)t$$

$$+ \frac{2\omega_{kD}}{\Omega_D}(\omega_{kD} + \Omega_D)\, e^{-2\zeta\Omega t}] \qquad (5.82)$$

B. $\underline{\omega_j = \Omega,\ \zeta_j = \zeta}$

Letting $\omega_{jD} = \Omega_D - \Delta$, where Δ is small, Eq. (5.81) becomes

$$\gamma_{kj}(t) \approx \frac{1}{8\omega_{kD}\Omega_D^2(2\Omega_D)\Delta(\Omega_D^2-\omega_{kD}^2)} [(3\Omega_D+\omega_{kD})e^{-(\zeta_k\omega_k+\zeta\Omega)t}$$

$$\cdot \cos(\omega_{kD}-\Omega_D+\Delta)t - (3\Omega_D + \omega_{kD})\, e^{-(\zeta_k\omega_k+\zeta\Omega)t}$$

$$\cdot \cos(\omega_{kD}-\Omega_D)t - (\frac{\omega_{kD}}{\Omega_D} 2\Omega_D + \Omega_D + \omega_{kD})\, e^{-2\zeta\Omega t} \cos \Delta t$$

$$+ (\frac{\omega_{kD}}{\Omega_D} 2\Omega_D + \Omega_D + \omega_{kD})\, e^{-2\zeta\Omega t}] + T(\Delta)$$

where $T(\Delta) \to 0$ as $\Delta \to 0$. Hence:

$$\lim_{\Delta\to 0} \gamma_{kj}(t) \approx \frac{3\Omega_D + \omega_{kD}}{16\omega_{kD}\Omega_D^3(\Omega_D^2-\omega_{kD}^2)}\, e^{-(\zeta_k\omega_k+\zeta\Omega)t}$$

$$\cdot \lim_{\Delta\to 0} \{\frac{1}{\Delta}(\cos(\omega_{kD}-\Omega_D+\Delta)t - \cos(\omega_{kD}-\Omega_D)t)\}$$

$$+ \frac{3\omega_{kD}+\Omega_D}{16\omega_{kD}\Omega_D^3(\Omega_D^2-\omega_{kD}^2)}\, e^{-2\zeta\Omega t} \lim_{\Delta\to 0} \{\frac{1}{\Delta}(1-\cos \Delta t)\}$$

$$= \frac{(3\Omega_D+\omega_{kD})\, e^{-(\zeta_k\omega_k+\zeta\Omega)t}}{16\omega_{kD}\Omega_D^3(\Omega_D^2 - \omega_{kD}^2)}\, t\sin(\Omega_D - \omega_{kD})t \qquad (5.83)$$

by L'Hospital's rule.

Hence, when the central frequency of the filter coincides with the damped natural frequency of the j^{th} mode:

$$\gamma_{kj}(t) \approx \frac{(3\Omega_D+\omega_{kD})\; e^{-(\zeta_k \omega_k + \zeta\Omega)t}}{16\omega_{kD}\Omega_D^3\;(\Omega_D^2 - \omega_{kD}^2)}\; t\; \sin(\Omega_D-\omega_{kD})t \qquad (5.84)$$

C. $\underline{\omega_j = \Omega}$, $\underline{\zeta_j = \zeta}$, $\underline{\omega_k = \Omega}$, $\underline{\zeta_k = \zeta}$

From Eq. (5.83); with $\omega_{kD} = \Omega_D - \Delta$

$$\gamma_{kj}(t) = \frac{1}{8\;\Omega_{OD}^4\,\Delta}\; t\; e^{-2\zeta\Omega t}\; \text{Sin}\;\Delta t + T(\Delta)$$

where $\lim_{\Delta\to 0} T(\Delta) = 0$. Hence,

$$\lim_{\substack{\zeta_k\to\zeta \\ \omega_k\to\Omega}} \gamma_{kj}(t) = \frac{t\; e^{-2\zeta\Omega t}}{8\;\Omega_{OD}^4} \lim_{\Delta\to 0} \left(\frac{\text{Sin}\Delta t}{\Delta}\right)$$

$$= \frac{t^2\; e^{-2\zeta\Omega t}}{8\;\Omega_{OD}^4}$$

Thus, for the case $\omega_j = \Omega$, $\zeta_j = \zeta$, $\omega_k = \Omega$, $\zeta_k = \zeta$:

$$\gamma_{kj}(t) \approx \frac{t^2\; e^{-2\zeta\Omega t}}{8\;\Omega_{OD}^4} \qquad (5.85)$$

CHAPTER VI

ESTIMATION OF PAYLOAD STRUCTURE FLIGHT LOADS

6.1 Introduction

In Chapter V the basic method of approximate calculation of the second moment response was presented. A variety of simple problems were considered in order to illustrate the basic application of the approximation technique. In this chapter the more realistic problem of the estimation of flight loads in a payload is treated. In particular, we return to a consideration of the problem that was introduced in Chapter I.

We take note of the following basic facts:

1. The dynamic inputs to the payload system are assumed to be the booster/payload interface accelerations of the six rigid body degrees of freedom.

2. These interface accelerations were obtained by a numerical calculation of the response of an analytical model (consisting of assemblage of discrete elements) of the booster structure. These booster models were driven by loads derived from chamber pressure measurements of the rocket engines during 27 test firings.

Thus the subensemble of excitations to the payload system consists of the 27 sets of six acceleration time histories.

We assume that each set is a sample of a random vector process whose vector components are the six rigid acceleration time histories at the interface.

In Section 6.2 we present the basic deterministic equations for the calculation of loads in the payload structure. In 6.3 and 6.4 we discuss the assumptions made about the interface accelerations and express these assumptions in analytical form. In sections 6.5 and 6.6 we show how the statistical parameters of the acceleration process can be estimated under the assumptions made. Finally in section 6.7 is given the approximate formulation of the second moment response of a payload to the booster/payload interface accelerations. This chapter concludes with the application of the approximate formulation to a simple two degree of freedom payload structure. This example exhibits all the main characteristics of the calculation of response of a real payload structure.

6.2 General Formulation: Equations of Motion

We assume that the modal frequencies and corresponding mode shapes of the payload structure are known. Then the problem may be formulated entirely in terms of the normal coordinates of the payload. According to assumption B, section 1.1, the rigid body displacements are prescribed at the booster/payload interface. This case is exactly that considered in Example 2.2, section 2.3. Thus, from Eq. (2.19) the equations of motion of the payload system are

$$\{\ddot{\eta}\} + 2[\diagdown \zeta_i \omega_i \diagdown]\{\dot{\eta}\} + [\diagdown \omega_i^2 \diagdown]\{\eta\} = [C]\{a\} \qquad (6.1)$$

where

$\{\eta\}$ ≡ normal coordinates of the payload

$= [\varphi]^T[m]\{u\}$, where $\{u\}$ is the payload relative displacement vector and $[m]$ is the mass matrix.

ω_i ≡ the modal frequencies

ζ_i ≡ the modal damping coefficients

$\{a\}$ ≡ the booster/payload interface accelerations

$= \{\ddot{x}, \ddot{y}, \ddot{z}, \ddot{\theta}_x, \ddot{\theta}_y, \ddot{\theta}_z\}$

$[C]$ is a matrix of constants

The member stresses, $\{S\}$, are related to the normal coordinates by

$$\{S\} = [G]\{\eta\} \qquad (6.2)$$

where $[G]$ is some matrix of constants.

In what follows, we shall assume that the modal damping coefficients are of order 10^{-2}.

Now, the interface accelerations are idealized as a random vector process. We wish to find the matrix of the second order central moments of the response, $\{\eta\}$, from a knowledge of the second-order statistics of the interface accelerations. Once this is accomplished, the second order moments of the member stresses are given by

$$[s_{ij}] = [G][\kappa_{k\ell}][G]^T \tag{6.3}$$

where

$[s_{ij}]$ ≡ the second order central moments of the member stresses

$[\kappa_{k\ell}]$ ≡ the second order central moments of the response $\{\eta\}$

Thus, the major problem is the determination of the statistics of the normal coordinate response. The first step in this task is the description of the interface accelerations considered as a random process. In this regard, we establish certain basic assumptions in the following section.

6.3 Idealization of the Booster/Payload Interface Accelerations

The basic assumptions about the nature of the booster/payload interface accelerations were presented in section 1.3. We restate them here in the following form.

A. (i) The interface accelerations are the response of a lumped parameter, second order, linear system to random excitation. We assume that this linear system is the structure of the booster itself.

(ii) The second time derivatives of the random excitation to the booster can be idealized as a non-stationary vector shot noise process.

B. The intensity functions of the components of the vector shot noise are the same to within multiplicative constants.

C. The excitations to different normal modes of the booster structure are uncorrelated.

These assumptions must be expressed in mathematical form. Consider assumption A(i), and suppose that the booster structure has N degrees of freedom. According to this assumption, the response of the booster to the initial random excitation may be expressed by an equation of the form of Eq. (3.29). That is,

$$\underset{(N\times N)}{[\hat{\Phi}_{\xi_b}(\omega_1,\omega_2)]} = \underset{(N\times N)}{[\diagdown H_{bk}(\omega_1)\diagdown]}\underset{(N\times N)}{[\hat{\Phi}_Q(\omega_1,\omega_2)]}\underset{(N\times N)}{[\diagdown H^*_{bk}(\omega_2)\diagdown]} \tag{6.4}$$

where

$[\hat{\Phi}_{\xi_b}(\omega_1,\omega_2)]$ = The matrix of generalized spectral densities of the normal coordinate response of the booster structure.

$H_{bk}(\omega) \equiv$ The frequency response function of the k^{th} booster normal coordinate, in the form given by Eq. (3.20).

$[\hat{\Phi}_Q(\omega_1,\omega_2)] \equiv$ The matrix of generalized spectral densities of the initial random excitation, assumed to have p non-zero normal coordinate components.

Employing the time differentiation theorem for Fourier integrals, we see that the generalized spectral density matrix of the acceleration response of the booster, $[\hat{\Phi}_{a_b}(\omega_1,\omega_2)]$ is given by

$$\underset{(N\times)}{[\hat{\Phi}_{a_b}(\omega_1,\omega_2)]} = \underset{(N\times N)}{[\diagdown H_{b_k}(\omega_1)\diagdown]}\underset{(N\times)}{[\omega_1^2\omega_2^2\,\hat{\Phi}_Q(\omega_1,\omega_2)]}\underset{(N\times)}{[\diagdown H^*_{b_k}(\omega_2)\diagdown]} \tag{6.5}$$

where $[\omega_1^2\omega_2^2\,\hat{\Phi}_Q(\omega_1,\omega_2)]$ is the generalized spectral density matrix of the second time derivatives of the random excitation components. Now the interface accelerations are related to the booster normal coordinate accelerations by a matrix transformation. Denoting the interface accelerations by $\{a_I\}$, and the booster accelerations by $\{a_b\}$, we have

$$\underset{(6\times 1)}{\{a_I\}} = \underset{(6\times N)}{[C]}\ \underset{(N\times 1)}{\{a_b\}} \tag{6.6}$$

where $[C]$ is a constant matrix, and the acceleration components are designated as follows:

$$\{a_I\} = \begin{Bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \\ \ddot{\theta}_x \\ \ddot{\theta}_y \\ \ddot{\theta}_z \end{Bmatrix} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}$$

Then from Eq. (6.6) and (6.5) and the definition of the generalized spectral density, we have

$$[\hat{\Phi}_{a_I}(\omega_1,\omega_2)] = [C][\diagdown H_{b_k}(\omega_1)\diagdown][\omega_1^2\omega_2^2\,\hat{\Phi}_Q(\omega_1\omega_2)][\diagdown H^*_{b_k}(\omega_2)\diagdown][C]^T$$

(6x6) (6xN) (NxN) (NxN) (NxN) (Nx6)

(6.7)

where $[\hat{\Phi}_{a_I}(\omega_1,\omega_2)]$ is the generalized spectral density matrix of the interface accelerations. Eq. (6.7) now embodies assumption A(i).

According to the definition of a general vector shot noise, Eq. (2.59); assumption A(ii) states that the covariance matrix of the second derivatives of the random excitation has the form $[I_{jk}(t)]\delta(t_2-t_1)$ where the $I_{jk}(t)$ are the general shot noise intensity functions. Hence, according to Eq. (4.43) of section 4.6:

$$[\omega_1^2\omega_2^2\,\hat{\Phi}_Q(\omega_1,\omega_2)] = [\hat{I}_{jk}(\omega_1-\omega_2)] \tag{6.8}$$

where $\hat{I}_{jk}(\omega)$ is the Fourier transform of $I_{jk}(t)$.

Eqs. (6.7) and (6.8) now yield the analytical statement of assumptions A:

$$[\hat{\Phi}_{a_I}(\omega_1,\omega_2)] = [C][\diagdown H_{b_k}(\omega_1)\diagdown][\hat{I}_{jk}(\omega_1-\omega_2)][\diagdown H^*_{b_k}(\omega_2)\diagdown][C]^T$$

(6xN) (NxN) (NxN) (NxN) (Nx6)

(6.9)

Now, assumption B states that

$$[\hat{I}_{jk}(\omega_1-\omega_2)] = \hat{I}(\omega_1-\omega_2)[K_{jk}] \tag{6.10}$$

where $\hat{I}(\omega_1-\omega_2)$ is the transform of the common intensity function and $[K_{jk}]$ is a matrix of constants. Thus, if assumptions A and B are valid, we have

$$[\hat{\Phi}_{a_I}(\omega_1,\omega_2)] = [C]\lceil H_{b_k}(\omega_1)\rfloor[K_{jk}]\lceil H^*_{b_k}(\omega_2)\rfloor[C]^T\cdot\hat{I}(\omega_1-\omega_2) \tag{6.11}$$

from Eqs. (6.9) and (6.10).

Finally, according to assumption C. the cross-covariances between different normal coordinate components of the initial random excitation are all zero. Thus, from Eq. (6.8) and the definition of the cross-spectral density,

$$[\hat{I}_{jk}(\omega_1-\omega_2)] = \lceil \hat{I}_{kk}(\omega_1-\omega_2)\rfloor$$

or,

$$[K_{jk}] = \lceil K_{jj}\rfloor \tag{6.12}$$

Substitution of this result into Eq. (6.11) results in

$$[\hat{\Phi}_{a_I}(\omega_1,\omega_2)] = [C]\lceil H_{b_k}(\omega_1)\rfloor\lceil K_{jj}\rfloor\lceil H^*_{b_k}(\omega_2)\rfloor[C]^T\cdot\hat{I}(\omega_1-\omega_2)$$

But since $\lceil K_{jj}\rfloor$ equals $\lceil\sqrt{K_{jj}}\rfloor\lceil\sqrt{K_{jj}}\rfloor$ we have

$$[\hat{\Phi}_{a_I}(\omega_1,\omega_2)] = [C]\lceil\sqrt{K_{jj}}\rfloor\lceil H_{b_k}(\omega_1)\rfloor\lceil H^*_{b_k}(\omega_2)\rfloor\lceil\sqrt{K_{jj}}\rfloor [C]^T\,\hat{I}(\omega_1-\omega_2) \tag{6.13}$$

Similarly, we can factor out the quantities $\frac{1}{m_k}$ in the $H_{b_k}(\omega)$ (see Eq. (3.20)). In summary, Eq. (6.13) may be written in the form

$$[\hat{\Phi}_{a_I}(\omega_1,\omega_2)] = [M][\diagdown H_{b_k}(\omega_1)\diagup][H^*_{b_k}(\omega_2)\diagup][M]^T \frac{\hat{I}(\omega_1-\omega_2)}{\hat{I}(0)} \tag{6.14}$$

(6×6) (6×N) (N×N) (N×N) (6×N)

where

$$[M] = \sqrt{\hat{I}(0)}[C][\diagdown \sqrt{K_i}\diagup][\frac{1}{m_k}] \tag{6.15}$$

$$H_{b_k}(\omega) = [\Omega_k^2 - \omega^2 + 2i\zeta_{0k}\Omega_k\omega]^{-1} \; * \tag{6.16}$$

$$\hat{I}(\omega_1-\omega_2) = \int_{-\infty}^{\infty} e^{-it(\omega_1-\omega_2)} I(t)\, dt \tag{6.17}$$

and where Ω_k and ζ_k are the natural frequency and damping ratio of the k^{th} booster normal mode.

Equations (6.14) through (6.17) completely specify our general assumptions concerning the nature of the booster interface accelerations. In the following section we present some of the data relevant to the interface accelerations and discuss the validity of assumptions A through C.

6.4 <u>Comparison of the Idealization With Relevant Data</u>

At our disposal we have two sets of data concerning the interface accelerations.

The first of these sets is composed of the acceleration time histories themselves. As stated in section 6.1 these

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* This differs from the form of Eq. (3.20) in that the reciprocal of the k^{th} component of the modal mass matrix has been factored out.

time histories were obtained from the calculated responses of analytical models of the booster structures to excitations derived from booster engine chamber pressure measurements. These measurements were obtained for 27 rocket engine test firings during the booster engine cut-off. Thus we have 27 sets of acceleration data, each set consisting of the 6 rigid body acceleration components at the booster payload interface. The coordinates for these components are as illustrated in Fig. 1 of section 1.2.

In Figures 13 through 18 we present these acceleration time histories. In each figure is shown a superposition of each of 27 cases of one acceleration component. It can be seen that the mean of the x components clearly indicates the thrust decay of the booster engines. The remaining components have approximately zero mean values. In the sections that follow, we shall be mainly interested in these zero mean acceleration components. This will imply no loss of generality in the remaining discussions since the mean of the x component may be easily calculated and subtracted from the original time histories to yield a set of time histories with a zero mean value.

The second kind of data obtained consists of the power spectral density estimates of the acceleration components. We define the power spectral density of the j^{th} and k^{th}

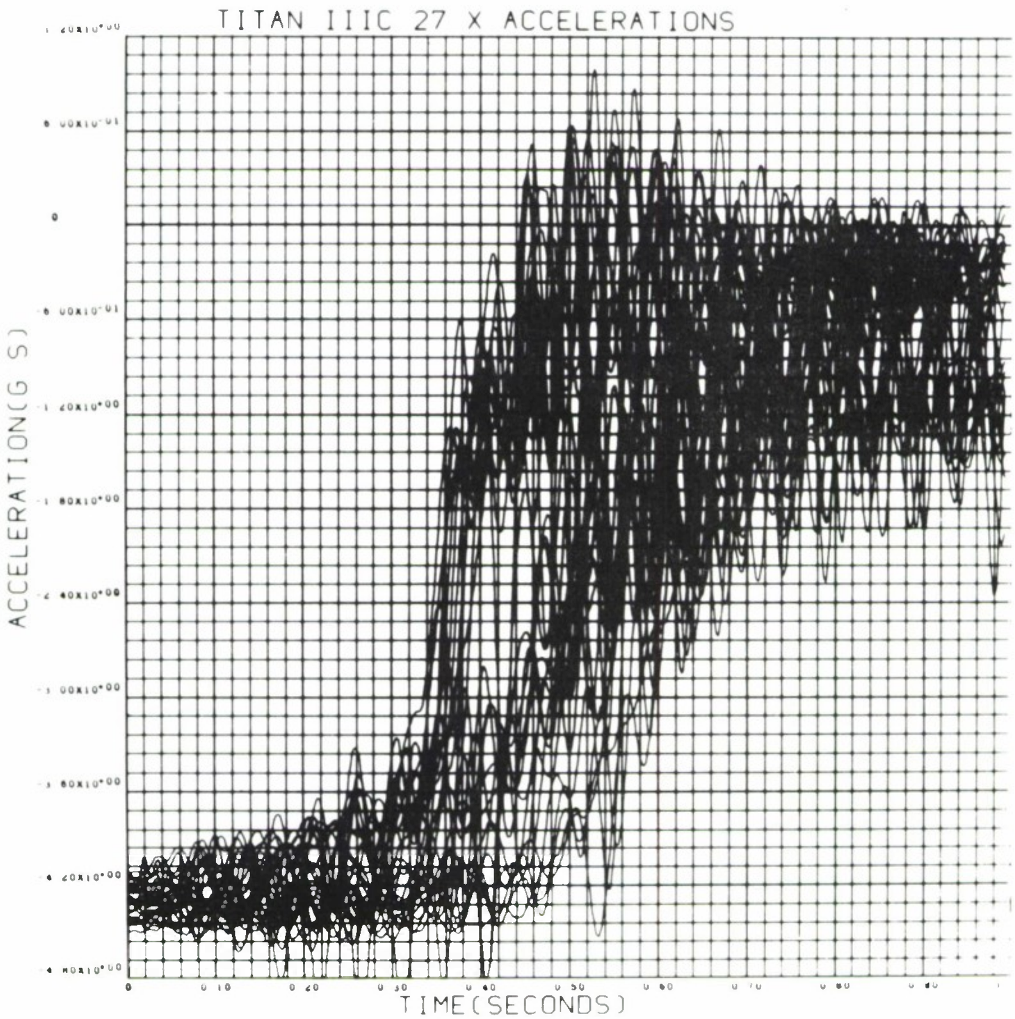
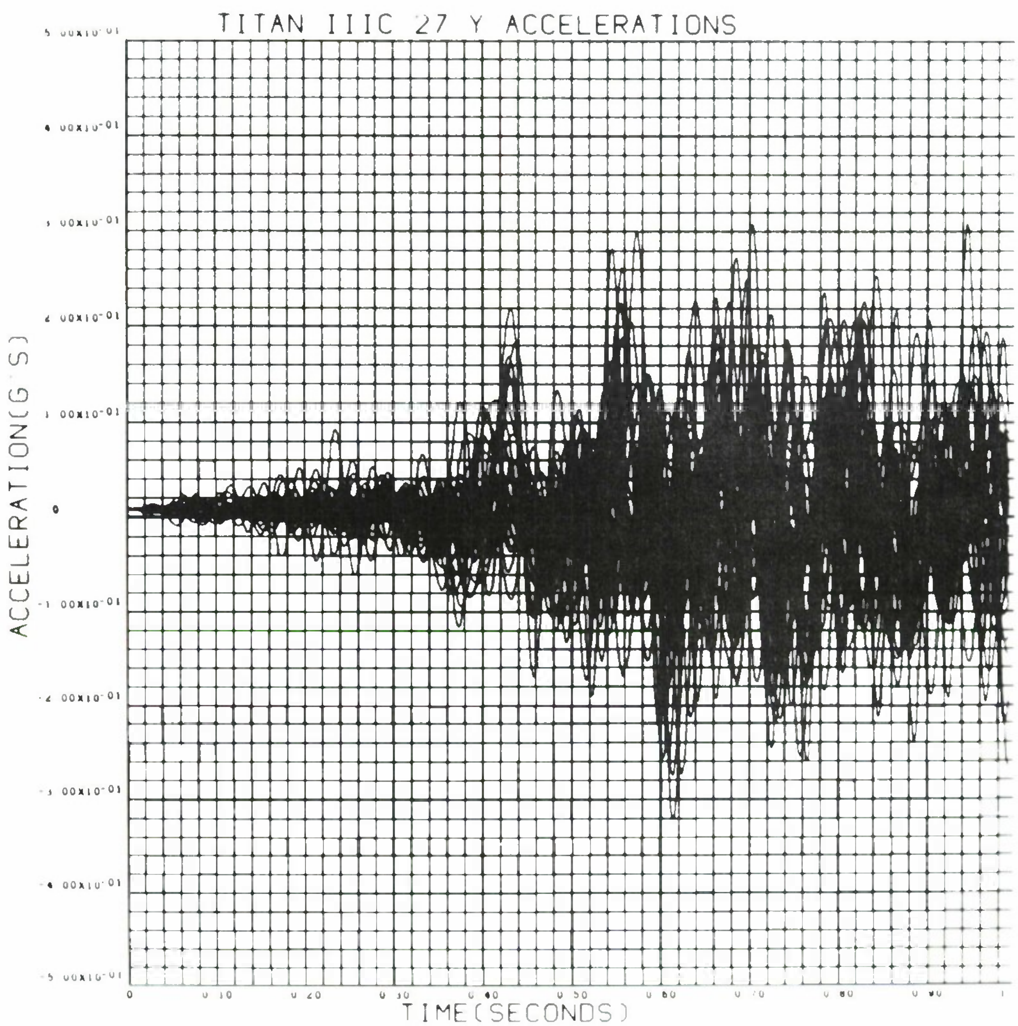

TITAN IIIC 27 X ACCELERATIONS
ACCELERATION(G S)
TIME(SECONDS)

Fig. 13



TITAN IIIC 27 Y ACCELERATIONS
5.00x10-01
4.00x10-01
3.00x10-01
2.00x10-01
1.00x10-01
0
-1.00x10-01
-2.00x10-01
-3.00x10-01
-4.00x10-01
-5.00x10-01
ACCELERATION(G'S)
0
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1
TIME(SECONDS)

FIG. 14

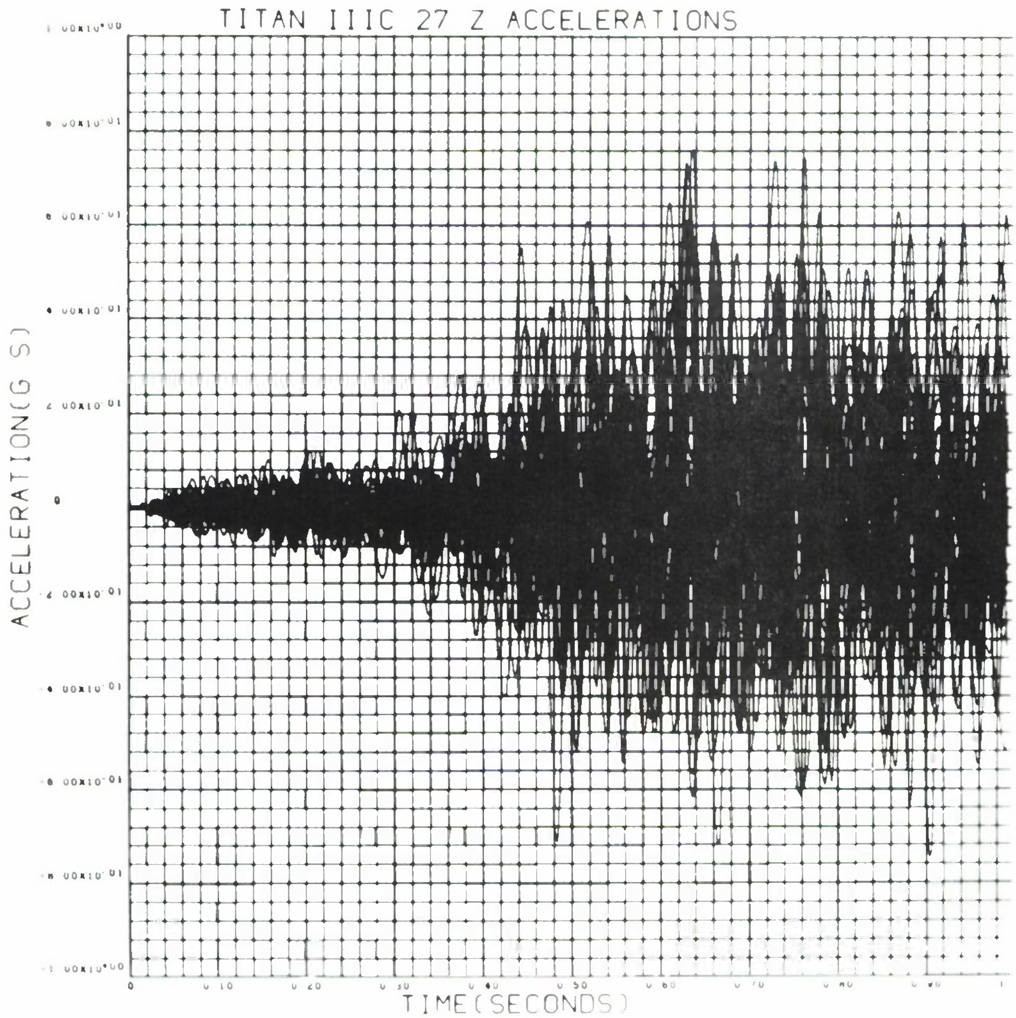

TITAN IIIC 27 Z ACCELERATIONS
ACCELERATION(G S)
TIME(SECONDS)

FIG. 15

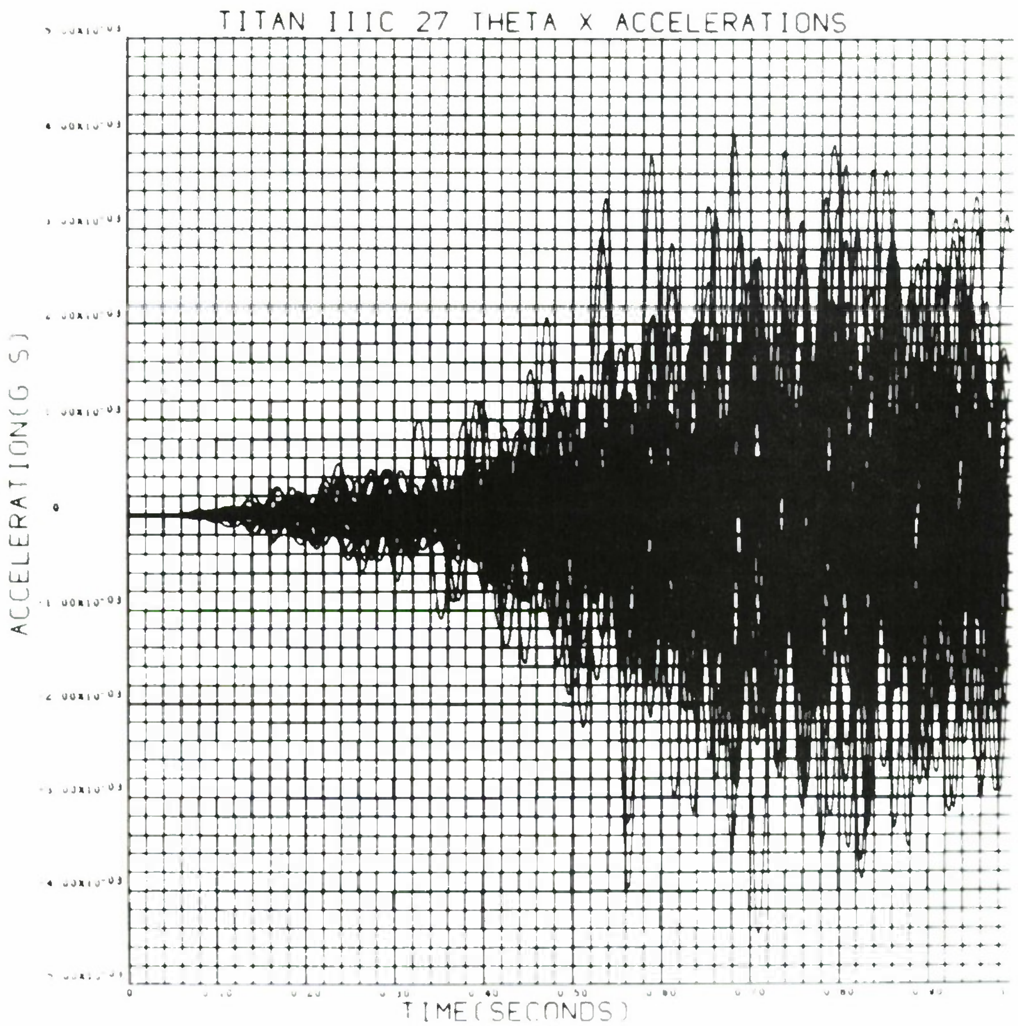

TITAN IIIC 27 THETA X ACCELERATIONS
ACCELERATION(G S)
TIME(SECONDS)

FIG. 16

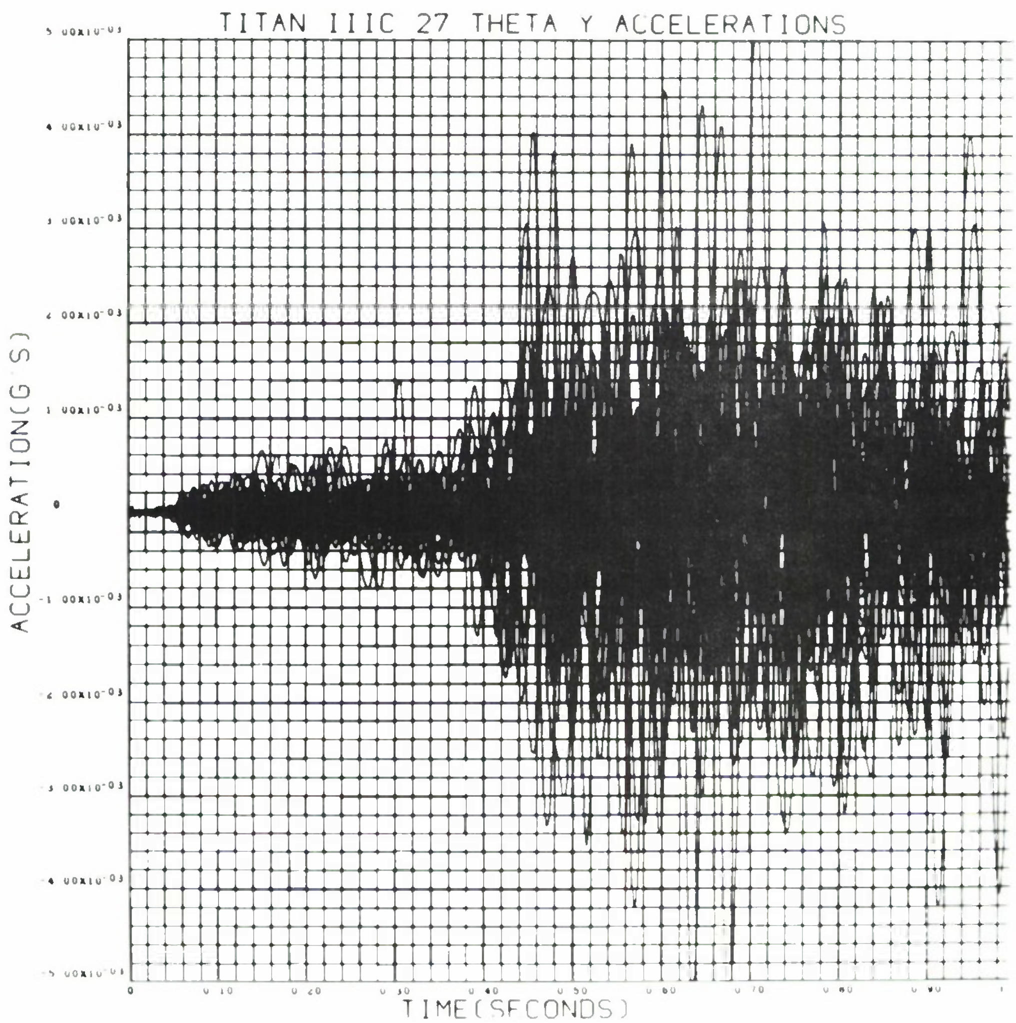

TITAN IIIC 27 THETA Y ACCELERATIONS
ACCELERATION(G'S)
TIME(SECONDS)

FIG. 17

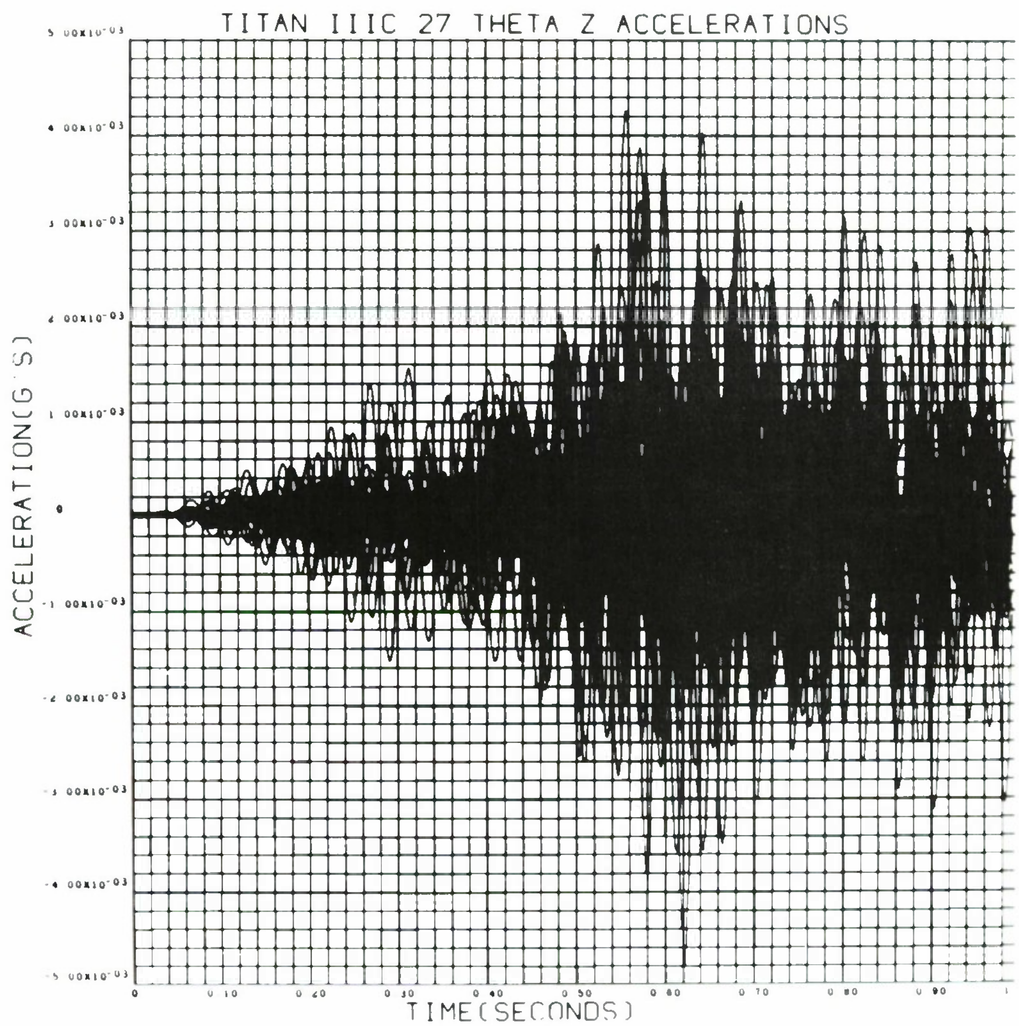

TITAN IIIC 27 THETA Z ACCELERATIONS
ACCELERATION(G'S)
TIME(SECONDS)

FIG. 18

acceleration components, $\Phi_{jk}(\omega)$ as

$$\Phi_{jk}(\omega) = E[A_j(\omega)A_k^*(\omega)] \tag{6.18}$$

where $A_j(\omega)$ is the Fourier transform of the j^{th} acceleration component. From elementary probability theory, an estimate of the quantity is[3]

$$\frac{1}{27}\sum_{k=1}^{27} A_j(\omega)A_k^*(\omega) = \underset{\sim}{\Phi}_{kj}(\omega) \tag{6.19}$$

where $\underset{\sim}{\Phi}_{kj}(\omega)$ is the estimated value of the quantity defined by equation (6.18). The Fourier transforms of all of the 6x27 acceleration time histories were obtained for frequencies in the 0 to 50 Hz. range and the six power spectral densities $\Phi_{kk}(\omega)$ (k=1,...,6) were estimated according to Eq. (6.19). These functions are presented in Figures 19 through 24. It can be seen that the non-zero mean value of the x component is reflected in the large peak at ω=0 on the spectral density plot. However, the x component power spectral density still provides useful spectral information at frequencies higher than ~ 10 Hertz.

There is a simple relationship between the power spectral densities and the generalized spectral densities. For the acceleration components with zero mean; the diagonal components of $[\hat{\Phi}_{a_I}(\omega_1,\omega_2)]$ evaluated at $\omega_1=\omega_2=\omega$ are just the power spectral densities defined by Eq. (6.18). Hence the estimates

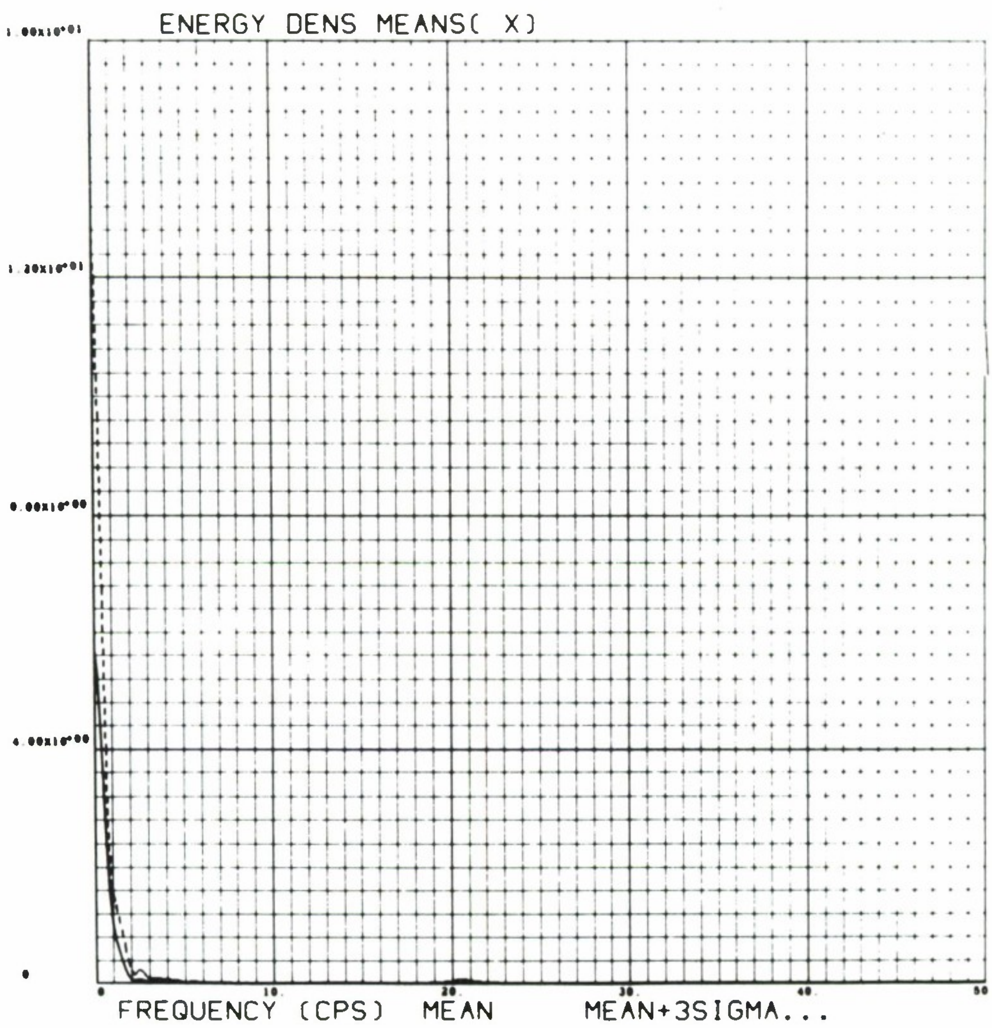

ENERGY DENS MEANS(X)
0
10.
20.
30.
40.
50
FREQUENCY (CPS) MEAN MEAN+3SIGMA...

Fig. 19

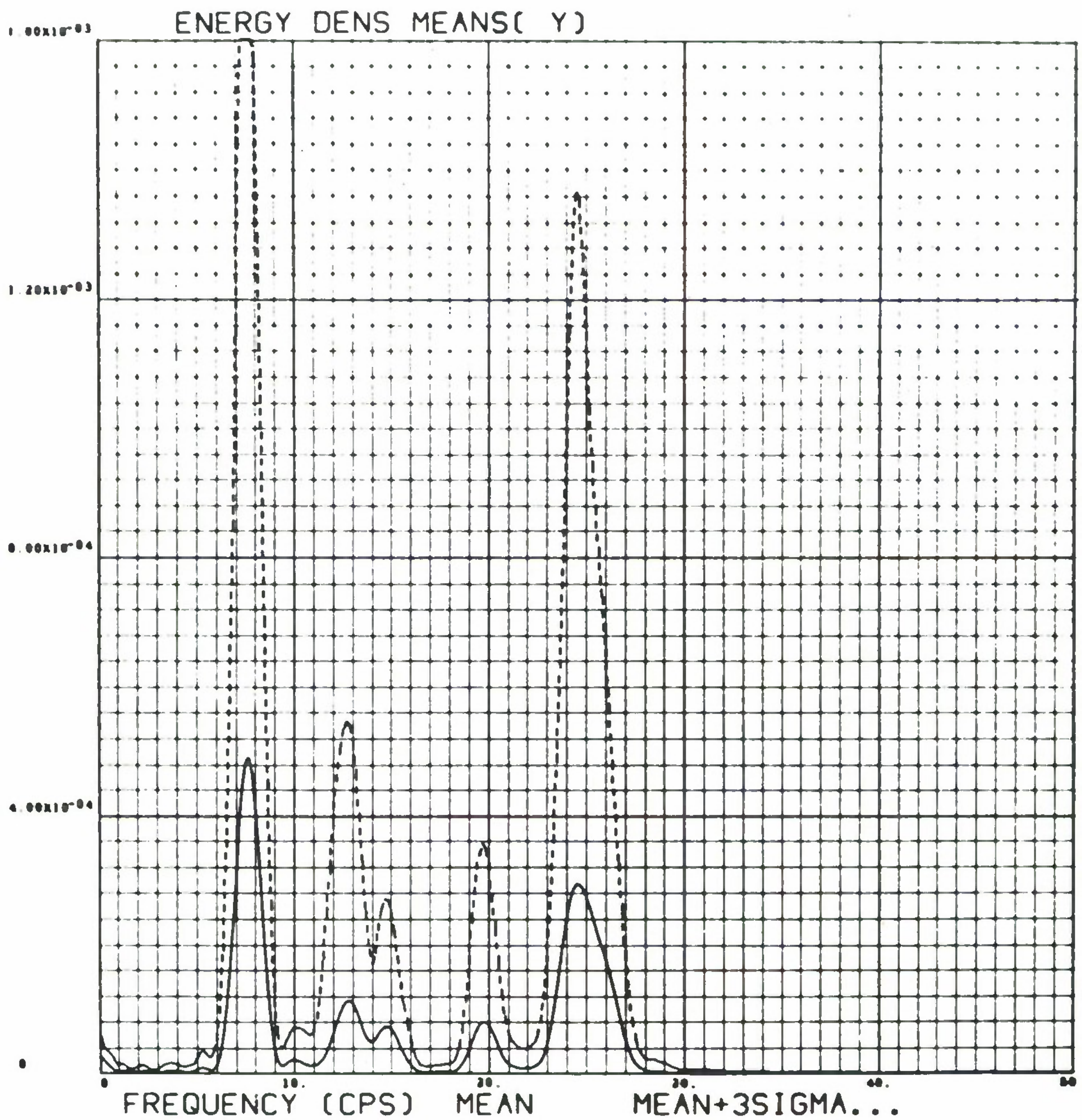

ENERGY DENS MEANS(Y)
1.20x10^-03
4.00x10^-04
0
0
10.
20.
30.
40.
FREQUENCY (CPS) MEAN MEAN+3SIGMA...

FIG. 20

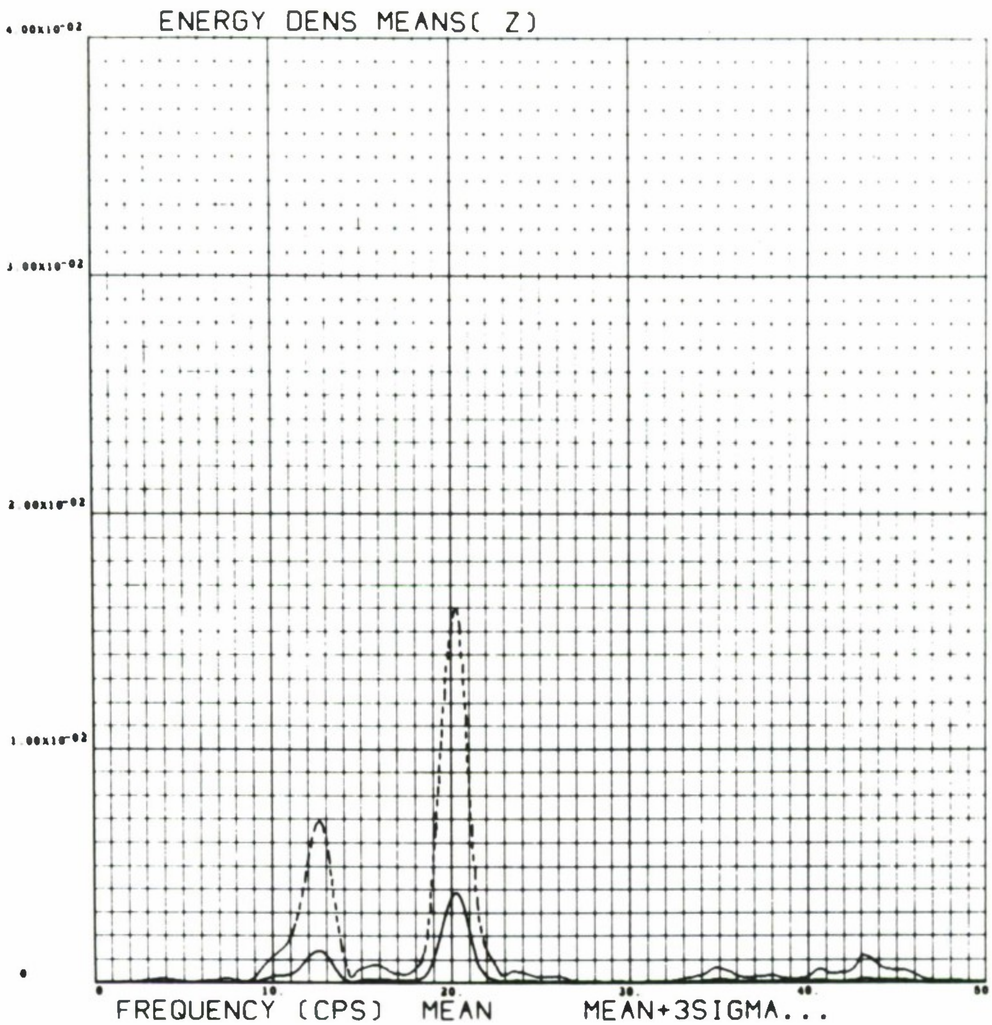

ENERGY DENS MEANS(Z)
4.00x10^-02
3.00x10^-02
2.00x10^-02
1.00x10^-02
0
10.
20.
30.
40.
50
FREQUENCY (CPS) MEAN MEAN+3SIGMA...

FIG. 21

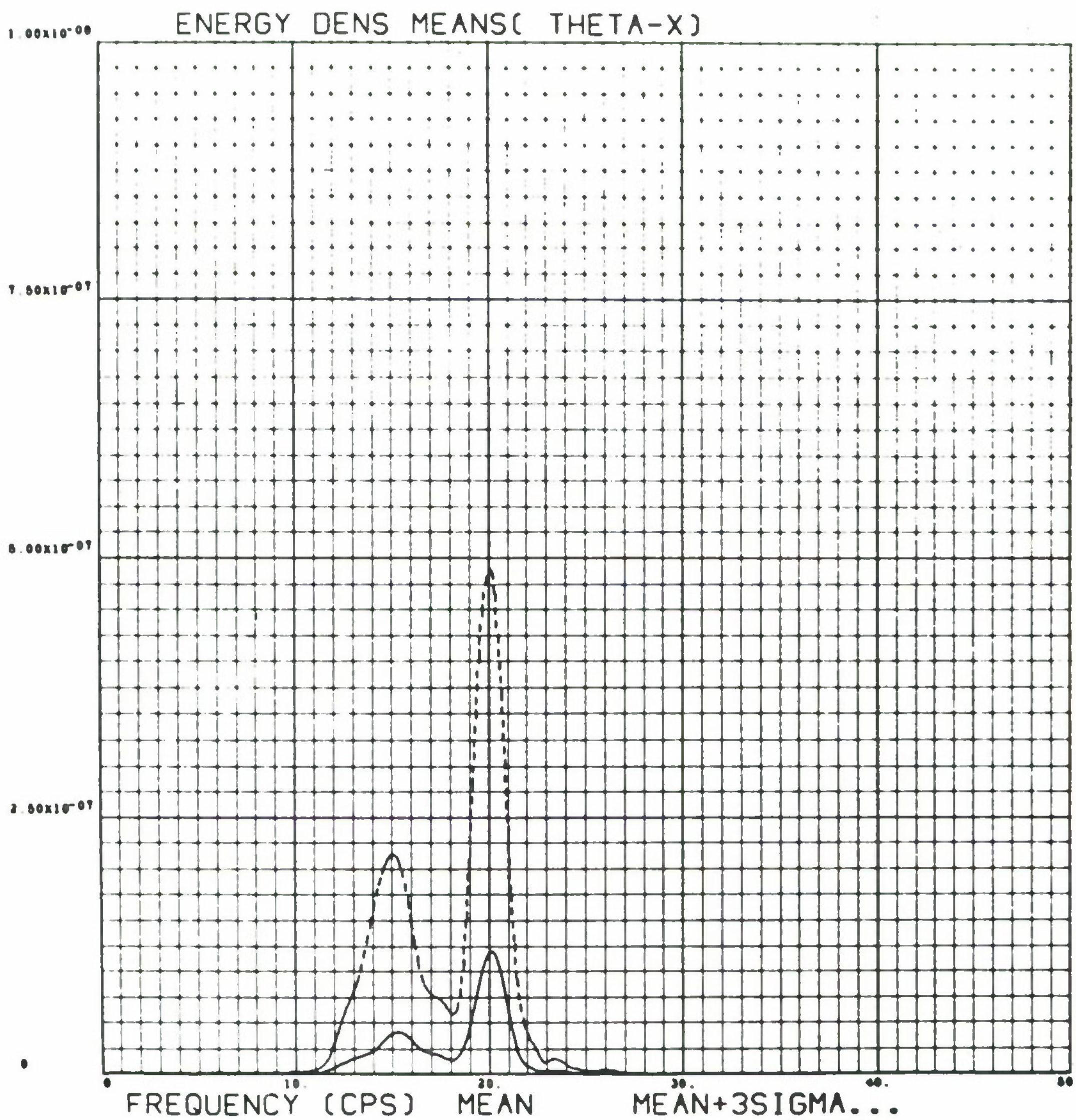


FIG. 22

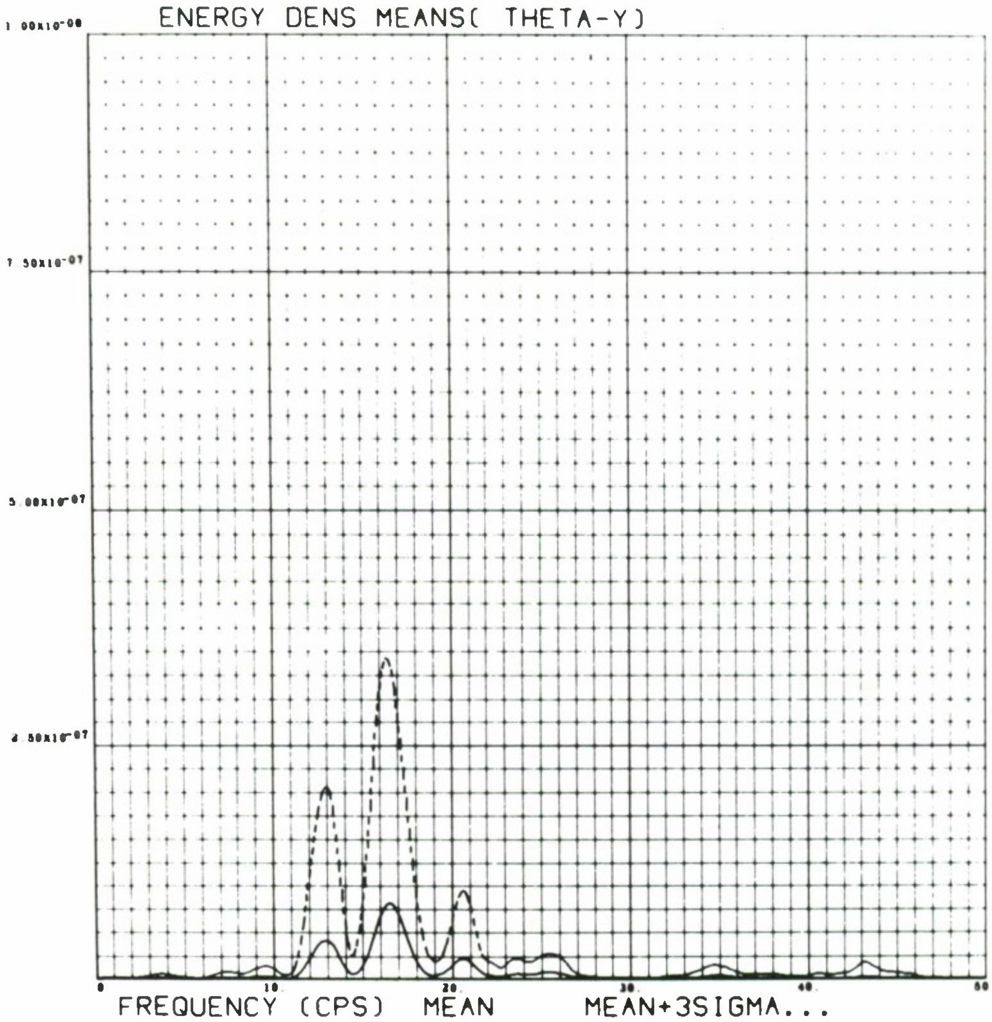

ENERGY DENS MEANS(THETA-Y)
1.00x10^-06
7.50x10^-07
5.00x10^-07
2.50x10^-07
0
10.
20.
30.
40.
50
FREQUENCY (CPS) MEAN MEAN+3SIGMA...

FIG. 23

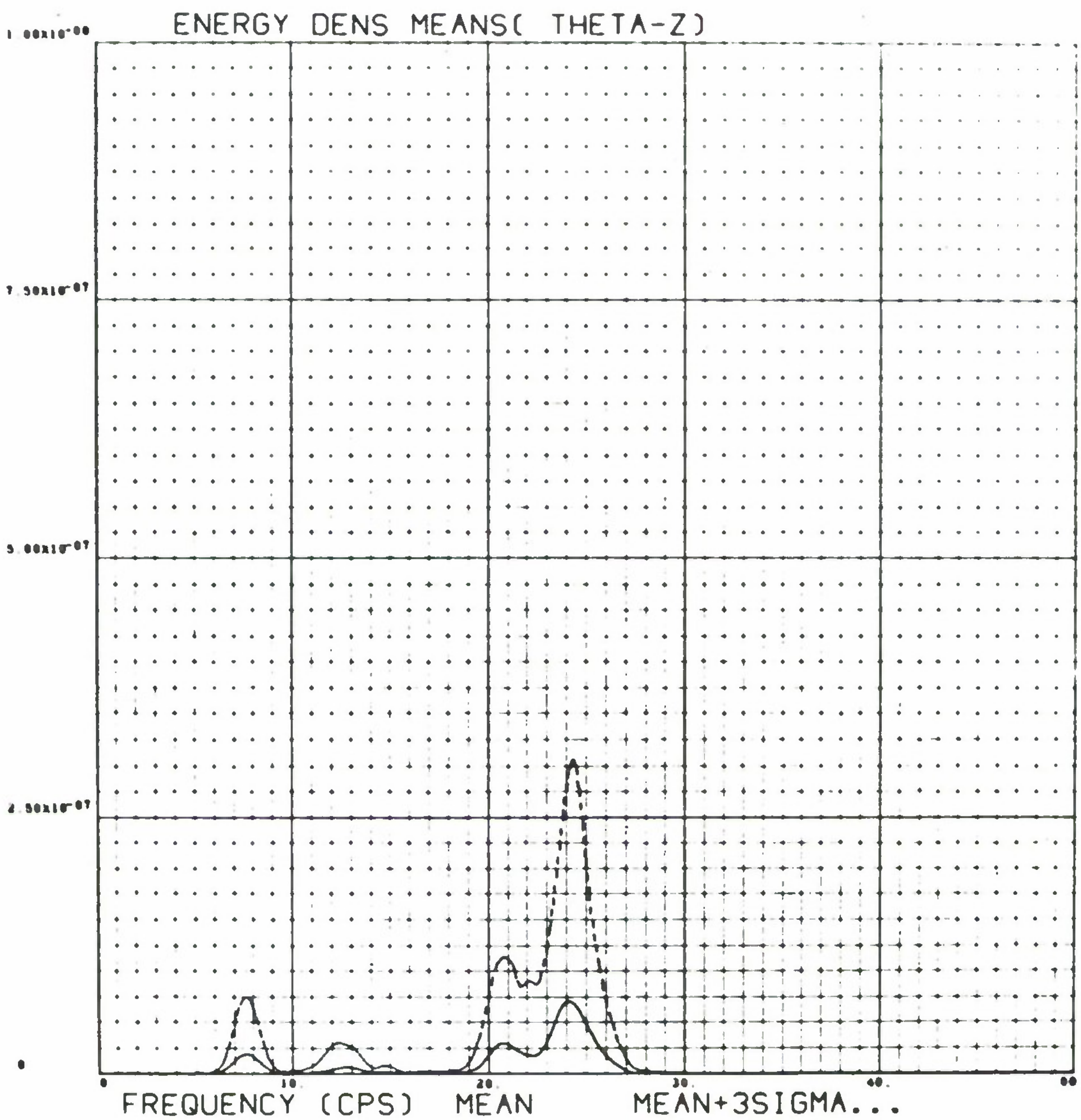

ENERGY DENS MEANS(THETA-Z)
7.50x10-07
5.00x10-07
2.50x10-07
0
0
10.
20.
30.
40.
FREQUENCY (CPS) MEAN MEAN+3SIGMA...

FIG. 24

of the power spectral densities are estimates of the diagonal elements of $[\hat{\Phi}_{a_I}(\omega_1,\omega_2)]_{\omega_1=\omega_2=\omega}$. These estimates are only first order statistics since we have provided no calculation of their confidence intervals. A detailed calculation of the higher order statistics of the spectral density measurements is beyond the scope of this work. So, for the present discussions we shall assume that the estimated power spectral densities are valid representations of the exact power spectra as defined by Eq. (6.18).

Having presented the data discussed above, we shall now discuss the validity of the assumptions stated in the last section. We shall examine these assumptions one by one in the light of the observed characteristics of the time histories and of the power spectral density measurements.

Assumption A(i):

The acceleration time histories clearly exhibit a random nature. Hence if these accelerations are to be regarded as the response of a linear system to some initial excitations, we must conclude that the excitations initially applied to the linear system are themselves random.

Examining the acceleration time histories more closely we see that, for the components with zero mean, the number of zero crossings is roughly equal to the number of peaks. This observation corresponds well to a description of a superposition of narrow band processes.[2] Such processes are typical of the response of lightly damped mechanical systems to broad-band random excitation.

Hence the characteristics of the time histories themselves suggest the validity of assumption A(i). This assumption is further strengthened by the fact mentioned in section 6.1; that the individual accelerations were obtained by calculating the responses of an analytical model of the booster structure to some initially applied excitation. We can then conclude that the linear system mentioned above is the booster structure itself. Of course, in the 27 cases, different booster models were used. We hypothesize a booster structure which has, in some sense, the "average" properties of all the booster structures used in the calculation of the interface accelerations. In the following discussions we shall use the term "booster structure" to denote this hypothesized booster structure.

From the foregoing observations we conclude that the interface accelerations are components of response of the booster structure (in the sense described above) to an initially applied random excitation. Thus, equation (6.7) is indeed appropriate.

Assumption A(ii):

Equation (6.7) predicts the following form for the power spectral densities of the interface accelerations:

$$[\Phi_{jk}(\omega)] = [C][\ulcorner H_{b_k}(\omega)\lrcorner][\omega^4\hat{\Phi}_Q(\omega,\omega)][\ulcorner H^*_{b_k}(\omega)\lrcorner][C]^T \tag{6.20}$$

Or, expanding the matrix product:

$$\Phi_{jk}(\omega) = \sum_{m=1}^{N}\sum_{n=1}^{N} C_{jm}H_{b_m}(\omega)(\omega^4\hat{\Phi}_{Q_{mn}}(\omega,\omega))H^*_{b_n}(\omega)C_{kn}$$

$$= \sum_{\ell=1}^{N} C_{j\ell}|H_{b_\ell}(\omega)|^2(\omega^4\hat{\Phi}_{Q_{\ell\ell}}(\omega,\omega)C_{k\ell}$$

$$+ \sum_{m=1}^{N}\sum_{n\neq m} C_{jm}H_{b_m}(\omega)(\omega^4\hat{\Phi}_{Q_{mn}}(\omega,\omega))H^*_{b_n}(\omega)C_{kn} \tag{6.21}$$

where $\hat{\Phi}_{Q_{mn}}(\omega_1,\omega_2)$ is the generalized cross-spectral density of the m^{th} and n^{th} components of $\{Q\}$. In particular for j=k, we have the spectral densities shown in Figs. 19 through 24:

$$\Phi_{kk}(\omega) = \sum_{\ell=1}^{N} C^2_{k\ell}|H_{b_\ell}(\omega)|^2(\omega^4\hat{\Phi}_{Q_{\ell\ell}}(\omega,\omega))$$

$$+ \sum_{m=1}^{N}\sum_{\substack{n=1\\n\neq m}}^{N} C_{km}C_{kn}\,\mathrm{Re}(H_{b_m}(\omega)H^*_{b_n}(\omega))(\omega^4\hat{\Phi}_{Q_{mn}}(\omega,\omega)) \tag{6.22}$$

Now, suppose that the $H_{b_\ell}(\omega)$ is the force input, displacement output frequency response function of the ℓ^{th} booster normal mode. That is, assume $H_{b_\ell}(\omega)$ in the form

$$H_{b_\ell}(\omega) = [\Omega_k^2 - \omega^2 + 2i\,\zeta_{0k}\Omega_k\omega]^{-1} \tag{6.23}$$

as in Eq. (6.16). With $\zeta_{0k} << 1$, $|H_{b_\ell}(\omega)|^2$ peaks sharply near $\omega = \Omega_k$ and drops off to negligible values for $|\omega - \Omega_k|$ equal to several multiples of the bandwidth, $2\,\zeta_{0k}\Omega_k$. Thus, when looking at the power spectral density measurements for frequencies in the vicinity of the ℓ^{th} modal frequency of the booster, we should see a peak of the form

$$\Phi_{kk}(\omega) \approx C_{k\ell}^2 |H_{b_\ell}(\omega)|^2 (\omega^4\, \hat{\Phi}_{Q_{\ell\ell}}(\omega,\omega))$$

$$\sum_{\substack{n=1 \\ n \neq m}}^{N} C_{k\ell} C_{kn}\, \mathrm{Re}(H_{b_\ell}(\omega) H_{b_n}^*(\omega)) (\omega^4\, \hat{\Phi}_{Q_{\ell n}}(\omega,\omega)) \qquad (6.24)$$

where we have assumed that the k^{th} modal frequency is several bandwidths distant from any other modal frequency.

Turning to the data we note that the power spectral density estimates exhibit sharp, distinct peaks, each of which closely resembles a function of the form of $|H_{b_\ell}(\omega)|^2$ or of the form $\mathrm{Re}(H_{b_\ell}(\omega) H_{b_n}^*(\omega))$. From (6.24) and this observation, we conclude that the quantities $(\omega^4\, \hat{\Phi}_{Q_{mn}}(\omega,\omega))$ must be nearly constant near modal frequencies of the booster. For mathematical simplicity we assume that the $(\omega^4\, \hat{\Phi}_{Q_{mn}}(\omega,\omega))$ $(m,n = 1,\ldots,N)$ are constants for all ω. Hence for $\omega_1 \neq \omega_2$:

$$(\omega_1^2\, \omega_2^2\, \hat{\Phi}_{Q_{mn}}(\omega_1,\omega_2)) = K_{mn}\, F(\omega_1 - \omega_2)/F(0) \qquad (6.25)$$

$$(m,n = 1,\ldots,N)$$

where the K_{mn} are constants and $F(\omega)$ is some function continuous at $\omega = 0$. But, it is obvious that the right hand side of Eq. (6.25) is just the cross-spectral density of two components of a general vector shot noise process.

Thus, we conclude that with the $H_{b_\ell}(\omega)$ given by (6.23) the properties of the spectral density measurements require that the random excitation to the idealized structure is a broad band nonstationary process. For convenience, we can assume, to a good degree of approximation, that the random excitation is a vector shot noise process.

Assumption B

It must be noted that the magnitudes of the acceleration time histories exhibit a general time trend which is slowly varying compared to the rapid periodic fluctuations. The accelerations with zero mean rise to maximum values in less than one second and rapidly decay to zero after one second (not shown in Figs. 13 through 18). This general time trend we associate with the variation of the intensity functions of the shot noise process which excites the booster structure.

Moreover, as the plots of the time histories show, the general time trends, as measured by functions which envelope the acceleration maxima, are of the same general form and duration. It can then be concluded that the intensity functions of the initial random excitation to the booster are approximately the same except for multiplicative constants. Thus Eq. (6.11) is approximately valid.

Assumption C

According to assumption B the general time trends of the interface accelerations are the result of the time variation of a nonstationary shot noise intensity function. If this is true it can be inferred from the time histories that this intensity function begins at zero at the time origin, increases to its maximum value after approximately one second and then drops back to zero a few seconds later. Thus it is apparent that the magnitude of the Fourier transform of the intensity function assumes a maximum value near a frequency of the order of one Hertz. This frequency is much less than the center frequencies of any of the peaks observed in the power spectra measurements presented above. Assuming the magnitude of the transform of the intensity function drops off rapidly enough from its maximum value, we conclude that the generalized spectral densities of the interface accelerations, as given by Eq. (6.11), take on significant values only when $|\omega_1-\omega_2|$ is small.

In the light of this assertion, examine Eq. (6.11). Expanding the matrix product we have

$$\hat{\Phi}_{a_{I_{kj}}}(\omega_1,\omega_2) = \sum_{\ell=1}^{N} C_{j\ell} C_{k\ell} K_{\ell\ell} H_{b\ell}(\omega_1) H^*_{b\ell}(\omega_2)$$

$$+ \sum_{m=1}^{N} \sum_{n\neq m} C_{jm} C_{kn} K_{mn} H_{bm}(\omega_1) H^*_{bn}(\omega_2) \qquad \hat{I}(\omega_1-\omega_2)$$

(6.26)

Now, from the spectral density measurements we see that, in almost all cases, the center frequencies of the peaks are separated by several band-widths. Thus the value of $|H_{bm}(\omega_1)|$ is quite small at the center frequency of a different booster normal mode. Therefore, with $|\omega_1-\omega_2|$ small the quantities $H_{bm}(\omega_1)H^*_{bn}(\omega_2)$ $(n\neq m)$ are much smaller than the quantities $H_{b\ell}(\omega_1)H^*_{b\ell}(\omega_2)$. Now, the quantities $C_{j\ell}C_{k\ell}$ are certainly comparable in magnitude to the quantities $C_{jm}C_{kn}$ $(m\neq n)$. We conclude that if the K_{mn} $(m\neq n)$ are less than or at most comparable to the $K_{\ell\ell}$ then the second term within the braces in Eq. (6.26) is negligible compared to the first term.

Now suppose that we calculate the second order central moment response of a payload structure using Eq. (4.12), with $[\hat{\Phi}_{aI}(\omega_1,\omega_2)]$ as the excitation. The second moments can be expressed as the sum of two terms. One of the terms arises from the first term of Eq. (6.26) and the other term arises from the second term of Eq. (6.26). From preceding discussions it can be concluded that the second group of terms in the second moment responses can be neglected in comparison with the first group of terms. That is, the second moments of the response of a payload to excitations given by Eq. (6.26) can be approximated by the response to excitations characterized by

$$\hat{\Phi}_{a_{I_{kj}}}(\omega_1,\omega_2) \approx \sum_{\ell=1}^{N} C_{j\ell} C_{k\ell} K_{\ell\ell} H_{b\ell}(\omega_1) H^*_{b\ell}(\omega_2) \tag{6.27}$$

Thus if the diagonal elements of $[K_{mn}]$ are not much less than the off diagonal terms. Then approximately correct results may be obtained with the use of the approximate spectral densities given by Eq. (6.27).

Writing Eq. (6.27) in the matrix form

$$[\hat{\Phi}_{aI}(\omega_1\omega_2)] \approx [C][\diagdown H_{bk}(\omega_1)\diagdown][\diagdown K_{\ell\ell}\diagdown][\diagdown H^*_{bk}(\omega_2)\diagdown][C]^T \hat{I}(\omega_1-\omega_2) \tag{6.28}$$

we see that the off diagonal elements of $[K_{\ell m}]$ are effectively zero. This assertion is equivalent to the statement of assumption C. We have thus demonstrated in some measure the applicability of this assumption.

Having established our basic assumptions about the nature of the interface accelerations we devote the next two sections to a more detailed estimation of the statistical parameters. As Eq. (6.14) shows, we have yet to determine

(a) the constants Ω_k and ζ_{0k} describing each booster normal mode

(b) the elements of the constant matrix [M], and

(c) the intensity function, I(t).

In section 6.5 we discuss the determination of the power spectral density parameters, (a) and (b), and in section 6.6 we estimate the intensity function, (c).

6.5 Estimation of the Spectral Density Parameters

According to Eq. (6.14) the power spectra presented in Figs. 19 through 24 have the form

$$\Phi_{kk}(\omega) = \sum_{\ell=1}^{N} M_{k\ell}^2 |H_{b\ell}(\omega)|^2 \qquad (6.29)$$

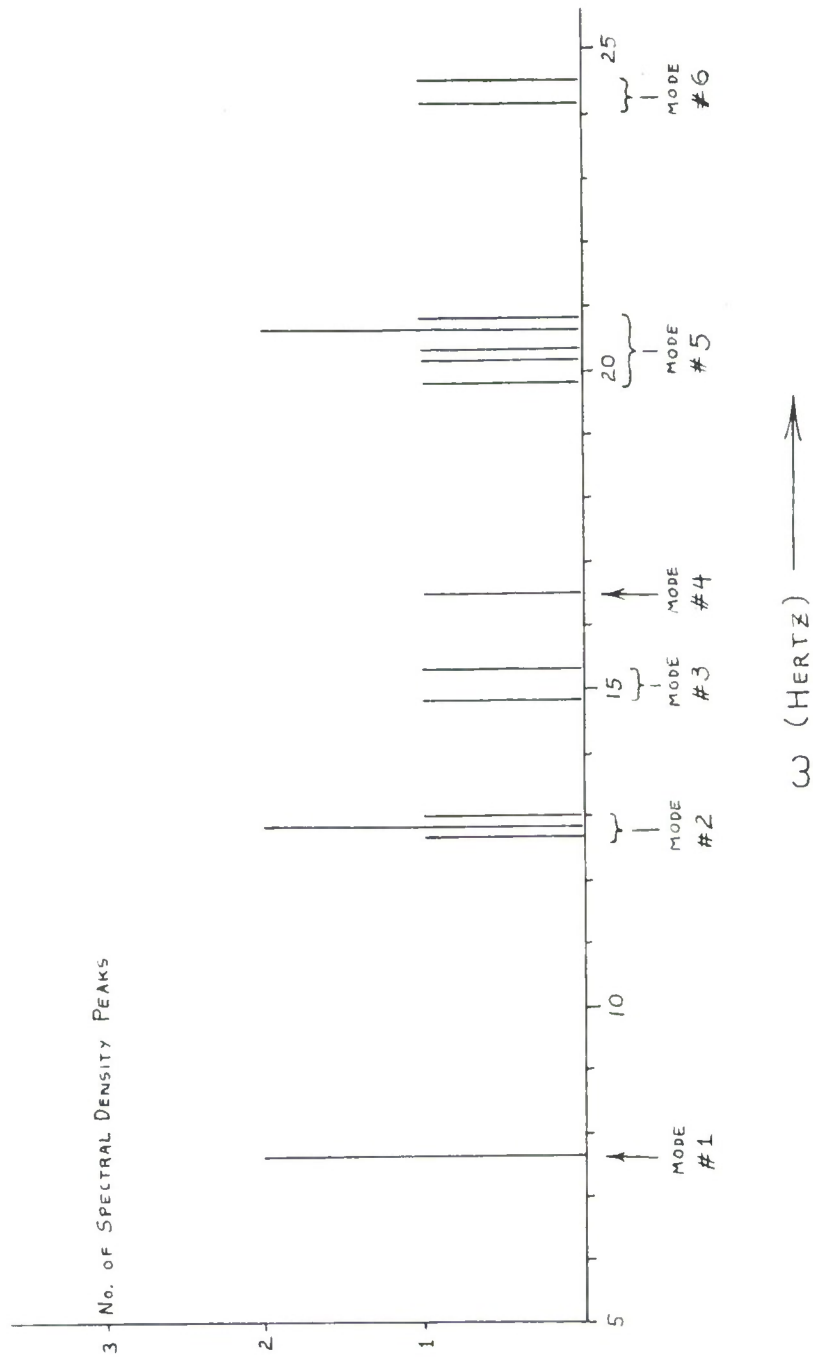
where

$$H_{bk}(\omega) = [\Omega_k^2 - \omega^2 + 2i\zeta_{0K}\Omega_k\omega]^{-1} \qquad (6.30)$$

From Eq. (6.29) and (6.30) we can show that from the shapes of the peaks in the power spectra the quantities Ω_k and ζ_{0k} $(k=1,\ldots,N)$ can be determined, and from the magnitudes of the peaks the absolute values of the elements of [M] can be found. But first we must identify and label the booster structure normal modes.

In Figure 25 is shown a plot of the number of the more significant spectral density peaks centered at any given frequency. We see that the positions of the peaks cluster at certain locations. We assume that the positions of the peaks within each grouping of peaks correspond to an estimate of one modal frequency of the booster structure. These groupings,

FIG. 25



No. of Spectral Density Peaks
3
2
1
5
10
15
20
25
MODE #1
MODE #2
MODE #3
MODE #4
MODE #5
MODE #6
ω (Hertz)

6 in number, are thus associated with particular booster normal modes. We have indicated this correspondence by numbering each grouping as shown. The peaks within the grouping labeled "mode 5", for example are, assumed to represent the response of the 5^{th} normal mode of the booster to the initial shot noise excitation.

With the booster mode identified as indicated above, we see that each significant peak in the estimated spectral densities corresponds to one of the frequency response functions of a booster normal coordinate. For a peak corresponding to the k^{th} booster normal mode in the j^{th} acceleration component spectral density the spectral density is given approximately by

$$\Phi_{jj}(\omega) \approx |H_{bk}(\omega)|^2 M_{jk}^2 = \frac{M_{jk}^2}{(\Omega_k^2-\omega^2)^2 + 4\zeta_{0k}^2 \Omega_k^2 \omega^2} \tag{6.31}$$

This is valid as long as the peak in question is not within a few band widths of another peak. Assuming that ζ_{0k} is small (of order 10^{-2}) we may write

$$\omega_{MAX} = \Omega_{kD} \tag{6.32}$$

$$\zeta_{0k} = \frac{1}{2\,\omega_{MAX}} \text{ (half height width of } \Phi_{jj}(\omega)) \tag{6.33}$$

$$|M_{jk}| \approx 2\,\zeta_{0k}\,\omega_{MAX}^2\,\Phi_{jj}(\omega)_{MAX} \tag{6.34}$$

to order ζ_{0k}^2, where $\Phi_{jj}(\omega)_{MAX}$ is the maximum value of

$\Phi_{jj}(\omega)$ and $\Omega_{kD} = \Omega_k \sqrt{1-\zeta_{0k}^2}$,

$$\Phi_{jj}(\omega)_{MAX} = \Phi_{jj}(\omega_{MAX})$$

The quantities given by Eqs. (6.32) to (6.34) were calculated for each peak in each of the spectral density estimates. The resulting numerical values are presented in Table I. The calculations for the x or q_1 component spectral density were also included since the large peak at $\omega = 0$ has little effect on the size and shape of the 20 Hertz peak. However a calculation of the M_{jk} associated with the x component is not included in Table I. As an example of the details of the calculation consider the determination of the parameters associated with the peak near 20 Hertz on the power spectral density of the z acceleration component. The position and magnitude of the maximum and of the half power points are indicated by Fig. 26. With the values of these as given, we proceed as follows:

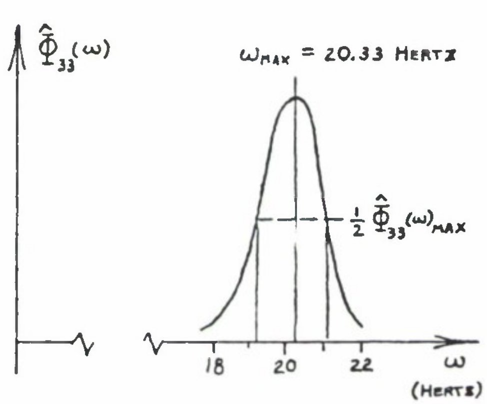


Fig. 26

(a) $\Omega_{kD} = \omega_{MAX} = 20.33$ Hz

$$= 127.76 \ \frac{\text{radians}}{\text{sec.}}$$

from Eq. (6.32).

TABLE I

Spectral Density Parameters

Acceleration Component j	Booster Mode K	Ω_{KD} (Eq. (6.31)) (radians/sec.)	ζ_{OK} (Eq. (6.32))	$\lvert M_{jK} \rvert$ (Eq. (6.33))
1	5	129.848	.043	-------------
2	1	48.167	.105	$\lvert M_{21} \rvert$ = 10.79
"	2	80.634	.070	$\lvert M_{22} \rvert$ = 5.11
"	3	93.201	.049	$\lvert M_{23} \rvert$ = 7.35
"	5	124.617	.038	$\lvert M_{25} \rvert$ = 10.58
"	6	153.938	.055	$\lvert M_{26} \rvert$ = 44.23
3	2	79.583	.076	$\lvert M_{32} \rvert$ = 35.62
"	5	127.756	.043	$\lvert M_{35} \rvert$ = 87.71
4	3	96.340	.104	$\lvert M_{43} \rvert$ = 0.394
"	5	126.711	.046	$\lvert M_{45} \rvert$ = 0.512
5	2	81.682	.069	$\lvert M_{52} \rvert$ = 0.190
"	4	103.673	.061	$\lvert M_{54} \rvert$ = 0.377
"	5	129.852	.039	$\lvert M_{55} \rvert$ = 0.201
6	1	48.171	.105	$\lvert M_{61} \rvert$ = 0.068
"	2	80.634	.071	$\lvert M_{62} \rvert$ = 0.074
"	5	130.899	.046	$\lvert M_{65} \rvert$ = 0.271
"	6	151.844	.050	$\lvert M_{66} \rvert$ = 0.616

(b) The half power points are located at 19.34 and 21.10 Hertz. Hence the half height width is (21.10-19.34) Hertz = 1.76 Hertz. Therefore, from Eq. (6.33):

$$\zeta_{0k} = \frac{1.76}{2(20.33)} = 0.043$$

(c) Finally from Eq. (6.34)

$$|M_{35}| \approx 2(0.043)(127.76)^2 \sqrt{0.387\times 10^{-2}} = 87.71$$

In a similar way, all the other values listed in Table I were obtained.

As would be expected, the center frequencies and damping coefficients of any one booster mode fluctuate randomly from one acceleration component spectral to another. In Table II, the average values of these center frequencies and the maximum values of the damping ratios are presented. Table III presents the matrix $[|M_{jk}|]$ with the booster modes designated as explained above. We shall use the contents of Tables II and III for the example at the end of the chapter.

It should be noted that we have obtained only the absolute values of the components of $[M]$. The signs of the components can be determined once the off diagonal terms of $[\Phi_{kj}(\omega)]$ have been estimated. However, this involves the calculation of 15 quantities similar to those given by (6.18). In some cases, this great mass of calculation can be avoided. In the example,

TABLE II

Booster Normal Modes

Booster Mode K	Average Modal Frequencies Ω_{KD} (radians/second)	Maximum Damping Coefficients $(\zeta_{0K})_{MAX}$
1	48.169	.105
2	80.633	.076
3	94.770	.104
4	103.673	.061
5	128.380	.046
6	152.891	.055

TABLE III

Elements of the Matrix $[|M_{jk}|]$

$[|M_{jk}|]$ =

	Mode No. 1	2	3	4	5	6
$\ddot{q}_1$	-----	-----	-----	-----	-----	-----
$\ddot{q}_2$	10.79	5.11	7.35	0.00	10.58	44.23
$\ddot{q}_3$	0.00	35.65	0.00	0.00	87.71	0.00
$\ddot{q}_4$	0.00	0.00	0.39	0.00	0.51	0.00
$\ddot{q}_5$	0.00	0.19	0.00	0.38	0.20	0.00
$\ddot{q}_6$	0.68	0.07	0.00	0.00	0.27	0.62

we show how the signs of the components of [M] can be obtained through physical considerations alone.

In the next section we consider the estimation of the intensity function, I(t).

6.6 Estimation of the Intensity Function

We here assume that the intensity function, I(t), is slowly varying in the sense described in section 5.6. Now, we have idealized the interface accelerations themselves as the response of an elastic structure to uncorrelated components of a vector shot noise. It follows that the response of the booster to one uncorrelated excitation component is uncorrelated with the response to another excitation component. Thus if $\kappa_{aj}^{(i)}$ is the variance of the j^{th} acceleration due to the i^{th} modal component of the vector shot noise, then the total variance is just the sum

$$\kappa_{aj}(t) = \sum_{i=1}^{N} \kappa_{aj}^{(i)}(t)$$

However, if the shot noise is slowly varying then Example 6 of section 5.5 shows that $\kappa_{aj}^{(i)}(t)$ is approximately proportional to the intensity function of the shot noise component. Since we have assumed these intensity functions to be the same within multiplicative constants the total variance of any one acceleration component will be proportional to the intensity function, I(t). That is

$$\kappa_{aj}(t) \propto I(t) \tag{6.35}$$

Now, for the acceleration components with a zero mean value, the Tchebychev inequality states that curves along which $\sqrt{\kappa_{ai}(t)}$ is constant are those for which the probability of reaching the magnitude of acceleration given by the curve is a constant. Hence, if we construct a function which closely envelopes all the ensemble members of each acceleration component, we can conclude that $\sqrt{\kappa_{aj}}$ (for all j) is proportional to the function represented by the curve. A suitable function for this purpose is

$$C(t) = \begin{cases} A(1-\cos \epsilon t) & 0 \leq t \leq \frac{2\pi}{\epsilon} \\ 0 & 0 > t > \frac{2\pi}{\epsilon} \end{cases} \tag{6.36}$$

ϵ was evaluated by noting the time position of the maxima attained by each time history, averaging these values and dividing 2π by the resulting average. The approximate value obtained in this way is

$$\epsilon = 4.833 \ \frac{\text{radians}}{\text{sec.}} \tag{6.37}$$

With this value for ϵ, the curve given by (6.36) was found to be a good estimate of the enveloping function.

Since $\sqrt{\kappa_{aj}} \propto C(t)$ (all j), relation (6.35) gives

$$I(t) \propto C^2(t)$$

or

$$I(t) = \begin{cases} C\,(1-\cos \epsilon t)^2 & 0 \le t \le \frac{2\pi}{\epsilon} \\ 0 & 0 > t > \frac{2\pi}{\epsilon} \end{cases} \qquad (6.38)$$

where C is an undetermined constant.

We shall need the transform of I(t) for the response formulation. A straightforward evaluation of the Fourier integral by elementary methods yields

$$\hat{I}(\omega) = C \frac{i(e^{-i\omega\frac{2\pi}{\epsilon}} - 1)\; 6\; \epsilon^4}{\omega(\omega^2 - \epsilon^2)(\omega^2 - 4\epsilon^2)}$$

and since

$$\hat{I}(0) = \int_0^{2\pi/\epsilon} C(1- \cos \epsilon t)^2 \, dt = \frac{3\pi}{\epsilon} C$$

we have

$$\frac{\hat{I}(\omega)}{\hat{I}(0)} = \frac{i(e^{-i\omega\frac{2\pi}{\epsilon}} - 1)\; \frac{2}{\pi}\; \epsilon^5}{\omega(\omega^2 - \epsilon^2)(\omega^2 - 4\epsilon^2)} \qquad (6.39)$$

From (6.39), it is apparent that I(t) is indeed slowly varying since the major components of its Fourier spectrum are centered at ~ 0.77 Hertz while the booster frequency response functions are centered in the region 7 to 35 Hertz.

Having obtained a crude but serviceable model of the interface accelerations, we now turn to the problem of calculating the response of a payload to these excitations.

6.7 Response Formulation

The two fundamental relations are

$$\{\ddot{\eta}\} + 2[\diagdown \zeta_i \omega_i \diagdown]\{\dot{\eta}\} + [\diagdown \omega_i^2 \diagdown]\{\eta\} = [C] \quad \{a\} \tag{6.1}$$

$$(M\times 1) \quad (M\times M) \quad (M\times M) \quad (M\times 6) \quad (6\times 1)$$

and

$$[\Phi_a(\omega_1,\omega_2)] = [M][\diagdown H_{bi}(\omega_1)\diagdown][\diagdown H^*_{bj}(\omega_2)\diagdown][M^T]\ \frac{\hat{I}(\omega_1-\omega_2)}{\hat{I}(a)} \tag{6.14}$$

According to Eq. (4.12), the Fourier transforms of the second order central moments of $\{\eta\}$ are given by

$$\hat{\kappa}_{kj}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_k(y)\hat{\Phi}_{Q_{kj}}(y,y-\omega)H_j(\omega-y)dy \tag{6.40}$$

In this case:

$$[\hat{\Phi}_{Q_{kj}}(y,y-\omega)] = [C] \quad [M] \quad [\diagdown H_{bi}(y)H^*_{bi}(y-\omega)\diagdown][M]^T \ [C]^T\ \frac{\hat{I}(\omega)}{\hat{I}(0)}$$

$$(M\times 6)\ (6\times N) \qquad (N\times N) \qquad (N\times 6)\ (6\times M)$$

so that

$$\hat{\Phi}_{Q_{kj}}(y,y-\omega) = \frac{\hat{I}(\omega)}{\hat{I}(0)} \sum_{i=1}^{N}\sum_{m=1}^{6}\sum_{n=1}^{6} C_{mj}M_{mi}M_{ni}C_{kn}(H_{bi}(y)H^*_{bi}(y-\omega)) \tag{6.41}$$

Substitution of (6.41) into (6.40) yields

$$\hat{\kappa}_{kj}(\omega) = \sum_{i=1}^{N}\sum_{m=1}^{6}\sum_{n=1}^{6} C_{mj}M_{mi}M_{ni}C_{kn}$$

$$* \frac{\hat{I}(\omega)}{\hat{I}(0)}\ \frac{1}{2\pi}\int_{-\infty}^{\infty} H_k(y)H_{bi}(y)H^*_{bi}(y-\omega)H_j(\omega-y)dy \tag{6.42}$$

where $H_k(\omega) = \frac{1}{m_k}(\omega_k^2 - \omega^2 + 2i\zeta_k\omega_k\omega)^{-1}$.

The integral on the right in Equation (6.42) is nothing but the integral of Eq. (5.69) of section 5.6. Hence the inversion of the Fourier integral gives

$$\kappa_{kj}(t) = \sum_{i=1}^{N} \sum_{m=1}^{6} \sum_{n=1}^{6} C_{mj} M_{mi} M_{ni} C_{kn} \{ \int_0^t \frac{I(t-\tau)}{\hat{I}(0)} \gamma_{kj}^{(i)}(\tau) d\tau \} \tag{6.43}$$

The quantity $\gamma_{kj}^{(i)}(t)$ is given by Eq. (5.81) with Ω and ζ replaced by Ω_i and ζ_{0i} respectively:

$$\left.\begin{aligned}
\gamma_{kj}^{(i)}(t) \approx\ & A[B_1 e^{-(\zeta_k \omega_k + \zeta_j \omega_j)t} \cos(\omega_{kD} - \omega_{jD})t \\
& - B_2 e^{-(\zeta_k \omega_k + \zeta_{0i} \Omega_i)t} \cos(\omega_{kD} - \Omega_{jD})t \\
& - B_3 e^{-(\zeta_j \omega_j + \zeta_{0i} \Omega_i)t} \cos(\omega_{jD} - \Omega_D)t + B_4 e^{-2\zeta_{0i}\Omega_i t}] \\
A =\ & [8\, \omega_{kD} \omega_{jD} \Omega_{iD} (\Omega_{iD}^2 - \omega_{jD}^2)(\Omega_{iD}^2 - \omega_{kD}^2)]^{-1} \\
B_1 =\ & 2\Omega_{iD} + \omega_{kD} + \omega_{jD} \\
B_2 =\ & \frac{\omega_{jD}}{\Omega_{iD}} (\Omega_{iD} + \omega_{kD}) + (\Omega_{iD} + \omega_{jD}) \\
B_3 =\ & \frac{\omega_{kD}}{\Omega_{iD}} (\Omega_{iD} + \omega_{jD}) + (\Omega_{iD} + \omega_{kD}) \\
B_4 =\ & \frac{\omega_{jD}}{\Omega_{iD}} (\omega_{kD} + \Omega_{iD}) + \frac{\omega_{kD}}{\Omega_{iD}} (\omega_{jD} + \Omega_D)
\end{aligned}\right\} \tag{6.44}$$

where $\Omega_{iD} = \Omega_i \sqrt{1-\zeta_{0i}^2}$

$$\omega_{kD} = \omega_k \sqrt{1-\zeta_k^2}$$

$$\omega_{jD} = \omega_j \sqrt{1-\zeta_j^2}$$

(6.44) yields a result for the convolution integral, $\int_0^t \frac{I(t-\tau)}{\hat{I}(0)} \gamma_{kj}^{(i)} (\tau)\, d\tau)$ which is correct except for the term $(T_1+T_2+T_3) \int_0^t \frac{I(t-\tau)}{\hat{I}(0)} \gamma_{kj}^{(i)} (\tau)\, d\tau)$ where, from (III.15) and section 5.6:

$$O\{T_1\} <$$

$$O\left\{\frac{2[\text{MAX}(|a_k^{(-)}|)]^{p+1}[\text{MAX}(|a_k^{(+)}-a_j^{(-)}|)]^2 \text{MAX}[|J(a_k^{(+)})|]}{(p+2)[\text{MIN}(|a_k^{(+)}-a_j^{(+)}|)][\text{MIN}(|a_k^{(+)}-a_j^{(-)}|)]^{p+2)} \text{MIN}[|J(a_k^{(-)})|]}\right\}$$

$$O\{T_2\} < O\left\{\frac{2\text{MAX}(|a_k^{(-)}|)}{\text{MIN}(|a_k^{(+)}|)}\right\}$$

$$O\{T_3\} < O\{\ \text{MAX}\ (\zeta_{0i}, \zeta_k, \zeta_j)\} \qquad (6.45)$$

It is understood that the locations of the poles of frequency response functions are denoted by a_k. In formulae (6.45), $a_k^{(+)}$ denotes all combinations of two a_j which involve sums of two of the frequencies ω_{kD}, ω_{jD} and Ω_{iD} while $a_k^{(-)}$ denotes all such combinations involving differences between two of these frequencies.

Equations (6.43), (6.44) and (6.45) summarize the approximate formulation of the second moment response of a payload structure to the interface accelerations as idealized by Eq. (6.14).

Once the approximate second moments have been determined, the member stresses can be found from a simple matrix transformation (e.g. Eq. (6.3)).

To illustrate the application of Eqs. (6.43) through (6.45) we present the following example.

Example 6.1

A two degree of freedom structure is driven by a combination of the z and θy components of the booster/payload interface accelerations. Suppose that the normal equations of motion are

$$\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + 2\zeta \begin{bmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} C_1\omega_1^2 & C_2\omega_1^2 \\ C_1\omega_2^2 & C_2\omega_2^2 \end{bmatrix} \begin{Bmatrix} \ddot{z} \\ \ddot{\theta}_y \end{Bmatrix} \tag{6.46}$$

where

x_1, x_2 ≡ payload normal coordinates

$$\left.\begin{aligned} \zeta &= 0.025 \\ \omega_1 &= 16.5 \text{ Hz.} \\ \omega_2 &= 20.5 \text{ Hz.} \\ C_1 &= 1.0 \\ C_2 &= 217.4 \end{aligned}\right\} \tag{6.47}$$

C_1 and C_2 were chosen so that the magnitude of the peaks centered near 16.5 Hertz on the θ_y spectral density equals the magnitude of the peak centered near 20 Hertz on the z spectral density. A comparison of Eq. (6.46) with (6.1) shows that

$$[C] = \begin{bmatrix} 0 & 0 & C_1\omega_1^2 & 0 & C_2\omega_1^2 & 0 \\ 0 & 0 & C_1\omega_2^2 & 0 & C_2\omega_2^2 & 0 \end{bmatrix}$$

And, from Table III:

$$[C][M] = \begin{bmatrix} 0 & M_{32}C_1\omega_1^2+M_{52}C_2\omega_1^2 & 0 & C_2\omega_1^2M_{54} & C_1\omega_1^2M_{35}+C_2\omega_1^2M_{55} & 0 \\ 0 & M_{32}C_1\omega_2^2+M_{52}C_2\omega_2^2 & 0 & C_2\omega_2^2M_{54} & C_1\omega_2^2M_{35}+C_2\omega_2^2M_{55} & 0 \end{bmatrix} \tag{6.48}$$

Let

$$[\diagdown H_j \diagdown] \equiv \frac{\hat{I}(\omega)}{\hat{I}(0)} [\diagdown H_{bi}(y) H^*_{bi}(y-\omega) \diagdown]$$

Then Eq. (6.13) becomes

$$[\hat{\Phi}_a(y,y-\omega)] = [M][\diagdown H_j \diagdown][M]^T$$

so that

$$[\hat{\Phi}_Q(y,y-\omega)] = [C][M][\diagdown H_j \diagdown][M]^T[C]^T$$

Or,

$$[\hat{\Phi}_Q(y,y-\omega)] = [H_2(M_{32}C_1+M_{52}C_2)^2 + H_4(C_2M_{54})^2 + H_5(C_1M_{35}+C_2M_{55})^2] \cdot \begin{bmatrix} \omega_1^4 & \omega_1^2\omega_2^2 \\ \omega_1^2\omega_2^2 & \omega_2^4 \end{bmatrix} \qquad (6.49)$$

To proceed further, we must ascertain the signs of the M's in Eq. (6.49). Let us focus our attention on the relative signs of M_{35} and M_{55} since the term $H_5(C_1M35 + C_2M_{55})^2$ will most affect the results.

First, we notice, after a consideration of Eq. (6.14) that the Fourier components of $\ddot{z}$ and $\ddot{\theta}_y$ in the vicinity of booster mode 5 are proportional to M_{35} and M_{55} respectively.

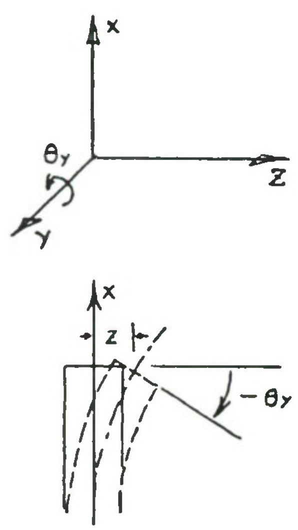


Fig. 27

It can be seen from Fig. 27 that the accelerations along the z and θ_y directions in the normal mode centered at ~ 20 Hz. can be considered a mode of vibration of a beam in the x-z plane. If we assume this to be the case, then it can be concluded that $\ddot{z}$ and $\ddot{\theta}_y$ at 20 Hz. are 180^o out of phase. Hence their Fourier components are opposite in sign; i.e., M_{35} and M_{55}

have opposite signs.

Assuming the above argument to be correct and referring to Table III, we can set

$$M_{32} = \pm\ 35.65$$

$$M_{52} = \pm\ 0.19$$

$$M_{54} = \pm\ 0.38$$

$$M_{35} = \pm\ 87.71$$

$$M_{55} = \mp\ 0.20$$

Then, with C_1 and C_2 given by (6.47):

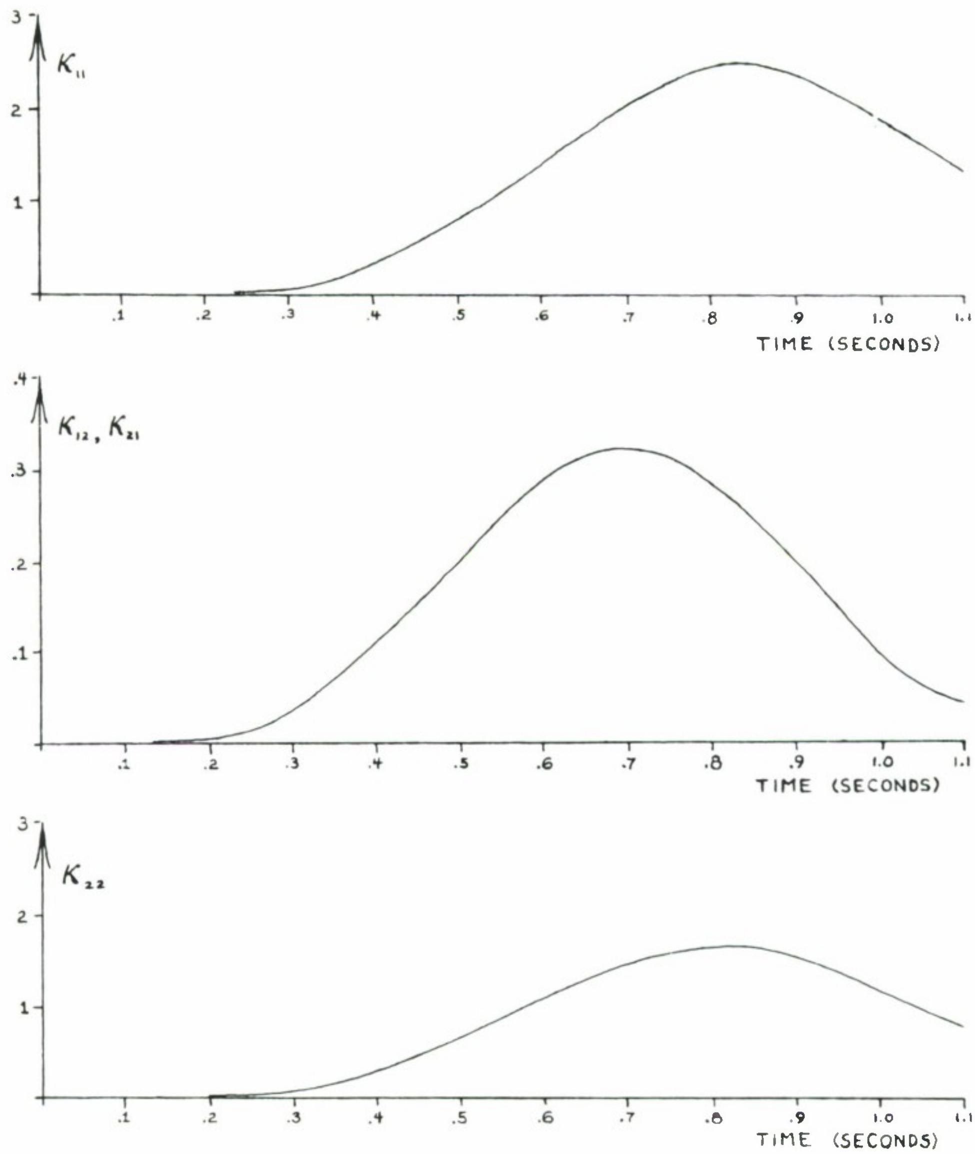
$$\left.\begin{aligned} (M_{32}C_1+M_{52}C_2)^2 &= 5922.84 \\ (C_2M_{54})^2 &= 6824.41 \\ (M_{35}C_1+M_{55}C_2)^2 &= 1956.29 \end{aligned}\right\} \qquad (6.50)$$

Finally, the result corresponding to Eq. (6.43) is

$$\kappa_{kj}(t) \approx \omega_k^2\omega_j^2 \Big[5922.84 \int_0^t I(t-\tau)\, \gamma_{kj}^{(2)}(\tau)d\tau + 6824.41 \int_0^t I(t-\tau)\, \gamma_{kj}^{(4)}(\tau)d\tau + 1956.29 \int_0^t I(t-\tau)\, \gamma_{kj}^{(5)}(\tau)d\tau\Big] \frac{1}{\hat{I}(0)} \qquad (6.51)$$

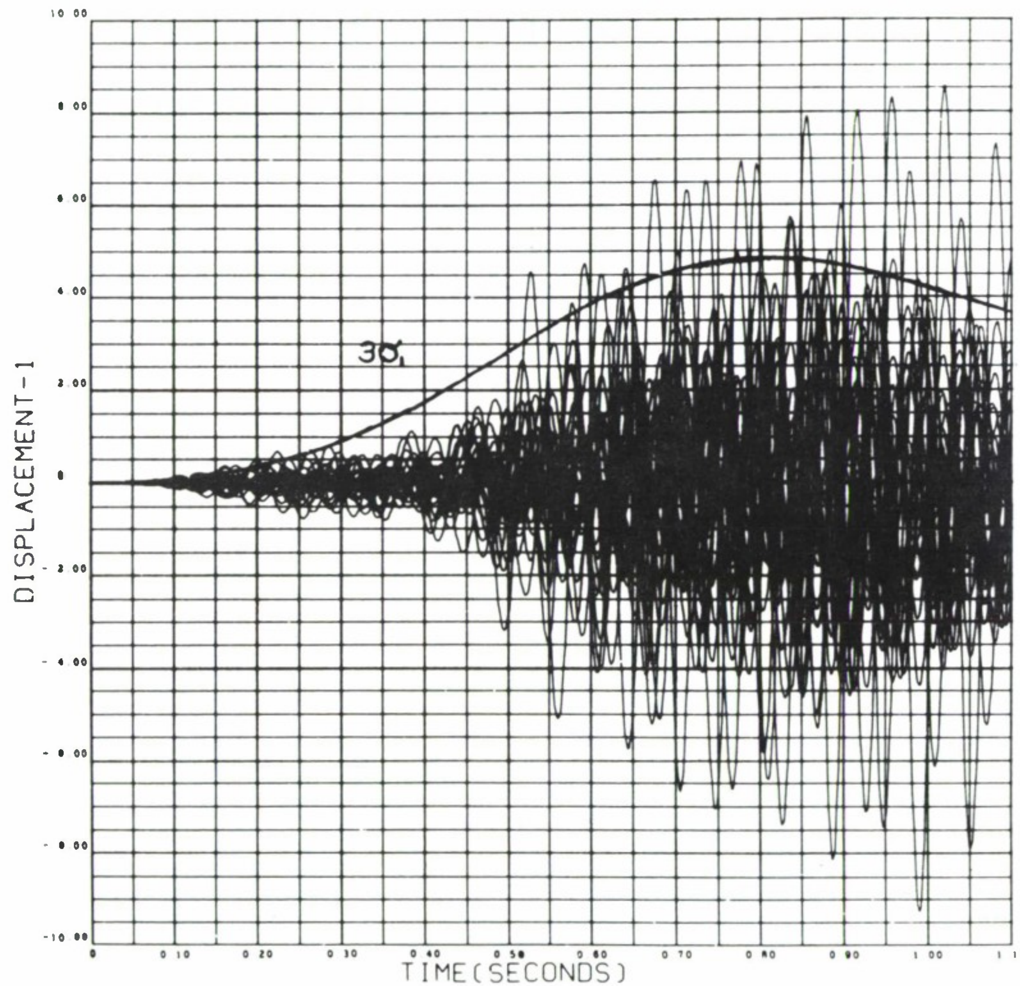
$$\left.\begin{aligned} \frac{I(t)}{\hat{I}(0)} &= \frac{\varepsilon}{3\pi}(1 - \cos\varepsilon t)^2 \\ \varepsilon &= 4.833 \end{aligned}\right\} \qquad (6.52)$$

FIG. 28



K_{11}
3
2
1
.1 .2 .3 .4 .5 .6 .7 .8 .9 1.0 1.1
TIME (SECONDS)
K_{12}, K_{21}
.4
.3
.2
.1
.1 .2 .3 .4 .5 .6 .7 .8 .9 1.0 1.1
TIME (SECONDS)
K_{22}
3
2
1
.1 .2 .3 .4 .5 .6 .7 .8 .9 1.0 1.1
TIME (SECONDS)

FIG. 29 (a)



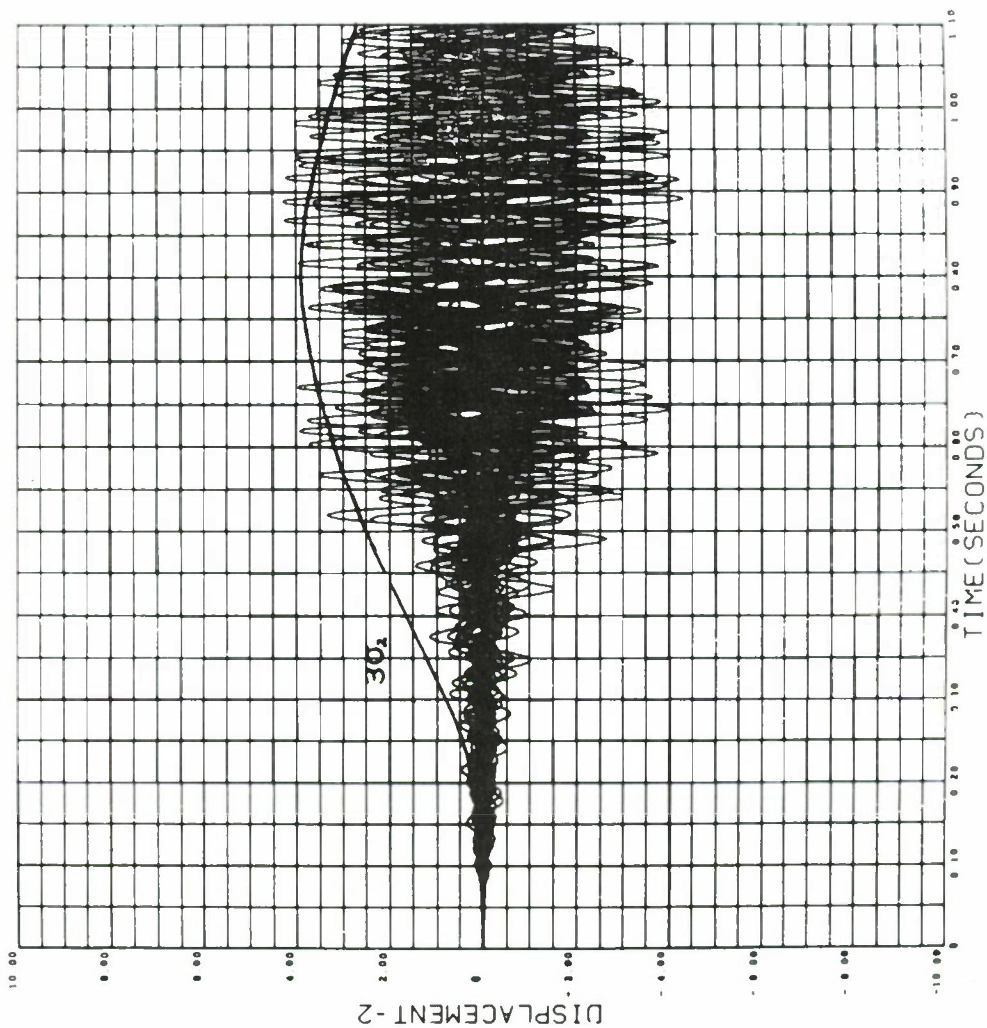


FIG. 29 (b)

where the $\gamma_{kj}^{(i)}$ are given by Eqs. (6.44).

With the numerical values given by Table II and Eq. (6.47), Eq. (6.51) may be integrated numerically to give the desired second order moments of the system response.

The integrals of (6.51) were evaluated by the trapazoidal rule. Fig. 28 shows the resulting time histories of κ_{11}, κ_{12} and κ_{22}. In Figs. 29(a) and (b) $3\sqrt{\kappa_{11}}$ and $3\sqrt{\kappa_{22}}$ (i.e., $3\sigma_1$ and $3\sigma_2$) were plotted along with a superposition of the system responses to each of the 27 sets of acceleration. These were obtained by a numerical integration of the deterministic equations of motion. It can be seen that $3\sqrt{\kappa_{11}}$ and $3\sqrt{\kappa_{22}}$ bound most of the deterministic solutions to the response. This is to be expected since by the Tchebycheff inequality, Eq. (2.28), there is a probability of approximately 0.89 that at any particular time, the responses are less than three times the corresponding standard deviations.

It is instructive to calculate the relative errors induced by the use of Eq. (6.51). That is we wish to calculate the orders of magnitude of the error terms associated with each of the three terms of (6.51) by Eq. (6.45). As an example, we shall calculate the error terms of $\kappa_{11}(t)$.

Since the first mode of the system is tuned to the center frequency of the 4th booster mode the second term in Eq. (6.51) gives the largest contribution to $\kappa_{11}(t)$. Hence, the relative errors associated with the quantity $\int_0^t \frac{I(t-\tau)}{\hat{I}(0)}$ $\gamma_{11}^{(4)}(\tau)d\tau$ will be calculated. The positions of the poles of $H_1(\omega)$ and $H_{b4}(\omega)$ are

$$a_1 = i\,\zeta\omega_1 + \omega_{1D}$$
$$a_2 = i\,\zeta\omega_1 - \omega_{1D}$$
$$a_3 = i\,\zeta_{04}\Omega_4 + \Omega_{4D}$$
$$a_4 = i\,\zeta_{04}\Omega_4 - \Omega_{4D}$$

And, we then have

$$\left.\begin{aligned}
a_1^{(+)} &= a_1+a_3 = i(\zeta\omega_1+\zeta_{04}\Omega_4) + (\omega_{1D}+\Omega_{4D}) \\
a_2^{(+)} &= 2\Omega_{4D} \\
a_4^{(+)} &= -2\Omega_{4D} \\
a_5^{(+)} &= a_2+a_4 = i(\zeta\omega_1+\zeta_{04}\Omega_4)-(\omega_{1D}+\Omega_{4D}) \\
\text{and}\quad a_1^{(-)} &= i(\zeta\omega_1 + \zeta_{04}\Omega_4) + (\omega_{1D}-\Omega_{4D}) \\
a_2^{(-)} &= i(\zeta\omega_1 + \zeta_{04}\Omega_4) + (\Omega_{4D}-\omega_{1D})
\end{aligned}\right\} \quad (6.53)$$

Hence:

$$\mathrm{MAX}(|a_k^{(-)}|) \approx \zeta\omega_1+\zeta_{04}\Omega_4 \approx \mathrm{MIN}(|a_k^{(+)} - a_j^{(+)}|)$$

$$\mathrm{MAX}(|a_k^{(+)} - a_j^{(-)}|) \approx \mathrm{MIN}(|a_k^{(+)} - a_j^{(-)}|) \approx 2\Omega_{4D}$$

$$\mathrm{MAX}(|a_k^{(-)}|) \approx \zeta\omega_1 + \zeta_{04}\Omega$$

(equation continued)

$$\mathrm{MIN}(|a_k^{(+)}|) \approx 2\Omega_{4D} \tag{6.54}$$

Now, consider the Fourier transform of $\frac{I(t)}{I(0)}$:

$$\frac{\hat{I}(\omega)}{\hat{I}(0)} = \frac{i(e^{-i\omega\frac{2\pi}{\epsilon}}-1)}{\omega} \cdot \frac{\frac{2}{\pi}\epsilon^5}{(\omega^2-\epsilon^2)(\omega^2-4\epsilon^2)}$$

Since the first factor has no singularities in the complex ω plane, we set

$$J(\omega) = \frac{i(e^{-i\omega\frac{2\pi}{\epsilon}}-1)}{\omega} \tag{6.55}$$

The remaining factor has 4 poles so that

$$p = 4 \tag{6.56}$$

And from (6.53) and (6.55), we have

$$\mathrm{MAX}\,[|J(a_k^{(+)})|] = \left|\frac{i(e^{-i(a_1^{(+)}\frac{2\pi}{\epsilon}}-1)}{a_1^{(+)}}\right|$$

$$\approx \frac{e^{(\zeta\omega_1+\zeta_{04}\Omega_4)\frac{2\pi}{\epsilon}}}{(\zeta\omega_1 + \zeta_{04}\Omega_4)}$$

$$\mathrm{MIN}\,[|J(a_k^{(-)})|] = \left|\frac{i(e^{-a_1^{(-)}\frac{2\pi}{\epsilon}}-1)}{a_1^{(-)}}\right|$$

$$\approx \frac{e^{(\zeta\omega_1+\zeta_{04}\Omega_4)\frac{2\pi}{\epsilon}}}{(\zeta\omega_1+\zeta_{04}\Omega_4)}$$

Hence,

$$\frac{\text{MAX}\,[\,|J(a_k^{(+)})|\,]}{\text{MIN}\,[\,|J(a_k^{(-)})|\,]} \approx 1 \tag{6.57}$$

Collecting the results (6.54), (6.56) and (6.57) and substituting them into (6.45) we obtain

$$O\,\{T_1\} < O\,\{\,\frac{1}{3}\;\frac{(\zeta\omega_1+\zeta_{04}\Omega_4)^4}{[2\Omega_{4D}]^4}\,\}$$

$$\approx \quad O\,\{\,\frac{1}{3}\,\zeta_{04}^4\,\}$$

$$O\,\{T_2\} < O\,\{\,\frac{2(\zeta\omega_1+\zeta_{04}\Omega_4)}{2\Omega_{4D}}\,\} \approx O\,\{2\,\zeta_{04}\}$$

$$O\,\{T_3\} < O\,\{\,\text{MAX}(\zeta,\zeta_{04})\} = O\,\{\zeta_{04}\}$$

Or, the order of magnitude of the total error is

$$O\,\{T_1+T_2+T_3\} < O\,\{3\zeta_{04}\}$$

$$\approx O\,\{0.183\} \tag{6.58}$$

Hence, with κ_{11} as given by (6.51) the exact variance of the first mode is $\kappa_{11}(1+T)$ where $O\{T\} < O\,\{0.183\}$. The upper bound to the variance is then

$$\text{S.U.P.}[\kappa_{11}] \approx \kappa_{11}(1.2)$$

In a similar way the errors due to the approximate formulation can be calculated for the other moments and upper bound estimates can be made. This will not be done in full since the calculations already made are sufficient to illustrate the application of the approximate techniques of Chapter V.

CHAPTER VII

SUMMARY AND CONCLUSIONS

Here we summarize the basic results of this work. We have presented an approximate evaluation of the second central moment response of a lumped parameter, second order, linear system to a class of random excitations. This class of random processes we have called "slowly varying" nonstationary processes, since the nonstationary variation of the correlation functions of these processes is small compared to the variation of the impulse response functions of the dynamic system under consideration. When the excitation falls under the above description we have demonstrated that the resulting response of a structural system may be conveniently approximated by relatively simple analytical expressions.

These analytical simplications make possible the estimation of response statistics from various statistics of the excitations. We have illustrated the possibility by a consideration of the problem of the estimation of flight loads in a payload during booster engine cutoff.

The scheme of analysis outlined in Chapter VI seems to offer the following advantages:

1. Integration of the equations of motion of a payload system for 27 forcing functions to obtain the design loads is avoided. The scheme proposed in Chapter VI, in comparison, involves only marginal computation time.

2. Under most of the present schemes the analysis is highly dependent on a prescribed engineering policy, i.e., the designer must first decide which of the 27 test data are to be used for the structural qualification of a proposed design. This practice can produce unreasonably conservative demands on the structural design. Under the proposed scheme, however, design policies can be formulated from the analysis itself.

APPENDIX I

Here we show how Eqs. (3.55) through (3.57) were obtained.

From the theory of the inversion of Fourier integrals I(t) is the sum of the residues of the integrand in the upper half plane multiplied by $2\pi i$. From Eq. (3.40) and assumptions a. and b. of section 3.6 it follows that

$$I(t) = i\Big[\sum_{k=1}^{n} \mathrm{Res}\Big(\frac{1}{F};a_k\Big)\frac{J(a_k)}{G(a_k)}e^{ia_k t} + \sum_{k=1}^{m} \mathrm{Res}\Big(\frac{1}{G};b_k\Big)\frac{J(b_k)}{F(b_k)}e^{ib_k t}\Big] \qquad (I.1)$$

Now, since a_k and b_k are in the upper half plane, their imaginary parts are positive and

$$|e^{ia_k t}| \leq 1 \quad , \quad |e^{ib_k t}| \leq 1$$

for all k.

Therefore:

$$\Big|\sum_{k=1}^{m} \mathrm{Res}\Big(\frac{1}{G};b_k\Big)\frac{J(b_k)}{F(b_k)}e^{ib_k t}\Big| \leq \sum_{k=1}^{m} \Big|\mathrm{Res}\Big(\frac{1}{G};b_k\Big)\frac{J(b_k)}{F(b_k)}\Big|$$

$$< m \cdot \mathrm{MAX}\Big[\Big|\mathrm{Res}\Big(\frac{1}{G};b_k\Big)\Big|\Big]\frac{\mathrm{MAX}[|J(b_k)|]}{\mathrm{MIN}[|F(b_k)|]} \qquad (I.2)$$

But, from (3.53) and (3.54) we see that

$$\text{MAX}\,[\,|\text{Res}(\tfrac{1}{G}\,,b_k)|\,] < \frac{1}{\epsilon_2^{m-1}} \tag{I.3}$$

$$\text{MIN}\,[\,|F(b_k)|\,] > \gamma_1^n \tag{I.4}$$

Hence,

$$\left|\sum_{k=1}^{m} \text{Res}(\tfrac{1}{G}\,;b_k)\,\frac{J(b_k)}{F(b_k)}\,e^{ib_k t}\right| < \frac{m}{\epsilon_2^{m-1}\,\gamma_1^n}\,\text{MAX}\,[\,|J(b_k)|\,] \tag{I.5}$$

The expression on the right is an estimate of the upper bound of that part of the integral derived from the residues of $\frac{1}{G}$.

Similarly:

$$\left|\sum_{k=1}^{n} \text{Res}(\tfrac{1}{F}\,;a_k)\,\frac{J(a_k)}{G(a_k)}\,e^{ia_k t}\right| \le \sum_{k=1}^{n} \left|\text{Res}(\tfrac{1}{F}\,;a_k)\,\frac{J(a_k)}{G(a_k)}\right|$$

$$\approx n \cdot \text{MIN}\,[\,|\text{Res}(\tfrac{1}{F}\,,a_k)|\,]\,\frac{\text{MIN}[\,|J(a_k)|\,]}{\text{MAX}[\,|G(a_k)|\,]} \tag{I.6}$$

From (3.52) and (3.54):

$$\text{MIN}\,[\,|\text{Res}(\tfrac{1}{F}\,;a_k)|\,] > \frac{1}{\delta_1^{n-1}} \tag{I.7}$$

$$\text{MAX}\,[\,|G(a_k)|\,] < \gamma_2^m \tag{I.8}$$

Therefore:

$$O\left\{\left|\sum_{k=1}^{n} \text{Res}(\tfrac{1}{F}\,;a_k)\frac{J(a_k)}{G(a_k)}\,e^{ia_k t}\right|\right\} = O\left\{\frac{n}{\gamma_1^{n-1}\gamma_2^m}\,\text{MIN}[\,|J(a_k)|\,]\right\} \tag{I.9}$$

With (I.5) and (I.9), we can rewrite (I.1) as

$$I(t) = [i \sum_{k=1}^{n} \mathrm{Res}(\frac{1}{F};a_k) \frac{J(a_k)}{G(a_k)} e^{ia_k t}] (1+T_1) \qquad (I.10)$$

where T_1 is a collection of terms such that

$$O\{|T_1|\} < \{ \frac{m\, \delta_1^{n-1}\, \gamma_2^{m}\, \mathrm{MAX}[|J(b_k)|]}{n\, \epsilon_2^{m-1}\, \gamma_1^{n}\, \mathrm{MIN}[|J(a_k)|]} \} \qquad (I.11)$$

Furthermore, suppose that $a_j = a+\Delta_j$, where $z = a$ is some point in the vicinity of the $a_k (k=1,\ldots,n)$. Then, we may write:

$$\frac{1}{G(a_j)} = \frac{1}{G(a)} - [\frac{1}{G(a)} - \frac{1}{G(a+\Delta_j)}]$$

where we note that $|\Delta_j| < \delta_1$. Hence:

$$\frac{1}{G(a_j)} = \frac{1}{G(a)} (1 + T_2) \qquad (I.12)$$

$$O\{T_2\} < O\{\mathrm{MAX}[\frac{G(a)}{G(a+\Delta_j)} - 1]\} \qquad (I.13)$$

and,

$$i(\sum_{k=1}^{n} \mathrm{Res}(\frac{1}{F};a_k)\frac{J(a_k)}{G(a_k)} e^{ia_k t}) = \frac{i}{g(a)} (\sum_{k+1}^{n} \mathrm{Res}(\frac{1}{F};a_k)J(a_k)e^{ia_k t})$$

$$\cdot\ (1+T_2) \qquad (I.14)$$

Substituting the right hand side of Eq. (I.14) into that of (I.10) we have

$$I(t) = \frac{1}{G(a)} \{ i \sum_{k=1}^{n} \text{Res}(\frac{1}{F}; a_k) J(a_k) e^{i a_k t} \} (1+T_1)(1+T_2)$$

But the term in braces is nothing but the integral $\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{J(\omega)}{F(\omega)} e^{i\omega t} d\omega$. Hence we may write I(t) as

$$I(t) = \{ \frac{1}{2\pi G(a)} \int_{-\infty}^{\infty} \frac{J(\omega)}{F(\omega)} e^{i\omega t} d\omega \} (1+T_1)(1+T_2) \qquad (I.15)$$

where

$$O\{T_1\} = O\{ \frac{m\, \delta_1^{n-1}\, \gamma_2^{m}\, \text{MAX}[|J(b_k)|]}{n\, \epsilon_2^{m-1}\, \gamma_1^{n}\, \text{MIN}[|J(a_k)|]} \qquad (I.16)$$

$$O\{T_2\} < O\{\text{MAX}[\frac{G(a)}{G(a+\Delta_j)} - 1]\} \qquad (I.17)$$

which was to be shown.

APPENDIX II

In section 4.4 we have presented two alternative expressions for the variances of response:

$$\hat{\kappa}_{kk}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_k(y)\,\hat{\Phi}_{Q_{kk}}(y,y-\omega)H_k(\omega-y) \tag{II.1}$$

$$\hat{\kappa}_{kk}(\omega) = \frac{\frac{1}{2\pi}\int_{-\infty}^{\infty}(\frac{\omega}{2}-2\zeta_k\omega_k i+y)H_k(y)\hat{\Phi}_{Q_{kk}}(y,y-\omega)dy}{m_k(\omega-2\zeta_k\omega_k i)(-\frac{\omega^2}{4}+i\zeta_k\omega_k\omega+\omega_k^2)} \tag{II.2}$$

Here, we show that these two results are equivalent. First, consider (II.1), and make the transformation

$$\tau = t_1 - t_2 \quad , \quad t^* = t_2 \tag{II.3}$$

Then, by (3.23); and since the Jacobian is one:

$$\hat{\kappa}_{kk}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_k(y)H_k(\omega-y)\iint_{-\infty}^{\infty} C_{Q_{kk}}(\tau,t^*)e^{-i(\tau y+t^*\omega)}d\tau dt^* dy$$

$$= \frac{1}{2\pi}\iint_{-\infty}^{\infty}\{\int_{-\infty}^{\infty} H_k(y)H_k(\omega-y)e^{-i\tau y}dy\}C_{Q_{kk}}(\tau,t^*)e^{-it^*\omega}dt^*d\tau$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\{\int_0^{\infty}[\int_{-\infty}^{\infty} H_k(y)H_k(\omega-y)e^{-i\tau t}dy]C_{Q_{kk}}(\tau,t^*)e^{-it^*\omega}d\tau$$

$$+\int_{-\infty}^{0}[\int_{-\infty}^{\infty} H_k(y)H_k(\omega-y)e^{-i\tau y}dy]C_{Q_{kk}}(\tau,t^*)e^{-it^*\omega}d\tau\}\,dt^* \tag{II.4}$$

where we have reversed the order of integration and have expanded the integral over τ. In the second integral in τ we let $\tau^* = -\tau$ so that

$$\hat{K}_{kk}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \{\int_0^{\infty} I_1 C_{Q_{kk}}(\tau,t^*)e^{-it^*\omega}\, d\tau$$

$$+ \int_0^{\infty} I_2\, C_{Q_{kk}}(\tau^*,t^*)e^{-it^*\omega}\, d\tau^* \}dt^* \qquad \text{(II.5)}$$

where

$$I_1 = \int_{-\infty}^{\infty} H_k(y)H_k(\omega-y)e^{-i\tau y}dy$$

and

$$I_2 = \int_{-\infty}^{\infty} H_k(y)H_k(\omega-y)e^{i\tau^* y}dy$$

Now, the coefficients of the exponentials in both of the above integrals can be written

$$H_k(y)H_k(\omega-y) = \frac{1}{(y-a_1)(y-a_2)(\omega-y-a_1)(\omega-y-a_2)} \qquad \text{(II.6)}$$

with $\quad a_1 = i\omega_k\zeta_k + \omega_{kD}\ ,\ a_2 = i\omega_k\zeta_k - \omega_{kD}$

so that I_1 and I_2 are to be calculated by the methods of section 3.5. According to The.'s 1 and 2 of that section, I_1 and I_2 are integrated over the upper and the lower half z planes respectively. Performing the contour integration, we have

$$\left.\begin{aligned} I_1 &= \frac{\pi i/\omega_{kD}}{\omega-a_1-a_2}\left[\frac{e^{-i\tau(\omega-a_1)}}{\omega-2a_1} - \frac{e^{-i\tau(\omega-a_2)}}{\omega-2a_2}\right] \\ I_2 &= \frac{\pi i/\omega_{kD}}{\omega-a_1-a_2}\left[\frac{e^{i\tau^* a_1}}{\omega-2a_1} - \frac{e^{i\tau^* a_2}}{\omega-2a_2}\right. \end{aligned}\right\} \qquad \text{(II.7)}$$

Substitution of (II.7) into (II.5) yields

$$\hat{K}_{kk}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\{\int_0^{\infty}\frac{\pi i}{\omega_{kD}(\omega-a_1-a_2)}\left[\frac{e^{-i\tau(\omega-a_1)}}{\omega-2a_1}-\frac{e^{-i\tau(\omega-a_2)}}{\omega-2a_2}\right]$$

$$C_{Q_{kk}}(\tau,t^*)e^{-it^*\omega}\,d\tau$$

$$+\int_0^{\infty}\frac{\pi i}{\omega_{kD}(\omega-a_1-a_2)}\left[\frac{e^{i\tau^*a_1}}{\omega-2a_1}-\frac{e^{i\tau^*a_2}}{\omega-2a_2}\right]$$

$$C_{Q_{kk}}(\tau^*,t^*)e^{-it^*\omega}d\tau^*\}\,dt^* \qquad \text{(II.8)}$$

But, by (I.3):

$$\iint_{-\infty}^{\infty} e^{-i\tau(\omega-a_1)}\,C_{Q_{kk}}(\tau,t^*)e^{-it^*\omega}d\tau\,dt^*$$

$$= \iint_{-\infty}^{\infty} e^{ia_1(t_1-t_2)}\,e^{-i(t_1-t_2)\omega}\,e^{-it_2\omega}\,C_{Q_{kk}}(t_1,t_2)dt_1dt_2$$

$$= \iint_{-\infty}^{\infty} e^{ia_1(t_1-t_2)}\,e^{it_1\omega}\,R_x(t_1,t_2)dt_1dt_2$$

If we now let $(t_1-t_2) = \tau$ as before but set $t_1=t^*$, $C_{Q_{kk}}(t_1,t_2)$ is the same as before since, by its definition, $C_{Q_{kk}}(t_1,t_2)$ is symmetrical in t_1 and t_2. Hence:

$$\iint_{-\infty}^{\infty} e^{-i\tau(\omega-a_1)}\,e^{-it^*\omega}\,C_{Q_{kk}}(\tau,t^*)d\tau dt^*$$

$$= \iint_{-\infty}^{\infty} e^{ia_1\tau}\,e^{-i\omega t^*}\,C_{Q_{kk}}(\tau,t^*)d\tau dt^*$$

(equation continued)

$$\iint_{-\infty}^{\infty} e^{-i\tau(\omega-a_2)} e^{-it^*\omega} C_{Q_{kk}}(\tau,t^*)d\tau dt^*$$

$$= \iint_{-\infty}^{\infty} e^{ia_2\tau} e^{-i\omega t^*} C_{Q_{kk}}(\tau,t^*)d\tau dt^* \qquad \text{(II.9)}$$

Employing relations (I.9) and setting $\tau^* = -\tau$ in Eq. (I.8) we have

$$\hat{\kappa}_{kk}(\omega) = \frac{1}{2}\iint_{-\infty}^{\infty} \frac{i/\omega_{kD}}{\omega-a_1-a_2}\left[\frac{e^{ia_1\tau}}{\omega-2a_1} - \frac{e^{ia_2\tau}}{\omega-2a_2}\right] C_{Q_{kk}}(\tau,t^*)$$

$$e^{-i\omega t^*} d\tau dt^* \qquad \text{(II.10)}$$

Now we consider Eq. (I.2) which can be written in the form

$$\left.\begin{aligned} \hat{\kappa}_{kk}(\omega) &= \frac{\Lambda(\omega)}{(\omega-a_1-a_2)(\omega-2a_1)(\omega-2a_2)} \\ \Lambda(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 4\left(-\frac{\omega}{2} + a_1+a_2-y\right)H_k(y)\,\hat{\Phi}_{Q_{kk}}(y,y-\omega)dy \end{aligned}\right\} \qquad \text{(II.11)}$$

where a_1 and a_2 are given by (II.6). We can express $\Lambda(\omega)$ as

$$\Lambda(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 4\left(-\frac{\omega}{2} + a_1+a_2-y\right)H_k(y)\hat{\Phi}_{Q_{kk}}(y,y-\omega)dy$$

$$= \frac{1}{2\pi}\iint_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} 4\left(-\frac{\omega}{2} + a_1+a_2-y\right)H_k(y)e^{-i\tau y}dy\right\}$$

$$C_{Q_{kk}}(\tau,t^*)e^{-i\omega t^*}d\tau dt^*$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_0^{\infty} I_1' \, C_{Q_{kk}}(\tau,t^*)d\tau + \int_0^{\infty} I_2' \, C_{Q_{kk}}(\tau^*,t^*)d\tau^*\right\}$$

$$e^{-it^*\omega}\, dt^* \qquad \text{(II.12)}$$

where τ and t^* are as in (II.3), $\tau^* = -\tau$ and

$$I_1' = \int_{-\infty}^{\infty} 4(-\frac{\omega}{2} + a_1+a_2-y)H_k(y)\, e^{-i\tau y}\, dy \qquad \text{(II.13)}$$

$$I_2' = \int_{-\infty}^{\infty} 4(-\frac{\omega}{2} + a_1+a_2-y)H_k(y)\, e^{i\tau^* y}\, dy \qquad \text{(II.14)}$$

I_1' is zero since it must be integrated over the lower half z plane and $H_k(y)$ has no singularities there.

We can write I_2' as

$$I_2' = \int_{-\infty}^{\infty} 4(\frac{\omega}{2} - a_1-a_2+y)\, \frac{e^{i\tau^* y}\, dy}{(y-a_1)(y-a_2)}$$

and a contour integration over the upper half z plane yields:

$$I_2' = \frac{2\pi i}{\omega_{kD}} [(\omega-2a_2)\, e^{ia_1\tau^*} - (\omega-2a_1)\, e^{ia_2\tau^*}] \qquad \text{(II.15)}$$

Finally, a substitution of (II.15) into (II.11) results in

$$\hat{\kappa}_{kk}(\omega) = \frac{1}{2}\,\frac{1}{2\pi} \iint_{-\infty}^{\infty} e^{-it^*\omega}\, \frac{\frac{2\pi i}{\omega_{kD}} [(w-2a_2)e^{ia_1\tau} - (\omega-2a_1)e^{ia_2\tau}]}{(\omega-a_1-a_2)(\omega-2a_1)(\omega-2a_2)}$$

$$\cdot\, C_{Q_{kk}}(\tau,t^*)d\tau dt^*$$

$$= \frac{1}{2} \iint_{-\infty}^{\infty} \frac{i/\omega_{kD}}{\omega-a_1-a_2} [\frac{e^{ia_1\tau}}{\omega-2a_1} - \frac{e^{ia_2\tau}}{\omega-2a_2}] C_{Q_{kk}}(\tau,t^*)e^{-i\omega t^*} d\tau dt^* \qquad \text{(II.16)}$$

which is identical to (I.10).

Hence the expressions (II.1) and (II.2) are equivalent formulations of the variances of the response.

APPENDIX III

In the following, the transform $\Gamma(\omega)$ (defined by (5.69)) is evaluated. The inverse transform is subsequently obtained in an approximate form by the methods of section 3.6. According to Eqs. (5.71) and (5.72), $\Gamma(\omega)$ can be expressed as

$$\Gamma(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1/m_k}{(y-a_1)(y-a_2)} \cdot \frac{1/m_j}{(\omega-y-a_3)(\omega-y-a_4)} \cdot \frac{1/m}{(y-a_5)(y-a_6)} \cdot \frac{dy/m}{(\omega-y-a_5)(\omega-y-a_6)} \qquad \text{(III.1)}$$

$$\left.\begin{aligned} a_1 &= i\zeta_k\omega_k+\omega_{kD}, & a_2 &= i\zeta_k\omega_k-\omega_{kD} \\ a_3 &= i\zeta_j\omega_j+\omega_{jD}, & a_4 &= i\zeta_j\omega_j-\omega_{jD} \\ a_4 &= i\zeta\Omega+\Omega_D, & a_6 &= i\zeta\Omega-\Omega_D \end{aligned}\right\} \qquad \text{(III.2)}$$

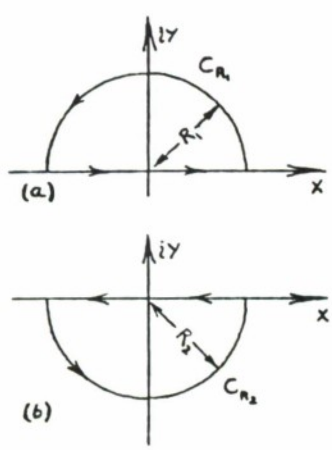


Fig. 30

According to the remarks made in section 4.4, $\Gamma(\omega)$ is

$$\frac{1}{2}(\Gamma_u(\omega) + \Gamma_\ell(\omega))$$

$\Gamma_u(\omega)$ is the integral obtained by a contour integration taken over the contour of Fig. 30(a) in the limit as $R_1 \to \infty$. Similarly, $\Gamma_\ell(\omega)$ is obtained by an integration in the lower half plane along the contour of Fig. 30(b) in the limit as $R_2 \to \infty$. Thus, replacing y in (III.1) by the complex number z and employing Eqs. (3.36), (3.38) and (3.40) we obtain:

$$\hat{I}\Gamma_u(\omega) = i\hat{I}\Big[\frac{-1}{2\omega_{kD}} \frac{1}{(\omega-a_1-a_3)(\omega-a_1-a_4)} H(a_1) \frac{1}{(\omega-a_1-a_5)(\omega-a_1-a_6)}$$

$$+ \frac{1}{2\omega_{kD}} \frac{1}{(\omega-a_2-a_3)(\omega-a_2-a_4)} H(a_2) \frac{1}{(\omega-a_2-a_5)(\omega-a_2-a_6)}$$

$$- \frac{1}{2\Omega_D} \frac{1}{(\omega-a_5-a_3)(\omega-a_5-a_4)} H_k(a_5) \frac{1}{(\omega-2a_5)(\omega-a_6-a_5)}$$

$$+ \frac{1}{2\Omega_D} \frac{1}{(\omega-a_6-a_3)(\omega-a_6-a_4)} H_k(a_6) \frac{1}{(\omega-2a_6)(\omega-a_6-a_5)}\Big] \qquad \text{(III.3)}$$

$$\hat{I}\Gamma_\ell(\omega) = i\hat{I}\Big[\frac{-1}{2\omega_{jD}} \frac{1}{(\omega-a_3-a_1)(\omega-a_3-a_2)} H(a_3) \frac{1}{(\omega-a_3-a_5)(\omega-a_3-a_6)}$$

$$+ \frac{1}{2\omega_{jD}} \frac{1}{(\omega-a_4-a_1)(\omega-a_4-a_2)} H(a_4) \frac{1}{(\omega-a_4-a_5)(\omega-a_4-a_6)}$$

$$- \frac{1}{2\Omega_D} \frac{1}{(\omega-a_5-a_1)(\omega-a_5-a_2)} H_j(a_5) \frac{1}{(\omega-2a_5)(\omega-a_6-a_5)}$$

$$+ \frac{1}{2\Omega_D} \frac{1}{(\omega-a_6-a_1)(\omega-a_6-a_2)} H_j(a_6) \frac{1}{(\omega-2a_6)(\omega-a_6-a_5)} \qquad \text{(III.4)}$$

Now, we examine the transform $\hat{I}(\omega)\ \frac{1}{2}[\Gamma_u(\omega) + \Gamma_\ell(\omega)] = \hat{\kappa}_{kj}$. Since I(t) is, by assumption, a slowly varying function, we may approximate the transform of $\kappa_{kj}(t)$ by the methods of section 3.6 so that upon inversion an approximately correct form of $\kappa_{kj}(t)$ is obtained. As was stated in section 5.6, ζ, ζ_j, and ζ_k are assumed to be small (of order 10^{-2}). We assume that the singularities of $\hat{I}(\omega)$ are no further than approximately MAX $\{|\omega_{kD}-\omega_{jD}|, |\omega_{kD}-\Omega_D|, |_{jD}-\Omega_D|\}$ from the origin of the complex plane and that $|\omega_{kD}-\omega_{jD}|$, $|\omega_{kD}-\Omega_D|$, $|\omega_{jD}-\Omega_D|$ are much smaller than the corresponding quantities $|\omega_{kD}+\omega_{jD}|$, $|\omega_{kD}+\Omega_D|$, $|\omega_{jD}+\Omega_D|$. In other words, the damped natural frequencies of both the k^{th} and j^{th} normal coordinates are in the vicinity of the centre frequency of the narrow band filter.

Consider the inverse Fourier transform of the first term of $\hat{I}(\omega)\Gamma_u(\omega)$ given by (III.3):

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{I}(\omega)\,\Gamma_u(\omega)e^{i\omega t}d\omega =$$

$$\frac{-1}{2\omega_{kD}}\,\frac{iH(a_3)}{2\pi}\int_{-\infty}^{\infty}\hat{I}(\omega)\frac{e^{i\omega t}d\omega}{(\omega-a_1-a_3)(\omega-a_1-a_4)(\omega-a_1-a_5)(\omega-a_1-a_6)} \qquad \text{(III.5)}$$

where from Eqs. (II.2):

$$\left.\begin{aligned} a_1+a_3 &= i(\zeta_j\omega_j+\zeta_k\omega_k)+(\omega_{kD}+\omega_{jD}) \\ a_1+a_4 &= i(\zeta_j\omega_j+\zeta_k\omega_k)+(\omega_{kD}-\omega_{jD}) \\ a_1+a_5 &= i(\zeta_k\omega_k+\zeta\Omega)+(\omega_{kD}+\Omega_D) \\ a_1+a_6 &= i(\zeta_k\omega_k+\zeta\Omega)+(\omega_{kD}-\Omega_D) \end{aligned}\right\} \qquad \text{(III.6)}$$

Using the notation of section 3.6, we can set

$$a \approx i(\zeta_j\omega_j+\zeta_k\omega_k)+(\omega_{kD}-\omega_{jD})$$

$$\left.\begin{aligned}
&\frac{1}{G(a)} = \frac{1}{(a-a_1-a_3)(a-a_1-a_5)} = \frac{1}{2\,\omega_{jD}(\Omega_D+\omega_{jD})} \approx \frac{1}{4\,\omega_{jD}\Omega_D} \\
&\frac{J(\omega)}{F(\omega)} = \frac{\hat{I}(\omega)}{(\omega-a_1-a_4)(\omega-a_1-a_6)} \\
&\varepsilon_2 \approx |\omega_{jD}-\Omega_D| \\
&\delta_1 < \mathrm{MAX}\,\{|\omega_{kD}-\omega_{jD}|, |\omega_{kD}-\Omega_D|\} \\
&\gamma_1 > \mathrm{MIN}\,\{2\omega_{jD}, 2\Omega_D, |\omega_{jD}\pm\Omega_D|, |2\omega_{kD}\pm(\omega_{jD}-\Omega_D)|, 2\omega_{kD}\} \\
&\gamma_2 < \mathrm{MAX}\,\{2\omega_{jD}, 2\Omega_D, |\omega_{jD}\pm\Omega_D|, |2\omega_{kD}\pm(\omega_{jD}-\Omega_D)|, 2\omega_{kD}\} \\
&n = p+2 \\
&m = 2
\end{aligned}\right\} \quad \text{(III.7)}$$

where $\hat{I}(\omega)$ is assumed to have p singularities and where we have neglected terms involving the ζ's. Hence:

$$\left|\frac{G(a)}{G(a+\Delta_j)} - 1\right| = \left|\frac{(a+\Delta_j-a_1-a_3)(a+\Delta_j-a_1-a_5)}{(a-a_1-a_3)(a-a_1-a_5)} - \frac{(a-a_1-a_3)(a-a_1-a_5)}{(a-a_1-a_3)(a-a_1-a_5)}\right|$$

$$= \left|\frac{(\Delta_j-2\omega_{jD})(\Delta_j-\omega_{jD}-\Omega_D) - 2\omega_{jD}(\omega_{jD}+\Omega_D)}{2\omega_{jD}(\omega_{jD}+\Omega_D)}\right|$$

$$= \left|\frac{\Delta_j^2 - 2\omega_{jD}\Delta_j-\Delta_j(\omega_{jD}+\Omega_D)}{2\omega_{jD}(\omega_{jD}+\Omega_D)}\right|$$

$$\approx \left|\frac{\Delta_j(3\omega_{jD}+\Omega_D)}{2\omega_{jD}(\omega_{jD}+\Omega_D)}\right| \approx \left|\frac{2\Delta_j}{\omega_{jD}+\Omega_D}\right|$$

since $\omega_{jD} \approx \Omega_D$ as stated earlier. Therefore,

$$\text{MAX} \left| \frac{G(a)}{G(a+\Delta_j)} - 1 \right| \approx \text{MAX} \left| \frac{2\Delta_j}{\omega_{jD}+\Omega_D} \right|$$

$$= \frac{2\delta_1}{\omega_{jD}+\Omega_D}$$

$$< \frac{2 \text{ MAX } \{|\omega_{kD}-\omega_{jD}|, |\omega_{kD}-\Omega_D|\}}{\omega_{jD}+\Omega_D} \qquad \text{(III.8)}$$

Hence, equations (3.69) to (3.71) give

$$\left.\begin{aligned} &\frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{I}(\omega)\Gamma(\omega)\, e^{i\omega t} d\omega \\ &\qquad = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{-i}{8\omega_{kD}\,\omega_{jD}\,\Omega_D} \frac{\hat{I}(\omega)}{(\omega-a_1-a_4)(\omega-a_1-a_6)} e^{i\omega t} d\omega \\ &\qquad\qquad * (1+T_1)(1+T_2) \\ &\text{where} \\ &O\{T_1\} = O\left\{ \frac{2\,\delta_1^{p+1}\,\gamma_2^2 \text{ MAX } [|J(b_k)|]}{(p+2)|\omega_{jD}-\Omega_D|\gamma_1^{p+2} \text{ MIN}[|J(a_k)|]} \right\} \\ &O\{T_2\} < O\left\{ \frac{2 \text{ MAX } \{|\omega_{kD}-\omega_{jD}|, |\omega_{kD}-\Omega_D|\}}{\omega_{jd}+\Omega_D} \right\} \end{aligned}\right\} \qquad \text{(III.9)}$$

where b_k are the poles of $\frac{1}{G(\omega)}$ and a_k are the poles of $\frac{1}{F(\omega)}$.

The result (III.9) can be generalized to include all the other terms of (III.3) and (III.4). Let us denote the combinations of two a_j (see III.2) which involve sums of two of the frequencies ω_{kD}, ω_{jD} and Ω_D by $a_k^{(+)}$ and those combinations involving differences by $a_k^{(-)}$. It is seen that each term of (III.3) and (III.4) involves two of the $a_k^{(+)}$ and two of the $a_k^{(-)}$ in the denominator. Hence, in each term, we can let the quantity involving the quantities $a_k^{(-)}$ be $\frac{1}{F(\omega)}$ and the quantity involving the $a_k^{(+)}$ be $\frac{1}{G(\omega)}$.

For the terms of (II.3) z=a is as given in (III.7) and

$$\frac{1}{G(a)} \approx \frac{1}{4\omega_{jD}\Omega_D} \qquad \text{(III.10)}$$

and for the terms of (III.4);

$$\left.\begin{aligned} a &= i\,(\zeta_j\omega_j + \zeta_k\omega_k) + (\omega_{jD} - \omega_{kD}) \\ \frac{1}{G(a)} &\approx \frac{1}{4\omega_{kD}\Omega_D} \end{aligned}\right\} \qquad \text{(III.11)}$$

And for both (III.3) and (III.4), we may set:

$$\left.\begin{aligned} &\epsilon_2 > \text{MIN}\,\{|a_k^{(+)} - a_j^{(+)}|\} \\ &\delta_1 < \text{MAX}\,\{|a_k^{(-)}|\} \\ &\gamma_1 > \text{MIN}\,\{|a_k^{(+)} - a_j^{(-)}|\} \\ &\gamma_2 < \text{MAX}\,\{|a_k^{(+)} - a_j^{(-)}|\} \\ &n = p+2 \\ &m = 2 \\ &\text{and} \\ &\left|\frac{G(a)}{G(a+\Delta_j)} - 1\right| < 2\,\frac{\text{MAX}\,\{a_k^{(-)}\}}{\text{MIN}\,\{a_k^{(+)}\}} \end{aligned}\right\} \qquad \text{(III.12)}$$

Hence, the approximate expressions for $\Gamma_u(\omega)$ and $\Gamma_\ell(\omega)$ are

$$\Gamma_u(\omega) = \frac{i}{8\omega_{kD}\omega_{jD}\Omega_D}\left[\frac{-H(a_1)}{(\omega-a_1-a_4)(\omega-a_1-a_6)} + \frac{H(a_2)}{(\omega-a_2-a_3)(\omega-a_2-a_5)}\right]$$

$$+ \frac{i}{8\Omega_D^2\Omega_{jD}}\left[\frac{-H_k(a_5)}{(\omega-a_4-a_5)(\omega-a_5-a_6)} + \frac{H_k(a_6)}{(\omega-a_3-a_6)(\omega-a_5-a_6)}\right]$$

$$\Gamma_\ell(\omega) = \frac{i}{8\omega_{kD}\omega_{jD}\Omega_D}\left[\frac{-H(a_3)}{(\omega-a_2-a_3)(\omega-a_3-a_6)} + \frac{H(a_4)}{(\omega-a_1-a_4)(\omega-a_4-a_5)}\right]$$

$$+ \frac{i}{8\Omega_D^2\Omega_{kD}}\left[\frac{-H_j(a_5)}{(\omega-a_2-a_5)(\omega-a_5-a_6)} + \frac{H_j(a_6)}{(\omega-a_1-a_6)(\omega-a_5-a_6)}\right]$$

(III.13)&(III.14)

which upon multiplication by $\hat{I}(\omega)$ and inversion gives the second moment response correct to within the terms $T_1{}^{\kappa}{}_{kj}(t)$ $+ T_2{}^{\kappa}{}_{kj}(t)$ where

$$O\{T_1\} <$$

$$O\left\{\frac{2[\mathrm{MAX}(|a_k^{(-)}|)]^{p+1}[\mathrm{MAX}(|a_k^{(+)}-a_j^{(-)}|)]^2\,\mathrm{MAX}[|J(a_k^{(+)})|]}{(p+2)[\mathrm{MIN}(|a_k^{(+)}-a_j^{(+)}|)][\mathrm{MIN}(|a_k^{(+)}-a_j^{(-)}|)]^{p+2}\,\mathrm{MIN}[|J(a_k^{(-)})|]}\right\}$$

$$O\{T_2\} < O\left\{\frac{2\mathrm{MAX}(|a_k^{(-)}|)}{\mathrm{MIN}(|a_k^{(+)}|)}\right\} \qquad \text{(III.15)}$$

We shall now obtain the inverse transform of $\Gamma_u(\omega)$ given by (III.13) and (III.14). By the methods of section 3.5 this can easily be accomplished. For example, the inverse transform of the first term of (III.13) becomes

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{i}{8\omega_{kD}\omega_{jD}\Omega_D} \frac{-H(a_1)\, e^{i\omega t} d\omega}{(\omega-a_1-a_4)(\omega-a_1-a_6)}$$

$$= \frac{H(a_1)}{8\omega_{kD}\omega_{jD}\Omega_D}\left[\frac{e^{i(a_1+a_4)t}}{a_4-a_6} + \frac{e^{i(a_1+a_6)t}}{a_6-a_4}\right] \qquad \text{(III.16)}$$

Doing this for all the other terms of (III.13) and (III.14) and noting the relations

$$H(a_2) = H^*(a_1)\ ,\ H_k(a_6) = H_k^*(a_5)\ ,\ H(a_4) = H^*(a_3)\ ,$$
$$H_j(a_6) = H_j^*(a_5) \qquad \text{(III.17)}$$

and

$$(a_3 - a_5) = -(a_4 - a_6)^*\ ,\ (a_1-a_5) = -(a_2-a_6)^* \qquad \text{(III.18)}$$

we obtain, after much algebra:

$$\gamma_u(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma_u(\omega)\, e^{i\omega t} d\omega$$

$$= \frac{1}{8\omega_{kD}\omega_{jD}\Omega_D}\left[2\mathrm{Re}\left(\frac{H(a_1)e^{i(a_1+a_4)t}}{a_4 - a_6}\right) - 2\mathrm{Re}\left(\frac{H(a_1)e^{i(a_1+a_6)t}}{a_4 - a_6}\right)\right]$$

(equation continued)

$$+ \frac{1}{8\Omega_D^2 \omega_{jD}} [2\,\mathrm{Re}(\frac{H(a_5)e^{i(a_4+a_5)t}}{a_4 - a_6}) - 2\mathrm{Re}(\frac{H_k(a_5)e^{i(a_5+a_6)t}}{a_4 - a_6})]$$

$$\gamma_\ell(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Gamma_\ell(\omega)\, e^{i\omega t} d\omega$$

$$= \frac{1}{8\omega_{kD}\omega_{jD}\Omega_D} [2\,\mathrm{Re}(\frac{H(a_3)e^{i(a_1+a_4)t}}{a_2 - a_6}) - 2\mathrm{Re}(\frac{H(a_3)e^{i(a_3+a_6)t}}{a_2 - a_6})]$$

$$+ \frac{1}{8\Omega_D^2 \omega_{kD}} [2\,\mathrm{Re}(\frac{H_j(a_5)e^{i(a_2+a_5)t}}{a_2 - a_6}) - 2\mathrm{Re}(\frac{H_j(a\ )e^{i(a_5+a_6)t}}{a_2 - a_6})]$$

And since $\Gamma(\omega) = \frac{1}{2}(\Gamma_u + \Gamma_\ell)$; $\gamma(t) = \frac{1}{2}(\gamma_u + \gamma_\ell)$ so that from the above we have

$$\gamma(t) = \{\frac{1}{8\omega_{kD}\omega_{jD}\Omega_D^2 m m_k m_j} [\mathrm{Re}(\frac{H(a_1)e^{i(a_1+a_4)t}}{a_4 - a_6})$$

$$+ \mathrm{Re}(\frac{H(a_3)e^{i(a_2+a_3)t}}{a_2 - a_6}) - \mathrm{Re}(\frac{H(a_1)e^{i(a_1+a_6)t}}{a_4 - a_6})$$

$$- \mathrm{Re}(\frac{H(a_3)e^{i(a_3+a_6)t}}{a_2 - a_6})]$$

$$+ \frac{1}{8\omega_{jD}\Omega_D^2} [\mathrm{Re}(\frac{H_k(a_5)e^{i(a_4+a_5)t}}{a_4 - a_6}) - \mathrm{Re}(\frac{H_k(a_5)e^{i(a_5+a_6)t}}{a_4 - a_6})]$$

(equation continued)

$$+ \frac{1}{8\nu_{kD}\Omega_D^2} \left[Re\left(\frac{H_j(a_5)e^{i(a_2+a_5)t}}{a_2 - a_6}\right) - Re\left(\frac{H_j(a_5)e^{i(a_5+a_6)t}}{a_2 - a_6}\right) \right] \}$$

$$* [1 + T_1 + T_2] \qquad (III.19)$$

where the orders of magnitude of T_1 and T_2 are given by Eqs. (III.15)

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13. ABSTRACT

In this work, the response of lumped parameter, second order systems to nonstationary random excitations is examined. Included is a brief introduction to the probabilistic theory of structural dynamics and various basic concepts required for subsequent work.

More specifically, the second order central moment (covariance) response of structural systems to random excitations is studied. In the course of the analysis, an approximate method for the calculation of system response to a class of nonstationary excitation processes is constructed. This class of excitations we have called "slowly varying" nonstationary random processes. By this is meant that the nonstationary variation of the correlation functions of the process is small compared with the time variation of the impulse response functions of the system considered.

It is shown how this approximation technique may be applied to the estimation of inertial loads in the structural members of a payload during the launch phase of flight. Employing previous rocket engine test data, the excitations to the payload are idealized as a "slowly varying" nonstationary random excitation. An approximation procedure is then developed for the calculation of the second-order central moments of the payload response.

14. KEY WORDS

nonstationary random processes
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