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SACLANT ASW RESEARCH CENTRE

(sin x)/x INTERPOLATION OF SAMPLED SIGNALS

by

JENS M. HOVEM

1 JULY 1971

NORTH
ATLANTIC
TREATY
ORGANIZATION

VIALE SAN BARTOLOMEO 400
I - 19026 - LA SPEZIA, ITALY

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Director

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TABLE OF CONTENTS

	Page
ABSTRACT	1
INTRODUCTION	2
1. REVIEW OF THE SAMPLING THEOREM	5
2. REALIZATION OF THE (sin x)/x INTERPOLATION USING THE FAST FOURIER TRANSFORM	8
2.1 Types of Application	8
2.2 Increasing the Sampling Frequency	8
2.3 Computing the Samples of a Time-Shifted Signal	12
3. APPLICATIONS AND EXAMPLES	14
3.1 Signal Reconstruction Using Straight Lines	14
3.2 Beam Steering with Linear Arrays	18
CONCLUSIONS	21
REFERENCES	22

List of Figures

1.	Fourier transform of a sampled signal.	6
2.	(sin x)/x interpolation by the frequency-domain method.	10
3.	Illustration of interpolation [Eq. 11].	11
4.	Sampling of a pure sine wave.	14
5.	Sampling of a Gaussian pulse.	15
6.	Interpolation of a sampled Gaussian pulse.	17
7.	Example of interpolation of the signal from an underwater explosion.	17
8.	Beam steering with linear array.	18
9.	Beam steering with sample signal.	19
10.	Time shifting of a sampled Gaussian pulse.	20

(sin x)/x INTERPOLATION OF SAMPLED SIGNALS

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ABSTRACT

A method for (sin x)/x interpolation of sampled signals is described. The method works by transformation to the frequency domain and therefore requires a digital computer with Fast Fourier Transform capability. Examples of how the (sin x)/x interpolation can be used for signal reconstruction and in connection with beam-steering are presented. The practical implication of this method for (sin x)/x interpolation is that it allows the use of a sampling frequency that is lower and closer to the theoretical minimum than would otherwise be possible. The complexity and cost of the electronic system used for sampling and recording can thereby be reduced.

INTRODUCTION

The sampling theorem states that a low-pass, band-limited function of time, having no frequency components outside the frequency interval from $-f_c$ to $+f_c$ may be described uniquely and completely for all time by a set of sample values taken at time instants separated by $^1/2f_c$ intervals or less.

The original continuous time function can be reconstructed by a (sin x)/x interpolation function, also called a band-limited interpolation. However, the accurate reproduction of a signal that is described by a series of sample values presents a certain number of practical problems.

In practice, the signals do not have a precisely-defined bandwidth, but a more or less gradual fall-off towards the high frequencies. Whatever sampling frequency, F_s is used, there will always be an error in the representation of the signal associated with the frequency components remaining outside the band $-\frac{F_s}{2}$ to $+\frac{F_s}{2}$. This error, called aliasing error or folding error, may be kept as small as desired by choosing a sufficiently high value of F_s. Technical and economical considerations, however, tend to limit the sampling frequency to a minimum value.

When the bandwidth of the signal is artificially limited by an analogue low-pass filter before sampling, the ratio between the sampling frequency and the bandwidth of the filter (such as can be defined by the 3 dB or 6 dB point for instance) depends on the steepness of the attenuation in the stop-band zone and on the accuracy required. In actual fact, with a proper design of the filter, such a ratio may be between 2.5 and 4, depending on the required precision. For the digital analysis system developed at SACLANTCEN, for instance, the sampling frequency has been chosen as three times the bandwidth of the analogue filter defined at the 6 dB point. This choice was compatible with the overall precision in amplitude of 1% required for the system.

Another aspect of the problem is the practical difficulties that may be encountered for the interpolation. The extension of the (sin x)/x function is from $-\infty$ to $+\infty$. For the numerical computation of intermediate values between the samples, this function has to be approximated by a time-limited filtering function. However, as the decreasing law of the (sin x)/x function is very slow, this filtering function involves a very large number of coefficients as soon as a rather high accuracy is required. In other words, a correct interpolation between two samples has to take account of the values of samples in the series that are far outside these two samples. The same problem exists when designing an analogue low-pass filter for the reconstruction of an analogue continuous signal from a series of pulses representing the sample values. Such a low-pass filter must have a very steep frequency fall-off, in other words, a very large number of poles, in order to give a correct reconstitution of the original signal.

The practical difficulties of interpolation have often led the system designers to choose a much higher sampling frequency than would be required for the limitation of the aliasing errors. Increasing the sampling frequency above its minimum value produces some redundancy in the numerical representation of the signal, which makes the interpolation much easier (when it is still necessary). The required number of coefficients of the numerical filters — or the required number of elements of the analogue low-pass filters — decreases when the sampling frequency increases.

The recent development of Fast Fourier Transform (FFT), however, may greatly simplify the interpolation problem when a digital computer (or a hardware FFT machine) is included in the processing system. An excellent approximation of the (sin x)/x interpolation function can then be obtained in a way that is easy to program and fast to execute. It should therefore no longer be necessary to increase the sampling rate of a processing equipment above the minimum value required for the correct representation of the signals.

This method of interpolation, by using the FFT, has been used at SACLANTCEN for several years but the method has not, however, been treated in current literature except for a brief paragraph in a recent book [Ref. 1, page 199]. It is however believed that the method is so important for practical applications that the more detailed discussion given in this paper is justified.

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1. REVIEW OF THE SAMPLING THEOREM

Consider a time signal $f(t)$ with Fourier transform $F(f)$. Let $f(t)$ be sampled at time intervals $T_s = 1/F_s$ to give a sequence of samples $f_n = f(t = nT_s)$. Defining the Fourier transform of the sampled signal as

$$\hat{F}(f) = T_s \sum_{n=-\infty}^{\infty} f_n \exp(-2\pi j\, f \cdot n \cdot T_s) \qquad \text{[Eq. 1]}$$

results in

$$\hat{F}(f) = \sum_{m=-\infty}^{\infty} F(f + m \cdot F_s) \ . \qquad \text{[Eq. 2]}$$

Equation 2 expresses the well-known fact that the Fourier transform of a sampled signal is an aliased version of the true Fourier transform.

If the signal $f(t)$ is band-limited such that $F(f) = 0$ for $|f| > f_c$ and the sampling frequency $F_s \geq 2f_c$, then the samples completely describe the signal and Eq. 2 becomes:

$$\hat{F}(f) = F(f) \quad \text{for} \quad |f| \leq \tfrac{1}{2} \cdot F_s \ . \qquad \text{[Eq. 3]}$$

When the conditions above are fulfilled, the signal can be reconstructed without errors by applying a (sin x)/x interpolation to the samples [Ref. 2]

$$f(t) = \sum_{n=-\infty}^{\infty} f_n \cdot \frac{\sin 2\pi f_2 \cdot (t - nT_s)}{2\pi f_2 (t - nT_s)} \ . \qquad \text{[Eq. 4]}$$

The frequency f_2 is arbitrary but must be such that

$$f_c \leq f_2 \leq F_s - f_c \ . \qquad \text{[Eq. 5]}$$

In practice, the signal $f(t)$ will not have a precisely defined bandwidth but will more likely have a gradual fall-off [Fig. 1].

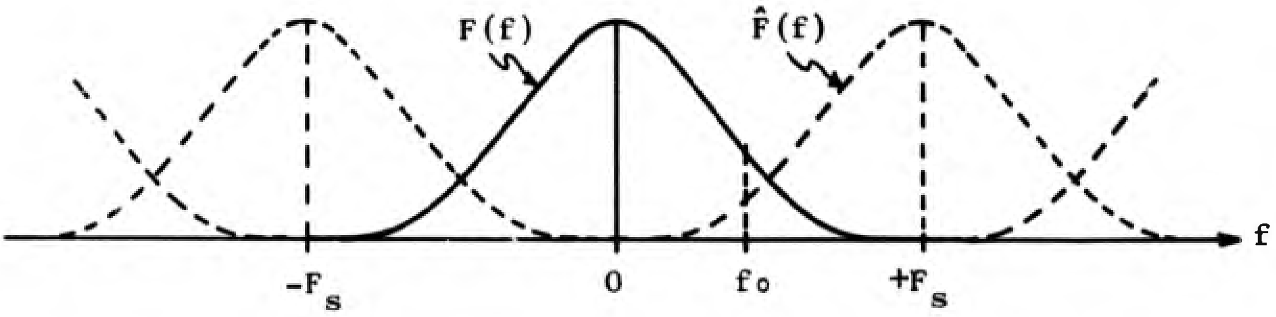

FIG. 1 FOURIER TRANSFORM OF A SAMPLED SIGNAL

There will therefore always be some errors because of frequency folding, but they can normally be kept as small as desired. If f_0 [Fig. 1] is the highest frequency of interest, the maximum relative error in the interesting frequency band is given by Eq. 2 (neglecting higher-order terms).

$$\varepsilon_F = \frac{|F(f_0) - \hat{F}(f_0)|}{F(f_0)} \quad \frac{F(f_0 - F_s)}{F(f_0)} \ . \qquad \text{[Eq. 6]}$$

In the time domain, when the signal is reconstructed by (sin x)/x interpolation, there will also be errors. An upper bound on this error is given in Ref. 2 where other types of sampling errors are also considered.

The errors resulting from sampling a signal that is not ideally band-limited depend, however, only on the signal spectrum outside the frequency band of interest. These errors can therefore be made as small as desired by proper analogue filtering of the signal before sampling. As an example of what can be achieved in practice, one can take the digital recording system described in Ref. 3. This uses a sampling frequency of three times the bandwidth defined by the -6 dB frequency. Specially designed filters [Ref. 4] with excellent time response suppress the higher frequencies such that the error ϵ_F [Eq. 6] is less than 1% when a broadband signal is recorded.

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2. REALIZATION OF THE (sin x)/x INTERPOLATION USING THE FAST FOURIER TRANSFORM

2.1 Types of Application

The preceding discussion indicates that in practice a sampling frequency only moderately higher than that given by the sampling theory is sufficient. The signal values can then be computed for any time instant, with negligible errors, using the (sin x)/x interpolation.

In practice, interpolation is used for two types of application.

In the first type of application, the requirement is to increase the sampling frequency by computing intermediate values at regular intervals between the original samples. This occurs, for example, when the signal has to be represented on a display; the density of points has to be such that it gives the appearance of continuity in the representation.

In the second class of application, the problem is to compute the value of the signal at some precise instants of time that do not coincide with the instants of sampling. Beam-steering, for instance, requires that the samples of a time-shifted signal be computed without increasing its original sampling frequency.

The problems of these two types of application will be briefly considered below.

2.2 Increasing the Sampling Frequency

The interpolation can be done in the time domain by the convolution described by Eq. 4. In general, however, this method requires a relatively long computation time. Another possibility is by transformation to the frequency domain. When this transformation is done by using the Fast Fourier Transform, the computation time may be substantially reduced with respect to the convolution method. This method becomes particularly attractive when transformation to the frequency domain has to be done anyway because of the rest of the processing.

The principle of this method can easily be understood by noticing that the Fourier transform of the sampled signal $\hat{F}(f)$ is a periodic function of frequency f, with period equal to the sampling frequency F_s. Therefore, if $\hat{F}(f)$ is known for one period, the sampling frequency can be increased by simply increasing the period of $\hat{F}(f)$.

Using the FFT, the method becomes as follows: Let the original sequence of samples be f_n with $n = 0, 1 \ldots\ldots N-1$. Applying the FFT gives

$$F_r = \sum_{n=0}^{N-1} f_n \cdot \exp\left(-2\pi j\, \frac{r \cdot n}{N}\right) \qquad \text{[Eq. 7]}$$

$$r = 0, 1 \ldots\ldots N-1 \quad .$$

The sequence F_r represents the sample values of $\hat{F}(f)$ [Eq. 1] such that

$$\begin{aligned} F_r &= \hat{F}\left(f = \tfrac{r}{N} \cdot F_s\right) && r = 0, 1 \ldots N/2 \\ F_{N-r} &= \hat{F}\left(f = -\tfrac{r}{N} \cdot F_s\right) && r = 1, 2 \ldots N/2-1 \end{aligned} \qquad \text{[Eq. 8]}$$

Increasing the period of $\hat{F}(f)$ is now equivalent to defining a new sequence G_r $r = 0, 1 \ldots M-1$ $(M > N)$ such that

$$\begin{aligned} G_r &= F_r && r = 0, 1 \ldots N/2-1 \\ G_{M-r} &= F_{N-r} && r = 1, 2 \ldots N/2-1 \\ G_{\frac{N}{2}} &= G_{M-\frac{N}{2}} = \tfrac{1}{2} F_{\frac{N}{2}} && \\ G_r &= 0 && r = N/2+1, \ldots M-N/2-1 \end{aligned} \qquad \text{[Eq. 9]}$$

This is indicated in Fig. 2

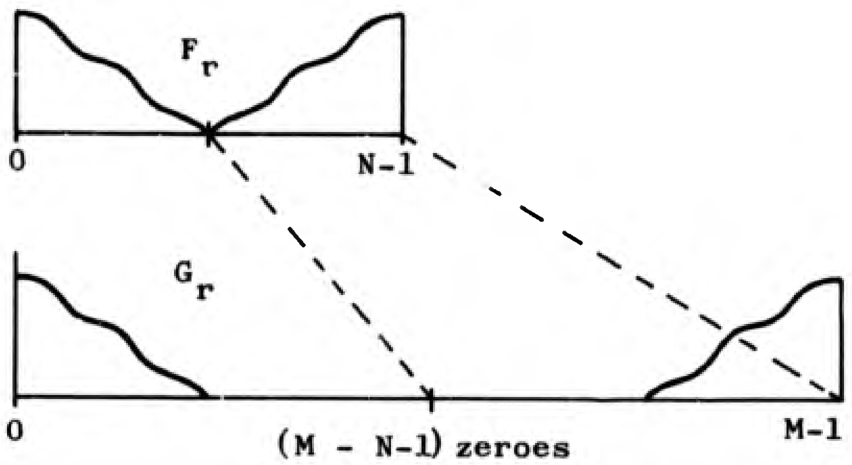

FIG. 2 (sin x)/x INTERPOLATION BY THE FREQUENCY-DOMAIN METHOD

Computing the inverse transform of G_r gives the new sequence of signal samples g_m

$$g_m = \frac{1}{N} \sum_{r=0}^{M-1} G_r \cdot \exp\left(+2\pi j \frac{r \cdot m}{M}\right) \qquad \text{[Eq. 10]}$$

$$m = 0, 1 \ldots M-1$$

Using Eq. 9 and Eq. 7 one can, after an easy but somewhat tedious calculation, express the new samples g_m by the original samples f_n:

$$g_m = \sum_{n=0}^{N-1} f_n \frac{\sin \pi \left(\frac{N}{M} m - n\right)}{N \cdot \tan \frac{\pi}{N} \left(\frac{N}{M} m - n\right)} . \qquad \text{[Eq. 11]}$$

Equation 11 represents the discrete version of the (sin x)/x interpolation, the difference being the tangent-function in the denominator, such that the kernel is

$$K_p = \frac{\sin \pi p}{N \cdot \tan \frac{\pi}{N} p} , \qquad \text{[Eq. 12]}$$

which is a periodic function with period N.

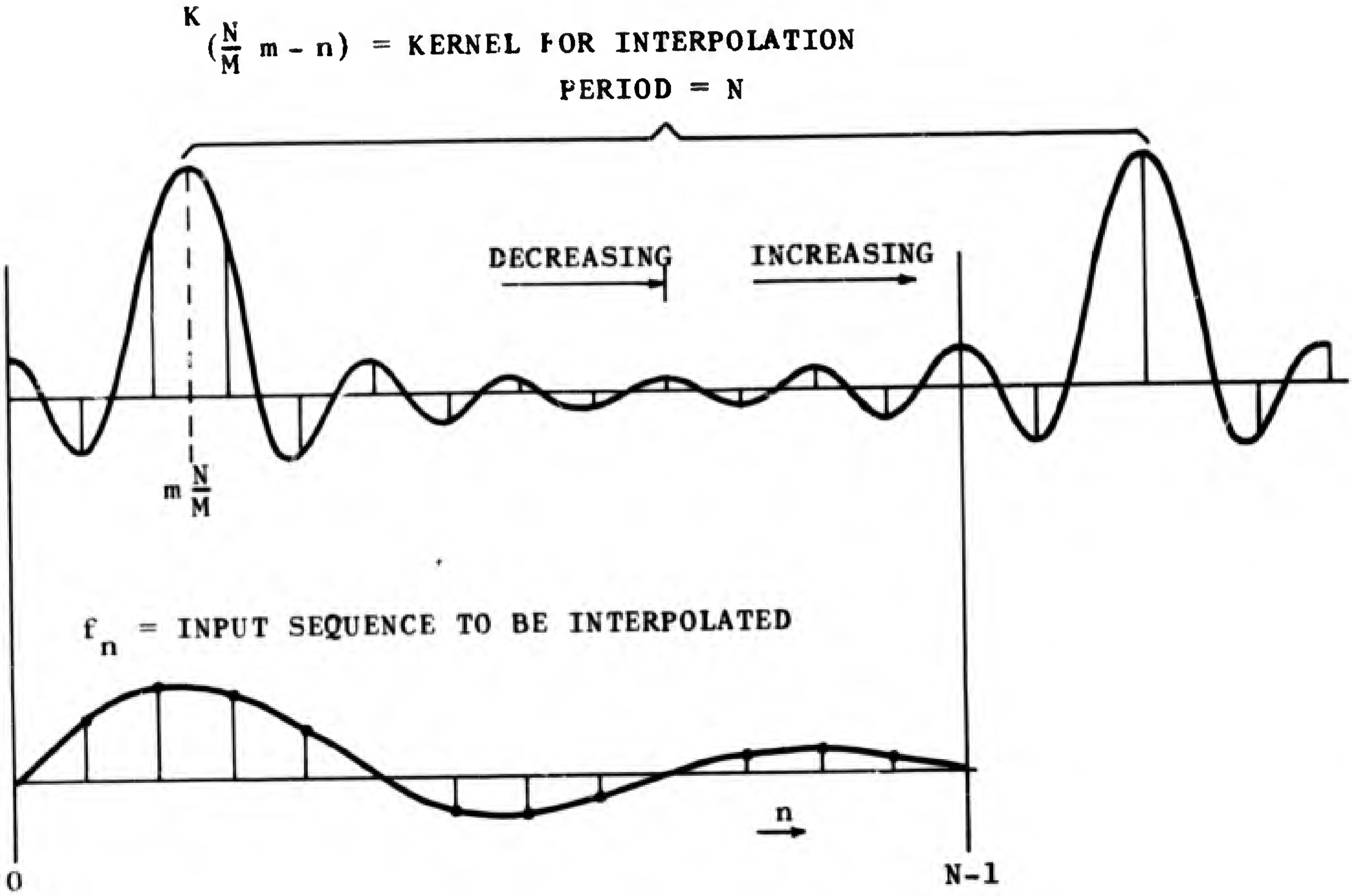

FIG. 3 ILLUSTRATION OF INTERPOLATION (Eq. 11)

Figure 3 illustrates Eq. 11 by showing the calculation of an interpolated value for the time instant $m\frac{N}{M}$. The original samples, which are close to the main lobe, will yield the highest contribution to the interpolated value. However, those samples that are in the vicinity of the next main lobe will also give a significant contribution. This contribution, which represents an aliasing error in the time domain, is typical for many applications where the discrete Fourier transform is used [Refs. 1 and 5] and can normally be reduced by redefining the input sequence. In this case, the original sequence should be defined so that a certain number N_z of zeroes are included in the end, the number depending on the desired accuracy of the reconstitution. If one wants to limit the aliasing contribution of a single sample to less than a fraction ϵ of the maximum, one can see from Eq. 12 that the number of zeroes required will be

$$N_z \geq \frac{1}{\pi\epsilon} . \qquad \text{[Eq. 13]}$$

If one also ensures that N is large enough, Eq. 11 becomes a good approximation of the desired (sin x)/x interpolation. After the interpolation an original sampling frequency of F_s becomes $\frac{M}{N} \cdot F_s$. When M/N is an integer, the original samples will remain unchanged by the interpolation. Since most FFT routines require both M and N to be powers of 2, this method increases the sampling rate by factors of 2, 4 ..... etc.

So far it has been tacitly assumed that the interpolation could be done for the whole signal in one operation. For practical reasons, this will very often not be the case and one is forced to do the interpolation piecewise. This can be done by exactly the same methods that are normally used for doing a convolution by using the FFT [Ref. 6]. One should then regard the effective length of the convolution kernel as being equal to $2N_z$, where N_z is determined by the desired accuracy [Eq. 13].

2.3 Computing the Samples of a Time-Shifted Signal

In some applications it is not necessary to increase the sampling frequency but only to compute the samples of the signal when it is shifted in time.

For the continuous case, it is well known that multiplication in the frequency domain with $\exp(-2\pi jfT)$ is equivalent to delaying the signal by a time T. This is true also in the discrete case, and when T is not a multiple of the sampling interval T_s the new samples will be related to the original samples by a (sin x)/x interpolation.

In the discrete case, a delay of $T = sT_s$ can be accomplished by multiplying F_r of Eq. 7 by $\exp(-2\pi j \frac{r}{N} s)$. (The relationship expressed by Eq. 8 has to be taken into account). After transforming back to the time domain the new sequence of samples will be

$$p_m = \sum_{n=0}^{N-1} f_n \frac{\sin \pi (m-n-s)}{N \cdot \tan \frac{\pi}{N} (m-n-s)} . \qquad \text{[Eq. 14]}$$

If the delay T is a multiple of the sampling interval, then s is an integer and Eq. 14 becomes

$$p_m = f_{m-s} \qquad m = s,\ s+1 \ldots N-1$$
$$p_m = f_{N+m-s} \qquad m = 0,1 \ldots s-1 \quad . \tag{Eq. 15}$$

When the delay is not a multiple of the sampling interval, Eq. 14 applies, and when a sufficient number of zeroes are included at the end of the original sequence then, by the same arguments as in the preceding ch pter, Eq. 14 becomes a close approximation to the (sin x)/x interpolation.

3. APPLICATIONS AND EXAMPLES

Having described methods for accomplishing the (sin x)/x interpolation, some applications of the methods will be indicated and the results demonstrated by some simple examples.

3.1 Signal Reconstruction Using Straight Lines

3.1.1 Applications

Consider the situation where a computer-driven display is used to present a signal specified by its sample values. In most cases, the display will do this by drawing straight lines between the sample points thereby obtaining a curve that may look quite different from the analogue signal it should represent. To find the errors in the display, one has to consider specific examples and for this consider first the very simple example of a pure sine wave.

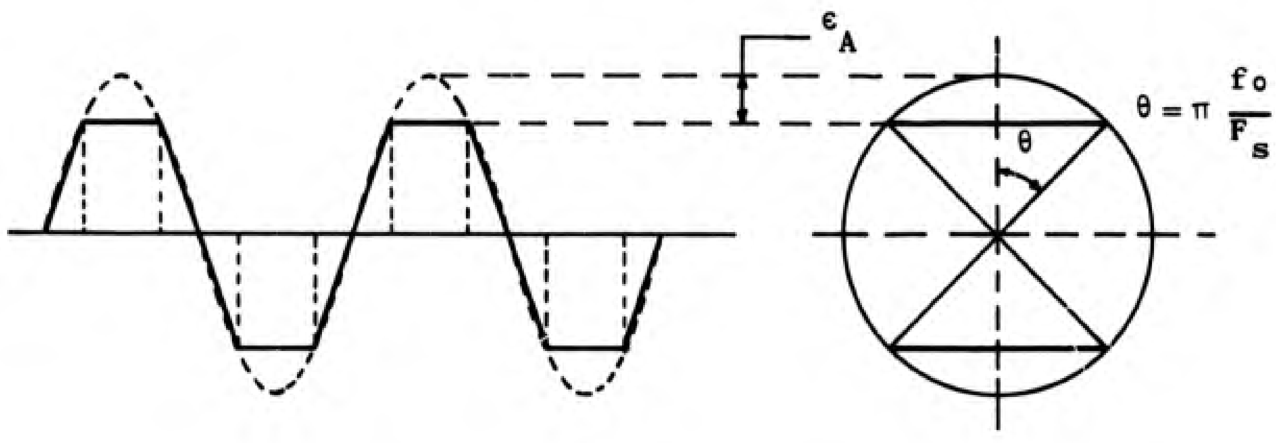

FIG. 4 SAMPLING OF A PURE SINE WAVE

With a sine wave of frequency f_0 and a sampling frequency of F_s, the maximum error ϵ_A in amplitude can easily be found from Fig. 4 as

$$\epsilon_A = 1 - \cos \pi \frac{f_0}{F_s} = \frac{1}{2} \sin^2 2\pi \frac{f_0}{F_s} . \qquad \text{[Eq. 16]}$$

For small errors the necessary sampling frequency becomes

$$F_s/f_0 = \frac{\pi}{\sqrt{2}} \frac{1}{\sqrt{\epsilon_A}} . \qquad \text{[Eq. 17]}$$

If, for example, $\epsilon_A \leq 0.05$ (assuming that the recording equipment has this precision) is wanted, then the sampling frequency must be ten times the frequency of the signal.

Another and more realistic example is when the signal is a Gaussian pulse [Fig. 5]

$$f(t) = \exp[-\tfrac{1}{2}(t/\tau)^2] \qquad \text{[Eq. 18]}$$

that has a Fourier transform

$$F(f) = \text{const} \cdot \exp[-\tfrac{1}{2}(2\pi f \tau)^2] . \qquad \text{[Eq. 19]}$$

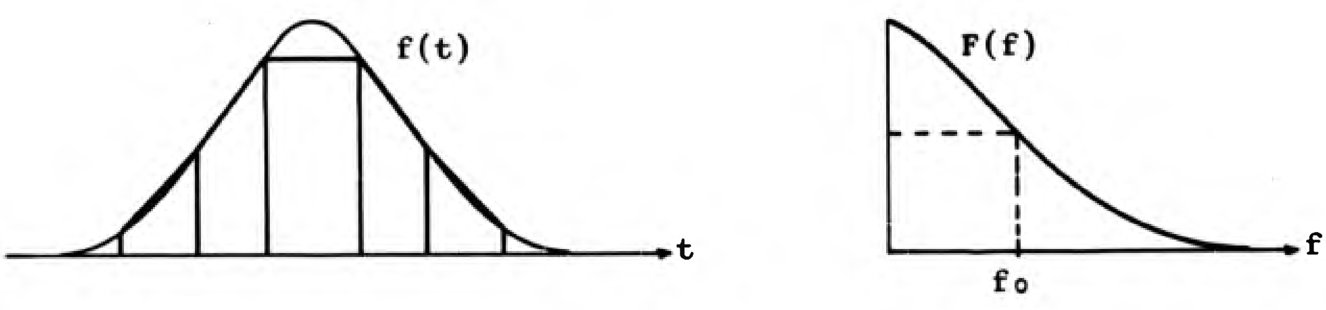

FIG. 5 SAMPLING OF A GAUSSIAN PULSE

The bandwidth of this pulse can be defined by the frequency f_0 where the amplitude spectrum is down 6 dB, giving

$$f_0 \approx \frac{1}{5\tau} \quad . \qquad \text{[Eq. 20]}$$

The maximum error in amplitude resulting from drawing straight lines between the sample points now becomes

$$\epsilon_A = 1 - \exp[-\tfrac{1}{2} (\frac{1}{2F_s\tau})^2] \qquad \text{[Eq. 21]}$$

which, when the error ϵ_A is small, gives the required sampling frequency

$$F_s/f_0 = \frac{5}{2\sqrt{2}\ \sqrt{\epsilon_A}} \quad . \qquad \text{[Eq. 22]}$$

With an error of 5%, Eq. 22 specifies a sampling frequency of 7.5 times the 6 dB frequency.

3.1.2 Example

Figure 6 shows an example where the (sin x)/x interpolation has been applied to samples from a Gaussian pulse. Curve A shows the analogue signal and Curve B the original result of sampling at a frequency of four times the 6 dB frequency. Curves C to E show the effect of the interpolation, with the sampling frequency being doubled for each step.

Another and more realistic example of interpolation is shown in Fig. 7, which presents the signal from an underwater explosion. The signal has been recorded by the digital recording system described in Ref. 3,using a sampling frequency of 48 kHz after it had been filtered by a 16 kHz linear phase filter of the type described in Ref. 4. When the signal is displayed by drawing straight lines between the samples the result is relatively poor, as can be seen in Fig. 7a. After interpolating so that the sampling frequency is increased four times, the representation is highly improved, as seen in Fig. 7b. The small ripple observed just before the shock

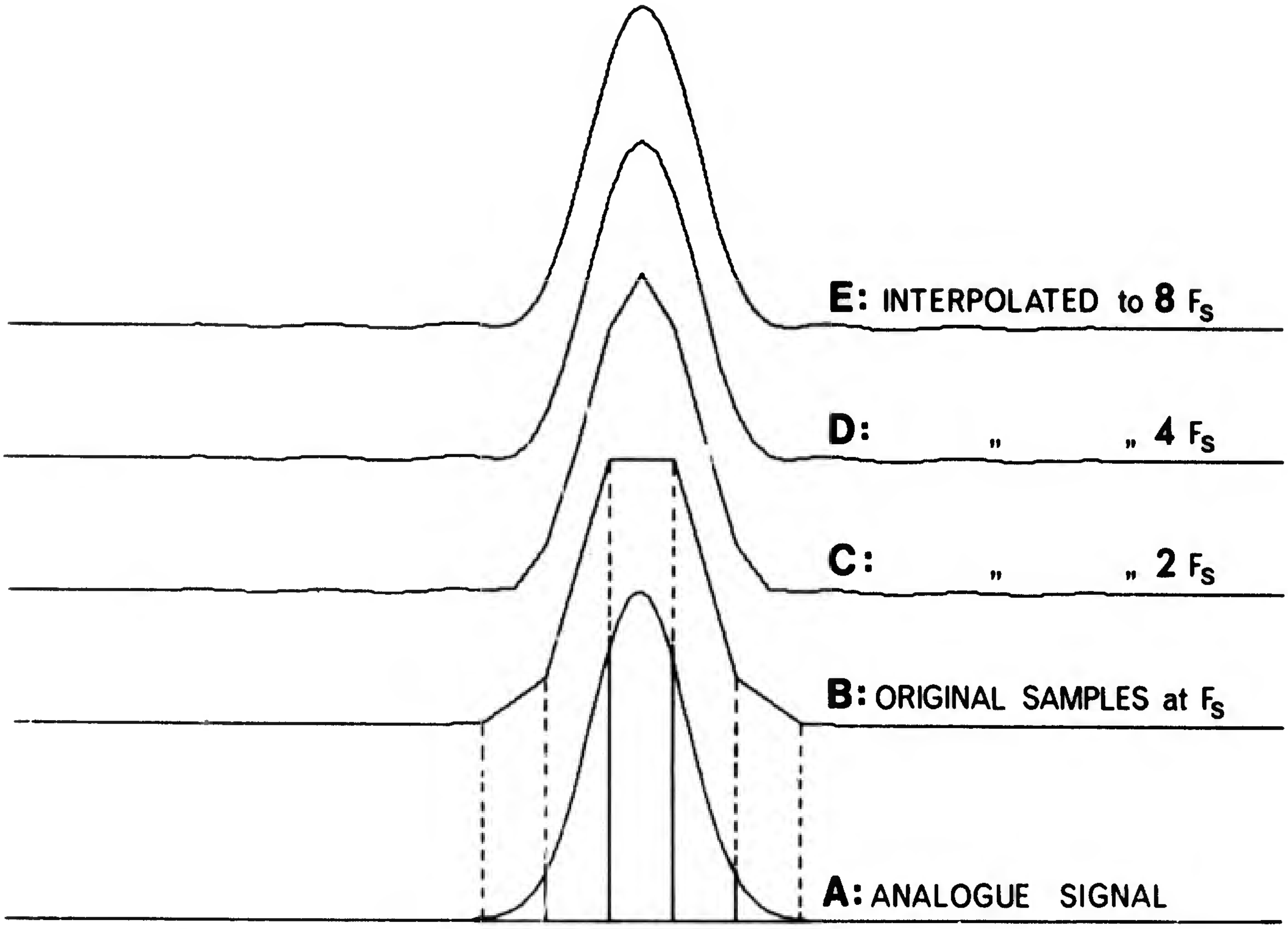

FIG. 6 INTERPOLATION OF A SAMPLED GAUSSIAN PULSE $\exp\left[-\frac{1}{2}(t/\tau)^2\right]$
Original sampling frequency $F_s = 0.8/\tau$
(4 times the 6 dB cut-off frequency)

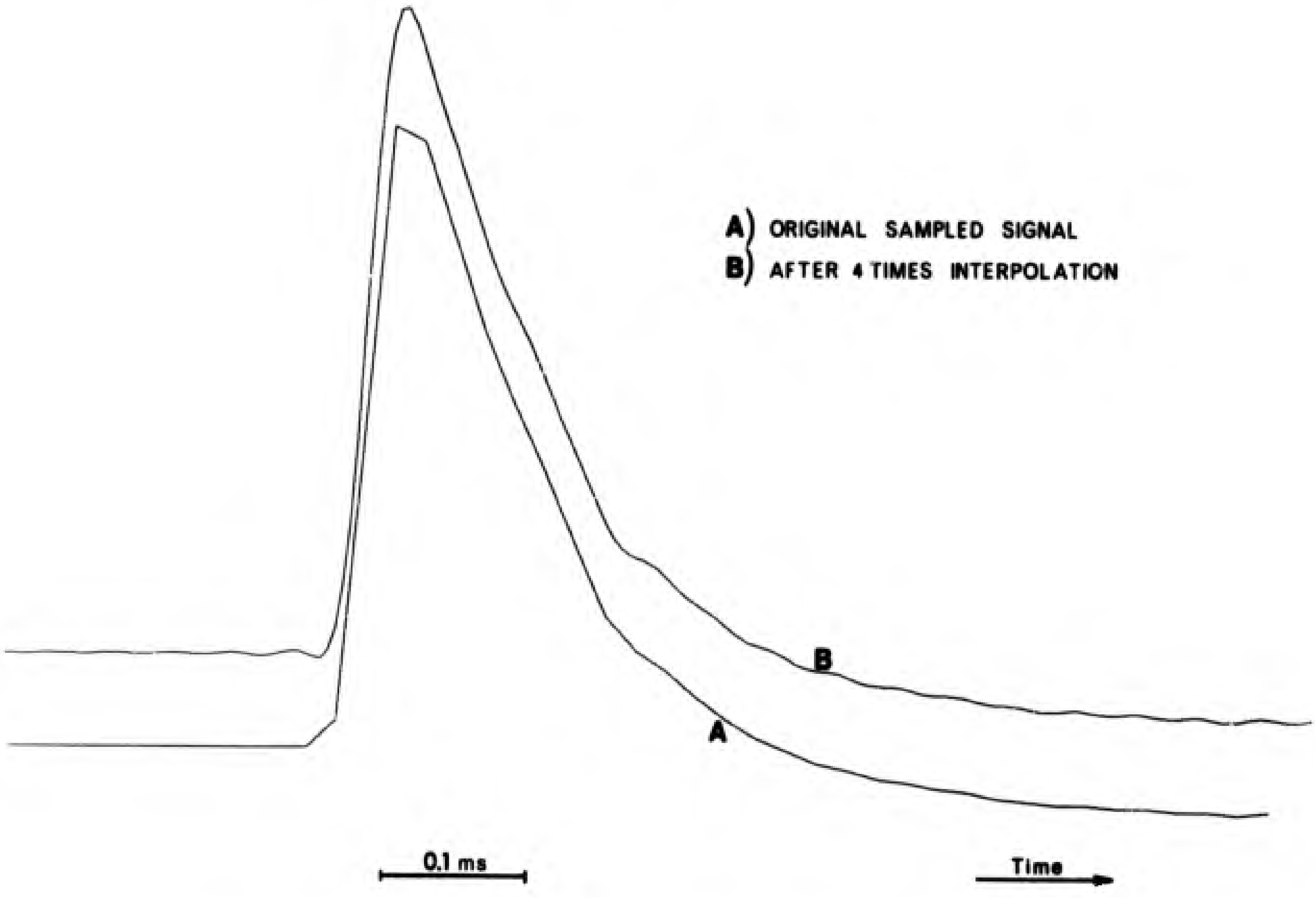

FIG. 7 EXAMPLE OF INTERPOLATION OF THE SIGNAL FROM AN UNDERWATER EXPLOSION

wave in Fig. 7b is probably due to frequency folding, because the analogue filter used before sampling does not have infinite attenuation above 24 kHz.

3.2 Beam Steering with Linear Arrays

Consider a linear array of hydrophones which, for simplicity, will be assumed to be equidistant. Steering the array through an angle θ requires a delay ΔT [Fig. 8] given by

$$\Delta T = \frac{d}{c} \sin \theta \, . \qquad \text{[Eq. 23]}$$

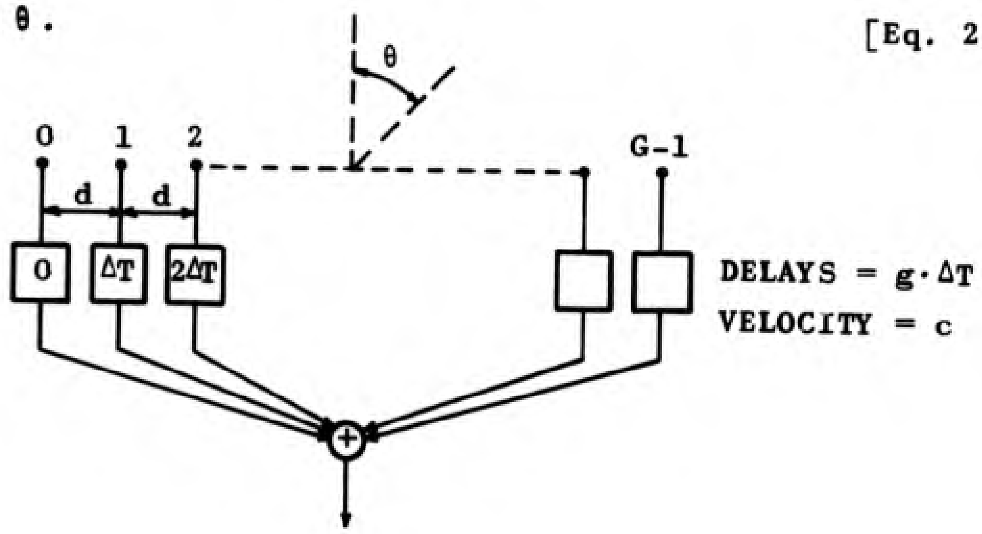

FIG. 8 BEAM STEERING WITH LINEAR ARRAY

If the steering is done in the time domain by adding delayed samples, only a delay, ΔT, that is a multiple of the sampling frequency can be obtained exactly. If one wants to steer in any other direction, this can only be done approximately, as indicated in Fig. 9. There will then be an error in the delays of up to half the sampling interval T_s. This error in the time delays will result in an error in the phase of the different frequency components of the signal. The maximum phase error for a frequency component at f_0 will be

$$\Delta\varphi = \pi \frac{f_0}{F_s} \, . \qquad \text{[Eq.24]}$$

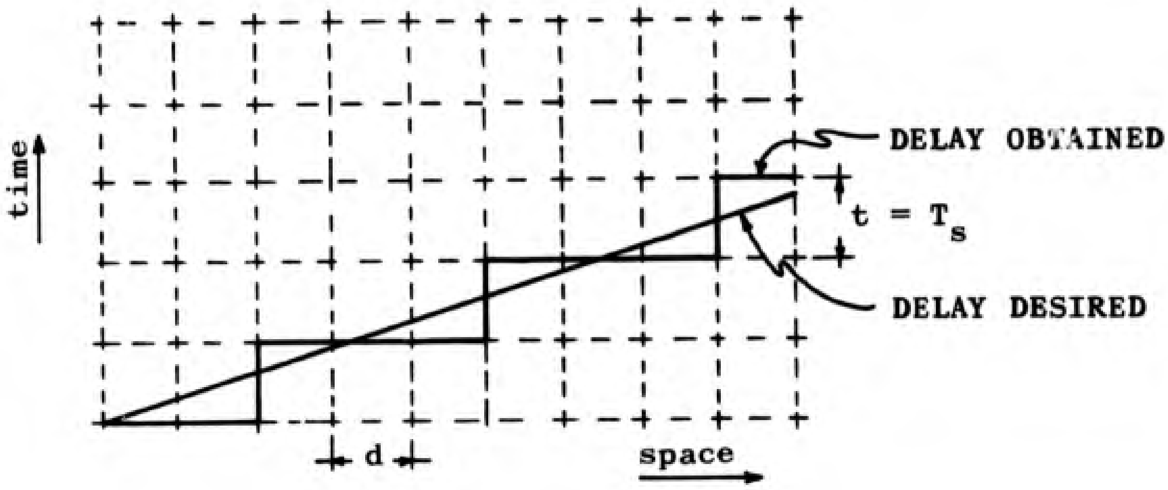

FIG. 9 BEAM STEERING WITH SAMPLE SIGNAL

The effect of the errors is that the different frequency components will not add exactly in phase, which may reduce the performances of the array, particularly with regards to its side lobes. Without studying in detail the consequences of this, one can at least state that the errors can be made as small as desired by increasing the sampling frequency, for example by the method described in Sect. 2.2.

Another and frequently more attractive possibility is to do the beam steering in the frequency domain by multiplying the Fourier transform of each hydrophone signal by $\exp[-2\pi jfg\Delta T]$. As explained in Sect. 2.3 there will then be no restriction on ΔT being a multiple of the sampling interval.

Figure 10 shows an example of a signal being delayed a fraction of the sampling period. The signal is again a Gaussian pulse, sampled at four times the 6 dB frequency [Eq. 21]. The delays between successive curves are 1/5 of the sampling interval such that after five steps the total delay is one interval. One notices the rather great differences in the shapes of the signals due to the fact that the positions of the sampling instants are different. The last signal is, however, identical to the first, only delayed by one sampling interval.

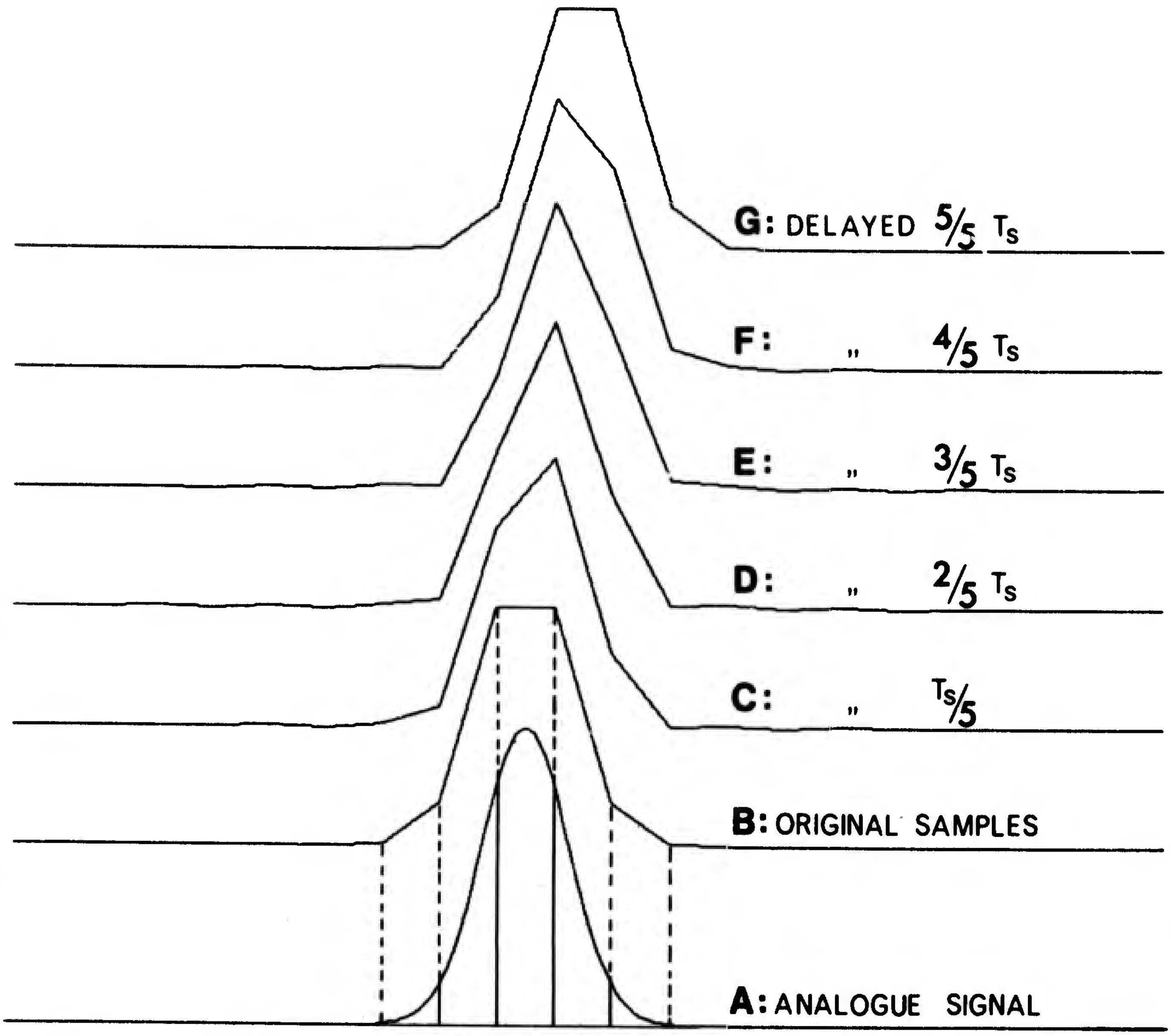

FIG. 10 TIME SHIFTING OF A SAMPLED GAUSSIAN PULSE $\exp\left[-\frac{1}{2}(t/\tau)^2\right]$
Sampling frequency $F_s = 0.8/\tau$
(4 times the 6 dB cut-off frequency)

CONCLUSIONS

In the processing of sampled signals it is often required to compute the values of the signal at intermediate time instants. As known from the sampling theorem, this should be done by applying the (sin x)/x interpolation (or band-limited interpolation). This report has presented practical methods by which this interpolation can be accomplished. The methods require the computation of Fourier transforms and can therefore only be used efficiently when the processing is done on a digital computer where the Fast Fourier Transform can be applied. As is often the case when the Fast Fourier Transform is used, one has to exercise some care in order to avoid aliasing errors, but it has been shown that these errors can be reduced to an acceptable level.

The practical implication of the (sin x)/x interpolation is that it allows the complexity of a recording system to be reduced by permitting the sampling frequencies to be kept to a minimum.

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