Error Prediction for the Implicit Numerical Solution of the Heat Conduction Equation

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for the
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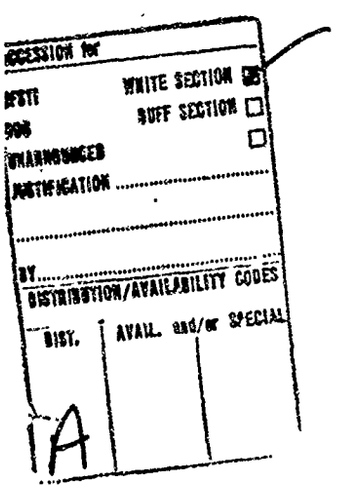

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NOMENCLATURE

C	Capacitance
g	Acceleration due to gravity
h	Film coefficient
k	Thermal conductivity
L	(1) Distance from surface to center of node, also (2) thickness of insulated slab
T	Transform variable
t	Temperature of body under consideration
t_f	Temperature of bounding fluid
X	Distance from surface to point under consideration in analytical solutions
α	Thermal diffusivity
ΔX	Thickness of node
ρ	Density
τ	Time
τ_1	Time at end of ramp

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FOREWORD

This report documents an investigation to fill an existing gap between the theory and the application of finite difference methods to transient thermal analyses. A methodology is developed that provides a means of predicting and modifying the analytical error associated with thermal response problems.

The work was performed at the University of Nevada during the period from September 1966 through September 1968 under two separate contracts with the Naval Weapons Center, China Lake, California. Contract N60530-67-C-0051 for the period from September 1966 to September 1967 was funded by the Bureau of Naval Weapons WepTask RMMO-42-008/216-1/F009-09-01 under the cognizance of W.K. Baker. Contract N60530-67-C-1278 for the period from September 1967 to September 1968 was funded by WepTask A32-320-008/216-1/F008-09-01 under the cognizance of W.C. Volz. The technical administrator of both contracts for the Naval Weapons Center was L.D. Schultz.

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CONTENTS

INTRODUCTION

Uncritical application of finite-difference procedures for solving transient aerodynamic heat-transfer problems has required 10 to 15 hours on the computer for each hour of actual flight. In addition, the complexity of problem, e.g., the variety in geometry, materials of construction, initial conditions, and boundary conditions, is such that an estimate of the error inherent in the analysis is difficult. Consequently, the introduction of errors of an intolerable magnitude into the problem's solution by adjustment of the spatial increment and/or time increment to minimize the excessive costs caused by long computer runs is difficult to avoid and even more difficult to evaluate. Of the many sources of potential significant error, such as inaccurate flight data, atmospheric conditions, aerodynamic heat transfer coefficient, and thermal properties, the error considered here is the analytical error due to the nature of the finite-difference approximation of the partial differential equations of transient thermal response. Because this error can be as great as 20% of the surface temperature rise, the necessity of maintaining the analytical error within acceptable limits is apparent.

The purpose of the investigation is to fill the existing gap between theory and practice by developing a methodology that provides the engineer with a means for predicting the analytical error associated with a specific thermal response problem. This ability allows the selection of spatial and time increments such that the minimum computation time is assured for a predetermined allowable analytical error limit. The benefits are two-fold: First, the analytical error can be controlled within limits governed by the problem under consideration; and second, the solution is obtained for the least cost.

A brief review of the implicit and explicit forms of the finite difference method for handling complex thermal transients would be in order at this point. In both methods, a system of equations is written describing the heat transfer processes taking place in a geometry which has been divided into discrete nodes. For each node, an energy balance is written: The sum of all forms of energy crossing the node boundaries are equated to the time rate of change of the heat capacity of the node. Thus, for both approaches, explicit and implicit, a transient thermal problem generates a large number of equations, specifically, one equation for each node in the system.

The set of these equations is to be solved at every time step for the temperature of each node in the system.

It is at this point that the difference between implicit and explicit becomes apparent. In the explicit or forward-difference method, the nodal temperature and the time rate of change of the node capacitance are referenced to the beginning of each time step. Thus, knowing all of the temperatures at the beginning of a time step for the entire system of nodes, one can predict the temperature each node will reach at the end of the time step. In this manner, each equation has a single unknown, the node temperature at the end of the time step. Thence comes the nomenclature explicit: each equation can be solved explicitly for its single unknown temperature.

The implicit approach expresses the nodal temperatures and references the time rate of change of the node capacitance to the end of each time step. This approach also results in a system of equations, one for each node in the system; but the individual equations in the system may contain several unknowns. Thus, the system is no longer explicit but is an implicit system in which the entire set of equations must be solved simultaneously.

In brief, the limitation for the explicit method is in the length of time step which may be taken before instability sets in. For stability, the length of the time step is a function of the thinnest dimension of any of the nodes in the system and, for practical problems, can be as small as thousandths of a second. This requirement can lead to an exceedingly large number of time steps to solve practical transient problems and can take an excessive amount of computer time. On the other hand, the implicit method has no limiting time step and has maximum stability compared to any of the other methods. A thorough discussion of the stability characteristics of the implicit and explicit methods is presented in the paper titled *The Stability of Three Finite Difference Methods of Solving for Transient Temperatures* by G.R. Gaumer. (See Entry 41, Appendix C).

In the one-dimensional problem shown by Fig. 1, the surface undergoes some form of a boundary condition change, and an energy balance is written for each of the nodes. The resulting system of equations in which all nodal temperatures are referenced to the end of the time step will form the implicit system. The next step is to solve the resulting matrix for the nodal temperatures either by a direct method such as the tridiagonal matrix solution or by an iterative method such as Gauss-Seidel. After determining the nodal temperatures at the end of a time step, a new set of equations is generated using the same principles and arriving at a second matrix which, in turn, will determine the temperatures at the end of the second time step. This process is repeated for each succeeding time step. In making comparisons of the effectiveness of several techniques for solving

the matrix, referred to previously, it was necessary to program the classical analytical solution for an infinite plate. This program, given in Appendix A for the convenience of the user, includes an eigenvalue generator.

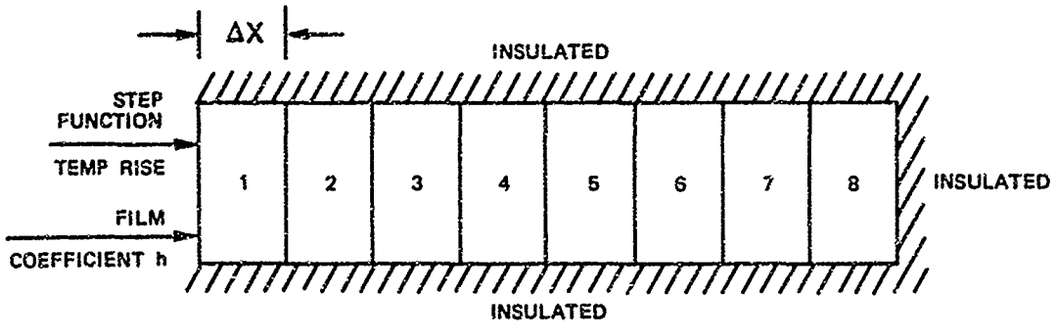

FIG. 1. Nodal Network for an Insulated Plate.

To improve accuracy and to provide rapid solutions, some form of compromise is necessary for intelligent use of the finite-difference method. In general but not under all conditions, the finer the subdivision of time steps in the time network or the finer the subdivision of geometrical spaces, i.e., the thickness of each layer, the closer the numerical analysis will come to approximating the analytical solution. However, the finer time network and the finer the spatial network for any given problem, the longer will be the computation time. In the explicit method round-off errors can become significant. In the implicit method, however, round-off errors are fairly well confined to an individual time step, and any error that is transmitted to the next time step is diffused among the nodes. If the user knows the effect a given spatial network will have in terms of spatial-truncation error and time-truncation error, a much more effective use of computer time may be made. Furthermore, acceleration techniques may be used to shorten the computer time needed. Round-off and truncation errors as well as acceleration techniques are discussed in Appendix B.

So that one can gain insight into the nature of the truncation errors, Fig. 2 shows a true temperature-distance plot for two adjacent nodes. Tangent to the true temperature curve are two lines, *True Slope 1-2* and *True Slope 2-3*. These lines represent the true slope of the temperature curve at the midpoint between node one and node two and at the midpoint between node two and node three. Compare these true slopes with the linear approximations that are shown as dashed lines in Figure 2; one sees that spatial-truncation error consists of the difference between the true slope and the linear approximation of the slope. Using this linear approximation of the temperature gradient results in an error for the rate of change of energy stored in the

node. This, in turn, causes an error in the prediction of the nodal temperature at the end of the time step.

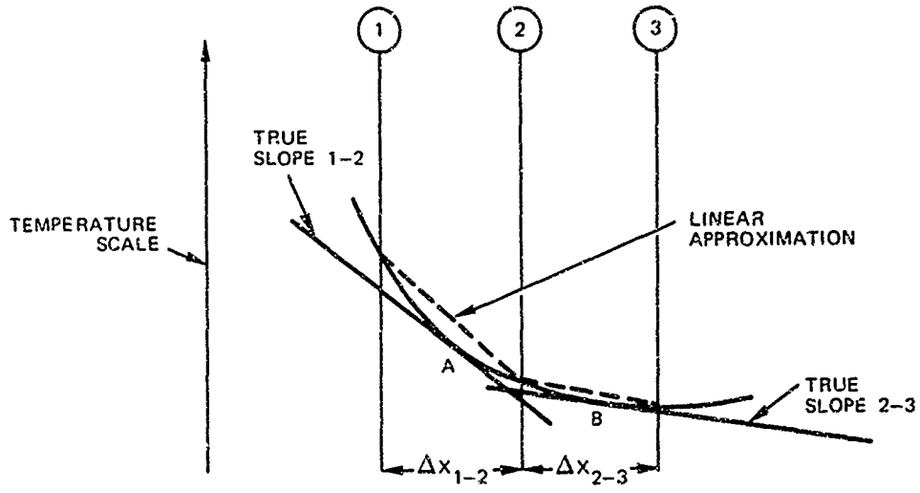

FIG. 2. Physical Concept of Spatial Truncation Error.

It has been found that by using the Fourier number and the Biot number, a correlation of the spatial truncation error may be made (Entry 7, Appendix C). In like manner, time truncation errors may also be correlated.

Commonly, the error analysis of transient problems is based on step-function boundary conditions; i.e., the driving potential undergoes an instantaneous change at the beginning of the transient. In actual aerodynamic heat transfer problems, step changes at the boundary are seldom realized; changes occur over time periods of significant duration. For this reason, ramp functions approximate the true boundary condition changes of practical aerodynamic problems more realistically than step functions do. Therefore, after the basic analytical error analysis is developed for step function changes at the boundary, the analysis is extended to determine the effect of duration and slope of ramp function boundary conditions on the spatial truncation error.

The summary of a literature survey, made in the initial months of this research effort, is included in this report as Appendix B. This literature survey compares a number of papers concerned with the various techniques for establishing iterative solutions, techniques for determining the truncation errors, both spatial and time, and techniques for accelerating iterative solutions. Appendix C is a bibliography on the general subject of numerical solution for transient heat transfer problems; Appendix D contains the same bibliography organized by categories. Appendix E is a derivation of the Laplace transform solution of the one-dimensional Fourier conduction equation for a semi-infinite solid (and for an insulated slab) with a ramp-function boundary condition.

MATRIX SOLUTION METHODS

Because the transient heat-transfer problems in this study are one-dimensional, a variety of solutions to the resulting matrix are available. For example, an iterative methods--Gauss-Seidel--and the various acceleration routines which may be used with it; the Runge-Kutta method; and a direct method utilizing the tridiagonal form of a matrix (TRIDAG) are available. The direct method bypasses the difficulty of convergence errors found in iterative solutions. As an example J.O. Wilkes of the University of Michigan reports on a solidification problem involving 11 nodes and 17 time steps for which the IBM 7090 has an execution time of 1.8 seconds. (See Entry 110, Appendix C.) Comparable problems using an iterative solution took an average of 18.0 seconds for execution.

For nonlinear problems, a simple direct solution such as TRIDAG is not applicable; therefore, a more general approach such as the accelerated Gauss-Seidel using a constant acceleration factor (usually referred to as over-relaxation) may be used. An alternative is to use Gauss-Seidel with an acceleration technique such as the Wegstein or the Steffensen methods.

A study of Steffensen's accelerating technique applied to a simple algorithm indicated possibilities. However, when this technique was used to accelerate the solution of a 20-node problem, the results were either the same as could be expected from the regular Gauss-Seidel or greater. The system did not seem to settle down even though the frequency of application of the technique was varied.

The Wegstein method requires more computer storage space and more computer operations than does the Steffensen method, but the Wegstein method effectively reduces the number of iterations needed for convergence. Appendix A presents Wegstein's basic equations and incorporates the Wegstein method into the ONE-D program for use with the Gauss-Seidel iteration technique. A FORTRAN IV listing of this program is given in Appendix A.

A note of caution: some acceleration techniques such as Steffensen or Wegstein work very well on the algorithm type of iterative solution, but when used on a system of equations, these techniques can actually slow down the rate of convergence. For comparison purposes, a 20-node

one-dimensional problem having a step input in the bounding fluid temperature was devised. Straight Gauss-Seidel using a 1/100-degree convergence limit took 46 iterations to determine the nodal temperatures at the end of the first time step. An unrestrained acceleration technique such as Wegstein's applied every third iteration caused such divergence that the automatic program stop of 100 iterations was reached. This difficulty was corrected by restricting the Wegstein technique only to positive values of acceleration. Only 17 iterations were required for convergence in a subsequent computer run using a combination of 10 initial iterations of Gauss-Seidel without over-relaxation, followed by the Wegstein acceleration method limited to positive values only, and then three additional iterations of standard Gauss-Seidel. As a further experiment, a run with 10 Gauss-Seidel over-relaxed iterations, followed by a Wegstein acceleration limited to positive values only, followed by standard Gauss-Seidel alternating with the Wegstein method took 20 iterations to converge the same problem. Mixing combinations of various acceleration techniques should be approached with cautior because some combinations actually increase the number of iterations over that required by a single technique.

SPATIAL-, TIME-, AND TOTAL-TRUNCATION ERRORS

One of the most confusing problems facing the engineer in attempting to utilize finite differences to solve transient heat-transfer problems is the determination of the spatial-network grid and the time interval to take in establishing a time network for dealing with a transient. In an effort to quantize this particular problem, the available background material was gleaned from the literature. (See Entry 7, 9 and 44 in Appendix B.) Truncation errors are caused by the strong second derivatives inherent in the beginning of the transient and near the surface of the geometry. The curves that are available have been found to be quite successful in predicting spatial- and time-truncation errors in problems consisting of homogeneous materials having uniform thicknesses of the individual slices. Some of these curves are given in Fig. 3, 4 and 5 in which the abscissa for the truncation-error curves are the Fourier number (the thermal diffusivity times the time span from the beginning of the transient up to the particular instant for which an error evaluation is desired, divided by the square of the distance from the surface to where the error is being evaluated). This family of curves is correlated by the Biot number (the film coefficient at the surface of the geometry, times the distance from the surface down to the point in question, divided by the thermal conductivity of the material). The measurement of the error itself is done in terms of the step-function temperature rise in the bounding fluid at the beginning of the transient. It is of interest to note in Fig. 3 that when the thickness of the slice is reduced, which is proportional to L for the first node, the Fourier number and the Biot number are both affected such that the spatial-truncation error is reduced. However, it may be noted also that it is possible in a Fourier number range from approximately 0 to 1.0 to reduce the thickness of a slice and have the spatial-truncation error increase.

With respect to variations in Biot number, one can easily see that the worst possible case occurs when the film coefficient at the surface is infinite, and the errors can range as high as 13% of the temperature rise in the bounding fluid. However, with decreasing values of the film coefficient, reflected in decreasing values of Biot number, the spatial-truncation errors are in turn decreased. As a result it is possible for certain classes of problems, that the spatial-truncation error will never get above 1 or 2% regardless of the thickness of the slice.

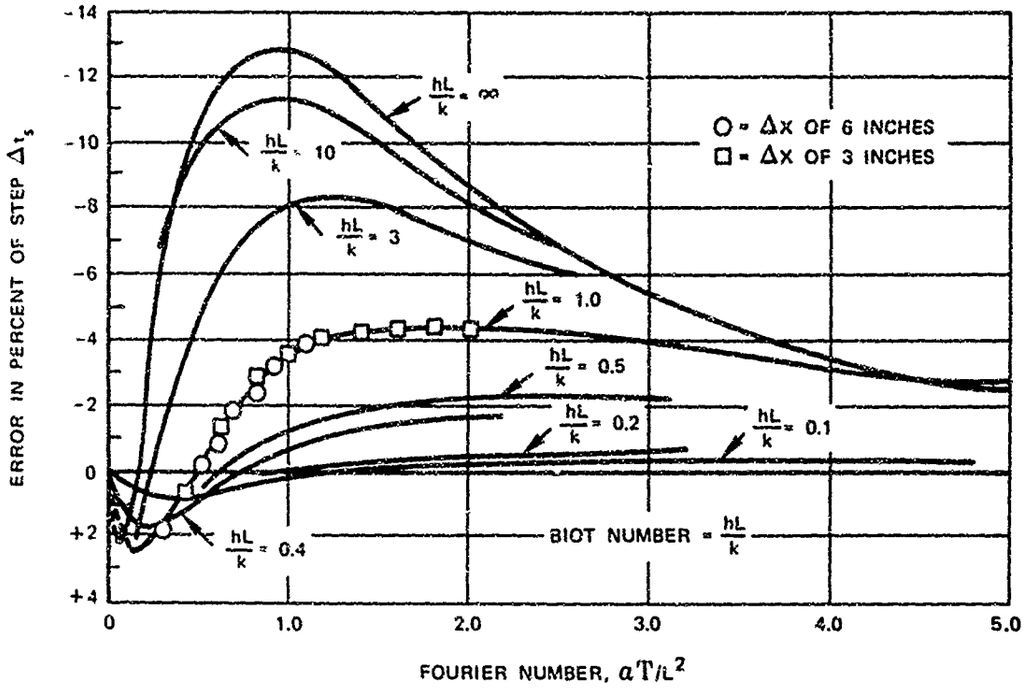

FIG. 3. Spatial-Truncation Error for Nodes Adjacent to the Surface.

In like manner, time-truncation errors can be evaluated using the curves shown in Fig. 4. Once again, the Fourier number is used as the abcissa for the curves, the Biot numbers for the family of curves, and the errors are expressed on the ordinate as a percentage of the step-function temperature change in the bounding fluid at the beginning of the transient. Once again, we see the effect of the Biot number; i.e., as the Biot number becomes smaller, the errors become smaller. In a final evaluation, one sees in Fig. 5 that the errors are accumulative and that the total error under the worst possible set of circumstances could be as high as 20% of the step function. One important difference between the time-truncation-error curves and the spatial-truncation-error curves should be realized. The time used in the Fourier number for the time-truncation error curves is the time interval between the beginning of the transient and the first evaluation of temperature in the problem; i.e., the first time step. The time in the Fourier number for the spatial-truncation error curves refers to the total time interval from the beginning of the transient to the time at which an error evaluation is made, regardless of how many time steps elapsed. This means that the time- and spatial-truncation errors may be superimposed only for the first time step. It is also important to realize that the distance, L, in the dimensionless groups of Fourier and Biot numbers would be half the thickness of the node slice when considering the first node adjacent to the surface and, subsequently, would be

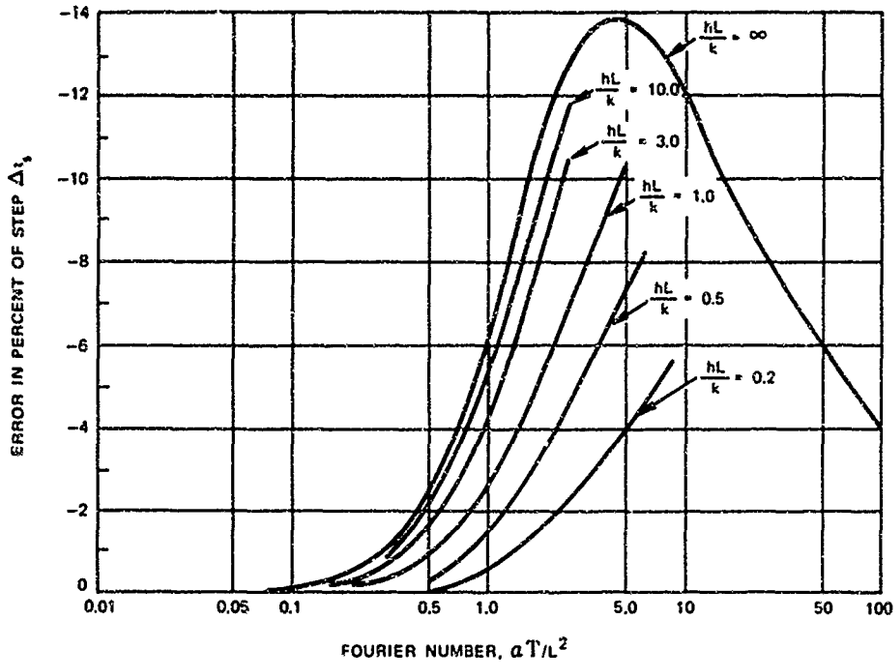

FIG. 4. Time-Truncation Error Versus Fourier Number for the First Node in Homogeneous Slab.

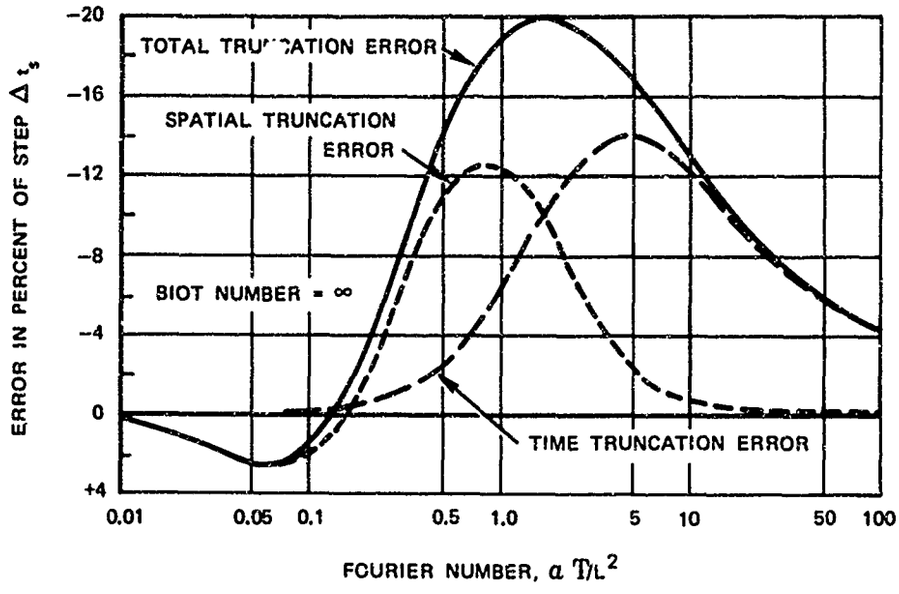

FIG. 5. Truncation Error Versus Fourier Number for First Node-First Time Step in Homogeneous Slab.

three halves the thickness of one node when considering the errors for the second node from the surface, five halves of the node thickness when dealing with the third node from the surface, etc.

When dealing with problems in which layers of nonhomogeneous materials are being considered, or in which a ramp function instead of a step function occurs on the surface, or when a nonflat plate geometry is encountered, the truncation errors are no longer exact. However, in the practical sense, valuable guidance may be obtained from these curves.

In considering the effect of successive nodes on spatial-truncation errors going into depth from the surface in a homogeneous flat plate, the trend is, in all cases, for the greatest truncation error to occur in the node adjacent to the surface; and with each succeeding node, the truncation error is reduced (Fig. 6). Generally, this reduction in spatial-truncation error is quite drastic; by the time the third or fourth node is reached, the truncation errors are of the order of magnitude of 1% or less. One can see that time-truncation errors are rapidly reduced if two or more time steps are utilized to complete the first time interval (Fig. 7, 8, and 9). Also the time-truncation error becomes less for successive nodes in depth (Fig. 10).

It is concluded that if the first time step is sized to keep the time-truncation error within reasonable limits in a homogeneous geometry, the time-truncation error for the nodes below the first node will automatically have lower error levels.

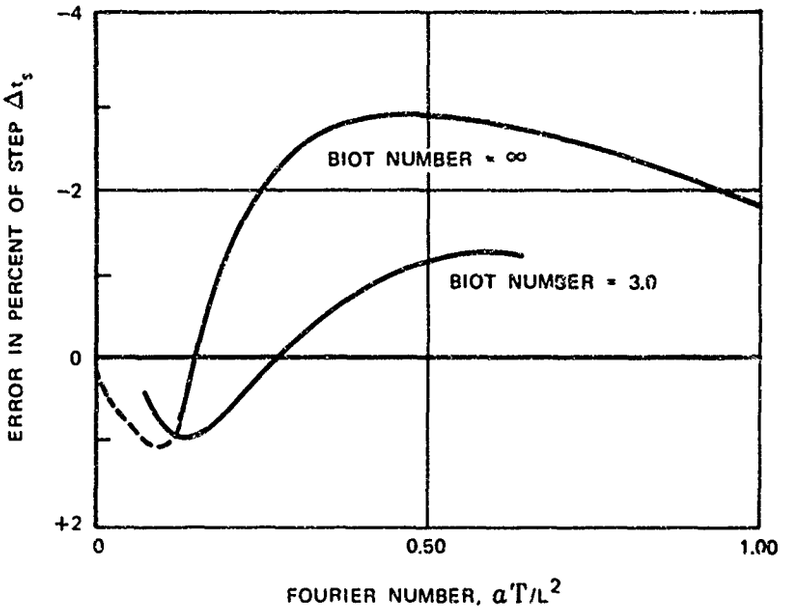

FIG. 6. Spatial-Truncation Error for Second Node From Surface.

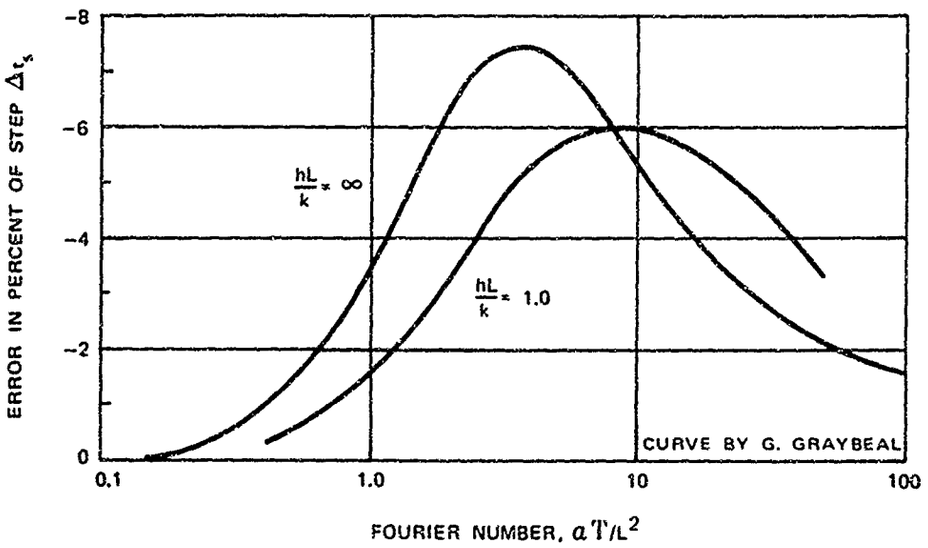

FIG. 7. Time-Truncation Error for Nodes Adjacent to the Surface; Two Time-Step Numerical Solution.

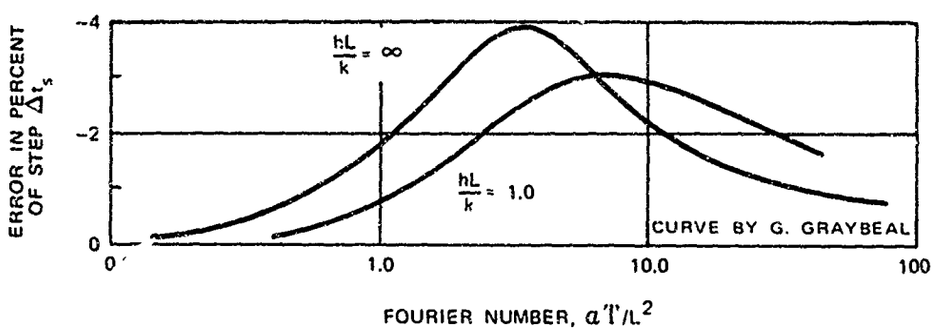

FIG. 8. Time-Truncation Error for Nodes Adjacent to the Surface; Four Time-Step Numerical Solution.

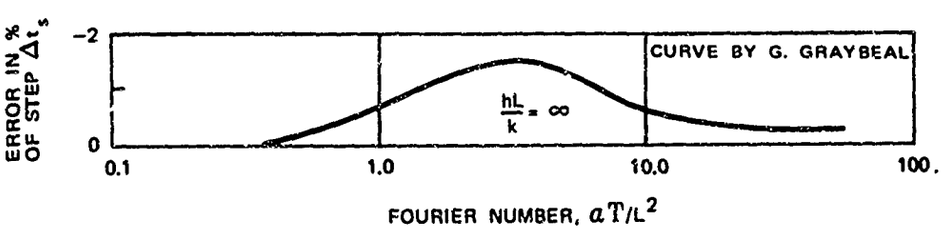

FIG. 9. Time-Truncation Error for Nodes Adjacent to the Surface; Ten Time-Step Numerical Solution.

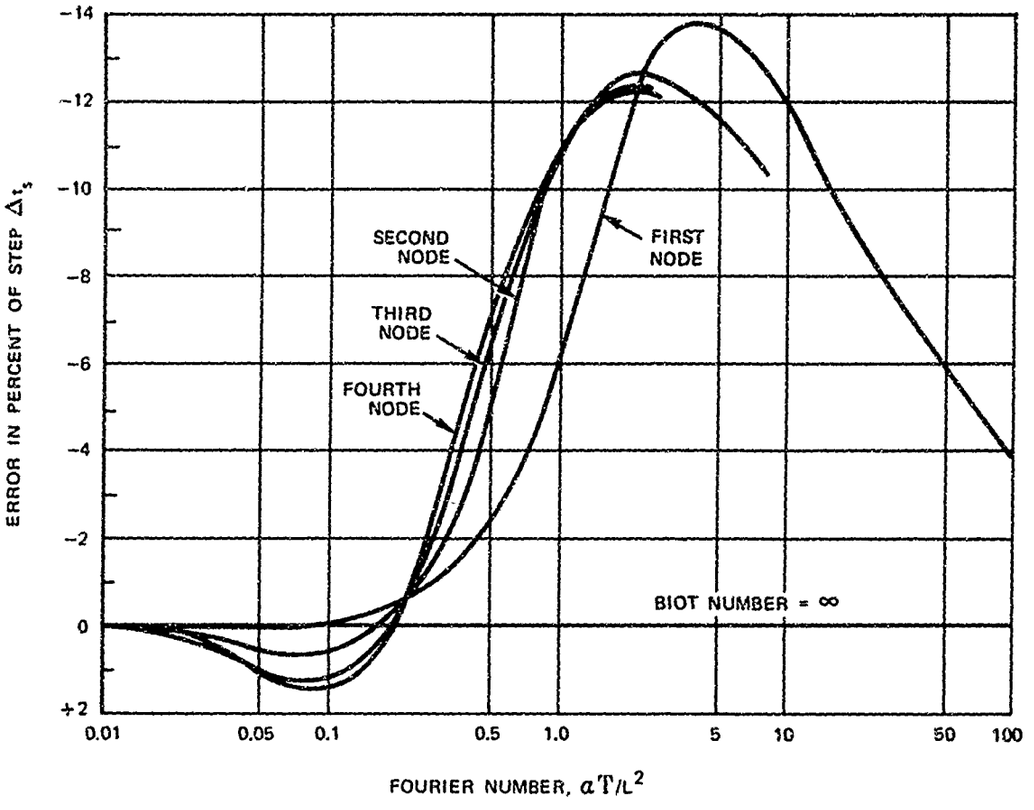

FIG. 10. Time-Truncation Error for Several Nodes.

NUMERICAL ERRORS IN THIN-THICK GEOMETRIES

A recurring type of problem of considerable interest is the transient thermal analysis of a two-layer geometry. The two layers consist of a thin, highly conductive layer, such as aluminum, exposed on one side to a bounding fluid and on the other side to a thick layer of low-conducting material acting as an insulator. The questions to be resolved are What are the numerical errors inherent in such a geometrical system? and What procedure might be followed to keep the errors to an acceptable level?

The truncation-error curves were prepared in much the same way as those presented by Graybeal (Entry 44, Appendix C). However, a somewhat different scheme was employed to obtain a reference solution. Whereas an analytical solution was employed directly by Graybeal as an errorless reference, such a solution was used indirectly here. This indirect reference solution was taken from a tabulation of an

analytical solution to the one-dimensional conduction problem involving a two-layer plate.[1] The tables were not used directly because the results listed for each set of parameters present a limited coverage of the time-space field, necessitated by a desire to offer solutions for a large number of combinations of parameters, yet keep the report from being excessively bulky. To obtain a usable reference solution, a set of parameters was chosen corresponding to one section of the table ($\alpha = 0.05$); a finite-difference solution was prepared using these parameters and also using very small ΔT and ΔX. This solution was then compared to the tabulated solution at points of correspondence. A truncation error of less than 0.1% (based on the size of the step change of temperature in the bounding fluid) was found at each point checked, and this finite-difference solution was then used as the standard for evaluating truncation error in other finite-difference approximations employing larger ΔT and ΔX. First, spatial-truncation error only was introduced by increasing ΔX and holding ΔT at the original small value, then evaluating spatial-truncation error for a given ΔX by comparison with the reference solution. Then, with ΔX held at the same value used in the reference solution, ΔT was varied to give a solution containing time-truncation error alone.

The problem considered here is one-dimensional conduction in a two-layer composite slab heated on one face by convection from a fluid which undergoes a step change of temperature. The fluid and slab are initially at the same uniform temperature, and the face of the slab not in contact with the fluid is perfectly insulated. All physical properties are assumed constant.

Two cases were studied: in one, the thickness of the thin layer next to the fluid was 5% of the total thickness of the slab; in the other, the thickness of the thin layer was 20%. The physical properties of this layer correspond roughly to those of aluminum, and the properties of the thick layer are similar to those of some common insulating materials:

Properties	Thin layer	Thick layer
k, BTU/(hr-ft-°F)	100	0.1
ρ, lb/(ft^3)	200	20
C, BTU/(lb-°F)	0.25	0.25

[1] Naval Ordnance Test Station. *Temperature Tables. Part 7, Vol. 1 and 2. Two-Layer Plate, One-Space Variable, Linear,* by H. N. Browne, Jr. and C. J. Thorn. China Lake, Calif., NOTS, 1 March 1960. (NAVORD Report 5562, Part 7; NOTS TP 2182, Vol. 1 and 2.)

RESULTS

The spatial-truncation errors for the first node of the first layer and the first and second node of the second layer for the thin-thick geometry are plotted in Fig. 11.

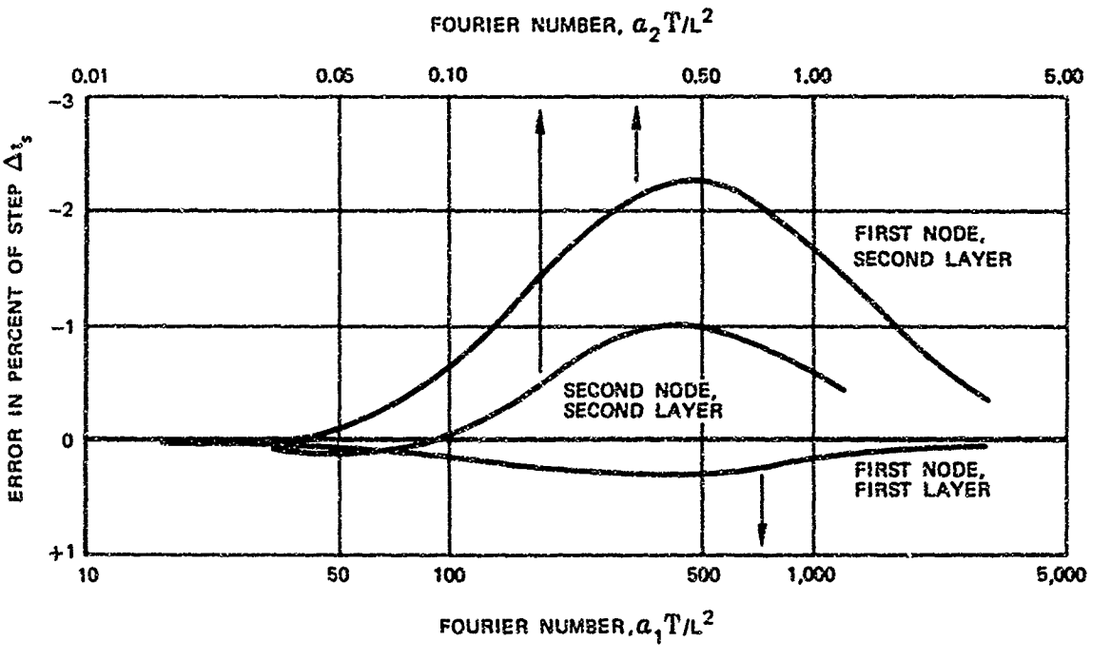

FIG. 11. Spatial-Truncation Error Vs. Fourier Number for Selected Nodes in Thin-Thick Configurations.

These curves contain spatial-truncation errors for the worst case considered: where the thin layer is represented by a single node, and all nodes are the same size. The maximum spatial-truncation error in the thin layer is 0.3%, while the maximum at the first node of the second layer is -2.3%. This drops to a maximum of -1.0% at the second node of the second layer.

It may be noted that the errors plotted in Fig. 11 are much smaller than the errors encountered in the curves describing truncation errors in homogeneous slabs: for example, a film coefficient of 1,000 and a thermal conductivity of 25.

In the thin-thick case, approximations of real problems always results in very small Biot numbers. For example, a 1/8-inch thick layer of steel could, in the limit, form a single node and under extreme circumstances, give rise to a Biot number of 0.2 (h - 1,000, k = 25, L = 0.0625). The spatial-truncation error associated with these small values of Biot number is less than 1% (Fig. 3).

To sum up then, the curve in Fig. 11 labeled first node of the first layer having only a fraction of a percent error certainly fits the trend discovered in homogeneous slabs. In a similar fashion the curve labeled first node second layer, the Biot number has a value of 7.5 based upon again a film coefficient of 50 and the thermal conductivity of 0.1, which represents the insulation involved. Once again, comparing this with the results for homogeneous slabs in Fig. 3, one sees that the homogeneous slab would have spatial-truncation errors of approximately 10% for this Biot number; whereas in the thin-thick case, the maximum error was under 2 1/2%. This is a logical extension of the homogeneous material problem because the aluminum layer on the surface acts as a buffer and tends to protect the insulating layer from the extreme spatial-truncation errors that would occur if the insulation were directly exposed to the bounding fluid. With reference to the third curve in Fig. 11 (that is, the second node of the second layer curve), its relationship to the first node of the second layer is approximately the same as second nodes normally have in homogeneous materials, as shown in Fig. 6. Thus, the data in Fig. 11 follows the same trends as the data for homogeneous slabs.

The time-truncation errors for the thin-thick class of problem are plotted in Fig. 12. The ordinate in Fig. 12 is the usual error measured in percent of the step-function temperature change in the bounding fluid, but the abscissa is the percent of the boundary temperature step-function change that equals the temperature change in the first node of either the first or second layer that occurs in the first time step. The time-truncation errors for a similar case involving a homogeneous slab are also displayed in Fig. 12.

It is surprising to find that the thin-thick layer has somewhat larger time-truncation errors than the homogeneous slab and at the present time, no explanation is available. The curves in Fig. 12, however, provide a very practical guide for those who are concerned with time-truncation errors because the temperature response of the first node of either the thin or the thick layer can be expressed as a percent of the boundary temperature step-function change. A reasonable criterion for limiting the time-function error at the beginning of a transient is to ensure that the temperature change of the first node is not greater than 20% of the step-function boundary temperature change during the first time step. At the 20% level or below, all of the time-truncation errors are less than 2%.

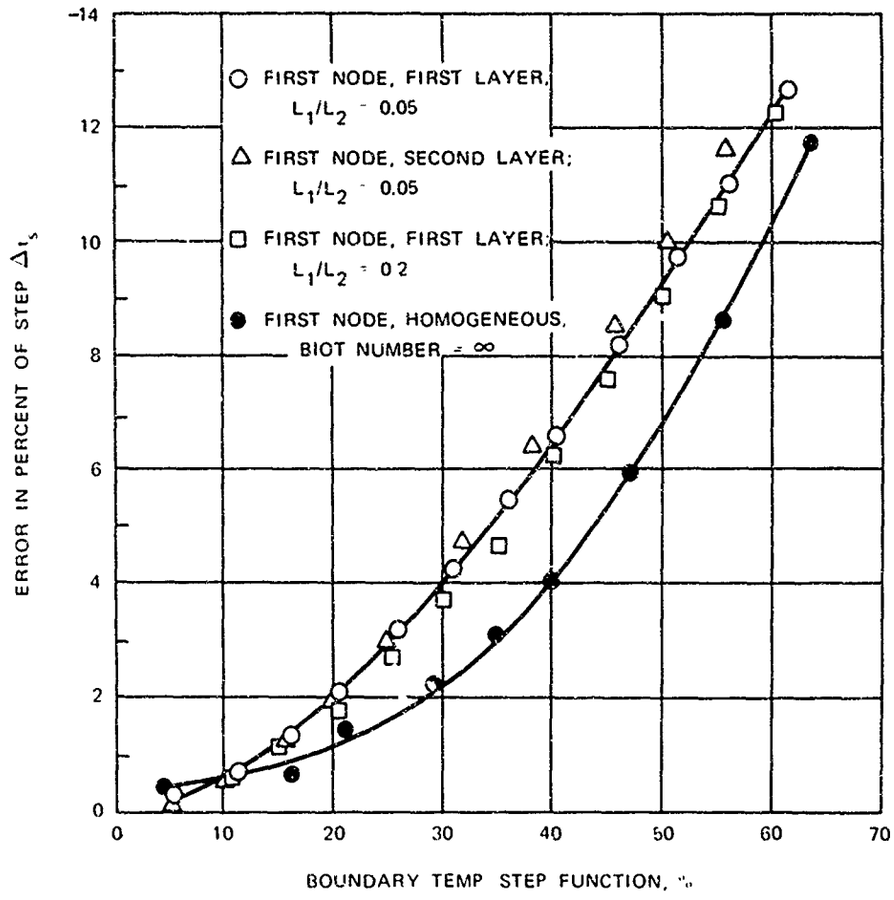

FIG. 12. Time Truncation Error Vs. Temperature Rise in Designated Nodes Expressed as a Percent of Boundary Temperature Step Function.

RAMP-FUNCTION SPATIAL-TRUNCATION ERRORS

Thus far, only errors for step-funccion boundary temperature changes have been considered. Since the ramp function is a closer approximation to the actual bounding-fluid temperature variations, such mathematical models of transient heat-transfer situations are more realistic, and the system response should result in a lower spatial-truncation error.

As a starting point, two approaches were used to determine the true temperature for use as a comparison basis for determining spatial-truncation errors. The first approach employs successively smaller and smaller time steps with smaller and smaller geometrical divisions in numerical solution for a transient one-dimensional homogeneous heat transfer problem. Thus an asymptotic approach to the true temperature distribution was obtained. The other approach uses the Laplace transform method in a classical analytical solution of the one-dimensional Fourier transient-conduction equation. Both approaches are successful, and a comparison of the results of the successive approximation method with the results from the analytical method is given in Table 1.

TABLE 1. Comparisons of Numerical & Analytical Solutions for Ramp Functions Boundary Conditions.

These data were generated by a problem having the following specifications: Semi-infinite solid, 2500°F/sec on bounding fluid. $\Delta X = 0.006$ in; $\Delta T = 0.0002$ sec; $h = 10^6$; $\alpha = 0.2$.

Time, Sec	Temperature, °F					
	Node 1		Node 2		Node 3	
	Analytic	Numerical	Analytic	Numerical	Analytic	Numerical
0.02	37.919	36.787	20.814	20.178	10.658	10.391
0.04	82.373	81.159	54.649	53.798	35.057	34.523
0.06	128.111	126.861	92.090	91.140	64.752	64.076
0.08	174.516	173.245	131.449	130.439	97.400	96.634
0.10	221.348	220.348	172.049	170.997	131.999	131.169

The Laplace transform approach is summarized below; the complete derivation is given in Appendix E.

The partial differential equation for a one-dimensional transient problem is

$$\frac{\partial t}{\partial T} = \alpha \frac{\partial^2 t}{\partial X^2} \tag{1}$$

where

t = temperature

T = time

The initial conditions are at T = 0, t = 0. From the Laplace transform of Eq. 1,

$$\frac{d^2 T}{dX^2} - \frac{s}{\alpha} = 0. \tag{2}$$

A general solution of this second order differential equation is

$$T = A \exp(\sqrt{s/\alpha}\, X) + B \exp(-\sqrt{s/\alpha}\, X) \tag{3}$$

where A and B are constants. Since a semi-infinite geometry is under consideration, A = 0. Constant B may be found from the boundary condition.

The bounding fluid temperature change is a ramp function (Fig. 13).

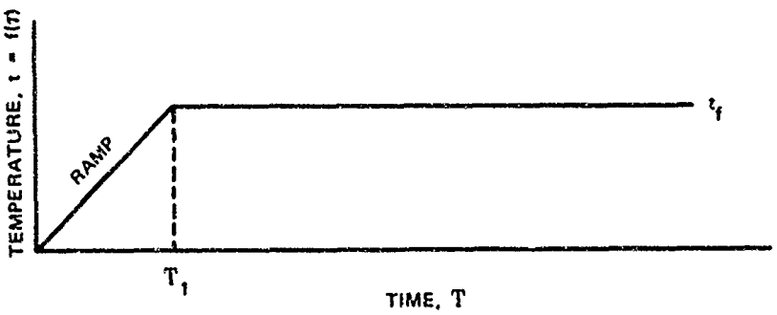

FIG. 13. Bounding Fluid Temperature Change.

At the surface, X = 0.

$$-k \frac{\partial t(0, T)}{\partial X} = h[f(T) - t(0, T)]. \tag{4}$$

If the Laplace transform of Eq. 4 is taken and if $\frac{dT}{dX}$ (0,s) and T(0,s) are evaluated from Eq. 3 the constant B may be expressed as

$$B = \frac{ht_f}{kT_1} \left[\frac{1 - \exp(-T_1 s)}{s^2(\sqrt{s/\alpha} + h/k)} \right] \tag{5}$$

Thus,

$$T = \frac{ht_f}{kT_1} \left\{ \frac{\exp(-\sqrt{s/\alpha}\, X) \left[1 - \exp(-T_1 s)\right]}{s^2(\sqrt{s/\alpha} + h/k)} \right\} \tag{6}$$

The inverse transform of Eq. 6 is

$$t = \frac{t_f}{T_1} \left\{ \left(T_1 + \frac{X^2}{2\alpha} + \frac{Xk}{\alpha h} + \frac{1}{\alpha} \left(\frac{k}{h}\right)^2 \right) \operatorname{erfc}\left(\frac{X}{2\sqrt{\alpha T}}\right) \right.$$

$$- \frac{1}{\alpha} \left(\frac{k}{h}\right)^2 \exp\left[\frac{h}{k} X + \left(\frac{h}{k}\right)^2 T\right] \operatorname{erfc}\left(\frac{X}{2\sqrt{\alpha T}} + \frac{h}{k}\sqrt{\alpha T}\right)$$

$$\left. - \sqrt{\frac{T}{\pi\alpha}} \left(X + \frac{2k}{h}\right) \exp\left[-\left(\frac{X}{2\sqrt{\alpha T}}\right)^2\right] \right\}, \quad 0 < T \le T_1 \tag{7}$$

$$t = \frac{t_f}{T_1} \left\{ \left[T + \frac{X^2}{2\alpha} + \frac{Xk}{\alpha h} + \frac{1}{\alpha} \left(\frac{k}{h}\right)^2 \right] \operatorname{erfc}\left(\frac{X}{2\sqrt{\alpha T}}\right) \right.$$

$$- \frac{1}{\alpha} \left(\frac{k}{h}\right)^2 \exp\left[\frac{h}{k} X + \alpha\left(\frac{h}{k}\right)^2 T\right] \operatorname{erfc}\left(\frac{X}{2\sqrt{\alpha T}} + \frac{h}{k}\sqrt{\alpha T}\right)$$

$$- \sqrt{\frac{T}{\pi\alpha}} \left(X + \frac{2k}{h}\right) \exp\left[-\left(\frac{X}{2\sqrt{\alpha T}}\right)^2\right] \left. \right\} - \frac{t_f}{T_1} \left\{ \left[(T - T_1) + \frac{X^2}{2\alpha} \right.\right.$$

$$\left. \left. + \frac{Xk}{\alpha h} + \frac{1}{\alpha} \left(\frac{k}{h}\right)^2 \right] \operatorname{erfc}\left[\frac{X}{2\sqrt{\alpha (T - T_1)}}\right] + \frac{1}{\alpha} \left(\frac{k}{h}\right)^2 \right.$$

$$\cdot \exp\left[\frac{hX}{k} + \alpha\left(\frac{h}{k}\right)^2\left(T - T_1\right)\right] \operatorname{erfc}\left[\frac{X}{2\sqrt{\alpha\,(T-T_1)}} + \frac{h}{k}\sqrt{\alpha\,(T-T_1)}\right]$$

$$+\sqrt{\frac{(T-T_1)}{\pi\alpha}}\left(X + \frac{2k}{h}\right)\exp\left(-\left[\frac{X}{2\sqrt{\alpha\,(T-T_1)}}\right]^2\right)\Bigg\}, \quad T > T_1 \qquad (8)$$

Note that since there is a discontinuity at time T_1, there are two solutions: one for T greater than zero, but equal to or less than T_1; and another for T greater than T_1. It may be further noted that at time greater than T_1, the first portion of the equation is the same as the equation for time less than T_1; and that the second portion of the equation is similar to the first portion except T is replaced by $(T- T_1)$. This, in effect, applies a negative slope at time T_1 as is illustrated in Fig. 14.

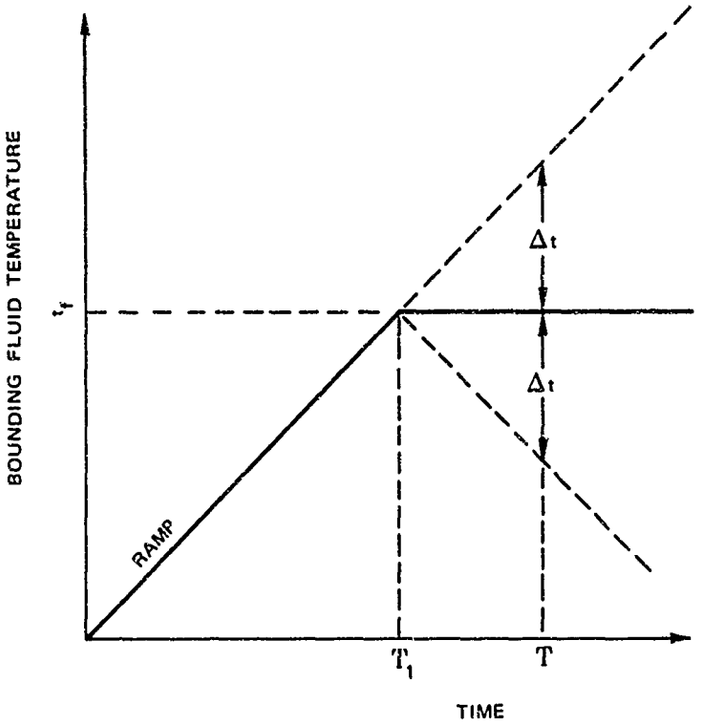

FIG. 14. Synthesis of Analytical Solution

Figure 14 illustrates the concept of subtracting Δt, which develops after T_1, from fluid temperature which would have developed after T_1 if the ramp had continued.

The effect is that the first portion of Eq. 8 will produce node temperatures analogous to an extension of the ramp after T_1 and the last half of Eq. 8 will produce a Δt generated in the nodes after T_1.

In similar manner, the analytical solution for the case of a ramp-function temperature change in the bounding fluid adjacent to the surface of an insulated slab was derived. The final equation for the insulated slab is

$$t = \frac{t_f}{T_1}\left\{T + \frac{4L^2}{\alpha}\sum_{n=1}^{\infty}\frac{\left[\exp\left(-\lambda_n{}^2\frac{\alpha T}{L^2}\right)-1\right]\sin\lambda_n\cos\left(\lambda_n\frac{X}{L}\right)}{\lambda_n{}^2\left(2\lambda_n+\sin 2\lambda_n\right)}\right\} \quad (9)$$

A complete derivation of Eq. 9 is given in Appendix E.

Computer programs were coded in FORTRAN for both of the ramp-function analytical solutions, the semi-infinite solid and the insulated slab. FORTRAN IV listings for these two programs are included in Appendix A of this report. These analytical solutions were used to calculate the temperatures for comparison with the temperatures for equivalent ramp-function problems solved by numerical methods. The differences between results of these two methods in terms of temperature were the spatial-truncation errors sought. The FORTRAN IV listing for the numerical method computer program--coded such that either step-function temperature rise in the bounding fluid or a ramp-function temperature rise in the he bounding fluid can be specified--is given in Appendix A. The attempt to make the same type of correlation that was successful in the step-function-error curve analysis failed, as shown in Fig. 15. That is, the assumptions that the errors could be correlated by the overall temperature rise in the fluid and that a further correlation would be available based upon the value of the Biot number is incorrect; the Biot number does not correlate the spatial-truncation error for ramp-function boundary conditions. Figure 16 shows that to correlate various ramp functions for the same Biot number using the overall temperature rise in the bounding fluid as the basis for percent error is not successful.

A correlation using the current fluid temperature rise as the basis for establishing the percent spatial-truncation error causes all the ramp-function errors to fall on the same curve (Fig. 17). The data presented in Fig. 17 were obtained from runs using a *long* ramp. A long ramp is one in which the maximum spatial-truncation error occurs well before the end of the ramp is reached. Of interest in Fig. 17 is the maximum error of about -9% which occurs at a Fourier number of about 2.0. These values are to be compared with the -13% error at a Fourier number of about 0.9 for the step function boundary condition. Maximum error is reduced because of the reduced rate of change at the model boundary, and the Fourier number is increased due to the increased time to reach terminal fluid temperatures.

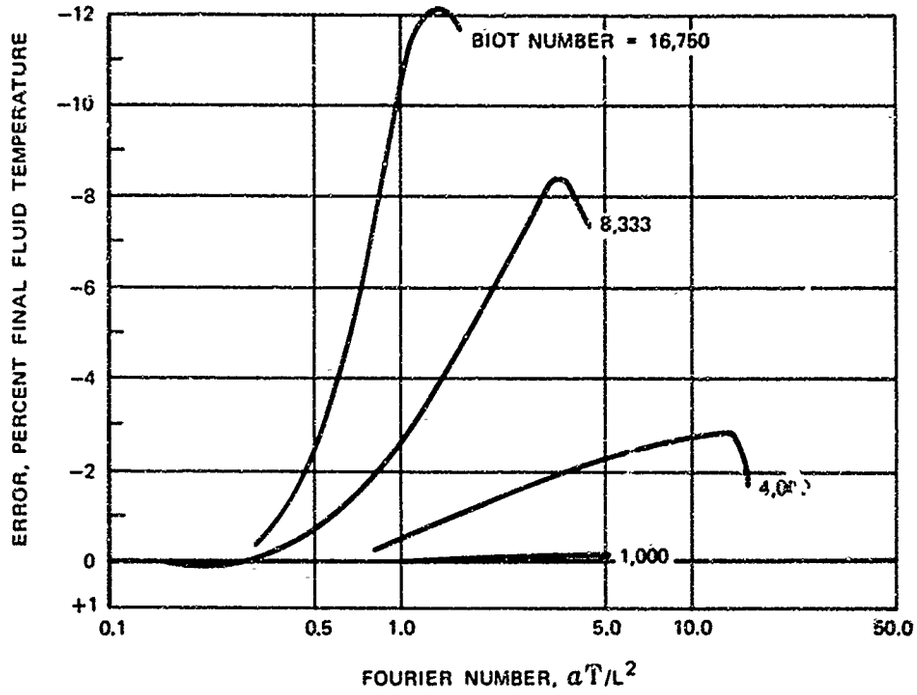

FIG. 15. Spatial-Truncation Error Vs. Fourier Number, Giving Various Biot Numbers.

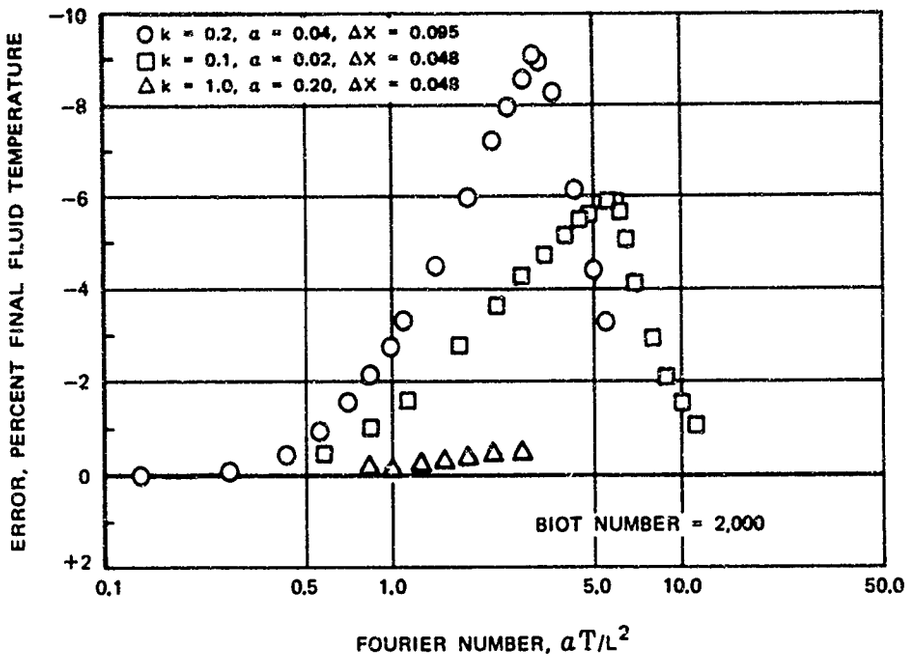

FIG. 16. Spatial-Truncation Error Vs. Fourier Number.

Using the current fluid temperature approach, an investigation was made in which the ramp temperature rise in the bounding fluid was completed before any spatial-truncation errors could develop in the first node. This situation is termed an *ultra-short* ramp. The ultra-short ramp curve is shown in Fig. 18; the dotted line represents the equivalent spatial-truncation error curve for a step-function temperature rise in the bounding fluid. Again, the ramp-function error curve is slightly delayed compared to the step-function error curve.

A further study, graphed in Fig. 19, shows the result when the end of the ramp occurs during the development of the spatial-truncation error, termed a *short* ramp. The arrow in Fig. 19 indicates the time of which the end of the ramp occurs; after the end of the ramp, a typical step-function curve develops. Thus Fig. 19 is an excellent example of how the nodal system shifts from a ramp-function error response to a step-function error response and how the Fourier number for the maximum error is increased due to the ramp-function effect.

In Fig. 20, the arrows again indicate the Fourier number corresponding to the end of the ramp temperature rise in the bounding fluid. For these curves the end of the ramp occurs shortly after the maximum spatial truncation error occurs. It may be seen that after the end of the ramp, the individual runs exhibit small increases in error showing a residual tendency to respond in a step function fashion.

These combined data on one set of coordinates are shown in Fig. 21, the dotted line represents the equivalent step-function error curve and the solid lines represent *short*-ramp functions and *long*-ramp functions. Of interest are the bumps or departures (dashed lines) visible in the short-ramp functions. These departures represent a progression of error increases that occur when the ramp portion of the rise in the bounding temperature ceases before the maximum spatial-truncation error occurs. At the end of the ramp, the error rapidly climbs toward the levels exhibited by a step-function rise in bounding fluid temperature. In this figure the long-ramp function (a ramp in which the maximum spatial-truncation error occurs while the ramp is still in progress) properly defines the error expressed as percent of current temperature rise in the bounding fluid. For low Biot numbers, it makes little practical difference whether or not a ramp function or a step function occurs in the bounding fluid because spatial-truncation errors for low Biot numbers are small in either case. As an example, the maximum possible spatial-truncation error is 2.2% for a Biot number of 0.5.

Whether or not a given problem has a short- or a long-ramp bounding fluid temperature rise is determined by the Fourier number ($\alpha T/L^2$). Since the maximum error for all cases falls between Fourier numbers of 0.9 and 2.0, node thicknesses can be chosen sufficiently small to force the Fourier number to a value such that the criteria for a long

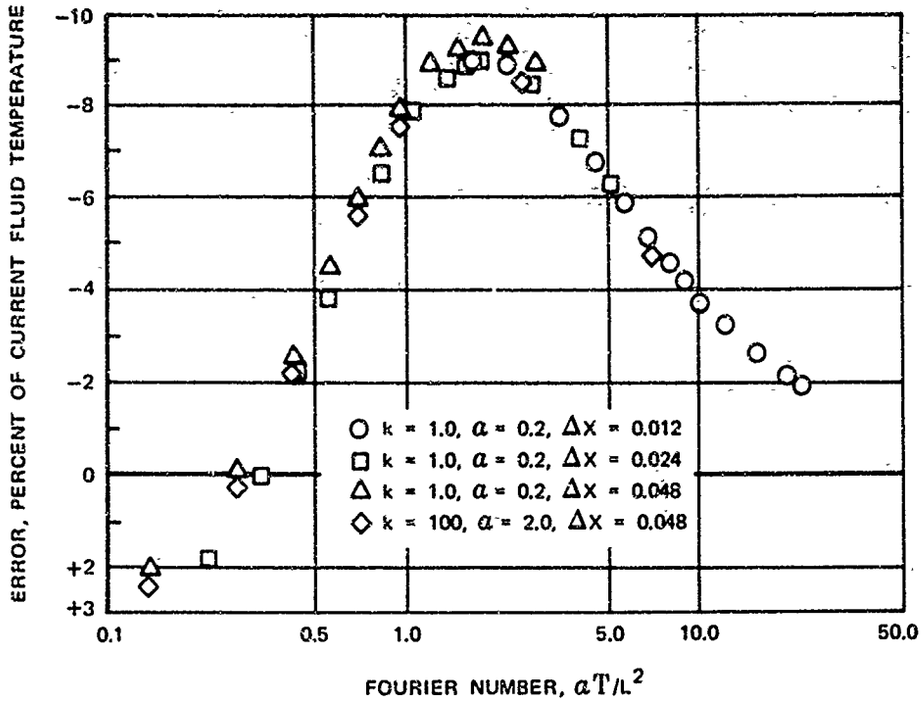

FIG. 17. Spatial-Truncation Error Vs. Fourier Number, Long Ramp.

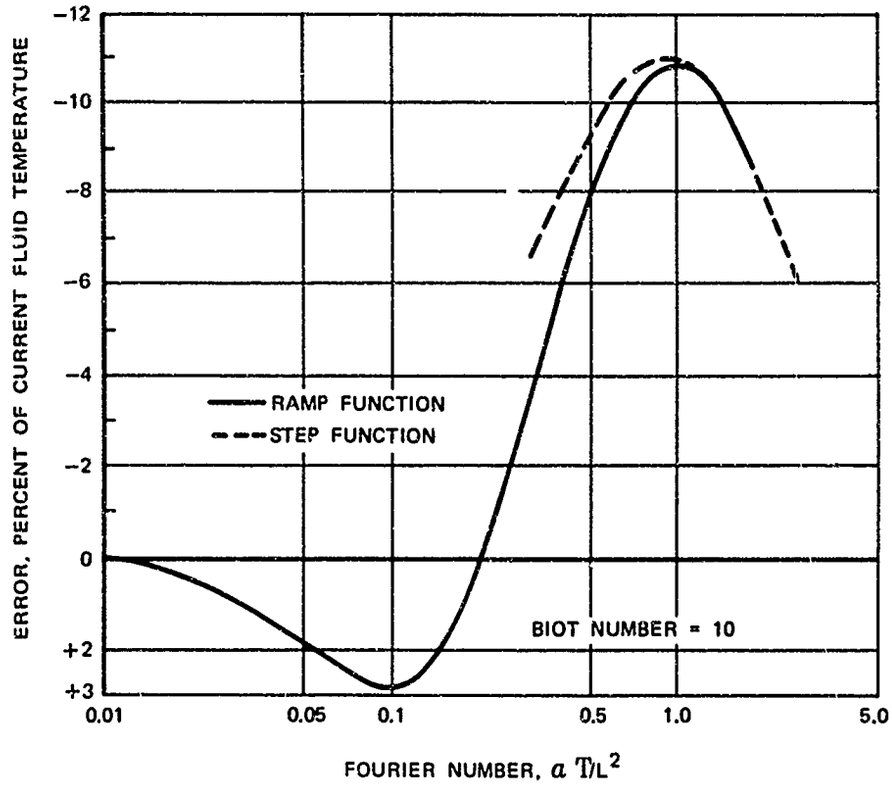

Fig. 18. Spatial-Truncation Error Vs. Fourier Number, Ultra Short Ramp.

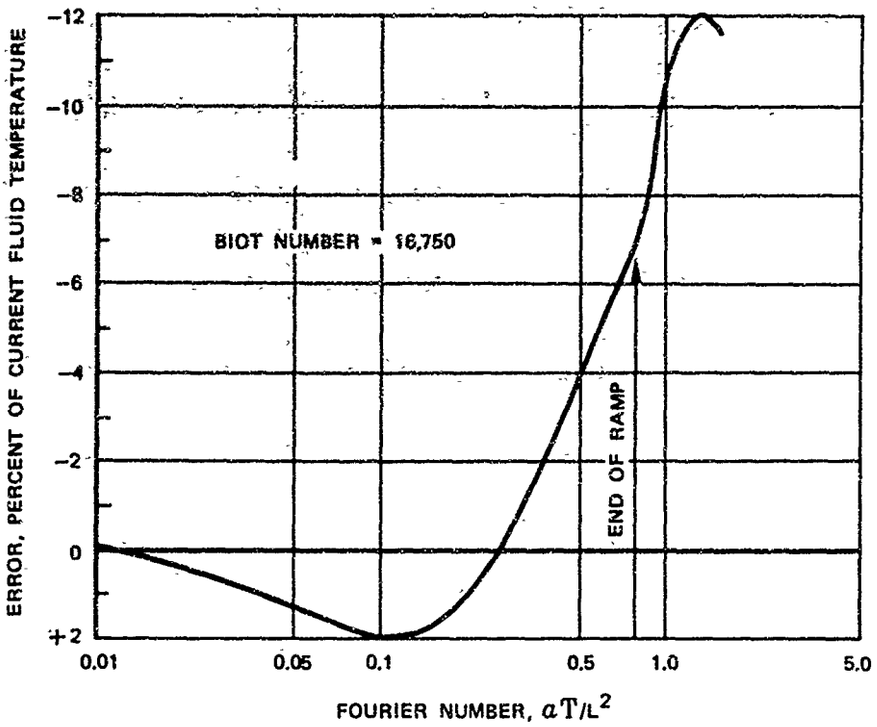

FIG. 19. Spatial-Truncation Error Vs. Fourier Number, Short Ramp.

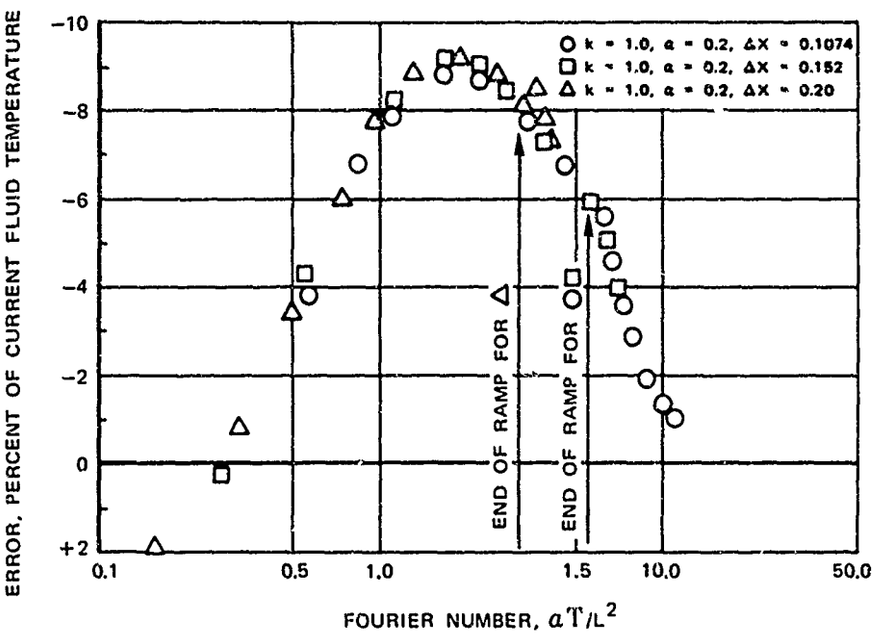

FIG. 20. Spatial-Truncation Error Vs. Fourier Number for Ramp-Function Temperature Rise in Bounding Fluid.

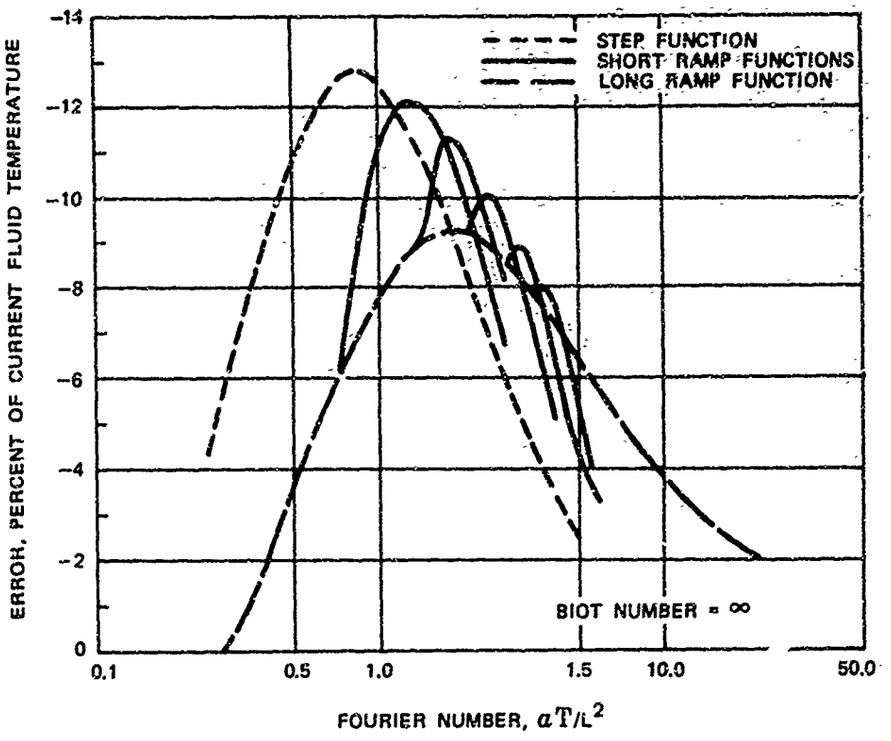

FIG. 21. Spatial-Truncation Error Vs. Fourier Number.

ramp exists. Conversely, by choosing a sufficiently large value for node thicknesses, the Fourier number will be less than 0.1 and the conditions of the short-ramp function can be produced.

As previously noted, the analytical solution for the ramp-function problem shows that the temperatures after the end of the ramp can be calculated by subtracting a Δt equal to the temperature rise from time zero for a time increment equal to the elapsed time after the end of the ramp from the temperature generated if the ramp were to continue indefinitely. To determine the temperatures after T_1 (the time representing the end of the ramp temperature rise of the bounding fluid), successive temperature responses are generated as if the ramp were continuing indefinitely. However, to correct for the fact that the ramp did cease at T_1, the analytical solution subtracts the values of a temperature response generated from time zero by using a time increment equal to the actual time increment after the end of the ramp.

If this approach is valid in the analytical solution, it should be valid in the equivalent numerical solution. Starting with the numerical temperature calculation as a function of time for a given ramp-function problem, extend the generation of nodal temperatures for an indefinitely long ramp by merely reversing the subtraction procedure outlined in the

analytical solution. That is, add the temperature rise found for a time increment after time zero to the temperature for a time equal to the time at the end of the ramp plus the time increment. Thus an extension of a ramp is synthesized by calculating the extensions from the single ramp run. This extended run was checked by an independent numerical run for the extended ramp and the two results were identical. Thus with a single computer run, ramps of any length can be synthesized.

A further examination of the analytical solution reveals that the final fluid temperature divided by the time length of the ramp is the controlling factor in the calculation of nodal temperatures. In other words, the crucial variable in determining node temperatures is the time slope of the temperature rise in the bounding fluid. Thus to calculate temperature responses for other time slopes, the ratio of the new slope to the old slope multiplied by the temperature rise of each node in the system would provide the correct node temperature rises for the new slope. For example, if an analytical solution for all the node temperatures caused by a fluid temperature rise of 2500 degrees per second has been calculated, node temperatures for a slope of 250 degrees per second may be determined by multiplying the node temperature rises from the 2500-degree-per-second results 0.1. Thus for any given problem, the node temperatures for the analytical solution may be found by multiplying each node temperature rise by the ratio of the new slope to the old slope.

It seemed reasonable that if temperature responses in the nodes could be synthesized for the analytical solutions by multiplying temperature responses by ratios of ramp slopes, then a similar synthesis method should prove valid for the numerical solution. Thus a number of runs for the same geometrical problem were made such that the only variable was the ramp slope. The length of the ramp in each case was held constant. As was expected the numerical results from this series of runs were all exactly proportional to the ratio of ramp slopes of temperature in the bounding fluid.

By combining these two separate synthesis methods, it is now possible for a one-dimensional equal-node geometry and given material properties to synthesize from a single numerical run what the node numerical temperature responses will be for either a longer or shorter ramp time and for any variation in ramp slope.

It may be noted that regardless of the ramp slope, the basic error is the same (Fig. 21). Whether or not the ramp is 25 or 2500 degrees per second, the percent error is a function of Fourier number only. It should be remembered, however, that the percentage error listed in Fig. 21 is a percentage of the current fluid temperature. Therefore, the absolute error in the 2500-degree-per-second ramp will be one hundred times the absolute value of the error in the 25-degrees-per-second ramp.

DELTA-FIRST-DERIVATIVE CORRELATION WITH SPATIAL-TRUNCATION ERROR

The possibility of establishing a correlation between the factors causing a spatial-truncation error and the error itself is of interest. The use of such a correlation would allow the computer to make a numerical calculation of eitner the second derivative or a change in the first derivative to determine the spatial-truncation error. This approach might be useful in practical, multilayer, three-dimensional problems.

It was first believed that an evaluation of the second derivative would be fruitful. Upon making some calculations of the second derivative during the early stages of a transient, it was realized that the second derivative was quite sensitive to the choice of the thickness of the individual nodes. In other words, as an increasingly thicker ΔX is specified, the nodes become increasingly less responsive to the input of energy, and the second derivative for the same transient has progressively smaller numerical values as ΔX is increased.

It was at this point that the decision was made to see if a difference in the first derivatives might be less sensitive to the effect of increasing ΔX. Calculations determining the magnitude of the first derivatives for the first node and the second node seemed to support the assumption that the change in first derivatives might be considerably less sensitive to node thickness than the numerical second derivative under the same conditions. The calculation of the first derivative of the surface node is made by (1) taking the temperature of the surface, (2) subtracting the temperature of the first node, and (3) dividing this difference by the distance from the surface to the center of the first node. The first derivative of the second node is made by (1) taking the temperature of the first node and (2) subtracting the temperature of the second node, and (3) then dividing the result by the distance between the first and second node centers. The delta-first-derivative is the difference between the two first derivatives. A plot of this delta-first-derivative as a function of Fourier number is shown in Fig. 22. The delta-first-derivative starts as a large value for time zero and progressively reduces to smaller values as time increases. Further thought suggested that the spatial-truncation error-response curve shape, i.e., a maximum error around a Fourier number of 1, could be developed for the delta-first-derivative data.

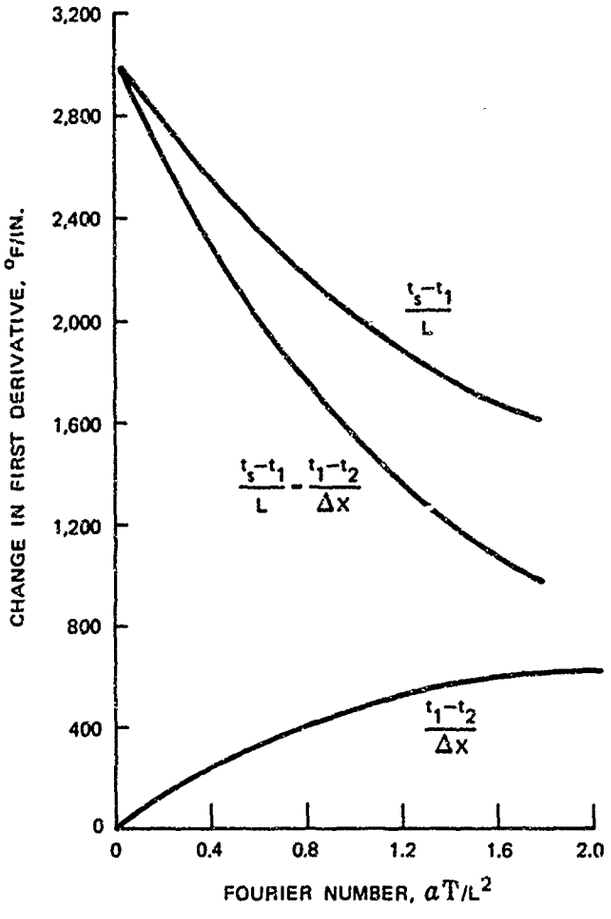

FIG. 22. Change in First Derivative as a Function of Fourier Number.

The absolute values of the temperatures during the early portions of the transient are quite small. Thus, the effect of a large change in delta-first-derivatives would be relatively small when multiplied by the small temperature rise. To compensate for the large delta-first-derivatives and for the small temperature rises in the early portion of the transient, these two factors were multiplied together to form a correlation number. Figure 23 shows the result of plotting this correlation number versus the Fourier number. For comparison purposes the spatial-truncation error expressed as a percentage of the step function in the bounding fluid is plotted as a dotted line. The Fourier number, at which a maximum occurred in the correlation number, coincides almost exactly with the maximum Fourier number for the spatial-truncation error curve.

Unfortunately, at the present time, there is not a good correlation in the values of the correlation number with respect to the percentage values of the spatial-truncation error curve. Presumably, this shortcoming of the present correlation could be overcome in a practical

computer program wherein the correlation numbers could be generated throughout the transient. After the computer had finished calculating the entire transient, a subroutine could cause the computer to go back and match the proportional correlation number curve with a spatial-truncation error curve, and to print appropriate temperature corrections at the various time steps. This appears to be a fruitful area for further work which could result in a generalized error treatment.

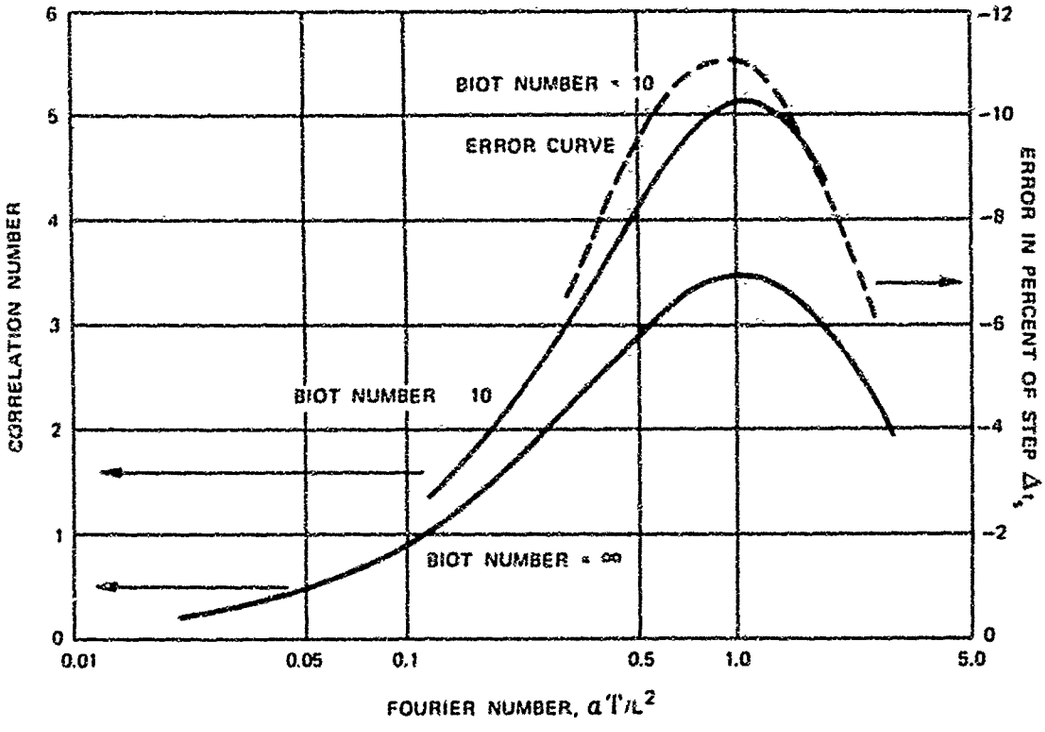

FIG. 23. Correlation Number as a Function of Fourier Number

PROGRAMMED ERROR PRINTOUT AND OPTIMIZATION

ERROR PRINTOUT

The ONE-D program contains a section that will compute the spatial- and time-truncation errors. Sufficient data have not been collected at this time to allow a similar error analysis for multilayer slabs. Some data are presented in Fig. 11 and 12.

Since it would take an infinite number of error curves similar to those in Fig. 3 and 4 to cover all possible nodal subdivisions and time increments, the error analysis can only be approximate.

Data from the error curves for Biot numbers of ∞, 10, 3, 1 and 0.5 were used in a Chebyshev-polynomial curve fitting program that converted the Chebyshev series to its equivalent power series. The power series is of the form,

$$Y_m = \sum_{j=0}^{j=m} A_j X^j$$

A 10-degree polynomial was used to fit the spatial error curves and a 6-degree polynomial for the time-truncation error curves. The coefficients for the power series are included with the ONE-D program, and are listed in Appendix A.

The power series are used to calculate the spatial-truncation error for a Fourier number between 0.25 and 7.0 except for Biot number of 0.5, where the Fourier number range is 0.75 through 7.0. For Fourier numbers less than 0.25 or 0.75, which ever the case may be, a value of 2.5% for the spatial error is printed. For Fourier numbers greater than 7.0, a value of -1.3% for the spatial error is printed.

In the case of the time-truncation error, the power series are used for Fourier numbers between 0.0 and 10.0. For Fourier numbers greater than 10.0, the Fourier number is printed along with a suggestion that the time increment (DTAU) be reduced.

The error program tests the Biot number against the Biot numbers for the error curves used. Any Biot number falling between the established numbers is replaced by the next higher number, and the error is calculated on that basis.

The error printout routine produces an error printout at each time step and provides an option for the user to apply the error printout to ramp functions. The spatial-truncation errors in ramp-function temperature responses in bounding fluids have been incorporated in the error printout. The base error curve for the long ramp has been established in terms of the Chebyshev series as a function of Fourier number and is stored in the subroutine for error printout. Thus in ramp-function cases where a long ramp is encountered, the user will receive a printout of the number of degrees Fahrenheit due to the spatial-truncation error in the first node. For ramps having lengths shorter than the long ramp, that is, for all ramps having Fourier numbers of less than five at the end of the ramp, the program is so devised that up to the end of the ramp, the program will print errors from the base curve. After the end of the ramp, the program will automatically shift over to the spatial-truncation error curve forming the envelope for the post-ramp deviations, as seen in Fig. 21.

A FORTRAN listing of the computer program for error printout is included in Appendix A.

AUTOMATIC TIME-STEP GENERATOR

Quite often the user of a transient heat-transfer computer program does not have an accurate idea of a time step to use to minimize time-truncation errors at the beginning of transient. In addition, the user may not know how many time steps to specify to completely cover the transient. In some cases, the user can over specify the number of time steps to be used and waste computer time by calling for printouts of temperatures after the transient has been completed; or conversely, and even worse, the user can specify only a fraction of the time steps necessary to complete the transient and thus be forced to go back and rerun the problem, with the resulting increased expense and delay before obtaining the results. Thus, if an automatic time-step generator were available that would keep the time-truncation errors within predetermined limits and would cause the program to cease generating temperature printouts when the transient had completed a predetermined level of response, a considerable savings in time and improvement of accuracy could result. To this end, an automatic time-step generator has been developed. Briefly, the basis for this routine is that the machine generates the first time step based upon the level of accuracy the user desires, as reflected in the Fourier number. Once the first time step has thus been determined, the program can then proceed at periodic time intervals to test each of three nodes specified: First,

a node adjacent to the surface; second, a node at an intermediate level, and third, a node at the deepest portion of the geometry. The test that the program runs is to determine the temperature change for each of the three nodes during the time step. If this number of degrees is less than 0.5% of the overall temperature rise in the bounding fluid for the transient, the program will automatically double the length of time for the succeeding time step. If the user so desires, the percentage that is used can be changed. The final control of the program will permit the user to specify the terminal temperature in any node desired.

ONE-D ACCELERATION OF THREE-DIMENSIONAL TRANSIENT PROBLEMS

The more complex three-dimensional heat-transfer problems involving very slow convergence pose the problems of excessive machine time and possible inaccuracies due to the convergence characteristics of the iterative processes used. These iterative processes depend upon some prescribed limit for termination, and the value of this iteration limit for maximizing accuracy and for minimizing machine time is not known a priori. To reduce machine time and to have less dependence upon the sensitivity of the iteration limit, a chain of nodes could be established in a psuedo ONE-D pattern extending from the surface of the three-dimensional geometry to the deepest portion of the geometry. These chains of nodes could be solved directly and could establish a temperature field as the beginning point for a computer program for three-dimensional thermal response, THT-B. This iterative process would then be reduced to essentially lateral heat transfer and, in many cases, be capable of convergence to the final solution within a few iterations. Preliminary work on this approach consisted of establishing a very slowly converging two-dimensional model. Temperatures for all nodes are evaluated at the beginning of each time step by the separate application of the one-dimensional TRIDAG technique to two columns of nodes. The entire temperature field of the model is then solved by using the Gauss-Seidel method. The results are summarized in Table 2. These preliminary results indicate that it may be profitable, from the dual standpoint of accuracy and reduction in machine time, to establish a subroutine wherein the user may elect in specific three-dimensional problems to establish one-dimensional chains of nodes for which a tridiagonal solution would automatically be used at the beginning of each time step to establish the approximate temperature field before an iterative technique is utilized.

TABLE 2. Comparison of Results of the THT-B Computer Program With Those of the THT-B and the ONE-D Programs Combined.

Run	Initial temp., °F	Time, sec	Iteration limit	Thermal conductivity, k		Temperature, °F								Number of iterations
				1	2	Node 1	Node 51	Node 20	Node 70	Node 40	Node 90	Node 50	Node 100	
ONE-D-1	0	600.0	...	...	1.05	...	938.50	...	141.86	...	21.71	...	13.60	...
ONE-D-2	0	600.0	...	1.00	...	937.57	...	135.22	...	19.20	...	12.03	...	...
THT-B-1	0	600.0	0.1	1.00	1.05	937.41	937.95	130.18	132.65	14.49	14.85	7.55	7.75	225
THT-B-2	0	600.0	0.1	1.00	1.00	937.21	937.22	128.12	128.17	13.82	13.85	7.08	7.11	220
THT-B-3	0	600.0	0.1	1.05	1.05	938.13	938.14	134.44	134.49	15.39	15.42	8.11	8.14	228
THT-B-4	0	600.0	0.001	1.05	1.05	938.50	938.50	141.80	141.80	21.62	21.62	13.52	13.52	628
THT-B-5	0	600.1	0.001	1.00	1.05	937.77	938.32	137.22	139.81	20.33	20.81	12.58	12.89	614
THT-B-6	From ONE-D	0.1	0.001	1.00	1.05	937.72	938.65	135.24	141.89	19.60	21.71	12.04	13.61	3
THT-B-7	From ONE-D	600.1	0.001	1.00	1.05	937.76	938.31	137.24	139.88	20.40	20.92	12.65	12.99	129
THT-B-8	0	600.0	0.001	1.00	1.05	937.76	938.31	137.19	139.79	20.32	20.81	13.58	12.88	614
THT-B-9	From ONE-D	600.0	0.1	1.00	1.05	937.58	938.50	135.29	141.80	19.62	21.69	12.07	13.57	1

SUMMARY AND CONCLUSIONS

To summarize the work described in this report, all of the component projects were directed (1) to improve the accuracy of the transient numberical solutions currently available, and (2) to reduce the machine time necessary for obtaining accurate transient-temperature histories.

This study, consisting of various projects devcted to different aspects of numerical solutions of transient heat-transfer problems, has arrived at an evaluation of spatial- and time-truncation errors for homogeneous slabs using uniform node spacing and being exposeu to bounding fluids that have either a step-function or a ramp-function change in temperature. The errors involved in thin-thick geometries of a thin layer of highly conductive material in contact with a thick layer of insulating material are presented. With the information provided by the curves, a user can ensure that the accuracy of his transient temperature analysis and design will fall within any predetermined level of accuracy. Whenever possible, the curves reflect the worst possible case; therefore, in practical problems where conditions not as severe as the study conditions occur, a conservative evaluation of the errors will automatically result. With respect to the speeding-up of the solutions, particularly with regard to computer time used, several acceleration techniques are presented with some evaluation as to their effectiveness. In addition, a direct solution applicable to one-dimensional problems is provided that reduces computer solution time when compared to the usual iterative solutions. As a fringe benefit of this study, a number of the small computer programs used in the error study, acceleration and other portions of the study, have been combined into one generalized transient heat-transfer one-dimensional program titled, ONE-D. This generalized program should prove to be of considerable help to the heat-transfer transient designer, for a number of options are available.

By examining the error response of ramp functions, the user now has a mathematical model that is considerably closer to practical problems than is the step-function model previously used. The error printout response both to the step function and to the ramp function at the choice of the user. The principle of utilizing the ONE-D solution for improving both the accuracy and the machine time aspects of slowly converging three-dimensional programs will be beneficial to the user.

In like manner, the automatic time-step generator will also afford the user convenience, accuracy and reduced machine time. The correlation of the delta-first-derivative with spatial-truncation error may prove to be the basis for a generalized error treatment in which the mathematical model will be the problem reduced to numerical terms instead of modeling real problems with the one-dimensional homogeneous equal ΔX geometry.

In conclusion, the user is now in a position (1) to define much more accurately the areas of real transient heat-transfer problems in which error may be considerable; (2) to change his input data to make sure that his results are reasonably accurate; and (3) to make maximum effective use of the transient heat-transfer programs available utilizing the techniques outlined here such as the ONE-D acceleration and the automatic time-step generator.

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Appendix A

COMPUTER PROGRAMS

Section 1

GENERAL HEAT-TRANSFER PROGRAM "ONE-D" AS APPLIED TO STEP-FUNCTION BOUNDARY CONDITIONS

A number of specialized small computer programs were developed in the course of this study to investigate various aspects of the step-function boundary conditions. These small programs, e.g., error printout, iteration-acceleration method, direct solutions, spatial- and time-truncation error, and thin-thick geometries, were combined into a single computer program called "ONE-D", coded in FORTRAN for the IBM 1620 computer.

This program is capable of handling a one-dimensional problem of fifty nodes; however, the number of nodes may be increased merely by changing the DIMENSION statement. The equations in the program are set up such that one surface of the slab is exposed to a fluid while the other surface of the slab is insulated.

The program contains a number of options that may be called by the user. As an example, the program can be used for a composite slab, made up of three materials, as well as for a homogeneous slab. The boundary conditions of fluid temperature distribution of the slab can be included in the input or can be considered constant. The solution of the equations in the program can be accomplished in a number of ways such as Gauss-Seidel, accelerated Gauss-Seidel, or TRIDAG, a direct solution for tridiagonal matrices.

CONTROL VARIABLES

The following variable names are used for the various options:

KODE — The variable name KODE is used to determine if the initial temperature of the body is constant or variable. If KODE is 1, the initial temperature is constant. If KODE is not 1, then the initial temperature must be specified for each node.

KODE1 The variable name KODE1 is used to determine if the fluid temperature and film coefficient are constant or variable. If KODE1 is 1, the fluid temperature and film coefficient are constant. If KODE1 is not 1, then the fluid temperature and film coefficient for 21 values of time must be specified in the input.

SEL1 The variable name SEL1 is used to determine if equations are solved by an iterative method or by Tridag. If SEL1 is 1, the solution is iterative. If SEL1 is not 1, then Tridag is used.

SEL2 The variable name SEL2 is used to determine the type if iterative solution to be used. If SEL2 is 1, then regular Gauss-Seidel iteration is used. If SEL2 is not 1, then accelerated Gauss-Seidel is used.

A comparison of NODES1 and NODES determines whether the slab is homogeneous or multilayer. If NODES1 equals NODES, the slab is homogeneous, and the properties only of the first layer are read. Node-spacing within a layer must be equal, but the node-spacing of the different layers need not be the same. This allows for small node-spacing in thin layers and larger node-spacing in thick layers.

Provisions have been made to allow for contact resistance between layers or between nodes. The variable ZH is used for this purpose. ZH1 is used as the conductance between the nodes of the three layers. ZH2 is the conductance between layer 1 and layer 2, while ZH3 is the conductance between layer 2 and layer 3.

The other variable names are defined at the beginning of the program listing.

Two matrix methods, iterative and tridiagonal are used in this study to solve the equations formed by the heat balance for each node in the slab.

The program contains two sections that use the iterative method. Section 1 uses regular Gauss-Seidel iteration to solve the set of simultaneous equations. Section 2 uses Gauss-Seidel iteration with either or both acceleration techniques, Wegstein and constant acceleration factor. The respective sections in the program listing are noted with comment cards.

The Wegstein technique causes the programming to be more complex than either TRIDAG or regular Gauss-Seidel since the temperatures for all nodes for the latest three iterations must be stored. In the Wegstein technique the following general equations are used:

$$\overline{T^n} = T^n + Q\ (T^{n-1} - T^n)$$

where

$\overline{T^n}$ = the accelerated temperature for the nth iteration of any node

T^n = the temperature for the nth iteration

T^{n-1} = the temperature for the previous iteration, and

$$A = \frac{T^n - T^{n-1}}{T^{n-1} - T^{n-2}}$$

$$Q = \frac{A}{A-1}$$

Values of Q are negative since values of A are restricted to $0<A<1$ to insure only positive values of acceleration. As A approaches 1, the value of Q can become quite large. Therefore Q is restricted to a maximum value of 100.

The Wegstein technique is applied to each node, so individual values of A and Q must be calculated. The alternative is to apply a constant acceleration factor to all nodes; this is done by using the variable FACTOR. The variable FACTOR is used in the same manner as is Q; therefore, it should be negative. Note that if FACTOR is zero, there is no acceleration; thus regular Gauss-Seidel results. The regular Gauss-Seidel section of the ONE-D program could have been eliminated and the FACTOR routine substituted by setting FACTOR = 0, but it was more convenient to include it in the program when acceleration study runs were made.

In the program a constant acceleration factor, FACTOR, is applied to all nodes until ISTOP is reached. NOGS iterations of Gauss-Seidel are made until IAPPLY is reached; at which iteration, the Wegstein technique is applied. The Wegstein technique is repeated after the interval INTER.

If the Wegstein technique is applied too soon or too often, the solution may be slowed down. Past experience has shown that the initial application should be made on about the iteration which is proportionate in number to the depth of the deepest node from the boundary. For example, in the 20-node problem, the initial application of the Wegstein technique should be on about the tenth iteration. The criterion suggested for frequency of application of the technique is given as about one-half the number of iterations used for the initial application of acceleration.

By using proper values of the variables FACTOR, ISTOP, NOGS, and INTER, it is possible to reduce the number of iterations compared to Gauss-Seidel by as much as 63%. Obviously there is a large number of combinations possible, some of which are undesirable. If a constant acceleration factor is to be used without Wegstein acceleration, the literature indicates that FACTOR should be between 0 and -1, possibly -0.5 or -0.6. When used with Wegstein, FACTOR may be a larger numerical number but still negative. From a series of twelve computer runs made on the sample 20-node problem, it was found that values from -11.4 to -1.5 for FACTOR gave the minimum number of iterations.

Results of the 20-node problem with constant initial temperature, constant fluid temperature, and film coefficient subjected to a step change in the boundary fluid are as follows:

Gauss-Seidel Iteration Technique

FACTOR[a]	ISTOP[b]	NOGS[c]	INTER[d]	Number of iterations
0	60	...	...	46
-1	60	...	...	35
-1	3	0	3	42
-1	5	0	5	24
-1	6	0	3	40
-1	10	3	6	24
-1.3	10	3	6	17
-1.4	10	3	6	17
-1.5	10	3	6	17
-1.5	10	0	6	20
-1.6	10	3	6	33
0	10	0	6	27
-0.5	50	...	...	31

[a]Acceleration factor for all nodes in system.

[b]Number of iterations that FACTOR is to be applied.

[c]Number of iterations between application of unaccelerated Gauss-Seidel iterations and Wegstein accelerations.

[d]Number of Gauss-Seidel iterations between applications of Wegstein accelerations.

The TRIDAG solution needs no special instructions or data cards. The solution is rapid and contains no convergence errors but is limited to linear equations.

ARRANGEMENT OF INPUT

Data Card 1:

FORMAT (514, 5F10.3), NODES1, NODES2, NODES, KODE, KODE1, SEL1, SEL2, DTAU, ZNUM, ZLIM. This card must be used for all options.

Data Card 2:

FORMAT (F10.0, 315), FACTOR, ISTOP, NOGS, INTER. This card is read only if accelerated Gauss-Seidel is to be used (SEL2 = 1).

Data Card 3:

FORMAT (4F10.3, E10.3), DELX1, ZK1, ZRHO1, ZC1, ZH1. This card must be used for all options.

Data Card 4:

FORMAT (4F10.3, E10.3), DELX2, ZK2, ZRHO2, ZC2, ZH2. This card is read if slab is made up of two or three layers (NODES7, NODES1).

Data Card 5:

FORMAT (4F10.3, E10.3), DELX3, ZK3, ZRHO3, ZC3, ZH3. This card is read if slab is made up of three layers (NODES7, NODES2).

Data Card 6-8:

FORMAT (8F10.3), TS. These cards are used for reading variable fluid temperature (KODE1 = 1). Twenty-one values of TS are read. These values are temperatures of the fluid for 20 equal time steps covering the time range for the transient.

Data Card 9-11:

FORMAT (8F10.3), HFVAR. These cards are used for reading variable film coefficient (KODE1 = 1). Twenty-one values of HFVAR are read. These values are film coefficients for 20 equal time steps covering the time range for the transient.

Data Card 12:

FORMAT (2F10.3) TC, HFC. This card is read if the fluid temperature and the film coefficient are constant (KODE1 = 1).

Data Card 13-19:

FORMAT (8F10.3), TPR. These cards are used to read the nodal temperatures if the slab has an initial temperature distribution (KODE = 1). The number of data cards is dependent on the number of nodes in the system with a maximum of 50 nodes.

Data Card 20:

FORMAT (F10.3) TAU. This card is read to indicate time for variable initial temperature distribution (KODE = 1).

Data Card 21:

FORMAT (F10.3) TIN. This card is read when slab has a constant initial temperature (KODE = 1).

Data Card 22:

FORMAT (15), ITERS. This card is read when the problem is solved by an iterative method (SEL1 = 1). This puts an upper limit on the number of iterations allowed to converge.

Data Card 23-47:

These cards are supplied with the program. Their purpose is to supply the coefficients of the power series that describe the spatial and time truncation error curves. These cards are read only if the slab is homogeneous (NODES1 = NODES) and for the first time step (TAU = DTAU).

```
C     ONE-DIMENSIONAL HEAT TRANSFER PROGRAM, ONE-D, CODED BY VAN TASSEL
C        AND STANLEY
C
C     AR,BR,CR,DR = COEFFICIENT ARRAYS CONTAINING THE SUB-DIAGONAL, DIA-
C                   GONAL, SUPER-DIAGONAL, AND RIGHT HAND ELEMENTS OF
C                   THE TRIDIAGONAL SYSTEM
C     BIOT = BIOT NUMBER
C     COEFF = COEFFICIENTS OF POWER SERIES USED IN ERROR ANALYSIS
C     DELX = SPATIAL INCREMENT, INCHES
C     DTAU = TIME STEP, SECONDS
C     FACTOR = CONSTANT OVER RELAXATION FACTOR(ZERO FOR REGULAR GAUSS-
C              SEIDEL).
C     FO = FOURIER NUMBER
C     HFC = CONSTANT FILM COEFFICIENT, BTU/HR-SQ FT-F
C     HFILM = FILM COEFFICIENT AT SPECIFIC TIME UNDER CONSIDERATION,
C             BTU/HR-SQ FT-F
C     HFVAR = TIME VARIABLE FILM COEFFICIENT, BTU/HR-SQ FT-F
C     INTER = NUMBER OF REGULAR GAUSS-SEIDEL ITERATIONS BETWEEN SUCCES-
C             SIVE APPLICATIONS OF THE WEGSTEIN ACCELERATION.
C     ISTOP = NUMBER OF ITERATIONS THAT THE CONSTANT OVER RELAXATION
C             FACTOR IS APPLIED.
C     ITERS = ALLOWABLE NUMBER OF ITERATIONS
C     KSWPS = NUMBER OF ITERATIONS PER TIME STEP
C     NODES1 = NUMBER OF NODES IN FIRST LAYER
C     NODES2 = NUMBER OF NODES IN SECOND LAYER
C     NODES = TOTAL NUMBER OF NODES IN THE SYSTEM
C     NOGS = NUMBER OF REGULAR GAUSS-SEIDEL ITERATIONS APPLIED BEFORE
C            WEGSTEIN ACCELERATION IS APPLIED.
C     TAU = TIME, SECONDS
C     T = TEMPERATURE OF NODE AT BEGINNING OF ITERATION, F
C     TC = CONSTANT FLUID TEMPERATURE, F
C     TF = FLUID TEMPERATURE AT SPECIFIC TIME UNDER CONSIDERATION, F
C     TIN = CONSTANT INITIAL SLAB TEMPERATURE, F
C     TNEW = TEMPERATURE OF NODE AT END OF ITERATION OR TIME STEP, F
C     TPR = TEMPERATURE OF NODE AT PREVIOUS TIME STEP, F
C     TS = TIME VARIABLE FLUID TEMPERATURE, F
C     ZC = SPECIFIC HEAT BTU/LBM-F
C     ZH = CONDUCTANCE BETWEEN NODES,TO ALLOW FOR CONTACT RESISTANCE
C     ZK = THERMAL CONDUCTIVITY, BTU/HR-FT-F
C     ZRHO = DENSITY, LBM/CU.FT.
C     ZLIM = ITERATION LIMIT
C     ZNUM = NUMBER OF TIME STEPS DESIRED
C
      DIMENSION TPR(50),T(50),TNEW(50),TT(50,3),FT(50,3),TOLD(50,3),
     1  C(50),D(50),AR(50),BR(50),CR(50),DR(50),BETA(50),GAMMA(50),ZK(50
     2),ZC(50),ZRHO(50),DELX(50),ZH(50),TS(21),HFVAR(21),COEFF(5,11)
   10 FORMAT(5I4,5F10.2)
   20 FORMAT(1H ,11X,7HFILM H=,F10.2,7X,11HBOUND TEMP=,F8.2,2H F)
   30 FORMAT (8F10.3)
   40 FORMAT(2F10.3)
   50 FORMAT(1H0,2X,4HNODE,11X,7HDELTA X,14X,1HK,16X,7HDENSITY,15X,1HC,
```

```
     1 16X,2HHC)
    50 FORMAT(1H ,1X,I3,5X,F14.3,5X,F14.3,5X,F14.3,5X,F14.3,9X,E10.3)
    70 FORMAT(1H ,11X,4HTIME,F12.3,10H  SECONDS ,/,11X,11HITERATIONS=,I3)
    80 FORMAT (1H ,5F11.3,5X,5F11.3)
    90 FORMAT(1H ,18X,I3,20X,F10.5)
   100 FORMAT(1H0,24HITERATION LIMIT EXCEEDED)
   110 FORMAT(1H0,11X,14HINITIAL TIME= ,F10.0,10H   SECONDS)
   120 FORMAT(1H0,78HFILM COEFFICIENT,BEGINNING AT TIME ZERO,WITH TIME IN
     1CREMENT=TOTAL TIME SPAN/20)
   130 FORMAT(1H0,73HFLUID TEMP.,BEGINNING AT TIME ZERO,WITH TIME INCREME
     1NT=TOTAL TIME SPAN/20)
   140 FORMAT(F10.0,3I5)
   150 FORMAT(I5)
   160 FORMAT(8E10.3)
   170 FORMAT(5E15.8)
   180 FORMAT(4F10.3,E10.3)
   190 FORMAT(1H0,49HMAXIMUM SPATIAL TRUNCATION ERROR IS +2.5 PER CENT)
   200 FORMAT(1H0,37HMAXIMUM SPATIAL TRUNCATION ERROR IS -,F6.1,9H PER CE
     T)
   210 FORMAT(1H0,77HTIME STEP IS TOO BIG, SUGGEST USING SMALLER FOURIER
     UMBER. FOURIER NUMBER IS,F10.2)
   220 FORMAT(1H0,34HMAXIMUM TIME TRUNCATION ERROR IS -,F6.1,9H PER CENT)
   230 FORMAT(1H0,49HMAXIMUM SPATIAL TRUNCATION ERROR IS -1.3 PER CENT)
   240 FORMAT(1H ,11X,11HNODE NUMBER,18X,17HTEMPERATURE DEG F)
   250 FORMAT(1H0,19HFILM COEFF.=CONST.=,F10.3)
   260 FORMAT(1H0,19HFLUID TEMP.=CONST.=,F10.3)
   270 FORMAT(1HO,11X,4HTIME,F12.3,10H   SECONDS)
   280 FORMAT(F10.3)
       READ 10,NODES1,NODES2,NODES,KODE,KODE1,SEL1,SEL2,DTAU,ZNUM,ZLIM
       IF(SEL2-1.)242,243,242
C      IF SEL2 IS 1, SOLUTION BY ORDINARY GAUSS-SEIDEL, IF NOT, SOLUTION
C      BY ACCELERATED GAUSS-SEIDEL,FACTOR,ISTOP,NOGS,AND INTER MUST BE
C      READ.
   242 READ 140,FACTOR,ISTOP,NOGS,INTER
   243 PRINT 50
C      READ PROPERTIES OF MATERIALS, MAXIMUM OF THREE MATERIALS
       READ 180,DELX1,ZK1,ZRHO1,ZC1,ZH1
       IF (NODES1-NODES)231,232,46
C      DETERMINE IF HOMOGENEOUS MATERIAL, IF NODES1=NODES -- HOMOGENEOUS.
   231 READ 180,DELX2,ZK2,ZRHO2,ZC2,ZH2
       IF(NODES2-NODES)233,232,46
   233 READ 180,DELX3,ZK3,ZRHO3,ZC3,ZH3
C      SET PROPERTIES OF EACH NODE IN EACH LAYER
   232 DO 93 N=1,NODES
       DELX(N)=DELX1
       ZK(N)=ZK1
       ZRHO(N)=ZRHO1
       ZC(N)=ZC1
    93 ZH(N)=ZH1
       IF(NODES1-NODES)234,236,46
   234 ZH(NODES1)=ZH2
       I=NODES1+1
       DO 102 N=I,NODES2
       DELX(N)=DELX2
       ZK(N)=ZK2
       ZRHO(N)=ZRHO2
```

```
      ZC(N)=ZC2
  102 ZH(N)=ZH1
      IF(NODES2-NODES)235,236,46
  235 ZH(NODES2)=ZH3
      I=NODES2+1
      DO 103 N=I,NODES
      DELX(N)=DELX3
      ZK(N)=ZK3
      ZRHO(N)=ZRHO3
      ZC(N)=ZC3
  103 ZH(N)=ZH1
  236 DO 4 N=1,NODES
      C(N)=300.*ZC(N)*ZRHO(N)*DELX(N)/DTAU
    4 PRINT 60,N,DELX(N),ZK(N),ZRHO(N),ZC(N),ZH(N)
      IF(KODE1-1)237,238,237
C     IF KODE1 IS 1, THE FLUID TEMPERATURE AND FILM COEFFICIENT ARE CON-
C     STANT. IF KODE IS NOT 1, VARIABLE FLUID TEMPERATURES AND FILM
C     COEFFICIENTS ARE READ.
C
C     READ AND PRINT VARIABLE FLUID TEMPERATURE AND FILM COEFFICIENT
  237 READ 30,(TS(I),I=1,21)
      PRINT 130
      PRINT 80,(TS(I),I=1,21)
      READ 30,(HFVAR(I),I=1,21)
      PRINT 120
      PRINT 80,(HFVAR(I),I=1,21)
      GO TO 241
  238 READ 40,TC,HFC
      DO 239 I=1,21
      TS(I)=TC
  239 HFVAR(I)=HFC
  241 NP=NODES-1
      DO 5 N=1,NP
    5 D(N)=1./(DELX(N)/(24.*ZK(N))+DELX(N+1)/(24.*ZK(N+1))+1./ZH(N))
      IF(KODE-1)1,3,1
C     IF KODE IS 1, ALL NODES ARE SAME TEMPERATURE, IF KODE IS NOT EQUAL
C     TO 1  NODAL TEMPERATURES ARE READ IN.
    1 DO 2 N=1,NODES
      READ 30,TPR(N)
      PRINT 80,TPR(N)
    2 T(N)=TPR(N)
      READ 280,TAU
      PRINT 110,TAU
      GO TO 33
    3 READ 280,TIN
      DO 7 N=1,NODES
      TPR(N) = TIN
    7 T(N)=TPR(N)
      TAU=0.0
   33 IF(SEL1-1.)35,36,35
C     IF SEL1 IS 1, THE SOLUTION IS BY EITHER GAUSS-SEIDEL OR BY ACCEL-
C     ERATED GAUSS-SEIDEL AND THE MAXIMUM ALLOWABLE NUMBER OF ITERATIONS
C     MUST BE READ. IF SEL1 IS NOT 1, SOLUTION IS BY TRIDAG.
   36 READ 150,ITERS
    6 TAU=TAU+DTAU
      KSWPS=0
```

```
      KOUNT=0
      J=1
      SK=1.+TAU*20./(ZNUM*DTAU)
      K=SK
      RK=K
      IF(TAU-ZNUM*DTAU)254,255,46
C     DETERMINE FILM COEFFICIENT AND FLUID TEMPERATURE FOR TIME UNDER
C     CONSIDERATION.
  254 TF=TS(K)+(TS(K+1)-TS(K))*(SK-RK)
      HFILM=HFVAR(K)+(HFVAR(K+1)-HFVAR(K))*(SK-RK)
      GO TO 256
  255 TF=TS(K)
      HFILM=HFVAR(K)
  256 A=1./((1./HFILM)+DELX(1)/(24.*ZK(1)))
      PRINT 20,HFILM,TF
      IF(SEL2-1.)61,8,61
C
C     REGULAR GAUSS-SEIDEL ITERATIVE SOLUTION
    8 IF(KSWPS-ITERS)9,9,28
    9 DO 25 N=1,NODES
      IF(N-1)12,12,11
   11 IF(N-NODES)14,13,13
   12 TNEW(1)=(A*TF+C(1)*TPR(1)+D(1)*T(2))/(A+C(1)+D(1))
      GO TO 15
   13 TNEW(NODES)=(C(NODES)*TPR(NODES)+D(NODES-1)*T(NODES-1))/(C(NODES)+
     1D(NODES-1))
      GO TO 15
   14 TNEW(N)=(C(N)*TPR(N)+D(N-1)*T(N-1)+D(N)*T(N+1))/(C(N)+D(N-1)+D(N))
   15 TEMP=TNEW(N)-T(N)
      IF(TEMP)16,17,17
   16 TEMP=(-TEMP)
   17 IF(TEMP-ZLIM)18,19,19
   18 KOUNT=KOUNT+1
      IF(KOUNT-NODES)25,26,28
   19 KOUNT=0
   25 T(N)=TNEW(N)
      KSWPS=KSWPS+1
      GO TO 8
C     END OF GAUSS-SEIDEL SOLUTION
   26 T(N)=TNEW(N)
   24 PRINT 70, TAU,KSWPS
      PRINT 240
      PRINT 90, (N, T(N),N=1,NODES)
   21 DO 27 N=1,NODES
   27 TPR(N)=T(N)
      IF(NODES1-NODES)47,198,46
   28 PRINT 100
      CALL EXIT
C
C     SPATIAL AND TIME TRUNCATION ERROR ANALYSIS
  198 IF(TAU-DTAU)227,227,226
C     READ COEFFICIENTS OF POWER SERIES FOR SPATIAL TRUNCATION ERROR
  227 DO 199 K=1,5
  199 READ 170, (COEFF(K,I),I=1,11)
      BIOT = HFILM*DELX1/24./ZK1
      FO = ZK1*TAU/(6.25*ZC1*ZRHO1*DELX1**2)
```

```
C     DETERMINE SPECIAL CONDITIONS
      IF(FO-7.)202,202,201
  201 PRINT 230
      GO TO 225
  202 IF(BIOT-.5)203,203,205
  203 IF(FO-.75)204,216,216
  204 PRINT 190
      GO TO 225
  205 IF(FO-.25)206,207,207
  206 PRINT 190
      GO TO 225
C     DETERMINE WHICH CURVE TO USE, BASED ON BIOT NUMBER
  207 IF(BIOT-10.)211,209,208
  208 K = 5
      GO TO 217
  209 K = 4
      GO TO 217
  211 IF(BIOT-3.)213,212,209
  212 K = 3
      GO TO 217
  213 IF(BIOT-1.)215,214,212
  214 K = 2
      GO TO 217
  215 IF(BIOT-.5)216,216,214
  216 K = 1
  217 SUM = COEFF(K,1)
      IF(I-1)223,218,218
  218 DO 219 I=1,10
  219 SUM =  SUM+COEFF(K,I+1)*FO**I
      PRINT 200 ,SUM
C     READ COEFFICIENTS OF POWER SERIES  FOR TIME TRUNCATION ERROR
  225 DO 221 K=1,5
  221 READ 170, (COEFF(K,I),I=1,7)
      FO = ZK1*DTAU/(6.25*ZC1*ZRHO1*DELX1**2)
      IF(FO-10.)207,207,222
  222 PRINT 210, FO
      GO TO 226
  223 DO 224 I=1,6
  224 SUM =  SUM+COEFF(K,I+1)*FO**I
      PRINT 220, SUM
C     END OF ERROR ANALYSIS
  226 IF(SEL1-1.)49,47,49
   47 IF(TAU-ZNUM*DTAU)6,46,46
C
C     ACCELERATED GAUSS-SEIDEL SOLUTION USING A CONSTANT ACCELERATION
C     FACTOR AND WEGSTEIN ACCELERATION
   61 DO 62 N=1,NODES
      TT(N,1) = T(N)
   62 TOLD(N,1) = T(N)
      IAPPLY=ISTOP*NOGS
   63 KSWPS = KSWPS+1
      INDEX = KSWPS-KSWPS/3*3
      IF(INDEX)64,64,65
   64 J = 3
      GO TO 66
   65 J = INDEX
```

```
 66 IF(KSWPS-ITERS)67,67,28
 67 DO 83 N=1,NODES
    IF(N-1)69,69,68
 68 IF(N-NODES)72,71,71
 69 FT(1,J)=(A*TF*C(1)*TPR(1)*D(1)*TT(2,J))/(A*C(1)*D(1))
    GO TO 73
 71 FT(NODES,J)=(C(NODES)*TPR(NODES)*D(NODES-1)*TT(NODES-1,J))/(C(NODE
   1S)*D(NODES-1))
    GO TO 73
 72 FT(N,J)=(C(N)*TPR(N)*D(N-1)*TT(N-1,J)*D(N)*TT(N+1,J))/(C(N)*D(N-1)
   1*D(N))
 73 TEMP=ABS(FT(N,J)-TT(N,J))
    IF(TEMP-ZLIM)74,74,75
 74 KOUNT = KOUNT+1
    IF(KOUNT-NODES)82,97,97
 75 KOUNT = 0
 82 IF(J-3)76,79,79
 76 TT(N,J) = FT(N,J)
    TT(N,J+1) = FT(N,J)
    TOLD(N,J+1) = FT(N,J)
    IF(KSWPS-2)83,77,77
 77 IF(KSWPS-ISTOP)78,83,83
 78 TT(N,J+1) = TT(N,J)+FACTOR*(TOLD(N,J)-TT(N,J))
    TOLD(N,J+1) = TT(N,J+1)
    GO TO 83
 79 TT(N,J) = FT(N,J)
    TT(N,1) = FT(N,J)
    TOLD(N,1) = FT(N,J)
    IF(KSWPS-ISTOP)81,83,83
 81 TT(N,1) = TT(N,J)+FACTOR*(TOLD(N,J)-TT(N,J))
    TOLD(N,1) = TT(N,1)
 83 CONTINUE
    IF(KSWPS-IAPPLY)63,85,85
 85 DO 96 N=1,NODES
    IF(J-2)86,87,88
 86 ALPHA = (TOLD(N,2)-TOLD(N,1))/(TOLD(N,1)-TOLD(N,3))
    GO TO 89
 87 ALPHA = (TOLD(N,3)-TOLD(N,2))/(TOLD(N,2)-TOLD(N,1))
    GO TO 89
 88 ALPHA = (TOLD(N,1)-TOLD(N,3))/(TOLD(N,3)-TOLD(N,2))
 89 IF(ALPHA)96,96,91
 91 IF(ALPHA-1.)92,96,96
 92 Q = ALPHA/(ALPHA-1.)
    IF(Q+100.)99,101,101
 99 Q = 100.
101 IF(J-3)94,95,95
 94 TT(N,J+1) = TT(N,J)+Q*(TOLD(N,J)-TT(N,J))
    TOLD(N,J+1) = TT(N,J)
    GO TO 96
 95 TT(N,1) = TT(N,J)+Q*(TOLD(N,J)-TT(N,J))
    TOLD(N,1) = TT(N,1)
 96 CONTINUE
    IAPPLY = IAPPLY+INTER
    GO TO 63
 97 TT(N,J) = FT(N,J)
    DO 98 N=1,NODES
```

```
   98 T(N) = TT(N,J)
C     END OF ACCELERATED ITERATIVE SOLUTION
      GO TO 24
C
C     TRIDAG SOLUTION
   35 TAU=TAU+DTAU
      SK=1.+TAU*20./(ZNUM*DTAU)
      K=SK
      RK=K
      IF(TAU-ZNUM*DTAU)251,252,46
  251 TF=TS(K)+(TS(K+1)-TS(K))*(SK-RK)
      HFILM=HFVAR(K)+(HFVAR(K+1)-HFVAR(K))*(SK-RK)
      GO TO 253
  252 TF=TS(K)
      HFILM=HFVAR(K)
  253 PRINT 20,HFILM,TF
      A=1./((1./HFILM)+DELX(1)/(24.*ZK(1)))
      AR(1)=0.
      BR(1)=A+D(1)+C(1)
      CR(1)=-D(1)
      DR(1)=C(1)*TPR(1)+A*TF
      N=NODES
      AR(N)=-D(N-1)
      BR(N)=D(N-1)+C(N)
      CR(N)=0.0
      DR(N)=C(N)*TPR(N)
      NP=NODES-1
      DO 31 N=2,NP
      AR(N)=-D(N-1)
      BR(N)=D(N-1)+D(N)+C(N)
      CR(N)=-D(N)
   31 DR(N)=C(N)*TPR(N)
      BETA(1)=BR(1)
      GAMMA(1)=DR(1)/BETA(1)
      DO 22 N=2,NODES
      BETA(N)=BR(N)-AR(N)*CR(N-1)/BETA(N-1)
   22 GAMMA(N)=(DR(N)-AR(N)*GAMMA(N-1))/BETA(N)
      N=NODES
      TNEW(N)=GAMMA(N)
      LAST=NODES-1
      DO 23 L=1,LAST
      N=NODES-L
   23 TNEW(N)=GAMMA(N)-CR(N)*TNEW(N+1)/BETA(N)
   44 N=NODES
      PRINT 270,TAU
      PRINT 240
      PRINT90, (N,TNEW(N),N=1,NODES)
       KOUNT=0
   32 DO 34 N=1,NODES
   34 TPR(N)=TNEW(N)
      IF(NODES1-NODES)49,198,46
   49 IF(TAU-ZNUM*DTAU)35,46,46
   46 CALL EXIT
      END
```

LISTING OF THE COEFFICIENTS FOR THE CHEBYSHEV POWER SERIES

```
C     COEFFICIENTS FOR SPATIAL TRUNCATION ERROR CURVE, BIOT NO.=0.5

 -.10085509E+02  .31313865E+02 -.40438398E+02  .32576778E+02 -.17195332E+02
  .60576895E+01 -.14294331E+01  .22260158E+00 -.21906165E-01  .12322493E-02
 -.30152627E-04

C     COEFFICIENTS FOR SPATIAL TRUNCATION ERROR CURVE, BIOT NO.=1.0

 -.47741800E+01  .10716588E+02  .39482730E+01 -.15347922E+02  .13378985E+02
 -.63696820E+01  .18748795E+01 -.34903101E+00  .39971559E-01 -.25676872E-02
  .70752864E-04

C     COEFFICIENTS FOR SPATIAL TRUNCATION ERROR CURVE, BIOT NO.=3.0

 -.10742661E+02  .57953052E+02 -.78197286E+02  .62807352E+02 -.33533087E+02
  .12135378E+02 -.29680640E+01  .48101017E+00 -.49327896E-01  .28919389E-02
 -.73736341E-04

C     COEFFICIENTS FOR SPATIAL TRUNCATION ERROR CURVE, BIOT NO.=10.0

 -.47343690E+01  .57069120E+02 -.84933970E+02  .72423840E+02 -.41121905E+02
  .15859021E+02 -.41168194E+01  .70284882E+00 -.75320501E-01  .45810170E-02
 -.12044014E-03

C     COEFFICIENTS FOR SPATIAL TRUNCATION ERROR CURVE, BIOT NO.=INFINITY

 -.13905012E+02  .10651562E+03 -.17496117E03  .15979799E+03 -.92659020E+02
  .35326802E+02 -.89197870E+01  .14726003E+01 -.15254262E+00  .89865613E-02
 -.22953966E-03

C     COEFFICIENTS FOR TIME TRUNCATION ERROR CURVE, BIOT NO.=0.5

 -.14942700E+00  .13063356E+01  .71703553E+00 -.27529148E+00  .39718724E-01
 -.26482984E-02  .67521740E-04

C     COEFFICIENTS FOR TIME TRUNCATION ERROR CURVE, BIOT NO.=1.0

 -.52064100E-01  .21030915E+01  .94564310E+00 -.41051716E+00  .63568391E-01
 -.45599169E-02  .12669898E-03

C     COEFFICIENTS FOR TIME TRUNCATION ERROR CURVE, BIOT NO.=3.0

 -.67415600E-01  .37673700E+01  .14662341E+01 -.87103151E+00  .16056679E+00
 -.13017263E-01  .39541555E-03

C     COEFFICIENTS FOR TIME TRUNCATION ERROR CURVE, BIOT NO.=10.0

 -.97815400E-01  .50510120E+01  .11366794E+01 -.86081270E+00  .16645989E+00
 -.13872656E-01  .43148691E-03

C     COEFFICIENTS FOR TIME TRUNCATION ERROR CURVE, BIOT NO.=INFINITY

 -.16628430E+00  .64759210E+01  .49463530E+00 -.74881550E+00  .15666910E+00
 -.13388434E-01  .41877831E-03
```

These coefficients are used in storing the spatial- and the time-truncation error curves in the ONE-D program.

Section 2

ANALYTICAL SOLUTION FOR AN INFINITE PLATE

```
C  SOLUTION OF TEMPERATURE DISTRIBUTION IN A FINITE SLAB - ONE DIMENSIONAL
C  FLOW - FINITE FILM COEFFICIENT ON ONE FACE, THE OTHER FACE INSULATED
C  ALPHA = THERMAL DIFFUSIVITY, SQ.FT./HR.
C  AM = EIGEN VALUES OF M*TAN(M)=CONSTANT
C  BIOT = BIOT NUMBER
C  DELS = LENGTH OF PLATE FROM SURFACE TO INSULATED FACE, INCHES.
C  DELX = SPATIAL INCREMENT, INCHES
C  DTAU = TIME STEP, SECONDS
C  HFILM = FILM COEFFICIENT, BTU/HR-SQ.FT.-F.
C  TAU = TIME, SECONDS
C  TF = FILM TEMPERATURE, F.
C  TI = INITIAL TEMPERATURE, F.
C  THETA = FOURIER NUMBER
C  X = DISTANCE FROM INSULATED FACE TO NODE CONSIDERED, INCHES
C  ZC = SPECIFIC HEAT, BTU/LBM-F.
C  ZK = THERMAL CONDUCTIVITY, BTU/HR.-FT.-F.
C  ZRHO = DENSITY, LBM/CU.FT.
C
      DIMENSION AM(50),DEXP(50),SIN1(50),SIN2(50)
   10 FORMAT(7F10.0,2I5)
   20 FORMAT(1H0,10X,43HTRANSIENT SOLUTION FOR INSULATED FLAT PLATE)
   30 FORMAT(1H0,10X,2HK=,F7.3,10X,2HC=,F7.4,5X,8HDENSITY=,F9.3)
   40 FORMAT(1H ,10X,18HFILM COEFFICIENT=,F10.3)
   50 FORMAT(1H ,10X,6HALPHA=,F8.4,2X,2HL=,F8.4)
   60 FORMAT(1H ,10X,13HINITIAL TEMP=,F10.4,2X,10HFILM TEMP=,F10.4)
   70 FORMAT(8F10.0)
   80 FORMAT(1H0,12X,12HEIGEN VALUES)
   90 FORMAT(1H ,10X,2HM(,I3,3H)= ,F10.5)
  100 FORMAT(F10.0)
  110 FORMAT(1H0,12X,4HTIME,8X,3HX/L,4X,10HTEMP RATIO,6X,4HTEMP)
  120 FORMAT(1H ,10X,F9.2,3X,F7.4,4X,F9.6,5X,F9.3)
  130 FORMAT(2H0,10X,5HDELX=,F7.4,3X,10HBIOT NO. =,F8.4,3X,12HFOURIER NO
     1.=,F8.4)
      READ 10,ZK,ZC,ZRHO,HFILM,DELS,TF,TI,KODE,N
      PRINT 20
      PRINT 30,ZK,ZC,ZRHO
      PRINT 40,HFILM
      BIOT=HFILM*DELS/(12.0*ZK)
      ALPHA=ZK/(ZC*ZRHO)
      PRINT 50,ALPHA,DELS
      PRINT 60,TI,TF
      PRINT 80
      IF(KODE-1) 2,1,2
    1 READ 70,(AM(I),I=1,N)
      GO TO 3
    2 CALL EIGEN(N,AM,BIOT)
    3 DO 4 I=1,N
    4 PRINT 90,I,AM(I)
      READ 100,DELX
      IF(SENSE SWITCH 1) 15,5
    5 READ 100,TAU
```

```
      THETA=ALPHA*TAU/(3600.0*DELS*DELS/144.)
      PRINT 130,DELX,BIOT,THETA
      DO 6 I=1,N
      SIN1(I)=SINF(AM(I))
      SIN2(I)=SINF(2.0*AM(I))
    6 DEXP(I)=EXPF(-AM(I)**2*THETA)
      X=0.
    7 SUM=0.
      DO 8 I=1,N
    8 SUM=SUM+(SIN1(I)*DEXP(I)/(2.0*AM(I)+SIN2(I)))*COSF(AM(I)*X/DELS)
      TRATIO=4.0*SUM
      TEMP=TF+(TI-TF)*TRATIO
      XRATIO=X/DELS
      PRINT 110
      PRINT 120,TAU,XRATIO,TRATIO,TEMP
      X=X+DELX
      IF(X-DELS) 7,7,5
   15 CALL EXIT
      END
C
C
      SUBROUTINE EIGEN(N,AM,BIOT)
      DOUBLE PRECISION T,TI,AM,ZK,ZC,ZRHO,TF,HFILM,X,DELX,DELS,A,TAU,DE,
     1BIOT
C  HALF INTERVAL SEARCH FOR ROOTS OF COT(M)-M/BIOT=0.
      DIMENSION AM(50)
      PI=3.14159265
      EPS=1.0E-3
      I=0
    1 I=I+1
      IEYE=I-1
      EYE=IEYE
      A=PI/180.*EYE*PI
      B=PI/2.*EYE*PI
      FA=DCOS(A)/DSIN(A)-A/BIOT
    2 X=(A+B)/2.
      COT=DCOS(X)/DSIN(X)
      XOB=X/BIOT
      F=COT-XOB
      IF(F) 3,11,4
    3 IF(F+EPS) 5,11,11
    4 IF(F-EPS) 11,11,5
    5 IF(F*FA) 6,11,7
    6 B=X
      FB=F
      GO TO 2
    7 A=X
      FA=F
      GO TO 2
   11 AM(I)=X
      IF(N-I) 12,12,1
   12 RETURN
      STOP
      END
```

Section 3

MODIFIED HEAT-TRANSFER PROGRAM "ONE-D" WITH ERROR ANALYSIS AS APPLIED TO RAMP-FUNCTION BOUNDARY CONDITIONS

The program used to generate data for the ramp-function boundary condition is an abbreviated form of ONE-D. Since the TRIDAG method of solution was the fastest and simplest to use, it was chosen and revised to handle only homogeneous solids. It was also revised so that the automatic time step generator subroutine (ATSG) could be used.

The error routine portion of ONE-D is also incorporated. Revisions were made in this routine so that the spatial error for either time step functions or ramp functions is given. Time-truncation error is printed only for step-function boundary conditions.

CONTROL VARIABLES

KODE(1) If KODE(1) is 1, the automatic time-step generator is used. Values for ANUMBE, PRCNTH, and PRCNTL must be read. PRCNTH and PRCNTL are given in decimal form.

KODE(2) If KODE(2) is 1, a Fourier number is read and the time step, DTAU, is calculated. If KODE(2) is not 1, DTAU is read.

KODE(3) If KODE(3) is 1, the initial temperature of the body is set at zero and the time, TAU, is set at zero. If KODE(3) is not 1, values for the temperature at the nodes and TAU are read.

KODE(4) If KODE(4) is 1, the temperature of the nodes may be found for a number of specified times. If KODE(4) is not 1, the temperatures will be printed according to DTAU.

TAU1 TAU1 may be used as a control variable in addition to specifying the time at the end of the ramp. If TAU1 is less than 1, a step function will be assumed.

TAU2 TAU2 may also be used as a control variable. TAU2 is ordinarily used to specify the time at which the run will stop. However, if TAU2 is 0, then the user can specify the terminal temperature in any node desired.

ARRANGEMENT OF INPUT

Data Card 1:

FORMAT (7I5), KODE(1), KODE(2), KODE(3), KODE(4), NODES, LTP, NTP. This card must be read for each set of data.

Data Card 2:

FORMAT (3F10.3), TAU1, TAU2, FT. This card must be read for each set of data.

Data Card 3:

FORMAT (F10.3, I5), TSTOP, NSTOP. This card is read if TAU2 is 0.

Data Card 4:

FORMAT (3F10.3), ANUMBE, PRCNTH, PRCNTL. This card is read if KODE(1) is 1.

Data Card 5:

FORMAT (4F10.3, E10.3, I5), DELX1, ZK1, ZRHO1, ZC1, ZH1, M2. This card must be read for each set of data.

Data Card 6:

FORMAT (E10.3) HFILM. This card must be read for each set of data.

Data Card 7:

FORMAT (F10.3) FO. This card is read if KODE(2) is 1.

Data Card 8:

FORMAT (F10.3), DTAU. This card is read if KODE(2) is not 1.

Data Card 9:

FORMAT (2F10.3) TAU, TF. This card is read if KODE(3) is not 1.

Data Card 10-22:

FORMAT (8F10.3), TPR. These cards are read if KODE(3) is not 1. The number of data cards is dependent on the number of nodes in the system with a maximum of 100 nodes.

Data Card 23:

FORMAT (I2), NO. This card is read if KODE(4) is 1.

Data Card 24-25:

FORMAT (8F10.3). These cards are read if KODE(4) is 1.

Data Cards 26-56:

FORMAT (5E15.8). These cards are used to supply the coefficients of the power series that describe the spatial- and time-truncation error curves. If more than one set of data is read (LTP greater than 1) these cards are read only once.

PROGRAM LISTING

```
C  TRIDAG SOLUTION OF ONE-DIMENSIONAL TRANSIENT HEAT TRANSFER
C  PROBLEM WITH ERROR ANALYSIS
C
C                              NOTATION
C  ANUMBE = THE NUMBER OF TIME STEPS BETWEEN CHECKS BY ATSG
C  AR,BR,CR,DR = COEFFICIENT ARRAYS CONTAINING THE SUB-DIAGONAL,
C                DIAGONAL, SUPER-DIAGONAL, AND RIGHT HAND ELEMENTS
C                OF THE TRIDIAGONAL SYSTEM
C  BIOT = BIOT NUMBER
C  DELX = SPATIAL INCREMENT, INCHES
C  DTAU = TIME STEP, SECONDS
C  FO = FOURIER NUMBER
C  FOI = FOURIER NUMBER AT THE END OF THE RAMP
C  FT = FINAL FLUID TEMPERATURE, F.
C  HFILM = FILM COEFFICIENT, BTU/HR.-SQ.FT.-F.
C  KODE(N) = CONTROL VARIABLES
C  LTP = NUMBER OF SETS OF DATA TO BE READ
C  M2= NUMBER OF NODES BETWEEN ZERO THICKNESS NODES
C  NO = NUMBER OF TIMES AT WHICH A TEMPERATURE PRINT-OUT IS REQUESTED
C  NODES = TOTAL NUMBER OF NODES IN THE SYSTEM
C  NTP = NUMBER OF TIME STEPS BETWEEN TEMPERATURE PRINTINGS
C  NSTOP = NODE SELECTED TO TERMINATE RUN
C  PRCNTH = LARGEST PERCENT INCREASE IN TEMPERATURE, EXPRESSES AS A DECIMAL
C  PRCNTL = SMALLEST PERCENT INCREASE IN TEMPERATURE, EXPRESSED AS A DECIMAL
C  SCOEFF = COEFFICIENTS OF POWER SERIES FOR SPATIAL ERROR ANALYSIS
C  TAU = TIME, SECONDS
C  TAU1 = TIME AT THE END OF THE RAMP, SECONDS
C  TAU2 = TIME AT TERMINATION, SECONDS
C  TAUT = TEMPORARY TIME STORAGE
C  TCOEFF = COEFFICIENTS OF POWER SERIES FOR TIME ERROR ANALYSIS
C  TF = FLUID TEMPERATURE AT SPECIFIC TIME UNDER CONSIDERATION, F.
C  TIME(I) = TIMES AT WHICH A TEMPERATURE PRINT-OUT IS REQUESTED, SEC.
C  TNEW = TEMPERATURE OF NODE AT END OF ITERATION OR TIME STEP, F.
C  TPR = TEMPERATURE OF NODE AT PREVIOUS TIME STEP, F.
```

```
C  TPRT = TEMPORARY STORAGE FOR PREVIOUS TEMPERATURES
C  TSTOP = TERMINATION TEMPERATURE FOR NODE NSTOP, F.
C  ZC = SPECIFIC HEAT, BTU/LBM-F.
C  ZH = CONDUCTANCE BETWEEN NODES, TO ALLOW FOR CONTACT RESISTANCE
C  ZK = THERMAL CONDUCTIVITY, BTU/HR.-FT.-F.
C  ZRHO = DENSITY, LBM/CU.FT.
C
      DIMENSION KODE(5),TPR(100),TNEW(100),C(100),CR(100),D(100),AR(100)
     1,BR(100),DR(100),BETA(100),GAMMA(100),ZK(100),ZC(100),ZRHO(100),
     2ZH(100),DELX(100),TPRT(100),TINE(10),SCOEFF(7,11),TCOEFF(5,7)
      COMMON TNEW,TPR,TAU,DTAU,NODES,TF,FT,TAUI,NUMB,TPRT,DTAUT,TAUT,
     1PRCNTH,PRCNTL,C,ZC,ZRHO,DELX
C
   10 FORMAT(7I5)
   20 FORMAT(3F10.3)
   30 FORMAT(8F10.3)
   40 FORMAT(F10.3)
   50 FORMAT(1H1,4HNODE,11X,7HDELTA X,14X,1HK,16X,7HDENSITY,15X,1HC,16X,
     12HHC)
   60 FORMAT(I2,I3,4(5X,F14.3),9X,E10.3)
   70 FORMAT(1H0,8HDELTA T=,F12.4,5H SEC.)
   80 FORMAT(I1,5F11.3,5X,5F11.3)
   90 FORMAT(E10.3)
  100 FORMAT(1H0,13HHFILM=CONST.=,F12.3)
  110 FORMAT(1H0,12HFLUID TEMP.=,F10.3)
  120 FORMAT(I2)
  130 FORMAT(1H0,11X,4HTIME,F12.4,10H   SECONDS)
  140 FORMAT(4F10.3,E10.3,I5)
  150 FORMAT(2F10.3)
  160 FORMAT(1H0, ,MAXIMUM SPATIAL TRUNCATION ERROR IS LESS THAN -2.0
     1 PERCENT,)
  170 FORMAT(5E15.8)
  180 FORMAT(F10.3,I5)
  190 FORMAT(1H0,49HMAXIMUM SPATIAL TRUNCATION ERROR IS +2.5 PERCENT )
  200 FORMAT(1H0,37HMAXIMUM SPATIAL TRUNCATION ERROR IS -,F6.1,9H PERCEN
     1T )
  210 FORMAT(1H0,77HTIME STEP IS TOO BIG, SUGGEST USING SMALLER FOURIER
     1NUMBER. FOURIER NUMBER IS, F10.2)
  220 FORMAT(1H0,34HMAXIMUM TIME TRUNCATION ERROR IS -,F6.1,9H PERCENT )
  230 FORMAT(1H0,49HMAXIMUM SPATIAL TRUNCATION ERROR IS -1.3 PERCENT )
C
      JKT=0
      KKK=0
    1 READ(105,10) KODE(1),KODE(2),KODE(3),KODE(4),NODES LPT,NTP
    READ(5,20) TAU1,TAU2,FT
      IF(TAU2.EQ.0.) READ(105,180) TSTOP,NSTOP
      IF(KODE(1).EQ.1) READ(105,20) ANUMBER,PRCNTH,PRCNTL
      NUMBER=ANUMBER
      NUMB=0
      READ(105,140) DELX1,AK1,ZRHO1,ZC1,ZH1,M2
      DO 42 N=1,NODES
      DELX(N)=DELX1
      ZK(N)=ZK1
      ZRHO(N)=ZRHO1
      ZH(N)=ZH1
   42 ZC(N)=ZC1
      DO 3 N=1,NODES,M2
```

```
    3 DELX(N)=0.
      WRITE(108,50)
      DO 4 N=1,NODES
    4 WRITE(108,60) N,DELX(N),ZK(N),ZRHO(N),ZC(N),ZH(N)
      READ(105,90) HFILM
      WRITE(108,100) HFILM
      IF(KODE(2)-1) 6,7,8
    7 READ(105,40) FO
      DTAU=FO*DELX1**2*ZRHO1*ZC1*6.25/ZK1
      GO TO 9
    8 READ(105,40) DTAU
    9 WRITE(108,70) DTAU
      DTAUT=DTAU
      IF(TAU1) 13,13,16
   13 TF=FT
      GO TO 17
   16 IF(DTAU.GE.TAU1) GO TO 13
      TF=FT*DTAU/TAU1
   17 WRITE(108,110) TF
      IF(KODE(3)-1) 14,12,14
   14 READ(105,150) TAU,TF
      WRITE(108,130) TAU
      READ(105,30) (TPR(N),N=1,NODES)
      WRITE(108,80) (TPR(N),N=1,NODES)
      GO TO 18
   12 DO 15 N=1,NODES
   15 TPR(N)=0.
      TAU=0.
   18 KOUNT=0
      NP=NODES-1
      DO 19 N=1,NP
   19 D(N)=1./(DELX(N)/(24.*ZK(N))+DELX(N+1)/(24.*ZK(N+1))+1./ZH(N))
      DO 39 N=1,NODES
      TPRT(N)=TPR(N)
   39 C(N)=300.*ZC(N)*ZRHO(N)*DELX(N)/DTAU
      A=1./(1./HFILM)+DELX(1)/(24.*ZK(1)))
      IF(KODE(4)-1) 35,95,35
   95 READ(105,120) NO
      READ(105,30) (TIME(N),N=1,NO)
      TAUT=TAU
      NN=1
   35 NUMB=NUMB+1
      TAU=TAU+DTAU
      IF(KODE(4)-1) 63,61,63
   61 IF(NN.EQ.NO+1) GO TO 63
      IF(TAU-TIME(NN)) 63,62,62
   62 TAU=TIME(NN)
      NN=NN+1
      IF(TAU-TAU1) 64,65,65
   64 TF=FT*TAU/TAU1
      WRITE(108,110) TF
      GO TO 63
   65 TF=FT
```

```
   63 KOUNT=KOUNT+1
      AR(1)=0.
      BR(1)=A+D(1)+C(1)
      CR(1)=-D(1)
      DR(1)=C(1)*TPR(1)+A*TF
      N=NODES
      AR(N)=-D(N-1)
      BR(N)=D(N-1)+C(N)
      CR(N)=0.
      DR(N)=C(N)*TPR(N)
      DO 21 N=2,NP
      AR(N)=-D(N-1)
      BR(N)=D(N-1)+D(N)+C(N)
      CR(N)=-D(N)
   21 DR(N)=C(N)*TPR(N)
      BETA(1)=BR(1)
      GAMMA(1)=DR(1)/BETA(1)
      DO 22 N=2,NODES
      BETA(N)=BR(N)-AR(N)*CR(N-1)/BETA(N-1)
   22 GAMMA(N)=(DR(N)-AR(N)*GAMMA(N-1))/BETA(N)
      TNEW(NODES)=GAMMA(NODES)
      DO 23 L=1,NP
      N=NODES-L
   23 TNEW(N)=GAMMA(N)-CR(N)*TNEW(N+1)/BETA(N)
      IF(KOUNT.LT.NTP) GO TO 33
   28 WRITE(108,130) TAU
      WRITE(108,80) (TNEW(N),N=1,NODES)
      KOUNT=0
C  SPATIAL AND TIME TRUNCATION ERROR ANALYSIS
      IF(KKK.GE.1) GO TO 229
C  READ COEFFICIENTS OF POWER SERIES FOR SPATIAL TRUNCATION ERROR
      DO 199 K=1,7
  199 READ(105,170) (SCOEFF(K,I),I=1,11
  229 BIOT=HFILM*DELX1/24./ZK1
      FO=ZK1*TAU/(6.25*ZC1*ZRHO1*DELX1**2)
C  DETERMINE IF RAMP FUNCTION OR STEP FUNCTION
      IF(TAU1.LT.1.) GO TO 222
      FO1=ZK1*TAU1/(6.25*ZC1*ZRHO1*DELX1**2)
      IF(FO.LE.FO1) K=6
      IF(FO.GE.FO1) K=7
      IF(FO.LE..75) K=6
      IF(FO.GT.10.) K=6
      IF(FO.LE.20.) GO TO 226
      WRITE(108,160)
      GO TO 225
  226 IF(FO.GT..25) GO TO 217
      WRITE(108,190)
      GO TO 225
C  DETERMINE SPECIAL CONDITIONS
  222 IF(FO.LE.7.) GO TO 202
      WRITE(108,230)
      GO TO 225
  202 IF(BIOT.GT..5) GO TO 205
      IF(FO.GE..75) GO TO 216
      WRITE(108,190)
      GO TO 225
```

```
  205 IF(FO.GE..25) GO TO 207
      WRITE(108,190)
      GO TO 225
C  DETERMINE WHICH CURVE TO USE, BASED ON BIOT NUMBER
  207 IF(BIOT-10.) 211,209,208
  208 K=5
      GO TO 217
  209 K=4
      GO TO 217
  211 IF(BIOT-3.) 213,212,209
  212 K=3
      GO TO 217
  213 IF(BIOT-1.) 215,214,212
  214 K=2
      GO TO 217
  215 IF(BIOT.GT..5) GO TO 214
  216 K=1
  217 IF(TAU.GT.DTAU) GO TO 218
      IF(I.LE.11) GO TO 223
  218 SUM=SCOEFF(K,1)
      DO 219 I=1,10
  219 SUM=SUM+SCOEFF(K,I+1)*FO**I
      IF(SUMLT.0.) SUM=-SUM
      WRITE(108,200) SUM
  225 IF(KKK.GE.1) GO TO 228
      KKK=KKK+1
C  READ COEFFICIENTS OF POWER SERIES FOR TIME TRUNCATION ERROR
      DO 221 K=1,5
  221 READ(105,170) (TCOEFF(K,I),I=1,7)
  228 IF(TAU1.GE.1.) GO TO 33
      IF(TAU.GT.DTAU) GO TO 33
      FO=ZK1*DTAU/(6.25*ZC1*ZRHO1*DELX1**2)
      IF(FO.LE.10.) GO TO 207
      WRITE(108,210) FO
      GO TO 33
  223 SUM=TCOEFF(K,1)
      DO 224 I=1,6
  224 SUM=SUM+TCOEFF(K,I+1)*FO**I
      WRITE(108,220) SUM
   33 IF(KODE(1)-1) 38,32,38
   32 IF(NUMB-NUMBER) 38,37,37
   37 CALL ATSG
      GO TO 29
   38 DO 24 N=1,NODES
   24 TPR(N)=TNEW(N)
      IF(TAU-TAU1) 25,29,29
   25 TF=FT*(TAU+DTAU)/TAU1
      IF(TF.GE.FT) TF=FT
   29 IF(KODE(4)-1) 67,66,67
   66 IF(TAU-TIME(N/)) 31,67,67
   67 IF(TAU2.EQ.0.) GO TO 68
      IF(TAU.GE.TAU2) GO TO 26
   31 WRITE(108,110) TF
      GO TO 35
   68 IF(TNEW(NSTOP).GE.TSTOP) GO TO 26
      GO TO 31
```

```
   26 JKT=JKT+1
      IF(JKT-LTP) 1,41,41
   41 STOP
      END
```

AUTOMATIC TIME-STEP GENERATOR SUBROUTINE

```
C
C
      SUBROUTINE ATSG
      DIMENSION TPR(100),TPRT(100),TNEW(100),C(100),ZC(100),ZRHO(100),
     1DELX(100)
      COMMON TNEW,TPR,TAU,DTAU,NODES,TF,FT,TAU1,NUMB,TPRT,DTAUT,TAUT,
     1PRCNTH,PRCNTL,C,ZC,ZRHO,DELX
   10 FORMAT(1H0,5X,55HDELTA TAU WAS HALVED, TIME SET BACK TO LAST CHECK
     1POINT.,1H ,5X,37HTPR,S ARE LISTED BELOW FOR THAT TIME.)
   20 FORMAT(1H0,10X,7HTIME IS,F12.5,5X,16HNEW DELTA TAU IS,F10.5)
   30 FORMAT(1H0,5X,26HPREVIOUS NODE TEMPERATURES)
   40 FORMAT(1H ,5F11.3,5X,5F11.3)
   50 FORMAT(1H0,5X,52HDELTA TAU WAS UNCHANGED. PROGRAM PROCEEDS AS BEFO
     1RE.)
   60 FORMAT(1H0,5X,48HDELTA TAU WAS DOUBLED. TEMPERATURES ABOVE ARE OK)
      NUMB=0
      MODES=NODES/2-2
      DTUP=PRCNTH*TF
      DTLOW=PRCNTL*TF
      DO 1 N=2,NODES,MODES
      DIFF=ABS(TNEW(N)-TPR(N))
      IF(DIFF.GE.DTUP) GO TO 2
      IF(DIFF.GE.DTLOW) GO TO 3
    1 CONTINUE
      DTAU=DTAU*2.
      WRITE(108,60)
      WRITE(108,20) TAU,DTAU
      GO TO 4
    3 WRITE(108,50)
    4 DO 5 N=1,NODES
      C(N)=300.*ZC(N)*ZRHO(N)*DELX(N)/DTAU
      TPR(N)=TNEW(N)
    5 TPRT(N)=TNEW(N)
      TAUT=TAU
      IF(TAU-TAU1) 6,7,7
    6 TF=FT*(TAU+DTAU)/TAU1
      IF(TF.GE.FT) TF=FT
    7 RETURN
    2 DTAU=DTAU/2.
      TAU=TAUT
      DO 8 N=1,NODES
      C(N)=300.*ZC(N)*ZRHO(N)*DELX(N)/DTAU
```

```
    8 TPR(N)=TPRT(N)
      WRITE(108,20) TAU,DTAU
      WRITE(108,10)
      WRITE(108,30)
      WRITE(108,40) (TPR(N),N=1,NODES)
      IF(TAU-TAU1) 9,11,11
    9 TF=FT+(TAU+DTAU)/TAU1
      IF(TF.GT.FT) TF=FT
      RETURN
   11 TF=FT
      RETURN
      END
```

Section 4

ANALYTICAL SOLUTION FOR AN INSULATED FLAT PLATE WITH RAMP-FUNCTION BOUNDARY CONDITIONS

```
    7 X=DELX+X
    8 IF(TAU.LE.TAU1) GO TO 13
    9 X=X1
      PROD=ALPHA*(TAU-TAU1)/3600./(DELS**2)/144.)
      DO 12 J=1,NODES
      SUM=0.
      DO 11 I=1,N
      ASIN1=DSIN(AM(I))
      ASIN2=DSIN(2.*AM(I))
      POWER=AM(I)**2*PROD
      IF(POWER.GT.174.) POWER=174.
   11 SUM=SUM+(ASIN1*(DEXP(-POWER)-1)/(AM(I)**2*(AM(I)*2.*ASIN2))*
     1DCOS(AM(I)*X/DELS))
      T1(J)=A*((TAU-TAU1)+B*SUM)
      T(J)=T(J)-T1(J)
      IF(T(J)-0.01) 13,13,12
   12 X=X+DELX
   13 TFRAMP=TF/TAU1*TAU
      IF(TF.LT.TFRAMP) TFRAMP=TF
   15 WRITE(108,100) TAU,TFRAMP
      WRITE(108,110) (T(K),K=1,J)
      TAU=TAU+DELTAU
      IF(TAU-TAU2) 5,5,16
   16 STOP
      END
C
C
      SUBROUTINE EIGEN(N,AM,BIOT)
      DOUBLE PRECISION T,T1,AM,ZK,ZC,ZRHO,TF,HFILM,X,DELX,DELS,A,TAU,DE,
     1BIOT
C  HALF INTERVAL SEARCH FOR ROOTS OF COT(M)-M/BIOT=0.
      DIMENSION AM(50)
      PI=3.14159265
      EPS=1.0E-3
      I=0
    1 I=I+1
      IEYE=I-1
      EYE=IEYE
      A=PI/180.*EYE*PI
      B=PI/2.*EYE*PI
      FA=DCOS(A)/DSIN(A)-A/BIOT
    2 X=(A+B)/2.
      COT=DCOS(X)/DSIN(X)
      XOB=X/BIOT
      F=COT-XOB
      IF(F) 3,11,4
    3 IF(F+EPS) 5,11,11
    4 IF(F-EPS) 11,11,5
    5 IF(F*FA) 6,11,7
    6 B=X
      FB=F
      GO TO 2
```

```
    7 A=X
      FA=F
      GO TO 2
   11 AM(I)=X
C                                  RAMP2
C  TRANSIENT SOLUTION FOR INSULATED FLAT PLATE, RAMP FUNCTION ON
C  BOUNDING FLUID.
      DOUBLE PRECISION T,T1,AM,ZK,ZC,ZRHO,TF,HFILM,X,DELX,DELS,TAU,DELTA
     1U,TAU1,TAU2,BIOT,ALPHA,X1,A,B,ASIN1,ASIN2,POWER,SUM,TFRAMP,PROD
      DIMENSION T(200),T1(200),AM(50)
   10 FORMAT(4F10.0)
   20 FORMAT(1H1,10X,76HTRANSIENT SOLUTION FOR INSULATED FLAT PLATE, RAM
     1P FUNCTION ON BOUNDING FLUID)
   30 FORMAT(1H1,T10,2HK=,F8.2,10X,2HC=,F7.4,10X,8HDENSITY=,F9.3)
   40 FORMAT(T11,17HFILM COEFFICIENT=,F13.2)
   50 FORMAT(T11,6HALPHA=,F8.4,5X,6HDELS= ,F8.3)
   60 FORMAT(3F10.0,3I10)
   70 FORMAT(8F10.0)
   80 FORMAT(1H0,T12,12HEIGEN VALUES)
   90 FORMAT(T10,2HM(,I3,3H)= ,F10.5)
  100 FORMAT(1H0,4HTIME,F12.3,9H SECONDS,10X,12HFLUID TEMP.=,F12.3)
  110 FORMAT(T2,5F11.3,5X,5F11.3)
  120 FORMAT(E10.1)
      READ(105,10) ZK,ZC,ZRHO,TF
      WRITE(108,20)
      WRITE(108,30) ZK,ZC,ZRHO
      READ(105,120) HFILM
      WRITE(108,40) HFILM
      READ(105,60) X,DELX,DELS,KODE,NODES,N
      READ(105,10) TAU,DELTAU,TAU1,TAU2
      BIOT=HFILM*DELS/(12.*ZK)
      ALPHA=ZK/(ZC*ZRHO)
      WRITE(108,50) ALPHA,DELS
      WRITE(108,80)
      IF(KODE-1) 2,1,2
    1 READ(105,70) (AM(I),I=1,N)
      GO TO 3
    2 CALL EIGEN(N,AM,BIOT)
    3 WRITE(108,90) (I,AM(I),I=1,N)
      X1=X
      A=TF/TAU1
      B=100.*DELS**2/ALPHA
    5 X=X1
      PROD=ALPHA*TAU/3600./(DELS**2/144.)
      DO 7 J=1,NODES
      SUM=0.
      DO 6 I=1,N
      ASIN1=DSIN(AM(I))
      ASIN2=DSIN(2.*AM(I))
      POWER=AM(I)**2*PROD
      IF(POWER.GT.174.) POWER=174.
    6 SUM=SUM+(ASIN1*(DEXP(-POWER)-1)/(AM(I)**2*AM(I)+2*ASIN2))*DCOS(AM(
     1I)*X/DELS)
      T(4)=A*(TAU+B*SUM)
      IF(T(J)-0.01) 8,8,7
      IF(N-1) 12,12,1
   12 RETURN
      STOP
      END
```

Appendix B

LITERARY SURVEY OF THE NUMERICAL SOLUTION OF THE ONE-DIMENSIONAL HEAT CONDUCTION EQUATION

prepared by

University of Nevada
Contract N60530-67-C-0051

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PREFACE

The bibliography for this summary comprises a relatively small part of the complete bibliography assembled under the contract. Only the articles and papers which appeared to deal directly with the questions at hand, and which were available, are discussed here. Many of the works were not available at this University. The remainder of the bibliography is intended to offer a somewhat wider range of references for information having possible application in the numerical treatment of heat conduction problems.

The bibliography is arranged alphabetically with a brief abstract for each title. The bibliography is then categorized under broad headings. Some titles may appear under more than one heading. Even though the contract requested a literature search covering the area of the implicit numerical solution of the one-dimensional transient heat transfer problem some explicit methods were considered.. Many works compared the methods so some titles covering explicit methods are included.

NOMENCLATURE

m = space index (node number)

n = time index ($n = 1,2,3,\cdots$

t = time

Δt = time increment

t_n = elapsed time = $n\Delta t$ (uniform Δt) = $\sum_{k=0}^{n-1} \Delta t_k$ (arbitrary Δt) $t_o = o$

T = temperature

Δx = distance increment

α = thermal diffusivity = $\frac{k}{cp}$

k = thermal conductivity

c = specific heat

p = density

θ = Fourier modulus = $\frac{\alpha \Delta t}{L}$

N_{Bi} = Biot number = $\frac{hL}{k}$

L = Length

RESULTS OF LITERATURE SEARCH

PART I

Errors in Implicit Finite Difference Solutions of the One Dimensional Heat Conduction Equation

Definitions

The following definitions will be used, consistent with Anderson and Botje[1].

Roundoff Error

This error is caused by the fact that all numbers used in computation must be rounded to a manageable number of digits. The error can become significant after long computations in which each calculation is dependent on the results of the previous calculation.

Convergence Error

This is the error caused by not completely satisfying the simultaneous equations when using an iterative solution. Increasing the number of iterations decreases this error.

Time Truncation Error

This error can be visualized as arising from the assumption that temperature is a linear function of time over each time step; or it can be seen to result from the fact that certain high order terms are neglected in the Taylor series type of development of the finite difference approximation to the time derivative in $\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$. Reducing the size of the time step reduces this error.

Space Truncation Error

This error can be thought of as arising from the assumption that temperature is a linear function of distance over each space increment; or it

can be visualized as a consequence of the fact that some high order terms are neglected in a Taylor series development of the finite difference approximation to the space derivative in $\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$.

An additional type of error is described by Schneider [16] and Fox[9]:

Error Caused by Boundary Discontinuity

The physical interpretation of this error is the same as that for truncation error, given above, except that near a boundary undergoing a step change of temperature, the assumption of linear temperature variation is poor, and the error increases sharply. It can be decreased by using a suitable average between the upper and lower step temperatures and by reducing the length of the time interval. This error decreases as time increases, in stable finite difference representations. In the Taylor series development, this error is seen to result from the fact that a step temperature increase represents a discontinuity in $\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$, whereas, the Taylor Series method assumes the derivatives to be continuous.

The foregoing definitions apply to both the explicit and implicit forms of finite difference representation.

SUMMARY

Roundoff

Roundoff error is not inherent in a finite difference approximation, but is a type of error associated with most numerical computations. It can become significant in calculations involving repeated use of rounded numbers. A fact to be considered when computing with finite differences is that roundoff error is nearly always present, so that even if truncation error is completely eliminated, the solution cannot be entirely free from error. The techniques to be described for reducing truncation error, consequently, have no value if the roundoff error is large enough to overshadow the effects of such techniques.

Truncation

Truncation error represents the difference between the analytical solution and the finite difference solution of a problem. Frequently, a Taylor series expansion is used to develop finite difference representations and numerical estimates of truncation error.[8,9,11]

The backward difference expression for $\frac{\partial T}{\partial t}$, developed from a Taylor series expansion, is $\frac{T_{m,n} - T_{m,n-1}}{\Delta t} + \frac{\Delta t}{2}\left(\frac{\partial^2 T}{\partial t^2}\right)_{m,n} - \frac{(\Delta t)^2}{6}\left(\frac{\partial^3 T}{\partial t^3}\right)_{m,n} + \cdots$

The finite difference expression for $\frac{\partial T}{\partial x^2}$, derived by use of a Taylor series expansion, is

$$\frac{T_{m+1,n} - 2T_{m,n} + T_{m-1,n}}{(\Delta x)^2} - \frac{(\Delta x)^2}{12}\left(\frac{\partial^4 T}{\partial x^4}\right)_{m,n} - \frac{(\Delta x)^4}{360}\left(\frac{\partial^6 T}{\partial x^6}\right)_{m,n} - \cdots$$

In forming the simple backward difference implicit approximation to the heat equation, the terms above which contain partial derivatives are neglected. If the neglected terms are considered to be the truncation error, the time truncation error in the approximation to the heat equation is:

$$-\frac{\Delta t}{2}\left(\frac{\partial^2 T}{\partial t^2}\right)_{m,n} + \frac{(\Delta t)^2}{6}\left(\frac{\partial^3 T}{\partial t^3}\right)_{m,n} - \cdots$$

and the space truncation error is

$$\frac{(\Delta x)^2}{12}\left(\frac{\partial^4 T}{\partial x^4}\right)_{m,n} + \frac{(\Delta x)^4}{360}\left(\frac{\partial^6 T}{\partial x^6}\right)_{m,n} + \cdots$$

Evaluation of these terms requires a knowledge of the analytical solution, but if they were to be computed for several cases having known analytical solutions and found to have similar values in each case, their use might be extended to cases having no known analytical solutions. Kardas[11] evaluates error terms for the case of the infinite plate with uniform initial temperature subjected to equal step changes of surface temperature at time zero. The error terms are found using the known analytical solution for the temperature distribution, and the results, given as "error parameter", are plotted against N_{Bi}, with θ as parameter. The curves presented, illustrate total truncation error only. An example is worked to illustrate the use of the curves, but no indication of the accuracy with which the derived correction approximates the true difference between analytical and numerical solutions is given.

Freed and Rallis[8] have expanded the method beyond consideration of the higher order derivative terms alone. Let δ^2_m = the central difference operator with respect to m = $T_{m+1,n} - 2T_{m,n} + T_{m-1,n}$ and let ∇_n = the backward difference operator with respect to n = $T_{m,n} - T_{m,n-1}$. Then, replacing $\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial x}$ by the backward difference implicit representation, including high order derivative terms:

$\delta^2_m T_{m,n} = \frac{1}{\theta} \nabla n \, T_{m,n} + U_{m,n}$ where $T_{m,n}$ represents the exact solution of the differential equation and $U_{m,n}$ represents the high order derivative terms of the Taylor series, which are neglected in finite difference calculations.

Also, consider $\delta^2_m W_{m,n} = 1 \nabla_n W_{m,n}$ which is the difference equation actually solved in finite difference calculations. Let $W_{m,n}$ be the exact solution of the difference equation; then the truncation error $V_{m,n}$ is defined to be the difference between the exact solution of the differential equation and the exact solution of the difference equation, or $V_{m,n}$ (truncation error) = $T_{m,n} - W_{m,n}$. This can be given by the difference between the two previous equations as $\delta^2_m V_{m,n} = \frac{1}{\theta} \nabla n \, V_{m,n} + U_{m,n}$. From this equation, error estimates can be made at each nodal point. The example used by Freed and Rallis to illustrate the method is the infinite plate at uniform temperature, subjected to identical step temperature changes at the surfaces. The analytical solution is compared to the backward difference implicit solution, and the errors predicted by the authors' method are compared to the errors predicted by computation of $U_{m,n}$ alone. The authors' method yields better error estimates than does consideration of $U_{m,n}$ only, except at the node nearest the surface. The lack of improvement at that point is attributed to boundary error. As in the previous paper, time and space truncation errors are not separated, and only total truncation error is considered.

A different approach to truncation error estimation has been used by another group. In this method, the analytical solutions for certain cases are compared directly with numerical solutions, in an effort to ascertain

truncation error relationships which can be extended to cases for which analytical solutions are not available. The time and space truncation errors are considered separately. Anderson and Botje[1] evaluate space truncation error for the case of the semi infinite slab with step temperature change at the surface. Time truncation error is reduced to an insignificant level by employing very small time steps in the numerical solution. The two nodes closest to the surface are considered, and spatial truncation error is shown to be significantly lower at the second node than at the node adjacent to the surface. Curves are presented illustrating truncation error plotted against Fourier modulus, using the Biot number as a parameter.

Anderson, Slonneger and Graybeal[3] study total and time truncation error for the semi-infinite solid and for the infinite plate with step temperature change at the surface. Spatial truncation error is determined by following the method used earlier by Anderson and Botje[1]. Then other numerical solutions are obtained using an arbitrary time increment in each solution. Subtracting the previously determined space truncation errors from the total truncation errors of these solutions yields the time truncation errors.

The conclusions of the study are, in summary:

1. By judicious use of the curves presented, values of space and time increments can be chosen in such a way that truncation error is minimized.
2. Truncation error curves for the semi-infinite body and the infinite plate closely approximate each other, indicating that the results of the paper can be applied directly to other configurations without causing large errors.
3. Time truncation errors of significant magnitude occur at the second, third, and fourth nodes from the surface, which contrasts with the limited influence of space truncation errors, mentioned above.
4. The curves illustrate the fact that decreasing the time increment will decrease the total truncation error.

This paper contains much data, in the form of curves, relating spatial, time, and total truncation error to time and space increment, to position of a node, and to elapsed time.

Space trunction error alone is treated in the paper of Murray and Landis[14]. In their analysis, only the space derivative of the conduction equation is replaced by a finite difference representation. When the resulting expression is applied to a nodal network, there is obtained a system of simultaneous first order ordinary differential equations. Solution of the system by an exact method gives a temperature distribution free from time truncation error. The method of solution employed by the authors involves reduction of the differential equations to algebraic equations by means of the Laplace transform, with the final solution obtained through matrix analysis. The case treated is a slab with equal temperature changes at the faces, and both step and ramp temperature changes are considered. The exact solution of the heat conduction equation is compared with the exact solution of the system of ordinary differential equations to determine the spatial truncation error. The results are plotted as truncation error versus Fourier modulus, with the number of nodes as parameter. The effect of the convective film coefficient is not considered in the example presented.

The methods of the papers mentioned above might be used to provide estimates of truncation error for a given set of constants used in a backward difference implicit solution of the conduction equation, or they might be used to select a set of constants which would minimize the truncation error. Another possible way to decrease truncation error is to use one of the other implicit difference equations, possessing lower inherent truncation error than does the simple backward difference equation. Two of these, the Crank-Nicolson and the Crandall equations, will be discussed briefly. The Crank-Nicolson equation is:

$$-T_{m-1,n+1} + \left(\frac{2}{\theta} + 2\right)T_{m,n+1} - T_{m+1,n+1} + T_{m-1,n} + \left(\frac{2}{\theta} - 2\right)T_{m,n} + T_{m+1,n},$$

This equation is discussed by Douglas[6], Fox[9], Gaumer[10], and Campbell, Kaplan and Moore[4]. Douglas and Fox show mathematically that the truncation error of this equation is less than that of the backward difference equation. Douglas shows the Crank-Nicolson equation to be convergent for any value of θ. He comments that the Crank-Nicolson equation provides considerably increased accuracy over the backward difference equation with a small increase in computation; but he cautions that a lack of smoothness in the solution of the heat equation will retard convergence of the Crank-Nicolson solution to a greater degree than convergence of the backward difference equation will be retarded.

Gaumer compares the backward difference and Crank-Nicolson equations, with regard to accuracy and stability, by applying both methods to a practical problem whose analytical solution is known. The problem considered is the infinite plate with equal step changes of temperature applied at each face. The results are presented as curves of temperature vs. time, with the reciprocal of the Fourier modulus as parameter. The curves illustrate several points:

1. Although both numerical solutions converge for $\theta = 4$ (largest value of θ used), convergence is not rapid, and neither of them is an accurate approximation to the analytical solution at early time for such a high modulus.
2. Using $\theta \leq 1/4$, both numerical solutions show rapid convergence.
3. With $\theta = 1/4$, the Crank-Nicolson equation offers slightly improved accuracy compared to the backward difference equation, but the drawings do not permit a precise evaluation to be made.

The Crandall equation is discussed by Crandall[5], Douglas[6], and Campbell, Kaplan and Moore[4]. It is related to the Crank-Nicolson equation, differing only in that different constants are used. The Crandall equation can be expressed as follows:

$$T_{m,n+1} - T_{m,n} = 1/2\ (\theta - 1/6)\ (T_{m+1,n+1} - 2T_{m,n+1} + T_{m-1,n+1}) + 1/2\ (\theta + 1/6)\ (T_{m+1,n} - 2T_{m,n} + T_{m-1,n})$$

Campbell, Kaplan and Moore compare the Crandall and Crank-Nicolson equations with the analytical solution for the temperature distribution in an infinite plate subjected to equal step temperature changes at the surfaces. Results are presented for only one value of Fourier modulus, which is given in the body of the paper as 1/2, but is reported as 2 in the conclusions section. The Crandall equation is shown to offer increased accuracy compared to the Crank-Nicolson equation, but no comparison of computation time is given. Another fact demonstrated by Campbell, Kaplan and Moore is that, as the time and space increments are decreased (θ maintained constant), the error of the Crandall equation is reduced more rapidly than that of the Crank-Nicolson equation.

As mentioned above, Campbell, et.al., present truncation error for only one value of Fourier modulus. Generally, truncation error is different for different values of Fourier modulus, and it is of interest to note that Crandall[5] recommends the use of $\theta = 0.2236$ in the Crandall difference equation as the value which should give the smallest truncation error.

Another approach to reducing the truncation error in finite difference approximation is an extension of the process known as Richardson's deferred approach to the limit, discussed by Douglas, Fox and Liebmann. Douglas[6], and Fox[9], demonstrate the mathematical validity of the method, and Liebmann[14], illustrates its application to a practical problem. In the example given by Liebmann, the Richardson technique is used to reduce time truncation error in the backward difference equation. The procedure will be described here: The backward difference solution is carried through using a given time increment, Δt. Then the problem is solved again using the same equations, but with the time increment doubled in length ($2\Delta t$). Then, for a given space node at a given time, the temperature $T(\Delta t)$ obtained by using Δt, is corrected by adding to it the difference between $T(\Delta t)$ and $T(2\Delta t)$, where $T(2\Delta t)$ is the temperature obtained using the time increment($2\Delta t$). Thus $T(\text{corrected}) = T(\Delta t) + [T(\Delta t) - T(2\Delta t)]$. Douglas[6] mentions that the technique can be used to reduce spatial truncation

error rather than time truncation error, by using a linear combination of two solutions, each of which uses a different value of space increment Δx, with Δt being the same in both solutions. He also comments that both spatial and time truncation error can be reduced by using a linear combination of three solutions, and, furthermore, that the basic idea can be applied to equations other than the backward difference equation.

Boundary Error

Boundary error is introduced at the start of calculations, and it decreases as the solution is advanced in the time dimension, provided the finite difference equation being used is stable. Implicit finite difference equations are stable for all values of the Fourier modulus; however, the boundary error is often excessive at early time steps unless the Fourier modulus is small. This requires that the ratio of the time step to the distance step be relatively small, which means that a large number of calculations is required to cover a given time interval. As time passes, the requirement for small $\frac{\Delta t}{\Delta x}$ is diminished because of the inherent decrease in boundary error. To reduce the amount of computation, the time increment can, therefore, be increased as the solution progresses, thus increasing θ also.

Douglas[6] and Douglas and Gallie[7] discuss variable time steps, and present two schemes for increasing the length of the time step in a systematic way. The first method results in a linear increase of the time increment as the solution of the difference equation advances in the time dimension. To determine the time interval to be used between time t_n and time t_{n+1}, use $\Delta t_n = (a + bt_n)(\Delta x)^2$ where $a>0$ and $b\geq 0$ and $t_n (n=1,2,3,) = \sum_{k=0}^{n-1} \Delta t_k$, $0<k<\infty$, $(t_0 = 0)$. The second method causes the length of the time step to grow exponentially. The time interval for use between t_n and t_{n+1} is given by $\Delta t_n = (\Delta x)^2 e^{at_n}$ where $0<a<\frac{\pi^2}{2}$ and t_n is given by the summation above. Douglas and Gallie discuss the use of these relations as applied to the backward difference implicit equation, and show that the rate of convergence of the solution is not reduced by their use, which implies that the accuracy at a given elapsed time is not reduced.

PART II

Some Methods of Solving Implicit Finite Difference Representations of the Heat Conduction Equation.

Two methods employed for solving the systems of equations resulting from use of the implicit difference equations are iteration and Gaussian elimination. Two papers comparing different variations of these processes and some references containing special techniques for use in solving the systems will be mentioned.

Anderson, Botje, and Koffel[2] employ the backward difference equation in a computer program, using Gauss-Seidel iteration to solve the simultaneous equations, for a two dimensional network of as many as 200 nodes. Two schemes are used in combination to accelerate convergence of the iteration, and they will be described later. Gaussian elimination was initially considered by these authors for solving the simultaneous equations, but it was found that excessive computation time would be required for such a large number of equations. The authors describe thoroughly the development and application of this program and give results for several industrial problems it has solved.

Several methods are available for accelerating the convergence of iterative processes, thus reducing computation time. Anderson, Botje and Koffel[2] discuss their experience with a combination of two such devices in connection with an implicit heat transfer program, commenting that time savings of as much as 75 percent have been obtained. The first of these schemes involves extrapolation, the initial value for an iteration process being obtained by extrapolating from the results of the two preceeding iteration steps. The second method is an adaptation of the technique described by Wegstein[17], which is very similar to the Aitken δ^2 process.[12,15] In the Wegstein method, the value of the unknown, x at the (k+1) iteration step, is corrected as follows: X_{k+1} (corrected) = $X_{k+1} - \frac{(X_{k+1} - X_k)^2}{X_{k+1} - 2X_k + X_{k-1}}$

The Wegstein technique accelerates convergence for simple iterative processes which exhibit either oscillatory or monotonic convergence, and also can be used to force the convergence of simple iterative processes which show oscillatory or monotonic divergence. Anderson, Botje and Koffel make two comments about this technique based on their experience with its use in their heat transfer computer program:

1. If this acceleration scheme is applied as often as every third iteration sweep, the extra machine time required to compute the correction may exceed the time saving accomplished by the acceleration.
2. A satisfactory method for determining the number of iteration sweeps between applications of the acceleration correction is to set the number of sweeps between corrections equal to the number of nodes from the boundary to the deepest node in the system.

A third acceleration technique is attributed to Steffensen. It is described briefly by Prager[15], who states that one application of this method has the same effect as three successive applications of Aitken's δ^2 process.

A modified form of Gaussian elimination, suitable for solving the tridiagonal system of equations obtained from application of the backward difference implicit equation, is described by Douglas[6], and Wilkes[18]. It is based on the fact that the difference equation at each interior node contains three unknowns, and those at the boundaries contain two unknowns. Using this property, a general expression is developed relating the unknown temperature at each node to the temperatures of adjacent nodes. An expression is obtained giving the temperature of the final node of the system explicitly. Then the temperature of each node in turn is calculated, beginning at the final node and proceeding toward the first node.

PART III

Truncation Error of Runge-Kutta Methods

If the second order (spatial) derivative alone is replaced by a finite difference expression, the one dimensional heat conduction equation becomes: $\frac{dT}{dt} = \frac{\alpha}{(\Delta x)^2}(T_{m+1,n} - 2T_{m,n} + T_{m-1,n})$. When this equation is applied to a system of heat transfer nodes, a system of simultaneous ordinary differential equations is obtained. Solution of the simultaneous system by an exact method would yield temperature values free from time truncation error. If the system is solved by a numerical scheme, truncation error will be introduced, because such methods are based on approximate relationships. For example, the Runge-Kutta method, a widely known device for solving ordinary differential equations, is developed from a Taylor series expansion, and contains a truncation error because of the fact that high order terms of the series are ignored. An estimate of the truncation error of one set of Runge-Kutta formulas is given by Prager[15].

To solve $\frac{dv}{du} = f(u,v)$ with the initial condition $v(u_o) = v_o$, let h = interval length, p = number of steps, V = approximation to v given by the Runge-Kutta equation; the most widely used Runge-Kutta equation is

$$V_{p+1} = V_p + 1/6(k_1 + 2k_2 + 2k_3 + k_4) \text{ where}$$

$$k_1 = hf(u_p, V_p);\ k_2 = hf\left(u_p + \frac{h}{2},\ V_p + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(u_p + \frac{h}{2},\ V_p + \frac{k_2}{2}\right);\ k_4 = hf(U_p + h,\ V_p + k_3)$$

The truncation error is estimated to be $\frac{1}{15}(V_{p+1} - V_p)$ and the corrected value of V_{p+1} is $V_{p+1} + \frac{1}{15}(V_{p+1} - V_p)$.

Lance [12], describes a modified Runge-Kutta procedure for digital computers which automatically adjusts the interval length to maintain a predetermined truncation error.

$$V_{p+1} = V_p + \tfrac{1}{2}(k_1 + 4k_4 + k_5)$$

$$k_1 = \tfrac{1}{3} hf(u_p, V_p);\ k_2 = \tfrac{1}{3} hf(u_p + \tfrac{1}{h_3},\ V_{p+} k_1)$$

$$k_3 = \tfrac{1}{3} hf\left(u_p + \frac{h}{3},\ V_p + \frac{k_1}{2} + \frac{k_2}{2}\right)$$

$$k_4 = \tfrac{1}{3} hf\left(u_p + \frac{h}{2},\ V_p + \frac{3k_1}{8} + \frac{9k_3}{8}\right)$$

$$k_5 = \tfrac{1}{3} hf\left(u_p + h,\ V_p + \frac{3k_1}{2} - \frac{9k_3}{2} + 6k_4\right)$$

The truncation error ε is estimated by

$5\varepsilon = k_1 - \frac{9k_3}{2} + 4k_4 - \frac{k_5}{2}$. If the right hand side of this equation is greater than five times the allowable error, h is halved and the computation for the step is repeated; but if the right side is less than 5/32 of the allowable error, the interval may be doubled and the computation repeated. Although this process would seem to require considerable extra time, Lance reports that it usually reduces the computation time required to attain a given accuracy of solution by about 20 per cent. He accounts for the reduction by mentioning that when a fixed interval length is used throughout the solution, this length is usually deliberately underestimated to insure accuracy, thus requiring the use of more steps than necessary. The self adjusting procedure eliminates more than enough of these extra steps to offset the time required for the extra numerical manipulations it requires, thus effecting an overall saving of computation time.

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15. Prager, W., "Introduction to Basic FORTRAN Programming and Numerical Methods", Blaisdell, New York, 1965, p. 171, 181

16. Schneider, P.J., "Conduction Heat Transfer," Addison-Wesley, Reading, Mass., 1955, p. 295

17. Wegstein, J.H., "Accelerating Convergence of Iterative Processes," Communications of the Association for Computing Machiner, Vol. 6, 1958, p. 9

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Appendix C

ANNOTATED BIBLIOGRAPHY

prepared by

University of Nevada
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1. Aitken, A.C., "Studies in Practical Mathematics: II, The Evaluation of the Latent Roots and Latent Vectors of a Matrix, "Proceedings of the Royal Society of Edinburgh, v. 57, p. 269, 1937

Aitken's δ^2 accelerated convergence method is described

2. Aitken, A.C., "Studies in Practical Mathematics: VI, On the Factorization of Polynomials by Iterative Methods," Proceedings of the Royal Society of Edinburg, Sect. A. 63, p. 174-91, 1951.

This article presents the theory of an iterative method for the approximation of an exact factor of a polynomial. The matrix governing the iterative process is obtained, and its latent roots and vectors are found. Convergence of the process discussed and processes are developed for the acceleration of convergence.

3. Albasiny, E.L., "On Numerical Solution of Cylindrical Heat Conduction Problem," Quarterly Journal of Mechanics and Applied Mathematics, Vol. 13, part 3, Aug. 1960, p. 374-84

This article treats the use of automatic computers for the numerical solution of the cylindrical heat conduction problem. It is shown that accurate solutions can be obtained easily and rapidly using the Crank-Nicolson implicit method. Attention is given to the adequacy of the finite-difference representation near a singularity at the boundary.

4. Allen, D.N. deG., and Severn, R.T., "The Application of Relaxation Methods to the Solution of Non-Elliptic Partial Differential Equations. I; the Heat Conduction Equation," The Quarterly Journal of Mechanics and Applied Mathematics, Vol. 4, p. 209-22, 1951

The equation considered is $\partial u/\partial t = k\partial^2 v/\partial x^2$. The authors make the transformation $v = \partial\omega/\partial t + k\,\alpha^2\omega/\alpha x^2$, which gives the equation $\partial^2\omega/\partial t^2 - k^2\partial^4\omega/\partial x^4 = 0$. The boundary conditions are also transformed and new ones added.

5. Allione, M.D., "Comparative Study of Runge-Kutta and Lanczos Numerical Integration Methods", Rept. No. U2421, Contract AF 19 628 562, ESD TDR63 662, Aeronutronic, Newport Beach, Calif., 9 Jan. 1964, 24 p. AD-429 958

The two methods are compared in solving the system of ordinary differential equations associated with the Variation of Paramenters formulation. Results of ephemeris calculations using each method are compared with a standard to determine the relative error growth. Conclusions are drawn regarding the relative merits of the two methods.

6. Anderson, J.T., "Review of Digital Computer Heat Transfer Programs" ASME Paper 65-WA/HT-48, 7 p.

This paper is a review of the available steady state and transient programs. Emphasis is given to the capabilities and limitations of general purpose programs, both explicit and implicit. Indications of computer time are given and discussion of the error magnitude is included.

7. Anderson, J. T., and Botje, J.M., "Spatial Truncation Error Analysis, "ASME paper No. 62-HT-27

A method is presented for evaluating spatial truncation errors in a finite difference solution of a parabolic partial differential equation.

A one-dimension transient heat transfer problem is used as an example. Curves are presented for rapid evaluation of the spatial truncation error.

8. Anderson, J.T., Botje, J.M., Koffel, W.K., "Digital Computer Solution of Complex Transient Heat Transfer Problems," W.Va. Univ. Bulletin, Engr. Experiment Station, Technical Bulletin No. 62, 26 p.

The authors describe a comprehensive computer program for heat transfer transient problems involving convection, conduction, contact resistance, solid and gaseous radiation, surface flux and internal heat generation. The Liebman backward time step approximation was used in developing the program, and the difference equations obtained by the Liebman method are presented with a discussion of methods used to solve them on a digital computer.

9. Anderson, J.T., Slonnegar, R.D., Graybeal, G.E., "Truncation Error Analysis for Transient Heat Transfer Calculations." unpublished

"A total and time truncation error analysis of numeric solutions of parabolic partial differential equations is herein reported. Mathematical models include transient heat conduction in an infinite body and an infinite plate with (1) convective heat transfer and (2) a step function temperature change on the surfaces. Results are tabulated as well as shown graphically." author's abstract

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Author's introduction: "An explicit finite difference approximation procedure which is unconditionally stable for the solution of the general multi-dimensional, non-homogeneous diffusion equation is presented. This method possesses the advantages of the implicit methods, i.e., no severe limitation on the size of time increment. Also it has the simplicity of the explicit methods and employs the same "marching" type technique of solution. Results obtained by this method are compared with the exact solution and with those obtained by other finite difference methods. For the examples solved the numerical results obtained by the present method are in closer agreement with the exact solution than are those obtained by other methods."

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This paper compares Hill's method (NACA Tech. Note 4105) with Dusinberre's method. In Hill's method, finite differences are taken only in the time variable, the equations used being already integrated with respect to distance. The authors conclude that Hill's method is, practically,

an exact method and is faster than Dusinberre's, only the two surface temperatures are needed. If the temperature distribution is needed, Hill's method is slower, but very accurate.

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This is an elementary text covering the solution of algebraic equations, finite differences, interpolation, numerical differentiation and integration, and the solution of ordinary differential equations.

14. Campbell, B.C, "An Investigation of the Accuracy of Numerical Solution's in the Diffusion Equation for Transient Heat Transfer," Master's thesis, AFIT GSF/Phys/64-1, A.F. Inst. of Technology, Wright-Patterson AFB, Ohio, Aug. 1964, 118 p. AD-605-489

This paper gives the results of a comparison of the accuracy obtained using two implicit finite difference representations of the transient heat conduction equation in obtaining solutions for two basic heat conduction problems. The representations compared are the Crank-Nicolson form and a theoretically more optimum form called the Optimum Implicit form. It is demonstrated that the major source of error occurs at the initial time/space node of the difference mesh. The Optimum Implicit form decreases this error and is shown to be as accurate as the Crank-Nicolson form in one problem and considerably more accurate in the other.

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The author exhibits a partial difference equation which is unstable but has stable characteristics locally. He points out that a stability analysis based on the procedure of O'Brien, Hyman, and Kaplan (J. Math. and Phys. V. 29, p. 223-51, 1951) is not valid for this type of equation.

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The article presents a comparison of three methods for the solution of the non-linear equation of heat flow in a medium where heat is being generated. The first method consists in a reduction to a system of ordinary differential equations by approximating the time derivatives with differences. The second method is the same except that the space derivatives are approximated by differences rather than the time derivatives. In the third method, all derivatives are approximated by differences, and the authors conclude this to be much faster and more satisfactory. The third method gives a system of non-linear algebraic equations, which is solved by a combination iterative and step by step method, and several variations of this method are discussed.

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This paper presents the results of an investigation of two methods for increasing the convergence rate for the two dimensional, five point implicit finite difference representation of the diffusion equation of transient heat transfer, the methods being the adapted Wegstein technique and successive overrelaxation. Various mesh scanning techniques are investigated. An example problem is used to show that successive overrelaxation with a technique of repeatedly scanning all boundary values into the finite difference mesh is the fastest method of solution for the equations used in this study. Solution of 28 other problems by this method shows an approximate saving of 1/3 in the number of iterations over successive overrelaxation with a conventional repetitive scanning technique. An "a priori" relaxation factor related to the maximum temperature gradients is obtained.

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Let $X_{n+1} = X_n + F(X_n)$ be a scalar iterative converging process. When X_n is close to the limit, $F(X_n)$ is small and can perhaps be obtained with higher absolute precision than X_n. Then the addition $X_n + F(X_n)$ involves a round off operation. The author shows that, for a fixed-point computer, an appropriate rounding method can impove the accuracy of solution. Appendix I gives analogous results for a floating-point computer. Appendix II deals with Aitken's δ^2 process.

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    The authors consider the numerical solution of $U_t = U_{xx}$, using a backward difference equation known to be stable for all Δx and Δt. Two cases of variable Δt are considered.

25. Douglas, Jr., J., "The Solution of the Diffusion Equation by a High Order Correct Difference Equation," Journal of Mathematics and Physics, Vol. 35, p. 145-51, 1956

    The author proposes a six-point implicit difference scheme for the solution of $U_t = U_{xx}$ with a smaller truncation error than the Crank-Nicolson form. Because the error is smaller, the new method allows larger ΔX and Δt to be used. (The method is described also in the book "Advances in Computers, Vol. II, p. 26.)

26. Douglas, Jr., J., "The Effect of Round-Off Error in the Numerical Solution of the Heat Equation," Association for Computing Machinery Journal, Vol. 6, No. 1, Jan. 1955, p. 48-58

    The artical presents an analysis of the approximation to the heat equation by the backward difference equation when boundary value problems are approximated by finite difference problems. The dependence on the method of solving the tri-diagonal equations is shown. It is shown that if linear equations are solved by a normalized form of Gaussian elimination, the procedure is stable against round-off error.

27. Dusinberre, G.M., "A Note on the 'Implicit' Method for Finite-Difference Heat-Transfer Calculations," ASME Journal of Heat Transfer, Vol. 83, p. 94 1961

    "The apparent advantage of the implicit method lies in the possibility of using relatively large time intervals. But this may be accompanied by (1) considerable sacrifice in accuracy and (2) no corresponding saving in digital time." This is the author's introduction to the article, which discusses points (1) and (2) above.

28. Dusinberre, G.M., "Numerical Methods for Transient Heat Flow," ASME Transactions, Vol. 67, p. 703-10, 1945

    A modulus is developed by choice of which a solution may be reached most rapidly or alternatively reached more slowly but with greater precision. Criteria are developed for choosing the modulus to insure convergence. A method is developed for handling thermal conductivity and heat capacity when they vary independently with temperature.

29. Elliot, D., "A Method for the Numerical Integration of the One-Dimensional Heat Equation Using Chebyshev Series," Proceedings of the Cambridge Philosophical Society, Vol. 57, p. 823-32, 1961

The equation used is $\partial\theta/\partial t = \partial^2\theta/\partial x^2$ $(-1<X<1; t>0$ with general linear boundary conditions along $x = \pm 1$. $\partial\theta/\partial t$ is replaced by a finite difference approximation and the resulting system of ordinary differential equations is solved by Clenshaw's method of θ in terms of Chebyshev polynomials. Two examples are worked and the results compared with exact results. The author concludes that the present method requires less computation than the usual finite difference methods, but is less versatile and not so well suited for complicated equations.

30. Elrod, Jr., H.G., "New Finite-Difference Technique for Solution of the Heat-Conduction Equation, Especially Near Surfaces with Convective Heat Transfer," ASME Transactions, Vol. 79, p. 1519-25, 1957

The success of most finite difference methods for transient heat conduction depends on the existence of a certain degree of uniformity of behavior of the temperature over the time and space intervals selected for computation. This often requires the use of inconveniently short time intervals. This paper represents an effort to develop a finite difference method not possessing such a defect.

31. Emmons, H. W., "The Numerical Solution of Heat Conduction Problems," ASME Transactions, Vol. 65, p. 607-12, 1943

The author discusses the application of the Southwell relaxation method to two and three dimension steady state heat conduction. A transient problem is also briefly considered. A short discussion on "Derivation of Difference Equations from Differential Equations," is included.

32. Emmons, H. W., "The Numerical Solution of Partial Differential Equations" Quarterly of Applied Mathematics, Vol. 2, p. 173-95, 1944

The author presents a detailed expository treatment of Southwell's relaxation process. Examples illustrate the application to the solution of boundary value problems for the Laplace, Poisson and other equations.

33. Enig, J.W., "A Method for the Rapid Numerical Solution of the Heat Conduction Equation for Composite Slabs," NAVORD Rept. 6666 Naval Ordnance Lab., White Oak, Md., 20 Aug 59, 22 p. PB 144 193

Two boundary conditions encountered in heating one dimensional double slabs are: heat flux given at inner surface and (a) heat flux or (b) temperature given at the outer surface. A method is developed which permits rapid calculation of (1) any interior point temperature and (2) the outer surface temperature or flux for (a) or (b) respectively, without computing other interior point temperatures. The partial differential equations are integrated in terms of an arbitrary outer surface temperature or flux by a simple numerical scheme. The method performs an exact integration over the space dimension, so once the outer surface temperature is determined the interior temperatures are computed by exact formulae. The numerical solutions are compared to the exact solution for accuracy and to other numerical schemes for speed.

34. Forsythe, G.E., and Wasow, W.R., "Finite-Difference Methods for Partial Differential Equations", John Wiley and Sons, Inc., New York-London 444 p., 1960

The heat equation $U_t = U_{xx}$ is considered (in part 2 of book) for $-\infty < x < \infty$. A forward difference equation is developed and studied from the standpoints of stability, convergence, and discretization error. Also in part 2, for linear problems on a finite interval, the forward difference, backward difference, Crank-Nicolson, and Dufort-Frankel schemes are considered with respect to stability, convergence, and discretization method.

35. Fowler, C.M., "Analysis of Numerical Solutions of Transient Heat Flow Problems," Quarterly of Applied Mathematics, Vol. 3, p. 361-76, 1946

Solutions of the difference equation for the temperatures $T_{x,t}$ for one dimensional conduction are considered. The author uses contour integrals of particular solutions of the difference equation to give the temperatures in terms of polynomials and trigonometric functions. The convergence of his solutions to the well-known solutions of the corresponding problems in partial differential equations as Δx and $\Delta t \to 0$ is investigated.

36. Fox, L. (editor), "Numerical Solution of Ordinary and Partial Differential Equations," Addison-Wesley Publ. Co., Inc., Palo Alto, California, 1962, 509 p.

Various finite difference schemes for partial differential equations are discussed with regard to convergence, stability and computational error.

37. Fox, L., "Some Improvements in the Use of Relaxation Methods for the Solution of Ordinary and Partial Differential Equations," Proceedings of the Royal Society of London, Series A, Vol. 190, p. 31-59, 1947

Boundary value problems associated with ordinary or partial differential equations are commonly solved by the use of difference equations which are solved by successive approximations. Usually a derivative is replaced by the leading term of a finite difference series for the derivative and a small interval is used to obtain the desired accuracy. The author proposes to use higher order differences and a larger interval, which gives a smaller number of unknowns to be found. Two examples are worked, one being Poisson's equation. Two further examples show the application of the method to curved boundaries.

38. Frankel, S.P., "Convergence Rates of Iterative Treatments of Partial Differential Equations, Mathematical Tables and Other Aids to Computation, Vol. 4, p. 65-75, 1950

Convergence rates are estimated for several iterative methods of solution for the Laplace and biharmonic equations. The methods used for the Laplace equations are (1) Richardson, (2) Liebmann, $\alpha = 1/4$, (3) Liebmann, optimum α, (4) second order Richardson. Quoted from the author's conclusions: "It is thus seen that with a fairly fine mesh the calculating time required with the slower machines is uncomfortably long for the Laplace equation ... if the normal Richardson method is used."

39. Freed, N.H., and Rallis, C.J., "Truncation Error Estimates for Numerical and Analog Solutions of the Heat Conduction Equation," ASME Journal of Heat Transfer, Vol. 83, p. 382-3, 1961

The authors describe a method for obtaining an estimate of truncation error for fully finite difference forms of the heat conduction equation. It may be used with manual and analog methods if the error due to mesh size is relatively large. Error estimates for a case of one dimensional flow are derived in an example, using the backward difference equation.

40. Frocht, M.M., "A New Approach to the Numerical Solution of Laplace's Equations," Numerical Methods of Analysis in Engineering, p. 75-80 The MacMillian Co., N.Y., 1949.

The author discusses a procedure for obtaining good initial values for interior grid points which is applicable to iterative and relaxation methods. The procedure requires known boundary values and is termed the Linear Rosetta Method.

41. Gaumer, G.R., "The Stability of Three Finite Difference Methods of Solving for Transient Temperatures," presented at the Fifth U.S. Navy Symposium on Aeroballistics on 18 Oct. 1961, 23 p.

The paper contains the results of stability investigations for three types of finite difference equations; which are the forward, mid and backward difference types. For each type, the one dimensional heat balance equations are presented for three combinations of heat transfer modes, which are (1) conduction, (2) conduction and convection, (3) conduction, convection and radiation. A stability criterion is developed for each type of equation with each combination of modes.

42. Gill, J., "A Process for the Step-by-Step Integration of Differential Equations in an Automatic Digital Computing Machine," Proceedings of the Cambridge Philosophical Society, Vol. 47, p. 96-108, 1951

The article presents a modified 4th order Runge-Kutta process for systems of 1st order ordinary differential equations. The process described requires a small number of storage spaces for each integration step. The effect of truncation and round-off error is discussed and illustrated by a numerical example.

43. Goodwin, E.T., Clenshaw, C.W., Martin, C.W., Miller, G.F., Olver, F.W.J., and Wilkinson, J.H., "Modern Computing Methods, ". Philosophical Library, N.Y., 170 p., 1961

The book includes five chapters on matrices, two each on ordinary and partial differential equations, and one on finite difference methods. Comments are made on adaptation to desk and automatic computation and careful attention is paid to the assessment of computational error for the methods described.

44. Graybeal, G. E., "Time and Total Truncation Error Analysis in Heat Transfer Calculations", Master's Thesis, West Virginia University, 1963, 59 p.

A method is presented for evaluating time and total truncation errors encountered in a finite difference solution of problems defined by a parabolic partial differential equation. The example problem is a one-dimensional heat transfer situation with a step function temperature change on the surface. Curves are presented to aid in the evaluation of time truncation and total truncation error.

45. Greenwood, J. A., "Implicit Numerical Methods for the Heat Conduction Equation," British Journal of Applied Physics, Vol. 13, No. 11, p. 571-2

The author shows that the Liebmann finite difference approximation is to be preferred to the Crank-Nicolson form in certain cases involving variable diffusivity.

46. Hamming, R. W., "Numerical Methods for Scientists and Engineers," McGraw-Hill Book Co.., Inc., San Francisco, 1962, 411 p.

The book includes a discussion of difference calculus and difference equations. Functions of more than one variable are not treated.

47. Hawkins, G.A., and Agnew, J. T., "Solution of Transient Heat Conduction Problems by Finite Differences," Purdue Univ. Eng. Experiment Station Research Series, No. 98, 1947, 38p.

The purpose of the authors is to bring the treatment for slabs, cylinders, and spheres together. Analytical methods are discussed in detail for unidirectional flow in slabs and radial flow in cylinders.

48. Henrici, P., "Elements of Numerical Analysis," J. Wiley and Sons, Inc., N. Y., 328 p., 1964.

This is an introductory book and it presents a small range of subjects with thorough coverage on each rather than treating a large number of techniques superficially. It contains much material on difference equations. Difference methods for ordinary differential equations are developed using Taylor's series. The Runge-Kutta method is treated only briefly.

49. Herriot, J.G., "Methods of Mathematical Analysis and Computation," J. Wiley and Sons, Inc., N.Y., 198 p. 1963

This book is intended for use by engineers, and it deals with only the best known numerical methods. The emphasis is on methods suitable for use on high speed computers. The subjects covered include: Numerical differentiation and integration, roots of equations, solution of simultaneous linear equations, solution of ordinary differential equations, solution of partial differential equations.

50. Hill, P.R., "A Method of Computing the Transient Temperature of Thick Walls from Arbitrary Variation of Adiabatic Wall Temperature and Heat Transfer Coefficient," NACA Tech. Note 4105, 5? p., Oct. 1957

Quoted from the author's introduction " ... simple method is developed for the calculation of the temperature history of the surfaces of a thick wall or of any plane within the wall. The procedure is to select from a table a set of coefficients which depend on the physical properties of the wall. These coefficients and other data are substituted into explicit algebraic formulas to determine the temperature of the heated wall surface. If the heat transfer coefficients are known, no guess or iteration procedure is required. .... For equal time step sizes, the method is more accurate than more laborious numerical methods." The method uses concepts called 'time series' and ' unit triangle variation of surface temperature'.

51. Hyman, M.A., "On the Numerical Solution of Partial Differential Equations," Thesis, Technishe Hogeschool te Delft, 108 p., 1953.

The paper consists of four chapters and an appendix. Stress is placed on methods suitable for use with computers. Ch. 1 treats convergence and stability of difference equation solutions; Chp. 2 treats convergence and extrapolation of difference solutions for parabolic equations, Ch. 3 discusses elliptic equations; Ch. 4, is on hyperbolic equations.

52. Juncosa, M.L., and Young, D.M., "On the Convergence of a solution of a Difference Equation to a Solution of the Equation of Diffusion," Proceedings of the American Mathematical Society, Vol. 5, p. 168-74, 1954.

Several sharp convergence theorems are proved. The equation treated in the principal theorem is $\partial u/\partial t = \partial^2 u/\partial x^2$. The approximation used is:

$\frac{U_m(X,t+\Delta t) - U_m(X,t)}{\Delta t} = \frac{U_m(X+\Delta X,t) - 2U_m(X,t) + U_m(X-\Delta X,t)}{(\Delta x)^2}$ with $0<\Delta t/\Delta x^2 \leq 1/2$, $M\Delta X = 1$. $U_m(X,t)$ is shown to converge uniformly to $u(x,t)$ in the region $0 \leq X \leq 1$ $t \geq t_0 > 0$ as $m \to \infty$.

53. Juncosa, M.L., and Young, D.M., "On the Order of Convergence of Solutions of a Difference Equation to a Solution of the Diffusion Equation," Journal of the Society for Industrial and Applied Mathematics, Vol. 1, p. 111-35, 1953

For $\partial u/\partial t = \partial^2 u/\partial x^2$ is substituted

$$\frac{v(X,t+\Delta t) - v(x,t)}{\Delta t} = \frac{v(x+\Delta x,t) - 2v(x,t) + v(x-\Delta x,t)}{(\Delta x)^2}.$$

The difference solution is discussed with regard to attempts to improve the solution by "extrapolation to zero grid-size."

54. Kaplan, B. and Clark, N., "Accuracy and Convergence Techniques for Implicit Numerical Solution of the Diffusion Equation for Transient Heat Transfer," Transactions of the American Nuclear Society, Vol. 4, No. 1, p. 80-81, June 1961

55. Kardas, A., "Errors in a Finite-Difference Solution of the Heat Flow Equation, ASME Journal of Heat Transfer, Vol. 86, p. 561-2, 1964

"This note gives magnitudes of discretization errors incurred in a finite difference solution of the heat flow equation in a symmetric slab with the boundary conditions of the third kind." author's abstract

56. Kunz, K.S., "Numerical Analysis," McGraw-Hill Book Co., Inc., New York, 381 p., 1957.

The book was written for engineers. It includes numerous, illustrative examples. Iterative methods receive only cursory attention. The book includes chapters on ordinary and on partial differential equations. An appendix on the estimation of errors in numerical computation is included.

57. Lance, G.N., "Numerical Methods for High Speed Computers," Iliffe and Sons, Ltd. London, 166 p., 1960

General text with descriptions of methods, often brief, some widely used methods are omitted. Discusses Runge-Kutta methods and description of Aitken's δ^2 process.

58. Lapidus, L, "Digital Computation for Chemical Engineers," McGraw-Hill Book Co., Inc., N.Y. 406 p., 1962.

General text which includes polynomial approximation, ordinary and partial differential equation, matrix solution of systems of linear algebraic equation, and etc.. Contains a description of the Tridag method.

59. Larkin, B.K., "Some Finite Difference Methods for Problems in Transient Heat Flow," Chemical Engineering Progress Symposium Series, Vol. 61, No. 59 1965, p. 1-11

Four explicit methods for digital solution of transient heat flow are compared. The superior stability of the newer methods is noted. The discussion is relevant to the design of booster vehicles, launch systems, vehicles for space travel, and re-entry heat shields.

60. Lax, P.D., and Richtmyer, R.D., "Survey of the Stability of Linear Finite Difference Equations," Communications on Pure and Applied Mathematics, Vol. 9, p. 267-93, 1956

The paper treats the numerical solution of initial value problems by finite difference methods by a sequence of calculations with increasingly finer mesh. The question is whether the solution converges to the true solution of the initial value problem as the mesh is made finer. A stability definition is given in terms of the uniform boundedness of a certain set of operators, and it is shown that, with suitable circumstances, for linear initial value problems, stability is necessary and sufficient for convergence in a certain uniform sense for arbitrary initial data. Two different approximations to the heat equation are considered, one being a general two level formula and the other the DuFort-Frankel equations.

61. Lea, R.N., "Stability of Multistep Methods in Numerical Integration," NASA TN D-2772, Manned Spacecraft Center, Houston, Texas, April, 1965, 16 p.

The paper contains a discussion of the stability of solutions of differential equations obtained by using difference equations. An original development leading to a definition of stability shows a relationship between stability and certain properties of the differential equation to be solved. The example worked in the paper is an ordinary differential equation.

62. Lees, M., "Approximate Solutions of Parabolic Equations," Journal of the Society for Industrial and Applied Mathematics, Vol. 7, p.167-83,1959

The author discusses the convergence of numerical solutions of partial differential equations. The analysis is based on energy methods. The author derives a priori bounds for solutions of linear parabolic difference equations, then applies them to establish the convergence of a difference solution to a non-linear parabolic equation. Crank-Nicolson type difference equations are also treated.

63. Lees, M., "A Priori Estimates for the Solutions of Difference Approximations to Parabolic Partial Differential Equations," Duke Mathematical Journal, Vol. 27, p. 297, 311, 1960.

The author derives, using energy methods a priori estimates for the solutions of several difference analogs of parabolic partial differential equations. All standard two level difference equations are discussed and two simple three level formula are also treated. Arguments are presented in detail for the heat equation, and generalizations are indicated.

64, Leppert, G., "Stable Numerical Solution for Transient Heat Flow," ASME Paper No. 53-F-4; also published Amer. Soc. Naval Engrs. Journal, Vol. 65, No. 4, Nov. 1953, p. 741-52

An implicit finite difference formula for numerical integration of the conduction equation is described. It is shown to offer a computing time saving over previously used methods. The method is a simple algebraic procedure for use on a desk calculator, which removes the necessity for iteration or substitution at each time step.

65. Leutert, W., "On the Convergence of Approximate Solutions of the Heat Equation to the Exact Solution, "Proceedings of the American Mathematical Society, Vol. 2, p. 433-39, 1951

The author discusses O'Brien, Hyman, and Kaplan's criticisms of the Richardson difference equation (criticisms contained in "A study of the Numerical Solution of Partial Differential Equations" by above named authors). He shows that the Richardson difference equation is always convergent if the initial values of the solution are chosen in a specifid way.

66. Leutert, W., "On the Convergence of Unstable Approximate Solutions of the Heat Equation to the Exact Solution," Journal of Mathematics and Physics, Vol. 30, p. 245-51, 1952.

The differential equation considered is $du/dt = du^2/dx^2$. The finite difference equation considered is $v(X,t+\Delta t) - v(X,t) = r[v(x+\Delta x,t) + v(X-\Delta X,t) - 2v(x,t)]$, where $(\Delta X)^2 r = \Delta t$. It is known that for $r>1/2$ the solution of the finite difference equation is unstable. The author proves that, even so, there exist for every fixed r, solutions of the difference equation that converge to the solution of the differential problem as $\Delta X \to 0$.

67. Liebmann, G., "Solution of Transient Heat Flow and Heat Transfer Problems by Relaxation," British Journal of Applied Physics, Vol. 6, No. 4, Apr. 1955, p. 129-35

This illustrates that by choosing a suitable finite difference approximation, parabolic partial differential equations can be converted into a series of boundary value problems of Poisson type, which can be solved by the Southwell relaxation technique. A very stable solution is obtained for all values of the time interval by using a backward difference approximation.

68. Lotkin, M., "On the Accuracy of Runge-Kutta's Method", Mathematical Tables, and Other Aids to Computation, Vol. 5, p. 128-33, 1951

The author obtains a bound for the error in Kutta's fourth order method (generally know as the Runge-Kutta method).

69. Lotkin, M., "The Numerical Integration of Heat Conduction Equations," Journal of Mathematics and Physics, Vol. 37, p. 178-87, 1958

The author discusses difference equation approximations to the equations of unsteady one-dimensional heat conduction in composite media: $\phi(M)\ \partial u/\partial f = \partial/\partial x\ (k^{1(m)}\ u/\partial u)$ where $k^{(m)}$ and $\phi^{(m)}$ denote known functions of u. The convergence is established for $k^{(m)}$ = constant and the convergence rate is estimated. An example is given, and numerical data compares the approximate and exact solutions.

70. Luke, Y.L., "Numerical Solution of the Heat Conduction Equation," Chemical Engineering, Vol. 68, No. 1, Jan. 9, 1961, p. 95, 102

The article discusses the numerical integration of the heat conduction equation, and the computation of flux and heat transfer. Included are tables listing numerical solutions for constant thermal coefficients. Variable diffusivity is also discussed.

71. Lynn, L.L., and Meyer, J.E., "A Numerical Comparison of the Implicit and Explicit Techniques for the Convective Boundary Condition," ASME Journal of Heat Transfer, Vol. 85, p. 280, 81, 1963.

The article compares the results of the Crank-Nicolson implicit method with those of Back's explicit method for 1000°F step change in ambient temperature. Some comments and conclusions are: (1) The implicit

calculations for surface temperature are, in most cases, more accurate than the Back method for a given time step size. (2) The implicit method permits variable mesh spacing with no concern for stability. (3) The use of Gauss elimination in the implicit method involves more effort than the explicit method per point per time step, but a decrease in the number of time steps needed for a given accuracy often offsets this.

72. Macon, N., "Numerical Analysis," John Wiley and Sons, Inc., N.Y., 1963, 161 p.

In this book, the emphasis is on methods for use with digital computers. Some of the topics covered are: Iterative methods for solving equations; matrices and systems of linear equations; difference equations.

73. Mann, W.R., Bradshaw, C.L., Cox, J.G., "Improved Approximations to Differential Equations by Difference Equations," Journal of Mathematics and Physics, Vol. 35, p. 408-15, 1957

This article shows that the truncation error in using a difference equation to approximate a differential equation can be reduced by modifying the coefficients of the difference equation from those normally used. The difference equation is expanded in a Taylor's series and the expansion compared with the leading terms of the original difference equation. The comparison provides a correction which allows a reduction in truncation error without increasing the order of the difference equation.

74. Martin, D.W., "Runge-Kutta Methods for Integrating Differential Equations on High Speed Digital Computers," The Computer Journal, Vol. 1, p. 118-123 1958

The author describes three adaptations of Runge-Kutta procedures for ordinary differential equations, due to Gill, Strachey, and Boulton; and he proposes an alternate method devised to save storage space and based on Kutta's Simpson rule method. Comparative errors and computational experience with the various methods are described.

75. Milne, W.E., "Note on the Runge-Kutta Method," Journal of Research, National Bureau of Standards, U44, p. 549-50, 1950

Comparison of the Runge-Kutta method with a step-by-step method of numerical quadrature shows the Runge-Kutta technique to be much less accurate in some cases.

76. Milne, W.E., "Numerical Solution of Differential Equations," J. Wiley and Sons, Inc., New York, 275 p., 1953.

The book offers general coverage, including both ordinary and partial differential equations. Notes on large scale computers occur in an appendix.

77. Milne, W.E., "Numerical Methods Associated With Laplace's Equation," Proceedings of a Second Symposium on Large Scale Digital Calculating Machinery, 1949, p. 152-63, Harvard University Press, Cambridge, Mass.,1951

This is a review of some difficulties which occur in solving partial differential equations by the method of differences, using large scale digital machines.

78. Mitchell, A.R., "Round-Off Errors in Solution of Heat Conduction Equation by Relaxation Methods," Applied Science Research, Sect. A, Vol. 4 n. 2, 1953, p. 109-19

A method is developed for assessing the magnitude of round-off errors, a stable six point finite difference approximation is used, which is relaxational in distance coordinate and step by step in time coordinate of the residuals. Formulas are derived for obtaining round-off errors for several different distributions of residuals.

79. Mitchell, A.R., Fairweather, G., "Improved Forms of the Alternating Direction Methods of Douglas, Peaceman, and Rachford for solving Parabolic and Elliptic Equations," Numerische Mathematik, Vol. 6, p. 285-92, 1964

For the heat conduction equation $U_t = U_{xx} + U_{yy}$ on a rectangular domain, the authors derive generalizations of the Peaceman-Rachford alternating direction difference schemes. A special choice of parameters leads to a stable scheme with fourth order accuracy.

80. Mitchell, D.B., "An Error Analysis of Numerical Solutions of the Transient Heat Conduction Equation," Master's Thesis, Rept. No. GA/PH/65-10, AF Inst. of Tech., Wright-Patterson AFB, Ohio, Aug. 1965, 111 p. AD-621 274

The paper presents a comparison of the Crank-Nicolson and Crandall methods in finding transient temperatures in a semi-infinite slab with convection. The results are compared with an exact analysis. The Crandall method gives more accurate results than the Crank-Nicolson under close node spacing conditions. Accuracy improvement factors are determined for the two methods.

81. Muchnik, G. F., "Solution of Heat Conduction Problems by the Grid Method," NASA TTF-151, April 1964, 15 p. (translated from Russian).

An approximate method is suggested for solving heat conduction problems using the grid method. The temperature of any point is related to the temperatures of other points by coefficients of relationship or "weights", which do not depend on the boundary conditions. The weights are found by a finite difference method. The author claims this method to be simpler and more exact than the usual finite difference method.

82. Murray, W.D., and Landis, Fred, "The Effect of Spacewise Lumping on the Solution Accuracy of the One-Dimensional Diffusion Equation," ASME Journal of Applied Mechanics, Vol. 29, p. 629-36, 1962.

The authors evaluate the truncation errors inherent in a spacewise difference formulation of the one-dimensional heat diffusion equation under general boundary conditions. The error between the semi-discrete and exact solutions is evolved by matrix algebra and the Laplace transform. An illustration shows the errors for the case of a symmetrically heated slab.

83. Nielsen, K.L., "Methods in Numerical Analysis," The MacMillan Co., New York, 382 p., 1956.

The author emphasizes methods suitable for desk calculating. One chapter treats ordinary and partial differential equations, but it presents only an outline of the topic. The methods of Euler, Milne, and Runge-Kutta are given for ordinary differential equations, Liebmann's method, relaxation, and step by step methods are given for partial differential equations.

84. O'Brien, G.G., Hyman, M.A., and Kaplan, S., "A Study of the Numerical Solution of Partial Differential Equations," Journal of Mathematics and Physics, Vol. 29, p. 223-51, 1951

This article is a discussion of methods for the analysis and improvement of the stability of finite difference equations used in the numerical solution of partial differential equations. A discussion of convergence is included. The heat conduction equation of the form $\partial\phi/\partial t = \partial^2\phi/\partial x^2$ is used as an illustration. The equivalent difference forms of Richardson, von Neumann and Hartree are discussed.

85. Peaceman, D.W., and Rachford, H.H., "The Numerical Solution of Parabolic and Elliptic Differential Equations," Journal of the Society for Industrial and Applied Mathematics, Vol. 3, p. 28-41, 1955.

The authors treat $U_t = U_{xx} + U_{yy}$ by implicit difference methods. They develop a process which is stable for all mesh ratios $((\Delta x)^2/t)$. They discuss the solution over a square, showing the work saving for their method compared to usual ones. For elliptic equations their method is a form of line relaxation rather than the usual point relaxation schemes.

86. Plunkett, R., "On the Rate of Convergence of Relaxation Methods," Quarterly of Applied Mathematics, V. 10, p. 263-66, 1952.

This article compares the convergence rate for the relaxation method of solving partial differential equations to the results obtained by Frankel (Math. Tables and Other Aids to Comp. v. 4, p. 65-75, 1950) for an iteration method. It is concluded that the relaxation method gives no saving for the Poisson and biharmonic equations and is more difficult to program then the iterative method.

87. Poppe, R.T., "An Investigation of Convergence Techniques for Implicit Numerical Solution of the Diffusion Equation for Transient Heat Transfer," Master's thesis, AFIT/GA/phys/63-8, AF Inst. of Technology, Wright-Patterson AF Base, Ohio, Aug. 1963, 163 p., AD-419 310.

This paper contains the results of an investigation of two techniques for increasing the rate of convergence of the Gauss-Siedel method of implicit numerical solution of the diffusion equation of transient heat flow. A sample problem is solved to provide the necessary comparison. The results provide a theoretical basis for the adapted Wegstein technique. This theoretical basis brings to light the fact that successive over-relaxation and the adapted Wegstein technique are based on same theoretical background. A procedure based on estimating the maximum eigenvalue of the method of successive displacements is used to make an approximation of the relaxation factor for successive over-relaxation.

88. Prager, W., "Introduction to Basic FORTRAN Programming and Numerical Methods" Blaisdell Publishing Company, New York, 202 p., 1965

This book contains information on Aitken's δ^2method and Steffensen's method for accelerating the convergence of iterative processes.

89. Price, P.H., Slack, M.R., "Stability and Accuracy of Numerical Solutions of Heat Flow Equation," British Journal of Applied Physics, Vol. 3, No. 12 Dec. 1952, p. 379-84

The authors describe a new method of deriving stability conditions. Its application to heat conduction with variable thermal diffusivity and heat transfer by convection at surface is treated. A new finite difference representation of surface heat flux equation is given.

90. Price, P.H., and Slack, M.R., "Effect of Latent Heat on Numerical Solutions of Heat Flow Equation," British Journal of Applied Physics, Vol. 5, No. 8, Aug. 1954, p. 285-7

The article treats the stability and accuracy of finite difference solutions of the heat flow equation with latent heat evolution. A dimensionless group is developed which governs the appearance of inaccuracies peculiar to numerical solutions involving latent heat.

91. Ralston, A., Wilf, H.S., "Mathematical Methods of Digital Computer," John Wiley & Sons, Inc., New York, 1960

A section on the solution of linear equations includes discussions of Gauss-Seidel iteration, the conjugate gradient method, Matrix Inversion by Rank Annihilation, Matrix Inversion by Monte Carlo Methods - flow chart and formula for estimating running time are included for each method.

92. Round, G.F., Newton, R., and Redberger, P.J., "Variable Mesh Size in Iteration Methods of Solving Partial Differential Equations and Application to Heat Transfer," Chemical Engineering Progress Symposium Series, Vol. 58, No. 37, 1962, p. 29-42

The article contains descriptions of some variable mesh systems. Computations are carried out for steady state heat transfer from a buried cylindrical heat source. The same accuracy is achieved in a shorter time than with a square mesh.

93. Saul'ev, V.K., "Integration of Parabolic Equations by the Method of Nets," translated from Russian; MacMillan, N.Y., 1964, Russian publication date 1960.

There are two sections, the first dealing primarily with stability and convergence and the second dealing with implicit methods and techniques for solving the algebraic equations. In the first section, a number of mesh schemes are considered and compared. The second section gives a fairly complete compendium of methods for solving linear algebraic systems.

94. Scarborough, J., "Numerical Mathematical Analysis," 5th Edition, The Johns Hopkins Press, Baltimore, 594, p., 1962

There is a chapter on partial differential equations which treats difference quotients, difference equations, the solution of differential equations by iteration, the inherent error in the solution of difference equations, and relaxation methods.

95. Schenck, H., Jr., "Fortran Methods in Heat Flow," The Ronald Press Co., N.Y., 289 p., 1963

As indicated by the title, the emphasis is on Fortran methods, with little discussion of theory. The chapter on one-dimensional transient flow follows the method of Dusinberre, but includes a brief discussion of the relative merit of the Liebmann implicit method. Sample Fortran programs are presented (using the Dusinberre method only.) The last chapter of the book is a short discussion of accuracy and of solution speed.

96. Schneider, P.J., "Conduction Heat Transfer," Addison-Wesley Publishing Co., 394, p. 1955

Chapter 12 of this book treats the transient numerical solution of conduction problems using the method of Dusinberre. A discussion of stability and convergence is included.

97. Schuh, H., "Finite Difference Method for Calculating Transient Temperature Distributions Due to One-Dimensional Heat Flow in Simple and Composite Bodies," RAE-Lib/Trans-750, translated from VDI - Forschungshoft, No. 459, 43. p.

The finite difference method is modified for finding transient temperatures due to heat flow normal or parallel to the surface of thick plates and thin walled bodies of high conductance. Details of the method are discussed extensively. The use of high speed computers for solving finite difference equations is discussed.

98. Southwell, R.V., "The Quest for Accuracy in Computations Using Finite Differences," Numerical Methods of Analysis in Engineering, p.66-74, The MacMillan Company, New York, 1949.

The author concludes that the best accuracy is obtained by reducing the interval size rather than using higher order difference equations and illustrates a labor saving device which facilitates "advance to a finer net".

99. Southworth, R.W., DeLeeuw, S.L., "Digital Computation and Numerical Methods," McGraw-Hill Book Co., N.Y., 508 p., 1965.

This book is intended for use as a textbook in a course combining FORTRAN programming, numerical methods, and engineering applications. Chapters 2, 3, and 4 deal with programming, and the remainder of book is concerned with numerical methods. Problems are given at the end of each chapter, and engineering applications appear throughout the text.

100. Stanton, R., "Numerical Methods for Science and Engineering," Prentice-Hall, Inc., Englewood Cliffs, N.J., 266 p., 1961

This book is intended as an undergraduate introduction to numerical analysis, and is short on precise theory. It stresses desk calculator methods. Included are discussions of ordinary finite differences, divided differences, and central differences. There is one chapter on the solution of differential equations by difference equation methods.

101. Strang, W.G., "On the Order of Convergence of the Crank-Nicolson Procedure," Journal of Mathematics and Physics, Vol. 38 p. 141-44, 1959-60

The author discusses the Crank-Nicolson difference equation for $U_t = U_{xx} + d(x,t)$. If the solution of the first boundary problem is sufficiently smooth, the solution of the difference equation converges point-wise with error $0((\Delta x)^2)$, if $\Delta t = 0(\Delta X)$. The proof makes use of explicitly known eigenfunctions of the process.

102. Thomas, L.H., "Numerical Solution of Partial Differential Equations of Parabolic Type," Proceedings, Seminar on Scientific Computation, Nov., 1949, p. 71-78, International Business Machines Corp., New York, N.Y. 1950

The article contains an expository treatment of some problems in the numerical solution of parabolic partial differential equations by finite differences. There are three major topics: (A) stability of the finite difference representation, (b) truncation errors, (c) round-off errors. Methods for improving the stability and reducting truncation errors are illustrated.

103. Thomas, L. H., "Stability of Solution of Partial Differential Equations," Rept. No. NOLR-1132, Naval Ord. Laboratory, White Oak, Md., 1950, p.83-94 title of report is "Symposium on Theoretical Compressible Flow, 28 June 1949"

The report is a survey of the present status of the art of stability analysis for finite difference equivalents of differential equations (both ordinary and partial). The author defines two types of instability, short range and long range.

104. Traub, J.F., "Iterative Methods for the Solution of Equations," Prentice-Hall, Inc., Englewood Cliffs, N.J., 310 p., 1964.

A large number of iteration functions are described and classified according to the efficiency of the algorithm and the amount of computational labor involved. Stress is placed on methods for constructing iteration functions and on determining their chief properties. Much attention is focused on the computational aspects of the topic.

105. Turner, L.R., "Improvement in the Convergence of Methods of Successive Approximation," Proceedings, Computation Seminar, Dec. 1949, p. 135-57 International Business Machines, Corp., New York, N.Y., 1951

An exposition on the well known procedure for improving the convergence of an iteration procedure when the steps form a geometric progression.

106. Turton, F.J., "The Errors in the Numerical Solution of Differential Equations," The Philosophical Magazine, Vol. 28, p. 359-63, 1939

The article contains a detailed analysis of the errors caused by (1) uncertainty of initial values, (2) intrinsic errors in formulae used in the step by step method, (3) round-off errors in (2), (4) random errors. The author's conclusion is that "to insure no errors to the desired number of significant figures.... requires that at least two formulae be used, in which the intrinsic errors are substantially different, to check each other."

107. Varga, R.S., "A Comparison of the Successive Overrelaxation Method and Semi-Iterative Methods Using Chebyshev Polynomials," Journal of the Society for Industrial and Applied Mathematics, Vol. 5, p. 39-46, 1957

The author shows that the successive overrelaxation method converges at least as fast as any semi-iterative method associated with the Jacobi method, Gauss-Siedel method, or with the successive overrelaxation method itself. Successive overrelaxation requires only the latest iterate at any stage, whereas semi-iterative methods require the simultaneous storage of several iterates; therefore, the author sees some advantage in using successive overrelaxation, instead of semi-iterative methods, with high speed computers.

108. Wasow, W., "On the Accuracy of Implicit Difference Approximations to the Equation of Heat Flow," Tech. Summary Rept. No. MRC-TSR-2, Contract DA11 022 ORD 2059, Math, Research Center, Univ. of Wisconsin, Madison, Wisc., 15 Apr. 1957, 22p. PB-167 605

The author discusses the convergence, stability and truncation error of implicit difference approximations to the initial value problem defined by the heat flow equation.

109. Wegstein, J.H., "Accelerating Convergence of Iterative Methods:" Communications of Computing Machinery, Vol. 1, No. 6, p. 9, 1958.

The article describes a method very similar to Aitken's δ^2 method. It is emphasized that the method can cause convergence in normally divergent cases. Several numerical examples are included.

110. Wilkes, J.O., "Chemical Engineering Workshop: Part II, Numerical Methods for Partial Differential Equations," paper presented at ASEE annual meeting, Pullman, Wash., June 1966

The author describes 3 types of finite difference approximations for partial differential equations, called the forward, backward and central difference types. The main part of the paper is a discussion of the solution of a heat transfer problem using finite differences. The FORTRAN II program employed is included in full, as is a special subroutine (Tridng) for solving the system of linear equations resulting from application of the finite difference approximation.

111. Zonneveld, J.A., "Automatic Numerical Integration," Mathematical Centre Tracts, No. 8, Mathematisch Centrum. Amsterdam, 1964, 110 p.

The author constructs a set of Runge-Kutta formulas suitable for automatic adjustment of step size. The function being integrated is evaluated for step size one or two additional times, and the size of the step is adjusted according to this evaluation, during the integration process. Also included are six ALGOL 60 programs for the integration of first and second order differential equations, and five numerical examples.

Appendix D

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Appendix E

DERIVATION OF THE LAPLACE TRANSFORM SOLUTION

DERIVATION OF THE LAPLACE TRANSFORM SOLUTION

This appendix is an application of the Laplace transform method to the solution of the one-dimensional Fourier conduction equation for a semi-infinite solid and for an insulated slab, each with a ramp-function boundary condition.

SEMI-INFINITE SOLID

The one-dimensional Fourier conduction equation is

$$\frac{\partial t}{\partial T} = \alpha \frac{\partial^2 t}{\partial X^2} \text{ at } T = 0,\ t = 0 \tag{1}$$

The Laplace transform is

$$\alpha \frac{d^2 T}{dX^2} = sT - t(0)$$

and for the conditions of Eq. 1, it becomes

$$\frac{d^2 T}{dX^2} - \frac{sT}{\alpha} = 0 \tag{2}$$

and has the general solution,

$$T(X, s) = A \exp(\sqrt{s/\alpha}\ X + B \exp(-\sqrt{s/\alpha}\ X \tag{3}$$

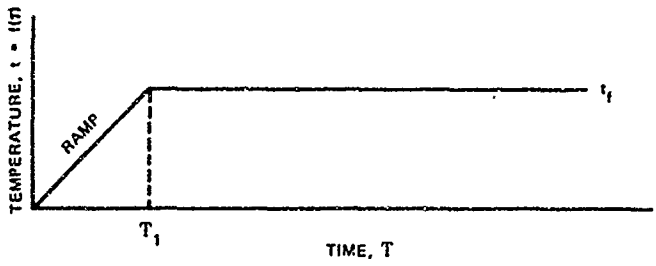
Since the body is a semi-infinite solid, $t \to 0$ as $X \to \infty$ implying that

$$A = 0$$

and the solution reduces to

$$T(X, s) = B \exp(-\sqrt{s/\alpha}\ X) \tag{4}$$

The coefficient B must be determined from the surface conditions. At $X = 0$, $t_f = f(T)$ as shown in the following figure.



The temperature of the bounding fluid expressed in terms of time, T, is

$$f(T) = \frac{t_f}{T_1}\left[Tu(T) - \left(T-T_1\right) u\left(T-T_1\right)\right] \tag{5}$$

A heat balance at the surface yields

$$-kA \frac{\alpha t(0,T)}{\alpha X} = hA\left[f(T) - t(0,T)\right] \tag{6}$$

which can be expressed as

$$- \frac{\alpha t(0,T)}{\alpha X} + \frac{h}{k} t(0,T) = \frac{ht_f}{kT_1}\left[Tu(T) - \left(T-T_1\right)u\left(T-T_1\right)\right]. \tag{7}$$

Transforming Eq. 7 gives

$$- \frac{dT(0,s)}{dX} + \frac{h}{k} T(0,s) = \frac{ht_f}{kT_1}\left[\frac{1}{s^2} - \frac{\exp(-T_1 s)}{s^2}\right] \tag{8}$$

but

$$\frac{dT(0,s)}{dX} = - B\sqrt{s/\alpha},\ T(0,s) = B$$

Thus,

$$B\sqrt{s/\alpha} + \frac{h}{k} B = \frac{ht_f}{kT_1}\left[\frac{1}{s^2} - \frac{\exp(-T_1 s)}{s^2}\right]$$

$$B = \frac{ht_f}{kT_1}\left[\frac{1}{s^2(\sqrt{s/\alpha} + h/k)} - \frac{\exp(-T_1 s)}{s^2(\sqrt{s/\alpha} + h/k)}\right]. \tag{9}$$

$$T = \frac{ht_f}{ki_1}\left[\frac{\exp(-\sqrt{s/\alpha}\ X)}{s^2\sqrt{s/\alpha}+h/k} - \frac{\exp(-\sqrt{s/\alpha}\ X)}{s^2(\sqrt{s/\alpha}+h/k)}\exp(-T_1 s)\right] \quad (10)$$

The inverse Laplace transform of

$$\left[\frac{\exp(-\ s/\alpha\ X)}{s(\ s/\alpha + h/k)}\right]$$

is

$$\frac{k}{h}\operatorname{erfc}\left(\frac{X}{2\sqrt{\alpha T}}\right) - \frac{k}{h}\exp\left[\frac{h}{k}\ X\ \alpha\left(\frac{h}{k}\right)^2 T\right]\operatorname{erfc}\left(\frac{X}{2\sqrt{\alpha T}} + \frac{h}{k}\sqrt{\alpha T}\right).$$

The inverse Laplace transform of

$$\left[\frac{\exp(-\sqrt{s/\alpha}\ X)}{s(\sqrt{s/\alpha}+h/k)}\ \frac{1}{s}\right]$$

is

$$1 * \left\{\frac{k}{h}\operatorname{erfc}\left(\frac{X}{2\sqrt{\alpha T}}\right) - \frac{k}{h}\exp\left[\frac{h}{k}\ X + \alpha\left(\frac{h}{k}\right)^2 T\right]\operatorname{erfc}\left(\frac{X}{2\sqrt{\alpha T}} + \frac{h}{k}\sqrt{\alpha T}\right)\right\} \quad (11)$$

and becomes

$$\frac{k}{h}\int_{\lambda=0}^{T}\operatorname{erfc}\left(\frac{X}{2\sqrt{\alpha\lambda}}\right)d\lambda - \frac{k}{h}\int_{\lambda=0}^{T}\exp\left[\frac{h}{k}\ X\alpha\left(\frac{h}{k}\right)^2\lambda\right]\operatorname{erfc}\left(\frac{X}{2\sqrt{\alpha\lambda}} + \frac{h}{k}\sqrt{\alpha\lambda}\right)d\lambda \quad (12)$$

The inverse Laplace transform of

$$\left[\frac{\exp(\sqrt{s/\alpha}\ X)}{s(\sqrt{s/\alpha}+h/k)}\ \frac{\exp(-T_1 s)}{s}\right]$$

is

$$\left\{\frac{k}{h}\int_{\lambda=0}^{T}\operatorname{erfc}\left[\frac{X}{2\sqrt{\alpha(\lambda-T_1)}}\right]d\lambda - \frac{k}{h}\int_{\lambda=0}^{T}\exp\left[\frac{h}{k}\ X + \alpha\left(\frac{h}{k}\right)^2(\lambda-T)\right]\operatorname{erfc}\left[\frac{X}{2\sqrt{\alpha(\lambda-T_1)}} + \frac{h}{k}\sqrt{\alpha(\lambda-T)}\right]d\lambda\right\} \cdot \left[u(T-T_1)\right] \quad (13)$$

Let

$$\beta = \frac{X^2}{4\alpha\lambda}$$

and the first term of Eq. 12 becomes

$$\int_{\beta=\infty}^{X^2/4\alpha T} \operatorname{erfc}(\sqrt{\beta}) \left(-\frac{X^2\beta-2}{4\alpha}\right) d\beta$$

which equals

$$\frac{X^2}{4\alpha}\int_{\beta=\infty}^{X^2/\alpha T} -\beta^{-2}(1-\operatorname{erf}\sqrt{\beta})\,d\beta$$

Integrating the preceeding integral by parts yields

$$T\ \operatorname{efrc}\left(\frac{X}{2\sqrt{\alpha T}}\right) - X\sqrt{T/\alpha\pi}\ \exp\left(-\frac{X^2}{4\alpha T}\right)$$

$$+ \frac{X^2}{4\alpha}\ \frac{2}{\sqrt{\pi}}\int_{\beta=X^2/4\alpha T}^{\infty} \beta^{-1/2}\ \exp(-\beta)\,d\beta$$

which, by letting $\beta = a^2$, becomes

$$T\ \operatorname{erfc}\left(\frac{X}{2\sqrt{\alpha T}}\right) - X\sqrt{T/2\pi}\ \exp\left(-\frac{X^2}{4\alpha T}\right)\frac{X^2}{2\alpha}\ \frac{2}{\sqrt{\pi}}\int_{a=X/2\sqrt{dT}}^{\infty} \exp(-a^2)\,da$$

and the first term of Eq. 12 becomes

$$\left(T + \frac{X^2}{2\alpha}\right)\operatorname{erfc}\left(\frac{X}{2\ \alpha T}\right) - X\sqrt{T/\alpha\pi}\ \exp\left(-\frac{X^2}{4\alpha T}\right) \tag{14}$$

The second term of Eq. 12 becomes

$$\int_{\lambda=0}^{T} \exp\left[X + \alpha\left(\frac{h}{k}\right)^2\lambda\right] \operatorname{erfc}\left(\frac{X}{2\sqrt{\alpha\lambda}} + \frac{h}{k}\sqrt{\alpha\lambda}\right) d\lambda$$

$$= \exp\left(\frac{h}{k}\,X\right)\int_{\lambda=0}^{T} \exp\left[\alpha\left(\frac{h}{k}\right)^2\lambda\right]\left[1 - \operatorname{erf}\left(\frac{X}{2\sqrt{\alpha\lambda}} + \frac{h}{k}\,\sqrt{\alpha\lambda}\right)\right]d\lambda\,.$$

Integrating by parts yields

$$\frac{1}{\alpha}\left(\frac{k}{h}\right)^2 \exp\left(\frac{h}{k}X\right)\left\{\exp\left[\alpha\left(\frac{h}{k}\right)^2 T\right]\ \operatorname{erfc}\left(\frac{X}{2\sqrt{\alpha T}} + \frac{h}{k}\sqrt{\alpha T}\right)\right.$$

$$+ \frac{h\sqrt{\alpha}}{k\sqrt{\pi}} \exp -\left(\frac{h}{k}X\right)\int_{\lambda=0}^{T} \lambda^{-1/2} \exp\left(-\frac{X^2}{4\alpha\lambda}\right)d\lambda$$

$$\left. - \frac{X}{4\sqrt{\alpha}}\ \exp\left(-\frac{h}{k}X\right)\frac{2}{\sqrt{\pi}}\int_{\lambda=0}^{T} \lambda^{-3/2} \exp\left(-\frac{X^2}{4\alpha\lambda}\right)d\lambda\right\} \qquad (15)$$

Integrate the first integral in Eq. 15 by parts and

$$\int_{\lambda=0}^{T} \lambda^{-1/2} \exp\left(-\frac{X^2}{4\alpha\lambda}\right) d\lambda = 2\sqrt{T} \exp\left(-\frac{X^2}{4\alpha T}\right)$$

$$- \frac{X^2}{2\alpha}\int_{\lambda=0}^{T} \lambda^{-3/2} \exp\left(-\frac{X^2}{4\alpha\lambda}\right)\ d\lambda$$

Substituting in Eq. 15 gives

$$\frac{1}{\alpha}\ \left(\frac{k}{h}\right)^2\left\{\exp\left[\frac{h}{k}X + \alpha\left(\frac{h}{k}\right)^2 T\right]\ \operatorname{erfc}\left(\frac{X}{2\sqrt{\alpha T}} + \frac{h}{k}\sqrt{\alpha T}\right) + \frac{2h}{k}\sqrt{\frac{\alpha T}{\pi}}\ \exp\left[-\left(\frac{X}{2\sqrt{\alpha T}}\right)^2\right]\right.$$

$$\left. - \frac{2}{\sqrt{\pi}}\ \frac{X}{4\sqrt{\alpha}}\int_{\lambda=0}^{T} \lambda^{-3/2} \exp\left[-\left(\frac{X}{2\sqrt{\alpha T}}\right)^2\right]d\lambda\ \left(\frac{Xh}{k} + 1\right)\right\} \qquad (16)$$

Let

$$Z = \frac{X}{2\sqrt{\alpha\lambda}}$$

and Eq. 16 becomes

$$\frac{1}{\alpha}\left(\frac{k}{h}\right)^2\left\{\exp\left[\frac{h}{k}X + \alpha\left(\frac{h}{k}\right)^2 T\right]\ \operatorname{erfc}\left(\frac{X}{2\sqrt{\alpha T}} + \frac{h}{k}\sqrt{\alpha T}\right)\right.$$

$$\left. + \frac{2h}{\sqrt{\pi}k}\sqrt{\alpha T}\ \exp\left[-\frac{X}{2\sqrt{\alpha T}}^2\right] - \left(\frac{Xh}{k} + 1\right)\ \operatorname{erfc}\left(\frac{X}{2\sqrt{\alpha T}}\right)\right\} \qquad (17)$$

Summing Eq. 14 and 17 and multiplying by ht_f/k_1 determines the temperature distribution for $0 < T \leq T_1$, which is

$$t = \frac{t_f}{T_1}\left\{\left[T + \frac{X^2}{2\alpha} + \frac{Xk}{\alpha h} + \frac{1}{c}\left(\frac{k}{h}\right)^2\right]\operatorname{erfc}\left(\frac{X}{2\sqrt{\alpha T}}\right)\right.$$

$$- \frac{1}{\alpha}\left(\frac{k}{h}\right)^2 \exp\left[\frac{h}{k}X + \alpha\left(\frac{h}{k}\right)^2 T\right]\operatorname{erfc}\left(\frac{X}{2\sqrt{\alpha T}} + \frac{h}{k}\sqrt{\alpha T}\right)$$

$$\left.-\sqrt{\frac{T}{\pi\alpha}}\left(X + \frac{2k}{h}\right)\exp\left[-\left(\frac{X}{2\sqrt{\alpha T}}\right)^2\right]\right\}. \qquad (18)$$

Similarly, the temperature distribution for $T > T_1$ is

$$t = \frac{t_f}{T_1}\left\{\left[T + \frac{X^2}{2\alpha} + \frac{Xh}{\alpha h} + \frac{1}{\alpha}\left(\frac{k}{h}\right)^2\right]\operatorname{erfc}\left(\frac{X}{2\sqrt{\alpha T}}\right)\right.$$

$$- \frac{1}{\alpha}\left(\frac{k}{h}\right)^2 \exp\left[\frac{h}{k}X + \alpha\left(\frac{h}{k}\right)^2 T\right]\operatorname{erfc}\left(\frac{X}{2\sqrt{\alpha T}} + \frac{h}{k}\sqrt{\alpha T}\right)$$

$$\left.-\sqrt{\frac{T}{\pi\alpha}}\left(X + \frac{2k}{h}\right)\exp\left[-\left(\frac{X}{2\sqrt{\alpha T}}\right)^2\right]\right\}$$

$$- \frac{t_f}{T_1}\left\{\left[\left(T-T_1\right) + \frac{X^2}{2\alpha} + \frac{Xk}{\alpha h} + \frac{1}{\alpha}\left(\frac{k}{h}\right)^2\right]\operatorname{erfc}\left[\frac{X}{\sqrt{\alpha(T-T_1)}}\right]\right.$$

$$+ \frac{1}{\alpha}\left(\frac{k}{h}\right)^2 \exp\left[\frac{h}{k}X + \alpha\left(\frac{h}{k}\right)^2\left(T-T_1\right)\right]\operatorname{erfc}\left[\frac{X}{2\sqrt{\alpha(T-T_1)}} + \sqrt{\alpha(T-T_1)}\right]$$

$$\left.+ \sqrt{\frac{T-T_1}{\pi\alpha}}\left(X + \frac{2k}{h}\right)\exp\left(-\left[\frac{X}{2\sqrt{\alpha(T-T_1)}}\right]^2\right)\right\} \qquad (19)$$

INSULATED SLAB

Again starting with the one-dimensional Fourier conduction equation, one sees that

$$\frac{\partial t}{\partial T} = \alpha\frac{\partial^2 t}{\partial X^2} \text{ at } t = 0, T = 0 \qquad (20)$$

and the Laplace transform is

$$\frac{d^2T(X,s)}{dX^2} - \frac{s}{\alpha} T(X,s) = 0 \tag{21}$$

A general solution is:

$$T(X,s) = A \cosh \sqrt{s/\alpha}\, X + B \sinh \sqrt{s/\alpha}\, X \tag{22}$$

The temperature of the bounding fluid expressed in terms of time, T, is

$$f(T) = \frac{t_f}{T_1}\left[Tu(T) - \left(T-T_1\right)u\left(T-T_1\right)\right] \tag{23}$$

A heat balance at the surface of the slab yields

$$-\frac{\partial t(0,T)}{\partial X} + \frac{ht(0,T)}{k} = \frac{ht_f}{kT_1}\left[Tu(T) - \left(T-T_1\right)u\left(T-T_1\right)\right] \tag{24}$$

The Laplace transform of Eq. 24 is

$$-\frac{dT(0,s)}{dX} + \frac{h}{k} T(0,s) = \frac{ht_f}{kT_1}\left[\frac{1}{s^2} - \frac{\exp\left(-T_1 s\right)}{s^2}\right] \tag{25}$$

but

$$\frac{dT(0,s)}{dX} = B\sqrt{s/\alpha}\ , \quad T(0,s) = A$$

Thus Eq. 25 becomes

$$-B\sqrt{s/\alpha} + A\frac{h}{k} = \frac{ht_f}{kT_1}\left[\frac{1}{s^2} - \frac{\exp\left(-T_1 s\right)}{s^2}\right] \tag{26}$$

Since the slab is insulated at X = L, $\frac{dT(L,s)}{dX} = 0$, and from Eq. 22,

$$\sqrt{s/\alpha}\left(A \sinh \sqrt{s/\alpha}\ L + B \cosh \sqrt{s/\alpha}\ L\right) = 0 \tag{27}$$

Solving Eq. 26 and 27 simultaneously yields

$$A = \frac{ht_f}{kT_1}\left(\frac{\cosh \sqrt{s/\alpha}\ L}{\sqrt{s/\alpha} \sinh \sqrt{s/\alpha}\ L + h/k\sqrt{\cosh \sqrt{s/\alpha}\ L}}\right)\left[\frac{1}{s^2} - \frac{\exp\left(-T_1 s\right)}{s^2}\right]$$

$$B = -\frac{ht_f}{kT_1}\left(\frac{\sinh\sqrt{s/\alpha}\ L}{\sqrt{s/\alpha}\ \sinh\sqrt{s/\alpha}\ L + h/k\ \cosh\sqrt{s/\alpha}\ L}\right)\left[\frac{1}{s^2} - \frac{\exp\left(-T_1 s\right)}{s^2}\right]$$

Therefore,

$$T(X, s) = \frac{ht_f}{kT_1}\left(\frac{\cosh\sqrt{s/\alpha}\ L\ \cosh\sqrt{s/\alpha}\ X - \sinh\sqrt{s/\alpha}\ L\ \sinh\sqrt{s/\alpha}\ X}{\sqrt{s/\alpha}\ \sinh\sqrt{s/\alpha}\ L + h/k\ \cosh\sqrt{s/\alpha}\ L}\right)$$

$$\cdot\left[\frac{1}{s^2} - \frac{\exp\left(-T_1 s\right)}{s^2}\right] \tag{28}$$

which can be put in the following form:

$$T(X, s) = \frac{t_f}{T_1}\left[\frac{\cosh\sqrt{s/\alpha}\ (L - X)}{k/h\ \sqrt{s/\alpha}\ \sinh\sqrt{s/\alpha}\ L + \cosh\sqrt{s/\alpha}\ L}\right]$$

$$\cdot\left[\frac{1}{s^2} - \frac{\exp\left(T_1 s\right)}{s^2}\right] \tag{29}$$

The inverse transform must be taken. A second-order pole exists at $s = 0$, and an infinite number of simple poles exist at the roots of

$$\frac{k}{h}\sqrt{s/\alpha}\ \sinh\sqrt{s/\alpha}\ L = -\cosh\sqrt{s/\alpha}\ L \tag{30}$$

$$\text{or } \cot\lambda = \frac{k}{hL}\lambda \quad \text{where } \lambda = i\sqrt{s/\alpha}\ L.$$

One can see from Eq. 29 that the inverse transform will consist of two parts, one the inverse transform of

$$\frac{1}{s}\left[\frac{\cosh\sqrt{s/\alpha}\ (L - X)}{s(k/h\sqrt{s/\alpha}\ \sinh\sqrt{s/\alpha}\ L + \cosh\sqrt{s/\alpha}\ L)}\right]; \tag{31}$$

the other part will have a similar inverse transform except that it will be translated along the time axis by T_1 units.

First, the inverse transform of

$$\frac{\cosh\sqrt{s/\alpha}\ (L - X)}{s(k/h\sqrt{s/\alpha}\ \sinh\sqrt{s/\alpha}\ L + \cosh\sqrt{s/\alpha}\ L)} \tag{32}$$

will be located and then integrated from $0 < T \leq T$.

The inverse transform may be calculated by the method of residues. The residues of the simple poles may be derived from

$$\Sigma\rho = \sum_{n=1}^{\infty} \frac{P(s_n)}{q'(s_n)} \exp(s_n T) \tag{33}$$

where $\frac{P}{q'}$ is a ratio of polynomials.

$$q' = s \frac{d}{ds} \left[\frac{k}{h} \sqrt{s/\alpha} \sinh \sqrt{s/\alpha} L + \cosh \sqrt{s/\alpha} L \right]$$
$$+ \left[\frac{k}{h} \sqrt{s/\alpha} \sinh \sqrt{s/\alpha} L + \cosh \sqrt{s/\alpha} L \right] \tag{34}$$

The second term is zero since the two parts of the second term are equated to determine the roots.

Performing the differentiation indicated in the first term of Eq. 34 yields

$$\frac{L}{2} \sqrt{s/\alpha} \left[-\left(\frac{k}{h}\right)^2 \frac{s}{\alpha} + \left(\frac{k}{hL} + 1\right) \right] \sinh \sqrt{s/\alpha} L$$

and changing to trignometric functions gives

$$- i \frac{L}{2} \sqrt{s/\alpha} \left[- \left(\frac{k}{h}\right)^2 \frac{s}{\alpha} + \left(\frac{k}{hL} + 1\right) \right] \sin i \sqrt{s/\alpha} L .$$

Substitution of

$$\lambda = i \sqrt{s/\alpha} L$$

into the preceeding expression yields

$$- \frac{\lambda}{2} \left[\left(\frac{k}{hL}\right)^2 \lambda^2 + \left(\frac{k}{hL} + 1\right) \right] \sin \lambda$$

The numerator of Eq. 33, P, may be changed to trignometric functions

$$P \cosh \sqrt{s/\alpha} X = \cos i \sqrt{s/\alpha} X = \cos \frac{\lambda X}{L} .$$

Substitution of the above expressions into Eq. 33 yields

$$\Sigma\rho = \sum_{n=1}^{\infty} \frac{\cos \lambda_n \, X/L \, \exp\left(-\lambda_n^2 \, \alpha T/L^2\right)}{\frac{\lambda_n}{2}\left[\left(\frac{k}{hL}\right)^2 \lambda_n^2 + \left(\frac{k}{hL} + 1\right)\right] \sin \lambda_n} \tag{35}$$

which may be simplified to

$$\Sigma\rho = -4 \sum_{n=1}^{\infty} \frac{\cos \lambda_n \, X/L \, \sin \lambda_n \, \exp\left(-\lambda_n^2 \, \alpha T/L^2\right)}{2\lambda_n + \sin 2\lambda_n} \tag{36}$$

The right side of Eq. 36 must be integrated between the limits of zero and T. Assume the order of integration and summation may be interchanged.

$$\Sigma\rho = -4 \sum_{n=1}^{\infty} \frac{\sin \lambda_n \cos \lambda_n \, X/L}{2\lambda_n + \sin 2\lambda_n} \int_{\theta=0}^{T} \exp\left(-\lambda_n^2 \, d\theta/L^2\right) d\theta$$

$$= \frac{4L^2}{\alpha} \sum_{n=1}^{\infty} \frac{\sin \lambda_n \cos \lambda_n \, X/L}{\lambda_n^2\left(2\lambda_n + \sin 2\lambda_n\right)} \left[\exp\left(-\lambda_n^2 \, \alpha T/L^2\right) - 1\right] \tag{37}$$

The residue of the second-order pole at s = 0 must be determined.

$$\rho = \lim_{s \to 0} \frac{\partial}{\partial s}\left[(s - 0)^2 \, T(X,s) \exp(sT)\right]$$

$$\cdot \frac{\partial}{\partial s}\left[\frac{\cosh \sqrt{s/\alpha} \; (L-X) \exp(sT)}{\frac{k}{h}\sqrt{s/\alpha} \sinh \sqrt{s/\alpha} \; L + \cosh \sqrt{s/\alpha} \; L}\right]$$

$$= \left(\frac{k}{h}\sqrt{s/\alpha} \sinh \sqrt{s/\alpha} \; L + \cosh \sqrt{s/\alpha} \; L\right)$$

$$\cdot \left[\frac{(L-X)}{2\sqrt{\alpha s}} \sinh \sqrt{s/\alpha} \; (L-X) \exp(sT) + T \exp(sT)\right.$$

$$\left.\cosh \sqrt{s/\alpha} \; (L-X)\right] - \left[\cosh \sqrt{s/\alpha} \; (L-X) \exp(sT)\right]$$

$$\left(\frac{L}{2\alpha}\ \frac{k}{h}\cosh\sqrt{s/\alpha}\ L + \frac{k}{h}\frac{1}{2\ \alpha s}\ \sinh\sqrt{s/\alpha}\ L + \frac{L}{2\sqrt{\alpha s}}\right.$$

$$\left.\cdot \sinh\sqrt{s/\alpha}\ L\right)$$

all divided by $\left(\frac{k}{h}\sqrt{s/\alpha}\ \sinh\sqrt{s/\alpha}\ L + \cosh\sqrt{s/\alpha}\ L\right)^2$. (38)

The limit of Eq. 28 as $s \to 0$ gives $\rho = T$

The summation of all residues yields the inverse transform:

$$t = \frac{t_f}{T_1}\left\{T + \frac{4L^2}{\alpha}\sum_{n=1}^{\infty}\frac{\left[\exp\left(-\lambda_n{}^2\ \alpha T/L^2\right)-1\right]\sinh\lambda_n\ \cosh\lambda_n\ X/L}{\lambda_n{}^2\left(2\lambda_n + \sin 2\ \lambda_n\right)}\right\} \quad (39)$$

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