TESTS FOR RANDOMNESS OF DIRECTIONS AGAINST EQUATORIAL AND BIMODAL ALTERNATIVES

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1. Introduction

Suppose a random sample of points P_i, $i=1, 2, \ldots, N$, is given on the surface of a sphere with center 0 and radius 1. The vectors $\underset{\sim\sim}{OP}_i$ may denote directions. In some geophysical problems vectorial data of this type occur in which the vectors $\underset{\sim\sim}{OP}_i$ are more or less coplanar, that is, clustered close to a plane through the center 0. Such a plane intersects the sphere in a great circle, which may be regarded as the equator of the sphere; the probability distribution over the spherical surface which is then appropriate to describe the distribution of points P (or equivalently of unit vectors $\underset{\sim\sim}{OP}$) must have high density near the equator and low density at the North and South Poles. Such a distribution is called an equatorial distribution. Suppose spherical polar coordinates (θ, ϕ) are used for P, and let $\theta = 0$ be the North Pole; here $0 \leq \phi \leq 2\pi$ and $0 \leq \theta \leq \pi$. The density

$$(1) \qquad f(\theta, \phi) = c_3(\kappa) \sin\theta \exp(-\kappa \cos^2 \theta) ,$$

with $\kappa \geq 0$ and $c_3(\kappa)$ a normalizing constant, was introduced as a suitable equatorial distribution by Watson (1965). If $\kappa < 0$, the density may be used to describe a bimodal population, with modes in the density at the poles and low density on the equator $\theta = \pi/2$. Clearly, suitable orthogonal transformations will place the polar axis along any vector $\underset{\sim}{A}$. When $\kappa = 0$, the density reduces to the uniform density over the sphere.

A common problem, given sample data, is first to test for uniformity, that is, to test the null hypothesis H_0: $\kappa = 0$ against the alternative H_A: $\kappa > 0$ or H_B: $\kappa < 0$. For the equatorial alternative, H_A, Watson (1965) showed that the likelihood ratio test of H_0, when the antimodal axis $\underset{\sim}{A}$ is known, is based on $S = N^{-1} \sum_{i=1}^{N} \cos^2 \alpha_i$, where α_i is the angle between $\underset{\sim}{A}$ and $\underset{\sim\sim}{OP}_i$, $i=1, 2, \ldots, N$. The vector A may point to either antimode. When S is significant in the lower tail, H_0 is rejected in favor of H_A. S may also be used to test for uniformity against the bimodal density given by (1) with $\kappa < 0$; then H_0 is rejected in favor of H_B if S is significant in the upper tail. Percentage points of S (there called T) are given in Stephens (1966).

In this paper we give significance points for the test statistic to be used for testing H_0 against H_A and for testing H_0 against H_B in the more realistic case when the antimodal and modal vectors $\underset{\sim}{A}$, respectively, are <u>not known</u>. We also give some indications of the power of the test. The problems posed in finding the distributions of the test statistics have interesting connections with principal component analysis and with an engineering application; they have not been solved exactly except in one special case, but significance points are given, based on a combination of asymptotic results and Monte Carlo studies. We conclude the paper with an explanation of the corresponding problem in two dimensions, because there the distribution theory may be solved exactly.

2. Tests for Randomness Against Equatorial or Bimodal Alternatives

2.1. The procedures. We now discuss the test of H_0 when the vector $\underset{\sim}{A}$, pointing to an antimode for the equatorial alternative and to a mode for the bimodal alternative, is not known. The likelihood ratio test statistic is found as follows (Watson, 1965). Suppose for a given vector $\underset{\sim}{A}'$, $S' = N^{-1} \sum_{i=1}^{N} \cos^2 \alpha_i$, where α_i is the angle between $\underset{\sim\sim}{OP}_i$ and $\underset{\sim}{A}'$.

As $\underset{\sim}{A}'$ is allowed to vary over the surface of the sphere, S' will take values between a minimum, S_{min}, and a maximum, S_{max}. S_{min} will be the likelihood ratio test statistic for the test of H_0 against H_A (equatorial alternative) and S_{max} will be used against H_B (bimodal alternative). S_{min} and S_{max} may easily be calculated from the latent roots (eigenvalues, characteristic roots) of a matrix Q to be given below. The tests may therefore be set out as follows:

(a) Suppose ℓ_i, m_i, n_i are the direction cosines of $\underset{\sim}{OP}_i$, referred to a fixed system of coordinates; that is, let $\ell_i = \sin\theta_i \cos\phi_i$, $m_i = \sin\theta_i \sin\phi_i$, $n_i = \cos\theta_i$, $i=1, \ldots, N$.

(b) Let $\underset{\sim}{Q}$ be the following matrix divided by N:

$$\begin{bmatrix} \sum_{i=1}^{N} \ell_i^2 & \sum_{i=1}^{N} \ell_i m_i & \sum_{i=1}^{N} \ell_i n_i \\ \sum_{i=1}^{N} \ell_i m_i & \sum_{i=1}^{N} m_i^2 & \sum_{i=1}^{N} m_i n_i \\ \sum_{i=1}^{N} \ell_i n_i & \sum_{i=1}^{N} m_i n_i & \sum_{i=1}^{N} n_i^2 \end{bmatrix}.$$

(c) Let the latent roots of $\underset{\sim}{Q}$ be $\lambda_1 \geq \lambda_2 \geq \lambda_3$, and let the corresponding latent vectors be $\underset{\sim}{u}_1$, $\underset{\sim}{u}_2$, and $\underset{\sim}{u}_3$. Then $S_{min} = \lambda_3$ and $S_{max} = \lambda_1$.

(d) If S_{min} is less than the appropriate entry for values of N and α in Table 1, reject H_0 at significance level α in favor of the equatorial alternative; in this event $\underset{\sim}{u}_3$ is the maximum likelihood estimate of the antimodal vector of the distribution. Alternatively, if S_{max} is greater than its appropriate entry, reject H_0 in favor of the bimodal alternative, with $\underset{\sim}{u}_1$ as the maximum likelihood estimate of the modal vector.

2.2 Related problems.

(a) Principal component analysis. Let the vectors $\underset{\sim\sim}{OP}_i$ be denoted

$$\underset{\sim\sim}{OP}_i = \underset{\sim}{z}_i = \begin{pmatrix} z_{1i} \\ z_{2i} \\ z_{3i} \end{pmatrix} = \begin{pmatrix} \ell_i \\ m_i \\ n_i \end{pmatrix}, \qquad i=1, \ldots , N .$$

A vector $\underset{\sim}{\beta} = (\beta_1, \beta_2, \beta_3)'$ normalized so

$$\underset{\sim}{\beta}'\underset{\sim}{\beta} = \sum_{k=1}^{3} \beta_k^2 = 1 \tag{2}$$

has direction cosines as components. Then

$$v_i = \underset{\sim}{\beta}' \underset{\sim}{z}_i = \sum_{k=1}^{3} \beta_k z_{ki}$$

is the signed length of the projection of $\underset{\sim\sim}{OP}_i$ on $\underset{\sim}{\beta}$; in other words it is the length of $\underset{\sim\sim}{OP}_i$ in the direction of $\underset{\sim}{\beta}$. The sum of squares of these lengths is

$$\sum_{i=1}^{N} v_i^2 = \sum_{i=1}^{N} (\underset{\sim}{\beta}' \underset{\sim}{z}_i)^2 = \sum_{i=1}^{N} \underset{\sim}{\beta}' \underset{\sim}{z}_i \underset{\sim}{z}_i' \underset{\sim}{\beta} = N\underset{\sim}{\beta}'\underset{\sim\sim}{Q\beta} . \tag{3}$$

The maximum of (3) with respect to $\underset{\sim}{\beta}$, subject to (2), is the largest characteristic root of $N\underset{\sim}{Q}$, which is $N\lambda_1 = N\,S_{max}$. Thus $N\,S_{max}$ is the sum of squared distances of the points $P_1, \ldots , P_N$ maximized with respect to direction. The maximizing direction $\underset{\sim}{\beta}^{(1)}$ defines the first (sample) principal component (from the origin) as $v^{(1)} = \underset{\sim}{\beta}^{(1)'} \underset{\sim}{z}$, and

$$S_{max} = \frac{1}{N} \sum_{i=1}^{N} v_i^{(1)2}$$

is the (sample) variance (about 0) of the first principal component.

The projection of $\underset{\sim}{OP}_i$ on the plane orthogonal to $\underset{\sim}{\beta}^{(1)}$ is $\underset{\sim}{z}_i - v_i^{(1)} \underset{\sim}{\beta}^{(1)}$. The signed length of the projection of this vector on a normalized vector $\underset{\sim}{\beta}$ in this plane is $\underset{\sim}{\beta}' \underset{\sim}{z}_i$ because $\underset{\sim}{\beta}' \underset{\sim}{\beta}^{(1)} = 0$. The sum of squares is (3) and the maximum is the second characteristic root of $N\underset{\sim}{Q}$, which is $N\lambda_2$. Let the maximizing direction define $\underset{\sim}{\beta}^{(2)}$ and let $\underset{\sim}{\beta}^{(3)}$ be orthogonal to $\underset{\sim}{\beta}^{(1)}$ and $\underset{\sim}{\beta}^{(2)}$. Then $N\underset{\sim}{\beta}^{(3)'} \underset{\sim}{Q} \underset{\sim}{\beta}^{(3)}$ is the sum of squares of lengths in the direction of $\underset{\sim}{\beta}^{(3)}$, and this is the minimum, namely $N\lambda_3 = N\, Q_{min}$.

The discussion here parallels the treatment of principal components in Section 10.2 of Anderson (1958), with expectation replaced by sample average (or sum), coordinates from the mean replaced by coordinates from the origin, and p set at 3.

The density (1) can be obtained from a three-dimensional normal distribution with means 0, covariances 0, and two variances equal. Then (1) is the conditional density of the two angles in polar coordinates for the radius fixed.

(b) Mechanical Engineering. If unit masses are located at each point P_i, the quantity $N(1-S)$ gives the moment of inertia about $\underset{\sim}{A}'$, and $N(1-S_{min})$ and $N(1-S_{max})$ are respectively the maximum and minimum moments of inertia obtainable by varying $\underset{\sim}{A}'$. Thus the distributions of S_{min} and S_{max} can be used to give the distributions of maximum and minimum moment of inertia obtainable from a random distribution of N unit masses on the sphere.

3. Distributions of S_{min} and of S_{max}

3.1. A large-sample result. The exact distributions of S_{min} and of S_{max} are very difficult to find. In this section we give large-sample results, leading to very good approximations to significance points, to the order of $N^{-\frac{1}{2}}$. As $N \to \infty$, the entries in $\underset{\sim}{Q}$ have an asymptotic joint normal distribution. From it we can obtain the asymptotic joint distribution of the latent roots; the marginal distributions of the largest and smallest roots are the asymptotic distributions of S_{max} and S_{min}. Since we want the asymptotic distribution for the two-dimensional case as well as the three-dimensional case and it may be useful for higher dimensional cases, we prove a more general result.

Suppose $\underset{\sim}{y} = (y_1, \ldots, y_p)'$ has a p-dimensional normal distribution with mean $\underset{\sim}{0}$ and covariance matrix $\underset{\sim}{I}$. Let $\underset{\sim}{z} = \underset{\sim}{y}/\sqrt{\underset{\sim}{y}'\underset{\sim}{y}}$. Then $\underset{\sim}{z}$ has the uniform spherical distribution in p dimensions, and $\underset{\sim}{z}$ and $w = \sqrt{\underset{\sim}{y}'\underset{\sim}{y}}$ are independent. Since w^2 has a χ^2-distribution with p degrees of freedom,

$$\mathcal{E} w^m = \mathcal{E}(\chi_p^2)^{\frac{1}{2}m} = 2^{\frac{1}{2}m} \frac{\Gamma[\frac{1}{2}(m+p)]}{\Gamma(\frac{1}{2}p)} .$$

Then $\underset{\sim}{0} = \mathcal{E}\underset{\sim}{y} = \mathcal{E}w\underset{\sim}{z} = \mathcal{E}w\,\mathcal{E}\underset{\sim}{z}$ implies $\mathcal{E}\underset{\sim}{z} = \underset{\sim}{0}$. Because $\underset{\sim}{z}$ and w are independent, a moment of the components of $\underset{\sim}{z}$ is the corresponding moment of the components of $\underset{\sim}{y}$ divided by the moment of w of the same degree. Let $\underset{\sim}{A} = (a_{k\ell}) = \underset{\sim}{z}\underset{\sim}{z}'$; then

$$\mathcal{E}\underset{\sim}{A} = \frac{1}{\mathcal{E}w^2}\,\mathcal{E}\underset{\sim}{y}\underset{\sim}{y}' = \frac{1}{p}\underset{\sim}{I} ,$$

$$\mathcal{E}a_{kk}^2 = \mathcal{E}\underset{\sim}{z}_k^4 = \frac{1}{\mathcal{E}w^2}\,\mathcal{E}\underset{\sim}{y}_k^4 = \frac{1}{p(p+2)} ,$$

$$\mathcal{E}\, a_{k\ell}^2 = \mathcal{E}\, a_{kk}a_{\ell\ell} = \mathcal{E}\, z_k^2 z_\ell^2 = \frac{1}{\mathcal{E}\, w^4}\, \mathcal{E}\, y_k^2 y_\ell^2 = \frac{1}{p(p+2)}\,, \qquad k \neq \ell\,,$$

$$\mathcal{E}\, a_{kk}a_{k\ell} = \mathcal{E}\, z_k^3 z_\ell = \frac{1}{\mathcal{E}\, w^4}\, \mathcal{E}\, y_k^3 y_\ell = 0\,, \qquad k \neq \ell\,.$$

The variances and covariances of the elements of $\underset{\sim}{A}$ are

$$\operatorname{Var} a_{kk} = \frac{3}{p(p+2)} - \frac{1}{p^2} = 2\,\frac{p-1}{p^2(p+2)}\,,$$

$$\operatorname{Cov}(a_{kk}, a_{\ell\ell}) = \frac{1}{p(p+2)} - \frac{1}{p^2} = -\frac{2}{p^2(p+2)}\,, \qquad k \neq \ell\,,$$

$$\operatorname{Var}(a_{k\ell}) = \frac{1}{p(p+2)}\,, \qquad k \neq \ell\,,$$

and 0 for other covariances.

Let $\underset{\sim}{z}_i$, $i=1, 2, \ldots, N$, be the sample values of $\underset{\sim}{z}$, and define

$$\underset{\sim}{Q} = (1/N) \sum_{i=1}^{N} \underset{\sim}{z}_i \underset{\sim}{z}_i'\,,$$

$$\underset{\sim}{X} = \sqrt{N}\,(\underset{\sim}{Q} - \frac{1}{p}\underset{\sim}{I})\,.$$

The trace of $\underset{\sim}{Q}$ is $(1/N)\sum_{i=1}^{N} \underset{\sim}{z}_i'\underset{\sim}{z}_i = 1$, so the trace of $\underset{\sim}{X}$ is zero.

Then $\underset{\sim}{X}$ has a limiting normal distribution with mean $\underset{\sim}{0}$ and the same covariances as $\underset{\sim}{A}$. In the limiting normal distribution of $\underset{\sim}{X}$, the (functionally independent) off-diagonal terms are mutually (statistically) independent and independent of the diagonal terms. Let $\underset{\sim}{x} = (x_{11}, x_{22}, \ldots, x_{pp})'$ be the vector of diagonal terms; its covariance matrix is

$$\frac{2}{p^2(p+2)} \begin{pmatrix} p-1 & -1 & \cdots & -1 \\ -1 & p-1 & \cdots & -1 \\ \vdots & \vdots & & \vdots \\ -1 & -1 & & p-1 \end{pmatrix} = \frac{2}{p^2(p+2)} [p\underset{\sim}{I}-\underset{\sim}{\varepsilon}\underset{\sim}{\varepsilon}'] ,$$

where $\underset{\sim}{\varepsilon} = (1, 1, \ldots , 1)'$. This is a singular matrix and the limiting distribution of $\underset{\sim}{X}$ is singular. In order to deal with densities, we study first the latent roots of the matrix $\underset{\sim}{U} = \underset{\sim}{X} + v\underset{\sim}{I}$, where v is normally distributed with mean 0 and variance $2/[p^2(p+2)]$. From the density of the latent roots of $\underset{\sim}{U}$, we can obtain the density of any $p-1$ roots of $\underset{\sim}{X}$.

Let $\underset{\sim}{u} = (u_{11}, u_{22}, \ldots , u_{pp})' = \underset{\sim}{x} + v\underset{\sim}{\varepsilon}$. With v chosen with the above variance, the covariance matrix of $\underset{\sim}{U}$ is diagonal:

$$\mathcal{E}\underset{\sim}{u}\underset{\sim}{u}' = \frac{2}{p^2(p+2)} [p\underset{\sim}{I}-\underset{\sim}{\varepsilon}\underset{\sim}{\varepsilon}'] + \frac{1}{p^2(p+2)} \underset{\sim}{\varepsilon}\underset{\sim}{\varepsilon}' = \frac{1}{p(p+2)} \underset{\sim}{I} .$$

The variance of u_{kk} is $2/[p(p+2)]$, which is twice the variance of $u_{k\ell}$, $k \neq \ell$; the latter is the variance of $a_{k\ell}$, given above. The density of this limiting distribution of $\underset{\sim}{U}$ is then proportional to

$$\exp\left\{-\tfrac{1}{2}\left[\frac{\sum_{k=1}^{p} u_{kk}^2}{\frac{2}{p(p+2)}} + \frac{\sum_{k<\ell} u_{k\ell}}{\frac{1}{p(p+2)}}\right]\right\} = \exp\left\{-\tfrac{1}{2}\,\frac{p(p+2)}{2} \sum_{k,\ell=1}^{p} u_{k\ell}^2\right\} ;$$

that is, the density is

$$\frac{[p(p+2)]^{p(p+1)/4}}{2^{\frac{1}{2}p}\,(2\pi)^{p(p+1)/4}}\, e^{-\frac{1}{2}[p(p+2)/2]\operatorname{tr}\underset{\sim}{U}^2} .$$

If the roots of $\underset{\sim}{U}$ with the above density are $s_1 > s_2 > \ldots > s_p$, their density is [by Theorem 13.3.1 of Anderson (1958)]

$$\frac{[p(p+2)]^{p(p+1/4}}{2^{p(p+3)/4}\prod_{i=1}^{p}\Gamma[\frac{1}{2}(p-i+1)]}\exp\left\{-\tfrac{1}{2}[p(p+2)/2]\sum_{i=1}^{p}s_i^2\right\}\prod_{\substack{i,j=1\\i<j}}^{p}(s_i-s_j)\ ,$$

$$s_p \leq \dots \leq s_1\ .$$

Now the roots $r_1 \geq r_2 \geq \dots \geq r_p$ of $\underset{\sim}{X}$ are related to those of $\underset{\sim}{U}$ $(s_1 > s_2 > \dots > s_p)$ by the relation

$$x_i = r_i + v\ .$$

Since $\operatorname{tr} \underset{\sim}{X} = 0$, we have $\sum_{i=1}^{p} r_i = 0$; then $pv = \sum_{i=1}^{p} s_i$. So $v = (s_1 + \dots + s_p)/p$ and

$$r_i = s_i - v = \frac{p-1}{p} s_i - \frac{1}{p}\sum_{\substack{j=1\\j\neq i}}^{p} s_j, \qquad i=1, \dots , p-1\ .$$

Then

$$\sum_{i=1}^{p} s_i^2 = \sum_{i=1}^{p} r_i^2 + pv^2\ ,$$

where $r_p = -(r_1 + \dots + r_{p-1})$. The joint density of v and $r_1, \dots , r_{p-1}$ is

$$\frac{p^{p(p+1)/4}(p+2)^{p(p+1)/4-\frac{1}{2}}}{2^{p(p+3)/4-1}\prod_{j=2}^{p}\Gamma(\frac{1}{2}j)}\exp\left\{-\tfrac{1}{2}[p(p+2)/2]\sum_{i=1}^{p}r_i^2\right\}\prod_{\substack{i,j=1\\i<j}}^{p}(r_i-r_j)$$

(4)

$$\times\ \frac{p(p+1)^{\frac{1}{2}}}{2\ \pi^{\frac{1}{2}}}\ e^{-\frac{1}{2}[p^2(p+2)/2]v^2}, \quad r_p = -r_1 - \dots - r_{p-1} < r_{p-1} < \dots < r_1\ .$$

Thus v is independent of $r_1, \dots , r_{p-1}$, and the density of the latter set is the first factor. The density of the limiting distribution of the $p-1$ largest characteristic roots of $\sqrt{N}(\underset{\sim}{Q} - \frac{1}{p}\underset{\sim}{I})$ is the first factor and the density of the limiting marginal distribution of any subset of these characteristic roots is obtained as a marginal density from the factor. [Justification follows from the Theorem on Limiting Distribution, p. 140, Anderson (1963).]

The three-dimensional case. When $p=3$, the above gives the density of the limiting distribution of r_1 and r_2. Our interest is in r_1 and $r_3 = -r_1 - r_2$, and the density of the limiting distribution of these variables is

$$\frac{15^2 \sqrt{45}}{4 \sqrt{2\pi}} \exp\left\{-\frac{15}{2}\left[r_1^2 + r_3^2 + r_1 r_3\right]\right\} (2r_1 + r_3)(r_1 - r_3)(-r_1 - 2r_3)$$

$$-2r_1 \leq r_3 \leq -\tfrac{1}{2}r_1 \leq 0 \quad .$$

Integration with respect to r_3 gives the density of r_1 as

$$(5) \qquad f(r_1) = \frac{\sqrt{45}}{\sqrt{2\pi}}\left[\frac{135}{8} r_1^2 e^{-ar_1^2} - e^{-ar_1^2} + e^{-4ar_1^2}\right], \qquad 0 \leq r \leq \infty ,$$

where $a = 45/8$. The density of r_3 is the same but for $r_3 \leq 0$. The limiting distribution of r_2, found from above, is normal with mean 0 and variance $1/45$.

For calculations, it is convenient to write the distribution of r_1 in terms of $u = (r_1 \sqrt{45})/2$; then integration of (5) gives

$$\Pr(u > z) = \Pr(r_1 > 2z/\sqrt{45}) = \frac{3}{\sqrt{2\pi}} z e^{-z^2/2} + 2 - \Phi(z) - \Phi(2z) ,$$

where $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} dt$, and $z > 0$. Similarly,

$$\Pr(r_3 < 2z/\sqrt{45}) = \frac{3}{\sqrt{2\pi}}\left(-ze^{-z^2/2}\right) + \Phi(z) + \Phi(2z) , \qquad z \leq 0 .$$

The critical values* of r_1 , at significance level α , are $(r_1)_\alpha$:

α:	.10	.08	.05	.025	.01	.005	.002
$(r_1)_\alpha$:	0.788	0.816	0.873	0.948	1.038	1.100	1.177

Critical values of r_3 are $(r_3)_\alpha = -(r_1)_\alpha$. Then critical values $(S_{min})_\alpha$ and $(S_{max})_\alpha$ are, to order $N^{-\frac{1}{2}}$:

$$(6) \qquad (S_{min})_\alpha = \frac{1}{3} + \frac{(r_3)_\alpha}{\sqrt{N}} = \frac{1}{3} - \frac{(r_1)_\alpha}{\sqrt{N}} \; ; \quad (S_{max})_\alpha = \frac{1}{3} + \frac{(r_1)_\alpha}{\sqrt{N}} \; .$$

3.2. *A small-sample result.* Let $\underset{\sim}{Z} = (\underset{\sim}{z}_1, \ldots, \underset{\sim}{z}_N)$, where $\underset{\sim}{z}_i$ is defined above. Then $N\underset{\sim}{Q} = \underset{\sim}{Z}\underset{\sim}{Z}'$ and the latent roots of $\underset{\sim}{Z}\underset{\sim}{Z}'$ are the latent roots of $\underset{\sim}{Z}'\underset{\sim}{Z}$ except for 0 roots, equal in number to the difference in dimensionalities of $\underset{\sim}{Z}\underset{\sim}{Z}'$ and $\underset{\sim}{Z}'\underset{\sim}{Z}$. Then the (nonzero) latent roots of $N\underset{\sim}{Q}$ are the (nonzero) latent roots of

$$\begin{aligned} \underset{\sim}{Z}'\underset{\sim}{Z} &= (\underset{\sim}{z}_i' \, \underset{\sim}{z}_j) \\ &= \left(\frac{\underset{\sim}{y}_i' \, \underset{\sim}{y}_j}{\sqrt{\underset{\sim}{y}_i' \, \underset{\sim}{y}_i} \sqrt{\underset{\sim}{y}_j' \, \underset{\sim}{y}_j}} \right) \\ &= \underset{\sim}{R} \; , \end{aligned}$$

say. $\underset{\sim}{R}$ is an $N \times N$ correlation matrix based on p observations from $N(\underset{\sim}{0}, \underset{\sim}{I})$, using deviations from $\underset{\sim}{0}$, instead of the mean.

For the particular case $p=3$, if $N=2$ or 3 , $\underset{\sim}{R}$ has a density; if $N > 3$, $\underset{\sim}{R}$ does not have a density because its rank is 3 which is less than its dimensionality N . If $N=2$

* We are indebted to R. L. Anderson for carrying out these computations.

$$\underset{\sim}{R} = \begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}$$

and under the null hypothesis r has the uniform density over $(-1, 1)$. If $N=3$, the density of r_{12}, r_{13}, r_{23} is $(1/4)|\underset{\sim}{R}|^{-\frac{1}{2}}$ for $\underset{\sim}{R}$ positive definite and 0 otherwise. [See Anderson (1958), pp. 64 and 175. The number of degrees of freedom n is 3 because here deviations from 0 are used, instead of from the mean; the dimensionality p is N.]

When $N=2$, the characteristic roots of $\underset{\sim}{R}$ are $1+r$ and $1-r$. Thus $S_{max} = \frac{1}{2}[1 + \max(r, -r]$ and $S_{min} = 0$ because $\underset{\sim}{Q}$ is singular. The cdf of S_{max} is (for $\frac{1}{2} \leq x \leq 1$)

$$\Pr\{S_{max} \leq x\} = \Pr\{\max(r, -r) \leq 2x - 1\} = 2x - 1$$

and the significance level based on a test with significance point x is $2(1 - x)$. Some values are given below.

Table 3.1

Probability	.90	.95	.975	.99
Significance level	.1	.05	.025	.01
Significance point	.95	.975	.9875	.995

3.3. Significance Points. For smaller values of N, exact critical values of S_{min} and S_{max} will increasingly depart from the values in (6). These exact values have been estimated by Monte Carlo studies, using mostly 10,000 samples, each of size N, for $N = 4(1)12$, 14, 15, 16, 20, 30, 50, 60, 80 for S_{min}, and 10,000 samples, for $N = 5, 10, 20, 50$ for S_{max} and the middle root λ_2. The Monte Carlo points, for given α were plotted against $N^{-\frac{1}{2}}$; both the lower tail points of S_{min} , and the upper tail points of S_{max} , approach smoothly, from above, the asymptotic (6). Smoothed curves through these points give the points in Table 1. Analysis of these curves suggests that these percentage points are accurate to within one or two units in the third decimal place. For N beyond the table the expressions in (6) give a very good approximation. The critical values of S_{min} and S_{max} , for small N , are not symmetric with respect to $1/3$ as are the asymptotic values; this is to be expected as S_{min} has range 0 to $1/3$, while S_{max} has range $1/3$ to 1 . For the middle latent root λ_2 of $\underset{\sim}{Q}$, Monte Carlo values agree asymptotically with $0.3333 - (r_2)_\alpha/\sqrt{N}$, where r_2 is distributed $N(0, 1/45)$; for smaller N , the distribution becomes gradually unsymmetric. Some critical values are given in Table 2.

3.4. Power Studies. Table 3 gives an indication of the power of the test for randomness, when applied at level α , and when the sample is of size N and in fact drawn from an equatorial distribution with a given κ . Thus, for example, when the test is applied at the 5% level, with $N = 20$ and $\kappa = 2$, the power (i.e., probability of declaring a sample significant, i.e., not random) is 0.400 ; if the sample is of size 10 and $\kappa = 3$, the power is 0.300 .

Monte Carlo results were used to find the proportion of 1,000 samples (for $N < 20$) or 400 samples (for $N = 20, 30, 60$) which were significant when the test was applied as described. Curves of proportion significant were plotted against $1/N$, for given κ and α , and used to construct the table.

4. The Two-Dimensional Problem

It is interesting to observe that the distributions of S_{min} and of S_{max} can be solved exactly when the points P_i are on the circumference of a circle. Suppose θ_i is the polar coordinate of P_i referred to a suitable origin, and let the radius be 1; in its most general form, the distribution corresponding to (1) is

$$(7) \qquad f(\theta) = c_2(\kappa) \exp[-\kappa \cos^2 (\theta - \theta_0)] , \qquad 0 \leq \theta \leq 2\pi .$$

This distribution has two modes and two antimodes symmetrically placed round the circle; when $\kappa \leq 0$, $\theta = \theta_0$ and $\theta = \theta_0 + \pi$ gives the antimodes, and when $\kappa > 0$, they give the modes. A change in sign of κ is exactly equivalent to a rotation of coordinates through $90°$, so that no distinction can be made between the equatorial and bimodal cases, and a test of uniformity against the alternative (7) should give the same result whether, for the the alternative, κ is positive or negative. We can see that this is so as follows. The test statistic, when $\underset{\sim}{A}$ is not known, is

$$S_{min} = \frac{1}{N} \min_{\theta_0} \sum_{i=1}^{N} \cos^2 (\theta_i - \theta_0)$$

$$= \frac{1}{2N} \min_{\theta_0} \left[\sum_{i=1}^{N} \cos 2(\theta_i - \theta_0) + N \right] .$$

Suppose Q_i is the point whose angular coordinate is $2\theta_i$, i=1, 2, ... , N . Let $\underset{\sim}{R}$, of length R , be the resultant of the unit vectors $\underset{\sim}{OQ_i}$. Then $X = \sum_{i=1}^{N} \cos 2(\theta_i - \theta_0)$ is the component of $\underset{\sim}{R}$ on $\underset{\sim}{A}^*$, the vector whose angular coordinate is $2\theta_0$. Now R is fixed for a given sample; clearly X is minimized when A^* is along the direction opposed to $\underset{\sim}{R}$, so that $X = -R$, and

$$X_{min} = \tfrac{1}{2}(1 - R/N) . \tag{8}$$

Similarly $S_{max} = \tfrac{1}{2}(1 + R/N)$, when A^* lies along $\underset{\sim}{R}$. If the set of points P_i is uniformly distributed over the circle, so is the set Q_i, and $\underset{\sim}{R}$ is then the resultant of a set of randomly distributed unit vectors; the distribution of its length R is well-known (see, e.g., Greenwood and Durand (1955), and Stephens (1969) for critical values), so that critical values of S_{min} and S_{max} can be found exactly. Because of the identity $S_{min} + S_{max} = 1$, tests based on these statistics will give the same result, as noted above. When A is known, the test statistic is $S = N^{-1} \sum_{i=1}^{N} \cos^2 (\theta_i - \theta_0) = \tfrac{1}{2}(N-X)$; critical points for X/N are given in Stephens (1969), where the tests are given in greater detail. Power tables are also given.

If we apply to the circle the arguments given above for the sphere, we find

$$S_{min} = \tfrac{1}{2} - \frac{r_1}{\sqrt{N}} , \quad S_{max} = \tfrac{1}{2} + \frac{r_1}{\sqrt{N}} , \tag{9}$$

where r_1 is the characteristic root of the matrix $\underset{\sim}{X}$ defined in Section 3.1 for $p=2$. Then the density of the limiting distribution of r_1 is obtained from Section 3.1 as

$$8r_1 \, e^{-4r_1^2} .$$

It then follows that $8r_1^2$ has a limiting χ_2^2-distribution. It has also been shown that for N large, $2R^2/N$ has the χ_2^2-distribution, if we write (8) as

$$S_{min} = \tfrac{1}{2} - \frac{1}{\sqrt{8N}}\left(\sqrt{\frac{2R^2}{N}}\right),$$

we see at once the equivalence, for large N, of (8) and (9). However, (8) leads to exact distributions, and we have no parallel exact results for the sphere. It is clear that since the distribution of (ℓ, m, n) does not depend on the coordinate system, the exact distribution of the entries of X in (3) does not depend on the coordinate system and one must be able to write the density in terms of the latent roots, which will not change with an orthogonal transformation of coordinates; if this could be found exactly, Theorem 13.3.1 of Anderson (1958) could again be used, but the problem appears to be a difficult one.

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Table 1

LOWER TAIL PERCENTAGE POINTS FOR S_{min} AND UPPER TAIL POINTS FOR S_{max} ON SPHERE

N \ α(%)	S_{min}: lower tail 1.0	2.5	5.0	10.0	S_{max}: upper tail 10.0	5.0	2.5	1.0
5	0.007	0.011	0.019	0.031	0.714	0.751	0.784	0.821
6	.016	.024	.034	.050	.678	.712	.743	.779
7	.026	.038	.050	.067	.651	.684	.712	.746
8	.037	.051	.064	.081	.630	.661	.687	.718
9	.048	.062	.075	.093	.610	.641	.667	.694
10	.058	.073	.087	.105	.596	.625	.650	.677
12	0.076	0.091	0.106	0.123	0.574	0.598	0.621	0.648
14	.091	.107	.120	.137	.554	.578	.599	.623
16	.103	.120	.133	.149	.538	.559	.581	.604
18	.114	.131	.144	.158	.526	.544	.566	.587
20	.124	.140	.152	.167	.515	.535	.553	.575
25	.144	.158	.170	.184	.496	.512	.530	.550
30	0.160	0.172	0.184	0.196	0.479	0.495	0.510	0.528
40	.183	.192	.203	.214	.459	.473	.487	.501
50	.198	.207	.216	.227	.447	.460	.471	.484
60	.208	.217	.226	.235	.438	.449	.458	.470
70	.216	.226	.234	.243	.429	.439	.448	.461
80	.223	.231	.239	.248	.423	.432	.441	.452
100	.233	.242	.248	.257	.413	.422	.430	.440
b:	-1.038	-0.948	-0.873	-0.788	0.788	0.873	0.948	1.038

For $N > 100$, use the approximation $1/3 + b/\sqrt{N}$, where b is given in the last row.

Table 2

PERCENTAGE POINTS FOR λ_2 (MONTE CARLO)

N α(%)	Lower Tail 1.0	2.5	5.0	10.0	50.0	Upper Tail 10.0	5.0	2.5	1.0
5	0.130	0.159	0.183	0.211	0.304	0.390	0.408	0.428	0.444
10	.203	.221	.238	.255	.320	.379	.395	.409	.422
20	.246	.258	.269	.282	.325	.368	.380	.389	.399
50	.279	.286	.294	.302	.329	.356	.364	.370	.377

Table 3

POWER OF TEST FOR RANDOMNESS

The table shows the proportion of Monte Carlo samples significant, when tested for randomness with test of size α, when the sample of size N, were in fact drawn from an equatorial distribution with given κ.

	$\alpha = 0.05$				$\alpha = 0.01$			
κ:	1	2	3	4	1	2	3	4
N								
6	0.060	0.09	0.15	0.21	0.01	0.03	0.04	0.05
10	0.072	.16	.30	.49	0.01	0.05	.09	.27
15	.091	.27	.53	.75	.03	.11	.25	.49
20	0.120	.40	.68	.92	.04	.16	.40	.71
30	.200	.58	.89	.99	.05	.30	.76	.95
50	.35	.82	.99	1.00	.11	.73	.97	.99

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This paper is concerned with tests of randomness of directions in three-dimensional space or equivalently tests of uniform distribution of points on the unit sphere. One test is against alternatives which concentrate probability density near an equator, and the other test is against alternatives which concentrate probability density near opposite poles; in each case the poles are unspecified. The tests are based on the latent roots of the matrix of sums of squares and cross-products of the coordinates of the observed points on the unit sphere. Against equatorial alternatives the null hypothesis is rejected if the smallest root is less than the appropriate significance point, and against bimodal alternatives the null hypothesis is rejected if the largest root is greater than the appropriate significance points. Tables of significance are given, based on Monte Carlo studies and the asymptotic distributions, which are derived in the paper. The two-dimensional problem is also discussed.

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