AFRCL-71-0049

PREDICTION AND ANALYSIS OF SOLAR ECLIPSE CIRCUMSTANCES

WENTWORTH WILLIAMS, JR.

ARTHUR D. LITTLE, INC.
ACORN PARK
CAMBRIDGE, MASSACHUSETTS 02140

Contract No. F19628-70-C-0087

FINAL REPORT

MARCH 25, 1971

Contract Monitor: Isabel M. Hussey
Analysis Simulation Branch

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131

Prepared for

AIR FORCE CAMBRIDGE RESEARCH LABORATORIES
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
BEDFORD, MASSACHUSETTS 01730

UNCLASSIFIED
Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Arthur D. Little, Inc. Acorn Park Cambridge, Massachusetts 02140	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP

3. REPORT TITLE
PREDICTION AND ANALYSIS OF SOLAR ECLIPSE CIRCUMSTANCES

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)
Final

5. AUTHOR(S) (First name, middle initial, last name)
Wentworth Williams, Jr.

6. REPORT DATE 25 March 1971	7a. TOTAL NO. OF PAGES 125 pages	7b. NO. OF REFS 13

8a. CONTRACT OR GRANT NO. F 19628-70-C-0087	9a. ORIGINATOR'S REPORT NUMBER(S) 71932-1
b. PROJECT NO. N/A	
c. DOD Element 61102F	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) AFCRL-71-0049
d. DOD Subelement 681300	

10. DISTRIBUTION STATEMENT
This document has been approved for public release and sale; its distribution is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Air Force Cambridge Research Laboratory (SUY) L.G. Hanscom Field Bedford, Massachusetts 01730

13. ABSTRACT

This report addresses the prediction and analysis of solar eclipse circumstances of interest to atmospheric, ionospheric and solar studies. In the first section specific algorithms for use in pre-eclipse planning and post-eclipse analysis are presented. In the second section, appendices, concerned with the calculation of solar and lunar ephemerides of requisite accuracy for prediction purposes, the calculation of ephemeris sidereal time, the prediction of the shadow outline on and above the earth, the calculation of local eclipse circumstances and the development of solar coordinate systems, provide the background for and the foundation of these algorithms.

DD FORM 1 NOV 65 1473 REPLACES DD FORM 1473, 1 JAN 64, WHICH IS OBSOLETE FOR ARMY USE.

UNCLASSIFIED
Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Solar Eclipse Prediction						
Besselian Elements						
Auxiliary Elements						
Maximum Eclipse Time						
Contact Times						
Magnitude and Obscuration						
Differential Corrections						
Solar Ephemeris						
Lunar Ephemeris						
Shadow Outline and Motion						
Heliographic Coordinate Systems						

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This report addresses the prediction and analysis of solar eclipse circumstances of interest to atmospheric, ionospheric, and solar studies. In the first section, specific algorithms for use in pre-eclipse planning and post-eclipse analysis are presented. In the second section, appendices present the calculation of solar and lunar ephemerides of requisite accuracy for prediction purposes, the calculation of ephemeris sidereal time, the prediction of the shadow outline on and above the earth, the calculation of local eclipse circumstances, and the development of solar coordinate systems, to provide the background for and the foundation of the preceding algorithms.

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E.S.T.	Ephemeris Sidereal Time
ΔT	difference between Ephemeris Time (E.T.) and Universal Time (U.T.)
J.D.	Julian Day Number of interest
$a_{\oplus}$	equatorial radius of the earth = 6378.160 km
e	ellipticity of the spheroid; for the earth $e^2 = 0.0066\ 9454$
ϕ	geodetic latitude of observer
ϕ'	geocentric latitude of observer
λ	geodetic longitude of observer (West taken as positive)
λ_e	ephemeris longitude of observer
h	height of observer above sea level
ξ, η, ζ	coordinates of observer w.r.t. fundamental plane
a	right ascension of shadow axis
μ	ephemeris hour angle of shadow axis
d	declination of shadow axis
x,y	shadow axis coordinates in fundamental plane
f_1	generator of penumbral cone
f_2	generator of umbral cone
ℓ_1	radius of penumbra on fundamental plane
ℓ_2	radius of umbra on fundamental plane
L_1	penumbral radius at observer height ζ above fundamental
L_2	umbral radius at observer height ζ above fundamental plane
m	observer-shadow axis separation at height ζ above fundamental plane
n	relative, observer-shadow axis speed at height ζ above fundamental plane
t_m	time of maximum eclipse
Q	position angle of point on sun measured eastwards from north point of sun
C,q	paraliactic angle

LIST OF SYMBOLS (Cont.)

V	position angle of point on sun measured eastwards from solar vertex (zenith point): V = Q - C
M_1	magnitude of a partial eclipse
M_2	magnitude of a total/annular eclipse
B_o	heliographic latitude of sub-terrestrial point (center of solar disc)
L_o	heliographic longitude of sub-terrestrial point
B	heliographic latitude of point on solar disc
L	heliographic longitude of point on solar disc
r	linear radial distance of point on solar disc of linear radius $r_\odot$ (the measure of r and $r_\odot$ being arbitrary)
ρ_1	angle subtended at earth's center by point at linear radius r and by center of solar disc
θ	position angle of point on solar disc measured eastwards from north point
$\alpha_\odot$	apparent right ascension of sun
$\delta_\odot$	apparent declination of sun
R	solar radius vector (in astronomical units)
$\pi_\odot$	solar parallax
$S_\odot$	solar semi-diameter
$\alpha'_\odot$	topocentric right ascension of sun
$\delta'_\odot$	topocentric declination of sun
$Az_\odot$	azimuth (east from north) of sun
$E\ell_\odot$	elevation of sun
$\alpha_{\leftmoon}$	apparent right ascension of moon
$\delta_{\leftmoon}$	apparent declination of moon
$\pi_{\leftmoon}$	lunar parallax (horizontal equatorial)
$r_{\leftmoon}$	geocentric distance of moon
$S_{\leftmoon}$	geocentric semi-diameter of moon
$\alpha'_{\leftmoon}$	topocentric right ascension of moon
$\delta'_{\leftmoon}$	topocentric declination of moon

LIST OF SYMBOLS (Cont.)

$r'_{☾}$	topocentric distance of moon
$S'_{☾}$	topocentric semi-diameter of moon (note: $S'_{☾} - S_{☾}$ is known as the augmentation)
$Az_{☾}$	azimuth (east from north) of moon
$E\ell_{☾}$	elevation of moon

1.0 INTRODUCTION

This report addresses the prediction and analysis of solar eclipse circumstances of interest to atmospheric, ionospheric, and solar studies. In the first section, specific algorithms for use in pre-eclipse planning and post-eclipse analysis are presented in a form suitable for manual or machine calculation. In the second section, five appendices provide the background for and the foundation of these algorithms.

Although much of the material in this report is familiar to the astronomical community, its dispersion in the literature or its rendition in a variety of forms do not make it immediately useful to the non-astronomer. This report's first section is, therefore, a compilation of familiar results; the second section is tutorial in nature.

Appendix A presents algorithms for calculating the geocentric coordinates of the sun and moon. In the solar case, the algorithms are based primarily upon the Newcomb *Tables* [7] and secondarily upon the more recently published *Tables* of Jean Meeus [5]. In the lunar case, the algorithms are based primarily upon the "Improved Lunar Ephemeris" [3] and secondarily upon Meeus. The series developments of solar and lunar longitude, latitude, and distance (parallax) have been truncated in view of the relaxed accuracy requirements. In all other respects, however, the developments of the solar and lunar ephemerides are identical to those given in the fundamental references.

Appendix B outlines the calculation of Ephemeris Sidereal Time.

Appendix C develops algorithms necessary to predict the outline and motion of the lunar shadow on an earth spheroid of arbitrary radius and flattening. It is based upon the exposition of Chauvenet [1] and the authoritative summary in *The Explanatory Supplement. . .*[2]. It is repeated here because the summary of requisite formula in [2], with a different ordering from that of [1], lacks both the justification and seeming inevitability with which elements related to the flattened spheroid and the shadow motion thereon were originally introduced. Furthermore, it seems appropriate to show that the relationships pertaining to ionospheric center line position and duration follow naturally from a development in which scale factors are explicitly employed.

Appendix D develops the algorithms required to predict all circumstances of an eclipse at a locale once the coordinates of the locale have been specified, to adjust these circumstances by means of differential correction procedures for modest departures from the locale specified, and to calculate other topocentric parameters of interest.

Finally, Appendix E describes various coordinate systems applicable to solar astronomy and defines their interrelationships. Particular attention is directed to the development of "pointing instructions" for observation of phenomena both on and above the solar surface.

2.0 PRELIMINARY REMARKS

Ephemeris Time (E.T.), which is related to Universal Time (U.T.) by means of the expression

$$\text{E.T.} = \text{U.T.} + \Delta T \qquad (1)$$

will be employed throughout this report, except where indicated. Similarly, the ephemeris longitude of a location λ_c, which is related to the geodetic longitude λ (West taken as positive) by means of the expression

$$\lambda_c = \lambda + 1.002738\ \Delta T, \qquad (2)$$

will also be utilized, except where indicated.

In either instance, conversion from one "system" to the other requires the explicit assignment of a value for ΔT. For the past years, values of ΔT are given in *The American Ephemeris and Nautical Almanac*, page vii, *Table of Time-Difference ΔT*. For the 1970's Table 1 – extracted from Table 67a, *Reduction from Universal to Ephemeris Time of [2]* – provides a useful extrapolation.

TABLE 1

ΔT VALUES FOR 1969-1980 INCLUSIVE

1969.5	$+42^s$	1975.5	$+51^s$
1970.5	44^s	1976.5	53^s
1971.5	45^s	1977.5	54^s
1972.5	47^s	1978.5	56^s
1973.5	48^s	1979.5	57^s
1974.5	50^s	1980.5	59^s

3.0 CALCULATION OF BESSELIAN AND AUXILIARY ELEMENTS

This section will follow the development presented in Appendix C and is based on the assumption that the following quantities are available at the time(s) of interest:

$\alpha_\odot$ $\delta_\odot$.=. apparent right ascension and declination of sun [cf (A21)]

R .=. the solar radius vector in A.U. [cf (A14)]

$\alpha_☾$, $\delta_☾$.=. apparent right ascension and declination of moon [cf (A46)]

$\pi_☾$.=. the horizontal equatorial parallax of the moon [cf (A45)]

E.S.T. .=. Ephemeris Sidereal Time [cf (B3)].

3.1 SHADOW AXIS PARAMETERS

Calculate the shadow axis right ascension a and declination d and the solar-lunar separation g (in terms of R) from

$$g \cos d \cos a = \cos \delta_\odot \cos \alpha_\odot - b \cos \delta_☾ \cos \alpha_☾ \qquad (3)$$

$$g \cos d \sin a = \cos \delta_\odot \sin \alpha_\odot - b \cos \delta_☾ \sin \alpha_☾ \qquad (4)$$

$$g \sin d = \sin \delta_\odot - b \sin \delta_☾ \qquad (5)$$

where

$$b = \frac{0.000\,042\,664}{R \sin \pi_☾} \qquad (6)$$

3.2 "FUNDAMENTAL" RECTANGULAR COORDINATES OF MOON

Calculate the rectangular coordinates x, y, z of the moon with respect to the fundamental plane (in units of the earth's equatorial radius $a_\oplus$ = 6378.160 km) from

$$x = r_☾ [\cos \delta_☾ \sin (\alpha_☾ - a)] \qquad (7)$$

$$y = r_☾ [\sin \delta_☾ \cos d - \cos \delta_☾ \sin d \cos (\alpha_☾ - a)] \qquad (8)$$

$$z = r_☾ [\sin \delta_☾ \sin d + \cos \delta_☾ \cos d \cos (\alpha_☾ - a)] \qquad (9)$$

where

$$r_{\mathbb{C}} = 1/\sin \pi_{\mathbb{C}} \tag{10}$$

3.3 EPHEMERIS HOUR ANGLE OF SHADOW AXIS

Calculate μ, the ephemeris hour angle of the shadow axis, from

$$\mu = \text{E.S.T.} - a \tag{11}$$

3.4 SHADOW CONE GENERATORS

Calculate f_1, the generator of the penumbral cone, and f_2, the generator of the umbral cone, from *

$$\sin f_1 = 0.0046\ 64018/gR \tag{12}$$

$$\sin f_2 = 0.0046\ 40783/g\,R \tag{13}$$

3.5 SHADOW CONE VERTEX DISTANCES

Calculate c_1, the distance of the penumbral cone vertex above the fundamental plane, and c_2, the distance of the umbral cone vertex above the fundamental plane, from

$$c_1 = z + 0.2724\ 880 \operatorname{cosec} f_1 \tag{14}$$

$$c_2 = z - 0.2724\ 880 \operatorname{cosec} f_2 . \tag{15}$$

3.6 SHADOW RADII ON FUNDAMENTAL PLANE

Calculate ℓ_1, the penumbral radius on the fundamental plane, and ℓ_2, the umbral radius on the fundamental plane, from

---

* If, as in the discussion leading to (E46), we wish to consider the radio sun whose radius is $\mathcal{H}_R$ times the optical radius where $\mathcal{H}_R = (1 + \mathcal{H}'_R)$, then

$$\sin f_{1,R} = (0.0046\ 64018 + 0.0046\ 52367\,\mathcal{H}'_R)/g\,R$$

and

$$\sin f_{2,R} = (0.0046\ 40783 + 0.0046\ 52367\,\mathcal{H}_R)/\,g\,R.$$

$$\ell_1 = c_1 \tan f_1 \qquad (16)$$

$$\ell_2 = c_2 \tan f_2 \qquad (17)$$

where $\ell_1 > 0$; $\ell_2 < 0$ for totality, $\ell_2 > 0$ for annularity.

3.7 HOURLY VARIATIONS

Calculate the hourly variations $\dot{d}$, $\dot{x}$, $\dot{y}$, $\dot{\mu}$, $\dot{\ell}_1$ and $\dot{\ell}_2$ by appropriate numerical differentiation of the Besselian elements sin d, cos d, x, y, μ, ℓ_1 and ℓ_2.

3.8 AUXILIARY ELEMENTS

Calculate the auxiliary elements ρ_1, ρ_2, sin d_1, sin d_2, sin $(d_1 - d_2)$ and cos $(d_1 - d_2)$ from

$$\rho_1 = (1 - e^2 \cos^2 d)^{½} \qquad (18)$$

$$\rho_2 = (1 - e^2 \sin^2 d)^{½} \qquad (19)$$

$$\sin d_1 = \sin d/\rho_1 \qquad (20)$$

$$\cos d_1 = (1 - e^2)^{½} \cos d/\rho_1 \qquad (21)$$

$$\sin (d_1 - d_2) = e^2 \sin d \cos d/\rho_1 \; \rho_2 \qquad (22)$$

$$\cos (d_1 - d_2) = (1 - e^2)^{½} / \rho_1 \; \rho_2 \qquad (23)$$

where the ellipticity e is given by $e = (0.0066\,9454)^{½}$ for the earth spheroid.

Also, calculate the additional auxiliary elements $\dot{a}_1$, $\dot{a}_2$, $\dot{b}$, $\dot{c}_1$ and $\dot{c}_2$ from

$$\dot{a}_1 = -\dot{\ell}_1 - \dot{\mu}\; x \tan f_1 \cos d \qquad (24)$$

$$\dot{a}_2 = -\dot{\ell}_2 - \dot{\mu}\; x \tan f_2 \cos d \qquad (25)$$

$$\dot{b} = -\dot{y} + \dot{\mu}\, x \sin d \qquad (26)$$

$$\dot{c}_1 = \dot{x} + \dot{\mu}\; y \sin d + \dot{\mu}\; \ell_1 \tan f_1 \cos d \qquad (27)$$

$$\dot{c}_2 = \dot{x} + \dot{\mu}\; y \sin d + \dot{\mu}\; \ell_2 \tan f_2 \cos d \qquad (28)$$

4.0 CALCULATION OF IONOSPHERIC CENTRAL LINE, DURATION, AND SHADOW OUTLINE

The following development, except where indicated, is based upon Sections C.9 and C.11 of Appendix C.

4.1 IONOSPHERIC CENTRAL LINE POINT

From the relations

$$y_1 = y/\rho_1 \tag{29}$$

$$\zeta_1 = [\mathcal{H}^2 - x^2 - y_1^2]^{1/2} \ , \tag{30}$$

where

$$\mathcal{H} = 1 + 0.15678503 \times 10^{-3}\, h_I \tag{31}$$

in which h_I is the ionospheric center line height above sea level in kilometers,* calculate a point on the central line from

$$\tan \textcircled{H} = \frac{x}{\zeta_1 \cos d_1 - y_1 \sin d_1} \tag{32}$$

and

$$\sin \phi_1 = \frac{\zeta_1 \sin d_1 + y_1 \cos d_1}{\mathcal{H}} \ . \tag{33}$$

The geodetic latitude ϕ of the point is given by

$$\tan \phi = 1.003364 \tan \phi_1 \ , \tag{34}$$

the ephemeris longitude λ_e is given by

$$\lambda_e = \mu - \textcircled{H} \tag{35}$$

in which $\textcircled{H}$ is the local hour angle of the shadow axis, and the longitude is given by

$$\lambda = \lambda_e - 1.002738\, \Delta T . \tag{36}$$

---

* Note the footnote following (D2).

4.2 ECLIPSE SEMI-DURATION AT CENTER LINE POINT

Calculate the semi-duration of totality at the ionospheric central line point ϕ and λ of (34) and (36) from

$$s = L_2/n \qquad (37)$$

where

$$L_2 = \ell_2 - \zeta_0 \tan f_2 \qquad (38)$$

in which

$$\zeta_0 = \rho_2 \, [\zeta_1 \cos(d_1 - d_2) - y_1 \sin(d_1 - d_2)] \qquad (39)$$

and where

$$n = [(\dot{c}_2 - \dot{\mu}\,\zeta_0 \cos d)^2 + (-\dot{b})^2]^{1/2}. \qquad (40)$$

4.3 SHADOW OUTLINE APPROXIMATION

The shadow outline, centered at the point specified by (34) and (36), can be approximated* by an ellipse whose semi-major axis is oriented toward the sun along azimuth $Az_{\odot}$. The semi-minor axis is given (in kilometers) by

$$\text{semi-minor axis} = L_2\ (\mathcal{H}a_{\oplus}) \qquad (41)$$

where $a_{\oplus}$ = the equatorial radius of the earth = 6378.160 km; the semi-major axis is given by

$$\text{semi-major axis} = L_2\ (\mathcal{H}a_{\oplus})/\sin E\ell_{\odot} \approx L_2(\mathcal{H}a_{\oplus})/\zeta \qquad (42)$$

in which $E\ell_{\odot}$ is the solar elevation. The approximation is derived in (D91).

Explicit formulae for the solar azimuth and elevation are given by (D90) and will be repeated in Section 11.3.

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* This approximation, avoids the tedious point-by-point, albeit more precise, outlined trace method developed (primarily for illustrative purposes) in Section C.8 of Appendix C. cf Comrie, L. J., "Some Computational Problems Arising in Eclipses," M.N.R.A.S. 87, 483 (1927).

5.0 CALCULATION OF OBSERVER COORDINATES

The observer is assumed to be located at geodetic latitude ϕ, longitude λ (West taken as positive) and height above sea level h (in meters).

5.1 GEOCENTRIC COORDINATES OF THE OBSERVER

Calculate the geocentric coordinates $\rho \sin \phi'$ and $\rho \cos \phi'$ from

$$\rho \sin \phi' = (S + 0.15678503 \times 10^{-6} h) \sin \phi \qquad (43)$$

$$\rho \cos \phi' = (C + 0.15678503 \times 10^{-6} h) \cos \phi \qquad (44)$$

where

$$S = 0.9949\,7418 - 0.0016\,7082 \cos 2\phi + 0.0000\,0210 \cos 4\phi \qquad (45)$$

$$C = 1.0016\,7997 - 0.0016\,8208 \cos 2\phi + 0.0000\,0212 \cos 4\phi \qquad (46)$$

5.2 "FUNDAMENTAL" RECTANGULAR COORDINATES OF THE OBSERVER

Calculate the rectangular coordinates ξ, η, ζ of the observer with respect to the fundamental plane from

$$\xi = \rho \cos \phi' \sin \textcircled{H} \qquad (47)$$

$$\eta = \rho \sin \phi' \cos d - \rho \cos \phi' \sin d \cos \textcircled{H} \qquad (48)$$

$$\zeta = \rho \sin \phi' \sin d + \rho \cos \phi' \cos d \cos \textcircled{H} \qquad (49)$$

where $\textcircled{H}$, the local hour angle of the shadow axis, is given by

$$\textcircled{H} = \mu - \lambda_e = \mu - \lambda - 1.002738\Delta T. \qquad (50)$$

5.3 SHADOW RADII AT HEIGHT ζ

Calculate the penumbral and umbral radii at the height ζ above the fundamental plane from

$$L_1 = l_1 - \zeta \tan f_1 \qquad (51)$$

$$L_2 = l_2 - \zeta \tan f_2 . \qquad (52)$$

5.4 OBSERVER COORDINATE VARIATIONS

Calculate the hourly variation of ξ and η from

$$\dot{\xi} = \dot{\mu} \rho \cos \phi' \cos \theta \tag{53}$$

$$\dot{\eta} = \dot{\mu} \xi \sin d - \zeta \dot{d} \approx \dot{\mu} \xi \sin d. \tag{54}$$

and calculate the following combinations:

$$u = x - \xi; v = y - \eta; m^2 = u^2 + v^2 \tag{55}$$

$$\dot{u} = \dot{x} - \dot{\xi}; \dot{v} = \dot{y} - \dot{\eta}; n^2 = \dot{u}^2 + \dot{v}^2 \tag{56}$$

$$D = u\dot{u} + v\dot{v}. \tag{57}$$

6.0 CALCULATION OF THE TIME OF MAXIMUM ECLIPSE AND OF CONTACT TIMES

In this section, which deals with calculation of the maximum eclipse and contact times for a specified observation site, an iterative procedure is used which is fully described in Sections D.1 and D.2 of Appendix D and then applied in Sections D.2, D.3 and D.4 of that appendix. In essence, one selects an ephemeris time "close to" the time of interest, and then calculates a formulated correction to this selected time. If necessary, this corrected time is employed as the value "close to" the time of interest, the cycle repeated and the time of interest ultimately established.

6.1 TIME OF MAXIMUM ECLIPSE

Select a time T_0 and – using elements appropriate to this time – calculate the correction.

$$\tau = -D/n^2 \qquad (58)$$

Iterate where necessary until a value has been settled upon; the maximum eclipse time, so determined, is designated t_m.

6.2 PENUMBRAL (FIRST AND FOURTH) CONTACTS

Select a time T_0 and – using elements appropriate to this time – calculate the correction

$$\tau_P = -\frac{D}{n^2} \pm \frac{L_1}{n} |\cos \psi_P| \qquad (59)$$

in which

$$\sin \psi_P = \frac{1}{L_1} \left[\frac{u\dot{v} - v\dot{u}}{n} \right] \qquad (60)$$

and where the negative sign is selected for first contact (immersion) and the positive sign is selected for fourth contact (emersion). Iterate where necessary.

6.3 UMBRAL (SECOND AND THIRD) CONTACTS

Employing elements appropriate to the time of maximum eclipse computed following (58), calculate

$$\tau_u = -\frac{D}{n^2} \pm \frac{L_2}{n} \mid \cos \psi_u \mid \qquad (61)$$

in which

$$\sin \psi_u = \frac{1}{L_2} \left[\frac{u\dot{v} - v\dot{u}}{n} \right] = \frac{K}{L_2} \; ; \qquad (62)$$

note that $L_2 \, |\cos \psi_u| / n$ is the semi-duration S_u as given in (D23). The positive sign is selected either for second contact (immersion) in the case of a total ($L_2 < 0$) eclipse or third contact (emersion) for an annular ($L_2 > 0$) eclipse; the negative sign is selected either for third contact in a total eclipse or second contact in an annular eclipse.

7.0 CALCULATION OF POSITION ANGLES

Calculate the position angle Q_i of the ith contact point on the solar limb measured eastwards from the north point (i.e., from the hour circle passing through the solar center) by means of

$$\tan Q_i = \frac{u_i}{v_i} \qquad (63)^*$$

where we note that for i = 1,4 the algebraic sign of sin Q_i is that of u_i whereas for i = 2,3 it is opposite that of u_i.

For some observational purposes, the position angle V_i measured eastwards from the vertex (zenith point) of the solar limb may prove more useful. This is given by

$$V_i = Q_i - C_i \qquad (64)$$

where the parallactic angle C_i is given approximately by

$$\tan C_i \approx \xi_i / \eta_i \qquad (65)$$

in which the sign of sin C_i is that of ξ_i.

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* At any time prior to, during, or after the eclipse, expression (63) yields the position angle $Q_{☾}$ of the line joining the solar and lunar centers and thus, the position angles of the contact points at the contact times.

8.0 CALCULATION OF DIFFERENTIAL CORRECTIONS

8.1 PRELIMINARY COEFFICIENTS

Calculate

$$A_1 = -\rho \cos \phi' \cos \textcircled{H} \qquad (66)$$

$$A_2 = -(SC^2 + 0.1568h \times 10^{-6}) \sin \phi \sin \textcircled{H} \qquad (67)$$

$$A_3 = \cos \phi \sin \textcircled{H} \qquad (68)$$

$$B_1 = -\xi \sin d \qquad (69)$$

$$B_2 = (SC^2 + 0.1568h \times 10^{-6}) (\cos \phi \cos d + \sin \phi \sin d \cos \textcircled{H}) \qquad (70)$$

$$B_3 = \sin \phi \cos d - \cos \phi \sin d \cos \textcircled{H} \qquad (71)$$

8.2 CORRECTION TO CALCULATED TIME OF MAXIMUM ECLIPSE

Apply the differential correction (in hours) to the time of maximum eclipse t_m (calculated in Section 6.1) in the form

$$\delta t_m = p_m \; \delta\lambda + q_m \; \delta\phi + r_m \; \delta h \qquad (72)$$

in which $\delta\lambda$ and $\delta\phi$ are expressed in minutes of arc, δh is expressed in meters and where*

$$p_m = (\dot{u} A_1 + \dot{v} B_1)/n^2 \cdot \sin 1' \qquad (73)$$

$$q_m = (\dot{u} A_2 + \dot{v} B_2)/n^2 \cdot \sin 1' \qquad (74)$$

$$r_m = (\dot{u} A_3 + \dot{v} B_3)/n^2 \; a_\oplus; \qquad (75)$$

$a_\oplus \approx 638 \times 10^4$ meters and $\sin 1' = 0.000\ 2909$.

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* p_m and q_m are written with sin 1' in the coefficients of $\delta\lambda$ and $\delta\phi$ in order to illustrate a convenient way of coping with both the "radian" nature of $p_m \; \delta\lambda$ and $q_m \; \delta\phi$ in the differential expression, on the one hand, and the requirement of having convenient units for the measure of differential displacements. This is done by noting that since 1 radian = $57^\circ 17' 45'' = 3437'.75 = 206{,}265''$, then $1' \approx 1/3438$ radian and $1'' = 1/206{,}265$ radian. Further, we note that to a good approximation, $\sin 1' = 1/3438$ and $\sin 1'' = 1/206{,}265$. Hence, if θ'' denotes the number of arc seconds in θ (radians) then θ (radians), $= \theta''/206{,}265 = \theta'' \sin 1'' = \sin \theta''$, and similarly, θ (radians) $= \theta'/3438 = \theta' \sin 1' = \sin \theta'$.

8.3 CORRECTIONS TO CALCULATED TIMES OF FIRST AND FOURTH CONTACTS

Apply the differential corrections (in hours) to the times of first and fourth contacts (calculated in Section 6.2) in the form

$$\delta t = p\delta\lambda + q\delta\phi + r\delta h \qquad (76)$$

in which $\delta\lambda$ and $\delta\phi$ are expressed in minutes of arc, δh is expressed in meters and where

$$p = (u A_1 + vB_1)/D \cdot \sin 1' \qquad (77)$$

$$q = (u A_2 + vB_2)/D \cdot \sin 1' \qquad (78)$$

$$r = (u A_3 + vB_3)/D\, a_{\oplus}. \qquad (79)$$

8.4 CORRECTIONS TO CALCULATED TIMES OF SECOND AND THIRD CONTACTS

Apply the differential correction (in hours) to the times of second and third contacts (calculated in Section 6.3) in the following way. Calculate

$$\delta K = p_s\, \delta\lambda + q_s\, \delta\phi + r_s\, \delta h \qquad (80)$$

in which

$$p_s = (\dot{u}\, B_1 - \dot{v}\, A_1)/n \cdot \sin 1' \qquad (81)$$

$$q_s = (\dot{u}\, B_2 - \dot{v}\, A_2)/n \cdot \sin 1' \qquad (82)$$

$$r_s = (\dot{u}\, B_3 - \dot{v}\, A_3)/na_{\oplus}. \qquad (83)$$

Next compute

$$\sin \psi'_u = \frac{(K + \delta K)}{L_2} \qquad (84)$$

where, as in (62), K and L_2 are evaluated at t_m and compute the corrected semi-duration S'_u from

$$S'_u = L_2 |\cos \psi'_u|/n. \qquad (85)$$

Apply (85) to the maximum eclipse time as corrected by (72).

9.0 CALCULATION OF ECLIPSE MAGNITUDE AND OBSCURATION

9.1 MAGNITUDE

The magnitude is defined as the fraction of the solar diameter covered by the lunar disc at the time of greatest phase in units of the solar diameter. Thus for a partial eclipse the magnitude M_1,[*] is given by

$$M_1 = \frac{L_1 - m}{L_1 + L_2} \tag{86}$$

or, if L_2 is unavailable, by

$$M_1 = \frac{L_1 - m}{2L_1 - 0.5464} \tag{87}$$

For a total eclipse ($L_2 < 0$) or an annular eclipse ($L_2 > 0$),

$$M_2 = \frac{L_1 - L_2}{L_1 + L_2} \tag{88}$$

9.2 OBSCURATION

Calculate the fraction S' of the solar disc obscured by the moon from

$$S' = (s^2 A + B - s \sin C)/\pi \tag{89}$$

in which

$$\cos C = (L_1^2 + L_2^2 - 2m^2)/(L_1^2 - L_2^2) \qquad 0 \leqslant C \leqslant \pi \tag{90}$$

$$\cos B = (L_1 L_2 + m^2)/m(L_1 + L_2) \qquad 0 \leqslant B \leqslant \pi \tag{91}$$

$$A = \pi - (B + C) \tag{92}$$

$$s = (L_1 - L_2)/(L_1 + L_2); \tag{93}$$

and where $S' = s^2$ during the annular phase and $S' = 1$ during totality.

---

[*] We note that M_1, as given in (86) and/or (87), is a useful expression for the fraction of the solar diameter eclipsed at any time prior to second contact and following third contact.

10.0 CALCULATION OF HELIOGRAPHIC COORDINATES OF ARBITRARY POINTS

10.1 HELIOGRAPHIC COORDINATES OF SUB-TERRESTRIAL POINT

The heliographic latitude B_0 and longitude L_0 of the sub-terrestrial point (center of solar disc) are gotten from

$$\sin B_0 = 0.12620 \sin(\lambda_\odot - \Omega) \tag{94}$$

$$\cos B_0 \cos(L_0 - M) = -\cos(\lambda_\odot - \Omega) \tag{95}$$

$$\cos B_0 \sin(L_0 - M) = -0.99200 \sin(\lambda_\odot - \Omega) \tag{96}$$

in which

$$\Omega = 73^\circ\ 40' + 50''.25\,t \tag{97}$$

$$M = 292^\circ.766 + 14^\circ.18439716\,(243\ 0000.5 - \text{J.D.}) \tag{98}$$

where

$\lambda_\odot$.=. the longitude of the sun which is given either by (A.16) or (A.18),

t .=. the time in years from 1850 A.D. to the date of observation,

J.D. .=. the Julian Day number of the date and time of observation.

10.2 POSITION ANGLE OF SUN'S AXIS OF ROTATION P

Calculate P, the position angle (measured eastwards from north point) of the sun's rotation axis from

$$P = X + Y \tag{99}$$

in which

$$\tan X = -\cos\lambda_\odot \tan\epsilon \tag{100}$$

$$\tan Y = -0.12722 \cos(\lambda_\odot - \Omega) \tag{101}$$

where

ϵ .=. the obliquity of the ecliptic given by (A8).

10.3 HELIOGRAPHIC COORDINATES OF ARBITRARY POINTS

The heliographic longitude L and latitude B of any point at position angle θ and linear radial distance r on the solar disc of linear radius $r_\odot$* and can be calculated from

$$\sin B = \sin B_0 \cos \rho + \cos B_0 \sin \rho \cos (P - \theta) \tag{102}$$

$$\cos B \sin (L - L_0) = \sin \rho \sin (P - \theta) \tag{103}$$

where ρ can be calculated from

$$\sin (\rho + \frac{r}{r_\odot} S_\odot) = \frac{r}{r_\odot} \tag{104}$$

in which $S_\odot$, the solar semi-diameter (in minutes of arc), is given by (A23).

Conversely, if L and B are given, θ and r can be calculated from the inversion of (102) in the form

$$\cos \rho = \sin B_0 \sin B + \cos B_0 \cos (L - L_0) \cos B, \tag{105}$$

and from the application of (103) and (104).

10.4 HELIOGRAPHIC COORDINATES OF CONTACT POINTS

The ith contact point at position angle Q_i and radial distance $r = r_\odot$, such that $\rho = \pi/2 - S_\odot$ and $\cos \rho \approx S_\odot \sin 1'$ and $\sin \rho \approx 1$, has the heliographic coordinates B_i and L_i given by the good approximation

$$\sin B_i \approx (S_\odot \sin 1') \sin B_0 + \cos (P - Q_i) \cos B_0 \tag{106}$$

$$\cos B_i \sin (L_i - L_0) \approx \sin (P - Q_i) \tag{107}$$

---

* r and $r_\odot$ are meant (see Figure E-2) as linear measures in some convenient scale on, say, a photograph.

11.0 TOPOCENTRIC LUNAR AND SOLAR QUANTITIES

11.1 TOPOCENTRIC RIGHT ASCENSION DECLINATION AND SEMI-DIAMETER OF MOON

The topocentric right ascension $\alpha'_{\leftmoon}$, declination $\delta'_{\leftmoon}$, and distance $r'_{\leftmoon}$ of the moon are given by the following exact relations

$$r'_{\leftmoon} \cos \delta'_{\leftmoon} \cos \alpha'_{\leftmoon} = r_{\leftmoon} \cos \delta_{\leftmoon} \cos \alpha_{\leftmoon} - \rho \cos \varphi' \cos \tau_{\ell_s} \tag{108}$$

$$r'_{\leftmoon} \cos \delta'_{\leftmoon} \sin \alpha'_{\leftmoon} = r_{\leftmoon} \cos \delta_{\leftmoon} \sin \alpha_{\leftmoon} - \rho \cos \varphi' \sin \tau_{\ell_s} \tag{109}$$

$$r'_{\leftmoon} \sin \delta'_{\leftmoon} = r_{\leftmoon} \sin \delta_{\leftmoon} \qquad \rho \sin \phi' \tag{110}$$

in which $\alpha_{\leftmoon}$ and $\delta_{\leftmoon}$ are the geocentric right ascension and declination, respectively; τ_{ℓ_s} is the local sidereal time (L.S.T.) at τ hours E.T. as given by (B4) and $r_{\leftmoon}$ is the geocentric lunar distance given by

$$r_{\leftmoon} = a_{\oplus}/\sin \pi_{\leftmoon} \tag{111}$$

where the lunar parallax $\pi_{\leftmoon}$ comes from (A45).

The topocentric lunar semi-diameter $S'_{\leftmoon}$ is gotten from

$$\sin S'_{\leftmoon} = \frac{r_{\leftmoon}}{r'_{\leftmoon}} \sin S_{\leftmoon} \tag{112}$$

where the sine of the geocentric lunar semi-diameter is given by (A48).*

11.2 TOPOCENTRIC RIGHT ASCENSION AND DECLINATION OF SUN**

The topocentric right ascension $\alpha'_{\odot}$ and declination $\delta'_{\odot}$ of the sun are given by the following approximate relations

$$\alpha'_{\odot} = \alpha_{\odot} - \Delta\alpha_{\odot} \tag{113}$$

$$\delta'_{\odot} = \delta_{\odot} - \Delta\delta_{\odot} \tag{114}$$

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* The difference $S'_{\leftmoon} - S_{\leftmoon}$ is called the augmentation.

** The semi-diameter of the sun is essentially indifferent to topocentric distance variations.

in which

$$\Delta \alpha_{\odot} = \pi_{\odot} \; [\rho \cos \phi' \sec \delta_{\odot} \sin (\tau_{\ell_s} - \alpha_{\odot})] \qquad (115)$$

$$\Delta\delta_{\odot} = \pi_{\odot} \; [\rho \sin \phi' \cos \delta_{\odot} - \rho \cos \phi' \sin \delta_{\odot} (\tau_{\ell_s} - \alpha_{\odot})] \qquad (116)$$

where the solar parallax $\pi_{\odot}$ is given by (A22).

11.3 SOLAR OR LUNAR AZIMUTH AND ELEVATION; SOLAR-LUNAR ANGULAR DISTANCE

The azimuth Az (measured east from north) and the elevation Eℓ of either the sun or moon can be calculated from

$$\cos E\ell \cos Az = \sin \delta' \cos \phi - \cos \delta' \sin \phi \cos (\tau_{\ell_s} - \alpha') \qquad (117)$$

$$\cos E\ell \sin Az = -\cos \delta' \sin (\tau_{\ell_s} - \alpha') \qquad (118)$$

$$\sin E\ell = \sin \delta' \sin \phi + \cos \delta' \cos \phi \cos (\tau_{\ell_s} - \alpha') \qquad (119)$$

in which $(\tau_{\ell_s} - \alpha')$ is recognized as the hour angle of the body in question.

The solar-lunar angular distance (the great circle separation of the centers of the sun and moon) ρ'_1 is given by

$$\cos \rho'_1 = \sin \delta'_{\odot} \sin \delta'_{\text{☾}} + \cos \delta'_{\odot} \cos \delta'_{\text{☾}} \cos (\alpha'_{\odot} - \alpha'_{\text{☾}}) \qquad (120)$$

or from (D.71) by

$$\rho'_1 = \frac{2m}{L_1 + L_2} \; S_{\odot} \; . \qquad (121)$$

12.0 PREDICTION OF SUN SPOT IMMERSION/EMERSION

In this section, three alternative methods will be presented for determining those times at which a particular point on the solar surface (a sun spot, for example) is immersed by the leading edge and emerges from the trailing edge of the moon. The first method employs the geocentric right ascension and declination of the solar location as coordinates of a fictitious star and proceeds with a "standard" stellar occultation calculation. The second method is based on the fact that at immersion or emersion the separation of the point on the sun and lunar center is equal to the semi-diameter of the moon. The third alternative describes a cinematic simulation composed from a temporal sequence of "stills" and the consequent "visual" observation of not only the occultation of selected solar points but also the visual determination of other local circumstances as well.

12.1 "STANDARD" OCCULTATION

If the heliographic coordinates of the spot L_s, B_s are given, these must be converted into the geocentric coordinates $\rho_{1,s}$ and θ_s by means of the relations

$$\cos \rho_s = \cos B_0 \cos B_s \cos (L_s - L_0) + \sin B_0 \sin B_s \tag{122}$$

$$\sin \rho_s \sin (P - \theta_s) = \cos B_s \sin (L_s - L_0) \tag{123}$$

and

$$\sin (\rho_s + \rho_{1,s}) = \frac{\rho_{1,s}}{S_\odot} \tag{124}$$

The right ascension α_s and declination δ_s of the spot are calculated according to

$$\alpha_s \approx \alpha_\odot + \rho_{1,s} \sin \theta_s \sec \delta_\odot \tag{125}$$

$$\delta_s \approx \delta_\odot + \rho_{1s} \cos \theta_s . \tag{126}$$

Calculate the rectangular coordinates x, y, z of the moon (in units of the equatorial radius of the earth) with respect to that fundamental plane, whose axis is maintained parallel to the earth center-sun spot vector, from

$$x = r_{\leftmoon} [\cos \delta_{\leftmoon} \sin (\alpha_{\leftmoon} - \alpha_s)] \tag{127}$$

$$y = r_{\leftmoon} [\sin \delta_{\leftmoon} \cos \delta_s - \cos \delta_{\leftmoon} \sin \delta_s \cos (\alpha_{\leftmoon} - \alpha_s)] \tag{128}$$

$$z = r_{\leftmoon} [\sin \delta_{\leftmoon} \sin \delta_s + \cos \delta_{\leftmoon} \cos \delta_s \cos (\alpha_{\leftmoon} - \alpha_s)] \tag{129}$$

in which

$$r_{\mathbb{C}} = 1/\sin \pi_{\mathbb{C}} \tag{130}$$

Calculate the ephemeris hour angle μ_s of the axis from

$$\mu_s = \text{E.S.T.} - \alpha_s \tag{131}$$

Calculate the hourly variations $\dot{x}, \dot{y}, \dot{\mu}_s$ and $\dot{\delta}_s$ of the Besselian elements x, y, μ_s and δ_s, respectively.

Calculate the rectangular coordinates of the observer with respect to the fundamental plane from

$$\xi = \rho \cos \phi' \sin \textcircled{H}_s \tag{132}$$

$$\eta = \rho \sin \phi' \cos \delta_s - \rho \cos \phi' \sin \delta_s \cos \textcircled{H}_s \tag{133}$$

$$\zeta = \rho \sin \phi' \sin \delta_s + \rho \cos \phi' \cos \delta_s \cos \textcircled{H}_s \tag{134}$$

in which $\textcircled{H}_s$, the local hour angle of the axis, is given by

$$\textcircled{H}_s = \mu_s - \lambda - 1.002738\, \Delta T \tag{135}$$

Calculate the hourly variations of ξ and η from

$$\xi = \dot{\mu}_s \rho \cos \phi' \cos \textcircled{H}_s \tag{136}$$

$$\dot{\eta} = \dot{\mu}_s \xi \sin \delta_s \tag{137}$$

and calculate the following combinations

$$u = x - \xi;\; v = y - \eta;\; m^2 = u^2 + v^2 \tag{138}$$

$$\dot{u} = \dot{x} - \dot{\xi};\; \dot{v} = \dot{y} - \dot{\eta};\; n^2 = \dot{u}^2 + \dot{v}^2 \tag{139}$$

$$D = u\dot{u} + v\dot{v}\,. \tag{140}$$

In the spirit of Section 6.0 make an "educated guess" as to the occultation time T_0 and calculate (138), (139), and (140) at that time. Next calculate the correction τ to T_0 from

$$\tau = -\frac{D}{n^2} \pm \frac{0.2724880}{n} \left|\cos \psi_s\right| \tag{141}$$

in which

$$\sin \psi_s = \frac{1}{0.2724880} \left[\frac{u\dot{v} - v\dot{u}}{n} \right] ; \tag*{(142}$$

the negative sign is selected for immersion and the positive sign is selected for emersion. The occultation time is then given by $T_0 + \tau$ which can be used to recalculate (138), (139), and (140) in preparation for a second "run" through (141) and (142) if higher precision is required.

The position angles, measured eastward from the north point of the moon (i.e., from the hour circle passing through the lunar center), of immersion and emersion on the lunar limb are given by

$$P_s = N + \psi_s \tag{143}$$

where

$$\tan N = \dot{u}/\dot{v} \tag{144}$$

in which the sign of sin N is that of $\dot{u}$.

12.2 SUN SPOT-LUNAR CENTER SEPARATION METHOD

Given a spot having a position angle θ_s and a linear radial distance from the solar center of r_s, the spot's distance from the center of the moon $\mathcal{R}$ is given at an arbitrary time following first contact and prior to fourth contact by

$$\mathcal{R} = [\rho_{1,s}^2 + b^2 - 2\rho_{1,s}\, b \cos(\theta_s - Q_{\leftmoon})]^{\frac{1}{2}} \tag{145}$$

in which

$$\rho_{1,s} = \left(\frac{r_s}{r_\odot}\right) S_\odot \tag{146}$$

$$b = \left(\frac{2m}{L_1 + L_2}\right) S_\odot \tag{147}$$

and where $Q_{\leftmoon}$, the position angle of the lunar center, is given, as in (63), by

$$\tan Q_{\leftmoon} = u/v \qquad (148)$$

The condition for immersion or for emersion is given by the condition that the spot-lunar separation $\mathcal{R}$ is equal to the lunar radius $r_{\leftmoon}$, or that

$$\mathcal{R} = r_{\leftmoon} = \left(\frac{L_1 - L_2}{L_1 + L_2} \right) S_{\odot} \qquad (149)$$

Hence, in order to determine the times at which the spot is occulted and then reappears, $\mathcal{R}$ is calculated at appropriate intervals over the time span of interest, followed by inverse interpolation to those times when the condition expressed in (149) obtains.

Along these same lines, it is interesting to note that the heliographic longitude and latitude of that portion of the lunar limb outlined on the solar disc can be readily calculated at any time as follows. For values of θ such that $\rho_1 \leqslant S_{\odot}$, calculate

$$\rho_1 = b \cos(\theta - Q_{\leftmoon}) - [r_{\leftmoon}^2 - b^2 \sin^2(\theta - Q_{\leftmoon})]^{1/2} . \qquad (150)$$

The (ρ_1, θ) pairs, so calculated can then be employed via (146), (104), (102) and (103) to determine the corresponding (L, B) pairs of the lunar limb outline.

12.3 CINEMATIC SIMULATION METHOD

In this method the eclipse is cinematically simulated by a temporal sequence of "stills" which can be drawn following calculation of the solar and lunar, topocentric right ascensions and declinations (or azimuths and elevations) and of the lunar semi-diameter (the solar semi-diameter assumed constant). If topocentric right ascensions and declinations are employed, the north point of the sun is easily identified by means of the hour circle "grid." If azimuths and elevations are employed, on the other hand, the north point is fixed by constructing the parallel of declination tangent to the "northern" limb of the sun from the apparent diurnal motion of the sun and by noting that the north point on each "still" is this tangent point. Then, from the appropriately scaled value of r_s (where the scale is determined by the selected right ascension/declination or azimuth/elevation grid) and the value of θ_s, the spot can be drawn on each "still's" solar disc and the occultation visualized.

13.0 MISCELLANEOUS ALGORITHMS

13.1 ANGLE OF SHADOW CONE SURFACE AT OBSERVATION SITE

The angle which the shadow (penumbral or umbral) cone surface makes at an observation site is, in fact, the angle which a generating ray of the cone (connecting the observation site to the appropriate contact point on the sun) makes at the observation site.

Thus, from the calculation of the ith contact position angle Q_i in (63) and the fact that $\rho_{1,i}$, the topocentric angle subtended at the earth by the contact point and the solar center, is the solar semi-diameter $S_\odot$, the topocentric right ascension and declination of the point are given, following (125) and (126), by

$$\alpha_i' = \alpha_\odot' + S_\odot \sin Q_i \sec \delta_\odot' \tag{151}$$

$$\delta_i' = \delta_\odot' + S_\odot \cos Q_i \, , \tag{152}$$

where $\alpha'_\odot$ and $\delta'_\odot$ are the topocentric right ascension and declination of the sun calculated in (113) and (114). The azimuth and elevation of the point, and thus of the shadow cone surface, are then readily calculated from (117), (118) and (119).

13.2 CENTROID OF THE UNOBSCURED SOLAR DISC

Pointing instructions for tracking the centroid of the unobscured solar disc follow directly from the development in Section D.9 of Appendix D. There (the solar semi-diameter being employed as the unit of distance), (D.83) gives as the angle subtended at the observation site by the centroid and the solar center

$$\rho_{1,c} = \frac{-s^2 (A - \sin A \cos A)}{\pi (1 - S')} \left(\frac{2m}{L_1 + L_2} \right) S_\odot \tag{153}$$

where the symbols are those of Section 9.2. Furthermore, the position angle θ_c is given by

$$\theta_c = Q_{\mathfrak{C}} - \pi \tag{154}$$

where $Q_{\mathfrak{C}}$, the position angle of the solar-lunar center line, is given by (148). Derivation of pointing instructions from (153) and (154) now follows the discussion of Section 13.1.

13.3 COORDINATES OF BORE-SIGHT VECTOR/SPHEROID "LAYER" INTERSECTION POINT

The following iterative procedure has been designed to calculate the latitude and longitude of that point in a "layer" at height h_L (in meters) above the earth spheroid, which is bore-sighted along a line of azimuth Az and elevation $E\ell$ from a location at latitude ϕ, longitude λ and height h (in meters) above the spheroid.

Step I: Calculate

$$X = (\rho \cos \phi') a_{\oplus} \qquad (155)$$

$$Y = 0 \qquad (156)$$

$$Z = (\rho \sin \phi') a_{\oplus} \qquad (157)$$

from (43) and (44) and calculate

$$\rho = [X^2 + Y^2 + Z^2]^{1/2} \qquad (158)$$

Step II: Calculate the nth approximation of slant range $\mathcal{R}_L$ from

$$\mathcal{R}_L^{(n)} = \mathcal{R}_L^{(n-1)} + \Delta^{(n-1)} \qquad (159)$$

where

$$\mathcal{R}_L^{(0)} = [(\rho + 0.15678503 \times 10^{-6} h_L)^2 - \rho^2 \cos^2 E\ell]^{1/2} - \rho \sin E\ell \qquad (160)$$

$$\Delta^{(0)} = 0 \qquad (161)$$

Step III: Calculate the nth approximation to the intersection point coordinates from

$$X_L^{(n)} = X + \mathcal{R}_L^{(n)} [- \cos E\ell \cos Az \sin \phi + \sin E\ell \cos \phi] \qquad (162)$$

$$Y_L^{(n)} = Y + \mathcal{R}_L^{(n)} [\cos E\ell \sin Az] \qquad (163)$$

$$Z_L^{(n)} = Z + \mathcal{R}_L^{(n)} [\cos E\ell \cos Az \cos \phi + \sin E\ell \sin \phi] \qquad (164)$$

and the nth approximation to its geocentric radius from

$$\rho_L^{(n)} = [(X^{(n)})^2 + (Y^{(n)})^2 + (Z^{(n)})^2]^{1/2} \qquad (165)$$

Step IV: Calculate the nth approximation to the geocentric latitude of the intersection point $\phi_L'^{(n)}$ from

$$\phi_L'^{(n)} = \tan^{-1}\left\{\frac{Z_L^{(n)}}{[(X_L^{(n)})^2 + (Y_L^{(n)})^2]^{1/2}}\right\} \tag{166}$$

Step V: Calculate the nth approximation to the geodetic latitude of the intersection point $\phi_L^{(n)}$ from

$$\phi_L^{(n)} = \tan^{-1}\left\{\frac{\tan \phi_L'^{(n)}}{0.99330 54 + 1.1 \times 10^{-9} h_L}\right\} \tag{167}$$

Step VI: Calculate the nth approximation to the geocentric radius of the sub-intersection point $\rho_P^{(n)}$ from

$$\begin{aligned}\rho_P^{(n)} = {} & 0.99832707 + 0.001167644 \cos 2\phi_L^{(n)} \\ & - 0.00000352 \cos 4\phi_L^{(n)}\end{aligned} \tag{168}$$

and calculate the nth approximation to the geocentric latitude of the sub-intersection point $\phi_P'^{(n)}$ from

$$\phi_P'^{(n)} = \tan^{-1}[0.9933054 \tan \phi_L^{(n)}] \tag{169}$$

Step VII: Calculate nth approximation to layer height $h_L^{(n)}$ from

$$\begin{aligned}h_L^{(n)} = {} & [(\rho_L^{(n)})^2 - (\rho_P^{(n)})^2 \sin^2(\phi_L^{(n)} - \phi_P'^{(n)})]^{1/2} \\ & - \rho_P^{(n)} \cos(\phi_L^{(n)} - \phi_P'^{(n)})\end{aligned} \tag{170}$$

Step VIII: Calculate $\mathcal{H}^{(n)} = h_L - h_L^{(n)}$. If $\mathcal{H}^{(n)} \geqslant \epsilon$, where ϵ is preselected, calculate

$$\Delta^{(n+1)} = \frac{\mathcal{H}^{(n)}}{\sin[EL + (\phi - \phi_L^{(n)})]} \tag{171}$$

and return to Step II for (n + 1)st approximation; if $\mathcal{H}^{(n)} < \epsilon$ proceed to Step IX.

Step IX: Calculate longitude change from nth approximation as

$$\delta\lambda^{(n)} = \tan^{-1} \frac{Y_L^{(n)}}{X_L^{(n)}} \qquad (172)$$

$\delta\lambda > 0$ is in an eastward direction. Hence, if original longitude is west, write $\lambda_L = \lambda\,(W) - \delta\lambda^{(n)}$; if east, write $\lambda_L = \lambda\,(E) + \delta\lambda^{(n)}$.

Step X: Final results, assuming nth approximation satisfactory, are:

$$\phi_L = \phi_L^{(n)};\ \lambda_L = \lambda_L \pm \delta\lambda^{(n)};\ \mathcal{R}_L = \mathcal{R}_L^{(n)}$$

13.4 COORDINATES OF BORE-SIGHTED OBJECT OF KNOWN SLANT RANGE

The following iterative procedure has been designed to calculate the latitude, longitude and height above the spheroid of an object with azimuth Az, elevation Eℓ, and slant range $\mathcal{R}$ observed from a location at latitude ϕ, longitude λ, and height h (in meters) above the spheroid.

Step I: Calculate

$$X = (\rho \cos\phi')\, a_{\oplus} + \mathcal{R}\,[-\cos E\ell \cos Az \sin\phi + \sin E\ell \cos\phi] \qquad (173)$$

$$Y = \mathcal{R}\,[\cos E\ell \sin Az] \qquad (174)$$

$$Z = (\rho \sin\phi')\, a_{\oplus} + \mathcal{R}\,[\cos E\ell \cos Az \cos\phi + \sin E\ell \sin\phi] \qquad (175)$$

Step II: Calculate the longitude change $\delta\lambda$ from

$$\delta\lambda = \tan^{-1} \frac{Y}{X} \qquad (176)$$

and, as in Step IX of preceding section, note that if original longitude is west, write $\lambda = \lambda(W) - \delta\lambda$; if east, write $\lambda = \lambda(E) + \delta\lambda$.

Step III: Calculate the geocentric latitude of object ϕ_0' from

$$\phi_0' = \tan^{-1} \left\{ \frac{Z}{[X^2 + Y^2]^{1/2}} \right\} \qquad (177)$$

and calculate the geocentric distance of object r_0 from

$$r_0 = [X^2 + Y^2 + Z^2]^{1/2} \tag{178}$$

Step IV: Calculate the nth approximation to geocentric latitude of sub-object point $\phi_s'^{(n)}$ from

$$\phi_s'^{(n)} = \phi_0' - \Delta^{(n-1)} \tag{179}$$

where $\Delta^{(0)} = 0$.

Step V: Calculate the nth approximation to the geodetic latitude of the sub-object point $\phi_s^{(n)}$ from

$$\phi_s^{(n)} = \tan^{-1}\left[\frac{\tan \phi_s'^{(n)}}{0.9933054}\right] \tag{180}$$

Step VI: Calculate the nth approximation to the geocentric radius of the sub-object point from

$$\begin{aligned}\rho_s^{(n)} = {} & 0.99832707 + 0.00167\,644 \cos 2\phi_s^{(n)} \\ & - 0.00000352 \cos 4\phi_s^{(n)}\end{aligned} \tag{181}$$

Step VII: Calculate the nth approximation to the height $h_s^{(n)}$ from

$$\begin{aligned}h_s^{(n)} = {} & [r_0^2 - (\rho_s^{(n)})^2 \sin^2 (\phi_s^{(n)} - \phi_s'^{(n)})]^{1/2} \\ & - \rho_s^{(n)} \cos (\phi_s^{(n)} - \phi_s'^{(n)})\end{aligned} \tag{182}$$

Step VIII: Calculate

$$\Delta^{(n+1)} = \sin^{-1}\left[\frac{h_s^{(n)} \sin (\phi_s^{(n)} - \phi_s'^{(n)})}{r_0}\right] \tag{183}$$

If $|\Delta^{(n)} - \Delta^{(n+1)}| \geq \epsilon$, return to Step IV for (n + 1)st approximation; if $|\Delta^{(n)} - \Delta^{(n+1)}| < \epsilon$, proceed to Step IX.

Step IX: Final results assuming nth approximation satisfactory are:

$\phi_0 = \phi_s^{(n)}$; $\lambda_0 = \lambda \pm \delta\lambda$; $h_0 = h_s^{(n)}$.

13.5 SOLAR/LUNAR VISIBILITY

The following simple criteria can be employed to determine whether the earth either partially or totally blocks the view of the sun or moon from a high-altitude platform (such as a rocket or satellite). We will assume that the right ascension α_p, declination δ_p, and geocentric radius r_p of the platform are known functions of time; that the earth spheroid can be approximated by a sphere whose radius is that of the actual earth spheroid at latitude 45° (i.e., 0.998331 $a_\oplus$); that the earth's atmosphere has the effect of increasing the earth radius by 2%; and that augmentation of the moon's semi-diameter is neglected. Further, let α, δ, S and π represent the right ascension, declination, semi-diameter and parallax, respectively, of either the sun or the moon.

Calculate

$$x = -\cos\delta_p \sin(\alpha_p - \alpha) \tag{184}$$

$$y = +\cos\delta \sin\delta_p - \sin\delta \cos\delta_p \cos(\alpha_p - \alpha), \tag{185}$$

and

$$\pi_p = \sin^{-1}\left(\frac{0.998331\ a_\oplus}{r_p}\right), \tag{186}$$

and

$$A_1 = 1.02\,(\pi_p + 0.998331\ \pi + S) \tag{187}$$

$$A_2 = 1.02\,(\pi_p + 0.998331\ \pi - S). \tag{188}$$

Then, if

$$x^2 + y^2 > \sin^2 A_1 \qquad \text{– no blocking}, \tag{189}$$

$$\sin^2 A_2 \leqslant x^2 + y^2 \leqslant \sin^2 A_1 \qquad \text{– partial blocking}, \tag{190}$$

$$x^2 + y^2 < \sin^2 A_2 \qquad \text{– total blocking.} \tag{191}$$

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APPENDICES

APPENDIX A

SOLAR AND LUNAR EPHEMERIDES

A.0 INTRODUCTION

This appendix presents algorithms by which geocentric coordinates of both the sun and the moon can be calculated given the date (Julian Day number) and ephemeris time (E.T.) of interest. These algorithms, in the case of the sun, are based primarily upon the "Tables" of Simon Newcomb[7] and, secondarily, upon the more recently published "Tables" of Jean Meeus[5]. In the case of the moon the algorithms are based primarily upon the "Improved Lunar Ephemeris"[3] and secondarily upon Meeus.

In Newcomb, the celestial longitude, latitude and distance of the sun is developed to high accuracy in a lengthy series of secular and periodic terms; in the "Improved Lunar Ephemeris" the celestial longitude, latitude and parallax of the moon is developed to high accuracy – following the Brown lunar theory – in an even more extensive series of secular and periodic terms. For the purposes of the current work, however, where the accuracy requirements are less demanding, conceptually modest, but numerically significant simplifications have been superimposed on these developments. These are:

a) All periodic terms with coefficients less than 0″.10 have been eliminated from both the solar and lunar longitude developments;

b) All periodic terms with coefficients less than 0″.025 have been eliminated from both the solar and lunar latitude developments;

c) All periodic terms with coefficients less than 20 units in the 8th decimal place have been eliminated from the solar distance development;

d) All periodic terms with coefficients less than 0″.005 have been eliminated from the lunar parallax development; and finally,

e) The corrections to coefficients of periodic terms given in Table IV of [3] have been eliminated.

In all other respects,* the developments of the solar and lunar ephemerides of this work are identical to those given in these fundamental references and thus could readily yield – at the expense of increased computational effort – the high accuracy associated with their presentation in *The American Ephemeris and Nautical Almanac* by restitution of the terms eliminated in a) through e).

A.1 PRELIMINARY CALCULATIONS: DATE AND TIME OF OBSERVATION

Given the Julian Day number of the date of observation J.D., calculate the number of days that have elapsed since January 0.5 E.T., 1900, the fundamental epoch with Julian Day number 2415020.0. This is denoted by d and is given by

$$d = \text{J.D.} - 2415020.0 \qquad \text{(A.1)}$$

Next, convert the Ephemeris Time of the observation into the decimal fraction of a day by means of the relation

$$\tau = \frac{\text{E.T.}}{86{,}400} \qquad \text{(A.2)}$$

in which E.T., the ephemeris time of the observation, is expressed in seconds.

Using (A.1) and (A.2), calculate the fraction of a Julian century of 36,525 days corresponding to the interval between the fundamental epoch and the date and time of observation by means of

$$T = \frac{t}{36{,}525} \qquad \text{(A.3)}$$

where

$$t = d + \tau \qquad \text{(A.4)}$$

---

* For multi-component terms, the retention of any component justifies retention of the entire, multi-component term.

A.2 PRELIMINARY CALCULATIONS: FUNDAMENTAL ARGUMENTS

Calculate the fundamental arguments L, Ω, L', V, J, M, S_n, T_e, L_0, ℓ, ℓ', F and D of Table A-1, in which

L .=. geocentric mean longitude of the moon
Ω .=. mean longitude of the moon's ascending node
L' .=. geocentric mean longitude of the sun
V .=. heliocentric mean longitude of Venus*
J .=. heliocentric mean longitude of Jupiter
M .=. heliocentric mean longitude of Mars
S_n .=. heliocentric mean longitude of Saturn
T_e .=. heliocentric mean longitude of Earth
L_0 .=. modified geocentric mean longitude of the moon
ℓ .=. mean anomaly of the moon
ℓ' .=. mean anomaly of the sun
F .=. mean distance of the moon from the ascending node
D .=. mean elongation of the moon from the sun.

In Table A-1 all of these arguments are given in the form $a + bt + ct^2 + dt^3$; t is given for the date and time of observation by (A.4) and the coefficients a, b, c and d are given in terms of revolutions** (denoted by a superscript, lower case r).

A.3 PRELIMINARY CALCULATIONS: NUTATION AND THE OBLIQUITY

A.3.1 Calculation of Nutation in Longitude and Nutation in Obliquity

Calculate the nutation in longitude $\Delta\psi$ from

$$\begin{aligned}\Delta\psi = {} & (-17''.2327 - 0''.01737T)\sin\Omega + \\ & + (0''.2086 + 0''.00002\,T)\sin 2\Omega + \\ & + (-1''.2729 - 0''.00013\,T)\sin(2\Omega + 2F - 2D) + \\ & + (0''.1261 - 0''.00031\,T)\sin\ell' + \\ & + (-0''.2037 - 0''.00002\,T)\sin(2\Omega + 2F)\ , \end{aligned} \qquad (A..5)$$

---

* V, J, M, S_n, T_e, and L_0 refer to the fixed equinox of 1850 in contradistinction to all other arguments which refer to the mean equinox of date; cf [3] p. 288 and [5] p. 14 regarding this point.

** We note that: $1^\circ = 0^r.00277\,77778$; $1' = 0^r.00004\,62963$; $1'' = 0^r.00000\,07716$.

TABLE A-1

FUNDAMENTAL ARGUMENTS

	a	b	c x 10^{20}	d x 10^{25}
L	0^r.75120 601080	+ 0^r.03660 11014 63356	− 235980^r	+ 1077^r
Ω	.71995 354167	− 14 70942 28332	+ 432630	+ 1266
L'	.77693 521605	+ 273 79092 64963	+ 63044	−
V	.95019 202160	+ 445 03624 51095		
J	.65931 199845	+ 23 08089 70898		
M	.81402 687500	+ 145 56470 68007		
S_n	.73852 641203	+ 9 29437 29984		
T_e	.27499 653549	+ 273 78030 94025		
L_0	.74926 733024	+ 3660 09952 92418		
ℓ	.82251 280093	+ 3629 16456 84716	+ 1913865	+ 8203
ℓ'	.99576 620370	+ 273 77785 19279	− 31233	− 1900
F	.03125 246914	+ 3674 81956 91688	− 668609	− 190
D	.97427 079475	+ 3386 31921 98393	− 299023	+ 1077

and calculate the nutation in obliquity from

$$\begin{aligned}\Delta\epsilon &= (9''.2100 + 0''.00091T)\cos\Omega + \\ &+ (-0''.0904 + 0''.00004\,T)\cos 2\Omega + \\ &+ (0''.5522 - 0''.00029\,T)\cos(2\Omega + 2F - 2D) + \\ &+ (0''.0884 - 0''.00005\,T)\cos(2\Omega + 2F)\end{aligned} \qquad (A.6)$$

in which all coefficients are given in arc seconds. It should be noted in passing that 64 smaller terms associated with $\Delta\psi$ and 36 smaller terms associated with $\Delta\epsilon$ in the complete developments of these quantities (cf. [2] pp. 44 and 45) have been eliminated.

A.3.2 Calculation of Obliquity

Calculate the mean obliquity of the ecliptic ϵ_M from

$$\epsilon_M = 23^\circ 27' 08''.26 - 46''.845T - 0''.0059T^2 + 0''.00181T^3 \qquad (A.7)$$

Calculate the true obliquity of the ecliptic ϵ from

$$\epsilon = \epsilon_M + \Delta\epsilon \qquad (A.8)$$

A.4 SOLAR RADIUS VECTOR, LONGITUDE AND LATITUDE

A.4.1 Calculation of Long Period Inequalities

Calculate the long period inequalities from

$$\begin{aligned}\delta L' &= 6''.40 \sin(231^\circ.19 + 20^\circ.2T) + \\ &+ (1''.882 - 0''.016T)\sin(57^\circ.24 + 150^\circ.27T) + \\ &+ 0''.266 \sin(31^\circ.8 + 119^\circ.0T) + \\ &+ 0''.202 \sin(315^\circ.6 + 893^\circ.3T)\end{aligned} \qquad (A.9)$$

Apply long period inequalities to the geocentric mean longitude of sun L' and mean anomaly of sun ℓ' to yield

$$L'_\odot = L' + \delta L' \qquad (A.10)$$

and

$$\ell'_\odot = \ell' + \delta L' \qquad (A.11)$$

A.4.2 Calculation of the Solar Radius Vector

Calculate the unperturbed solar radius vector R_u from

$$
\begin{aligned}
R_u \;=\;& 1.0001\,4033 \quad 0.0000\,0070\,T\; + \\
+\;& (\;0.0167\,4928 + 0.0000\,4179\,T + 0.0000\,0126\,T^2)\cos \ell'_c\; + \\
+\;& (\;0.0001\,4027 + 0.0000\,00\,70\,T)\cos 2\ell'_c \\
+\;& (\;0.0000\,0176 + 0.0000\,0001\,T)\cos 3\ell'_c
\end{aligned}
\qquad (A.12)
$$

Calculate the planetary perturbations in the solar radius vector from the cosine terms presented in Table A-2. Each term is of the form

$$K \cos (i\ell'_c + j\,[\mathrm{Te} - \mathrm{Planet}] + \mathrm{Angle})$$

in which the coefficient K is given in units of the 8th aecimal place, i and j are tabulated positive or negative integers, Planet stands for V, M, J or S_n, and Angle is the tabulated phase angle in degrees. The sum of these planetary terms is denoted by ΔR_p.

Calculate the lunar perturbations in the solar radius vector ΔR_ℓ from

$$
\begin{aligned}
\Delta R_\ell =\;& 3076 \cos D + 85 \cos (D + \ell) - 306 \cos (D - \ell)\; - \\
-\;& 32 \cos (D + \ell'_c) + 83 \cos (D - \ell'_c)
\end{aligned}
\qquad (A.13)
$$

where the coefficients are given in units of the 8th decimal place.

Calculate the solar radius vector R from the expression

$$R = R_u\,(1 + \Delta R_p + \Delta R_\ell). \qquad (A.14)$$

A.4.3 Calculation of the Apparent Solar Longitude

Calculate the equation of the center from the expression

$$
\begin{aligned}
C \;=\;& (6910''.057 - 17''.240\,T - 0''.052\,T^2)\sin \ell'_c\; + \\
+\;& (72''.338 - 0''.361\,T)\sin 2\ell'_c\; + \\
+\;& (1''.054 - 0''.001\,T)\sin 3\ell'_c
\end{aligned}
\qquad (A.15)
$$

TABLE A-2

PLANETARY PERTURBATIONS IN SOLAR RADIUS VECTOR

Coefficient*	ℓ'_c	$(T_e - V)$	Angle
543.2	0	− 1	180°
1575.4	0	− 2	0.12
204.6	0	− 3	0.27
86.6	0	− 4	− 0.11
37.5	0	− 5	−
21.6	+ 1	− 1	175.9
36.8	+ 1	− 2	0.2
200.1	− 1	− 2	168.5
344.7	− 1	− 3	167.95
45.1	− 1	− 4	348.8
44.7	− 2	− 3	322.2
21.6	− 2	− 4	138.4
32.5	− 2	− 5	319.9
		$(T_e - M)$	
34.5	0	+ 1	0°.6
473.6	0	+ 2	− 0.3
38.7	0	+ 3	+182.3
34.8	− 1	+ 2	+ 40.9
49.5	− 1	+ 3	+227.8
110.1	− 1	+ 4	226.92
24.6	− 1	+ 5	49.3
24.2	− 2	+ 4	277.6
20.5	− 2	+ 5	95.7
32.0	− 2	+ 6	94.9
		$(T_e - J)$	
1627.3	0	+ 1	1°.10
927.0	0	+ 1	180.22
64.7	0	+ 3	175.9
47.9	+ 1	+ 1	23.6
23.7	+ 1	+ 2	173.6
56.2	− 1	+ 1	250.2
336.0	− 1	+ 2	202.58
184.9	− 1	+ 3	87.23
40.1	− 2	+ 3	103.3
26.0	− 2	+ 4	353.9
		$(T_e - S_n)$	
98.8	0	+ 1	0°.36
37.3	0	+ 2	180.1
25.8	− 1	+ 2	182.6

*In units of the 8th decimal place.

Calculate the planetary perturbations in the solar longitude from the *cosine* terms presented in Table A-3 in which the coefficient of each term is given in arc seconds; the sum of these *cosine* terms is denoted by ΔL_P.

Calculate the lunar perturbations in the solar longitude ΔL_ℓ from

$$\Delta L_\ell = 6''.454 \sin D + 0''.177 \sin (D + \ell) - 0''.424 \sin (D - \ell) + 0''.172 \sin (D - \ell'_c) \qquad (A.16)$$

Calculate the solar longitude referred to the mean equinox of date L'_m from

$$L'_M = L'_c + C + \Delta L_P + \Delta L_\ell \qquad (A.17)$$

Calculate the apparent solar longitude (referred to the true equinox of date and corrected for aberration) from

$$\lambda' = L'_M + \Delta \psi - 20''.496/R. \qquad (A.18)$$

A.4.4 Calculation of the Apparent Solar Latitude

Calculate the planetary perturbations in the solar latitude from the cosine terms presented in Table A-4 in which the coefficient of each term is given in arc seconds; the sum of these terms is denoted by $\Delta\beta'_P$.

Calculate the lunar perturbations in the solar latitude $\Delta\beta'_\ell$ from

$$\Delta\beta'_\ell = 0''.576 \sin F - 0''.047 \sin (F - \ell). \qquad (A.19)$$

Calculate the apparent solar latitude from the following expression*

$$\beta' = \Delta\beta'_P + \Delta\beta'_\ell. \qquad (A.20)$$

---

* The expression (A.13) yields the latitude for the mean ecliptic of date directly; since the aberrative correction is negligible and the latitude is unaffected by nutation, the expression is also the apparent latitude.

TABLE A-3

PLANETARY PERTURBATIONS IN SOLAR LONGITUDE

Coefficient	ℓ'_c	$(T_e - V)$	Angle
4″.858	0	− 1	270°
5.526	0	− 2	90.12
0.666	0	− 3	90.41
0.210	0	− 4	89.8
0.116	+ 1	− 2	90.7
2.497	− 1	− 2	257.75
1.559	− 1	− 3	257.96
0.144	− 1	− 4	79.0
1.024	− 2	− 3	230.85
0.152	− 2	− 4	227.4
0.123	− 2	− 5	49.8
0.154	− 3	− 5	214.1
		$(T_e - M)$	
0″.273	0	+ 1	90°.6
2.043	0	+ 2	89.76
0.129	0	+ 3	273.0
1.770	− 1	+ 2	306.27
0.425	− 1	+ 3	317.70
0.500	− 1	+ 4	316.94
0.585	− 2	+ 4	185.82
0.204	− 2	+ 5	185.5
0.154	− 2	+ 6	185.0
0.101	− 3	+ 6	53.9
0.106	− 3	+ 7	53.3
		$(T_e - J)$	
7″.208	0	+ 1	91°.09
2.731	0	+ 2	270.25
0.164	0	+ 3	265.2
0.163	+ 1	+ 1	110.2
2.600	− 1	+ 1	174.77
1.610	− 1	+ 2	292.60
0.556	− 1	+ 3	177.31
0.210	− 2	+ 3	193.2
		$(T_e - S_n)$	
0″.419	0	+ 1	90°.34
0.108	0	+ 2	270.1
0.320	− 1	+ 1	259.22
0.112	− 1	+ 2	273.1

TABLE A-4

PLANETARY PERTURBATIONS IN SOLAR LATITUDE

Coefficient	ℓ'_c	$(T_e - V)$	Angle
0″.029	+ 1	− 1	296°
0.092	− 1	− 1	244.6
0.067	− 1	− 2	244.8
0.210	− 1	− 3	244.5
0.031	− 1	− 4	65.4
		$(T_e - J)$ + 2	
0″.166	− 1		268°.6

A.5 SOLAR RIGHT ASCENSION, DECLINATION PARALLAX AND SEMI-DIAMETER

A.5.1 Calculation of Solar Right Ascension and Declination

Calculate the solar right ascension $\alpha_\odot$ and declination $\delta_\odot$ from

$$\begin{aligned} \cos\delta_\odot \cos\alpha_\odot &= \cos\lambda' \\ \cos\delta_\odot \sin\alpha_\odot &= \sin\lambda' \cos\epsilon - 19.29\,\beta' \times 10^{-7} \\ \sin\delta_\odot &= \sin\lambda' \sin\epsilon + 44.48\,\beta' \times 10^{-7}. \end{aligned} \tag{A.21}$$

A.5.2 Calculation of Solar Parallax and Semi-Diameter

Calculate the solar parallax $\pi_\odot$ from

$$\pi_\odot = \pi_0/R = 8''.794/R. \tag{A.22}$$

Calculate the solar semi-diameter *appropriate for eclipse calculations** from

$$S_\odot = S_0/R = 15'59''.63/R \tag{A.23}$$

A.6 LUNAR LONGITUDE, LATITUDE AND PARALLAX

A.6.1 Calculation of Additive Terms

Calculate the additive terms** of Table A-5; these are designated as:

δL .=. the sum of the 8 sine terms in L,
$\delta\tilde{\omega}$.=. the sum of the 6 sine terms in $\tilde{\omega}$,
$\delta\Omega$.=. the sum of the 5 sine terms in Ω,
$\delta T_c \equiv \delta\ell'$.=. the sum of the 4 sine terms in T_c, ℓ',***
δJ .=. the single sine term of J,
δS_n .=. the single sine term of S_n,
$\delta\gamma_c$.=. the sum of the 3 cosine terms in γ.

---

* For other than eclipse calculations, the adopted value of the semi-diameter at unit distance S_0 is $16'01''.18$.

** Each term is of the form $K \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(a + bt + ct^2)$ with its coefficient K listed both in arc seconds and revolutions; when applied to the fundamental arguments of Table A-1, the latter unit is the more useful.

*** These are, in fact, the long period inequalities of (A.9).

TABLE A-5

ADDITIVE TERMS

Add to:	Serial No.	Coefficient	Coefficient x 10^{12}	a	b x 10^{12}	c x 10^{16}
L	1628	+ 0″.84	64 8148^{r}	0^{r}.1422 2222	+ 153 6238^{r}	+ 191^{r}
L	1629	+ 0.31	23 9197	.2336 3774	+ 123 2723	+ 191
L	1636	+ 14.27	1101 0802	.5373 3431	− 1010 4982	+ 43
L	1638	+ 7.261	560 2623	.7199 5354	− 1 4709 4228	+ 43
L	1639	+ 0.282	21 7592	.4839 8132	− 1 4726 9147	
L	1645	+ 0.237	18 2870	.8453 6324	− 1145 9387	
L	1646	+ 0.108	8 3333	.4035 3088	− 2148 8317	
L	1648	+ 0.126	9 7222	.6554 4893	− 7864 5335	
ϖ	1631	− 2″.10	162 0370^{r}	.1422 2222	+ 153 6238^{r}	
ϖ	1663	− 0.118	9 1049	.5373 3431	− 1010 4982	+ 191
ϖ	1664	− 2.076	160 1851	.7199 5354	− 1 4709 4228	+ 43
ϖ	1665	− 0.840	64 8148	.4839 8132	− 1 4726 9147	+ 43
ϖ	1666	− 0.10	7 7160	.5875 0000	+ 905 0118	
ϖ	1667	− 0.593	45 7562	.8453 6324	− 1145 9387	
Ω	1632	+ 0″.63	48 6111^{r}	.1422 2222	+ 153 6238^{r}	
Ω	1669	+ 0.17	13 1172	.5373 3431	− 1010 4982	+ 191
Ω	1670	+ 95.96	7404 3210	.7199 5354	− 1 4709 4228	+ 43
Ω	1671	+ 15.58	1202 1605	.4839 8132	− 1 4726 9147	+ 43
Ω	1672	+ 1.86	143 5185	.5245 3688	− 1 4716 2675	+ 43
Te, ℓ'	1633	− 6″.40	493 8271^{r}	.1422 2222	+ 153 6238^{r}	
Te, ℓ'	1673	− 0.27	20 8333	.5875 0000	+ 905 0118	
Te, ℓ'	1674	− 1.89	145 8333	.8453 6324	− 1145 9387	
Te, ℓ'	1675	+ 0.20	15 4321	.6104 3085	− 6771 8733	
γ	1676	− 4″.318	333 1790^{r}	.7199 5354	− 1 4709 4228^{r}	+ 43
γ	1677	− 0.698	53 8580	.4839 8132	− 1 4726 9147	+ 43
γ	1678	− 0.083	6 4043	.5245 3688	− 1 4716 2675	+ 43
J	1634	+ 0°.33	9 1666 6667^{r}	.3729 1667	+ 292 7979^{r}	
S_n	1635	− 0″.83	23 0555 5556^{r}	.3729 1667	+ 292 7979^{r}	

Apply these terms to the fundamental arguments as follows:

$$L_c = L + \delta L \tag{A.24}$$

$$\ell'_c = \ell' + \delta \ell'; \; T_{e_c} = T_e + \delta T_e \tag{A.25}$$

$$J_c = J + \delta J \tag{A.26}$$

$$S_{nc} = S_n + \delta S_n \tag{A.27}$$

$$\ell_c = \ell + \delta L - \delta \tilde{\omega} \tag{A.28}$$

$$F_c = F + \delta L - \delta \Omega \tag{A.29}$$

and

$$D_c = D + \delta L \tag{A.30}$$

A.6.2 Calculation of the Apparent Lunar Longitude

Calculate the *Code 0: Solar Terms in Longitude* from Table A-6. Each term is of the form

$$Kq \sin [i\,\ell_c + j\ell'_c + kF_c + m\,D_c]$$

in which the coefficient K in arc seconds is given in column 1, the multiples of the fundamental arguments $|i| \leqslant 6; j \leqslant 4; |k| \leqslant 5$ and $|m| \leqslant 8$ are given in columns 3 through 6, respectively, and

$$q = (1 + 2.208 \times 10^{-6})^{|i|} (1 - 6.832 \times 10^{-8} t)^{|j|}$$
$$(1 + 2.708 \times 10^{-6} + 139.978\, \delta \gamma_c)^{|k|} \tag{A.31}$$

where $\delta\gamma_c$ (expressed in revolutions) comes from Section 6.1. The sum of the 117 *sine* terms will be denoted by ΔL_s.

Calculate the *Code 0: Planetary Terms in Longitude* from the sine terms presented in the latter section of Table A-6. The sum of the 26 *sine* terms will be denoted by ΔL_P.

Calculate the lunar longitude referred to the mean equinox of date from

$$L_M = L_c + \Delta L_s + \Delta L_P \, . \tag{A.32}$$

Calculate the apparent lunar longitude (referred to the true equinox of date)* from

$$\lambda_A = L_M + \Delta\psi - 0''189 \sin \Omega + 0''168 \sin D. \tag{A.33}$$

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* To within the accuracy of this ephemeris, (A.33) also includes aberration; for higher accuracy, specific aberrative corrections listed in [2] p. 109 are required.

TABLE A-6

CODE 0: SOLAR TERMS IN LONGITUDE

Ser. No.	Coeff."	ℓ_c	ℓ'_c	F_c	D_c
1	+ 0.127	0	0	0	+6
2	+ 13.902	0	0	0	+4
3*	+ 2369.912	0	0	0	+2
5	+ 1.979	+1	0	0	+4
6*	+ 191.953	+1	0	0	+2
7	+22639.500	+1	0	0	0
8*	− 4586.465	+1	0	0	−2
9*	− 38.428	+1	0	0	−4
10	− 0.393	+1	0	0	−6
13	− 0.289	0	+1	0	+4
14*	− 24.420	0	+1	0	+2
15*	− 668.146	0	+1	0	0
16*	− 165.145	0	+1	0	−2
17	− 1.877	0	+1	0	−4
20	+ 0.403	0	0	0	+3
21*	− 125.154	0	0	0	+1
23	+ 0.213	+2	0	0	+4
24	+ 14.387	+2	0	0	+2
25*	+ 769.016	+2	0	0	0
26*	− 211.656	+2	0	0	−2
27	− 30.773	+2	0	0	−4
28	− 0.570	+2	0	0	−6
31	− 2.921	+1	+1	0	+2
32*	− 109.673	+1	+1	0	0
33*	− 205.962	+1	+1	0	−2
34	− 4.391	+1	+1	0	−4
38	+ 0.283	+1	−1	0	+4
39	+ 14.577	+1	−1	0	+2
40*	147.687	+1	−1	0	0
41	28.475	+1	−1	0	−2
42	0.636	+1	−1	0	−4
45	− 0.189	0	+2	0	+2
46	− 7.486	0	+2	0	0
47	− 8.096	0	+2	0	−2
48	− 0.151	0	+2	0	−4
52	− 5.741	0	0	+2	+2
53*	− 411.608	0	0	+2	0
54	− 55.173	0	0	+2	−2
58	− 8.466	+1	0	0	+1
59	+ 18.609	+1	0	0	−1
60	+ 3.215	+1	0	0	−3
63	+ 0.150	0	+1	0	+3
64	+ 18.023	0	+1	0	+1
65	+ 0.560	0	+1	0	−1

TABLE A-6 (Cont.)

Ser. No.	Coeff.*	ℓ_c	ℓ'_c	F_c	D_c
69	+ 1.060	+3	0	0	+2
70	+36.124	+3	0	0	0
71	−13.193	+3	0	0	−2
72	− 1.187	+3	0	0	−4
73	− 0.293	+3	0	0	−6
76	− 0.290	+2	+1	0	+2
77	− 7.649	+2	+1	0	0
78	− 8.627	+2	+1	0	−2
79	− 2.740	+2	+1	0	−4
83	+ 1.181	+2	−1	0	+2
84	+ 9.703	+2	−1	0	0
85	− 2.494	+2	−1	0	−2
86	+ 0.360	+2	−1	0	−4
89	− 1.167	+1	+2	0	0
90	− 7.412	+1	+2	0	−2
91	− 0.311	+1	+2	0	−4
94	0.757	+1	−2	0	+2
95	+ 2,580	+1	−2	0	0
96	+ 2.533	+1	−2	0	−2
98	− 0.103	0	+3	0	0
99	− 0.344	0	+3	0	−2
102	− 0.992	+1	0	+2	+2
103	−45.099	+1	0	+2	0
104	− 0.179	+1	0	+2	−2
105	− 0.301	+1	0	+2	−4
108	− 6.382	+1	0	−2	+2
109*	+39.528	+1	0	−2	0
110	+ 9.366	+1	0	−2	−2
111	+ 0.202	+1	0	−2	−4
115	+ 0.415	0	+1	+2	0
116	− 2.152	0	+1	+2	−2
118	− 1.440	0	+1	−2	+2
120	+ 0.384	0	+1	−2	−2
123	− 0.586	+2	0	0	+1
124	+ 1.750	+2	0	0	−1
125	+ 1.225	+2	0	0	−3
129	+ 1.267	+1	+1	0	+1
130	+ 0.137	+1	+1	0	−1
131	+ 0.233	+1	+1	0	−3
134	− 0.122	+1	−1	0	+1
135	− 1.089	+1	−1	0	−1
136	− 0.276	+1	−1	0	−3
143	+ 0.255	0	0	+2	+1
144	+ 0.584	0	0	+2	−1
145	+ 0.254	0	0	+2	−3
149	+ 1.938	+4	0	0	0

TABLE A-8 (Cont.)

Ser. No.	Coeff."	ℓ_c	ℓ'_c	F_c	D_c
150	− 0.952	+4	0	0	−2
155	− 0.551	+3	+1	0	0
156	− 0.482	+3	+1	0	−2
157	− 0.100	+3	+1	0	−4
162	+ 0.681	+3	−1	0	0
163	− 0.183	+3	−1	0	−2
167	− 0.297	+2	+2	0	−2
168	− 0.161	+2	+2	0	−4
172	+ 0.197	+2	−2	0	0
173	+ 0.254	+2	−2	0	−2
177	− 0.250	+1	+3	0	−2
186	− 0.123	+2	0	+2	+2
187	− 3.996	+2	0	+2	0
188	+ 0.567	+2	0	+2	−2
192	− 0.459	+2	0	−2	+2
193	− 1.298	+2	0	−2	0
194	+ 0.538	+2	0	−2	−2
195	+ 0.173	+2	0	−2	−4
198	+ 0.263	+1	+1	+2	0
205	+ 0.426	+1	+1	−2	−2
209	− 0.304	+1	−1	+2	0
213	− 0.372	+1	−1	−2	+2
224	+ 0.418	0	0	+4	0
227	+ 0.130	+3	0	0	−1
237	− 0.352	+2	−1	0	−1
264	+ 0.113	+5	0	0	0
295	− 0.330	+3	0	+2	0

CODE 0: PLANETARY TERMS IN LONGITUDE

Ser. No.	Coeff."	ℓ_c	D_c	Angle	T_{ec}	$Planet_c$
986	0.822	0	0	0°	+1	−1V
987	0.307	0	0	179.8	+2	−2V
1001	0.348	0	0	272.9	+3	−2V
1002	0.176	0	0	271.7	+4	−3V
1021	0.129	+1	0	180	−1	+1V
1022	0.152	+1	0	0	+1	−1V
1024	0.127	+1	0	180	+3	−3V
1061	0.136	0	+2	179.5	+2	−2V
1097	0.662	−1	+2	180	−3	+3V
1099	0.137	−1	+2	0	−2	+2V
1101	0.133	−1	+2	0	+1	−1V
1102	0.157	−1	+2	179.6	+2	−2V

TABLE A-6 (Cont.)

Ser. No.	Coeff."	ℓ_c	D_c	Angle	T_{ec}	Planet_c
1172	0.643	0	0	178°8	−1	+1J
1173	0.187	0	0	359.6	−2	+2J
1178	0.165	0	0	241.5	−1	+2J
1187	0.144	+1	0	1.0	+1	−1J
1188	0.158	+1	0	179.0	−1	+1J
1189	0.190	+1	0	180	−2	+2J
1208	0.167	0	+2	178.5	−1	+1J
1225	1.137	−1	+2	180.3	+2	−2J
1227	0.211	−1	+2	178.4	−1	+1J
1236	0.436	−1	+2	7.5	+2	−3J
1269	0.240	+2	−2	179.9	−2	+2J
1270	0.284	+2	−2	172.5	−2	+3J
1279	0.195	0	0	180°2	−2	+2M
1283	0.327	0	0	224.4	−1	+2M

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*In Tables A-6, A-7 and A-8, the coefficients of those terms whose serial numbers carry asterisks are subject to small corrections given in Table IV of [3]; these corrections have been eliminated in this report.

A.6.3 Calculation of the Apparent Lunar Latitude

Calculate the *Code 1 Solar Terms in Latitude*, S from Table A-7 where each term has the form

$$Kq \sin [i\ell_{☾} + j\ell'_{☾} + kF_{☾} + mD_{☾}]$$

and again q is given by (A.31). The sum of the 178 *sine* terms will be denoted by $\Delta\beta_s$.

Calculate

$$S = F_{☾} + \Delta\beta_s \qquad (A.34)$$

and calculate sin S, sin 3S and sin 5S.

Next, calculate

$$[\sin S] = \eta_1 \sin S \qquad (A.35)$$

$$[\sin 3S] = (\eta_1)^3 \sin 3S \qquad (A.36)$$

and

$$[\sin 5S] = (\eta_1)^5 \sin 5S \qquad (A.37)$$

where

$$\eta_1 = (1 + 2.708 \times 10^{-6} + 139.978\ \delta\gamma_{☾}) \qquad (A.38)$$

Calculate the *Code 2 Solar Terms in Latitude*, $\gamma_1 C$ from Table A-7 where each term has the form

$$Kq \cos [i\ell_{☾} + j\ell'_{☾} + kF_{☾} + mD_{☾}].$$

The sum of the 60 *cosine* terms will be denoted by $\gamma_1 C$.

Next calculate*

$$A = 18{,}519''.700 + \gamma_1 C. \qquad (A.39)$$

$$B = -33.6992 \times 10^{-5} A \qquad (A.40)$$

$$C = +216 \times 10^{-9} A \qquad (A.41)$$

$$D = 5.3996 \times 10^{-6} A. \qquad (A.42)$$

---

* We note in passing that $A = \gamma_1 + \gamma_1 C$, $B = \frac{\gamma_2}{\gamma_1} A$, $C = \frac{\gamma_3}{\gamma_1} A$ and $D = \frac{1}{\gamma_1} A$ where γ_1, γ_2 and γ_3 are the values of the coefficients given in Table A-7 *Code 6 Solar Terms in Latitude, Principal Terms* having Serial Numbers 797, 798 and 799, respectively

TABLE A-7

CODE 1: SOLAR TERMS IN LATITUDE, β

Ser. No.	Coeff."	ℓ_c	ℓ'_c	F_c	D_c
397	− 112.79	0	0	0	+1
398	+ 2373.36	0	0	0	+2
399	− 4.01	0	0	0	+3
400	+ 14.06	0	0	0	+4
401	− 0.13	0	0	0	+5
402	+ 0.60	0	0	0	+6
404	+ 0.25	+1	0	0	+6
406	+ 6.98	+1	0	0	+4
407	− 0.74	+1	0	0	+3
408	+ 192.72	+1	0	0	+2
409	− 13.51	+1	0	0	+1
410	+22609.07	+1	0	0	0
411	+ 3.59	+1	0	0	−1
412	− 4678.13	+1	0	0	−2
413	+ 5.44	+1	0	0	−3
414	− 38.64	+1	0	0	−4
415	+ 0.25	+1	0	0	−5
416	− 1.43	+1	0	0	−6
417	− 0.03	+1	0	0	−8
418	+ 0.03	+2	0	0	+6
419	+ 1.02	+2	0	0	+4
420	− 0.10	+2	0	0	+3
421	+ 14.78	+2	0	0	+2
422	− 1.20	+2	0	0	+1
423	+ 767.96	+2	0	0	0
424	+ 2.01	+2	0	0	−1
425	− 152.53	+2	0	0	−2
426	+ 0.91	+2	0	0	−3
427	− 34.07	+2	0	0	−4
428	+ 0.12	+2	0	0	−5
429	− 1.40	+2	0	0	−6
430	− 0.07	+2	0	0	−8
431	+ 0.16	+3	0	0	+4
432	+ 2.96	+3	0	0	+2
433	− 0.09	+3	0	0	+1
434	+ 50.64	+3	0	0	0
435	+ 0.19	+3	0	0	−1

TABLE A-7 (Cont.)

Ser. No.	Coeff."	ℓ_c	ℓ'_c	F_c	D_c
436	− 16.40	+3	0	0	−2
437	+ 0.05	+3	0	0	−3
438	− 0.74	+3	0	0	−4
439	+ 0.03	+3	0	0	−5
440	− 0.31	+3	0	0	−6
443	+ 0.30	+4	0	0	+2
444	+ 3.60	+4	0	0	0
445	− 1.58	+4	0	0	−2
447	− 0.03	+4	0	0	−6
448	+ 0.04	+5	0	0	+2
449	+ 0.28	+5	0	0	0
450	− 0.14	+5	0	0	−2
452	− 0.06	0	+1	0	+6
454	− 1.59	0	+1	0	+4
455	+ 0.53	0	+1	0	+3
456	− 25.10	0	+1	0	+2
457	+ 17.93	0	+1	0	+1
458	−126.98	0	+1	0	0
459	+ 0.32	0	+1	0	−1
460	−150.06	0	+1	0	−2
461	+ 0.29	0	+1	0	−3
462	− 6.46	0	+1	0	−4
463	− 0.22	0	+1	0	−6
464	− 0.04	0	+2	0	+4
465	− 1.68	0	+2	0	+2
466	− 0.04	0	+2	0	+1
467	− 0.66	0	+2	0	0
468	− 0.04	0	+2	0	−1
469	− 16.35	0	+2	0	−2
471	− 0.65	0	+2	0	−4
472	− 0.57	0	+2	0	−2
475	− 0.50	+1	+1	0	+4
476	+ 0.08	+1	+1	0	+3
477	− 11.75	+1	+1	0	+2
478	+ 1.52	+1	+1	0	+1
479	−115.18	+1	+1	0	0
480	− 0.12	+1	+1	0	−1
481	−182.36	+1	+1	0	−2
482	+ 0.36	+1	+1	0	−3
483	− 9.66	+1	+1	0	−4
485	− 0.37	+1	+1	0	−6
486	− 0.09	+2	+2	0	0
487	− 0.27	+2	+2	0	−2
488	− 0.16	+2	+2	0	−4
490	− 0.09	−1	+1	0	+6
492	− 2.27	−1	+1	0	+4

TABLE A-7 (Cont.)

Ser. No.	Coeff.*	ℓ_c	ℓ'_c	F_c	D_c
493	+ 0.38	−1	+1	0	+3
494	− 23.59	−1	+1	0	+2
495	− 0.55	−1	+1	0	+1
496	−138.76	−1	+1	0	0
497	+ 0.33	−1	+1	0	−1
498	− 31.70	−1	+1	0	−2
499	+ 0.04	−1	+1	0	−3
500	− 1.53	−1	+1	0	−4
501	− 0.06	−1	+1	0	−6
502	− 0.04	−2	+2	0	+4
503	− 0.21	−2	+2	0	+2
504	− 0.22	−2	+2	0	0
505	− 0.22	−2	+2	0	−2
506	− 0.07	+2	+1	0	+4
507	− 1.45	+2	+1	0	+2
508	+ 0.14	+2	+1	0	+1
509	− 10.56	+2	+1	0	0
511	− 7.59	+2	+1	0	−2
512	+ 0.07	+2	+1	0	−3
513	− 2.54	+2	+1	0	−4
514	− 0.25	+2	+1	0	−6
515	+ 0.22	+2	−1	0	+4
516	+ 3.32	+2	−1	0	+2
517	− 0.04	+2	−1	0	+1
518	+ 11.67	+2	−1	0	0
519	− 0.37	+2	−1	0	−1
520	− 1.17	+2	−1	0	−2
521	+ 0.04	+2	−1	0	−3
522	+ 0.20	+2	−1	0	−4
523	+ 0.06	+2	−1	0	−6
524	− 0.17	+3	+1	0	+2
526	− 0.94	+3	+1	0	0
527	− 0.67	+3	+1	0	−2
528	− 0.08	+3	+1	0	−4
529	− 0.06	+3	+1	0	−6
531	+ 0.36	+3	−1	0	+2
532	+ 0.96	+3	−1	0	0
533	− 0.23	+3	−1	0	−2
542	− 0.13	+1	+2	0	+2
543	− 1.25	+1	+2	0	0
544	− 6.12	+1	+2	0	−2
545	− 0.65	+1	+2	0	−4
546	− 0.03	+1	+2	0	−6
547	− 0.07	−1	+2	0	+4
548	− 2.40	−1	+2	0	+2
549	− 2.32	−1	+2	0	0

TABLE A-7 (Cont.)

Ser. No.	Coeff."	ℓ_c	ℓ'_c	F_c	D_c
550	− 1.82	−1	+2	0	−2
551	− 0.12	−1	+2	0	−4
553	− 0.22	+1	+3	0	−2
556	− 0.04	−1	+3	0	0
558	− 0.06	−1	+3	0	−2
560	− 0.04	0	0	+2	+2
561	− 0.20	0	0	+2	0
562	+ 0.84	0	0	+2	−1
563	− 52.14	0	0	+2	−2
564	+ 0.25	0	0	+2	−3
565	− 1.67	0	0	+2	−4
566	− 0.03	0	0	+2	−6
569	+ 0.07	+1	0	+2	−1
570	− 9.52	+1	0	+2	−2
571	+ 0.04	+1	0	+2	−3
572	− 0.33	+1	0	+2	−4
574	− 0.04	+1	0	+2	−6
577	− 0.71	−1	0	+2	+2
578	+ 0.06	−1	0	+2	+1
579	− 85.13	−1	0	+2	0
580	+ 0.04	−1	0	+2	−1
581	+ 3.37	−1	0	+2	−2
583	+ 0.04	−1	0	+2	−4
585	− 0.75	+2	0	+2	−2
588	− 1.14	−2	0	+2	+2
589	− 0.74	−2	0	+2	0
590	+ 0.38	−2	0	+2	−2
593	− 0.04	+3	0	+2	0
594	− 0.07	+3	0	+2	−2
595	− 0.08	−3	0	+2	+4
597	− 0.11	−3	0	+2	0
598	+ 0.04	−3	0	+2	−2
603	+ 0.10	0	+1	+2	0
604	− 2.26	0	+1	+2	−2
606	− 0.17	0	+1	+2	−4
607	+ 0.04	0	−1	+2	+2
608	+ 0.16	0	−1	+2	0
609	− 0.06	0	−1	+2	−1
610	+ 1.30	0	−1	+2	−2
611	+ 0.08	0	−1	+2	−4
612	− 0.09	0	+2	+2	−2
617	− 0.35	+1	+1	+2	−2
618	− 0.03	+1	+1	+2	−4
619	− 0.07	−1	−1	+2	+2
620	+ 0.31	−1	−1	+2	0
624	+ 0.03	+1	−1	+2	0

TABLE A-7 (Cont.)

Ser. No.	Coeff."	ℓ_c	ℓ'_c	F_c	D_c
625	+ 0.07	+1	−1	+2	−2
627	− 0.33	−1	+1	+2	0
629	+ 0.19	−1	+1	+2	−2

CODE 2: SOLAR TERMS IN LATITUDE, $\gamma_1{}^{C}$

Ser. No.	Coeff."	ℓ_c	ℓ'_c	F_c	D_c
641	− 0.725	0	0	0	+1
642	+ 0.601	0	0	0	+2
643	+ 0.394	0	0	0	+3
646	− 0.042	0	0	0	+6
650	− 0.445	−1	0	0	+4
651	+ 0.068	+1	0	0	+3
652	+ 0.029	+1	0	0	+2
653	+ 0.455	+1	0	0	+1
654	+ 0.079	+1	0	0	0
655	− 0.034	+1	0	0	−1
656	− 0.077	+1	0	0	−2
657	− 0.192	+1	0	0	−3
660	− 0.092	+1	0	0	−6
663	− 0.074	+2	0	0	+4
666	+ 0.054	+2	0	0	+1
667	+ 0.107	+2	0	0	0
669	+ 5.679	+2	0	0	−2
670	− 0.030	+2	0	0	−3
671	− 0.308	+2	0	0	−4
673	− 0.074	+2	0	0	−6
676	− 0.166	+3	0	0	+2
678	− 1.300	+3	0	0	0
680	+ 0.253	+3	0	0	−2
682	+ 0.042	+3	0	0	−4
687	− 0.145	+4	0	0	0
688	+ 0.052	+4	0	0	−2
696	+ 0.123	0	+1	0	+4
697	− 0.032	0	+1	0	+3
698	+ 0.040	0	+1	0	+2
700	− 1.302	0	+1	0	0
702	+ 0.054	0	+1	0	−2
703	+ 0.031	0	+1	0	−3
704	− 0.416	0	+1	0	−4
707	+ 0.131	0	+2	0	+2
708	− 0.037	0	+2	0	0
709	− 0.740	0	+2	0	−2
711	− 0.044	0	+2	0	−4
712	− 0.025	0	+3	0	−2
715	+ 0.041	+1	+1	0	+4

TABLE A-7 (Cont.)

Ser. No.	Coeff.″	ℓ_c	ℓ'_c	F_c	D_c
717	+ 0.787	+1	+1	0	+2
719	+ 0.461	+1	+1	0	0
721	+ 2.056	+1	+1	0	−2
723	− 0.371	+1	+1	0	−4
725	− 0.027	+1	+1	0	−6
731	+ 0.146	−1	+1	0	+4
733	− 0.443	−1	+1	0	+2
735	+ 0.670	−1	+1	0	0
737	− 1.540	−1	+1	0	−2
739	− 0.111	−1	+1	0	−4
744	+ 0.116	+2	+1	0	+2
746	+ 0.259	+2	+1	0	0
747	+ 0.078	+2	+1	0	−2
752	− 0.212	+2	−1	0	+2
753	− 0.151	+2	−1	0	0
760	+ 0.032	+3	+1	0	0
766	− 0.026	+3	−1	0	0
777	+ 0.117	+1	+2	0	−2
778	− 0.032	+1	+2	0	−4
782	+ 0.027	−1	+2	0	0
783	− 0.105	−1	+2	0	−2

CODE 3: SOLAR TERMS IN LATITUDE, N

787*	−526.069	0	0	+1	−2
788	− 3.352	0	0	+1	−4
789*	+ 44.297	+1	0	+1	−2
790	− 6.000	+1	0	+1	−4
791*	+ 20.599	−1	0	+1	0
792	− 30.598	−1	0	+1	−2
793	− 24.649	−2	0	+1	0
794	− 2.000	−2	0	+1	−2
795	− 22.571	0	+1	+1	−2
796	+ 10.985	0	−1	+1	−2

CODE 6: SOLAR TERMS IN LATITUDE, PRINCIPAL TERMS

797	+ 18518″.511 sin S
798	+ 1.189 sin S
799	− 6.241 sin 3S

TABLE A-7 (Cont.)

CODE 4: PLANETARY TERMS IN LATITUDE

Ser. No.	Coeff."	ℓ_c	F_c	D_c	L_o	Ω	Angle	T_{ec}	Planet_c
1428	+0.045	0	−1	+2	0	0	0°	−3	+3V
1437	+0.068	0	−1	+2	0	0	270	−6	+5V
1442	+0.029	−1	+1	+2	0	0	180	−3	+3V
1450	+0.031	−1	−1	+2	0	0	180	−3	+3V
1468	+0.027	0	0	0	+1	0	285	−1	+1V
1472	+0.077	0	0	0	+1	0	215.6	+5	−3V
1476	+0.025	0	0	0	+1	0	255	−6	+4V
1477	+0.074	0	0	0	+1	0	51.6	−5	+3V
1483	+0.030	0	0	0	+1	0	125	+8	−5V
1534	+0.051	+1	+1	−2	0	0	0°	−2	+2J
1535	+0.051	+1	−1	−2	0	0	0	−2	+2J
1546	+0.035	0	0	0	+1	0	168	0	+2J
1552	+0.083	0	+1	0	0	+2	0	0	0

Calculate the *Code 3: Solar Terms in Latitude, N* from Table A-7 where each term has the form

$$Kq \sin [i\ell_c + j\ell'_c + k F_c + m D_c].$$

The sum of the 10 *sine* terms will be denoted by N.

Calculate the *Code 4: Planetary Terms in Latitude* from Table A-7; the sum of the 13 *sine* terms (each similar in form to the planetary perturbations of Section A.4.2) will be denoted by $\Delta\beta_P$.

Calculate the apparent lunar latitude (referred to the true ecliptic of date) of the center of mass of the moon from

$$\beta = A [\sin S] + B [\sin 3S] + C [\sin 5s] + DN + \Delta\beta_P + \\ + 0''.215 \sin L \qquad (A.43)$$

For use in eclipse calculations only, calculate the apparent latitude of the center of figure of the moon β (F) from the expression

$$\beta (F) = \beta - 0''.6 \qquad (A.44)$$

A.6.4 CALCULATION OF THE LUNAR PARALLAX

Calculate the *Code 5: Solar Terms in Parallax* and the *Code 5: Planetary Terms in Parallax* from Table A-8. The sum of the 70 cosine solar terms and the 2 cosine planetary terms will be denoted by $\sin \pi$ (in arc seconds).

Calculate the horizontal parallax of the moon $\pi_{\leftmoon}$ from

$$\pi_{\leftmoon} = \sin \pi [0.9999\,53253 + (3.9168 \times 10^{-12} (\sin \pi)^2] - \\ - 0''.0890 - 0''.0049 \cos \ell. \qquad (A.45)$$

where $\pi_{\leftmoon}$ is in arc seconds.

A.7 LUNAR RIGHT ASCENSION, DECLINATION AND SEMI-DIAMETER

A.7.1 Calculation Lunar of Right Ascension and Declination

Calculate the lunar right ascension $\alpha_{\leftmoon}$ and declination $\delta_{\leftmoon}$ from

$$\begin{aligned} \cos \delta_{\leftmoon} \cos \alpha_{\leftmoon} &= \cos \beta \cos \lambda \\ \cos \delta_{\leftmoon} \sin \alpha_{\leftmoon} &= \cos \beta \sin \lambda \cos \epsilon - \sin \beta \sin \epsilon \\ \sin \delta_{\leftmoon} &= \cos \beta \sin \lambda \sin \epsilon + \sin \beta \cos \epsilon \end{aligned} \qquad (A.46)$$

TABLE A-8

CODE 5: SOLAR TERMS IN PARALLAX

Ser. No	Coeff."	ℓ_c	ℓ'_c	F_c	D_c
802	+ 0.2607	0	0	0	+ 4
803*	+ 28.2333	0	0	0	+ 2
804	+3422.7000	0	0	0	0
806	+ 0.0433	+ 1	0	0	+ 4
807	+ 3.0961	+ 1	0	0	+ 2
808	+ 186.5398	+ 1	0	0	0
809*	+ 34.3117	+ 1	0	0	− 2
810	+ 0.6008	+ 1	0	0	− 4
811	+ 0.0086	+ 1	0	0	− 6
813	− 0.0053	0	+ 1	0	+ 4
814	− 0.3000	0	+ 1	0	+ 2
815	− 0.3997	0	+ 1	0	0
816	+ 1.9178	0	+ 1	0	− 2
817	+ 0.0339	0	+ 1	0	− 4
820	− 0.9781	0	0	0	+ 1
821	+ 0.0054	+ 2	0	0	+ 4
822	+ 0.2833	+ 2	0	0	+ 2
823	+ 10.1657	+ 2	0	0	0
824	− 0.3039	+ 2	0	0	− 2
825	+ 0.3722	+ 2	0	0	− 4
826	+ 0.0109	+ 2	0	0	− 6
829	− 0.0484	+ 1	+ 1	0	+ 2
830	− 0.9490	+ 1	+ 1	0	0
831	+ 1.4437	+ 1	+ 1	0	− 2
832	+ 0.0673	+ 1	+ 1	0	− 4
834	+ 0.0060	+ 1	− 1	0	+ 4
835	+ 0.2302	+ 1	− 1	0	+ 2
836	+ 1.1523	+ 1	− 1	0	0
837	− 0.2257	+ 1	− 1	0	− 2
838	− 0.0102	+ 1	− 1	0	− 4
841	− 0.0085	0	+ 2	0	0
842	+ 0.0916	0	+ 2	0	− 2
845	− 0.0124	0	0	+ 2	0
846	− 0.1052	0	0	+ 2	− 2
849	− 0.1093	+ 1	0	0	+ 1
850	+ 0.0118	+ 1	0	0	− 1
851	− 0.0386	+ 1	0	0	− 3
854	+ 0.1494	0	+ 1	0	+ 1
858	+ 0.0243	+ 3	0	0	+ 2
859	+ 0.6215	+ 3	0	0	0
860	− 0.1187	+ 3	0	0	− 2
861	+ 0.0074	+ 3	0	0	− 4
864	− 0.0051	+ 2	+ 1	0	+ 2
865	− 0.1038	+ 2	+ 1	0	0
866	− 0.0192	+ 2	+ 1	0	− 2
867	+ 0.0324	+ 2	+ 1	0	− 4
870	+ 0.0213	+ 2	− 1	0	+ 2
871	+ 0.1268	+ 2	− 1	0	0
875	− 0.0106	+ 1	+ 2	0	0
876	+ 0.0484	+ 1	+ 2	0	− 2

TABLE A-8 (Continued)

Ser. No.	Coeff."	ℓ_c	ℓ'_c	F_c	D_c
880	+ 0.0112	+ 1	− 2	0	+ 2
881	+ 0.0196	+ 1	− 2	0	0
882	− 0.0212	+ 1	− 2	0	− 2
888	− 0.0833	+ 1	0	+ 2	− 2
892	− 0.0481	+ 1	0	− 2	+ 2
893	− 0.7136	+ 1	0	− 2	0
894	− 0.0112	+ 1	0	− 2	− 2
896	− 0.0066	0	+ 1	+ 2	− 2
900	− 0.0100	+ 2	0	0	+ 1
901	+ 0.0155	+ 2	0	0	− 1
902	− 0.0088	+ 2	0	0	− 3
905	+ 0.0164	+ 1	+ 1	0	+ 1
911	+ 0.0071	0	0	+ 2	− 1
914	+ 0.0401	+ 4	0	0	0
915	− 0.0130	+ 4	0	0	− 2
918	− 0.0097	+ 3	+ 1	0	0
923	+ 0.0115	+ 3	− 1	0	0
939	− 0.0090	+ 2	0	+ 2	− 2
941	− 0.0053	+ 2	0	− 2	+ 2
943	− 0.0141	+ 2	0	− 2	− 2

CODE 5: PLANETARY TERMS IN PARALLAX

Ser. No.	Coeff"	ℓ_c	D_c	Angle	T_{ec}	Planet_c
1580	+ 0.0055	− 1	+ 2	180°	− 3	+ 3V
1610	+ 0.0095	− 1	+ 2	180°.3	+ 2	− 2J

in which β is given either by (A.43) or (A.44) depending upon whether "usual" or eclipse calculations are contemplated.

A.7.2 Calculation of Lunar Semi-Diameter

Calculate the lunar semi-diameter (in arc seconds) from the expression

$$S_{☾} = 0.0799 + 0.272453\,\pi_{☾} \tag{A.47}$$

or from

$$\sin S_{☾} = 0.272488 \sin \pi_{☾} \tag{A.48}$$

APPENDIX B

EPHEMERIS SIDEREAL TIME

The Ephemeris Sidereal Time (i.e., the right ascension of the Ephemeris Meridian) at 0^h E.T. on the date of the observation is calculated from the expression

$$\begin{aligned} \text{E.S.T.}(0^h) &= 0^r\!.27691\ 939765 + 0^r\!.00273\ 79153\ 97842d + \\ &\quad + 80556^r \times 10^{-20}\ d^2 \qquad \text{(B.1)} \end{aligned}$$

where d is given by (A.1). The increment of sidereal time associated with the interval extending from 0^h E.T. to the time of observation is given by

$$\Delta\ \text{E.S.T.} = \{1^r\!.00273\ 79092\ 65 + 1^r\!.6126 \times 10^{-18}\ d\}\ \tau \qquad \text{(B.2)}$$

where τ is defined by (A.2).

The "equation of the equinoxes" ($\Delta\psi$ cos ϵ) is added next to the sum of (B.1) and (B.2) to yield the apparent ephemeris sidereal time on the date and at the (ephemeris) time of observation

$$\text{E.S.T.} = \text{E.S.T.}\ (0^h) + \Delta\ \text{E.S.T.} + 0.9174\ \Delta\psi \qquad \text{(B.3)}$$

where $\Delta\psi$ is given by (A.5).

Finally, for an observer at longitude λ (West positive; East negative) the local sidereal time (L.S.T.) at τ hours E.T. is given by

$$\text{L.S.T} = \text{E.S.T.} - \lambda. \qquad \text{(B.4)}^{*}$$

---

[*] We note that the Ephemeris Sidereal Time of (B.3) at τ hours Ephemeris Time is numerically equal to Greenwich Sidereal Time (i.e., the right ascension of the Greenwich Meridian) at τ hours Universal Time; the same comment obtains for (B.4) also.

APPENDIX C

PREDICTION OF SHADOW OUTLINE AND MOTION ON THE SPHEROID

C.0 INTRODUCTION

This appendix develops the algorithms necessary to predict the outline and motion of the moon's shadow on an earth spheroid of arbitrary radius and flattening. The development is based upon Chauvenet's expositon of Bessel's original treatment [1] and later summarized authoritatively in "The Explanatory Supplement" [2].

The development is being repeated here in detail because the summary of requisite formula in [2] with an ordering quite different from that of [1] lacks both the justification and seeming inevitability with which elements related to the flattened spheroid and the shadow motion thereon were originally introduced. Furthermore, it seems appropriate to show (in view of today's interest in upper atmospheric and ionospheric research and the employment of high altitude and orbiting instrument platforms) that the relationships pertaining to center line position and duration in the ionosphere – originally cited by Lewis* in 1940 – follow naturally from a development in which scale factors are explicitly employed.

C.1 SHADOW AXIS COORDINATES AND THE FUNDAMENTAL PLANE

Given the right ascension $\alpha_{\odot}$, declination $\delta_{\odot}$ and parallax $\pi_{\odot}$ of the sun and the right ascension $\alpha_{☾}$, declination $\delta_{☾}$ (calculated from the lunar latitude of the center of figure) and parallax $\pi_{☾}$, of the moon, we will calculate the right ascension a and declination d of the solar-lunar shadow axis from the equations.

$$\begin{aligned} G \cos d \cos a &= r_{\odot} \cos \delta_{\odot} \cos \alpha_{\odot} - r_{☾} \cos \delta_{☾} \cos \alpha_{☾} \\ G \cos d \sin a &= r_{\odot} \cos \delta_{\odot} \sin \alpha_{\odot} - r_{☾} \cos \delta_{☾} \sin \alpha_{☾} \\ G \sin d &= r_{\odot} \sin \delta_{\odot} - r_{☾} \sin \delta_{☾} \end{aligned} \tag{C.1}$$

where G is the solar-lunar separation. Following division by $r_{\odot}$, (C.1) can be written in the form,

$$\begin{aligned} g \cos d \cos a &= \cos \delta_{\odot} \cos \alpha_{\odot} - b \cos \delta_{☾} \cos \alpha_{☾} \\ g \cos d \cos a &= \cos \delta_{\odot} \sin \alpha_{\odot} - b \cos \delta_{☾} \sin \alpha_{☾} \\ g \sin d &= \sin \delta_{\odot} - b \sin \delta_{☾} \end{aligned} \tag{C.2}$$

---

* Lewis, I. M. "Formulas for the Ionospheric Track in Eclipses," Astron. J. *40*. 4. (1940).

in which $g = G/r_{\odot}$, is given by

$$b = \frac{r_{\text{☾}}}{r_{\odot}} = \frac{a_{\oplus}/\sin \pi_{\text{☾}}}{a_{\oplus}/\sin \pi_{\odot}} = \frac{\sin \pi_{\odot}}{\sin \pi_{\text{☾}}} , \qquad \text{(C.3)}$$

and $a_{\oplus}$ is the equatorial radius of the earth.

C.2 COORDINATES OF THE MOON RELATIVE TO THE FUNDAMENTAL PLANE

Next, following Bessel, we introduce a right handed cartesian system with its origin at the earth's center and its z-axis maintained parallel to the (moving) shadow axis of right ascension a and declination d as determined by (C.2) and (C.3). The xy-plane, so determined, is the fundamental plane; the x-axis, which is positive to the east, is traced out by the intersection of this fundamental plane and the equatorial plane; the y-axis, which is positive toward the north, completes the right handed triad of this system portrayed in Figure C-1.

The cartesian coordinates of the moon in this system follow immediately from application of an Euler Angle transformation* to the lunar coordinates in the geocentric equatorial system. After some trigonometric reduction, these coordinates become

$$\begin{aligned} X &= x a_{\oplus} \\ Y &= y a_{\oplus} \\ Z &= z a_{\oplus} \end{aligned} \qquad \text{(C.4)}$$

in which

$$\begin{aligned} x &= [\cos \delta_{\text{☾}} \sin (\alpha_{\text{☾}} - a)]/\sin \pi_{\text{☾}} \\ y &= [\sin \delta_{\text{☾}} \cos d - \cos \delta_{\text{☾}} \sin d \cos (\alpha_{\text{☾}} - a)]/\sin \pi_{\text{☾}} \\ z &= [\sin \delta_{\text{☾}} \sin d + \cos \delta_{\text{☾}} \cos d \cos (\alpha_{\text{☾}} - a)]/\sin \pi_{\text{☾}} \end{aligned} \qquad \text{(C.5)}$$

---

[*] In this transformation we successively rotate positively (counterclockwise) about the original z-axis by ϕ, positively about the new x-axis by θ, and positively about the new z-axis by ψ; the rotation matrix $R(\psi,\theta,\phi)$ so obtained relates the old (x,y,z) system to the new (x',y',z') system via the matrix equation

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = R(\psi,\theta,\phi) \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

for the case at hand $\psi = 0$, $\theta = \frac{\pi}{2} - d$, and $\phi = a + \frac{\pi}{2}$.

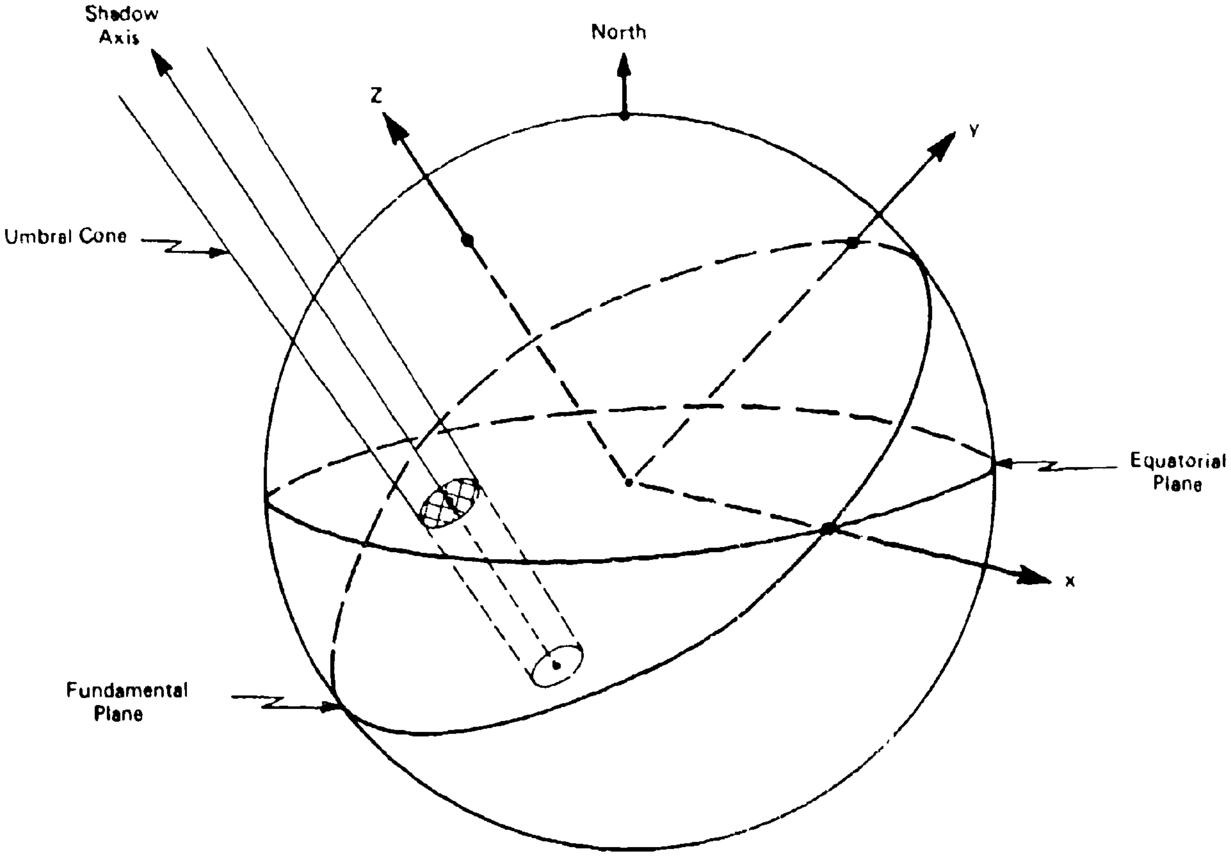

FIGURE C–1 FUNDAMENTAL PLANE GEOMETRY

and where it should be noted that the x and y coordinates are the coordinates of the shadow axis' intersection with the fundamental plane.

C.3 PENUMBRAL AND UMBRAL RADII ON THE FUNDAMENTAL PLANE

In Figure C.2 two shadow cones have been drawn in cross section, the interior tangent cone defining the penumbra and having the vertex V_p a distance c_1 above the fundamental plane and the exterior tangent cone defining the umbra and having a vertex V_u a distance c_2 above the fundamental plane. We may write immediately that

$$\sin f_1 = \frac{d_\odot + d_\leftmoon}{G} \tag{C.6}$$

$$\sin f_2 = \frac{d_\odot - d_\leftmoon}{G} \tag{C.7}$$

where f_1 and f_2 are the penumbral and umbral cone half-angles, respectively, and where $d_\odot$ and $d_\leftmoon$ are the linear semi-diameters of the sun and moon, respectively. But, we note from (C.2) and (A.22) that

$$G = g r_\odot = g \frac{a_\oplus}{\sin \pi_\odot} = g \frac{a_\oplus}{\sin \pi_0} R, \tag{C.8}$$

from the definition of the solar semi-diameter that $\sin S_\odot = d_\odot / r_\odot$, and further from (A.23) that

$$d_\odot = r_\odot \sin S_\odot = (\sin S_0 / \sin \pi_0)\ a_\oplus; \tag{C.9}$$

hence, (C.6) and (C.7) can be written in the usual form as

$$\sin f_1 = (\sin S_0 + k \sin \pi_0)/gR \tag{C.10}$$

$$\sin f_2 = (\sin S_0 - k \sin \pi_0)/gR \tag{C.11}$$

where $k = d_\leftmoon / a_\oplus$ is an adopted constant.

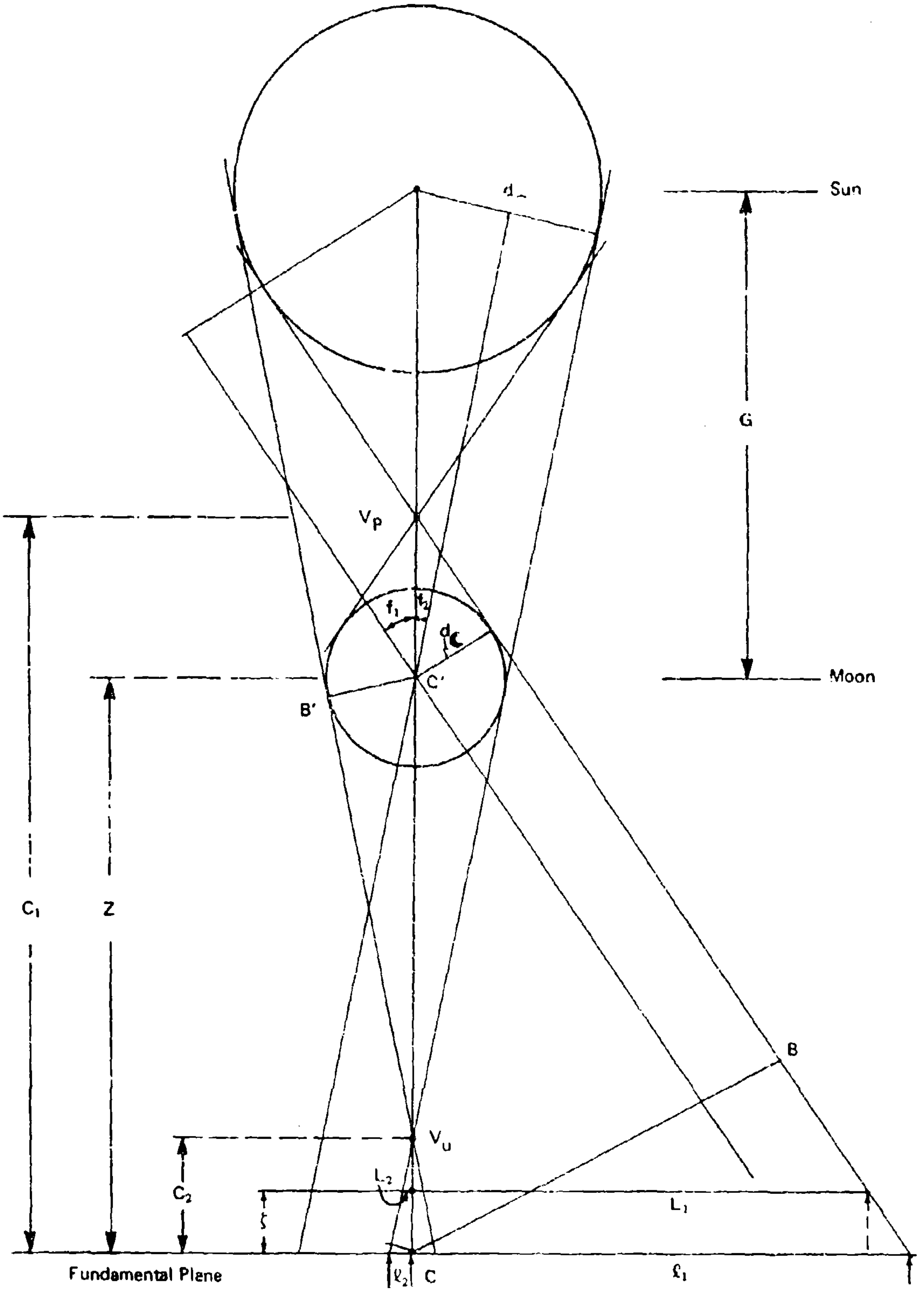

FIGURE C–2 PENUMBRAL AND UMBRAL SPATIAL RELATIONSHIPS

From ΔV_pBC we may write that

$$c_1 = \mathrm{z} + d_{\leftmoon} \operatorname{cosec} f_1 \tag{C.12}$$

and similarly from $\Delta V_uB'C'$

$$c_2 = \mathrm{z} - d_{\leftmoon} \operatorname{cosec} f_2, \tag{C.13}$$

hence,

$$c_1 = (z + k \operatorname{cosec} f_1)\, a_{\oplus} \tag{C.14}$$

and

$$c_2 = (z - k \operatorname{cosec} f_2)\, a_{\oplus} \tag{C.15}$$

where, from (C.4)* and (C.5) respectively,

$$\mathrm{z} = z\,a_{\oplus}$$

and

$$z = [\sin \delta_{\leftmoon} \sin d - \cos \delta_{\leftmoon} \cos d \cos (\alpha_{\leftmoon} - a)]/\sin \pi_{\leftmoon}.$$

Thus, the penumbral radius on the fundamental plane can be found from

$$\ell_1 = c_1 \tan f_1 \tag{C.16}$$

and the umbral radius can be found from

$$\ell_2 = c_2 \tan f_2. \tag{C.17}$$

The sign convention introduced in (C.15) is such that ℓ_2 is negative for total eclipses and positive for annular eclipses.

C.4 SUMMARY – THE BESSELIAN ELEMENTS

In the developments of the prior sections the geometric position of the shadow axis and radii of the penumbral and umbral cones on the fundamental plane have been described. The elements appropriate to this description are the Besselian elements: x, y, sin d, cos d, ℓ_1, ℓ_2 and μ, the ephemeris hour angle of the shadow axis. This hour angle is used in place of the a, the right ascension of the shadow axis, and is calculated from

$$\mu = \text{E.S.T.} - a \tag{C.18}$$

where the ephemeris sidereal time E.S.T. is given by (B.3).

---

* We note that $a_{\oplus}$, the earth's equatorial radius is generally set equal to unity. In such instances, of course, (C.4) immediately reduces to the trivial identity $\mathrm{z} = z$.

These elements and their hourly variations, which are independent of position on the spheroid, are tabulated in the major almanacs and the major eclipse canons, such as [4].

C.5 COORDINATES OF THE OBSERVER RELATIVE TO THE FUNDAMENTAL PLANE

We will consider a spheroid of flattening f (or, equivalently, of ellipticity or first eccentricity e)* centered at the geocenter and having an equatorial radius a_s where

$$a_s = \mathcal{H}\, a_{\oplus} \qquad (C.19)$$

with $\mathcal{H}$ an arbitrary constant; positions will be specified on the spheroidal surface by assignment of longitude (West taken positive) and geodetic latitude. Hence, by employing the same transformation used to effect (C.4) and (C.5), an observer at ephemeris longitude λ_e and latitude ϕ will have the fundamental plane coordinates

$$\begin{aligned} \xi &= [\rho \cos \phi' \sin \textcircled{H}\,]\, \mathcal{H}\, a_{\oplus} \\ \eta &= [\rho \sin \phi' \cos d - \rho \cos \phi' \sin d \cos \textcircled{H}\,]\, \mathcal{H} a_{\oplus} \\ \zeta &= [\rho \sin \phi' \sin d + \rho \cos \phi' \cos d \cos \textcircled{H}\,]\, \mathcal{H} a_{\oplus} \qquad (C.20) \end{aligned}$$

The hour angle $\textcircled{H}$ is given by

$$\textcircled{H} = \mu - \lambda_e \qquad (C.21)$$

where the ephemeris longitude λ_e is related to the longitude λ by means of the relation

$$\lambda_e = \lambda + 1.002738\ \Delta T. \qquad (C.22)$$

ΔT is the difference (either extrapolated, in the case of the future or distant past, or measured, in the case of the immediate past) between Ephemeris Time and Universal Time. The geocentric latitude ϕ' and geocentric distance of the observer ρ are related to the geodetic latitude ϕ through

$$\rho \sin \phi' = S \sin \phi;\ \rho \cos \phi' = C \cos \phi \qquad (C.23)$$

---

[*] We note from the definitions $f = (a - b)/a$ and $e = [1 - b^2/a^2]^{1/2}$, where a is the equatorial radius and b the polar radius of the spheroid, that $e^2 = 2f - f^2$.

where

$$S = (1 - f)^2 C; \quad C = [\cos^2 \phi + (1 - f)^2 \sin^2 \phi]^{-1/2} \tag{C.24}$$

or, equivalently

$$S = (1 - e^2)C; \quad C = (1 - e^2 \sin^2 \phi)^{-1/2} \tag{C.25}$$

C.6 PENUMBRAL AND UMBRAL RADII ON THE OBSERVER'S PLANE

We now construct a plane through the observer parallel to and at a distance ζ above the fundamental plane. Referring once again to Figure C-2, it is seen that the penumbral and umbral radii on this plane are given by

$$L_1 = (c_1 - \zeta) \tan f_1 = \ell_1 - \zeta \tan f_1 \tag{C.26}$$

and

$$L_2 = (c_2 - \zeta) \tan f_2 = \ell_2 - \zeta \tan f_2 \tag{C.27}$$

respectively.

With these two equations and some additional analysis, it becomes possible to calculate all circumstances of the eclipse appropriate to the locale of a specified observer; these calculations will be discussed extensively in a subsequent appendix. But, if one wishes to predict the observation site or sites appropriate for a given circumstance – such as the outline of the umbra on the spheroid – then, the spheroidal flattening introduces the complication that, in order to calculate L_2 from (C.27), for example, ζ must be known which requires, in turn, that ϕ' and ϕ, a desired end result, must be known. This difficulty is minimized by the following treatment of Bessel.

C.7 INTRODUCTION OF THE AUXILIARY ELEMENTS

The development is initiated by introduction of the parametric latitude ϕ_1, through the relation

$$\cos \phi_1 \equiv \rho \cos \phi' = \cos \phi / (1 - e^2 \sin^2 \phi)^{1/2} \tag{C.28}$$

from which, using (C.25),

$$\sin \phi_1 = [1 - \cos^2 \phi_1]^{1/2} = (1 - e^2)^{-1/2} \rho \sin \phi' \ . \tag{C.29}$$

Consequently, (C.20) becomes

$$\xi = [\cos\phi_1 \sin \textcircled{H}]\, \mathcal{H}a_{\oplus}$$

$$\eta = [(1 - e^2)^{1/2} \sin\phi_1 \cos d - \cos\phi_1 \sin d \cos \textcircled{H}]\, \mathcal{H}a_{\oplus}$$

$$\zeta = [(1 - e^2)^{1/2} \sin\phi_1 \sin d + \cos\phi_1 \cos d \cos \textcircled{H}]\, \mathcal{H}a_{\oplus} \qquad (C.30)$$

We now make the following substitutions in η,

$$\sin d = \rho_1 \sin d_1 \, ; (1 - e^2)^{1/2} \cos d = \rho_1 \cos d_1 \qquad (C.31)$$

and the following substitutions in ζ,

$$\cos d = \rho_2 \cos d_2 \, ; (1 - e^2)^{1/2} \sin d = \rho_2 \sin d_2 \qquad (C.32)$$

so that (C.30) now becomes

$$\xi = [\cos\phi_1 \sin \textcircled{H}]\mathcal{H}a_{\oplus}$$

$$\eta_1 = \eta/\rho_1 = [\sin\phi_1 \cos d_1 - \cos\phi_1 \sin d_1 \cos \textcircled{H}]\mathcal{H}a_{\oplus}$$

$$\zeta = \rho_2 [\sin\phi_1 \sin d_2 + \cos\phi_1 \cos d_2 \cos \textcircled{H}]\mathcal{H}a_{\oplus}. \qquad (C.33)$$

Next, we define the variable ζ_1 from

$$\zeta_1 \equiv (\mathcal{H}a_{\oplus})^2 - \xi^2 - \eta_1^2 \qquad (C.34)$$

a useful form of which – after substitution of ξ and η from (C.33) and some reduction – is

$$\zeta_1 = [\sin\phi_1 \sin d_1 + \cos\phi_1 \cos d_1 \cos \textcircled{H}]\, \mathcal{H}a_{\oplus} \qquad (C.35)$$

Further, we note from (C.33) that ζ_1 and η_1 may be readily manipulated to yield,

$$\mathcal{H}a_{\oplus} (\sin\phi_1) = \eta_1 \cos d_1 + \zeta_1 \sin d_1 \qquad (C.36)$$

and

$$\mathcal{H}a_{\oplus} (\cos\phi_1 \cos \textcircled{H}) = -\eta_1 \sin d_1 + \zeta_1 \cos d_1 \qquad (C.37)$$

which upon insertion into ζ of (C.33) yields

$$\zeta = \rho_2 [\zeta_1 \cos(d_1 - d_2) - \eta_1 \sin(d_1 - d_2)]. \qquad (C.38)$$

This form of ζ has the useful property that the assignment of specific values to ξ and η - *both of which are in the fundamental plane* - allows the immediate calculation of ζ_1 by (C.35) and thus the third coordinate ζ by (C.38).

Finally, from the first equation of (C.33) written in the form

$$\mathcal{H} a_{\varphi} (\cos \phi_1 \sin \textcircled{H}) = \xi \tag{C.39}$$

and from (C.36) and C.37), it follows that

$$\tan \textcircled{H} = \frac{\xi}{\zeta_1 \cos d_1 - \eta_1 \sin d_1} \tag{C.40}$$

and

$$\sin \phi_1 = \frac{1}{(\mathcal{H} a_{\varphi})} (\zeta_1 \sin d_1 + \eta_1 \cos d_1). \tag{C.41}$$

Thus, if definite values of ξ and η are available from the application of specific geometric conditions, ξ, η_1 and ζ_1 can be calculated and, hence $\textcircled{H}$ (and thus λ) and ϕ_1 (and thus ϕ) can be obtained from (C.40) and (C.41), respectively.

These calculations, and the calculation for ζ in (C.38), fully incorporate the flattening and are based upon the auxiliary elements ρ_1, ρ_2, $\sin d_1$, $\sin d_2$, $\sin (d_1 - d_2)$ and $\cos (d_1 - d_2)$ which, after manipulating (C.31) and (C.32), follow directly from the Besselian element d and the ellipticity of the spheroid e, in the form

$$\rho_1 = (1 - e^2 \cos^2 d)^{1/2}$$

$$\rho_2 = (1 - e^2 \sin^2 d)^{1/2}$$

$$\sin d_1 = \sin d / \rho_1$$

$$\cos d_1 = (1 - e^2)^{1/2} \cos d / \rho_1$$

$$\sin (d_1 - d_2) = e^2 \sin d \cos d / \rho_1 \ \rho_2$$

$$\cos (d_1 - d_2) = (1 - e^2)^{1/2} / \rho_1 \ \rho_2 \,. \tag{C.42}$$

C.8 OUTLINE OF THE SHADOW

In general, eclipse phenomena are described by specifying the distance and position angle Q (measured east from north) of the shadow axis with respect to the "observation" point in question. Thus, in the plane drawn through the "observer" and maintained perpendicular to the shadow axis, we can write that the in-plane components of the observer coordinates (ξ, η) are related to the shadow axis coordinates (x, y) by means of

$$\xi = x - m \sin Q$$

and

$$\eta = y - m \cos Q \qquad (C.44)$$

where

$$m^2 = (x - \xi)^2 + (y - \eta)^2 \qquad (C.45)$$

Consequently, it follows immediately that the outline of the umbra* on the spheroidal surface at a specific time is given by those (ξ,η) pairs which circumscribe (x,y) at the distance $m = L_2$.

However, before allowing Q to assume all values from 0° to 360° in calculating the shadow outline, it is useful to determine whether the entire umbral cone intersects the spheroidal surface or, equivalently, whether the entire umbral circle falls within a circle of (nearly constant) radius $\mathcal{H}a_{\phi}$ on the fundamental plane.

Figure C-3 portrays the situation when a portion of the umbral circle falls outside this circle.

If we now set $x = \gamma \sin m$ and $y = \gamma \cos m$ (where M is the position angle of the shadow axis) and note that at the extreme points $L_2 = \ell_2$ (since these points are in the fundamental plane and $\zeta = 0$), then

$$(\mathcal{H}a_{\phi})^2 = \gamma^2 + \ell_2^2 - 2\gamma \ell_2 \cos (Q - M)$$

or

$$\cos (Q - M) = \frac{\gamma^2 + \ell_2^2 - (\mathcal{H}a_{\phi})^2}{2\ell_2 \gamma} \qquad (C.46)$$

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* The outline of the penumbra can be calculated by replacing ℓ_2 and L_2 by ℓ_1 and L_1 respectively.

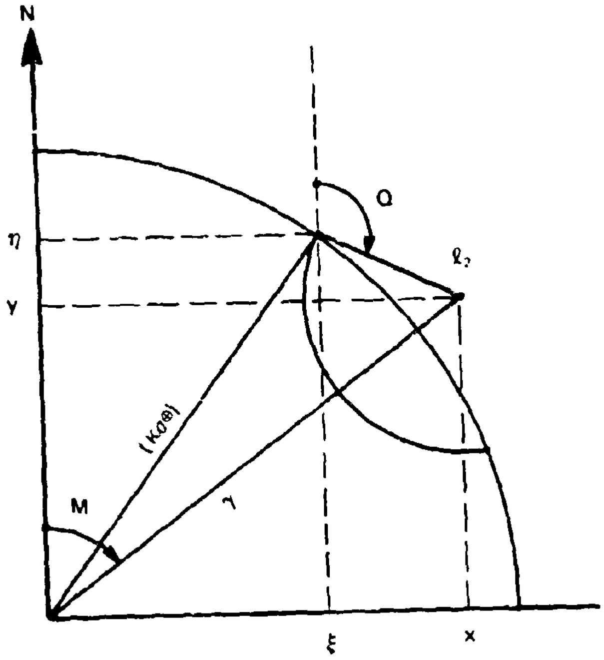

FIGURE C–3 LIMITS OF THE UMBRAL CIRCLE

where $\gamma^2 = x^2 + y^2$. When the evaluation of the right-hand member of (C.40) leads to the condition that $\cos(Q - M) \leqslant 1$, the two consequent values of Q define two circumferential segments: one within the fundamental plane circle and (running) values of Q producing $\zeta > 0$; the other outside the fundamental plane circle and (running) values of Q leading to $\zeta < 0$. When (C.46) results in $\cos(Q - M) > 1$, the entire shadow lies within the fundamental plane circle and all values of Q from 0° to 360° are permissible.

Assuming that the appropriate range of values has been established, the shadow outline can now be calculated by the following iterative procedure for each of the permissible Q.

Step I: Assume $L_2 = \ell_2$ and calculate

$$\xi^{(0)} = x - \ell_2 \sin Q$$

$$\eta_1^{(0)} = (y - \ell_2 \cos Q)/\rho_1$$

$$\zeta_1^{(0)} = [(\mathcal{H} a_\oplus)^2 - (\xi^{(0)})^2 - (\eta_1^{(0)})^2]^{1/2} \qquad (C.47)$$

Step II: From $\eta_1^{(0)}$ and $\zeta_1^{(0)}$ calculate $\zeta^{(0)}$ from

$$\zeta^{(0)} = \rho_2 \, [\zeta_1^{(0)} \cos(d_1 - d_2) - \eta_1^{(0)} \sin(d_1 - d_2)] \qquad (C.48)$$

and $L_2^{(0)}$ from

$$L_2^{(0)} = \ell_2 - \zeta^{(0)} \tan f_2 \qquad (C.49)$$

Step III: Using this value of L_2 calculate

$$\xi^{(1)} = x - L_2^{(0)} \sin Q$$

$$\eta^{(1)} = (y - L_2^{(0)} \cos Q)/\rho_1$$

$$\zeta_1^{(1)} = [(\mathcal{H} a_\oplus)^2 - (\xi^{(1)})^2 - (\eta_1^{(1)})^2]^{1/2} \qquad (C.50)$$

Step IV: Assuming that an additional iteration of Step II is not required by the results of (C.50), use (C.50) to calculate – following (C.40) and (C.41) –

$$\tan \textcircled{H} = \frac{\xi^{(1)}}{\zeta_1^{(1)} \cos d_1 - \eta_1^{(1)} \sin d_1} \qquad (C.51)$$

and

$$\sin \phi_1 = \frac{\zeta_1^{(1)} \sin d_1 + \eta_1^{(1)} \cos d_1}{\mathcal{H} a_{\oplus}} \qquad \text{(C.52)}$$

which leads, following (C.21) to

$$\lambda_e = \mu - \textcircled{H} \qquad \text{(C.53)}$$

and, following (C.23), (C.25), (C.28) and (C.29) to

$$\tan \phi = (1 - e^2)^{-1/2} \tan \phi_1 \qquad \text{(C.54)}$$

C.9 THE CENTRAL LINE

Since the central line is the locus of points of intersection of the shadow axis with the spheroid, it follows immediately that the results of the previous section may be utilized by shrinking the shadow radius ℓ_2 to zero. Thus, the equations of (C.50) become

$$\begin{aligned} \xi &= x \\ \eta_1 &= y/\rho_1 = y_1 \\ \zeta_1 &= [(\mathcal{H} a_{\oplus})^2 - x^2 - y_1^2]^{1/2} \end{aligned} \qquad \text{(C.55)}$$

which leads immediately, upon substitution in (C.51) and (C.52) to

$$\tan \textcircled{H} = \frac{x}{\zeta_1 \cos d_1 - y_1 \sin d_1} \qquad \text{(C.56)}$$

and

$$\sin \phi_1 = \frac{\zeta_1 \sin d_1 + y_1 \cos d_1}{\mathcal{H} a_{\oplus}} \qquad \text{(C.57)}$$

where again from (C.54)

$$\tan \phi = (1 - e^2)^{-1/2} \tan \phi_1 \quad . \qquad \text{(C.58)}$$

If (C.4) and (C.34) are employed, then (C.56) and (C.57) are transformed to

$$\tan \textcircled{H} = \frac{x}{\mathcal{H}\left[1 - \dfrac{x^2 + y_1^2}{\mathcal{H}^2}\right]^{1/2} \cos d_1 - y_1 \sin d_1} \tag{C.59}$$

and

$$\sin \phi_1 = \frac{1}{\mathcal{H}} \left(\mathcal{H}\left[1 - \frac{x^2 + y_1^2}{\mathcal{H}^2}\right]^{1/2} \sin d_1 + y_1 \cos d_1 \right) \tag{C.60}$$

which is the form employed by Lewis* in calculating the ionospheric central line.

C.10 NORTHERN AND SOUTHERN LIMITS OF THE SHADOW – ADDITIONAL AUXILIARY ELEMENTS

From the discussion immediately following (C.45), it is readily seen that the examination of the time variation of the observer/shadow-cone separation $m^2 - L^2$ (where L is either the penumbral or umbral radius) can be employed either to calculate circumstances given the observation location or, conversely, to calculate the observer location given the circumstances. Here we are concerned with the latter type and, more specifically, with calculating the northern and southern limits of the shadow directly rather than by inferring these values from the complete shadow outline calculated in Section C.8.

In Figure C-4, a time variation of $m^2 - L^2$ has been hypothesized for two locations. At location A the particular phase of the eclipse under consideration (partial for $L \rightarrow L_1$, total for $L \rightarrow L_2$) begins at $t^{(i)}$ and ends at $t^{(f)}$, i.e., at the two solutions of $m^2 - L^2 = 0$. At location B, on the other hand, the eclipse both begins and ends at the same time, an occurrence identified by the coalescence of the two roots of $m^2 - L^2 = 0$ to t_B and the presence of the horizontal tangent at t_B. Hence, at t_B we have that

$$(x - \xi)^2 + (y - \eta)^2 - (\ell - \zeta \tan f)^2 = 0 \tag{C.61}$$

and, using dots to signify differentiation with respect to time, that

$$(x - \xi)(\dot{x} - \dot{\xi}) + (y - \eta)(\dot{y} - \dot{\eta}) - (\ell - \zeta \tan f)(\dot{\ell} - \dot{\zeta} \tan f) = 0 \tag{C.62}$$

where f, given by (C.10) or (C.11), is sensibly constant.

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* Lewis, *op. cit.*

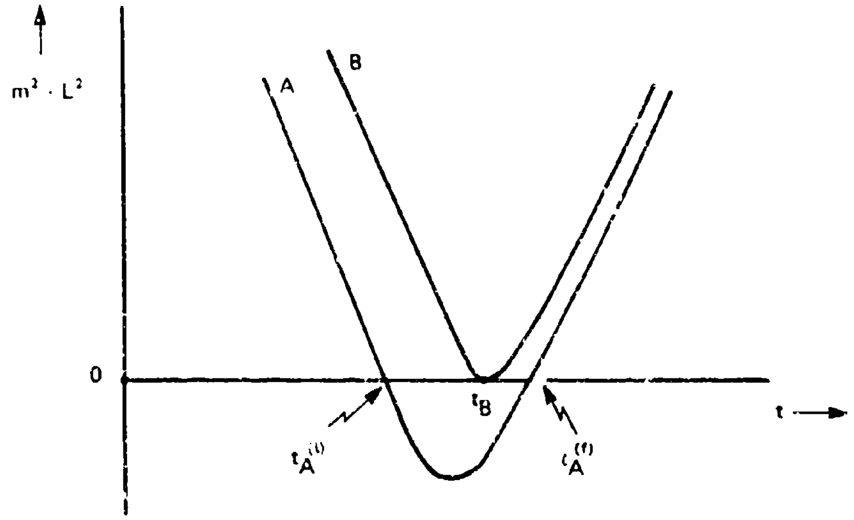

FIGURE C–4 HYPOTHETICAL TIME VARIATION OF $m^2 - L^2$ FOR TWO LOCATIONS

We now note that differentiation of (C.20) leads, after a modest reduction, to

$$\dot{\xi} = \dot{\mu} (- \eta \sin d + \zeta \cos d)$$

$$\dot{\eta} = \dot{\mu} \xi \sin d - \dot{d} \zeta$$

$$\dot{\zeta} = \dot{\mu} \xi \cos d + \dot{d} \eta \qquad \text{(C.63)}$$

($\dot{\mu}$ and $\dot{d}$ are presented as hourly variations in the literature) which become, upon substitution of (C.43 and (C.44)

$$\dot{\xi} = \dot{\mu} (- y \sin d + \zeta \cos d + m \sin d \cos Q)$$

$$\dot{\eta} = \dot{\mu} (x \sin d - m \sin d Q) - \dot{d} \zeta$$

$$\dot{\zeta} = \dot{\mu} (- x \cos d + m \cos d \sin Q) + \dot{d} (y - m \cos Q). \qquad \text{(C.64)}$$

Hence, (C.62) becomes, upon substitution of (C.64), and upon noting that at $t = t_B$, $m = L = \ell - \zeta \tan i$,

$$\sin Q\,[\dot{x} + \dot{\mu}\, y \sin d + \dot{\mu}\, \ell \tan f \cos d - \dot{\mu}\, \zeta \cos d\,(1 + \tan^2 f)] +$$

$$+ \cos Q\,[\dot{y} - \dot{\mu}\, x \sin d + \dot{d}\, \zeta - \dot{d} \tan f\,(\ell - \zeta \tan f)] +$$

$$+ [- \dot{\ell} - \dot{\mu}\, x \cos d \tan f + \dot{d} \tan f\, y] = 0. \qquad \text{(C.65)}$$

or, upon defining the auxiliary elements

$$\dot{a} = - \dot{\ell} - \dot{\mu}\, x \tan f \cos d$$

$$\dot{b} = \dot{y} + \dot{\mu}\, x \sin d$$

$$\dot{c} = \dot{x} + \dot{\mu}\, y \sin d + \dot{\mu}\, \ell \tan f \cos d \qquad \text{(C.66)}$$

(where $\dot{a}_1$ and $\dot{c}_1$ are penumbral auxiliaries; $\dot{a}_2$ and $\dot{c}_2$ are umbral auxiliaries)

$$\sin Q\,[\dot{c} - \dot{\mu}\, \zeta \cos d\,(1 + \tan^2 f)] +$$

$$\cos Q\,[- \dot{b} + \dot{d}\, \zeta - \dot{d} \tan f\,(\ell - \zeta \tan f)] + \dot{a} + \dot{d} \tan f\, y = 0. \qquad \text{(C.67)}$$

However, following Chauvenet, terms involving $\tan^2 f$ and $\dot{d} \tan f$ will be neglected so that we have

$$\tan Q = \frac{\dot{b} - \dot{d}\, \zeta - \dot{a} \sec Q}{\dot{c} - \dot{\mu}\, \zeta \cos d} \qquad \text{(C.68)}$$

as the condition for the northern and southern limits. For umbral limits, cos Q is positive for the northern limit of a total eclipse and the southern limit of an annular eclipse; cos Q is negative for the southern limit of a total eclipse and the northern limit of an annular eclipse. For penumbral limits, cos Q is positive for the southern limit and negative for the northern limit.

The calculations for the limits are similar to those of the shadow outline

Step I: Assume $L_2 = \ell_2$ (or $L_1 = \ell_1$) and calculate

$$\tan Q^{(0)} = \frac{\dot{b}}{\dot{c}}. \qquad \text{(C.69)}$$

from which

$$\xi^{(0)} = x - \ell_2 \sin Q^{(0)}$$

$$\eta_1^{(0)} = (y - \ell_2 \cos Q^{(0)})/\rho_1$$

$$\zeta_1^{(0)} = [(\mathcal{H}a_\oplus)^2 - (\xi^{(0)})^2 - (\eta_1^{(0)})^2]^{1/2} \qquad \text{(C.70)}$$

Step II: From $\eta_1^{(0)}$ and $\zeta_1^{(0)}$, calculate $\zeta^{(0)}$

$$\zeta^{(0)} = \rho_2 \; [\xi_1^{(0)} \cos(d_1 - d_2) - \eta_1^{(0)} \sin(d_1 - d_2)] \qquad \text{(C.71)}$$

and thence

$$\tan Q^{(1)} = \frac{\dot{b} - \dot{d}\,\zeta^{(0)} - \dot{a}_2 \sec Q^{(0)}}{\dot{c}_2 - \dot{\mu}\,\zeta^{(0)} \cos d} \qquad \text{(C.72)}$$

and

$$L_2^{(1)} = \ell_2 - \zeta^{(0)} \tan f_2 \qquad \text{(C.73)}$$

Step III: Using these values of $Q^{(1)}$ and $L_2^{(1)}$, calculate

$$\xi^{(1)} = x - L_2^{(1)} \sin Q^{(1)}$$

$$\eta_1^{(1)} = (y - L_2^{(1)} \cos Q^{(1)})/\rho_1$$

$$\zeta_1^{(1)} = [\mathcal{H}a_\oplus)^2 - (\xi^{(1)})^2 - (\eta_1^{(1)})^2]^{1/2} \qquad \text{(C.74)}$$

Step IV: Assuming that an additional iteration of Step II is not required by the results of (C.74), proceed as in (C.51), (C.52), and (C.53).

C.11 DURATION ON CENTRAL LINE

Given an observation point ξ_0, η_0 and a center line point x_0, y_0 both evaluated at the time T_0, the two components of the position of the shadow axis *relative* to this observation point can be expanded to first order at the time $T_0 + \tau$ as

$$(x - \xi) = [x_0 + \dot{x}\,\tau + \ldots] - [\xi_0 + \dot{\xi}\,\tau + \ldots] = m \sin Q \qquad \text{(C.75)}$$

and

$$(y - \eta) = [y_0 + \dot{y}\tau + \ldots] - [\eta_0 + \dot{\eta}\tau + \ldots] = m \cos Q \qquad (C.76)$$

or, upon squaring and adding the components to yield the relative separation,

$$m^2 = (x_0 - \xi_0)^2 + (y_0 - \eta_0)^2 + 2[(x_0 - \xi_0)(\dot{x} - \dot{\xi}) + (y_0 - \eta_0)(\dot{y} - \dot{\eta})]\tau +$$

$$+ [(\dot{x} - \dot{\xi})^2 + (\dot{y} - \dot{\eta})^2]\tau^2 + \ldots \qquad (C.77)$$

We will now place the observer on the center line, such that $\xi_0 = x_0$ and $\eta_0 = y_0$ and such that the time $\tau = T_0 - T_i$ will be taken as the beginning of the umbral eclipse when $m = L_2$.

Thus (C.77) becomes

$$L_2^2 = [(\dot{x} - \dot{\xi})^2 + (\dot{y} - \dot{\eta})^2](T_0 - T_i)^2 \qquad (C.78)$$

and hence, the semi-duration $s = T_0 - T_i$ of the eclipse is immediately given by the solution of (C.80) in the form

$$s = \frac{L_2}{n} \qquad (C.79)$$

where n, the speed of the shadow relative to the observer, is given by

$$n = [(\dot{x} - \dot{\xi})^2 + (\dot{y} - \dot{\eta})^2]^{1/2}. \qquad (C.80)$$

L_2 is assumed to be constant throughout the calculation and is evaluated from

$$L_2 = \ell_2 - \zeta_0 \tan f_2 \qquad (C.81)$$

in which

$$\zeta_0 = \rho_2 [\zeta_1 \cos(d_1 - d_2) - \eta_1 \sin(d_1 - d_2)] \qquad (C.82)$$

where ζ_1 and η_1 are evaluated from (C.55) with x_0 and y_0 replacing x and y. η can be evaluated by noting that since $\xi = x$ and $\eta = y$, (C.63) can be manipulated to yield

$$\dot{x} - \dot{\xi} = \dot{x} + \dot{\mu}\, y \sin d - \dot{\mu}\, \zeta \cos d \qquad (C.83)$$

$$\dot{y} - \dot{\eta} = \dot{y} - \dot{\mu}\, x \sin d + \dot{d}\, \zeta \qquad (C.84)$$

or, by employing the auxiliary elements of (C.66),

$$\dot{x} - \dot{\xi} = \dot{c} - \dot{\mu} \cos d\, (\zeta + \ell \tan f) \tag{C.85}$$

$$\dot{y} - \dot{\eta} = - \dot{b} + \dot{d}\, \zeta \tag{C.86}$$

which, consistent with the level of approximation of (C.75) through (C.80), reduce to the approximate relations,

$$\dot{x} - \dot{\xi} \approx \dot{c} - \dot{\mu}\, \zeta \cos d \tag{C.87}$$

$$\dot{y} - \dot{\eta} \approx - \dot{b}. \tag{C.88}$$

APPENDIX D

LOCAL CIRCUMSTANCES

D.0 INTRODUCTION

As pointed out in Section C.5 of the previous appendix, it is possible to predict all circumstances of an eclipse at a locale once the coordinates of that locale have been specified. The purpose of this appendix is to develop the basis of algorithms employed to calculate these circumstances, to develop differential correction procedures for adjusting these circumstances to modest departures from the specified locale and to develop the calculations of other topocentric parameters of potential interest to solar,* atmospheric and ionospheric research.

D.1 PRELIMINARY CONSIDERATIONS

Following (C.20) we may write immediately for a site with geodetic latitude ϕ, longitude λ (West taken positive) and height h (in meters) above the spheroid that

$$\begin{aligned} \xi &= \rho \cos \phi' \sin \textcircled{H} \\ \eta &= \rho \sin \phi' \cos d - \rho \cos \phi' \sin d \cos \textcircled{H} \\ \zeta &= \rho \sin \phi' \sin d + \rho \cos \phi' \cos d \cos \textcircled{H} \end{aligned} \qquad \text{(D.1)}$$

in which the equatorial radius of the earth $a_\oplus$ will be assumed henceforth as the unit of distance and where

$$\rho \sin \phi' = (S + H) \sin \phi; \ \rho \cos \phi' = (C + H) \cos \phi \qquad \text{(D.2)}$$

with $H = 0.15678503 \times 10^{-6} h$;** S and C are given by (C.24) and (C.25). The hour angle $\textcircled{H}$ is, following (C.21) and (C.22),

$$\textcircled{H} = \mu - \lambda - 1.002738\, \Delta T \qquad \text{(D.3)}$$

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*Appendix E is devoted to the development of various coordinate systems (and their interrelationships) for use in observation of optical and radio phenomena on and above the solar surface.

**We recognize, with reference to (C.20), the difference between the change of scale implied by $\mathcal{H} a_\oplus$ of (C.20), on the one hand, and the addition of the height term in D.2 normal to the spheroid, if added at each point of the spheroid, on the other. However, for a scale change (or height addition) of $0.1 a_\oplus$, the difference between the two approaches leads to a difference of ~ 2km at the poles and will be neglected.

where μ, the ephemeris hour angle of the shadow axis, is given by (C.18) and ΔT, is the difference of Ephemeris Time and Universal Time for the year in question.

If these coordinates and their time variations, given by (C.63) as

$$\dot\xi = \dot\mu\,(-\eta \sin d + \zeta \cos d)$$

$$\dot\eta = \dot\mu\,\xi \sin d - \dot d\,\zeta$$

$$\dot\zeta = -\dot\mu\,\xi \cos d + \dot d\,\eta, \qquad (D.4)$$

are combined as

$$u = x - \xi;\ v = y - \eta;\ m^2 = u^2 + v^2 \qquad (D.5)$$

and

$$\dot u = \dot x - \dot\xi;\ \dot v = \dot y - \dot\eta;\ n^2 = \dot u^2 + \dot v^2\ , \qquad (D.6)$$

then (C.77), the first order expression relating the distance m between the shadow axis (x,y) and an observation site (ξ, η) at the time $T_0 + \tau$ to the shadow axis and observation site positions (x_0, y_0; ξ_0, η_0) and velocities ($\dot x$, $\dot y$; $\dot\xi$, $\dot\eta$) at the time T_0, becomes

$$m^2 = m_0^2 + 2(u_0\,\dot u + v_0\,\dot v)\,\tau + n^2\,\tau^2\ . \qquad (D.7)$$

This equation has the solution

$$\tau = -\frac{D}{n^2} \pm \frac{D}{n^2}\left[1 - \frac{n^2}{D^2}\,(m_0^2 - m^2)\right]^{1/2} \qquad (D.8)$$

where $D = u_0\,\dot u + v_0\,\dot v$, or, following rearrangement of the second term

$$\tau = -\frac{D}{n^2} \pm \frac{m}{n}\left[1 - \frac{1}{m^2}\left(\frac{u_0\,\dot v - v_0\,\dot u}{n}\right)^2\right]^{1/2}$$

which leads to the form commonly encountered in the literature

$$\tau = -\frac{D}{n^2} \pm \frac{m}{n}\cos\psi \qquad (D.9)$$

where $\sin\psi \equiv (u_0\,\dot v - v_0 \dot u)/n\,m$.

Thus, to determine the time at which m assumes a specified value, one estimates an Ephemeris Time T_0 "reasonably" close to the desired time, evaluates all component quantities of (D.5) and (D.6) and solves – after insertion of the specified value of m – either (D.8) or (D.9) for the correction τ to the estimated time.

If great accuracy is required, a second iteration may be required in which (D.5) and (D.6) are re-evaluated at $T_0 + \tau = T_0'$ and the correction τ' to T_0' is calculated from (D.8) or (D.9).

Similarly, by differentiating (D.7) and rearranging the result to yield

$$\tau = \left(\frac{m}{n^2}\right) \dot{m} - \frac{D}{n^2} \qquad \text{(D.10)}$$

one can determine the increment τ to the time T_0 at which $\dot{m}$ assumes a specified value, all other "component" quantities of (D.10) being evaluated at the "reasonably" close time T_0.

D.2 TIME OF GREATEST PHASE

The time of greatest phase occurs when the eclipse magnitude $M_1 = (L_1 - m)/(L_1 + L_2)$, derived in Section D.7, is precisely a maximum. But since the shadow radii variations $\dot{L}_1 \approx \dot{L}_2 = \dot{L}$ are small,* the time of greatest phase is *usually* taken as the time at which the distance m between the shadow axis and the observer is minimum, or when $\dot{m} = 0$.

If this specified value of $\dot{m}$ is substituted in (D.10), then the correction τ to the "reasonable" time of greatest phase T_0 is immediately given by

$$\tau = -\frac{D}{n^2} \qquad \text{(D.11)}$$

which leads to the interpretation of the first terms in both (D.8) and (D.9) as that correction to the assumed time T_0 which effects the minimum separation between the shadow axis and the observer.

If, on the other hand, accuracies of tens of milliseconds are required, then the maximum magnitude definition employed by Gossner**

$$\dot{M}_1 = \frac{d}{dt}\left(\frac{L_1 - m}{L_1 + L_2}\right) = 0 \qquad \text{(D.12)}$$

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* cf. (D.39) and associated footnote.

** Gossner, S. D., "A Correction to the Time of Maximum Obscuration in Solar Eclipses," A. J. *60*, 383 (1955).

which can be written, following substitution of $\dot{L}_1 \approx \dot{L}_2 = \dot{L}$, in the form,

$$\dot{m} = \frac{(L_1 + L_2)\,\dot{L}_1 - (L_1 - m)(\dot{L}_1 + \dot{L}_2)}{(L_1 + L_2)} \approx \frac{(L_2 - L_1 + 2m)}{L_1 + L_2}\,\dot{L} \qquad \text{(D.13)}$$

leads, upon substitution of (D.13) in (D.10) to Gossner's Eq. (5)

$$\tau = \left(\frac{L_2 - L_1 + 2m}{L_1 + L_2}\right)\dot{L} - \frac{D}{n^2} \qquad \text{(D.14)}$$

which reduces immediately to (D.11) when shadow radii variations are neglected.

D.3 PENUMBRAL (FIRST AND FOURTH) CONTACTS AND DURATION

In the case of the penumbral (first and fourth) contacts we set $m = L_1$ in (D.9), following the choice of some initial time T_0, and write immediately that

$$\tau_P = -\frac{D}{n^2} \pm \frac{L_1}{n}\left|\cos\psi_P\right| \qquad \text{(D.15)}$$

in which

$$\sin\psi_P = \frac{L}{L_1}\left[\frac{u_0\,\dot{v} - v_0\,\dot{u}}{n}\right]\;; \qquad \text{(D.16)}$$

the negative sign is selected in calculating the correction to the Ephemeris Time T_0 for first contact (immersion) and the positive sign is selected for fourth contact (emersion). Since $L_1 |\cos\psi_P|/n$ is either added to or subtracted from the time of maximum eclipse $T_0 - D/n^2$ to give either the first or fourth contact, then assuming T_0 has been selected to be reasonably close to maximum, $L_1 |\cos\psi_P|/n$ can be regarded as the semi-duration of the partial phase.

Because the times of penumbral contacts are generally not required with great accuracy, one can expand the square root of (D.8) and keep only the first term, thus yielding as the approximate correction

$$\tau_P \approx \pm\frac{(L_1^2 - m_0^2)}{2D} \qquad \text{(D.17)}$$

where again the negative and positive signs are selected for first and fourth contacts, respectively.

D.4 UMBRAL (SECOND AND THIRD) CONTACTS AND DURATION

For the second and third (umbral) contacts m = L_2 and (D.9) becomes

$$\tau_u = -\frac{D}{n^2} \pm \frac{L_2}{n} \left| \cos \psi_u \right| \tag{D.18}$$

in which

$$\sin \psi_u = \frac{1}{L_2} \left[\frac{u_0 \dot{v} - v_0 \dot{u}}{n} \right] . \tag{D.19}$$

The positive sign is selected either for second contact (immersion) in the case of a total eclipse ($L_2 < 0$) or third contact (emersion) for an annular eclipse; the negative sign is selected either for third contact (emersion) for a total eclipse or second contact (immersion) for an annular eclipse. Because of accuracy requirements a second iteration is often required.

The semi-duration of the umbral phase is given immediately as

$$S_u = \frac{L_2}{n} \left| \cos \psi_u \right| \tag{D.20}$$

which reduces, as expected, to the center line value of L_2/n given by (C.81) when $u_0 = v_0 = 0$ and $\cos \psi_u = 1$.

D.5 POSITION ANGLES

The angle Q which can be calculated -- following (C.43), (C.44), (D.5) and (D.6) - from

$$\tan Q = \frac{u}{v} \tag{D.21}$$

is at all times the topocentric position angle of the moon's center measured eastward from the north point of the solar limb.* When u and v are evaluated at the contact times of (D.15) and (D.18), Q is the angle of the points of contact of the apparent disc of the moon with the apparent disc of the sun.

In instances when altitude-azimuth instrumentation is being employed, the position angle with respect to the vertex (defined as that point of the solar limb nearest the zenith) may prove more useful. This angle is determined by noting that

---

* The north point of the sun is readily determined observationally as that point on the apparent solar disc tangent to the disc's apparent (diurnal) motion since it is the intersection point of the northern limb and the hour circle through the solar center.

the distance from the north point of the solar limb to the vertex is equal to the parallactic angle q which can be gotten from the equation

$$\tan q = \frac{\cos \phi \sin \theta}{\sin \phi \cos d - \cos \phi \sin d \cos \theta} \qquad (D.22)$$

or, since $\cos \phi \approx \rho \cos \phi'$ and $\sin \phi \approx \rho \sin \phi'$ in (D.1), from the approximate relation

$$\tan q \approx \frac{\xi}{\eta} \quad . \qquad (D.23)$$

Hence, the position angle V, as measured from the vertex, is given by

$$V = Q - q \quad . \qquad (D.24)$$

D.6 DIFFERENTIAL CORRECTIONS TO LOCAL CIRCUMSTANCES

In this section all differential corrections requires to adjust calculated local circumstances to modest changes in locale will be developed.

D.6.1 Development of Differential Coefficients

We begin by casting the spatial variation of a function in the operator form

$$\delta = \delta\lambda \frac{\partial}{\partial\lambda} + \delta\phi \frac{\partial}{\partial\phi} + \delta H \frac{\partial}{\partial H} \qquad (D.25)$$

where $\delta\lambda$ and $\delta\phi$ are variations in longitude and latitude, respectively, and are expressed in radians while δH is the variation in height expressed in units of the earth's radius.

From (D.3), (D.5) and (D.6) it follows that

$$\delta u = -\delta\xi, \ \delta v = -\delta\eta, \ \delta \textcircled{H} = -\delta\lambda \qquad (D.26)$$

and from (C.25) that

$$\frac{\partial C}{\partial H} = \frac{\partial S}{\partial H} = 0; \ \frac{\partial C}{\partial\lambda} = \frac{\partial S}{\partial\lambda} = 0 \qquad (D.27)$$

and

$$\frac{\partial C}{\partial \phi} = \frac{\partial}{\partial \phi} (1 - e^2 \sin^2 \phi)^{1/2} = C^3 e^2 \sin \phi \cos \phi \qquad (D.28)$$

$$\frac{\partial S}{\partial \phi} = (1 - e^2) \frac{\partial C}{\partial \phi} = SC^2 e^2 \sin \phi \cos \phi \qquad (D.29)$$

Consequently, by applying (D.25) to ξ of (D.1) we have

$$\delta\xi = A_1 \delta\lambda + A_2 \delta\phi + A_3 \delta H \qquad (D.30)$$

in which

$$A_1 = \frac{\partial \xi}{\partial \lambda} = -\rho \cos \phi' \cos \textcircled{H}$$

$$A_2 = \frac{\partial \xi}{\partial \phi} = -(SC^2 + H) \sin \phi \sin \textcircled{H}$$

$$A_3 = \frac{\partial \xi}{\partial H} = \cos \phi \sin \textcircled{H} \qquad (D.31)$$

and, similarly, for η of (D.1)

$$\delta\eta = B_1 \delta\lambda + B_2 \delta\phi + B_3 \delta H \qquad (D.32)$$

in which

$$B_1 = \frac{\partial \eta}{\partial \lambda} = -\xi \sin d$$

$$B_2 = \frac{d\eta}{\partial \phi} = (SC^2 + H)(\cos \phi \cos d + \sin \phi \sin d \cos \textcircled{H})$$

$$B_3 = \frac{\partial \eta}{\partial H} = \sin \phi \cos d - \cos \phi \sin d \cos \textcircled{H} \qquad (D.33)$$

However, since many calculations do not require the accuracy of (D.31) and (D.33), the following approximations are sometimes used,

$$A_2 \approx -C^2 \rho \sin \phi' \sin \textcircled{H}$$

$$A_3 \approx \rho \cos \phi' \sin \textcircled{H} = \xi \qquad (D.34)$$

and

$$B_2 \approx CS\,\rho\cos\phi'\cos d + C^2\rho\sin\phi'\sin d\cos\textcircled{H}$$

$$B_3 \approx \rho\sin\phi'\cos d - \rho\cos\phi'\sin d\cos\textcircled{H} = \eta \qquad \text{(D.35)}$$

In the balance of this sub-section, those differential coefficients associated with $\delta\zeta$, δL; $\delta\dot{\xi}$ $\delta\dot{\eta}$, $\delta\dot{\zeta}$, $\delta(\eta^2)$ and δn will be developed for completeness.

We begin by applying (D.25) to ζ of (D.1) thereby yielding

$$\delta\zeta = C_1\,\delta\lambda + C_2\,\delta\phi + C_3\,\delta H \qquad \text{(D.36)}$$

in which

$$C_1 = \frac{\partial\zeta}{\partial\lambda} = \xi\cos d$$

$$C_2 = \frac{\partial\zeta}{\partial\phi} = (SC^3 + H)(\cos\phi\sin d - \sin\phi\cos d\cos\textcircled{H})$$

$$C_3 = \frac{\partial\zeta}{\partial H} = \sin\phi\sin d + \cos\phi\cos d\cos\textcircled{H} \qquad \text{(D.37)}$$

which, like the approximations of (D.34) and (D.35), can be approximated as

$$C_2 \approx CS\,\rho\cos\phi'\sin d - C^2\rho\sin\phi'\cos d\cos\textcircled{H}$$

$$C_3 \approx \rho\sin\phi'\sin d + \rho\cos\phi'\cos d\cos\textcircled{H} = \zeta \qquad \text{(D.38)}$$

Further, by employing (C.26) or (C.27) which define L_1 and L_2, respectively, we note immediately that

$$\delta L = -\tan f\,\delta\zeta = -\tan f\,[C_1\,\delta\lambda + C_2\,\delta\phi + C_3\,\delta H] \qquad \text{(D.39)}^{*}$$

Next, we apply (D.25) to $\dot{\xi}$ given by

$$\dot{\xi} = \dot{\mu}\,(C + H)\cos\phi\cos\textcircled{H}$$

---

[*] $\delta\zeta$ is usually ignored for the simple reason that ζ-variations can enter local circumstances only through the shadow radii and thus through δL of (D.39). Hence, even though $O(\xi) = O(\eta) = O(\zeta)$, $\tan f \approx 0.005$ and thus δL is small by comparison.

and find that

$$\delta\dot{\xi} = A_1' \; \delta\lambda + A_2' \; \delta\phi + A_3' \; \delta H \tag{D.40}$$

in which (recalling that $\delta \textcircled{H} = \delta\lambda$)

$$A_1' = \frac{\partial\dot{\xi}}{\partial\lambda} = \dot{\mu}\,\rho \cos\phi' \sin \textcircled{H}$$

$$A_2' = \frac{\partial\dot{\xi}}{\partial\phi} = -\dot{\mu}\,(SC^2 + H) \sin\phi \cos \textcircled{H}$$

$$A_3' = \frac{\partial\dot{\xi}}{\partial H} = \dot{\mu} \cos\phi \cos \textcircled{H} \tag{D.41}$$

Similarly, (D.25) applied to $\dot{\eta}$ of (D.4) yields

$$\delta\dot{\eta} = B_1' \; \delta\lambda + B_2' \; \delta\phi + B_3' \; \delta H \tag{D.42}$$

in which

$$B_1' = \frac{\partial\dot{\eta}}{\partial\lambda} = \dot{\mu} \sin d \; A_1 - \dot{d}\; C_1$$

$$B_2' = \frac{\partial\dot{\eta}}{\partial\phi} = \dot{\mu} \sin d \; A_2 - \dot{d}\; C_2$$

$$B_3' = \frac{\partial\dot{\eta}}{\partial H} = \dot{\mu} \sin d \; A_3 - d\; C_3 \tag{D.43}$$

and when applied to $\dot{\zeta}$ of (D.4) yields

$$\delta\dot{\zeta} = C_1' \; \delta\lambda + C_2' \; \delta\phi + C_3' \; \delta H \tag{D.44}$$

in which

$$C_1' = \frac{\partial\dot{\zeta}}{\partial\lambda} = -\dot{\mu} \cos d \; A_1 + \dot{d}\; B_1$$

$$C_2' = \frac{\partial\dot{\zeta}}{\partial\phi} = -\dot{\mu} \cos d \; A_2 + \dot{d}\; B_2$$

$$C_3' = \frac{\partial\dot{\zeta}}{\partial H} = -\dot{\mu} \cos d \; A_3 + \dot{d}\; B_3 \tag{D.45}$$

Hence, we can write immediately (noting that $\delta\dot{u} = -d\dot{\xi}$ and that $\delta\dot{v} = -\delta\dot{\eta}$) that

$$\delta(n^2) = N_1\ \delta\lambda + N_2\ \delta\phi + N_3\delta H \tag{D.46}$$

in which

$$\begin{aligned} N_1 &= -2(\dot{u}\ A_1' + \dot{v}\ B_1') \\ N_2 &= -2(\dot{u}\ A_2' + \dot{v}\ B_2') \\ N_3 &= -2(\dot{u}\ A_3' + \dot{v}\ B_3') \end{aligned} \tag{D.47}$$

and that

$$\delta(n) = \frac{\delta(n^2)}{2n}\ . \tag{D.48}$$

D.6.2 Development of Differential Corrections to Local Circumstances

In subsequent calculations of the differential corrections to local circumstances, δL, $\delta\dot{\mu} = -\delta\dot{\xi}$, $\delta\dot{v} = -\delta\dot{\eta}$, $\delta(n)$ and $\delta(n^2)$ will be neglected.

Because as stated previously, the penumbral contact times are not required with great accuracy, the correction will be based on (D.20) and thus

$$\delta t = \delta\left(\frac{L_1^2 - m^2}{2D}\right) = -\frac{\delta(m^2)}{2D} - \frac{(L_1^2 - m^2)}{2D^2}\delta D \tag{D.49}$$

or, since near the contact time $(L_1^2 - m^2) \approx 0$,

$$\delta t \approx -\frac{\delta(m^2)}{2D} = \frac{u\ \delta\xi + v\ \delta\eta}{D} \tag{D.50}$$

Hence, the correction to the penumbral contact time can be written as

$$\delta t = p\delta\lambda + q\ \delta\phi + r\ \delta H \tag{D.51}$$

in which

$$\begin{aligned} p &= (uA_1 + vB_1)/D \\ q &= (uA_2 + vB_2)/D \\ r &= (uA_3 + vB_3)/D \end{aligned} \tag{D.52}$$

Similarly, the correction to the time of greatest phase from (D.14) becomes

$$\delta t_m = \delta\left(-\frac{D}{n^2}\right) = \frac{\dot{u}\,\delta\xi + \dot{v}\,\delta\eta}{n^2} \tag{D.53}$$

or

$$\delta t_m = p_m\,\delta\lambda + q_m\,\delta\phi + r_m\,\delta H \tag{D.54}$$

in which

$$p_m = (\dot{u}A_1 + \dot{v}B_1)/n^2$$

$$q_m = (\dot{u}\,A_2 + \dot{v}B_2)/n^2$$

$$r_m = (\dot{u}\,A_3 + \dot{v}B_3)/n^2 \tag{D.55}$$

The correction to the umbral contact times will be calculated, under the assumption that (D.54) has been calculated, by correcting the semi-duration S_u of (D.20) for changes in locale. This is accomplished by noting, from (D.19), that

$$\sin\psi_u = \frac{1}{L_2}\left(\frac{u\dot{v} - v\dot{u}}{n}\right) = \frac{K}{L_2} \tag{D.56}$$

and that

$$\delta K = p_s\delta\lambda + q_s\delta\phi + r_s\delta H \tag{D.57}$$

in which

$$p_s = (\dot{u}\,B_1 - \dot{v}\,A_1)/n$$

$$q_s = (\dot{u}\,B_2 - \dot{v}\,A_2)/n$$

$$r_s = (\dot{u}\,B_3 - \dot{v}\,A_3)/n \tag{D.58}$$

Hence, we recalculate the semi-duration S_u from the formula

$$S'_u = \frac{L_2}{n}\left|\cos\psi'_u\right| \tag{D.59}$$

where

$$\sin\psi'_u = \frac{K + \delta K}{L_2} \quad . \tag{D.60}$$

Finally, we note that a change in ΔT from ΔT to $\Delta T + \delta T$ can be readily incorporated within the existing framework by noting its complete equivalence to a fictitious change in longitude ($\delta\phi = \delta H = 0$) c. the form

$$\delta\lambda = 1.002738\,\delta T \qquad (D.61)$$

and thus can be accommodated via p, p_m and p_s (the differential coefficients of $\delta\lambda$) in computing δt, δt_m and s'_u from (D.50), (D.54) and (D.57), respectively. The corrections to these previous times (when expressed in U.T.) will be $\delta t - \delta T$ and $\delta t_m - \delta T$.

D.7 ECLIPSE MAGNITUDE

The magnitude of an eclipse is defined as the fraction of the solar diameter covered by the moon at the time of greatest phase. Consequently, referring to Figure D-1 – a redrawn and simplified version of Figure C-2 – we see that for an observer O_p located in the penumbral shadow PP′ that the magnitude M_1 of the partial eclipse is given by

$$M_1 = \frac{S'A}{S'S}$$

which, employing the similarity of triangles PMO_p and S′MA; and PMU and S′MS leads to

$$M_1 = \frac{PO_P}{PU} = \frac{PX - XO_P}{PX - XU} \quad .$$

But $PX = L_1$, the penumbral radius, $XO_p = m$ and $XU = -L_2$ during totality (as pictured), hence

$$M_1 = \frac{L_1 - m}{L_1 + L_2} \qquad (D.62)$$

which is the desired result valid for any time during the eclipse.

Similarly for an observer located within the umbra at O_u the magnitude is given by

$$M_2 = \frac{B'B}{S'S}$$

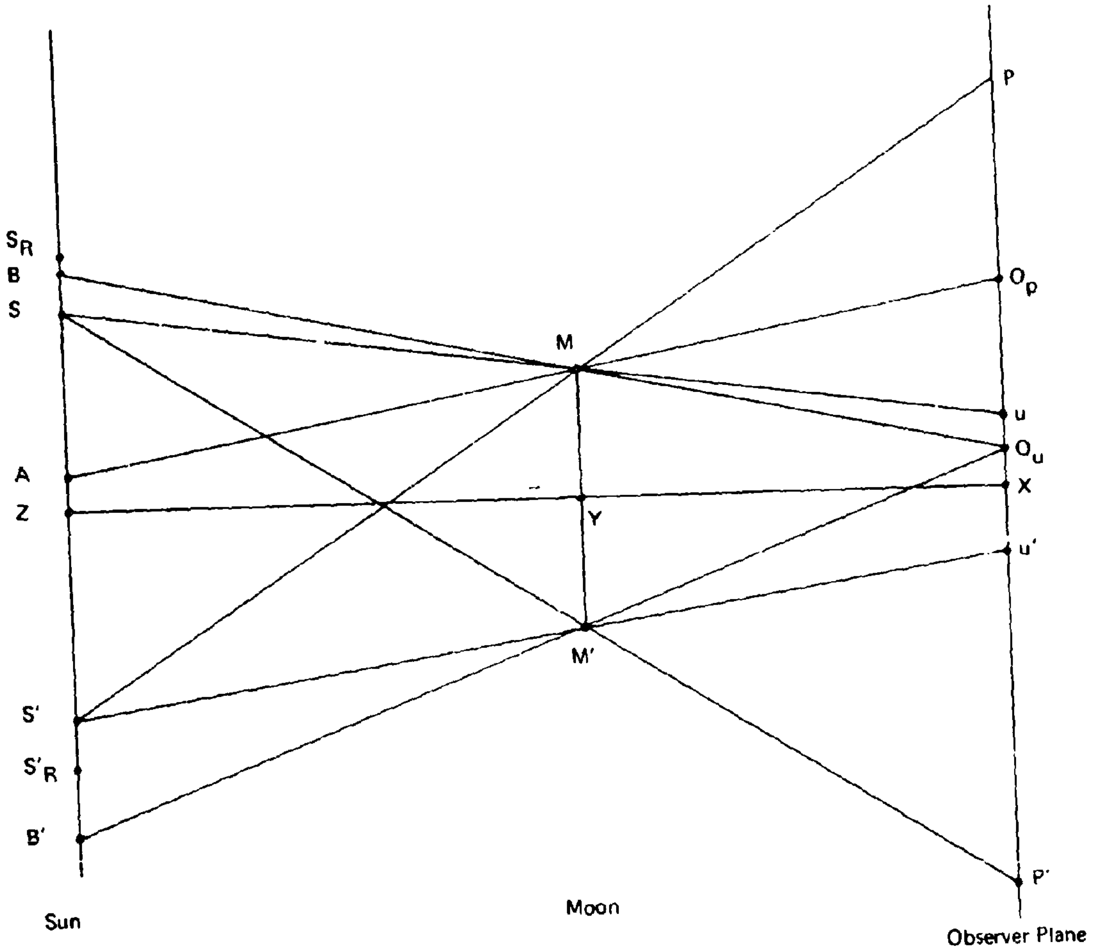

FIGURE D–1 ECLIPSE MAGNITUDE

But $B'B = SB + SB'$ and from the similar triangles UMO_u and SMB; and PMO_u and $S'MS$ we have $SB/S'S = UO_u/PU \equiv UO_u/P'U$, from the similar triangles $SM'B'$ and $P'M'O_u$, and $S'M'S$ and $U'M'P'$ we have $S'B/SS' = P'O_u/P'U'$, hence

$$\frac{SB + SB'}{SS} = \frac{UO_u + P'O_u}{P'U'} = \frac{P'U}{P'U'} = \frac{P'X + XU}{P'X - XU'} ,$$

or

$$M_2 = \frac{L_1 - L_2}{L_1 + L_2} \tag{D.63}$$

Alternatively, we can write that

$$\frac{B'B}{S'S} = \frac{\frac{B'B}{2} \Big/ XZ}{\frac{S'S}{2} \Big/ XZ} = \frac{\frac{M'M}{2} \Big/ XY}{\frac{S'S}{2} \Big/ XZ} = \frac{\sin S_{\leftmoon}}{\sin S_{\odot}} \approx \frac{S_{\leftmoon}}{S_{\odot}}$$

where $S_{\leftmoon}$ and $S_{\odot}$ are the semi-diameters of the moon and sun, respectively. Hence

$$\frac{S_{\leftmoon}}{S_{\odot}} \approx M_2 \tag{D.64}$$

D.8 DEGREE OF OBSCURATION

The degree of obscuration at a particular location is defined as that fraction of the solar disc obscured by the moon during the partial phase and is – in the notation of Figure D-2 and based upon the treatment in [2] – given by

$$S' = \frac{(a_A + a_B)}{\pi r_{\odot}^2} . \tag{D.65}$$

To develop S' in a form suitable for calculation we note first that

$$a_B = 2 \int_{r_{\odot} \cos B}^{r_{\odot}} \int_0^{\sqrt{r_{\odot} - x^2}} dy\, dx$$

or

$$a_B = \left[x\sqrt{r_{\odot}^2 - x^2} + r_{\odot}^2 \sin^{-1}\left(\frac{x}{r_{\odot}}\right)\right] \Bigg|_{r_{\odot} \cos B}^{r_{\odot}}$$

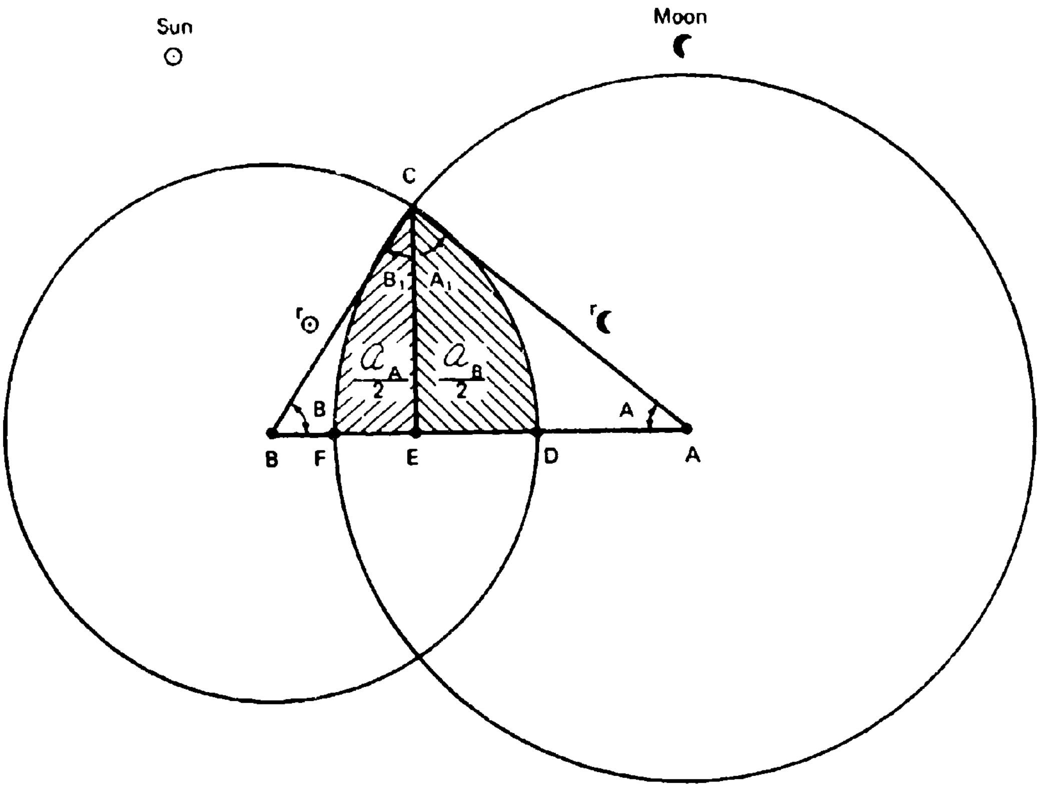

FIGURE D–2 OBSCURATION GEOMETRY

which becomes

$$a_B = r_\odot^2 \frac{\pi}{2} - \left[r_\odot \cos B \sqrt{r_\odot^2 - r_\odot^2 \cos^2 B} + r_\odot^2 \sin^{-1}(\cos B) \right] ,$$

and, since
$$A = \frac{\pi}{2} - A_1 ,$$

$$a_B = r_\odot^2 B - r_\odot^2 \sin B \cos B. \tag{D.66}$$

Similarly, we can write that

$$a_A = r_\leftmoon^2 A - r_\leftmoon^2 \sin A \cos A \tag{D.67}$$

and hence that (D.55) becomes

$$S' = \left[\left(\frac{r_\leftmoon}{r_\odot} \right)^2 A + B - \left(\frac{r_\leftmoon}{r_\odot} \right)^2 \sin A \cos A - \sin B \cos B \right] \Big/ \pi. \tag{D.68}$$

Second, we note from (D.13) that

$$\left(\frac{r_\leftmoon}{r_\odot} \right) = \left(\frac{S_\leftmoon}{S_\odot} \right) = M_2 .$$

Hence, if we set the solar semi-diameter* $r_\odot = \overline{BC} = 1$, we can write – referring once again to Figure D-2 –

$$r_\leftmoon = \overline{AC} = s = \frac{(L_1 - L_2)}{(L_1 + L_2)} . \tag{D.69}$$

From (D.11) we note, since the solar *semi*-diameter is the unit of distance, that

$$\overline{FD} = 2M_1 = \frac{(L_1 - m)}{(L_1 + L_2)} \tag{D.70}$$

and thus that $\overline{BA} = \overline{BF} + \overline{AF} = \overline{BD} - \overline{FD} + \overline{AF} = 1 - 2M_1 + s$, or that

$$\overline{BA} = \frac{2m}{L_1 + L_2} . \tag{D.71}$$

---

* It should be noted that the semi-diameter of the sun can either be that of the "usual" optical sun or that of a specific radio sun, the particular semi-diameter chosen will manifest itself through the value of L_2.

Next, application of the law of cosines twice, yields

$$\overline{AC}^2 = \overline{BC}^2 + \overline{BA}^2 - 2\overline{BC}\ \overline{BA} \cos B$$

and

$$\overline{BA}^2 = \overline{AC}^2 + \overline{BC}^2 - 2\overline{AC}\ \overline{BC} \cos C$$

or, using (D.69), D.70) and (D.71)

$$\cos B = (L_1 L_2 + m^2)/m (L_1 + L_2) \qquad (D.72)$$

and

$$\cos C = (L_1^2 + L_2^2 - 2m^2)/(L_1 + L_2) \qquad (D.73)$$

from which

$$A = \pi - (B + C). \qquad (D.74)$$

Thus, S′ of (D.68) becomes

$$S' = [s^2 A + B - (s^2 \sin A \cos A + \sin B \cos B)]/\pi \qquad (D.75)$$

or, since the third term is twice the area of ΔBCA, which is readily shown to be ½ s sin C,

$$S' = [s^2 A + B - s \sin C]\pi \qquad (D.76)$$

where, during the annular phase, $S' = s^2$ and during the total phase $S' = 1$.

D.9 CENTROID OF UNOBSCURED SOLAR DISC

From symmetry it is seen that the position of the unobscured solar disc centroid lies on an extension of the line joining the solar and lunar centers which, as in the previous section, is taken as the x-axis of a system having its origin at the solar center. Hence, we may write – referring to Figures D-2 and D-3 – that the centroid of area B (a_B) is given by

$$a_B \cdot \overline{x}_B = 2 \int_{r_\odot \cos B}^{r_\odot} \int_0^{\sqrt{r_\odot^2 - x^2}} dy\, x\, dx$$

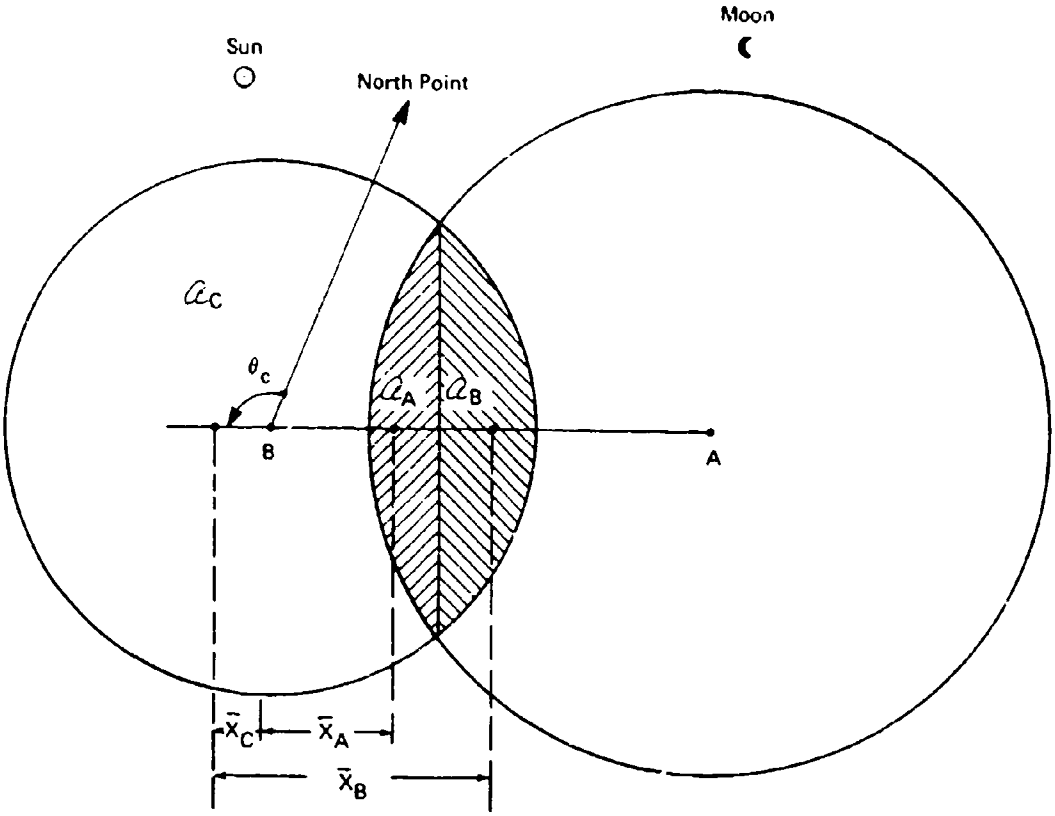

FIGURE D–3 CENTROID GEOMETRY

or

$$a_B \cdot \overline{x}_B = -\frac{2}{3}\left[r_{\odot}^2 - x^2\right]^{3/2}\Big|_{r_{\odot}\cos B}^{r_{\odot}}$$

which becomes

$$\overline{x}_B = \frac{2}{3}\,(r_{\odot}\sin B)^3/a_B \qquad \text{(D.77)}$$

Similarly, we can write that

$$\overline{x}_A = \overline{BA} - \frac{2}{3}\,(r_{☾}\sin A)^3/a_A \qquad \text{(D.78)}$$

Next, we note that the centroid of the combined areas $\overline{x}_{A+B}$ is given by

$$(a_A + a_B)\cdot\overline{x}_{A+B} = a_A \cdot \overline{x}_A + a_B \cdot \overline{x}_B \qquad \text{(D.79)}$$

or

$$(a_A + a_B)\cdot\overline{x}_{A+B} = \overline{BA}\;a_A - \frac{2}{3}\,(r_{☾}\sin A)^3 + \frac{2}{3}\,(r_{\odot}\sin B)^3$$

which, since $r_{☾}\sin A = r_{\odot}\sin B$, reduces to

$$\overline{x}_{A+B} = \overline{BA}\;\frac{a_A}{a_A + a_B} \qquad \text{(D.80)}$$

Furthermore, we note that

$$(a_A + a_B + a_C)\cdot\overline{x}_{A+B+C} = (a_A + a_B)\,\overline{x}_{A+B} + a_C \cdot \overline{x}_C \qquad \text{(D.81)}$$

or since $\overline{x}_{A+B+C} = 0$,

$$\overline{x}_C = -\frac{(a_A + a_B)\cdot\overline{x}_{A+B}}{a_C}$$

which reduces, upon substitution of (D.80) to

$$\overline{x}_C = -\,\overline{BA}\;\frac{a_A}{\pi r_{\odot}^2 - (a_A + a_B)} \qquad \text{(D.82)}$$

or, if the results of the previous section are employed

$$\bar{x}_{\mathbb{C}} = \frac{s^2 (A \quad \sin A \cos A)}{\pi (1 - S')} \left(\frac{2m}{L_1 + L_2} \right) \qquad (D.83)$$

Hence, by specifying its position angle $\theta_{\mathbb{C}} = P - \pi$ and radial distance relative to the solar center $x_{\mathbb{C}}$, the centroid may be located easily on the solar disc.

D.10 TOPOCENTRIC PARAMETERS

The topocentric right ascension $\alpha'_{\leftmoon}$ declination $\delta'_{\leftmoon}$ and distance $r'_{\leftmoon}$ of the moon can be calculated from the following exact relationships

$$r'_{\leftmoon} \cos \delta'_{\leftmoon} \cos \alpha'_{\leftmoon} = r_{\leftmoon} \cos \delta_{\leftmoon} \cos \alpha_{\leftmoon} - \rho \cos \phi' \cos \tau_{\ell_s}$$

$$r'_{\leftmoon} \cos \delta'_{\leftmoon} \sin \alpha'_{\leftmoon} = r_{\leftmoon} \cos \delta_{\leftmoon} \sin \alpha_{\leftmoon} - \rho \cos \phi' \sin \tau_{\ell_s}$$

$$r'_{\leftmoon} \sin \delta'_{\leftmoon} = r_{\leftmoon} \sin \delta_{\leftmoon} - \rho \sin \phi' \qquad (D.84)$$

in which $\alpha_{\leftmoon}$ and $\delta_{\leftmoon}$ are the geocentric lunar right ascension and declination of (A.46), respectively; $r_{\leftmoon}$ is the geocentric distance given by

$$r_{\leftmoon} = a_{\oplus}/\sin \pi_{\leftmoon} \qquad (D.85)$$

where the lunar parallax $\pi_{\leftmoon}$ comes from (A.45); and τ_{ℓ_s} is the local sidereal time (L.S.T.) at τ hours E.T. given by (B.4).

Next, the azimuth $Az_{\leftmoon}$ (measured east from north) and the elevation $E\ell_{\leftmoon}$ of the moon can be calculated from

$$\cos E\ell_{\leftmoon} \cos Az_{\leftmoon} = + \sin \delta'_{\leftmoon} \cos \phi - \cos \delta'_{\leftmoon} \sin \phi \cos (\tau_{\ell_s} - \alpha'_{\leftmoon})$$

$$\cos E\ell_{\leftmoon} \sin Az_{\leftmoon} = - \cos \delta'_{\leftmoon} \sin (\tau_{\ell_s} - \alpha'_{\leftmoon})$$

$$\sin E\ell_{\leftmoon} = + \sin \delta'_{\leftmoon} \sin \phi + \cos \delta'_{\leftmoon} \cos \phi \cos (\tau_{\ell_s} - \alpha'_{\leftmoon}) \qquad (D.86)$$

in which $(\tau_{\ell_s} - \alpha'_{\leftmoon})$ is recognized as the hour angle of the moon.

And finally, the topocentric lunar semi-diameter $S'_{\leftmoon}$ can be calculated from the expression

$$\sin S'_{\leftmoon} = \frac{r_{\leftmoon}}{r'_{\leftmoon}} \sin S_{\leftmoon} \tag{D.87}$$

where the sine of the geocentric lunar semi-diameter $S_{\leftmoon}$ can be gotten directly from (A.48) we note in passing that $S'_{\leftmoon} - S_{\leftmoon}$ is known as the augmentation.

In the case of the sun, an approximate form of (D.84) suffices; thus, the topocentric right ascension $\alpha'_{\odot}$ and the declination $\delta'_{\odot}$ are given by

$$\begin{aligned} \alpha'_{\odot} &= \alpha_{\odot} - \Delta\alpha_{\odot} \\ \delta'_{\odot} &= \delta_{\odot} - \Delta\delta_{\odot} \end{aligned} \tag{D.88}$$

in which

$$\begin{aligned} \Delta\alpha_{\odot} &= \pi_{\odot} \, [\rho \cos \phi' \sec \delta_{\odot} \sin (\tau_{\ell_s} - \alpha_{\odot}) \\ \Delta\delta_{\odot} &= \pi_{\odot} \, [\rho \sin \phi' \cos \delta_{\odot} - \rho \cos \phi' \sin \delta_{\odot} \cos (\tau_{\ell_s} - \alpha_{\odot})] \end{aligned} \tag{D.89}$$

and where $\pi_{\odot}$ is given by (A.22). As in (D.86), the azimuth $Az_{\odot}$ and $E\ell_{\odot}$ are given by

$$\begin{aligned} \cos E\ell_{\odot} \cos Az_{\odot} &= + \sin \delta'_{\odot} \cos \phi - \cos \delta'_{\odot} \sin \phi \cos (\tau_{\ell_s} - \alpha'_{\odot}) \\ \cos E\ell_{\odot} \sin Az &= - \cos \delta'_{\odot} \sin (\tau_{\ell_s} - \alpha'_{\odot}) \\ \sin E\ell_{\odot} &= + \sin \delta'_{\odot} \sin \phi + \cos \delta'_{\odot} \cos \phi \cos (\tau_{\ell_s} - \alpha'_{\odot}) \end{aligned} \tag{D.90}$$

where it should be noted, for locations near the center line and at or around eclipse maximum, that the useful approximation

$$\sin E\ell_{\odot} \approx \xi \tag{D.91}$$

follows from an examination of (C.20).

Since the solar semi-diameter is almost totally insensitive to topocentric variations; the value given by (A.23) can be used throughout the eclipse.

APPENDIX E

SOLAR COORDINATE SYSTEMS AND RELATIONSHIPS

E.0 INTRODUCTION

In this appendix various coordinate systems applicable to solar astronomy will be described and their interrelationships defined. Particular attention will be directed to the development of both precise and approximate "pointing instructions" for observation of phenomena both on and above the solar surface.

E.1 PRELIMINARY DISCUSSION OF SOLAR COORDINATE SYSTEMS

Points are located on or above the solar surface by specifying both their heliographic longitude L and heliographic latitude B, once the prime meridian of the rotating* sun (the x-axis of the heliographic system) and the rotation axis of the sun (the z-axis of the heliographic system) have been specified for the date and time of interest.

This specification uses as reference an ecliptic plane having no secular motion and furthermore, the latitude of the sun is neglected. Hence, one requires, first, the inclination of the solar equator with respect to the ecliptic I, or, equivalently, the angle between the ecliptic north pole z_c and the north pole of the sun z_H (this is shown in Figure E-1); second, the longitude of the ascending node of the solar equator on the ecliptic Ω; and finally, because of the gaseous nature of the sun and, hence, the lack of a recognizable reference point on the solar equator with which to measure longitudes, the location of the prime meridian by specification of the heliographic longitude of the ascending node as a given function of time, M. These are given by

$$\begin{aligned} I &= 7^{\circ}15' \\ \Omega &= 73^{\circ}40' + 50''.25\,(t - 1850.0) \\ M - 180^{\circ} &= 112^{\circ}.766 - 14^{\circ}.18439716\,(\text{J.D.} - 243\,0000.5) \end{aligned} \qquad \text{(E.1)}$$

in which t represents the number of years since 1850 and J.D. represents the Julian Day number of the date and time of observation.

In "actual" appearance, of course, the sun resembles a disc onto which points both on and above the solar surface are apparently projected. The normal to the disc (x', x'' in Figure E-1) lies along the earth-sun line at the celestial longitude $\lambda_{\odot} + \pi$ where $\lambda_{\odot}$ is given by (A.17) or (A.18). A projected point is located on the disc by specifying two polar coordinates: first, its radial distance

---

* In what follows we neglect the differential (heliographic latitude dependent) rotation of the sun. Longitude is measured westward (in the direction of rotation) from the prime meridian and ranges from 0° to 360°; latitude is measured from the solar equator, positive towards the north.

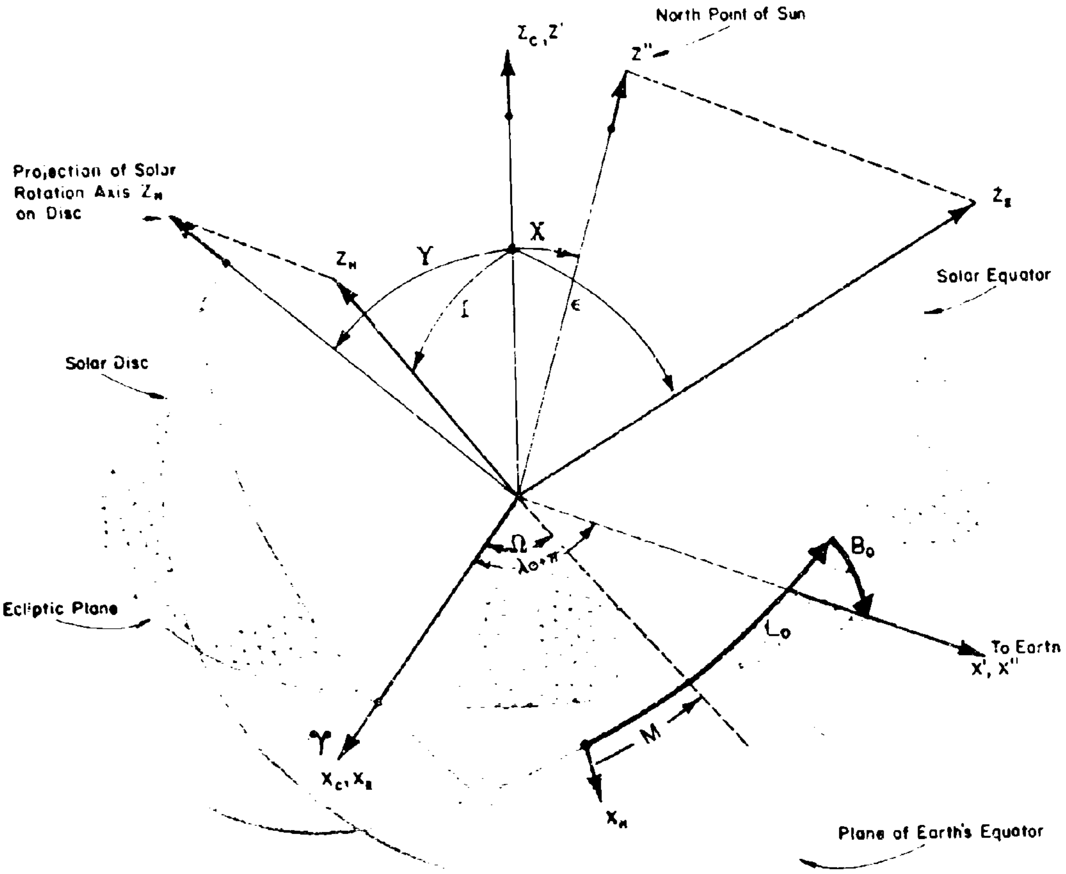

FIGURE E-1 SOLAR COORDINATE SYSTEMS

from the disc center, equivalent to the angle ρ subtended at the center of the sun by this point and the earth's center; and second, its position angle θ measured eastward from the north point (the z″ axis of Figure E-1) which is the projection of the earth's rotation axis z_E on the disc*. The relationship between this "observational" disc coordinate system and the heliographic systems can be fixed by specification of the position angle P of the projection of the solar rotation axis z_H on to the disc

E.2 PRELIMINARY CONSIDERATIONS

In the remainder of this appendix we assume that a positive rotation with respect to an axis proceeds in the right handed sense, and furthermore, in the following three relations, we assume that the rotation operator operates upon the coordinate basis x, y, z to produce the transformed x′, y′, z′ basis.

By positive rotations with respect to x, y and z axes we mean

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} , \tag{E.2}$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} , \tag{E.3}$$

and

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \text{E.4)}$$

respectively.

---

* As discussed in Appendix D, the north point is observationally defined as that point on the solar limb which is the tangent point of the parallel of declination defined by the apparent (diurnal) motion of the sun.

E.3 THE HELIOGRAPHIC/ECLIPTIC COORDINATE TRANSFORMATION

The heliographic system (x_H, y_H, z_H) is transformed into the ecliptic system (x_c, y_c, z_c) by the following sequence of rotations: first, a rotation around z_H by M; second, a rotation around the new x-axis by − I; and finally, a rotation around the new z-axis by −Ω. This leads to

$$\begin{pmatrix} x_c \\ y_c \\ z_c \end{pmatrix} = [R^{CH}] \begin{pmatrix} x_H \\ y_H \\ z_H \end{pmatrix} \tag{E.5}$$

in which

$$\begin{aligned}
R_{11}^{CH} &= \cos\Omega \cos M + \sin\Omega \sin M \cos I \\
R_{12}^{CH} &= -[\sin\Omega \cos M \cos I - \cos\Omega \sin M] \\
R_{13}^{CH} &= \sin\Omega \sin I \\
R_{21}^{CH} &= \sin\Omega \cos M - \cos\Omega \sin M \cos I \\
R_{22}^{CH} &= \cos\Omega \cos M \cos I + \sin\Omega \sin M \\
R_{23}^{CH} &= -\cos\Omega \sin I \\
R_{31}^{CH} &= -\sin I \sin M \\
R_{32}^{CH} &= \sin I \cos M \\
R_{33}^{CH} &= \cos I
\end{aligned} \tag{E.6}$$

E.4 THE 'DISC'/HELIOGRAPHIC COORDINATE TRANSFORMATION AND RELATIONS

The 'disc'/heliographic transformation can be developed by either of two routes, namely: that the ecliptic system (x_c, y_c, z_c) is transformed into the 'disc' system (x', y', z') by a rotation around z_c through ($\lambda_\odot + \pi$) and, in turn, that the ecliptic system is transformed by use of (E.5) or, equivalently, that the same development as that leading to (E.5) is used but with $-\Omega$ replaced by $-\Omega + (\lambda_\odot + \pi)$ in the third rotation. In either case,

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = [R'^{H}] \begin{pmatrix} x_H \\ y_H \\ z_H \end{pmatrix} \tag{E.7}$$

with

$$\begin{aligned}
R'^{H}_{11} &= -\cos(\lambda_\odot - \Omega)\cos M + \sin(\lambda_\odot - \Omega)\sin M \cos I \\
R'^{H}_{12} &= -[\sin(\lambda_\odot - \Omega)\cos M \cos I + \cos(\lambda_\odot - \Omega)\sin M] \\
R'^{H}_{13} &= \sin(\lambda_\odot - \Omega)\sin I \\
R'^{H}_{21} &= \sin(\lambda_\odot - \Omega)\cos M + \cos(\lambda_\odot - \Omega)\sin M \cos I \\
R'^{H}_{22} &= -\cos(\lambda_\odot - \Omega)\cos M \cos I + \sin(\lambda_\odot - \Omega)\sin M \\
R'^{H}_{23} &= \cos(\lambda_\odot - \Omega)\sin I \\
R'^{H}_{31} &= -\sin I \sin M \\
R'^{H}_{32} &= \sin I \cos M \\
R'^{H}_{33} &= \cos I
\end{aligned} \tag{E.8}$$

Next, we note that the x'-axis of the 'disc' system, i.e., the 'disc' center, defined by the subterrestrial point located at heliographic longitude L_0 and latitude B_0, can be specified by transposing (E 7) in the form

$$\begin{pmatrix} \cos B_0 \cos L_0 \\ \cos B_0 \sin L_0 \\ \sin B_0 \end{pmatrix} = [R'^{H}]^T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} R'^{H}_{11} \\ R'^{H}_{12} \\ R'^{H}_{13} \end{pmatrix} \tag{E.9}$$

or

$$\begin{aligned}
\cos B_0 \cos L_0 &= -\cos(\lambda_\odot - \Omega)\cos M + \sin(\lambda_\odot - \Omega)\sin M \cos I \\
\cos B_0 \sin L_0 &= -[\sin(\lambda_\odot - \Omega)\cos M \cos I + \cos(\lambda_\odot - \Omega)\sin M] \\
\sin B_0 &= \sin(\lambda_\odot - \Omega)\sin I
\end{aligned} \tag{E.10}$$

These, in turn, can be transformed, by means of some straightforward manipulation into the more commonly employed form for calculating L_0 and B_0,

$$\sin B_0 = \sin (\lambda_\odot - \Omega) \sin I$$

$$\cos B_0 \cos (L_0 - M) = - \cos (\lambda_\odot - \Omega)$$

$$\cos B_0 \sin (L_0 - M) = - \sin (\lambda_\odot - \Omega) \cos I. \qquad \text{(E.11)}$$

Finally we have that the angle Y between the projection onto the 'disc' of the sun's rotation axis z_H and the ecliptic pole can be gotten in two steps: first acquire the disc coordinates of z_H through

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = [R'^H] \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} R'^H_{13} \\ R'^H_{23} \\ R'^H_{33} \end{pmatrix} \qquad \text{(E.12)}$$

or

$$x' = \sin (\lambda_\odot - \Omega) \sin I$$

$$y' = \cos (\lambda_\odot - \Omega) \sin I$$

$$z' = \cos I \qquad \text{(E.13)}$$

and second, note that

$$\tan Y = - \frac{y'}{z'} = - \cos (\lambda_\odot - \Omega) \tan I. \qquad \text{(E.14)}$$

E.5 THE 'DISC'/EQUATORIAL COORDINATE TRANSFORMATION AND RELATIONS

The 'disc'/equatorial transformation is developed by noting that the equatorial system is transformed into the ecliptic system by a rotation around x_E through the obliquity ϵ and that, in turn, the ecliptic system is transformed into the disc system by a rotation around z_c by the angle $(\lambda_\odot + \pi)$. Hence,

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = [R'^E] \begin{pmatrix} x_E \\ y_E \\ z_E \end{pmatrix} \qquad \text{(E.15)}$$

with

$$R'^{E}_{11} = -\cos\lambda_{\odot}$$

$$R'^{E}_{12} = -\sin\lambda_{\odot}\ \cos\epsilon$$

$$R'^{E}_{13} = -\sin\lambda_{\odot}\ \sin\epsilon$$

$$R'^{E}_{21} = \sin\lambda_{\odot}$$

$$R'^{E}_{22} = -\cos\lambda_{\odot}\cos\epsilon$$

$$R'^{E}_{23} = -\cos\lambda_{\odot}\ \sin\epsilon$$

$$R'^{E}_{31} = 0$$

$$R'^{E}_{32} = -\sin\epsilon$$

$$R'^{E}_{33} = \cos\epsilon \tag{E.16}$$

Furthermore, we also note that the angle X between the north point defining the projection of the earth's rotation axis z_E on the 'disc' and the ecliptic pole can, like the angle Y be gotten in two steps: first, acquire the disc coordinates of z_E through

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = [R'^{E}] \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} R'^{E}_{13} \\ R'^{E}_{23} \\ R'^{E}_{33} \end{pmatrix} \tag{E 17}$$

or

$$x' = -\sin\lambda_{\odot}\ \sin\epsilon$$

$$y' = -\cos\lambda_{\odot}\ \sin\epsilon$$

$$z' = \cos\epsilon \tag{E.18}$$

and second, note that

$$\tan X = \frac{y'}{z'} = -\cos\lambda_{\odot}\ \tan\epsilon. \tag{E.19}$$

Finally, we see that the position angle P of the solar rotation axis is given by

$$P = X + Y \qquad (E.20)$$

where Y was defined in (E.14).

E.6 THE 'DISC'/"DISC" COORDINATE TRANSFORMATION

The 'disc'/"disc" transformation is developed by rotating the ecliptic pole (the z'-axis) around the x'-axis by the angle $-X$ into the north point (the z″-axis), or that

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & \cos X & -\sin X \\ 0 & \sin X & \cos X \end{pmatrix} \begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} \qquad (E.21)$$

E.7 THE HELIOGRAPHIC/"DISC" COORDINATE TRANSFORMATION AND RELATIONS

This "disc" system (x″, y″, z″) is transformed into the heliographic system (x_H, y_H, z_H) by the following rotations: first, a rotation around x″ by P; second, a rotation around the new y-axis by B_0; and finally, a rotation around the z_H axis by $-L_0$. This leads to

$$\begin{pmatrix} x_H \\ y_H \\ z_H \end{pmatrix} = [R^{H''}] \begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} \qquad (E.22)$$

in which

$$\begin{aligned}
R_{11}^{H''} &= \cos B_0 \cos L_0 \\
R_{12}^{H''} &= -[\cos P \sin L_0 - \sin P \cos L_0 \sin B_0] \\
R_{13}^{H''} &= -[\sin P \sin L_0 + \cos P \cos L_0 \sin B_0] \\
R_{21}^{H''} &= \cos B_0 \sin L_0 \\
R_{22}^{H''} &= \cos P \cos L_0 + \sin P \sin L_0 \sin B_0 \\
R_{23}^{H''} &= \sin P \cos L_0 - \cos P \sin L_0 \sin B_0 \\
R_{31}^{H''} &= \sin B_0 \\
R_{32}^{H''} &= -\sin P \cos B_0 \\
R_{33}^{H''} &= \cos P \cos B_0 .
\end{aligned} \qquad (E.23)$$

Consequently, a point on the surface of the sun (of radius $R_\odot$) at latitude B and longitude L, thereby having the heliographic coordinates

$$\begin{aligned} x_H &= R_\odot \cos B \cos L \\ y_H &= R_\odot \cos B \sin L \\ z_H &= R_\odot \sin B, \end{aligned} \tag{E.24}$$

will be transformed into a point in the "disc" system with heliocentric angle ρ and position angle θ thereby having the "disc" coordinates

$$\begin{aligned} x'' &= \quad R_\odot \cos \rho \\ y'' &= -R_\odot \sin \rho \sin \theta \\ z'' &= \quad R_\odot \sin \rho \cos \theta \end{aligned} \tag{E.25}$$

If these are substituted in (E.22), and appropriate operations and manipulations performed, we have that

$$\sin B = \sin B_0 \cos \rho + \cos B_0 \sin \rho \cos (P - \theta) \tag{E.26}$$

$$\cos B \sin (L - L_0) = \sin \rho \sin ((P - \theta) \tag{E.27}$$

which permits the immediate calculation of the heliographic latitude and longitude of a point once its disc coordinates ρ (whose determination will be discussed in Section E.10) and θ have been specified. Equivalently, we have

$$\cos \rho = \cos B_0 \cos B \cos (L - L_0) + \sin B_0 \sin B \tag{E.28}$$

$$\sin \rho \sin (P - \theta) = \cos B \sin (L - L_0) \tag{E.29}$$

if B and L are specified and ρ and θ are required.

E.8 RIGHT ASCENSION AND DECLINATION OF AN OBSERVED POINT ON THE "DISC"

This development is initiated by employing the transpose of (E.21) operating on the 'disc' components of the point in the form

$$\begin{aligned} x' &= \quad R_\odot \cos \rho \\ y' &= -R_\odot \sin \rho \sin (\theta - X) \\ z' &= -R_\odot \sin \rho \cos (\theta - X) \end{aligned} \tag{E.30}$$

to yield

$$
\begin{aligned}
x_E &= R_\odot\,[-\cos\rho\cos\lambda_\odot - \sin\rho\sin\lambda_\odot\sin(\theta - X)] \\
y_E &= R_\odot\,[-\cos\rho\sin\lambda_\odot\cos\epsilon + \sin\rho\cos\lambda_\odot\cos\epsilon\sin(\theta - X) \\
&\qquad - \sin\rho\sin\epsilon\cos(\theta - X)] \\
z_E &= R_\odot\,[-\cos\rho\sin\lambda_\odot\sin\epsilon + \sin\rho\cos\lambda_\odot\sin\epsilon\sin(\theta - X) \\
&\qquad + \sin\rho\cos\epsilon\cos(\theta - X)]\,. \qquad \text{(E.31)}
\end{aligned}
$$

Next, because the geocentric, equatorial x, y, z components of the solar center (neglecting the solar latitude) are given by (A.21) as

$$
\begin{aligned}
x_\odot &= R\cos\delta_\odot\cos\alpha_\odot = R\cos\lambda_\odot \\
y_\odot &= R\cos\delta_\odot\sin\alpha_\odot = R\sin\lambda_\odot\cos\epsilon \\
z_\odot &= R\sin\delta_\odot = R\sin\lambda_\odot\sin\epsilon \qquad \text{(E.32)}
\end{aligned}
$$

then the geocentric, equatorial components of the disc point can be written by

$$
\begin{aligned}
x = x_\odot + x_E = R &\left[\left(1 - \frac{R_\odot}{R}\cos\rho\right)\cos\lambda_\odot \right. \\
&\left. - \left(\frac{R_\odot}{R}\sin\rho\right)\sin(\theta - X)\sin\lambda_\odot\right] \\
y = y_\odot + y_E = R &\left[\left(1 - \frac{R_\odot}{R}\cos\rho\right)\sin\lambda_\odot\cos\epsilon \right. \\
&\left. + \left(\frac{R_\odot}{R}\sin\rho\right)(\sin(\theta - X)\cos\epsilon\cos\lambda_\odot - \cos(\theta - X)\sin\epsilon)\right] \\
z = z_\odot + z_E = R &\left[\left(1 - \frac{R_\odot}{R}\cos\rho\right)\sin\lambda_\odot\sin\epsilon + \right. \\
&\left. + \left(\frac{R_\odot}{R}\sin\rho\right)(\cos(\theta - X)\cos\epsilon + \sin(\theta - X)\sin\epsilon\cos\lambda_\odot)\right]
\end{aligned}
$$

(E.33)

Consequently, if we employ the reasonable approximation that $R_s^2 \equiv x^2 + y^2 + z^2 \approx R^2$, we can write immediately that the right ascension α_s and declination δ_s of the point are given by

$$\begin{aligned} R \cos \delta_s \cos \alpha_s &= x \\ R \cos \delta_s \sin \alpha_s &= y \\ R \sin \delta_s &= z \end{aligned} \qquad \text{(E.34)}$$

where x, y, and z are given by (E.33).

But (E.34) and (E.33) can be transformed into a more familiar form by expanding the sin $(\theta - X)$ and cos $(\theta - X)$ terms and identifying sin x and cos x with the ecliptic and equatorial coordinates of the sun. For from (E.18) and (E.32) we have that

$$(y')^2 + (z')^2 = (-\cos\lambda_\odot \sin\epsilon)^2 + \cos^2\epsilon = \cos^2\delta_\odot \qquad \text{(E.35)}$$

so that

$$\sin X = \frac{y'}{\sqrt{(y')^2 + (z')^2}} = -\cos\lambda_\odot \sin\epsilon / \cos\delta_\odot \qquad \text{(E.36)}$$

and

$$\cos X = \frac{z'}{\sqrt{(y')^2 + (z')^2}} = \cos\epsilon / \cos\delta_\odot \qquad \text{(E.37)}$$

which, when employed – together with (E.32) – in (E.34), yield (after considerable manipulation)

$$\begin{aligned} \cos\delta_s \cos\alpha_s &= \left(1 - \frac{R_\odot}{R} \cos\rho\right) \cos\delta_\odot \cos\alpha_\odot - \\ &\quad \frac{R_\odot}{R} \sin\rho \, [\sin\theta \sin\alpha_\odot + \cos\theta \cos\alpha_\odot \sin\delta_\odot] \\ \cos\delta_s \sin\alpha_s &= \left(1 - \frac{R_\odot}{R} \cos\rho\right) \cos\delta_\odot \sin\alpha_\odot + \\ &\quad + \frac{R_\odot}{R} \sin\rho \, [\sin\theta \cos\alpha_\odot - \cos\theta \sin\alpha_\odot \sin\delta_\odot] \\ \sin\delta_s &= \left(1 - \frac{R_\odot}{R} \cos\rho\right) \sin\delta_\odot + \frac{R_\odot}{R} \sin\rho \cos\theta \cos\delta_\odot] , \end{aligned}$$

(E.38)

or

$$\frac{R_\odot}{R}\sin\rho\sin\theta = \sin(\alpha_s - \alpha_\odot)\cos\delta_s$$

$$\frac{R_\odot}{R}\sin\rho\cos\theta = \cos\delta_\odot\sin\delta_s - \sin\delta_\odot\cos\delta_s\cos(\alpha_s - \alpha_\odot)$$

$$\left(1 - \frac{R_\odot}{R}\cos\rho\right) = \sin\delta_\odot\sin\delta_s + \cos\delta_\odot\cos\delta_s\cos(\alpha_s - \alpha_\odot). \qquad \text{(E.39)}$$

But reference to Figure E-2 shows that

$$\sin\rho_1 = \frac{R_\odot}{\mathcal{R}_s}\sin\rho \approx \frac{R_\odot}{R}\sin\rho \qquad \text{(E.40)}$$

and

$$\cos\rho_1 = \frac{R - R_\odot\cos\rho}{\mathcal{R}_s} \approx 1 - \frac{R_\odot}{R}\cos\rho \qquad \text{(E.41)}$$

so that (E.39) reduces to the familiar set of relations

$$\sin\rho_1\sin\theta = \sin(\alpha_s - \alpha_\odot)\cos\delta_s$$

$$\sin\rho_1\cos\theta = \cos\delta_\odot\cos\delta_s - \sin\delta_\odot\sin\delta_s\cos(\alpha_s - \alpha_\odot)$$

$$\cos\rho_1 = \sin\delta_\odot\sin\delta_s + \cos\delta_\odot\cos\delta_s\cos(\alpha_s - \alpha_\odot) \qquad \text{(E.42)}$$

which also follow directly from Figure E-3 and the solution of the astronomical triangle on the celestial sphere of infinite radius (wherein the approximations of (E.40) and (E.41) are rendered unnecessary).*

If the separation of $(\alpha_\odot, \delta_\odot)$ and (α_s, δ_s) is sufficiently small that square and higher order terms can be neglected, then

$$\rho_1\sin\theta \approx (\alpha_s - \alpha_\odot)\cos\delta_s$$

$$\rho_1\cos\theta \approx \sin(\delta_s - \delta_\odot) \approx \delta_s - \delta_\odot\ .$$

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* See [9] pp. 25-26.

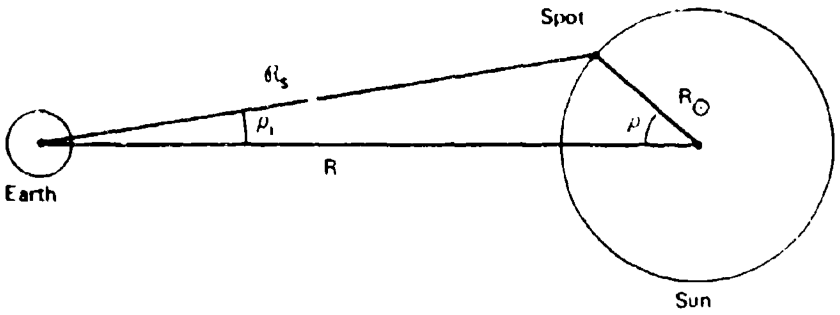

FIGURE E-2 EARTH-SUN-SPOT SIDE VIEW

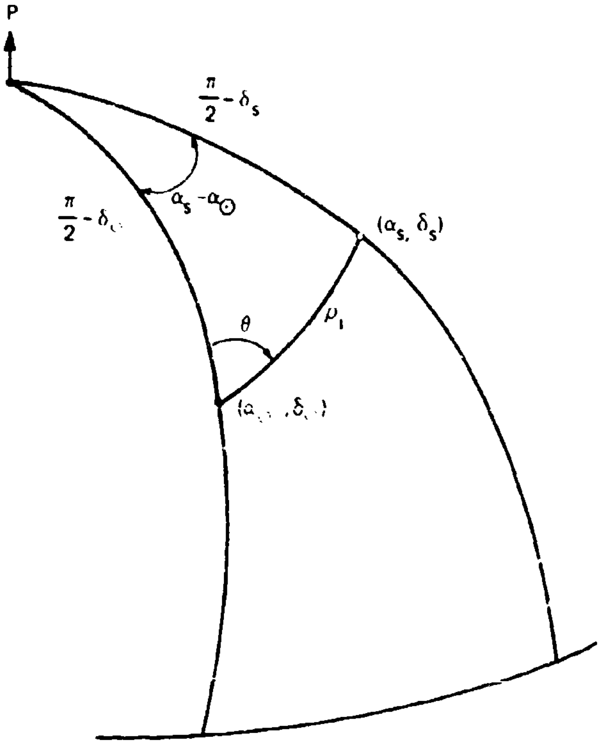

FIGURE E-3 THE ASTRONOMICAL TRIANGLE

or, to within this level of approximation

$$\alpha_s - \alpha_\odot \approx \rho_1 \sin\theta \sec\delta_\odot \qquad \text{(E.43)}$$

$$\delta_s - \delta_\odot \approx \rho_1 \cos\theta. \qquad \text{(E.44)}$$

E.9 DETERMINATION OF ρ FROM OBSERVATIONS

The determination of ρ is based on the geometry of Figure E-4, a more detailed version of Figure E-2, which also provides for a radio sun of radius $R_R = \mathcal{H}_R R_\odot$ with $\mathcal{H}_R \geqslant 1$ as well as for the optical sun of radius R. r and $r_\odot$ are meant to symbolize linear observational measures in some convenient scale on say, a photograph.

Because of its small magnitude (~ 16′) we can write that the solar semi-diameter of A.23 is given by

$$\sin S_\odot = \frac{R_\odot}{R} \approx S_\odot \qquad \text{(E.45)}$$

and, to the same level of approximation that

$$\frac{\rho_1}{S_\odot} \approx \frac{r}{r_0} \qquad \text{(E.46)}$$

Hence, ρ_1 can be determined from the three known or measured quantities $S_\odot$, r and r_0.

But we see that

$$\frac{\sin A}{R} = \frac{\sin(\rho_1 + \rho)}{R} = \frac{\sin\rho_1}{R_\odot} \approx \frac{\rho_1}{R_\odot} \qquad \text{(E.47)}$$

or, employing (E.45), that

$$\sin(\rho_1 + \rho) \approx \rho_1 / S_\odot \qquad \text{(E.48)}$$

where it should be pointed out that the optical radius has been tacitly employed.

If on the other hand we wish to calculate the ρ appropriate to a radio sun of radius $R_R = \mathcal{H}_R R_\odot$ from the same ρ_1, we modify (E.48) such that

$$\sin(\rho_1 + \rho_R) \approx \rho_1 / S_K = \frac{1}{\mathcal{H}_R}\left(\frac{\rho_1}{S_\odot}\right). \qquad \text{(E.49)}$$

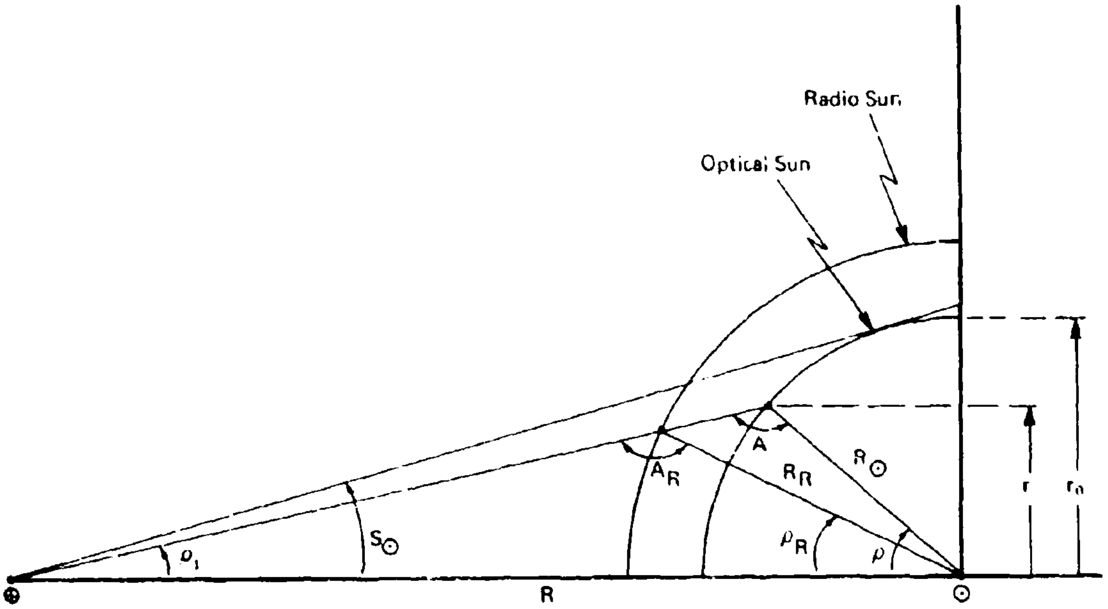

FIGURE E–4 EARTH-SUN OBSERVATIONAL GEOMETRY