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A LaGRANGE MULTIPLIER METHOD
FOR CERTAIN CONSTRAINED MIN-MAX PROBLEMS

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13. ABSTRACT

Constrained min-max problems are constant-sum two-person games in which the maximizing player enjoys the advantage of moving last and both players select strategies subject to separate side conditions. In this paper a LaGrange multiplier method is presented for solving such problems where the maximizing player is permitted to probabilistically mix strategies. A simple ABM/shelter deployment problem is solved to illustrate the essential features of the method.

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PREFACE

The essential ideas of this paper, the introduction of mixed
strategies and the modification of Everett's [5] and Pugh's [9]
double Lagrange multiplier method for constrained min-max problems,
have had a checkered career.

These ideas arose from a computational device that completed
the solution (or appeared to complete the solution) of the ABM/
Shelter deployment problem formulated by Robert Kupperman. The
device was a powerful one and I promptly applied it to a much more
complex and interesting version of the same problem. The resulting
offense and defense solutions were thoroughly convincing.

What has followed this initial discovery and application has
been a succession of attempts to explain why the device, now
christened the modified double Lagrange multiplier method, apparently
works. The first attempt appears to me now to have been quite
naive. It foundered with the discovery that a presumed minimum was,
in fact, a local maximum. The second trip to the drawing board
produced a saddle-point theorem which works if the number of feasible
strategies for the maximizing player is finite. However, all of
my attempts to extend this theorem to a continuum of strategies
exploded in one way or another. Worse, the saddle-point theorem
could not provide a basis for my method unless it could be so
extended.

This paper follows an entirely different route from these
earlier attempts. I am frankly indebted to Owen [7] whose paper
contains the essential step I have borrowed and put to my own use.
The results that I have (I hope) proved are not as strong as I had
initially expected. Nevertheless they should provide a sufficient

ii

basis for the more grandiose applications of the modified double
Lagrange multiplier method that I have attempted elsewhere.

I am greatly indebted to a number of my colleagues who
patiently studied the confused logic and opaque language of several
previous drafts of this paper. In particular, the comments of
James T. McGill of IDA and James E. Falk of RAC led to the correction
of several errors and helped me to improve the exposition throughout.
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May 1971

CONTENTS

A LAGRANGE MULTIPLIER METHOD FOR
CERTAIN CONSTRAINED MIN-MAX PROBLEMS

I

INTRODUCTION

Constrained min-max problems are constant-sum two-person games in
which the maximizing player enjoys the advantage of moving last
and both players select strategies subject to separate side condi-
tions. Problems of this general form are natural archetypes for a
variety of defense resource allocation models. Probably the most
common defense application is to the allocation among cities of
shelters and antiballistic missiles. The objective of the defense
is to minimize the casualties that may be inflicted by an enemy
with a given arsenal of ballistic missiles and a full view of the
defense allocation. Among the authors who have formulated such
models as constrained min-max problems are Robert Kupperman,
Smith [10], Eisen [4], and Owen [7]. An excellent mathematical
survey of min-max problems in general has been provided by
Danskin [3].

Since the minimizing player is deprived of the means to imple-
ment a mixed strategy, constrained min-max problems do not always
possess equilibrium pairs, i.e., strategies that are simultaneously
optimal against each other. This inconvenient characteristic
separates min-max problems from the well-developed body of theory
for games with saddle-point solutions. Methods designed to locate
saddle-points will succeed only fortuitously in solving min-max
problems.

Equally important, the absence of saddle-point solutions pre-
vents the straightforward use of Lagrange multipliers in dealing
with the players' constraints. Nevertheless, several procedures
employing Lagrange multipliers (or, equivalently, a convex duality
theorem) have been proposed for min-max problems with constraints.
Bram [2] and Danskin [3] prove Lagrange multiplier theorems in

1

connection with their use of directional derivatives. However, the constraints pertain only to the minimizing player and directional derivatives typically lead to cumbersome processes for calculating solutions in applied work. A more workable approach is the "double" Lagrange multiplier method proposed by Everett [5] and Pugh [8]. Unfortunately, the circumstances under which the method might yield optimal solutions are not described by Everett and Pugh. Moreover, their method is clearly capable of yielding strategies that are not optimal solutions to constrained min-max problems. Pugh attempts to deal with this difficulty by offering "verification" techniques that, in effect, provide bounds for solutions. However, these bounds are of doubtful value in applied work. Most recently, Owen [7] has employed a convex duality theorem to transform a constrained min-max problem into a pure minimization problem. Unfortunately, Owen fails to exhibit solution strategies for the maximizing player.

In this paper, we shall employ Owen's procedure to obtain a specialization of Everett and Pugh's double Lagrange multiplier method. The method applies to problems in which the maximizing player's strategies may be segregated by tactics and probabilistically mixed. For example, in attacking a city defended by antiballistic missiles, the maximizing player might employ strategies designed to leak through, suppress (by destroying the radars), or exhaust the defense--or he may choose not to attack the city at all. Normally, the maximizing player would not be indifferent between two strategies associated with the same tactic, i.e., having decided to, say, exhaust the defense, the maximizing player will typically find that there is a single best number of warheads for this purpose. On the other hand, a clever defense may leave the maximizing player indifferent between the best strategies associated with two or more distinct tactics. For example, antiballistic missiles may have been installed in the city in numbers just sufficient to leave the maximizing player indifferent between exhausting the defense and not attacking the city at all. We shall allow the

2

maximizing player to resolve such dilemmas by probabilistically mixing strategies associated with different tactics.

This is a novel formulation for constrained min-max problems and requires some elaboration which we shall provide in Section II. In Section III a convex duality theorem is employed to derive a set of sufficient conditions for solutions to a class of constrained min-max problems with mixed strategies. An even stronger set of sufficient conditions may be derived if certain functions are differentiable and quasi-convex. These conditions are derived in Section IV and provide the basis for the presentation of the modified double Lagrange multiplier method (Section V). In the concluding Sections the characteristics of constrained min-max problems to which the method is best suited are described and a simple ABM/shelter deployment problem is solved to provide an illustration.

II

THE CONSTRAINED MIN-MAX PROBLEM (WITH MIXED STRATEGIES)

By convention, the outcome of a constant-sum two-person game is
measured as a payoff to one of the players from which the other's
loss differs, at most, by a constant amount. If we let X and Y
denote strategies for the maximizing and minimizing players, re-
spectively, then this payoff (to the maximizing player) may be
represented by the real-valued function $H(X,Y)$. The constraints
on the players' uses of resources may be represented by the inequality
systems $C(X) \leq c$ and $B(Y) \leq b$ where $C(X)$ and $B(Y)$ denote real-valued
vector functions and c and b are vectors of the available quantities
of the players' resources. Moreover, the players are restricted in
their selection of strategies to the elements X and Y of the sets
S and T. The function H is defined over the product set S x T, the
vector function C is defined over S, and B is defined over T.
(Throughout the paper vectors are taken to be row vectors unless
marked with an apostrophe to indicate the transpose.)

A general statement of the constrained min-max problem is to
find a strategy X^o for the maximizing player and a strategy Y^o
for the minimizing player that satisfy the requirements:

1. The strategies are feasible for both players: $X^o \in S$, $C(X^o) \leq c$
   and $Y^o \in T$, $B(Y^o) \leq b$.

2. The strategy X^o maximizes the payoff function given the strategy
   Y^o subject to $X \in S$ and $C(X) \leq c$:
   $H(X^o, Y^o) \geq H(X, Y^o)$ for all $X \in S$, $C(X) \leq c$.

3. The strategy Y^o minimizes the maximum payoff in $X \in S$, $C(X) \leq c$
   subject to $Y \in T$ and $B(Y) \leq b$:
   $H(X^o, Y^o) \leq \mathrm{Sup}_X \left\{ H(X,Y) \mid X \in S, \ C(X) \leq c \right\}$ for all $Y \in T$, $B(Y) \leq b$.

4

(Note that a constrained max-min problem may always be converted to a min-max problem with the same constraints simply by reversing the sign of the payoff function.)

Mixed strategies are not conventionally believed to serve any useful purpose in the mathematical theory of min-max problems. The minimizing player cannot implement probabilistic mixes of strategies and the maximizing player appears to gain nothing by using them. In short, it is impossible to visualize any useful function of mixed strategies in an actual play of the game. From this observation it is easy to presume that mixed strategies for the maximizing player can have no role in the mathematical treatment of min-max problems. This has been the presumption of virtually all previous work. (Note, however, that Pugh [8] and Owen [7] in their work replace the payoff function with its concave envelope. This device implicitly admits mixed strategies since there may be points on the concave envelope that correspond only to weighted averages of points on the real payoff function. These payoffs can be achieved only by probabilistically mixing the maximizing player's strategies.)

In standard game theory the introduction of mixed strategies for both players serves to insure the existence of a saddle-point solution. That is, there exist mixed strategies that constitute an equilibrium pair. The introduction of mixed strategies for just the maximizing player is not alone sufficient. The resultant game will not possess a saddle-point unless the min-max problem had an equilibrium pair of strategies in the first place.

Nevertheless, mixed strategies for the maximizing player may be made to serve a useful purpose by the interesting expedient of re-defining the concept of an equilibrium pair and then exhibiting a correspondence between such redefined equilibrium pairs and solutions to constrained min-max problems. A slightly modified version of the "seesaw" problem provides a simple illustration of this use of mixed strategies. The maximizing player chooses a point between the right end, $X=1$, and the left end $X=-1$, of a seesaw but he may not sit directly over the fulcrum, i.e., $-1 \leq X \leq 1$ and $X \neq 0$.

5

The minimizing player chooses the angle of the seesaw, $-\Pi/2 \leq Y \leq \Pi/2$. The payoff is the height of the maximizing player's seat above the fulcrum, $H(X,Y) = X \sin Y$. This problem has no equilibrium pair of strategies (X^o, Y^o). However, a mixed strategy may be used to demonstrate that $Y^o = 0$ is optimal for the minimizing player. Two tactics may be descerned for the maximizing player by observing that either $X = -1$ or $X = 1$ is optimal for the maximizing player for any Y (both are optimal if $Y = 0$). Let $H_L^*(Y)$ and $H_R^*(Y)$ be the payoffs that result from a choice of the left and right ends of the seesaw, respectively. Then, $H_L^*(Y) = -\sin Y$ and $H_R^*(Y) = \sin Y$. A probabilistic mix of these two strategies yields the expected payoff $u(-\sin Y) + (1 - u) \sin Y = (1 - 2u) \sin Y$ where $0 \leq u \leq 1$. The probability $u^o = .5$ and the strategy $Y^o = 0$ constitute an equilibrium pair in the restricted sense that:

$$(1 - 2u^o) \sin Y^o \geq (1 - 2u) \sin Y^o \quad \text{for all } 0 \leq u \leq 1$$

and

$$(1 - 2u^o) \sin Y^o \leq (1 - 2u^o) \sin Y \quad \text{for all } -\Pi/2 \leq Y \leq \Pi/2.$$

The maximizing player's mixed strategy is to sit on the right end of the seesaw ($X^o = 1$) with probability .5 ($u^o = .5$) and to sit on the opposite end of the seesaw ($X^o = -1$) with the same probability .5 ($1 - u^o = .5$). That $Y^o = 0$ satisfies Statement 3 of the definition of a solution to the min-max problem follows from the fact that:

$$(1 - 2u^o) \sin Y \leq \text{Max}_X \left\{ X \sin Y \mid -1 \leq X \leq 1, X \neq 0 \right\}$$

$$\text{for all } -\Pi/2 \leq Y \leq \Pi/2.$$

Either $X^o = -1$ or $X^o = 1$ satisfy Statement 2 and both ($X^o = -1$, $Y^o = 0$) and ($X^o = 1$, $Y^o = 0$) satisfy the feasibility requirement, Statement 1. Now, the mixed strategy we have derived for the maximizing player is completely superfluous so far as playing the seesaw game is concerned. Its function in the argument above is solely to permit us to identify an optimal strategy Y^o for the minimizing player. This approach to min-max problems will be

6

formalized and extended to problems with constraints in the remainder of the paper (The solution to the seesaw problem is derived with directional derivatives in Danskin [3].)

For the special class of problems dealt with in this paper, we assume that the maximizing player's strategies may be assembled by tactics and that strategies associated with different tactics may be probabilistically mixed. Let $j = 1,\ldots,n$ be an index of tactics. For each tactic j we denote:

S_j a set of strategies X_j

$H_j(X_j,Y)$ a payoff function

$C_j(X_j)$ a resource vector function for the maximizing player.

μ_j a probability of use.

We also assume:

1. The strategies X_j are representable as nonnegative row vectors of real numbers, i.e., $S_j = \left\{ X_j \text{ in } E^n \mid X_j \geq 0 \right\}$.

2. The resource vector functions are linear, i.e., $C_j(X_j) = X_j A_j$ where A_j is a real matrix of appropriate dimensions.

3. The systems $X_j \geq 0$, $X_j A_j \leq c$ all have at least one solution.

4. The functions $H_j(X_j,Y)$ are finite-valued and concave in $X_j \geq 0$ for all $Y \in T$, $B(Y) \leq b$.

5. The functions $H_j^*(V,Y)$ defined by:

$$H_j^*(V,Y) = \text{Sup}_{X_j} \left\{ H_j(X_j,Y) - V A_j' X_j' \mid X_j \geq 0 \right\}$$

are continuous in $V \geq 0$ [A_j' and X_j' denote the transposes of A_j and X_j.] for all $Y \in T$, $B(Y) \leq b$.

6. $c > 0$.

The purpose in assembling strategies by tactics is to provide functions $H_j^*(V,Y)$ $j=1,\ldots,n$ with convenient mathematical properties when Y is defined as a nonnegative real vector. Specifically, it is generally necessary to define tactics in such a way that the functions

7

$H_j^*(V,Y)$ $j=1,\ldots,n$ are continuous and differentiable for $V \geq 0$ and $Y \geq 0$. Sorting the maximizing player's strategies by tactics usually does little conceptual violence to a constrained min-max problem. Indeed, useable divisions of the strategy set S are often suggested by the structure of the problem itself and probabilistically mixing the strategies associated with different tactics causes few difficulties in the interpretation of solutions.

The constrained min-max problem with mixed strategies is to find strategies X_j^0 $j=1,\ldots,n$ for each of the maximizing player's tactics, probabilities μ_j^0 $j=1,\ldots,n$, and a strategy Y^0 for the minimizing player such that:

1. $X_j^0 \geq 0$ $j=1,\ldots,n$, $\mu_j^0 \geq 0$ $j=1,\ldots,n$, $\sum_{j=1}^{n} \mu_j^0 = 1$, $\sum_{j=1}^{n} \mu_j^0 X_j^0 A_j \leq c$

    and $Y^0 \in T$, $B(Y^0) \leq b$.

2. $\sum_{j=1}^{n} \mu_j^0 H_j(X_j^0, Y^0) \geq \sum_{j=1}^{n} \mu_j H_j(X_j, Y^0)$ for all $X_j \geq 0$ $j=1,\ldots,n$,

    $\mu_j \geq 0$ $j=1,\ldots,n$, $\sum_{j=1}^{n} \mu_j = 1$, $\sum_{j=1}^{n} \mu_j X_j A_j \leq c$.

3. $\sum_{j=1}^{n} \mu_j^0 H_j(X_j^0, Y^0) \leq \operatorname{Sup}_{X_j, \mu_j} \left\{ \sum_{j=1}^{n} \mu_j H_j(X_j, Y) \,\middle|\, X_j \geq 0 \ j=1,\ldots,n, \right.$

    $\mu_j \geq 0$ $j=1,\ldots,n$, $\sum_{j=1}^{n} \mu_j = 1$, $\left. \sum_{j=1}^{n} \mu_j X_j A_j \leq c \right\}$

    for all $Y \in T$, $B(Y) \leq b$.

This definition is an obvious analogue of the definition of a solution provided earlier for constrained min-max problems without mixed strategies.

The definition incorporates expectational statements of the payoff function, $\sum_{j=1}^{n} \mu_j H_j(X_j, Y)$, and the maximizing player's resource constraints, $\sum_{j-1}^{n} \mu_j X_j A_j \leq c$. That is, the solution to the constrained min-max problem with mixed strategies establishes a random process in

8

which the strategies X_j^0 j=1,...,n are played with probabilities μ_j^0 j=1,...,n. The outcome of the game is the expected payoff $\sum_{j=1}^{n} \mu_j^0 H_j(X_j^0,Y^0)$ and the expected use of resources by the maximizing player $\sum_{j=1}^{n} \mu_j^0 X_j^0 A_j$ does not exceed the vector of available quantities c. However, there is no guarantee that the expected payoff will be achieved and that the maximizing player's resource constraints will be respected in any single play of the game. In most applications, therefore, it would be preferable to solve for strategies X_j^0 j=1,...,n such that if $\mu_j^0 > 0$ then $X_j^0 A_j \leqq c$ and $H_j(X_j^0,Y^0) = \sum_{j=1}^{n} \mu_j^0 H_j(X_j^0,Y^0)$.

With this condition observed any play of the game must result in the same actual payoff at a cost in resources to the maximizing player that never exceeds the amounts available. Unfortunately, the inclusion of this additional restriction in the definition of a solution makes mixed strategies superfluous and leaves us with a mathematically less tractable problem.

A sensible second-best procedure, then, is to solve for strategies that satisfy only Statements 1 through 3 of our definition. Whether or not this procedure produces acceptable approximations can be determined after the fact. Variations in payoff and resource use may be computed and if they are small relative to the expected values, little distortion is likely to result from accepting them as approximations. In any case the expected payoff, $\sum_{j=1}^{n} \mu_j^0 H_j(X_j^0,Y^0)$, provides a useful upper bound to the solution value of the payoff function if we add the condition $X_j^0 A_j \leqq c$ and $H_j(X_j^0,Y^0) = \sum_{j=1}^{n} \mu_j^0 H_j(X_j^0,Y^0)$ if $\mu_j^0 > 0$ for all tactics j=1,...,n.

III

SUFFICIENT CONDITIONS FOR A SOLUTION

Let V denote a vector of nonnegative real numbers whose elements correspond to the elements of the resource vector function $\sum_{j=1}^{n} \mu_j X_j A_j$. Sufficient conditions for a solution to the constrained min-max problem with mixed strategies are:

1. $X_j^0 \geqq 0$ $j=1,\ldots,n$, $\mu_j^0 \geqq 0$ $j=1,\ldots,n$, $\sum_{j=1}^{n} \mu_j^0 = 1$, $Y^0 \epsilon T$, $V^0 \geqq 0$.

2a. $H_j(X_j^0,Y^0) - V^0 A_j' X_j^{0'} \geqq H_j(X_j,Y^0) - V^0 A_j' X_j'$ for all $X_j \geqq 0$ $j=1,\ldots,n$.

2b. $\sum_{j=1}^{n} \mu_j^0 [H_j(X_j^0,Y^0) - V^0 A_j' X_j^{0'}] \geqq \sum_{j=1}^{n} \mu_j [H_j(X_j^0,Y^0) - V^0 A_j' X_j^{0'}]$

　　for all $\mu_j \geqq 0$ $j=1,\ldots,n$, $\sum_{j=1}^{n} \mu_j = 1$.

2c. $V^0 \left[\sum_{j=1}^{n} \mu_j^0 X_j^0 A_j - c \right]' \geqq V \left[\sum_{j-1}^{n} \mu_j^0 X_j^0 A_j - c \right]'$ for all $V \geqq 0$.

3a. $\sum_{j=1}^{n} \mu_j^0 H_j^*(V^0,Y^0) + V^0 c' \leqq \sum_{j=1}^{n} \mu_j^0 H_j^*(V,Y) + V c'$

　　　　for all $V \geqq 0$, $Y \epsilon T$, $B(Y) \leqq b$.

3b. $B(Y^0) \leqq b$

where the functions H_j^* are defined as:

$$H_j^*(V,Y) = \text{Sup}_{X_j} \left\{ H_j(X_j,Y) - V A_j' X_j' \,\middle|\, X_j \geqq 0 \right\}$$

and $[\ ]'$ denotes the transpose of a matrix. The numbering of the conditions has been chosen to roughly correspond to the numbered statements in the definition of a solution.

Condition 1 largely repeats Statement 1 of the definition. Non-negative conditions on the solution value of the vector V have been added and the inequalities $\sum\limits_{j=1}^{n} u_j^O x_j^O A_j \leq c$ and $B(Y^O) \leq b$ have been deleted. The reasons for these alterations will become apparent as we proceed to show the origins of Conditions 2 and 3.

Condition 2, together with the relevant components of Condition 1 may be used to show that Statement 2 of the definition of a solution and the inequality $\sum\limits_{j=1}^{n} u_j^O x_j^O A_j \leq c$ are satisfied. For this demonstration we require the Saddle-Point Theorem of mathematical programming. For our purposes a convenient statement of the Theorem is the following: Let F(X) be a real-valued function and G(X) a real-valued vector function of the strategies X in S. If the strategy $X^O \in S$ and a real vector $V^O \geq 0$ constitute a saddle-point of the Lagrangian function $F(X) - V[G(X)]'$ in $X \in S$ and $V \geq 0$, i.e.,

$$F(X^O) - V^O[G(X^O)]' \geq F(X) - V^O[G(X)]' \text{ for all } X \in S$$

and

$$F(X^O) - V^O[G(X^O)]' \leq F(X^O) - V[G(X^O)]' \text{ for all } V \geq 0$$

then X^O maximizes F(X) in S subject to $G(X) \leq 0$, i.e.,

$$F(X^O) \geq F(X) \text{ for all } X \in S, G(X) \leq 0$$

and $G(X^O) \leq 0$. The Theorem can be found in most texts on mathematical programming such as Karlin [6]. The proof is not dependent on the characteristics of the set S.

To apply the Theorem let:

$$S = \left\{ X_j, u_j \ j=1,\ldots,n \mid X_j \geq 0 \ j=1,\ldots,n, \ u_j \geq 0 \ j=1,\ldots,n, \right.$$
$$\left. \sum_{j=1}^{n} u_j = 1 \right\}$$

$$F(X) = \sum_{j=1}^{n} \mu_j H_j(X_j, Y^o)$$

and

$$G(X) = \sum_{j=1}^{n} \mu_j X_j A_j - c.$$

Conditions 2a and 2b may be used to construct:

$$\sum_{j=1}^{n} \mu_j^o H_j(X_j^o, Y^o) - V^o[\sum_{j=1}^{n} \mu_j^o X_j^o A_j - c]'$$

$$\geq \sum_{j=1}^{n} \mu_j H_j(X_j, Y^o) - V^o[\sum_{j=1}^{n} \mu_j X_j A_j - c]'$$

for all $X_j \geq 0$ $j=1,\ldots,n$, $u_j \geq 0$ $j=1,\ldots,n$, $\sum_{j=1}^{n} u_j = 1$.

From Condition 2c:

$$\sum_{j=1}^{n} u_j^o H_j(X_j^o, Y^o) - V^o[\sum_{j=1}^{n} u_j^o X_j^o A_j - c]'$$

$$\leq \sum_{j=1}^{n} \mu_j^o H_j(X_j^o, Y^o) - V[\sum_{j=1}^{n} u_j^o X_j^o A_j - c]' \text{ for all } V \geq 0.$$

Since

$$X_j^o \geq 0 \; j=1,\ldots,n, \; u_j^o \geq 0 \; j=1,\ldots,n, \; \sum_{j=1}^{n} u_j^o = 1 \text{ and } V^o \geq 0$$

we must have a saddle-point. By the Saddle-Point Theorem:

$$\sum_{j=1}^{n} u_j^o H_j(X_j^o, Y^o) \geq \sum_{j=1}^{n} u_j H_j(X_j, Y^o) \text{ for all } X_j \geq 0 \; j=1,\ldots,n,$$

$$u_j \geq 0 \; j=1,\ldots,n, \; \sum_{j=1}^{n} u_j = 1, \; \sum_{j=1}^{n} \mu_j X_j A_j \leq c$$

which is Statement 2 of the definition of a solution and

$$\sum_{j=1}^{n} \mu_j^o X_j^o A_j \leq c.$$

12

The first step of the demonstration that Statement 3 of the definition of a solution is implied by Conditions 1 through 3 is to exhibit a saddle-point of the quantity $\sum_{j=1}^{n} \mu_j H_j^*(V,Y) + VC'$. From Conditions 3a and 3b:

$$\sum_{j=1}^{n} \mu_j^0 H_j^*(V^0,Y^0) + V^0 c' \leq \sum_{j=1}^{n} \mu_j^0 H_j^*(V,Y) + Vc'$$

for all $Y \in T$, $B(Y) \leq b$, $V \geq 0$

and $B(Y^0) \leq b$.

Condition 2b provides:

$$\sum_{j=1}^{n} \mu_j^0 H_j^*(V^0,Y^0) + V^0 c' \geq \sum_{j=1}^{n} \mu_j H_j^*(V^0,Y^0) + V^0 c'$$

for all $\mu_j \geq 0$ $j=1,\ldots,n$, $\sum_{j=1}^{n} \mu_j = 1$.

So we apparently have a saddle-point of the expression $\sum_{j=1}^{n} \mu_j H_j^*(V,Y) + Vc'$ where the minimization proceeds over $Y \in T$, $B(Y) \leq b$, $V \geq 0$ and the maximization is taken over $\mu_j \geq 0$ $j=1,\ldots,n$, $\sum_{j=1}^{n} \mu_j = 1$. This is the equilibrium pair in a restricted sense that we referred to earlier. Since the order in which the minimization and maximization operations are performed is immaterial when dealing with saddle-points we have:

$$\sum_{j=1}^{n} \mu_j^0 H_j^*(V^0,Y^0) + V^0 c' \leq \text{Min}_V \text{Max}_{\mu_j} \left\{ \sum_{j=1}^{n} \mu_j H_j^*(V,Y) + Vc' \right.$$

$$\left| V \geq 0, \mu_j \geq 0 \; j=1,\ldots,n, \sum_{j=1}^{n} \mu_j = 1 \right\} \text{ for all } Y \in T, B(Y) \leq b.$$

The next step is to show that $\text{Min}_V \text{Max}_{\mu_j}$ equals $\text{Max}_{\mu_j} \text{Min}_V$. This is done by appealing to the familiar Min-Max Theorem of

13

game theory. The following statement of the Theorem approximates that given in Karlin [6]. Let $F(X,Y)$ be a real-valued function of X in S and Y in T where both S and T are closed, bounded and convex sets in E^n. If F is continuous, convex in Y for each X, and concave in X for each Y, then:

$$\text{Min}_Y \text{Max}_X \left\{ F(X,Y) | X \in S, \; Y \in T \right\} = \text{Max}_X \text{Min}_Y \left\{ F(X,Y) | X \in S, \; Y \in T \right\}.$$

To apply the Min-Max Theorem define:

$$S = \left\{ \mu_j \; j=1,\ldots,n \; \Big| \; \mu_j \geq 0 \; j=1,\ldots,n, \; \sum_{j=1}^{n} \mu_j = 1 \right\},$$

$$T = \left\{ V \; \Big| \; V \geq 0, \; Vc' \leq \text{Max}_j \left\{ H_j^*(0,Y) - H_j(0,Y) \right\} \right\},$$

and

$$F(X,Y) = \sum_{j=1}^{n} u_j H_j^*(V,Y) + Vc'.$$

It is obvious that S is closed, bounded and convex and that T is closed and convex for any $Y \in T$, $B(Y) \leq b$. To show that T is also bounded we must recall that $c > 0$ and that the functions $H_j(X_j,Y)$ $j=1,\ldots,n$ are finite valued in $X_j \geq 0$ $j=1,\ldots,n$ for $Y \in T$, $B(Y) \leq b$. Then $\text{Max}_j \left\{ H_j^*(0,Y) - H_j(0,Y) \right\} < \infty$ and the set T must be bounded. The function $\sum_{j=1}^{n} u_j H_j^*(V,Y) + Vc'$ is clearly continuous and concave in the variables u_j $j=1,\ldots,n$ for all values of the vector V. The function is continuous in V for any u_j $j=1,\ldots,n$ because the functions $H_j^*(V,Y)$ $j=1,\ldots,n$ are each assumed to be continuous in V.

Since $u_j \geq 0$ $j=1,\ldots,n$ the function $\sum_{j=1}^{n} u_j H_j^*(V,Y) + Vc'$ is convex in V if the functions $H_j^*(V,Y)$ $j=1,\ldots,n$ are each convex functions of V. Choose $V_1 \geq 0$ and $V_2 \geq 0$, then for any $0 \leq \tau \leq 1$:

$$H_j^*(tV_1 + (1-t)V_2, Y) = \text{Sup}_{X_j}\left\{H_j(X_j, Y) - (tV_1 + (1-t)V_2)A_j'X_j' \mid X_j \geqq 0\right\}$$

$$\leqq \text{Sup}_{X_j}\left\{tH_j(X_j, Y) - tV_1 A_j'X_j' \mid X_j \geqq 0\right\} + \text{Sup}_{X_j}\left\{(1-t)H_j(X_j, Y)\right.$$

$$\left. - (1-t)V_2 A_j'X_j' \mid X_j \geqq 0\right\} = tH_j^*(V_1, Y) + (1-t)H_j^*(V_2, Y)$$

which proves convexity.

Since the conditions of the Min-Max Theorem are met for all $Y \in T$, $B(Y) \leqq b$:

$$\text{Min}_V \text{ Max}_{u_j}\left\{\sum_{j=1}^{n} \mu_j H_j^*(V,Y) + Vc' \mid \mu_j \geqq 0 \; j=1,\ldots,n, \; \sum_{j=1}^{n}\mu_j = 1, \right.$$

$$\left. V \geqq 0, \; Vc' \leqq \text{Max}_j\left\{H_j^*(0,Y) - H_j(0,Y)\right\}\right\}$$

$$= \text{Max}_{u_j} \text{ Min}_V\left\{\sum_{j=1}^{n} \mu_j H_j^*(V,Y) + Vc' \mid \mu_j \geqq 0 \; j=1,\ldots,n, \; \sum_{j=1}^{n}u_j = 1, \right.$$

$$\left. V \geqq 0, \; Vc' \leqq \text{Max}_j\left\{H_j^*(0,Y) - H_j(0,Y)\right\}\right\}.$$

The bound $Vc' \leqq \text{Max}_j\left\{H_j^*(0,Y) - H_j(0,Y)\right\}$ is superfluous. Consider any saddle-point $(\bar{u}_j \; j=1,\ldots,n, \; \bar{V})$. At such a point we must have:

$$\sum_{j=1}^{n}\bar{u}_j H_j^*(\bar{V},Y) + \bar{V}c' \leqq \sum_{j=1}^{n}\bar{u}_j H_j^*(V,Y) + Vc'$$

for any $V \geqq 0$, $Vc' \leqq \text{Max}_j\left\{H_j^*(0,Y) - H_j(0,Y)\right\}$.

Since the inequality must hold at $V = 0$:

$$\sum_{j=1}^{n}\bar{u}_j H_j^*(\bar{V},Y) + \bar{V}c' \leqq \sum_{j=1}^{n}\bar{u}_j H_j^*(0,Y).$$

Also,

$$H_j^*(\bar{V},Y) = \text{Sup}_{X_j}\left\{H_j(X_j,Y) - \bar{V}A_j'X_j' \mid X_j \geqq 0\right\} \geqq H_j(0,Y) \; j=1,\ldots,n.$$

We may substitute and rearrange terms to obtain the inequality:

$$\bar{V}c' \leqq \sum_{j=1}^{n}\bar{u}_j [H_j^*(0,Y) - H_j(0,Y)].$$

On the right-hand side, $\bar{\mu}_j \geq 0$ $j=1,\ldots,n$ and $\sum_{j=1}^{n} \bar{\mu}_j = 1$, therefore, $\bar{V}c' \leq \text{Max}_j \left\{ H_j^*(0,Y) - H_j(0,Y) \right\}$. Thus the bound $Vc' \leq \text{Max}_j \left\{ H_j^*(0,Y) - H_j(0,Y) \right\}$ is always respected and its inclusion to meet the conditions of the Min-Max Theorem is purely formal. Omitting the bound we have:

$$\sum_{j=1}^{n} \mu_j^0 H_j^*(V^0,Y^0) + V^0 c' \leq \text{Max}_{\mu_j} \text{Min}_V \left\{ \sum_{j=1}^{n} \mu_j H_j^*(V,Y) + Vc' \right|$$

$$V \geq 0, \; \mu_j \geq 0 \; j=1,\ldots,n, \; \sum_{j=1}^{n} \mu_j = 1 \right\} \text{ for all } Y \in T, \; B(Y) \leq b.$$

Our next step is to show that:

$$\text{Min}_V \left\{ \sum_{j=1}^{n} \mu_j H_j^*(V,Y) + Vc' \right| V \geq 0 \right\}$$

$$= \text{Sup}_{X_j} \left\{ \sum_{j=1}^{n} \mu_j H_j(X_j,Y) \right| X_j \geq 0 \; j=1,\ldots,n \; \sum_{j=1}^{n} \mu_j X_j A_j \leq c \right\}$$

for all $\mu_j \geq 0$ $j=1,\ldots,n$, $\sum_{j=1}^{n} \mu_j = 1$ and $Y \in T$, $B(Y) \leq b$. Our procedure is similar to that followed by Owen [7] and makes use of the same Convex Duality Theorem. The Theorem is due to Rockafellar [9] who proves it in a somewhat more general form than that employed by Owen. Let $F(X)$ be a concave function of a vector X in E^n. The conjugate of F is the function $F^*(U)$ of U in E^n defined by:

$$F^*(U) = \text{Sup}_X \left\{ F(X) - UX' \right\} .$$

Let $G(Z)$ be a convex function of a vector Z in E^n. The conjugate of G is the function $G^*(V)$ of V in E^n defined by:

$$G^*(V) = \text{Inf}_Z \left\{ G(Z) - VZ' \right\} .$$

16

If the system $X \geq 0$, $XA \leq Z$ (A is a real matrix of appropriate dimensions) has a solution in the domains (of finiteness) of F and G, then:

$$\text{Sup}_{X,Z}\left\{F(X) - G(Z) \mid X \geq 0, XA \leq Z\right\}$$

$$= \text{Min}_{U,V}\left\{F^*(U) - G^*(V) \mid V \geq 0, VA' \geq U\right\}.$$

To apply Rockafellar's Theorem we define:

$$F(X) = \begin{cases} \sum_{j=1}^{n} \mu_j H_j(X_j, Y) & \text{if } X_j \geq 0 \; j=1,\ldots,n \\ -\infty & \text{otherwise} \end{cases}$$

and

$$G(Z) = \begin{cases} 0 & \text{if } Z = c \\ \infty & \text{otherwise.} \end{cases}$$

To verify that the conditions of the theorem are met we observe, first, that $F(X)$ is concave in X_j $j=1,\ldots,n$ for $\mu_j \geq 0$ $j=1,\ldots,n$ provided that the functions $H_j(X_j,Y)$ are concave in $X_j \geq 0$ for all $Y \in T$, $B(Y) \leq b$. But this has already been assumed. Second, the function $G(Z)$ is trivially convex in Z. Lastly we must show that the system $X_j \geq 0$ $j=1,\ldots,n$, $\sum_{j=1}^{n} \mu_j X_j A_j \leq Z$ has a solution in the set of points for which F and G are finite for any $u_j \geq 0$ $j=1,\ldots,n$, $\sum_{j=1}^{n} \mu_j = 1$. Since $G(Z)$ is finite only at $Z = c$ this condition is met only if $X_j \geq 0$, $X_j A_j \leq c$ is a solvable system and $H_j(X_j,Y)$ is finite-valued for at least one solution for each tactic $j=1,\ldots,n$. We have assumed that $X_j \geq 0$, $X_j A_j \leq c$ is solvable and that $H_j(X_j,Y)$ is finite-valued in $X_j \geq 0$ for all $Y \in T$, $B(Y) \leq b$ for every tactic $j=1,\ldots,n$. The conjugate functions $F^*(U)$ and $G^*(V)$ are:

17

$$F^*(U) = \mathrm{Sup}_{X_j} \left\{ \sum_{j=1}^{n} \mu_j H_j(X_j, Y) - \sum_{j=1}^{n} U_j X_j' \,\middle|\, X_j \geqq 0 \; j=1,\ldots,n \right\}$$

and

$$G^*(V) = - Vc'.$$

By the Convex Duality Theorem:

$$\mathrm{Sup}_{X_j, Z} \left\{ \sum_{j=1}^{n} \mu_j H_j(X_j, Y) - G(Z) \,\middle|\, X_j \geqq 0 \; j=1,\ldots,n, \; \sum_{j=1}^{n} \mu_j X_j A_j \leqq Z \right\}$$

$$= \mathrm{Min}_{U_j, V} \left\{ \mathrm{Sup}_{X_j} \left\{ \sum_{j=1}^{n} \mu_j H_j(X_j, Y) - \sum_{j=1}^{n} U_j X_j' \,\middle|\, X_j \geqq 0 \; j=1,\ldots,n \right\} \right.$$

$$\left. + Vc' \,\middle|\, V \geqq 0, \; V\mu_j A_j' \geqq U_j \; j=1,\ldots,n \right\} \text{ for all } \mu_j \geqq 0 \; j=1,\ldots,n,$$

$$\sum_{j=1}^{n} \mu_j = 1 \quad \text{and} \quad Y \in T, \; B(Y) \leqq b.$$

Each side of this equation can be simplified. On the left-hand side a supremum always is attained at $Z = c$. On the right-hand side the supremum within the brackets is a nonincreasing function of the vectors U_j $j=1,\ldots,n$. Therefore, a minimum can always be found among the vectors V, U_j $j=1,\ldots,n$ such that $V\mu_j A_j' = U_j$ $j=1,\ldots,n$.

On the right-hand side of the equation we may omit the inequalities $V\mu_j A_j' \geqq U_j$ $j=1,\ldots,n$ and make the substitution:

$$\mathrm{Sup}_{X_j} \left\{ \sum_{j=1}^{n} \mu_j H_j(X_j, Y) - \sum_{j=1}^{n} U_j X_j' \,\middle|\, X_j \geqq 0 \; j=1,\ldots,n \right\}$$

$$= \mathrm{Sup}_{X_j} \left\{ \sum_{j=1}^{n} \mu_j H_j(X_j, Y) - \sum_{j=1}^{n} \mu_j V A_j' X_j' \,\middle|\, X_j \geqq 0 \; j=1,\ldots,n \right\}.$$

This can be simplified still further by inserting the functions $H_j^*(V, Y)$ $j=1,\ldots,n$:

$$\text{Sup}_{X_j} \left\{ \sum_{j=1}^{n} \mu_j H_j(X_j,Y) - \sum_{j=1}^{n} u_j VA'_j X'_j \Big| X_j \geqq 0 \ j=1,\ldots,n \right\}$$

$$= \sum_{j=1}^{n} u_j H_j^*(V,Y).$$

After all of these simplifications have been incorporated the equation provided by the Convex Duality Theorem becomes:

$$\text{Sup}_{X_j} \left\{ \sum_{j=1}^{n} \mu_j H_j(X_j,Y) \Big| X_j \geqq 0 \ j=1,\ldots,n, \ \sum_{j=1}^{n} \mu_j X_j A_j \leqq c \right\}$$

$$= \text{Min}_V \left\{ \sum_{j=1}^{n} \mu_j H_j^*(V,Y) + Vc' \Big| V \geqq 0 \right\} \text{ for all}$$

$\mu_j \geq 0 \ j=1,\ldots,n, \ \sum_{j=1}^{n} \mu_j = 1$ and $Y \in T$, $B(Y) \leqq b$.

Combining this result with that of the previous step yields:

$$\sum_{j=1}^{n} u_j^o H_j^*(V^o,Y^o) + V^o c' \leqq \text{Sup}_{X_j,\mu_j} \left\{ \sum_{j=1}^{n} \mu_j H_j(X_j,Y) \right.$$

$$\left| X_j \geqq 0 \ j=1,\ldots,n, \ u_j \geqq 0 \ j=1,\ldots,n, \ \sum_{j=1}^{n} \mu_j = 1, \ \sum_{j=1}^{n} \mu_j X_j A_j \leqq c \right\}$$

for all $Y \in T$, $B(Y) \leqq b$.

To obtain Statement 3 of the definition of a solution it remains only to show that:

$$\sum_{j=1}^{n} u_j^o H_j^*(V^o,Y^o) + V^o c' = \sum_{j=1}^{n} u_j^o H_j(X_j^o,Y^o).$$

From Condition 2a we obtain:

$$H_j^*(V^o,Y^o) = H_j(X_j^o,Y^o) - V^o A'_j X_j^{o'} \ j=1,\ldots,n.$$

Multiplying by the probabilities u_j^0 $j=1,\ldots,n$, summing and adding $V^0 c'$ to each side:

$$\sum_{j=1}^{n} u_j^0 H_j^*(V^0, Y^0) + V^0 c' = \sum_{j=1}^{n} u_j^0 H_j(X_j^0, Y^0) - V^0 \left[\sum_{j=1}^{n} u_j^0 X_j^0 A_j - c \right]'.$$

From $V^0 \geq 0$, Condition 2c and the previously derived inequality $\sum_{j=1}^{n} u_j^0 X_j^0 A_j \leq c$ we may infer:

$$V^0 \left[\sum_{j=1}^{n} u_j^0 X_j^0 A_j - c \right]' = 0.$$

Statement 3 of the definition of a solution to the constrained min-max problem follows since $B(Y^0) \leq b$ is simply Condition 3b.

THE SUFFICIENT CONDITIONS WITH DIFFERENTIABLE QUASI-CONVEX FUNCTIONS

A somewhat stronger set of sufficient conditions may be derived for problems with the characteristics (in addition to those already assumed--see page 7.)

7. $T = \left\{ Y \text{ in } E^n \mid Y \geq 0 \right\}$

8. $H_j(X_j, Y)$ $j=1,\ldots,n$ are differentiable in $X_j \geq 0$ $j=1,\ldots,n$ and $Y \geq 0$.

9. $H_j^*(V,Y)$ $j=1,\ldots,n$ are differentiable in $V \geq 0$ and $Y \geq 0$.

10. $B(Y)$ is differentiable and quasi-convex in $Y \geq 0$.

If Assumptions 7 through 10 are met, the sufficient conditions of the previous section may be replaced by the following stronger set. Let W denote a vector of nonnegative real numbers whose elements correspond to the elements of the resource vector function $B(Y)$:

1. $X_j^0 \geq 0$ $j=1,\ldots,n$, $u_j^0 \geq 0$ $j=1,\ldots,n$, $\sum_{j=1}^{n} u_j^0 = 1$, $Y^0 \geq 0$, $V^0 \geq 0$ $W^0 \geq 0$.

2a. For every tactic j and each element X_{jk} of X_j:

$$\frac{\partial H_j(X_j^0, Y^0)}{\partial X_{jk}} - V^0 a_{jk}' \leq 0 \quad \text{and}$$

$$\frac{\partial H_j(X_j^0, Y^0)}{\partial X_{jk}} - V^0 a_{jk}' = 0 \quad \text{if } X_{jk}^0 > 0,$$

where a_{jk} denotes the kth row of A_j.

2b. For every tactic j:

$$\sum_{j=1}^{n} \mu_j^o [H_j(X_j^o, Y^o) - V^o A_j' X_j^{o\prime}] \geqq H_j(X_j^o, Y^o) - V^o A_j' X_j^{o\prime} \quad \text{and}$$

$$\sum_{j=1}^{n} \mu_j^o [H_j(X_j^o, Y^o) - V^o A_j' X_j^{o\prime}] = H_j(X_j^o, Y^o) - V^o A_j' X_j^{o\prime} \quad \text{if } \mu_j^o > 0.$$

2c. $V^o [\sum_{j=1}^{n} \mu_j^o X_j^o A_j - c]' \geqq V [\sum_{j=1}^{n} \mu_j^o X_j^o A_j - c]' \quad \text{for all } V \geqq 0.$

3a. For each element Y_k of Y:

$$\sum_{j=1}^{n} \mu_j^o \frac{\delta H_j(X_j^o, Y^o)}{\delta Y_k} + W^o \left[\frac{\delta B(Y^o)}{\delta Y_k} \right]' \geqq 0 \quad \text{and}$$

$$\sum_{j=1}^{n} \mu_j^o \frac{\delta H_j(X_j^o, Y^o)}{\delta Y_k} + W^o \left[\frac{\delta B(Y^o)}{\delta Y_k} \right]' = 0 \quad \text{if } Y_k^o > 0.$$

3b. $W^o [B(Y^o) - b]' \geqq W[B(Y^o) - b]' \quad \text{for all } W \geqq 0.$

4. $\sum_{j=1}^{n} \mu_j^o H_j^*(V, Y) + Vc'$ is quasi-convex in $V \geqq 0$ and $Y \geqq 0$ and

one of the following:

a. $\sum_{j=1}^{n} \mu_j^o X_j^o A_j - c \neq 0$

b. $\sum_{j=1}^{n} \mu_j^o \frac{\delta H_j(X_j^o, Y^o)}{\delta Y_k} > 0$ for at least one element Y_k of Y.

c. $\sum_{j=1}^{n} \mu_j^o \frac{\delta H_j(X_j^o, Y^o)}{\delta Y_k} < 0$ for at least one element Y_k of Y

and there exists a $\bar{Y} \geqq 0$, $B(\bar{Y}) \leqq b$ such that $\bar{Y}_k > 0$.

d. $\sum_{j=1}^{n} \mu_j^o \frac{\delta H_j(X_j^o, Y^o)}{\delta Y_k} \neq 0$ for at least one element Y_k of Y

and the functions $H_j^*(V, Y)$ $j=1, \ldots, n$ are twice differentiable

in the neighborhood of V^o, Y^o.

If the functions $H_j^*(V,Y)$ $j=1,\ldots,n$ are all convex in $V \geq 0$ and $Y \geq 0$, Condition 4 may be omitted.

In Condition 1, $Y^O \epsilon T$ has been replaced by $Y^O \geq 0$ and the non-negativity condition $W^O \geq 0$ has been added. Conditions 2b and 2c are unchanged. The new Condition 2a may be readily derived by applying a familiar lemma. Let $F(X)$ be a differentiable concave function of a vector X in the region $X \geq 0$. Let $X^O \geq 0$ be chosen such that for each element X_k of X:

$$\frac{\partial F(X^O)}{\partial X_k} \leq 0 \qquad \text{and} \qquad \frac{\partial F(X^O)}{\partial X_k} = 0 \quad \text{if } X_k^O > 0.$$

Then X^O maximizes $F(X)$ in $X \geq 0$. To obtain our new Condition 2a, let:

$$F(X) = H_j(X_j, Y^O) - V^O A_j' X_j' \ .$$

According to the lemma X_j^O maximizes $H_j(X_j, Y^O) - V^O A_j' X_j' \geq 0$, i.e.,

$$H_j(X_j^O, Y^O) - V^O A_j' X_j^{O\prime} \geq H_j(X_j, Y^O) - V^O A_j' X_j' \quad \text{for all } X_j \geq 0.$$

Since this must hold for all tactics $j=1,\ldots,n$, the new Condition 2a implies the old one.

To derive the remaining conditions (and the inequality $W^O \geq 0$) we must first evaluate the derivatives of the functions $H_j^*(V,Y)$ $j=1,\ldots,n$ in the region of (V^O, Y^O). This is done by making use of an inference from Condition 2a. For every tactic j and each element X_{jk} of X_j:

$$\left[\frac{\partial H_j(X_j^O, Y^O)}{\partial X_{jk}} - V^O a_{jk}' \right] X_{jk}^O = 0.$$

Take the total derivative of $H_j^*(V,Y)$ at (V^O, Y^O):

$$dH_j^* = \left[\sum_k \frac{\partial H_j(X_j^O, Y^O)}{\partial X_{jk}} - V^O a_{jk}' \right] dX_{jk} + \frac{\partial H_j(X_j^O, Y^O)}{\partial Y} dY' - X_j^O A_j \, dV'$$

23

where $\dfrac{\partial H_j(X_j^O, Y^O)}{\partial Y}$ denotes a vector of partial derivatives with respect to the elements of Y. It is clear that the maximum is preserved when dX_{jk} is nonzero only if $\dfrac{\partial H_j(X_j^O, Y^O)}{\partial X_{jk}} - V^O a'_{jk} = 0.$

Therefore, $dH_j^* = \dfrac{\partial H_j(X_j^O, Y^O)}{\partial Y} dY' - A'_j X_j^{O'} dV',$

$$\dfrac{\partial H_j^*(V^O, Y^O)}{\partial Y} = \dfrac{\partial H_j(X_j^O, Y^O)}{\partial Y} \quad j=1,\ldots,n.$$

and

$$\dfrac{\partial H_j^*(V^O, Y^O)}{\partial V} = - X_j^O A_j \quad j=1,\ldots,n.$$

The nonnegativity condition $W^O \geq 0$ added to Condition 1 and our new Conditions 3 and 4 all proceed from a Theorem by Arrow and Enthoven [1]. The Theorem is stated here in a form appropriate for constrained minima and quasi-convex functions. Let F(X) be a differentiable quasi-convex function of the vector X, and let G(X) be a differentiable quasi-convex vector function, both defined for $X \geq 0$. Let X^O and λ^O (a vector) satisfy the Kuhn-Tucker Conditions, specifically:

$$X^O \geq 0, \quad \lambda^O \geq 0$$

$$\dfrac{\partial F(X^O)}{\partial X} + \lambda^O \left[\dfrac{\partial G(X^O)}{\partial X} \right]' \geq 0$$

$$X^O \left[\dfrac{\partial F(X^O)}{\partial X} + \lambda^O \left[\dfrac{\partial G(X^O)}{\partial X} \right]' \right] = 0$$

$$\lambda^O [G(X^O)]' = 0,$$

where $\dfrac{\partial F(X^O)}{\partial X}$ denotes a vector and $\dfrac{\partial G(X^O)}{\partial X}$ a matrix of derivatives with respect to the elements of X, and let one of the following conditions be satisfied:

24

a. $\dfrac{\partial F(X^O)}{\partial X_k} > 0$ for at least one element X_k of X;

b. $\dfrac{\partial F(X^O)}{\partial X_k} < 0$ for at least one element X_k of X where there

   exists an $\bar{X} \geq 0$ with $\bar{X} > 0$ such that $G(\bar{X}) \leq 0$;

c. $\dfrac{\partial F(X^O)}{\partial X} \neq 0$ and $F(X)$ is twice differentiable in the neighbor-

   hood of X^O;

d. $F(X)$ is convex.

Then X^O minimizes $F(X)$ subject to $X \geq 0$ and $G(X) \leq 0$. To apply
Arrow and Enthoven's Theorem let:

$$F(X) = \sum_{j=1}^{n} u_j^O H_j^*(V,Y) + Vc'$$

and $G(X) = B(Y) - b$. Both $F(X)$ and $G(X)$ are assumed to be quasi-convex
in $V \geq 0$ and $Y \geq 0$. The Kuhn-Tucker Conditions for a constrained
minimum are:

$$V^O \geq 0, \ Y^O \geq 0, \ W^O \geq 0$$

$$\sum_{j=1}^{n} u_j^O \frac{\partial H_j^*(V^O,Y^O)}{\partial V} + c \geq 0, \quad \sum_{j=1}^{n} u_j^O \frac{\partial H_j^*(V^O,Y^O)}{\partial Y} + W^O \left[\frac{\partial B(Y^O)}{\partial Y}\right]' \geq 0$$

$$V^O \left[\sum_{j=1}^{n} u_j^O \frac{\partial H_j^*(V^O,Y^O)}{\partial V} + c\right]' = 0, \quad Y^O \left[\sum_{j=1}^{n} u_j^O \frac{\partial H_j^*(V^O,Y^O)}{\partial Y} + W^O \right.$$

$$\left. \left[\frac{\partial B(Y^O)}{\partial Y}\right]'\right]' = 0$$

$$W^O \left[B(Y^O) - b\right]' = 0.$$

Substituting for the derivatives of $H_j^*(V,Y)$ $j=1,\ldots,n$ the Kuhn-Tucker
Conditions may be rewritten as:

$$V^O \geq 0, \ Y^O \geq 0, \ W^O \geq 0$$

$$\sum_{j=1}^{n} u_j^O X_j^O A_j - c \leq 0 \quad \text{and} \quad V^O \left[\sum_{j=1}^{n} u_j^O X_j^O A_j - c\right]' = 0$$

25

$$\sum_{j=1}^{n} \mu_j^o \frac{\delta H_j(X_j^o, Y^o)}{\delta Y} + W^o \left[\frac{\delta B(Y^o)}{\delta Y} \right]' \geq 0$$

and

$$Y^o \left[\sum_{j=1}^{n} \mu_j^o \frac{\delta H_j(X_j^o, Y^o)}{\delta Y} + W^o \left[\frac{\delta B(Y^o)}{\delta Y} \right]' \right]' = 0$$

$$W^o [B(Y^o) - b]' = 0.$$

Conditions 1 through 3 of our new set imply that the Kuhn-Tucker Conditions for a constrained minimum are satisfied. Condition 1 includes the nonnegativity conditions $V^o \geq 0$, $Y^o \geq 0$. Condition 2c is equivalent to:

$$\sum_{j=1}^{n} \mu_j^o X_j^o A_j - c \leq 0 \quad \text{and} \quad V^o [\sum_{j=1}^{n} \mu_j^o X_j^o A_j - c]' = 0.$$

Condition 3a may be used to derive:

$$\sum_{j=1}^{n} \mu_j^o \frac{\delta H_j(X_j^o, Y^o)}{\delta Y} + W^o \left[\frac{\delta B(Y^o)}{\delta Y} \right]' \geq 0$$

and

$$Y^o \left[\sum_{j=1}^{n} \mu_j^o \frac{\delta H_j(X_j^o, Y^o)}{\delta Y} + W^o \left[\frac{\delta B(Y^o)}{\delta Y} \right]' \right]' = 0,$$

and $W^o [B(Y^o) - b]' = 0$ follows from Condition 3b.

The remaining stipulations of Arrow and Enthoven's Theorem apply to the constrained minimization problem as follows:

a. $\sum_{j=1}^{n} \mu_j^o X_j^o a_{hj}' - c_h < 0 \quad \text{or} \quad \sum_{j=1}^{n} \frac{\delta H_j(X_j^o, Y^o)}{\delta Y_k} > 0$

for at least one element V_h of V or Y_k of Y [a_{hj} denotes the hth row of A_j];

b. $\sum_{j=1}^{n} u_j^O X_j^O a'_{hj} - c_h > 0$ or $\sum_{j=1}^{n} u_j^O \dfrac{\partial H_j(X_j^O, Y^O)}{\partial Y_k} < 0$

for at least one element V_h of V or Y_k of Y where there exists a $\bar{Y} \geq 0$ with $\bar{Y}_k > 0$ such that $B(\bar{Y}) \leq b$;

c. $\sum_{j=1}^{n} u_j^O X_j^O a'_{jh} - c_k \neq 0$ or $\sum_{j=1}^{n} u_j^O \dfrac{\partial H_j(X_j^O, Y^O)}{\partial Y_k} \neq 0$

for at least one element V_h of V or Y_k of Y and the functions $H_j^*(V, Y)$ $j=1,\ldots,n$ are twice differentiable in the neighborhood of (V^O, Y^O);

d. The expression $\sum_{j=1}^{n} u_j^O H_j^*(V, Y) + Vc'$ is convex in $V \geq 0$ and

$Y \geq 0$.

Condition 4 is easily shown to be equivalent to a, b, and c providing

the expression $\sum_{j=1}^{n} \mu_j^O H_j^*(V, Y) + Vc'$ is quasi-convex. The fact that

Condition 4 is expendable if the functions $H_j^*(V, Y)$ $j=1,\ldots,n$ are convex is implied by d.

Therefore, our new set of conditions imply that (V^O, Y^O) minimizes

$\sum_{j=1}^{n} u_j^O H_j^*(V, Y) + Vc'$ subject to $B(Y) \leq b$, $V \geq 0$ and $Y \geq 0$, i.e.,

$$\sum_{j=1}^{n} u_j^O H_j^*(V^O, Y^O) + V^O c' \leq \sum_{j=1}^{n} \mu_j^O H_j^*(V, Y) + Vc'$$

for all $V \geq 0$, $Y \geq 0$ and $B(Y) \leq b$

and

$B(Y^O) \leq b$

which is Condition 3 of the previous Section.

Before concluding this Section some observations should be made

about the properties of the expression $\sum_{j=1}^{n} \mu_j^O H_j^*(V, Y) + Vc'$ in the

27

region $V \geq 0$, $Y \geq 0$. Condition 4 requires that this expression be quasi-convex, i.e., every set $R_{\bar{z}}$ defined as

$$R_{\bar{z}} = \left\{ V,Y \ \middle| \ \sum_{j=1}^{n} \mu_j^O H_j^*(V,Y) + Vc' \leq \bar{z} \right\}$$

must be convex. If the functions $H_j^*(V,Y)$ $j=1,\ldots,n$ are all convex, then by virtue of the fact that $\mu_j^O \geq 0$ $j=1,\ldots,n$, the expression $\sum_{j=1}^{n} \mu_j^O H_j^*(V,Y) + Vc'$ must also be convex. However, the expression is not necessarily quasi-convex if the functions $H_j^*(V,Y)$ $j=1,\ldots,n$ are each quasi-convex. To establish quasi-convexity of $\sum_{j=1}^{n} \mu_j^O H_j^*(V,Y) + Vc'$ the following result from Arrow and Enthoven [1] may be used. A function $F(X)$ of an n-dimensional vector X is quasi-convex if $D_r \leq 0$ for all $r=1,\ldots,n$ and for all X where D_r is the bordered determinant:

$$D_r = \begin{vmatrix} 0 & \partial F(X)/\partial X_1 & \ldots \partial F(X)/\partial X_r \\ \partial F(X)/\partial X_1 & \partial^2 F(X)/\partial X_1 \partial X_1 & \ldots \partial^2 F(X)/\partial X_1 \partial X_r \\ \vdots & \vdots & \vdots \\ \partial F(X)/\partial X_r & \partial^2 F(X)/\partial X_r \partial X_1 & \ldots \partial^2 F(X)/\partial X_r \partial X_r \end{vmatrix}.$$

If it is possible to show that the bordered determinants for the function $F(X) = \sum_{j=1}^{n} \mu_j^O H_j^*(V,Y) + Vc'$ have the desired sign everywhere in the region $V \geq 0$, $Y \geq 0$ then our Conditions 1 through 4 are sufficient for a global solution to the constrained min-max problem. If quasi-convexity can only be demonstrated in the vicinity of (V^O,Y^O), then a local solution has been found that may or may not also be a global solution.

A MODIFIED DOUBLE LAGRANGE MULTIPLIER METHOD

In general, Lagrange multiplier methods replace constrained problems with unconstrained problems by reversing the computational roles of resource levels and imputed resource prices. Such methods are "fail-safe" if it is possible to show that a solution to the unconstrained problem always solves a constrained problem for an ascertainable pair of resource vectors. The value of the method as a procedure for arranging computations is entirely attributable to the fact that explicit constraints can frequently be an expensive computational nuisance.

A "fail-safe" double Lagrange multiplier method for constrained min-max problems with mixed strategies that conform to assumptions 1 through 10 begins with an arbitrary choice of the multiplier vectors $\bar{V} \geqq 0$ and $\bar{W} \geqq 0$. Next, strategies X_j^O $j=1,\ldots,n$ for the maximizing player's tactics, probabilities μ_j^O $j=1,\ldots,n$ and a strategy Y^O for the minimizing player are found such that:

1. $X_j^O \geqq 0$ $j=1,\ldots,n$, $\mu_j^O \geqq 0$ $j=1,\ldots,n$, $\sum\limits_{j=1}^{n} \mu_j^O = 1$, $Y^O \geqq 0$

2a. For every tactic j and each element X_{jk} of X_j:

$$\frac{\partial H_j(X_j^O, Y^O)}{\partial X_{jk}} - \bar{V} a_{jk}' \leqq 0 \quad \text{and}$$

$$\frac{\partial H_j(X_j^O, Y^O)}{\partial X_{jk}} - \bar{V} a_{jk}' = 0 \quad \text{if } X_{jk}^O > 0$$

where a_{jk} denotes the kth row of A_j.

2b. For every tactic j:

$$\sum_{j=1}^{n} \mu_j^O \left[H_j(X_j^O, Y^O) - \bar{V} A_j' X_j^{O\prime} \right] \geqq H_j(X_j^O, Y^O) - \bar{V} A_j' X_j^{O\prime} \quad \text{and}$$

$$\sum_{j=1}^{n} \mu_j^O \left[H_j(X_j^O, Y^O) - \bar{V} A_j' X_j^{O\prime} \right] = H_j(X_j^O, Y^O) - \bar{V} A_j' X_j^{O\prime} \quad \text{if } \mu_j^O > 0.$$

3a. For each element Y_k of Y:

$$\sum_{j=1}^{n} \mu_j^O \frac{\delta H_j(X_j^O, Y^O)}{\delta Y_k} + W^O \left[\frac{\delta B(Y^O)}{\delta Y_k} \right]' \geqq 0 \quad \text{and}$$

$$\sum_{j=1}^{n} \mu_j^O \frac{\delta H_j(X_j^O, Y^O)}{\delta Y_k} + W^O \left[\frac{\delta B(Y^O)}{\delta Y_k} \right]' = 0 \quad \text{if } Y_k^O > 0.$$

The resource vectors c and b are determined after the fact according to the formulae:

$$c = \sum_{j=1}^{n} \mu_j^O X_j^O A_j \quad \text{and} \quad b = B(Y^O)$$

and the solution is examined to verify that the problem is not de-
generate in the sense that Condition 4 is not satisfied. It should
be perfectly apparent that the strategies X_j^O j=1,...,n, probabilities
μ_j^O j=1,...,n, the strategy Y^O, and the vectors $\bar{V}$ and $\bar{W}$ jointly satisfy
the strong set of sufficient conditions of the previous Section.

The procedure is an extrapolation to constrained min-max problems
of Everett's generalized Lagrange multiplier method. In the special
case of a single tactic the procedure reduces to the double Lagrange
multiplier method proposed by Pugh. The probabilities then drop
from view and Condition 2b may be omitted.

At this point a rough intuitive description of the method may
be helpful. The Lagrange multiplier vectors $\bar{V}$ and $\bar{W}$ bear interpre-
tation as prices to the maximizing and minimizing players on their
resources. These vectors provide a priori rates at which the players
may convert additional units of resources to changes in the expected
value of the payoff function. The solution value of the Lagrange
function:

30

$$\sum_{j=1}^{n} \mu_j^O H_j(X_j^O, Y^O) - V^O \sum_{j=1}^{n} \mu_j^O A_j' X_j^{O'} + W^O [B(Y^O)]'$$

also has an economic interpretation. The quantities $V^O \sum_{j=1}^{n} \mu_j^O A_j' X_j^{O'}$

and $W^O [B(Y^O)]'$ are the opportunity costs in payoff units of the resources used by the maximizing and minimizing players, respectively. The value of the Lagrange function is the expected payoff to the maximizing player modified by subtracting the opportunity cost of the resources he must expend to gain this payoff and by adding the value of the defense resources used to limit the payoff.

The strategies X_j^O $j=1,\ldots,n$, the probabilities μ_j^O $j=1,\ldots,n$, and the strategy Y^O may be viewed as an equilibrium pair in a restricted sense. The maximizing player's mixed strategy is optimal against the minimizing player's strategy but the reverse relationship holds in only a partial fashion. The strategy Y^O is not necessarily strictly optimal against the specific strategies X_j^O mixed with probabilities μ_j^O. Rather, Y^O is optimal against a specific probabilistic mix of efficient tactical responses. That is, the fixed probabilities μ_j^O $j=1,\ldots,n$ apply to strategies X_j $j=1,\ldots,n$ that are continuously altered in response to changes in the strategy choice Y and the imputed prices of the minimizing player's resources V. These alterations have the effect of maintaining the strategies X_j $j=1,\ldots,n$ as economically efficient uses of resources against the minimizing player's strategies. The specific strategies X_j^O $j=1,\ldots,n$ are efficient given the vector $\bar{V}$ and the solution strategy Y^O.

VI

ON SUITABLE PROBLEMS

The modified double Lagrange multiplier method does not constitute
a uniformly powerful approach to all constrained min-max problems
with the requisite characteristics. Since resource levels are
treated in computations as an output rather than as an input, the
method is best suited to problems with only small numbers of con-
straints and to investigations of solutions over a range of resource
levels. Generally, the computations required to yield an acceptable
body of results for a problem will tend to increase geometrically with
the number of constraints.

The method is principally useful as a means for addressing
problems that may be partitioned by "cells". Let $h = 1,\ldots,m$ be
an index of cells. Each cell is assumed to possess, individually,
all of the components of a constrained min-max problem with mixed
strategies. For the hth cell, then, we denote these components
with superscripts as follows:

$j=1,\ldots,n^h$ - an index of tactics for the maximizing player.

χ_j^h - a strategy for the maximizing player's tactic j.

μ_j^h - a probability of use of χ_j^h.

Y^h - a strategy for the minimizing player.

$H_j^h(\chi_j^h, Y^h)$ - a payoff function for the tactic j.

$\chi_j^h A_j^h$ - resource use by the maximizing player for tactic j.

$B^h(Y^h)$ - resource use by the minimizing player.

A tactic for the maximizing player over all cells is a specific
combination of tactics for each cell. A complete list of such
overall tactics would consist of every possible combination of cell

32

tactics and can be quite long even for very simple problems. Probabilities for these overall tactics may be retrieved from the probabilities assigned to the cell tactics in any way that preserves the weights μ_j^h as conditional probabilities. For example,

$$\mu_j = \prod_{h=1}^{m} \mu_j^h \quad \text{for} \quad X_j = \left\{ X_j^1, X_j^2, \ldots, X_j^m \right\}$$

corresponds to an independent random drawing of the cell strategies X_j^h according to the probabilities μ_j^h. An overall strategy for the minimizing player is the composite strategy $Y = \left\{ Y^1, Y^2, \ldots, Y^m \right\}$. The expected payoff over all cells is the sum of the expected payoff in each cell:

$$\sum_{j=1}^{n} \mu_j H_j(X_j, Y) = \sum_{h=1}^{m} \sum_{j=1}^{n^h} \mu_j^h H_j^h(X_j^h, Y^h)$$

and the resource constraints of the overall problem are:

$$\sum_{j=1}^{n} \mu_j X_j A_j = \sum_{h=1}^{m} \sum_{j=1}^{n^h} \mu_j^h X_j^h A_j^h \leqq c$$

and

$$B(Y) = \sum_{h=1}^{m} B^h(Y^h) \leqq b.$$

The modified double Lagrange multiplier method makes it possible to solve partitionable problems cell by cell. That is, the derivation of the strategies $X_j^{h^O}$ $j=1,\ldots,n^h$, the probabilities $\mu_j^{h^O}$ $j=1,\ldots,n^h$, and the strategy Y^{h^O} for any one cell is completely separated from the derivation for any other cell. This fact often makes it possible to solve even very large constrained min-max problems by partitioning them into cells such that the application of the method in any single cell is straightforward.

Equally important, an expectational interpretation of the payoff function and the maximizing player's resource constraints is least likely to be objectionable with problems that may be partitioned.

33

Suppose that the maximizing player's solution implies an independent random drawing of strategies in each cell. This random process establishes probability distributions for the actual payoff and resource use by the maximizing player that would result from a play of the game with the solution strategies. In general the standard deviations of these probability distributions will tend to diminish relative to their means as the number of cells is increased. This consideration militates strongly in favor of highly separable problems in which the probable variations in payoff and resource use become insignificant in relation to the expected values.

The modified double Lagrange multiplier method is a general approach to formulating and solving constrained min-max problems. It is not a computational algorithm in the same sense as the simplex algorithm or Newton's method. In deriving solutions that satisfy the conditions of the previous section, the analyst is left to his own devices. The method applies equally well with clever mathematical derivations and unsubtle searches on a digital computer. The choice of a computational route for any particular problem must be made on the basis of the specific characteristics of the problem. Moreover, we have provided no grounds to warrant the belief that solutions to all suitable constrained min-max problems can be found by applying the method. The conditions we have derived are sufficient but they may not all be necessary.

VII

AN ABM/SHELTER DEPLOYMENT PROBLEM

A simple ABM/shelter deployment problem can be solved as an illustration of the essential features of the modified double Lagrange multiplier method. The game pits warheads against terminal interceptors and blast shelters allocated to cities. The maximizing player's strategies are targeting plans in which each of the cities is assigned a specific number of warheads. Strategies for the minimizing player consist of assignments of interceptors and shelters to cities. The interceptors are taken to be perfect in the sense that any warhead engaged by the defense is certain to be destroyed. We shall also assume that each shelter provides one resident of the city with complete protection from all weapons effects.

The payoff of the game is the total number of fatalities anticipated from an attack. In any single city fatalities occur among the unsheltered population and are computed as an exponential function of the number of detonating warheads. Both players are constrained by a single scarce resource. In the case of the maximizing player, his arsenal is presumed to consist solely of ballistic missiles capable of delivering only a limited number of warheads. The defense is restricted to deployments that do not require expenditures in excess of a given budget. His cost function is linear. Since the maximizing player observes the defense before choosing a targeting plan we have a constrained min-max problem. (This formulation of the ABM/shelter deployment problem is not intended to be more than illustrative. An application of the method to a more sophisticated and realistic version of the problem has been made elsewhere by the author.)

The problem may be partitioned by cities. Let $h=1,\ldots,m$ be an index of cities. For the hth city the maximizing player exercises

one of two tactical options. He may choose not to attack the city at all. This we shall call the null attack. The second tactic may be labeled an exhaustion attack. In an efficient use of this tactic the defense's interceptors are exhausted and the maximizing player continues to target warheads until the last warhead is just worth the damage it will cause. Mathematically, the components of the game for the hth city are:

X^h - number of warheads for the exhaustion attack,

μ^h - probability of using the exhaustion attack,

Y^h - a two-element vector as follows:

Y_1^h - the number of terminal interceptors

Y_2^h - the number of blast shelters,

P^h - the population of the city,

s^h - a damage function parameter,

α - cost per interceptor, and
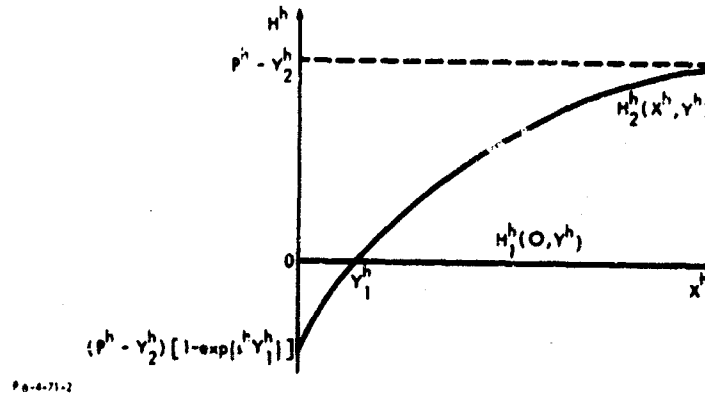
β - cost per shelter.

The payoff function for the null attack is:

$$H_1^h(0, Y^h) = 0$$

and for the exhaustion attack:

$$H_2^h(X^h, Y^h) = \begin{cases} 0 & \text{if } P^h - Y_2^h \leq 0 \\ (P^h - Y_2^h)[1 - \exp\{-s^h(X^h - Y_1^h)\}] & \text{if } P^h - Y_2^h > 0. \end{cases}$$

These two functions are graphed in the figure below for $P^h - Y_2^h > 0$ and $Y_1^h > 0$:

We have no vector X_1^h for the null attack, so:

$$H_1^*(V, Y^h) = 0.$$

For the exhaustion attack we have:

$$H_2^*(V, Y^h) = \begin{cases} 0 & \text{if } P^h - Y_2^h \leq 0 \\ \text{Sup}_{X^h} \left\{ (P^h - Y_2^h)[1 - \{\exp \ -s^h(X^h - Y_1^h)\}] - VX^h \,\big|\, X^h \geq 0 \right\} & \\ & \text{if } P^h - Y_2^h > 0 \end{cases}$$

If $V = 0$, the supremum takes the value $(P^h - Y_2^h)$; if $V > 0$, the supremum is a maximum. Differentiating with respect to X^h yields:

$$\frac{\partial H_2^h(X^h, Y^h) - VX^h}{\partial X^h} = s^h(P^h - Y_2^h) \exp\left\{-s^h(X^h - Y_1^h)\right\} - V.$$

At a maximum:

$$s^h(P^h - Y_2^h) \exp\left\{-s^h(X^h - Y_1^h)\right\} - V \leq 0 \quad \text{and}$$

$$s^h(P^h - Y_2^h) \exp\left\{-s^h(X^h - Y_1^h)\right\} - V = 0 \text{ if } X^h > 0.$$

37

Therefore:

$$H_2^*(V, Y^h) = \begin{cases} 0 & \text{if } P^h - Y_2^h \leqq 0 \\ (P^h - Y_2^h)[1 - \exp\{s^h Y_1^h\}] & \text{if } 0 < P^h - Y_2^h \leqq V\exp\{-s^h Y_1^h\}/s^h \\ (P^h - Y_2^h) - V[1 + \log\{s^h P^h - Y_2^h)/V\}]/s^h - VY_1^h \\ \qquad\qquad \text{if } P^h - Y_2^h \geqq V\exp\{-s^h Y_1^h\}/s^h. \end{cases}$$

The expected payoff in the nation as a whole is the expression:

$$\sum_{h=1}^{m} \mu^h \begin{cases} 0 & \text{if } P^h - Y_2^h \leqq 0 \\ (P^h - Y_2^h)[1 - \exp\{-s^h(X^h - Y_1^h)\}] & \text{if } P^h - Y_2^h > 0. \end{cases}$$

The maximizing player's expected use of warheads is:

$$\sum_{h=1}^{m} \mu^h X^h$$

and the cost of the ABM/shelter deployment is the linear expression:

$$\sum_{h=1}^{m} [\alpha Y_1^h + \beta Y_2^h].$$

It may be readily verified that the problem has all of the character-istics (Assumptions 1 through 10) that are necessary for an appropriate application of the modified double Lagrange multiplier method.

To apply the modified double Lagrange multiplier method we select multipliers $V > 0$ and $W > 0$ and then choose strategies and probabili-ties that constitute an equilibrium pair in the restricted sense of the previous sections. Omitting the superscript h, Conditions 1, 2a, 2b, and 3a reduce to the following system of conditional equations and inequalities for each city:

$$\chi^O = 0 \qquad\qquad\qquad\qquad \text{if } (P-Y_2^O)-\bar{V}/s \leqq 0$$

$$\chi^O = \log\left\{s(P-Y_2^O)/\bar{V}\right\}/s + Y_1^O \qquad \text{if } (P-Y_2^O)-\bar{V}/s > 0$$

$$\mu^O = 0 \text{ if } (P-Y_2^O) - \bar{V}[1 + \log\left\{s(P-Y_2^O)/\bar{V}\right\}]/s - \bar{V}Y_1^O < 0$$

$$0 \leqq \mu^O \leqq 1 \text{ if } (P-Y_2^O)-\bar{V}[1+\log\left\{s(P-Y_2^O)/\bar{V}\right\}]/s - \bar{V}Y_1^O = 0$$

$$\mu^O = 1 \text{ if } (P-Y_2^O)-\bar{V}[1+\log\left\{s(P-Y_2^O)/\bar{V}\right\}]/s - \bar{V}Y_1^O > 0$$

$$Y_1^O = 0 \text{ if } \mu^O[-V] + \bar{W}\alpha > 0$$

$$Y_1^O \geqq 0 \text{ if } \mu^O[-V] + \bar{W}\alpha = 0$$

$$Y_2^O = 0 \text{ if } \mu^O[\bar{V}/s(P-Y_2^O) - 1] + \bar{W}\beta > 0$$

$$Y_2^O \geqq 0 \text{ if } \mu^O[\bar{V}/s(P-Y_2^O) - 1] + \bar{W}\beta = 0.$$

There are five distinct types of solutions to this system discussed below as cases 1 through 5.

Case 1: $sP < \bar{V}$

$$\chi^O = 0$$

$$\mu^O = 0$$

$$Y_1^O = 0$$

$$Y_2^O = 0$$

$$\mu^O H_c(\chi^O, Y^O) = 0$$

$$\mu^O \chi^O = 0$$

$$\alpha Y_1 + \beta Y_2^O = 0.$$

The undefended city is not an attractive target to the maximizing player. Therefore, there is no reason to install interceptors or build shelters. In all of the remaining cases, the undefended city presents a worthwhile target to the maximizing player.

Case 2: $sP \geqq \bar{V}$, $\bar{W}\alpha > \bar{V}$, $\bar{W}\beta > 1 - \bar{V}/sP$

$$\chi^O = \log\left\{sP/\bar{V}\right\}/s$$

$$\mu^O = 1$$

$$Y_1^O = 0$$

$$Y_2^O = 0$$

$$\mu^O H_2(\chi^O, Y^O) = P - \bar{V}/s$$

$$\mu^O \chi^O = \log\left\{sP/\bar{V}\right\}/s$$

$$\alpha Y_1^O + \beta Y_2^O = 0.$$

Although the city is subject to attack, the minimizing player finds that it is best to leave the city undefended. The reasons for this are, first, that the opportunity cost in lives of an interceptor, $\bar{W}\alpha$, exceeds the value in lives of the single warhead it destroys, $\bar{V}$. Second, the opportunity cost in lives of a shelter, $\bar{W}\beta$, exceeds the expected saving in lives of installing the shelter, $1 - \bar{V}/sP$.

Case 3: $sP \geq \bar{V}$, $\bar{W}\alpha > \bar{V}$, $\bar{W}\beta \leqq 1 - \bar{V}/sP$

$$\chi^O = \log\left\{1/(1 - \bar{W}\beta)\right\}$$

$$\mu^O = 1$$

$$Y_1^O = 0$$

$$Y_2^O = P - \bar{V}/s(1 - \bar{W}\beta)$$

$$\mu^O H_2(\chi^O, Y^O) = \bar{V}\,\bar{W}\beta/s(1 - \bar{W}\beta)$$

$$\mu^O \chi^O = \log\left\{1/(1 - \bar{W}\beta)\right\}$$

$$\alpha Y_1^O + \beta Y_2^O = \beta P - \bar{V}\beta/s(1 - \bar{W}\beta).$$

In this case the blast shelters, but not the interceptors, are worth the cost. The optimum number of shelters minimizes the Lagrangian:

$$\mu^O H_2^*(\bar{V}, Y) + \bar{W}\beta Y_2.$$

Differentiating at $Y = Y^O$ we find that:

$$\mu^O[-1 + \bar{V}/s(P - Y_2^O)] + \bar{W}\beta = 0.$$

At the margin no increase or decrease in the number of shelters leads ultimately to a decrease in the overall payoff. Since the offense may move warheads among cities to maintain a marginal return of $\bar{V}$ per warhead, the net number of lives saved by an additional shelter in the city is $[1 - \bar{V}/s(P - Y_2^O)]$. This just balances the opportunity cost in lives of the shelter, $\bar{W}\beta$.

Case 4: $sP \geq \bar{V}$, $\bar{W}\alpha \leq \bar{V}$, $\bar{W}\beta > (\bar{W}\alpha/\bar{V})(1 - \bar{V}/sP)$

$$X^O = P/\bar{V} - 1/s$$

$$\mu^O = \bar{W}\alpha/\bar{V}$$

$$Y_1^O = P/\bar{V} - [1 + \log\left\{sP/\bar{V}\right\}]/s$$

$$Y_2^O = 0$$

$$\mu^O H_2(X^O, Y^O) = (\bar{W}\alpha/\bar{V})[P - \bar{V}/s]$$

$$\mu^O X^O = (\bar{W}\alpha/\bar{V})[P/\bar{V} - 1/s]$$

$$\alpha Y_1^O + \beta Y_2^O = \alpha P/\bar{V} - \alpha[1 + \log\left\{sP/\bar{V}\right\}]/s.$$

The city is allocated interceptors but not shelters since the opportunity cost of a shelter, $\bar{W}\beta$, exceeds the anticipated saving in lives from an attack, $1 - \bar{V}/sP$, multiplied by the probability of an attack, $\bar{W}\alpha/\bar{V}$. The optimum number of interceptors, Y_1^O, is determined not by a margin calculation as in case 3, but is set

41

instead at a level that makes attacking the city at all an indifferent proposition to the maximizing player. That is:

$$H_1^*(\bar{V}, Y^O) = H_2^*(\bar{V}, Y^O) = 0.$$

The null attack and the exhaustion attack tactics are probabilistically mixed with probabilities $(1 - \bar{W}\alpha/\bar{V})$ and $\bar{W}\alpha/\bar{V}$ such that neither an increase nor a decrease in the number of interceptors will reduce the value of the Lagrangian:

$$(1 - \mu^O)H_1^*(\bar{V}, Y^O) + \mu^O H_2^*(\bar{V}, Y^O) + \bar{W}\alpha Y_1^O .$$

Differentiating with respect to Y_1 and evaluating the derivative at Y^O we have:

$$\mu^O \frac{\partial H_2^*(\bar{V}, Y^O)}{\partial Y_1} = \left(\frac{\bar{W}\alpha}{\bar{V}}\right) [- \bar{V}] + \bar{W}\alpha = 0.$$

A rough intuitive explanation for the choice of $\mu^O = \bar{W}\alpha/\bar{V}$ may be provided by considering the role of the probability μ^O in ascertaining the resource level c. If we choose $\mu < \bar{W}\alpha/\bar{V}$, then the resulting arsenal is not sufficiently large to discourage a defender from decreasing the number of interceptors in every defended city. On the other hand $\mu > \bar{W}\alpha/\bar{V}$ yields a total number of warheads so large that an increase in the number of interceptors at every defended city must produce a return in the imputed value of intercepted warheads that on the average exceeds $\bar{W}\alpha$ per additional interceptor.

Case 5: $sP \geq \bar{V}$, $\bar{W}\alpha \leq \bar{V}$, $\bar{W}\beta \leq (\bar{W}\alpha/\bar{V})(1 - \bar{V}/sP)$

$$\chi^O = \alpha/s(\alpha - \bar{V}\beta) - 1/s$$

$$\mu^O = \bar{W}\alpha/\bar{V}$$

$$Y_1^O = \alpha/s(\alpha - \bar{V}\beta) - 1/s - \log\left\{\alpha/(\alpha - \bar{V}\beta)\right\}/s$$

42

$$Y_2^O = P - \bar{V}\alpha/s(\alpha - \bar{V}\beta)$$

$$\mu^O H_2(X^O, Y^O) = (\bar{W}\alpha/\bar{V})[\bar{V}\alpha/s(\alpha - \bar{V}\beta) - \bar{V}/s]$$

$$\mu^O X^O = (\bar{W}\alpha/\bar{V})[\alpha/s(\alpha - \bar{V}\beta) - 1/s]$$

$$\alpha Y_1^O + \beta Y_2^O = \beta P - \alpha \log\left\{\alpha/(\alpha - \bar{V}\beta)\right\}/s.$$

This last and most interesting type of solution is a mixture of active and passive defense systems. An heuristic explanation of the solution is a straightforward composite of the descriptions provided for cases 3 and 4.

The various values of the multipliers and their association with the five types of solutions are depicted in the figure below.
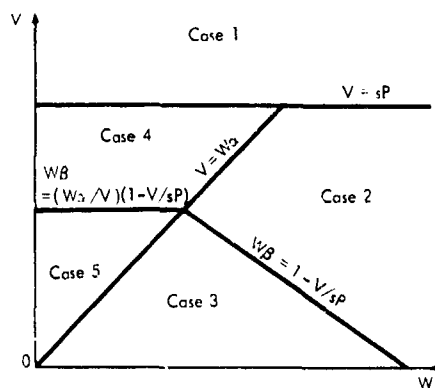
Once a number of warheads, X^{ho}, probability, μ^{ho}, number of interceptors, Y_1^{ho}, and number of shelters, Y_2^{ho}, have been determined for every city, the last step of the modified double Lagrange multiplier method is a simple summation of expected payoffs, expected use of warheads, and defense expenditures. In the normal course of an operations research exercise this would be repeated for many combinations of values for the multipliers V and W.

43

Unfortunately, it is only possible to show that the strategies derived using the modified double Lagrange multiplier method are local solutions to the ABM/shelter problem. An examination of the bordered determinants for the expression $F(X) = \sum_{j=1}^{n} \mu_j^O H_j^*(V,Y) + Vc'$ indicates they will all be negative wherever:

$$\frac{\partial F}{\partial V} = \sum_h \mu^{ho}\left\{- \log\left\{s^h(P^h - Y_2^h)/V\right\}/s^h - Y_1^h + \chi^{ho}\right\} \leqq 0.$$

This derivative is equal to zero at the solution point $V = \bar{V}$, $Y = Y^O$ but is not generally nonpositive over the entire region $V \geqq 0$, $Y \geqq 0$. It can be easily shown that part c of Condition 4 is met if at least one city is both targeted and defended.

VIII

CONCLUDING REMARKS

There are three major criticisms that can be made of the approach
to constrained min-max problems offered in this paper:

1. The wrong problem has been solved. A solution to a con-
strained min-max problem that implies a mixed strategy for the
maximizing player can have no direct application.

2. Ten strong assumptions are needed to obtain the set of
sufficient conditions that finally provide the basis for the modified
double Lagrange multiplier method. Also, quasi-convexity of the
expression $\sum_{j=1}^{n} \mu_j^0 H_j^*(V,Y) + Vc'$ (Condition 4) is not usually easy
to establish, especially over the entire region $V \geq 0$, $Y \geq 0$.

3. As a practical matter, the use of the method is limited to
partitionable problems with few constraints and to crude explorations
of solutions over a range of resource levels.

My response to these criticisms is:

1. The right problem has proved to be quite difficult to
address on a practical level with existing computational algorithms.
The introduction of mixed strategies and the redefinition of an
equilibrium pair appears to me to be the least damaging alteration
that leaves a readily solveable version of the constrained min-max
problem. Solutions to this wrong problem will often provide close
approximations or at least bounds for solutions to the right problem.

2. The ten assumptions we have used are probably not the only
assumptions on which a Lagrange multiplier method for constrained
min-max problems can be based. For example, a fail-safe version of
the method could proceed directly from the first set of sufficient
conditions providing Condition 3a of Section III can be weakened to:

45

$$\sum_{j=1}^{n} \mu_j^O H_j^*(V^O, Y^O) + W^O[B(Y^O)]'$$

$$\leqq \sum_{j=1}^{n} \mu_j^O H_j(V^O, Y) + W^O[B(Y)]' \text{ for all } Y \in T.$$

I doubt that there are many problems for which this cannot be done. Also, it may be possible in many circumstances to define the functions H_j^* $j=1,\ldots,n$ more generally as:

$$H_j^*(V, Y) = Sup_{X_j} \left\{ H_j(X_j, Y) - V[C_j(X_j)]' \mid X_j \in S_j \right\} .$$

Further research may also reveal that Condition 4 of Section IV may be weakened particularly with respect to the quasi-convexity of

$$\sum_{j=1}^{n} \mu_j^O H_j^*(V, Y) + Vc' \text{ over the entire region } V \geqq 0, \ Y \geqq 0.$$

3. The practical limits on the usefulness of our method are about the same as those that apply to Everett's original generalized Lagrange multiplier method. Many problems of interest are not well-suited for applications of Lagrange multiplier methods. However, some of the problems that are well-suited are just about impossible to address in any other way. For example, the author has been engaged in a study of mixes of area ABM, terminal ABM, and blast shelters to defend the nation's urban population from strategic missile attacks. The admissible offense strategies involve a variety of weapons and decoys deployed to leak through, suppress, exhaust, and detonate above the terminal defenses. Despite its apparent complexity the problem can be formulated as a min-max problem with constraints on the offense use of throw weight and the defense use of budget funds. The problem can be partitioned by cities and rapidly and repeatedly solved for different offense and defense resource levels simply by varying two multipliers. For entirely practical reasons no other approach to this constrained min-max problem is currently feasible.

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