ROOMY SCATTERING MATRIX SYNTHESIS

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13. ABSTRACT

The synthesis of a scattering matrix is a problem both of dynamical system theory and of operator theory. Properties relevant for synthesis are investigated based on dynamical principles as concatenation and Nerode equivalence. Then, use of operator theory in the sense of Helson and Lowdenslager is made to achieve complete decomposition of the scattering system into simple subsystems for a major class of scattering matrices, which are called "roomy". Hence, roomy scattering matrices prove to be completely decomposable non-normal contractive operators.

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1. ABSTRACT

The synthesis of a scattering matrix is a problem both of dynamical system theory and of operator theory. It transcends the context of finite dimensional system theory, as developed by Kalman in ([1], Chapter 10) in that its state space is an infinite dimensional Hilbert space, and so requires a slightly different approach based however on similar principles as concatenation and Nerode equivalence, and also in that it needs the use of operator theory, mainly as developed by Helson-Lowdenslager [2]. We give a precise description of the scattering matrix as a dynamical system in paragraph 3. Then we start to investigate properties relevant for synthesis purposes, a synthesis being a "complete decomposition of a system into simple subsystems". A major class of systems is discerned and defined as the class of "roomy" scattering matrices, for which complete synthesis procedures are deduced, essentially with the help of the factorization theory for J-contractive matrices by Potapov [3]. From the point of view of operator theory, a synthesis becomes some sort of spectral decomposition. The class of roomy scattering matrices -- which is a class of non-normal contractive operators -- proves to be completely decomposable through the use of rather unconventional devices: analytic transformations and non-associative multiplication, due in the first place -- in a very physical setting -- to Belevitch [4]. Several salient synthesis procedures are discussed in paragraphs 4 to 7. Physical insight from the finite situation as described in ([5], [6], [7]) is often a guide in the sometimes tricky developments from one stage to another. There has been consistent effort to use invariant subspace theory -- the major clarifying concept in the theory -- in most contexts. This proves to produce a successful generalization of the notion of 'degree' except somewhat in one critical instance discussed in

paragraph 7. Many descriptions of the physical implications of the synthesis procedures are kept very sketchy and have to be augmented with above mentioned technical papers.

2. INTRODUCTION

A Scattering Matrix is an Input/Output description of a physical system which we regard for instance as consisting of electrical circuitry. Input and Output are considered to be "ingoing" and "reflected" waves (for precise meaning and definition see [8]). We will be primarily concerned with systems having a finite set of entries, "ports", carrying each an incoming and a reflected wave, represented by time dependent vectors $a(t)$ and $b(t)$ respectively. There are essentially two types of scattering matrices as pictured in figure 1: in the first type, the scattering matrix describes the functional relationship between in- and outgoing waves at the same set of ports, while in the second type the set of ports is split into two subsets, and the scattering matrix describes the functional relationship between incoming waves at the first set and outgoing at the second.

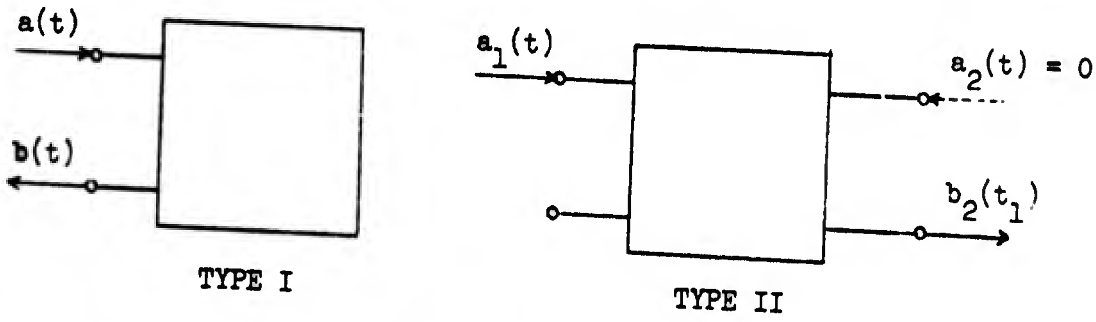

Figure 1: Different types of Scattering

Without loss of generality (see [7]), we may suppose that the two sets consist of an equal number of ports. The definition of incoming and outgoing waves is dependent on an analytic transformation A of the unit matrix circle, a fact that we will have to use in the sequel. More precisely, if $\begin{bmatrix} a(t) \\ b(t) \end{bmatrix}$ are a set of in and outgoing waves, then any other set is given by:

$$\begin{bmatrix} a_1(t) \\ b_1(t) \end{bmatrix} = H(S_o) \begin{bmatrix} a(t) \\ b(t) \end{bmatrix} \tag{1.1}$$

in which S_o is a constant, strictly contractive matrix (i.e. $1_n - \tilde{S}_o S_o > 0_n$, $\tilde{S}_o$ being the Hermitian conjugate of S_o) and

$$H(S_o) = \begin{bmatrix} (1_n - \tilde{S}_o S_o)^{-1/2} & 0_n \\ 0_n & (1_n - S_o \tilde{S}_o)^{-1/2} \end{bmatrix} \begin{bmatrix} 1_n & -\tilde{S}_o \\ -S_o & 1_n \end{bmatrix} \tag{1.2}$$

In fact S_o "induces" an analytic transformation A of the set of contractive matrix functions in the open right half complex plane $\{S\}$ into itself by:

$$A(S) = (1_n - S_o\tilde{S}_o)^{-1/2}(S - S_o)(1_n - \tilde{S}_o S)^{-1}(1_n - \tilde{S}_o S_o)^{1/2} \tag{1.3}$$

a fact that we will use later.

Since we are in the first place interested in the problem of synthesis, we will not be concerned any longer with these physical considerations (which are dealt with at length in [5] and in [7]), and we introduce at once following axioms:

An n dimensional scattering matrix is an operator

$$\mathcal{S} : L^2_{\mathbb{R}^n}(o,\infty) \to L^2_{\mathbb{R}^n}(o,\infty) \; : \; a \to b = \mathcal{S}a$$

such that:

(1) $\mathcal{S}$ is bounded in the $L^2_{\mathbb{R}^n}$ norm.

(2) $\mathcal{S}$ is linear.

(3) $\mathcal{S}$ commutes with translations:

$$T_\tau b = \mathcal{S}(T_\tau a) \tag{1.4}$$

in which $T_\tau f(t) = f(t - \tau)$ and it is assumed that $T_\tau a \in L^2_{\mathbb{R}^n}$.

It should be noted that a direct consequence of such a set of axioms is that $\mathcal{S}$ is "causal" (see e.g. [8]).

By Bochners L^2 - theorem ([9], p.142 : Bochners proof is literally valid in this situation and even in the more general one in which we have any Hilbert space $\mathcal{H}$ instead of $L^2_{\mathbb{R}^n}$), we have:

$$B(j\omega) = S(j\omega)A(j\omega) \tag{1.5}$$

in which $A(j\omega)$ and $B(j\omega)$ are Fourier transforms of $a(t)$ and $b(t)$, $S(j\omega)$ is a contractive matrix function (i.e. contractive for each ω), which is moreover analytic, i.e. the non-tangential limit a.e. of matrix $S(p)$ holomorphic in the open right half complex plane. Also

$$\text{ess.}_{\omega}\text{sup.} \|S(j\omega)\|_{\mathbb{R}^n} = \|\mathcal{S}\|_{L^2_{\mathbb{R}^n}} \tag{1.6}$$

in which $\|\cdot\|_{\mathbb{R}^n}$ is the usual Hilbert norm of constant operators in $\mathbb{R}^n$ and $\|\cdot\|_{L^2_{\mathbb{R}^n}}$ is the norm of operators in the Hilbert space of L^2 vector functions.

Bochners theorem leads us right into the theory of vector valued functions, its fourier transforms and the theory of Invariant Subspaces. Much of the content of this paper bears on this and we will use the results of [2] freely. Especially important is the Beurling-Lax theorem [2, p.61] which will be used in several instances.

The synthesis problem consists in actually realizing a given scattering matrix by means of a "desirable" physical device. In this context we define "desirable" by "using a cascade structure each of which blocks performs a simple physical function". (Figure 2).

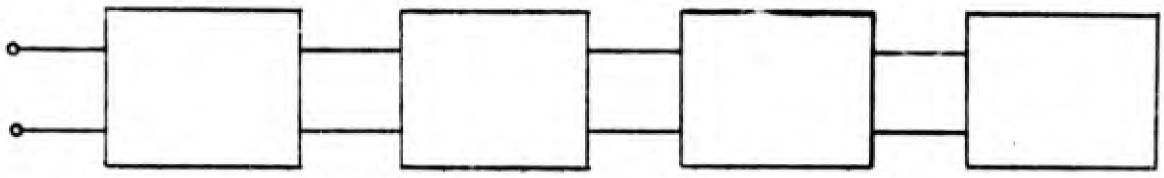

Figure 2: A Cascade Structure

This means algebraically (see [4], [5], [7]):

(a) For a Scattering Matrix of type I, that an analytic transformation A (defined by 1.3) must be found so that

$$A(S) = S_1 \cdot S_2 \tag{1.7}$$

in which S_1 is a unitary analytic matrix function (an "inner function") and S_2 is contractive and "simpler" than S (to be made precise later).

(b) For a Scattering Matrix of type II, that, if it has dimension $n\times n$, it has to be embedded in a unitary, analytic matrix function Σ of dimension $2n\times 2n$:

$$\Sigma = \begin{bmatrix} S & \Sigma_{22} \\ \Sigma_{11} & \Sigma_{12} \end{bmatrix} \tag{1.8}$$

and that the "Chain Scattering Matrix"

$$\Theta = (P + P^{\perp}\Sigma)(P\Sigma + P^{\perp})^{-1} \tag{1.9}$$

defined by means of the projection P of $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$ on its first component, with $P^{\perp} = I - P$, has to be factored into factors Θ_j satisfying: ($*$ denoting adjoints)

$$\Theta_j^* T \Theta_j = T \quad T = P - P^{\perp} \tag{1.10}$$

It should be noted that these physical structures simply stand for specific types of spectral decompositions of the operators involved, and these will be made precise in the sequel without reference to the situation described here only as motivation.

The simple observation that the response of the system at a given time $t > o$ is dependent only on the values of the input for times from zero up to t, induces one to view the system as a "memory bank" (state space) in which information is stored to be continuously released. In other words, $\mathcal{S}$ defines something like the "Dynamical System" defined in [1], and our first task will be to make this idea precise for the specific, scattering context. Heuristically, one might guess that each of the simple blocks in the cascade synthesis are to contain some piece of the state space. Pulling out such pieces, and/or deciding whether such pulling can be done will become the major concern of this paper.

3. THE CONSTRUCTION OF THE STATE SPACE

The main problem in dynamical systems theory has to do with the construction of the state space and the action induced in it from the Input/Output data. In this context, this is performed by the following theorem:

Theorem 1.

A Scattering Matrix $\mathcal{S}$ induces a Hilbert space as state space ($\mathcal{H}$) naturally embedded in $L^2_{\mathbb{R}^n}(o,\infty)$, a projection $P : L^2_{\mathbb{R}^n}(o,\infty) \to \mathcal{H}$, a uniformily continuous semigroup $\sigma(t)$ in $\mathcal{H}$ and bounded maps $C : \mathcal{H} \to \mathbb{R}^n$, $D : \mathbb{R}^n \to \mathbb{R}^n$ such that, for $b = \mathcal{S}a$:

(1) $$x(t) = \sigma(t-\tau)x(\tau) + P(i_{\tau t}a) \tag{3.1}$$

(2) $$b(t) = Cx(t) + Da(t) \tag{3.2}$$

in which: $x(t)$ is a continuous map $[o,\infty) \to \mathcal{H}$, $i_{\tau t}a(t) = a(t-\tau)\big|_{[o,\infty)}$ (the cut and flip operator, $\big|_{[o,\infty)}$ meaning: restricted to the interval $[o,\infty)$).

The proof proceeds in steps:

Step 1: The definition of $\mathcal{H}$: Nerode Equivalence classes.
Consider $a_i(t) \in L^2_{R^n}(o,\infty)$ such that $a_i(t) = 0$ for $t > \tau_i$, $i = 1,2$. We will say that a_1 is Nerode equivalent to a_2, or $a_1 \overset{N}{\sim} a_2$ iff for $\tau > \tau_i$ we have that $\mathcal{S}(i_{o\tau}a_1) = \mathcal{S}(i_{o\tau}a_2)$ a.e. on $t \geq \tau$. Hence, $a(t) \overset{N}{\sim} 0$ iff $\mathcal{S}(i_{o\tau}a) = 0$ for $t \geq \tau$. Note first that this definition is not dependent on τ, any $\tau_1 > \tau$ would do, and also that the "a.e." is essential in order to avoid the difficult problem of concatenation of distributions (see e.g. [10]). The Nerode zero-equivalence defines a set of closed linear subspaces $\mathcal{M}_t$ in the different $L^2(o,t)$'s, and they satisfy in a natural way $\mathcal{M}_t \supset \mathcal{M}_\tau$ for $t > \tau$. We consider $L^2_{R^n}(o,t)$ also as a natural subspace of

$L^2_{\mathbb{R}^n}(o,\tau)$, $\tau > t$ and define a map:

$$\mathcal{H} : \bigcup_{m=1}^{\infty} L^2_{\mathbb{R}^n}(o,m) \to L^2_{\mathbb{R}^n}(o,\infty)$$

by:

$$\mathcal{H}(a(t)) = T_{-m}\mathscr{S}(i_{om}a(t))\Big|_{(o,\infty)}$$

for $a(t) \in L^2_{\mathbb{R}^n}(o,m)$.

$\mathcal{H}$ is uniquely defined and $a_1 \overset{N}{\sim} a_2$ iff $\mathcal{H}(a_1) = \mathcal{H}(a_2)$. The map is seen to be continuous in the $L^2_{\mathbb{R}^n}(o,\infty)$ topology since:

$$\begin{aligned}\|\mathcal{H}(a_1) - \mathcal{H}(a_2)\| &\le \|T_{-m}\mathscr{S}(i_{om}a_1)\,_{(o,s)} - T_{-m}\mathscr{S}(i_{om}a_2)\,_{(o,\infty)}\| \\ &\le \|T_{-m}\mathscr{S}(i_{om}a_1) - T_{im}\mathscr{S}(i_{om}a_2)\|_{L^2(-\infty,\infty)} \\ &\le \|\mathscr{S}(i_{om}a_1) - \mathscr{S}(i_{om}a_2)\|_{L^2(-\infty,\infty)} \\ &\le \|\mathscr{S}\|\cdot\|a_1 - a_2\|\end{aligned}$$

Hence, $\mathcal{H}$ is a bounded transformation on a subspace dense in $L^2_{\mathbb{R}^n}(o,\infty)$. It can be extended on the whole space, in a unique way. We denote the extended operator also by $\mathcal{H}$ and consider its kernel $\mathcal{M}$. Let $\mathscr{H}$ be the orthogonal complement of $\mathcal{M}$. Hence $L^2_{\mathbb{R}^n}(o,\infty) = \mathcal{M} \oplus \mathscr{H}$, the "Nullspace" and the "State Space" respectively. We note that $\mathcal{M}_t$ consists of the restrictions of $\mathcal{M}$ to $L^2_{\mathbb{R}^n}(o,t)$.

<u>Step 2</u>: The state trajectory.

If the input is $a(t)$, then the state trajectory $x(t)$ is defined by: $x(\tau) = P(i_{o\tau}a(t))$ where P projects $L^2_{\mathbb{R}^n}(o,\infty)$ on $\mathscr{H}$. Associated with this is a semigroup on $\mathscr{H}$:

$$\sigma(t) = PT_t \tag{3.3}$$

where T_t is a positive shift and $\mathcal{H}$ is considered embedded in $L^2_{\mathbb{R}^n}(o,\infty)$ -- the notation being slightly improper.

It is easily verified that this is a semigroup: we must only show that $PT_t = PT_tP$ on $L^2_{\mathbb{R}^n}(o,\infty)$, which follows from the fact that $\mathcal{M}$ is an invariant subspace, $T_t\mathcal{M} \subset \mathcal{M}$ for positive t, a property that is, by the way, physically obvious. Equation 3.1 now follows directly by linearity. The representation 3.3 for $\sigma(t)$ shows that it is a uniformily continuous semigroup.

Step 3. The output map.

By linearity and the definition of the state we have that for some (any) $t > \tau$:

$$b(t) = \gamma(t,\tau)\cdot x(\tau) + \mathcal{A}[(T_{-\tau}a(t))\big|_{(o,\infty)}](t)$$

where $\gamma(t,\tau): \mathcal{H} \to \mathbb{R}^n$ such that

$$\gamma(t,\tau) = \gamma(t,\theta)\sigma(\theta-\tau) \quad \text{for any} \quad t < \theta < \tau,$$

and hence, using properties of uniformily continuous semigroups (see [11])

$$\begin{aligned}\gamma(t,\tau) &= [\gamma(t,\theta)\sigma^{-1}(t-\theta)]\sigma(t-\tau) \\ &= C\sigma(t-\tau) \quad \text{with}\end{aligned}$$

$$C = \gamma(t,\theta)\sigma^{-1}(t-\theta) \; : \; \mathcal{H} \to \mathbb{R}^n.$$

Note that C is independent of θ, and is a continuous map.

Next, consider the system $\mathcal{S}_1$ defined by equation 3.1 and $b(t) = Cx(t)$. The system $\mathcal{S} - \mathcal{S}_1$ has empty state space since the outputs for $\mathcal{S}$ and $\mathcal{S}_1$ are equal once the excitation has ceased. Hence the proof of equation 3.2 will be completed with the following lemma:

<u>Lemma</u>.

If the state space of the system $\mathcal{S}$ is empty, then $\mathcal{S}$ is represented by $b = Da$ where D is a constant map $\mathbb{R}^n \to \mathbb{R}^n$.

Proof of the Lemma:

The essential support of the response $b(t)$ is contained in the essential support of $a(t)$. Hence $S(j\omega)$ is both analytic and conjugate analytic, and thus a constant.

This also proves Theorem 1.

In the course of the proof of Theorem 1 we have only used the fact that $\mathcal{S}$ is a bounded map. From now on however, especially since this seems more practically interesting, we will add the property that $S(j\omega)$ or $\mathcal{S}$ is contractive.

Several avenues seem now open to obtain a reduction of the state space. The simplest one would be to observe that $\sigma(t)$ being uniformily continuous, has a bounded generator, say A, such that $\sigma(t) = e^{At}$. If A is normal, then a spectral decomposition can be performed on A. This leads to an additive decomposition of $S(j\omega)$, an unphysical and undesired feature. If A is nonnormal, then one could try to embed it, but this would increase the complexity of the state space, an undesired feature equally. Another way altogether would be to consider the "characteristic function" of A in the sense of Nagy and Livšič (see e.g. [12]). The matrix $S(j\omega)$ itself however is easily seen to be very closely related this characteristic function. We will thus have to exploit the very close connection between state space decomposition and harmonic analysis of operators. This will be the subject of the next paragraph.

4. STATE CHARACTERIZATION AND INPUT/OUTPUT MAP

$L^2_{\mathbb{R}^n}(o,\infty)$ is isomorphic to $H^2_{\mathbb{R}^n}$ through the Fourier transform (indicated by means of a $\hat{}$). We denote by $\hat{\mathcal{M}}$ the image of $\mathcal{M}$ under this isomorphism. Denoting further by $K^2_{\mathbb{R}^n}$ the complement of $H^2_{\mathbb{R}^n}$ in $\hat{L}^2_{\mathbb{R}^n}(-\infty,\infty)$ and by $S^*(j\omega)$ the adjoint of $S(j\omega)$ in $L^2_{\mathbb{R}^n}(-\infty,\infty)$ which is also a matrix function, we have:

<u>Proposition</u>.

$A(j\omega) \in \hat{\mathcal{M}}$ if and only if $S(-j\omega)A(j\omega) \in H^2_{\mathbb{R}^n}$ or equivalently $A(j\omega) \perp S^*(-j\omega)K^2_{\mathbb{R}^n}$.

<u>Proof</u>.

Suppose $a(t) \in L^2_{\mathbb{R}^n}(o,\theta)$. Then $a(t) \overset{N}{\sim} 0$ if and only if $T_{-\theta}\mathcal{S}(i_{o\theta}a(t)) \in L^2_{\mathbb{R}^n}(-\infty,o)$. But: $\mathcal{S}(i_{o\theta}a(t)) = S(j\omega)$. $(i_{o\theta}a(t))\hat{}$, and $(i_{o\theta}a(t))\hat{} = A(-j\omega)e^{-j\omega\theta}$. Next: $T_{-\theta}\mathcal{S}(i_{o\theta}a(t)) = A(-j\omega)$, and the fact that the set of $\mathcal{M}_\theta$'s is dense in $\mathcal{M}$ shows the first part. Further, if $S(-j\omega)A(j\omega) \perp H^2_{\mathbb{R}^n}$, then $S(-j\omega)A(j\omega) \perp K^2_{\mathbb{R}^n}$ and $A(j\omega) \perp S^*(-j\omega)K^2_{\mathbb{R}^n}$. This proves the proposition.

<u>Corollary</u>. If $\mathcal{S}$ is unitary (i.e. if $S(j\omega)$ is an inner function), then $\hat{\mathcal{M}} = S^*(-j\omega)H^2_{\mathbb{R}^n}$.

<u>Proof</u>.

$$S^*(-j\omega)K^2_{\mathbb{R}^n} \perp S^*(-j\omega)H^2_{\mathbb{R}^n},$$

for since $S(j\omega)$ is inner, so is $S^*(-j\omega)$.

Thus, $\hat{\mathcal{M}}$ consists of vector functions which push $S(-j\omega)$ into analyticity. There may not be any such vector. Most important is the case where $\hat{\mathcal{M}}$ has "full range", which means that for each $j\omega$ there are vectors in $\hat{\mathcal{M}}$ which span $\mathbb{R}^n$.

It should be noted that it is enough that there are vectors in $\hat{\mathcal{M}}$ which span $\mathbb{R}^n$ only at one ω.

<u>Definition 1</u>. $\mathcal{S}$ is said to be "roomy" if its null-space $\hat{\mathcal{M}}$ has full range.

<u>Definition 2</u>. A system is said to be finitely reachable if, for any state x there exists an input $a(t) \in L^2_{\mathbb{R}^n}(o,\theta)$ for some θ, such that $x = Pa(t)$ (In other words: if every state can be reached in finite time).

<u>Definition 3</u>. A system is said to be finitely controllable if, for every state x there exists an input $a(t) \in L^2_{\mathbb{R}^n}(o,\theta)$ for some θ, such that $a(t)$ brings x to zero at θ.

<u>Theorem 2</u>.

A system is roomy if and only if every non-zero one dimensional system $x^*\mathcal{S}a$, $x^* \in (\mathbb{R}^n)^*$, $a \in \mathbb{R}^n$, $(x^*\mathcal{S}a)g(t) = x^*\mathcal{S}(ag(t))$ is roomy.

<u>Proof</u>.

<u>Only if</u>: If $\mathcal{S}$ is roomy, then $\hat{\mathcal{M}}$ has full range, and by the Beurling Lax theorem (see [2]), $\hat{\mathcal{M}} = UH^2_{\mathbb{R}^n}$, where U is an inner function. Since we have $(\det U)H^2_{\mathbb{R}^n} \subset \hat{\mathcal{M}}$, we also have that $(\det U)\cdot x^*S(-j\omega)a$ is analytic, and the one dimensional system $x^*\mathcal{S}a$ is roomy.

<u>If</u>: Using a basis $\{a_j^*\}$, $\{a_k\}$ for $(\mathbb{R}^n)^*$, $(\mathbb{R}^n)$ we have for every $a_j^*\mathcal{S}a_k$ an inner function q_{kj} such that $q_{kj}(a_j^*S(-j\omega)a_k) \in H^2$. Let $q = \prod_{k,j}^{n} q_{kj}$. Then $q\cdot S(-j\omega) \in H^2$, and $q\cdot I\cdot H^2_{\mathbb{R}^n} \subset \hat{\mathcal{M}}$, which thus has full range.

<u>Corollary</u>. A system is roomy if any subsystem $x^*\mathcal{S}a$ is finitely reachable and controllable.

Proof. Any $x \in \mathcal{S}_a$ is either zero, or there exists a finite non-zero input producing a zero state, in which case it is roomy. The theorem shows that roominess is essentially a property of the entries of the matrix $S(j\omega)$. We will use this fact later.

Theorem 3. A system $\mathcal{S}$ is roomy if there exist n states x_k which can be brought to zero respectively by means of n inputs of the form $g_k(t)a_k$, $g_k(t) \in L^2_{\mathbb{R}^n}(o,\theta_k)$, a_k being a base for $\mathbb{R}^n$.

Proof. The n states x_k can be represented by n functions in $\mathcal{M}$: $(\{x_k\})$. One constructs n $h_k(t) = g_k(t)a_k + T_{\theta_k} \cdot x_k$ which are zero state equivalent. By Titchmarsh theorem ([13], p.166), the function $g_1(t)*g_2(t)*\ldots*g_n(t) \neq 0$ on $(0,\min_k \theta_k)$ -- the $*$ indicating convolution. Hence, the $\hat{h}_k(j\omega)$ form a full rank basis of $\hat{\mathcal{M}}$.

The notion of roominess is a very important one for the further development of the theory. There are systems which are not roomy. An eash way to construct such systems is indicated by Cambern and Helson ([2], Ch.IX): Suppose that $S(j\omega)$ is such that $U \cdot S(-j\omega) \in H^2_{R^n}$. Since it is automatic that $U \cdot S^*(-j\omega) = U \cdot \tilde{S}(-j\omega) \in H^2_{R^n}$, we also have that $U \cdot (S(-j\omega) + \tilde{S}(-j\omega)) = U \cdot S^H(-j\omega) \in H^2_{Rn}$, where $S^H(j\omega)$ stands for the Hermitian part of $S(j\omega)$. Hence $\det(U \cdot S^H(-j\omega)) \in H^p_{R^n}$ for some p since the determinant is a sum of a product of H^2 functions. Hence $\log(\det S^H(-j\omega))$ must be integrable by a corollary of Szegö's theorem (see e.g. [14] p.53). We will thus have constructed a non-roomy system by choosing a non-trivial $S(j\omega)$ such that $\det S^H(j\omega)$ is zero on a set of positive measure, using a Poisson integral to determine S.

Before taking on the cascade synthesis of a roomy system (it will be shown that for non-roomy systems there is no lossless cascade synthesis) we point to following useful consequence of the Beurling-Lax Theorem:

If $\mathcal{S}$ is roomy, then and only then there exists a $n \times n$ Unitary matrix function $U(j\omega)$ such that $U(j\omega)S^*(j\omega)$ is analytic. In fact, with $\mathcal{M} = U_1(j\omega)H^2_{R^n}$, we have by a previous theorem that $S(-j\omega)U_1(j\omega)$ is analytic, and so is $U_1^*(-j\omega)S^*(j\omega)$. The converse statement is obvious.

We end up with essentially two invariant subspaces:

(1) $\mathcal{N} = \overline{S(j\omega)H^2_{R^n}}$, the closure of the range of $S(j\omega)$.

(2) $\mathcal{M}$, the nullspace.

Because of the intimate relationship between $\mathcal{N}$ and $\mathcal{M}$ in the Unitary case as exemplified by a previous corollary, the synthesis of such a system proves easiest, and as the synthesis proceeds the nullspace increases. More generally however, we will show that a common feature of all synthesis procedures is the increase of the nullspace through admissible operations: factorizations and analytic transformations.

5. CASCADE SYNTHESIS OF A UNITARY MATRIX FUNCTION

In the unitary case, a product decomposition of $S(j\omega)$ corresponds exactly to an increase of the nullspace of the system. This is seen as follows (following [2], p.81):

With
$$\hat{\mathcal{M}} = S^*(-j\omega)H^2_{\mathbb{R}^n},$$
we have that $(\det S^*(-j\omega))H^2_{\mathbb{R}^n} \subset S^*(-j\omega)H^2_{\mathbb{R}^n}$, in which $q = \det S^*(-j\omega)$ is an inner function. It can be decomposed as $q = q_1q_2$ in which q_1 is the "Blashke part" and q_2 the "singular part":

$$q_1 = \prod_{k=0}^{\infty}\left[\frac{p-a_k}{p+a_k}\cdot\frac{|1-a_k^2|}{1-a_k^2}\right]^{\delta_k}, \quad a_j \neq a_k \text{ if } j \neq k.$$

$$q_2 = \exp\int_{-\infty}^{\infty}\frac{p(j\omega)-1}{p-j\omega}\,d\mu_s(\omega)$$

q_1 determines a set of zeros a_k with rank δ_k, while q_2 determines a singular positive measure on the $j\omega$-axis which is finite for the measure. A comprehensive treatment of such inner functions is to be found in ([14], Ch.8). In the finite case (i.e. when $S(j\omega)$ is rational) we have that the ordinary notion of degree coincides with $\sum_k \delta_k$. q_1 and q_2 represent some kind of generalization of the notion of degree, as is indicated by the following.

We have:
$$\hat{\mathcal{M}} = \hat{\mathcal{M}}_{q_1} \cap \hat{\mathcal{M}}_{q_2} \quad ([2], \text{p.82})$$
in which:
$$\hat{\mathcal{M}}_{q_1} = \{F \in H^2_{\mathbb{R}^n},\ q_2F \in \hat{\mathcal{M}}\}$$
$$\hat{\mathcal{M}}_{q_2} = \{F \in H^2_{\mathbb{R}^n},\ q_1F \in \hat{\mathcal{M}}\}$$

Suppose
$$\hat{\mathcal{M}}_{q_1} = U_1\cdot H^2_{\mathbb{R}^n}, \text{ then}$$

(1) $\det U_1 = q_1$

(2) $\det(U^{-1}U_1) = q_2$.

Hence $\hat{\mathcal{M}} = U_1 \hat{\mathcal{M}}_2$ with $\hat{\mathcal{M}}_{q_1} = U_1 H^2_{\mathbb{R}^n}$ and $\hat{\mathcal{M}}_2 = U_2 \cdot H^2_{\mathbb{R}^n}$ for which det $U_2 = q_2$.

We will say that reduction in degree occurs, whenever such a reduction in encountered. There seems to be no simple relationship between $\hat{\mathcal{M}}$ and $\hat{\mathcal{M}}_{q_2}$, the process being essentially non-associative. However $\hat{\mathcal{M}}_2$ as nullspace of the system with lable "2" is strictly bigger than $\hat{\mathcal{M}}$. The same decomposition can be performed for subproducts of q_1 and q_2, the only condition being that they are prime. For the Blashke part the basic step hence becomes the extraction of a single zero of maximal order δ_k. The further decomposition of such a zero is essentially a finite procedure and is discussed at length in [7]. To see the finiteness ([2], p.87) we remark that the invariant subspace $\hat{\mathcal{M}}_k = U_k H^2_{\mathbb{R}^n}$ corresponding to any finite Blashke product has rational U_k. Hence, using the theory of finite scattering matrix synthesis ([5], [7]) we have:

$$U_k = \prod_{\ell=1}^{\delta_k} \left[\left(1_n - \frac{2\alpha_k u_\ell \tilde{u}_\ell}{p + p_k^*} \right) A_k \right]$$

where: $\alpha_k = \frac{1}{2}(p_k + p_k^*)$, and A_k is choosen so that the resulting Blashke product will converge, i.e.:

$$\det A_k = \frac{|1 - p_k^2|}{1 - p_k^2}$$

The convergence of the Blashke product on compact subsets of the open right half complex plane is well known ([3], p.138). However, this result is not very attractive from the systems point of view. Better is the following result (inspired from the scalar case as treated in [14], p.65):

<u>Proposition</u>. The partial Blashke products formed with the first n factors (B_n) converge to the Blashke product B in the uniform operator topology on $L^2\left(\frac{d\omega}{1+\omega^2}\right)$.

<u>Proof</u>. For $m > n$ we have:

$$\|(B_m - B_n)F\|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \|[B_m - B_n]F\|^2_{\mathbb{R}^n} \frac{d\omega}{1+\omega^2}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} ([B_m - B_n]F,[B_m - B_n]F)_{\mathbb{R}^n} \frac{d\omega}{1+\omega^2}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} ([B_m^* - B_n^*][B_m - B_n]F, F)_{\mathbb{R}^n} \frac{d\omega}{1+\omega^2}$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} (F,F)_{\mathbb{R}^n} \frac{d\omega}{1+\omega^2} - \frac{1}{\pi}\,\mathrm{Re}\int_{-\infty}^{\infty} (B_n^* B_m F,F) \frac{d\omega}{1+\omega^2}$$

Denoting by β_{m-n} the analytic Blashke product $B_n^* B_m$, consisting of the m-n last factors in B_m we have, with

$$F \sim \sum_{k=0}^{\infty} a_k\left(\frac{p-1}{p+1}\right)^k, \quad \sum_{k=0}^{\infty} \|a_k\|^2_{\mathbb{R}^n} < \infty$$

$$\int_{-\infty}^{\infty} (\beta_{m-n}F,F) = \int_{-\infty}^{\infty} \left(\beta_{m-n}\sum_k a_k\left[\frac{p-1}{p+1}\right]^k, \sum_k a_k\left[\frac{p-1}{p+1}\right]^k\right)$$

$$= \sum_k \int_{-\infty}^{\infty} (\beta_{m-n}a_k,a_k)$$

$$= \sum_k (\beta_{m-n}(0)a_k,a_k)$$

(Note: we use the fact that the $\left[\frac{p-1}{p+1}\right]^k$, $k = 0,1\ \ldots$ form a basis in $H^2\left(\frac{d\omega}{1+\omega^2}\right)$).

The infinite product $\beta_{\infty-n}(0)$ is close to one, since $\beta_m(0)$ converges ([3], p.138), hence $\beta_{m-n}(0) = 1 + a_{m,n}$, $\|a_{m,n}\| < \varepsilon(m,n)$ and

$$\sum_k (\beta_{m-n}(0)a_k, a_k) = \sum_k (a_k, a_k) + \sum_k (a_{mn}a_k, a_k)$$

$$= 2\pi(F,F)_{L^2_{\mathbb{R}^n}} + \sum_k (a_{mn}a_k, a_k)$$

Hence
$$\|(B_m - B_n)F\|^2_{L^2_{\mathbb{R}^n}} = \frac{1}{\pi}\left|\sum_k (a_{mn}a_k, a_k)\right|$$

$$\leq 2\varepsilon(m,n)\cdot\|F\|^2_{L^2_{\mathbb{R}^n}}$$

and
$$\|(B_m - B_n)\|^2 \leq 2\varepsilon(m,n) \to 0$$

The $L^2\left(\frac{d\omega}{1+\omega^2}\right)$ topology ensures in fact time domain convergence of $f(t)*e^{-t}$'s in the L^2 sense. It also ensures plain L^2 convergence on finite segments. Each Blashke factor results in a cascade section containing a specific circuitry. Since the number of sections has to be finite it is essential that only a finite number of Blashke factors be realized.

Concerning the singular part we refer to Potapov's messy procedure of obtaining a product integral by means of approximating Blashke products ([3], p.140). It may be noted that, by our first result and a close look at Potapov's proof we have here also $L^2\left(\frac{d\omega}{1+\omega^2}\right)$ convergence of Riemann sums in the product integral. Potapov's result states that:

$$U_s = \int_0^{\Omega} \exp\left[\frac{p(j\omega(t)) - 1}{p - j\omega(t)}\right] dE(t)$$

where: (1) U_s is the to-be-synthesized singular inner function.

(2) $\omega(t)$ is a monotone increasing function from $-\infty$ to ∞.

(3) $E(t)$ is a monotone increasing family of Hermitian matrices,

t = trace of $E(t)$ and $\mu((-\infty,\omega]) = \int_{-\infty}^{\omega(t)_+} dt$ defines a singular

measure μ_s, which is such that $\frac{\mu_s}{1+\omega^2}$ is finite.

(4) The product integral has the meaning of a Riemann-Stieltjes integral, i.e. the limit of the product:

$$\exp\left[\frac{pj\omega(\theta_1) - 1}{p - j\omega(\theta_1)} E(t_1 - t_0)\right] \cdots \exp\left[\frac{pj\omega(\theta_n) - 1}{p - j\omega(\theta_n)} E(t_n - t_{n-1})\right]$$

in the correct order.

It may be interesting to point out that the product integral for U_s consists of variations on three basic structures (for real U_s):

(1) when μ_s is concentrated at ∞, and $E(t)$ is a commuting family:

$$U_s = \exp(-p \cdot M) = U \exp(-p\Lambda)U^{-1}$$

in which $M = U\Lambda U^{-1}$ is constant Hermitian, U constant unitary and Λ a diagonal matrix consisting of the (non-negative) eigenvalues of M. One recognizes easily a circuit consisting of a transformer and an LC transmission line.

(2) when μ_s is concentrated at 0, and $E(t)$ is a commuting family:

$$U_s = \exp(-\frac{1}{p} M) = U \exp(-\frac{1}{p} \Lambda)U^{-1}$$

where M, U and Λ have same meaning as above. Here we have a (not-very physical but conceivable) CL-transmission line.

(3) when μ_s is concentrated at, say, $p = -j$ and, since supposed real also at $p = +j$ with identical commuting families at the two points:

$$U_s = \exp\left(\frac{-4p}{p^2+1} M\right) = U \exp\left(\frac{-4p}{p^2+1} \Lambda\right) U^{-1}$$

A CL-LC line structure, combination of (1) and (2).

The product integral adds to this following features:

(1) at a fixed point of singularity in μ_s : a gradual rotation in U and a gradual change in transmission line characteristic through change in Λ, hence produces a non uniformily coupled, non uniform set of n transmission lines of given type.

(2) a possibly infinite sequence of transmission lines of different types, since μ_s needs not to be concentrated on a finite number of points.

We will say that a network is finitely generated if (1) it has only a finite number of Blashke factors, (2) μ_s is concentrated on a finite number of points, and (3) for each of these points $E(t)$ is a step function. Potapov's factorization hence gives a complete characterization of type I cascade synthesis for Unitary functions. This fact was first seen by Dominguez ([15]). However, it is useful to draw attention here to following facts:

(1) The Riesz-Herglotz-Potapov factorization of a non-unitary matrix function (e.g. a contractive $S(j\omega)$) is not useful in this context since it does not reduce the degree. It ends up with an outer function which is "as complicated" as the original as is made precise by comparing nullspaces: a factorization is expected to reduce the codimension of the nullspace, which is not necessarily done in the Riesz-Herglotz formula: the nullspace of $\frac{p-1}{p+2}$ has codimension one, and so has the nullspace of its outer part $\frac{p+1}{p+2}$ (actually, they have identical nullspaces).

(2) A type I unitary synthesis is not physically interesting. Much more important is either a type II synthesis, or a general contractive synthesis. We turn to these in following paragraphs.

6. CASCADE TYPE II SYNTHESIS

We will restrict ourselves to the case where $S(j\omega)$ is strictly contractive, i.e. when ess. sup. $\|S(j\omega)\|_{\mathbb{R}^n} \leq \rho < 1$. This is no serious restriction in generality since every bounded transformation can be brought to this form followed by a trivial constant map.

Proposition. If $S(j\omega)$ is strictly contractive, then there exists an analytical transformation A of the $n\times n$ matrix circle into itself such that $A(S(j\omega))$ is a homeomorphism of $H^2_{\mathbb{R}^n}$.

Proof. We have indicated in the introduction that an analytic transformation of the $n\times n$ matrix circle is induced by the set of strictly contractive constant matrices S_o through the formula

$$A(S(j\omega)) = (1_n - S_o\tilde{S}_o)^{-1/2}(S - S_o)(1_n - \tilde{S}_oS)^{-1}(1_n - \tilde{S}_oS_o)^{1/2} \quad (6.1)$$

$A(S(j\omega))$ will be a homeomorphism iff S_o lies outside the closed range of matrix values of $S(p)$, $\mathrm{Re}\,p > 0$ in any $\mathbb{R}^{n^2}$ topology. Because $S(p)$ is assumed to be strictly contractive in $\mathrm{Re}\,p > 0$, its range is not dense in the unit matrix circle. Hence it suffices to choose S_o outside the closed range of $S(p)$.

Proposition. If $S(j\omega)$ is roomy, then so is $A(S(j\omega))$.

Proof. By formula (6.1) we have

$$(1_n - \tilde{S}_oS(-j\omega))^{-1}A(j\omega) \in H^2_{\mathbb{R}^n}$$

whenever $A(j\omega) \in (1_n - \tilde{S}_oS(-j\omega))\hat{\mathcal{M}} = \hat{\mathcal{N}}_1$.

$\hat{\mathcal{N}}_1$ is a full range invariant subspace, and if $(1_n - \tilde{S}_o S(-j\omega))^{-1} \in H^2_{\mathbb{R}^n}$ so does $\tilde{S}_o S(-j\omega)(1_n - \tilde{S}_o S(-j\omega))^{-1}$. This proves the proposition in case S_o is non singular. When it is singular, we observe that, if $\hat{\mathcal{M}} = U \cdot H^2_{\mathbb{R}^n}$, it is enough to take $A(j\omega) \in \det U \cdot \hat{\mathcal{N}}_1$ to ensure analyticity, proving again that $A(S(j\omega))$ is roomy.

__Theorem 4.__ If S is a strict contraction, then it can be embedded in a $2n \times 2n$ unitary matrix function if and only if it is roomy.

__Proof.__ We perform first an analytic transformation A on S so that $A(S)$ is a homeomorphism. This is possible by Proposition 1. $A(S)$ is roomy also by Proposition 2. The analytic transformation can easily be reversed on the result.

__If:__ We have to construct the unitary matrix function:

$$\Sigma' = \begin{bmatrix} S & \Sigma_{22} \\ \Sigma_{11} & \Sigma_{12} \end{bmatrix} \tag{6.2}$$

in which S is embedded as the 1,1 element (labeled here 2,1 because S is conceived as being a transfer, Type II matrix). The other Σ_{ij} are also $n \times n$ matrices.

First we determine minimal Σ_{11} and Σ_{22} through spectral factorizations (see [2], p.111):

$$\Sigma_{11}^* \Sigma_{11} = 1_n - S^*S$$

$$\Sigma_{22} \Sigma_{22}^* = 1_n - SS^*$$

Σ' will be a unitary matrix function $\hat{L}^2_{\mathbb{R}^n} \to \hat{L}^2_{\mathbb{R}^n}$ if and only if

$$\Sigma_{12} = -\Sigma_{11}^{*-1} S^* \Sigma_{22} \tag{6.3}$$

In order to make Σ' analytic it is necessary to push Σ_{12} into analyticity by means of a unitary -- to be choosen minimal -- analytic matrix function V. In other words, we construct from (6.3) the inner function

$$\Sigma = \begin{bmatrix} S & \Sigma_{22}V \\ \Sigma_{11} & \Sigma_{12}V \end{bmatrix} \qquad (6.4)$$

with $\Sigma_{12}V$ analytic.

We must show that such a V exists. To this end, consider the maximal invariant subspace $\mathcal{N}$ which is such that $\Sigma_{12}\mathcal{N} \subset H^2_{\mathbb{R}^n}$, and we will show that $\mathcal{N}$ has full range. In fact, $\mathcal{N}$ contains all analytic $F \in H^2_{R^n}$ such that $F = \Sigma^*_{22}G$, G and S^*G analytic. This is seen by computing:

$$\begin{aligned} \Sigma^{*-1}_{11}S^*\Sigma_{22}F &= \Sigma^{*-1}_{11}S^*\Sigma_{22}\Sigma^*_{22}G \\ &= \Sigma^{*-1}_{11}S^*(1_n - SS^*)G \\ &= \Sigma_{11}S^*G \end{aligned}$$

Note that Σ_{22} is an outer homeomorphism since it is assumed minimal and S is a contractive homeomorphism. It follows that $\Sigma^*_{22}G = \Sigma^{-1}_{22}G - \Sigma^{-1}_{22}SS^*G$ is analytic together with S^*G. Since S is assumed to have a roomy nullspace, the set G such that S^*G is analytic has full range. This produces a V for $\mathcal{N}$ by the Beurling Lax theorem.

<u>Only if</u>: We have: $S^*[S\Sigma^{-1}_{11}\Sigma_{12} - \Sigma_{22}] = \Sigma^{-1}_{11}\Sigma_{12}$. Hence the Hermitian transpose of S is roomy, and so is S, since roominess is a property of the entries. This concludes the proof.

<u>Remarks</u>. Roominess, according to paragraph 4 is a property of the entries. Hence, if S cannot be embedded in a $2n\times 2n$ inner function, then it cannot

be embedded in any bigger one of finite dimension. The remaining possibility is to use an (necessarily infinite dimensional) dilation (see [12]). This indicates a sharp division between networks: (1) those which can be realized with a finite number of resistors, and (2) those that cannot. Thus: "A passive scattering matrix can be realized with a finite number of resistors if and only if it is roomy".

It was indicated in paragraph 2 that a type II synthesis requires the factorization of the Chain Scattering Matrix which in this case is:

$$\Theta = [P + P^{\perp}\Sigma][P\Sigma + P^{\perp}]^{-1}$$

with

$$P = \begin{bmatrix} 1_n & 0_n \\ 0_n & 0_n \end{bmatrix}$$

When Σ is unitary, then Θ is J-unitary with $J = P - P^{\perp}$. We fall in the domain of factorization of J-unitary matrices where again the factorization of Potapov [3] is applicable. However, again, except for the strictly numerical results, we will use the theory of invariant subspaces. In fact, Θ determines two invariant subspaces:

(1) $\mathcal{M}$, the subspace of $H^2_{\mathbb{R}^{2n}}$ mapped by Θ into $H^2_{\mathbb{R}^{2n}}$.

(2) $\mathcal{N}$, the subspace of $H^2_{\mathbb{R}^{2n}}$ image of $\mathcal{M}$ under Θ.

<u>Proposition</u>. $\mathcal{M}$ and $\mathcal{N}$ are closed, full range invariant subspaces of $H^2_{\mathbb{R}^{2n}}$, whenever S is a homeomorphism on its closed range.

<u>Proof</u>. The assertion that $\mathcal{M}$ and $\mathcal{N}$ are invariant subspaces is obvious. That they have full range follows from the fact that $[P\Sigma + P^{\perp}]H^2_{\mathbb{R}^{2n}}$ is mapped into $[P + P^{\perp}\Sigma]H^2_{\mathbb{R}^{2n}}$. Both have full range since S is assumed to be strictly

contractive. In fact:

$$\mathcal{M} = [P\Sigma + P^{\perp}]H^2_{\mathbb{R}^{2n}} \tag{6.5}$$

$$\mathcal{N} = [P + P^{\perp}\Sigma]H^2_{\mathbb{R}^{2n}}$$

which shows directly that $\mathcal{M}$ and $\mathcal{N}$ are closed.

Proposition. If Θ maps $H^2_{\mathbb{R}^{2n}}$ onto $H^2_{\mathbb{R}^{2n}}$ then Θ is a constant J-unitary operator.

Proof. By the hypothesis it follows that S is an outer homeomorphism, and also, by (6.3) and (6.4), $\Sigma_{11}^{*-1}S^*\Sigma_{22}V$. Hence it follows that $\Sigma_{22}VH^2_{\mathbb{R}^n} = S^{*-1}\Sigma_{11}^*H^2_{\mathbb{R}^n}$. Thus $S^{*-1}\Sigma_{11}^*\Sigma_{11}S + I_n = \Sigma_{11}^{*-1}\Sigma_{11}^{-1}$ is analytic, meaning that Σ_{11} must be constant en hence S because of its outer homeomorphism.

The synthesis procedure is then reduced to representing $\mathcal{M}$ and $\mathcal{N}$ as $\mathcal{M} = AH^2_{\mathbb{R}^{2n}}$, $\mathcal{N} = BH^2_{\mathbb{R}^{2n}}$, with A and B J-unitary, A (-J)-contractive and B, J-contractive. Then

$$\Theta_1 = A\Theta B^{-1}$$

maps $H^2_{\mathbb{R}^{2n}}$ onto $H^2_{\mathbb{R}^{2n}}$, and is thus a constant J-unitary matrix (which can obviously be choosen to be one). Hence: $\Theta = A^{-1}B$ in which A carries the poles of Θ, together with a singular part and B carries its zeros together with a singular part. A^{-1} and B must be such that they originate from unitary scattering matrices, hence A must be (-J)-contractive and B, J-contractive, so that the synthesis stays in the same category.

When S is an outer homeomorphism, then of course $\mathcal{M} = H^2_{\mathbb{R}^{2n}}$. It is always advantageous to choose an analytic transformation on S resulting in this situation. This will ensure minimality in the synthesis since $\mathcal{N}$ proves to be the same for either S or its outer part. It would be tempting to attribute the occurence of an $\mathcal{M}$ strictly smaller than $H^2_{R^{2n}}$ to a non-minimality in phase shift. This is not necessarily correct -- and the notion of phase-minimality does not carry over easily to the general situation. Nor is the occurence of $\mathcal{N} = H^2_{\mathbb{R}^{2n}}$ an indication that the system is an all-pass in the conventional meaning -- although this makes sense for individual sections.

It should also be noted that not every full range invariant subspace can be represented as $A \cdot H^2_{\mathbb{R}^{2n}}$, where A is J-contractive. Using a $2n$ constant unit vector u and $\alpha_o = \mathrm{Re}\, p_o$, if $\mathcal{L} = UH^2_{\mathbb{R}^{2n}}$ with $U = 1_{2n} - \frac{2\alpha_o u\tilde{u}}{p + p_o^*}$, then $\mathcal{L}$ can be represented as $A_1 H^2_{\mathbb{R}^{2n}}$ where A_1 is J-contractive if and only if $\tilde{u}Ju > 0$, and as $A_2 H^2_{\mathbb{R}^{2n}}$ where A_2 is (-J)-contractive if and only if $\tilde{u}Ju < 0$ and when $\tilde{u}Ju = 0$, not at all in terms of J and -J contractive matrices.

However, in this case, because of the minimality of the spectral factorizations and the minimality of V in (6.4) it is clear that

$$\mathcal{N} = [P + P^{\perp}\Sigma]H^2_{\mathbb{R}^{2n}} = [P + P^{\perp}\Sigma']H^2_{\mathbb{R}^{2n}}$$

where Σ' is built on the outer part of Σ, for which $\mathcal{M}' = H^2_{\mathbb{R}^{2n}}$. Hence $\mathcal{N} = B \cdot H^2_{\mathbb{R}^{2n}}$ with B analytic and J-contractive. Similarly $\mathcal{M} = A \cdot H^2_{\mathbb{R}^{2n}}$ with A, (-J)-contractive and analytic.

It is possible to determine step by step the J-unitary matrix A from its corresponding inner function. In the case that $\mathcal{M}$ is finitely generated we have: $(\lambda_\ell > o)$:

$$\mathcal{M} = \prod_k \left(1_{2n} - \frac{2\alpha_k u_k \tilde{u}_k}{p + p_k^*}\right) . \prod_\ell \exp\left[\lambda_\ell \frac{pj\omega_\ell - 1}{p - j\omega_\ell} u_\ell \tilde{u}_\ell\right]$$

It is easy to see (by direct computation) that:

$$U_k = 1_{2n} - \frac{2\alpha_k u_k \tilde{u}_k}{p + p_k^*} \quad \text{or} \quad U_\ell = \exp\left[\lambda_\ell \frac{pj\omega_\ell - 1}{p - j\omega_\ell} u_\ell \tilde{u}_\ell\right]$$

and

$$A_k = 1_{2n} - \frac{2\alpha_k}{\tilde{u}_k J u_k} \frac{J u_k \tilde{u}_k}{p + p_k^*} \quad \text{or} \quad A_\ell = \exp\left[\lambda_\ell \frac{pj\omega_\ell - 1}{p - j\omega_\ell} \cdot \frac{J u_\ell \tilde{u}_\ell}{\tilde{u}_\ell J u_\ell}\right]$$

define the same invariant subspaces, it being understood that $\tilde{u}_k J u_k > 0$ and $\tilde{u}_\ell J u_\ell > 0$ for all k and l. We will not prove this latter assertion which follows fairly easily from the above remarks. At this point it is necessary to introduce Potapovs factorization ([3], p.133) which here reduces to:

$$B = \left[\prod_{k=1}^{\infty} \left(1_{2n} - \frac{2\alpha_k u_k \tilde{u}_k J}{p + p_k^*}\right) A_k\right] \cdot \int_0 \exp\left[\frac{pj\omega(t) - 1}{p - j\omega(t)} \, dE(t)\right]$$

in which:

(1) $\alpha_k = \operatorname{Re} p_k > 0$, $\tilde{u}_k J u_k = 1$

(2) A_k is J-unitary, insuring the convergence of the Blashke product, i.e.

$$\det A_k = \frac{|1 - p_k^2|}{1 - p_k^2}$$

(3) $\omega(t)$ and $E(t)J$ are as in section 5, t being the trace of $E(t)J$, with same integrability as in section 5.

(4) $E(t) = A(t) \left[\begin{array}{c:c} \Lambda(t)^2 & 0_n \\ \hdashline 0_n & 0_n \end{array}\right] [A(t)]^{-1}$

in which $\Lambda(t)$ is diagonal and $A(t)$ is J-unitary.

The last assertion needs some proof, but we shall only indicate that one ([3], P.169) generally would expect

$$E(t) = A(t)\left[\begin{array}{c|c|c|c} \Lambda(t)^2 & & & \\ \hline & 1_s & -1_s & \\ \hline & 1_s & -1_s & \\ \hline & & & -M(t)^2 \end{array}\right][A(t)]^{-1}$$

this would produce infinitesimals of the forms:

(1)

$$A\left[\begin{array}{c|c|c|c} 0 & & & \\ \hline & 1_s & -1_s & \\ \hline & 1_s & -1_s & \\ \hline & & & 0 \end{array}\right]A^{-1}\Delta t$$

an idempotent making the operator A not onto on its range (a 'Brune' section) and

(2)

$$A\left[\begin{array}{c|c} 0_n & \\ \hline & -M^2 \end{array}\right]A^{-1}\Delta t$$

which would make A not analytic. Both are unacceptable leaving only (4).

This Riesz-Herglotz-Potapov product can hence be approximated by a cascade of sections of the forms:

(1)

$$1_{2n} - \frac{2\alpha_o a\tilde{a}J}{p + p_o^*}$$

(2)

$$\exp\left[\frac{pj\omega_o - 1}{p - j\omega_o}\, a\tilde{a}J\right]$$

wherein $\tilde{a}Ja = 1$, $\alpha_o = \mathrm{Re}\,p_o > 0$ and ω_o is real.

Both types of sections can easily be synthesized and essentially produce the same type of circuits as in Chapter 5, which however a different synthesis procedure described in [7].

Hence, we obtain a cascade of sections containing either a reactive element or various types of transmission lines. Non uniformities and infinite cascades are obtained as in Chapter 5.

7. SYNTHESIS OF A ROOMY STRICTLY CONTRACTIVE SCATTERING MATRIX

Suppose $S_o(j\omega)$ is a roomy, non singular (to avoid trivialities) scattering matrix and that

$$S_o(j\omega) = S_1(j\omega)V(j\omega) \tag{7.1}$$

with V an inner function and $S_1(j\omega)$ analytic. Suppose $\hat{\mathcal{M}}_0$ and $\hat{\mathcal{M}}_1$ are the nullspaces of S_o and S_1 respectively. Then:

<u>Proposition</u>. $V(-j\omega)\hat{\mathcal{M}}_1 \subset \hat{\mathcal{M}}_o$

<u>Proof</u>. $S(-j\omega)V^*(-j\omega) = S_1(-j\omega)$

Hence $S(-j\omega)V^*(-j\omega)U_1(j\omega)$ is analytic, and thus

$$V^*(-j\omega)U_1(j\omega)H^2_{\mathbb{R}^n} \subset UH^2_{R^n}$$

or

$$V^*(-j\omega)\hat{\mathcal{M}}_1 \subset \hat{\mathcal{M}}_o .$$

<u>Remark</u>. The factorization $S_o(j\omega) = S_1(j\omega)V(j\omega)$ may or may not reduce the nullspace since it is perfectly conceivable that $\det U_1 = \det U_o$. An easy example: $S_o = \frac{p-1}{p+1}$ has identical $\hat{\mathcal{M}}_o$ and $\hat{\mathcal{M}}_1$. Hence, in this case, a factorization does not reduce the "degree" defined, e.g. by means of the determinant of the nullspace.

<u>Definition</u>. The factorization (7.1) is to be called 'minimal' if $\det V(j\omega)$. $\det U_1(j\omega) = \det U_o(j\omega)$.

Given $S(j\omega)$, then a factorization (7.1) requires at least that $S^*(-j\omega)$ (and hence also S) has an inner part. This is, as indicated above, not sufficient. Also, we touch here the main reason why a Riesz-Herglotz-Potapov factorization does not produce a synthesis. The way out of this difficulty, in the case of roomy, strictly contractive scattering matrices is the subject of the remainder.

It has been indicated earlier that an analytic transformation A (induced by a constant contractive matrix S_o) may be performed on S. The pair $(A(S),A)$ will describe the same system as (S,I). Actually, a scattering matrix defines a partition in the set $\mathcal{S}\times\mathcal{A}$, where $\mathcal{S}$ stands for the set of contractive analytic matrix functions, and $\mathcal{A}$ for the set of analytic transformations. The partition is described by $(S_1,A_1) = (S,I)$ if and only if $S_1 = A_1(S)$. This is clearly an equivalence relation, which does not respect neither the additive, nor the multiplicative structures on $\mathcal{S}$. If $S(j\omega)$ is inner, with corresponding nullspace $\hat{\mathcal{M}} = S^*(-j\omega)H^2_{\mathbb{R}^n}$, then $A(S)$ has nullspace $A^*(S^*(-j\omega))H^2_{\mathbb{R}^n}$, where A^* is induced by S_o, supposing that A is induced by S_o. However, if $S(j\omega)$ is not inner with the nullspace $\hat{\mathcal{M}} = U(j\omega)H^2_{\mathbb{R}^n}$, then it is not true that the nullspace of $A(S)$ is $A^*(U(j\omega))H^2_{\mathbb{R}^n}$.

It may be noted that the set $\mathcal{A}$ has the structure of a non-commutative group with an involution. We have for instance:

$$[A(S_o)]^{-1} = A(-S_o)$$

and define:

$$A^*(S_o) = A(\tilde{S}_o).$$

(Note: it is necessary to identify S with USV, U and V constant unitary.) To discuss the decomposition of S we adopt the following notation: Let σ_i denote the equivalence class of (S_i,I). Then we will say:

$$\sigma_o = \sigma_1 \overset{A}{\Box} \sigma_2 \tag{7.2}$$

if
$$A(S_o) = A(S_1)\cdot A(S_2).$$

This composition is neither commutative nor associative. Hence it is not necessary that:

$$\sigma_1 \overset{A_1}{\square} (\sigma_2 \overset{A_2}{\square} \sigma_3) = (\sigma_1 \overset{A_1}{\square} \sigma_2) \overset{A_2}{\square} \sigma_2$$

(it might be, e.g. when $A_1 = A_2$, or when the S_i are unitary). We define the following compact notation:

$$\overset{\xrightarrow{n}}{\underset{i=1}{\boxed{A_i}}} \sigma_i = \sigma_1 \overset{A_1}{\square}(\sigma_2 \overset{A_2}{\square} (\ldots\sigma_{n-1} \overset{A_{n-1}}{\square} \sigma_n))\ldots)$$

or even the integral:

$$\overset{\xrightarrow{\ell}}{\underset{o}{\boxed{A(t)}}} f(t)\, d\gamma(t) = \lim \overset{\xrightarrow{N}}{\underset{o}{\boxed{A(\theta_i)}}} f(\theta_i)\Delta\gamma_i$$

in the obvious way (we will not use these integrals for lack of a handy theory of convergence).

The remainder of this paragraph will be used to show that a roomy, non-singular, strictly contractive scattering matrix can be decomposed by means of this composition rule. Suppose S as stated. Then it can be embedded in a $2n\times 2n$ unitary

$$\Sigma = \begin{bmatrix} \Sigma_{21} & \Sigma_{22} \\ S & \Sigma_{12} \end{bmatrix}$$

with a corresponding Θ as in formula 6.4. S can, as customary, be choosen to be a homeomorphism on its closed range, which can be choosen $H^2_{\mathbb{R}^n}$. Then Σ_{21} will have same properties, since it is choosen minimal. Hence the subspace $\mathcal{M}$ of section 6 is $H^2_{\mathbb{R}^{2n}}$ and $\mathcal{N}$ can be represented as $\Theta H^2_{\mathbb{R}^{2n}}$.

Suppose $\mathcal{M}$ is finitely generated (we will generalize later). Then Θ is, by section 6 again, essentially composed of factors of the form:

$$1_{2n} - BG(p)\tilde{B}J$$

in which:

(1)

$$G(p) = \begin{bmatrix} g_1(p) & & 0 \\ & \ddots & \\ 0 & & g_n(p) \end{bmatrix}$$

and $\tilde{G}(p)G(p) \geq 1_n$ everywhere in $\operatorname{Re} p > 0$.

(2) $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ is a $2n \times 2n$ constant scattering matrix such that:

$$\tilde{B}_1 B_1 - \tilde{B}_2 B_2 = 1_n$$

For example, in the case of a simple Blashke factor:
$g_1(p) = 2\alpha_o/(p+p_o^*)$, $g_i(p) = 0$, $(i > 1)$ and in case of a simple singular factor: $g_1(p) = 1 - \exp \dfrac{pj\omega_o - 1}{p - j\omega_o}$, $g_i = 0$, $(i > 1)$.

<u>Proposition 7.1</u>.

If

$$\Theta = [1_{2n} - BG(p)\tilde{B}]\Theta_1$$

then $\sigma = \sigma' \overset{A}{\Box} \sigma''$ in which:

(1)

$$\sigma' = ([1_n - WG\tilde{W}]^{-1}, A)$$

$$W = (1_n - \tilde{S}_o S_o)^{-1/2}[B_1 - \tilde{S}_o B_2]$$

(2) A is induced by S_o with $S_o = B_2 B_1^{-1}$

(3) $\sigma'' = (S'', I)$ where S'' originates from Θ_1 in the natural way.

<u>Proof</u>. The proof is purely algebraic and the reader is referred to the appendix for the computation. It should be noted that σ' is a genuine scattering operator since it is produced through a J-contractive matrix.

The proposition disposes at once with the synthesis of a finitely generated case. Also, we have for Blashke products:

Theorem 5. Suppose S is roomy and a strictly contractive homeomorphism of $H^2_{R^n}$. Then:

$$S = \sigma_1 \left[\overset{A}{\square} \ \sigma_2 \right]$$

in which:

(1)
$$\sigma_1 = \overset{\infty}{\underset{i=1}{\overrightarrow{\boxed{A_i}}}} \ \sigma_i$$

$$\sigma_i = [(S_i, A_i)]$$

S_{2i+1}, $i = 0,\ldots,\infty$ are Blashke factors with zeros at the zeros for $\mathcal{N} : (\{p_i\})$.

S_{2i}, $i = 1,\ldots,\infty$, are convergence producing constant factors. The A_i belong of course to $\mathcal{O}\!\mathcal{l}$.

(2) $\sigma_2 = (S_2, I)$ is such that $A(S_2)$ does not contain any Blashke factor for arbitrary A (in the meaning of reduction in $\mathcal{N}$).

(3) The notation $S = \sigma_1 \left[\overset{A}{\square} \ \sigma_2 \right]$ means:

$$S = \lim_{n\to\infty} \sigma_1 \overset{A_1}{\square} \left[\sigma_2 \overset{A_2}{\square} \left[\ldots \overset{A_{n-1}}{\square} \left[\sigma_n \overset{A_n}{\square} \sigma_2 \right] \ldots \right] \right].$$

the limit being uniform convergence on compact subsets.

Proof. We have $\Theta = \Theta_1 \Theta_2$ in which Θ_1 is Blashke, and thus a product of factors of the form discussed in Proposition 7.1. It was shown there that for each such factor an A_0 existed producing the composition law $\square$. At each step the remaining σ corresponds exactly to the input scattering matrix of the remaining Θ. Hence the same procedure can be repeated. Convergence producing factors originate from the same in the decomposition of Θ, and do not produce any difficulty. The composition converges because the product in Θ converges and both produce the same (2,1) element in the corresponding Σ.

<u>Remark</u>. It is not true that from $\Theta_0 \underset{A}{=} \Theta_1\Theta_2$ for some J-contractive Θ_1 and Θ_2, one can deduce that $\sigma_0 = \sigma_1 \,\Box\, \sigma_2$ with the σ_i corresponding to the Θ_i. Only the very specific form for the Θ's in the proposition allows it.

<u>Corollary</u>. Given a strictly contractive and roomy $H^2_{R^n}$ homeomorphism S, we have that for a at most countable 'Blashke' set of points and corresponding transformations A_i, $A_i(S)$ is factorable in the sense of a reduction in η.

<u>Definition</u>. The zeros of η are called points of local losslessness of S.

<u>Theorem 6</u>. If S is such that η is singular, then S is the limit in the sense of convergence on compact subsets, of

$$\overset{n}{\underset{i=1}{\boxed{A_i}}} \; [1_n - W_i G_i(p) \tilde{W}_i]^{-1}$$

with W_i defined as in Proposition 7.1 and:

$$G_i(p) = \begin{bmatrix} g_1(p) & & \\ & \ddots & \\ & & g_n(p) \end{bmatrix}, \quad g_k(p) = 1 - \exp\left[\lambda_k\left(\frac{pj\omega_i - 1}{p - j\omega_i}\right)\right], \quad \lambda_k > 0.$$

<u>Proof</u>. The proof rests again on Proposition 7.1 and on the approximation of Θ by means of a finite product: since

$$\Theta = \overset{\ell}{\underset{0}{\int\!\!\!\!\curvearrowright}} \exp\left(\frac{pj\omega(t)-1}{p-j\omega(t)}\right) dE(t)$$

we have, choosing partitions of t corresponding to constant values of $\omega(t)$:

$$\Theta = \lim \prod_{i=1}^{N} \exp\left(\frac{pj\omega_i - 1}{p - j\omega_i}\right) M_i$$

in which, because of the special form of $E(t)$:

$$M_i = \begin{bmatrix} B_1^{(i)} \\ \\ B_2^{(i)} \end{bmatrix} \Lambda [B_1^{(i)*} \; B_2^{(i)*}] J$$

Hence, the proposition is again applicable. The convergence on compact subsets follows again from convergence on compacts in Θ.

Remark 1. The preceding theorem could lead to an integral representation for S. Pending a complete theory of convergence for such integrals, it should be noted that the theorem does show that a particular finite decomposition converges, but not that any decomposition resulting from the integral representation does indeed converge.

Remark 2. Undoubtedly, there is more than convergence on compacts, in fact, there is $L^2\left(\frac{d\omega}{1+\omega^2}\right)$ convergence as was the case in Chapter 4. (We give no proof for this, but note that it follows from the representation of $\mathcal{N}$ by means of a product of Θ's converging in this topology, the Θ's being deduced from the inner function U with $\mathcal{N} = UH^2_{R^n}$.)

Remark 3. The synthesis produces a continuous reduction of the space $\mathcal{N}$. The author has not been able to connect directly this space to the nullspace of S: the two spaces are in some sense equivalent, but what sense?

8. CONCLUSION

The previous developments produce several new problems both of fundamental and practical interest:

(1) The synthesis of non-roomy systems, or 'dilation-synthesis'. Many new results are available in the theory of dilations (see [12]), but nothing similar to a synthesis has emerged yet.

(2) The theory of equivalence of invariant subspaces: although it is possible to construct a substitute for the Smith form in case of inner functions, not much is known of its properties. Such a theory could lead to improvements in the synthesis of par. 7.

(3) The theory of non-associative integrals: a technical point indicated in par. 7.

(4) The generalization of the several syntheses from $\mathbb{R}^n$ to an arbitrary Hilbert space $\mathcal{H}$ (some results are e.g. in [16]).

From the technical viewpoint also a lot remains to be done. A good theory of transcendal approximation is lacking, for instance.

Hence, although it appears to the author that the theory is very promising, still many problems are there to be solved and it is hoped that the first step taken here will lead to an efficient theory for scattering synthesis in general.

APPENDIX

Analytic transformations and the proof of Proposition 7.1.

It is shown in ([7], Chapter 2, par. 3) that $\Theta_0 = \Theta_1\Theta_2$ induces the composition law $\sigma_0 = \sigma_1 \overset{A}{\Box} \sigma_2$ with A induced by S_o, if and only if Θ_1 can be diagonalized by means of an analytic transformation of dimension 2n 2n induced by:

$$\begin{bmatrix} S_o & 0_n \\ 0_n & \tilde{S}_o \end{bmatrix}$$

(or, in other words: $A \oplus A^*$ with obvious meaning for $\oplus$). Such an analytic transformation on the 2n×2n system induces a transformation on Θ given by:

$$\Theta' = H(S_o)\cdot\Theta\cdot X\cdot H(-\tilde{S}_o)X$$

in which $H(S_o)$ is given by (1.2) and

$$X = \begin{bmatrix} 0_n & 1_n \\ 1_n & 0_n \end{bmatrix}.$$

With

$$\Theta = \begin{bmatrix} 1_n - B_1G\tilde{B}_1 & B_1G\tilde{B}_2 \\ -B_2G\tilde{B}_1 & 1_n + B_2G\tilde{B}_2 \end{bmatrix}$$

we have:

$H(S_o)\Theta XH(-\tilde{S}_o)X =$

$$= \begin{bmatrix} (1_n-\tilde{S}_oS_o) & 0_n \\ 0_n & (1_n-S_o\tilde{S}_o) \end{bmatrix}^{-1/2}$$

$$\cdot \begin{bmatrix} (1_n-\tilde{S}_oS_o) - (B_1-\tilde{S}_oB_2)G(\tilde{B}_1-\tilde{B}_2S_o) & (B_1-\tilde{S}_oB_2)G(\tilde{B}_2-\tilde{B}_1\tilde{S}_o) \\ -(B_2-S_oB_1)G(\tilde{B}_1-\tilde{B}_2S_o) & (1_n-S_o\tilde{S}_o) + (B_2-S_oB_1)G(\tilde{B}_2-\tilde{B}_1\tilde{S}_o) \end{bmatrix}$$

$$\cdot \begin{bmatrix} (1_n-\tilde{S}_oS_o) & 0_n \\ 0_n & (1_n-S_o\tilde{S}_o) \end{bmatrix}^{-1/2}$$

Hence, choosing $S_o = B_2B_1^{-1}$, we obtain:

$$\Theta' = \begin{bmatrix} 1_n - WG\tilde{W} & 0_n \\ 0_n & 1_n \end{bmatrix}$$

with

$$W = [1_n-\tilde{S}_oS_o]^{-1/2}(B_1-\tilde{S}_oB_2).$$

The corresponding Σ is then:

$$\Sigma = \begin{bmatrix} (1_n-WG\tilde{W})^{-1} & 0_n \\ 0_n & 1_n \end{bmatrix}$$

producing

$$\sigma_1 = ([1_n-WG\tilde{W}]^{-1},A)$$

where A is induced by S_o.

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