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SVM-4

Dynamics of Rotating Shafts

DYNAMICS OF ROTATING SHAFTS

The Shock and Vibration Information Center
United States Department of Defense

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PREFACE

In engineering circles it is common to speak of "inventing the wheel" to imply the oldest of developments. And it is true that rotational motion was employed—for example, to achieve translation as with the wheel and axle, or to store energy as in the sling—in truly ancient devices.

Much later, in transferring power from one point to another, the use of drive belts and related mechanisms and multistep cogwheel (and subsequently, gear) trains gave way to the use of drive shafts because of their advantages as regards efficiency, wear, and adjustment. Since strength requirements are related to the torque carried by such a shaft, and the relationship between torque and rotational speed is inverse—for a given level of transmitted power—there has been a continuing trend toward higher and higher shaft speeds. The dynamic forces of rotation, which act on drive shafts, of course, also increase with rotational speed. Thus, the problems associated with rotational shaft dynamics have been increasingly important in the field of engineering for the last century.

The proliferation of devices rotating at high speeds in recent years has brought both new problems and a concentration of attention which, in solving one problem, has thrown new light on a host of others—sometimes on a puzzling phenomenon or aspect of long standing. The "fly-ball governor," probably the first recognizable component with no function other than to provide feedback control, depended on rotational dynamics; ultrahigh speed gyros may be considered their modern counterpart. Gas turbines have rotational speeds that were unheard of only a short time ago, and the closer to those speeds a drive shaft can function, the less gearing is required to go from one to the other. Airplane propellers and helicopter rotors provide still another class of examples where limber structures are subject to problems arising from the complexities of rotational dynamics.

In preparing this monograph, the authors and the Shock and Vibration Information Center intended to provide a stock-taking of the knowledge and techniques accumulated to date and the rotating shaft dynamics problems remaining at this time. Because the problems are familiar, in the nonscientific sense, to virtually everyone in the scientific and technological fields, but understood rather poorly by most—including many mechanical designers directly involved with the creation of new rotating systems—the approach taken has been comprehensive without attempting to be complete. The early chapters have the objective of providing physical descriptions of the simplest examples which could be thought of (both conceptually and mathematically) to illustrate the phenomena in question. Later, the mathematical foundations are presented with, hopefully, sufficient rigor to provide insight into realistic cases and a jumping-off point from which one familiar with engineering analysis but not necessarily with rotational dynamics can develop a useful capability in this field. Throughout, references to the extensive literature are made, along with (mostly unproven) statements as to the important phenomena encountered with typical shaft systems.

While the time spent in preparing this monograph did not allow a really exhaustive treatment of any area (references to the literature are depended on for that), the chapter on balancing is perhaps the farthest from being complete. The authors certainly did not mean to imply, by leaving the chapter in this state, that the subject is unimportant. On the contrary, no matter how well the theory is developed and how refined the manufacturing procedures, it seems likely that some test procedure will always be required, perhaps for every high-speed shaft intended for operation where safety is paramount—as, for example, in aircraft.

This subject, however, is undergoing more rapid transition today than any other covered in the monograph. For example, the whole concept of a "balancing machine" may be abandoned, or at least drastically revised, for supercritical shafts. The importance of the supporting structure to satisfactory operation may require that production shafts be balanced in situ or on an installation which closely simulates all the important structural and dynamic properties of the frame on which the shaft system is to be supported during its useful life. Neither the associated theoretical methods nor experimental techniques are now at a stage where they can be summarized with the same confidence as the other material in this book. Such developments are, however, "just around the corner"; no fundamental advances are required, and analysts, designers, and manufacturers in the field are at work to arrive at the needed procedures.

An extensive bibliography is presented, which includes more than 540 entries. These are cross referenced in several forms; for example, by author and by subject. In discussing the important phenomena encountered in typical shaft systems, references to the extensive literature are made throughout the text to allow the reader to pursue the complete, detailed treatments which could not be included in this review of the subject.

The authors gratefully acknowledge the support of W. W. Mutch and H. C. Pusey, of The Shock and Vibration Information Center, for enabling this monograph to be produced and for furnishing advice and guidance throughout the progress of the task. Acknowledgment is also due the Foreign Technology Division of the Air Force Systems Command, the Defense Documentation Center, and the Scientific and Technical Information Facility of the National Aeronautics and Space Administration, for their help in preparing the bibliography.

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CHAPTER 1
INTRODUCTION

1.1 Historical Perspective

The first published work on the dynamics of rotating shafts was presented in 1869 by Rankine [1]. According to Gunter [2], Rankine's neglect of Coriolis acceleration led to erroneous conclusions which confused engineers for half a century. Despite its long history—for a technological subject—confusion exists even today in otherwise knowledgeable circles regarding aspects of these phenomena, whose names are familiar to almost all engineers.

Figure 1 shows the chronological trend begun by Rankine. The rate of publication has gone from less than one per year (on the average) for the pre-1935 period to 65 per year in 1965 and 1966. This is evidence that the rate of growth of interest in the dynamics of rotating shafts has at least kept pace with the general growth in technology.

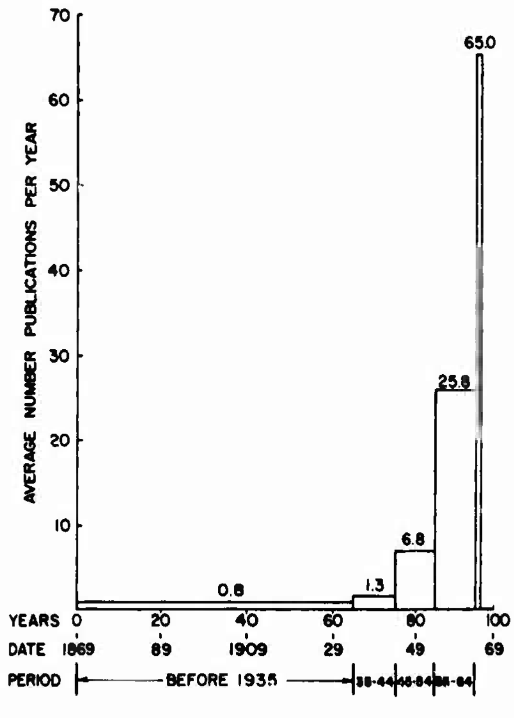

Fig. 1. Chronological trend of publishing activity.

The open-literature publications on the subject now total more than 550 references. The number of contributing authors stands at more than 425. Approximately half are not American, the two most prolific authors on the subject being Dimentberg and Tondl.

1.2 General Topical Perspective

The most extensive portion of the literature on the dynamics of rotating shafts is concerned with determining critical speeds and natural frequencies. Over the years, most rotating machinery has been designed to operate below the first critical speed. One could usually obtain reliable operation by ensuring that the highest operating shaft speed would be below the first natural transverse frequency of the shaft. (Eliminating the ambiguities in this very statement constitutes an important objective of this monograph.) It follows that the earliest papers were concerned with predicting first critical speeds and with balancing shafts for subcritical operations.

Modern rotating machinery, however, often must operate at very high speeds, far in excess of first critical. The more recent literature, therefore, treats a greater range of problems and phenomena. Topics such as the proximity of operating speed in relation to higher criticals, the extent of unstable regions and stresses during transition through lower criticals are all of practical interest to the designers of modern rotating equipment.

The analytical studies of rotor dynamics presented in current papers usually deal with systems with a very few degrees of freedom. These papers serve to illustrate the basic mechanisms involved in the various phenomena, providing generalized, qualitative information on the effects of damping and of other important parameters. Their significance should not be underestimated; they not only provide rules of thumb for designers, but often are essential to the diagnosis of shaft problems in real systems.

That part of the literature involving more complex mathematical models usually concerns the prediction of critical speeds for a particular configuration. These analyses are, therefore, more specific and quantitative in nature; they are, in general, representative of the analyses which are virtually a necessity in the design of most modern rotating systems.

1.3 Monograph Overview

Excluding this introductory chapter and the concluding chapter, the monograph contains six topical chapters. These are devoted to three main topics: lateral motion, coupled lateral and torsional motion, and balancing. The latter two are separate chapters. Because the preponderance of existing literature treats the first topic, and because several important subphenomena must be treated within the context of lateral motion, four chapters are devoted to the subject of lateral motion. These attempt to present physical descriptions of the phenomena, review important results, introduce the reader to the mathematical foundations, and discuss forced response briefly, in Chapters 2, 3, 4, and 5, respectively.

The scope of this monograph was such that not all of the many references could be exhaustively digested and reported. An attempt was made to deal with the most important, and these are cited in the text. All references, whether perused or not, are listed to produce as complete a bibliography on the subject as possible within the time allowed.

CHAPTER 2
INTRODUCTION TO THE PHENOMENA: LATERAL SHAFT PROBLEMS

2.1 Fixed and Rotating Reference Frames

There has always been some confusion as to what the term *critical speed* means. In the early work by Rankine, Dunkerley, and others, it was observed that a rotating shaft had certain speed ranges in which deflections of large amplitudes were developed. The shaft, rotating in a nonoscillatory—or simultaneously rotating and oscillating—deflected position, initiated vibrations of the whole supporting structure and often caused catastrophic failure of some part of the system. Hence, those particularly dangerous operating speeds became known as critical speeds.

From an engineering viewpoint, it is necessary to be more precise in defining critical speed. Some physical insight is gained by examining simplified cases. Consider, for example, a disk rotating on infinitely stiff bearings and shaft, but with a mass imbalance. This is shown in Fig. 2. One can place an arbitrary set of Cartesian axes (y, z) fixed in space so that the origin is at the bearing-shaft axe and another, less arbitrary, set (V, W) with the same origin but having one axis through the center of mass. The latter set obviously must rotate with the shaft and disk. If the disk is rotated at a constant speed Ω in a horizontal plane, so that gravity is normal to the disk, only one force will exist transverse to the shaft, namely that due to centrifugal force, $m\Omega^2 e$. The shaft will experience this as a steady force in the (rotating) direction V. The outer nonrotating races of the supporting bearings will experience sinusoidally varying forces

$$(m\Omega^2 e)\cos\Omega t \qquad \text{in the } y \text{ direction.}$$

$$(m\Omega^2 e)\sin\Omega t \qquad \text{in the } z \text{ direction.} \tag{2.1}$$

This differing view of the same phenomena, depending on whether the vantage point is the fixed or the rotating system, is illustrative of the complicating factors in shaft problems and probably accounts for much of the confusion regarding critical speeds.

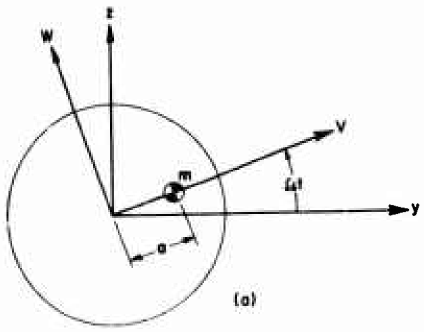

Fig. 2a. Disk of mass m, with mass offset distance a, on perfectly rigid shaft and bearings.

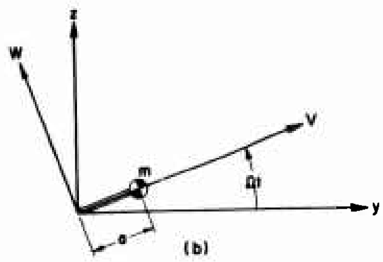

Fig. 2b. Mathematical model of disk in Fig. 2a for problems where polar mass moment of inertia is unimportant.

2.2 Instability

To emphasize the foregoing points further, consider now that the disk of Fig. 2 is perfectly balanced but contains a frictionless, radial slot, in which there is a mass m, restrained by a spring with rate k as shown in Fig. 3. The radial equilibrium of forces on the mass could result from a balance of centrifugal and elastic forces, thus

$$m\Omega^2(e+v) = kv$$

or

$$v = \frac{m\Omega^2 e}{k - m\Omega^2} = \frac{e}{\dfrac{k}{m\Omega^2} - 1}. \tag{2.2}$$

From this it can be seen that the elastic deflection v will be unbounded for any finite initial deflection e if the rotational speed $\Omega_d = \sqrt{k/m}$. It is noted that this rotational speed happens to coincide with the nonrotating natural frequency of the mass on its restraining spring ω_n. The absence of radial acceleration terms in Eq. (2.2), however, makes it clear that this divergence is a static phenomenon. One is likely to agree that Ω_d is a dangerous speed, perhaps even "critical"; resonance, however, is clearly not involved.

If the mass is considered to be oscillating radially about a mean position $e = 0$ in a free vibration, Newton's second law yields*

*All tangential accelerations and forces (Coriolis and others) are put in equilibrium by the "slot."

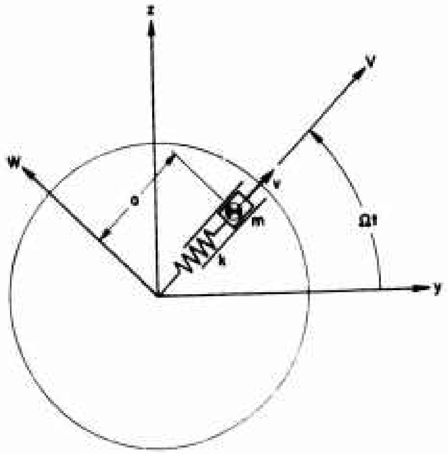

Fig. 3. Rotating disk containing a spring-restrained mass in a radial slot.

$$m\ddot{v} = -kv + m\Omega^2 v$$

or

$$\ddot{v} + \left(\frac{k}{m} - \Omega^2\right) v = 0.$$

The rotating natural frequency (viewed in the rotating system) is

$$\omega_{n_R} = \sqrt{\frac{k}{m} - \Omega^2} = \Omega^2 \sqrt{\left(\frac{\omega_n}{\Omega}\right)^2 - 1}\,. \qquad (2.3)$$

Hence, the divergence speed Ω_d occurs when the rotating natural frequency—viewed in the rotating system—is zero.

The solution for the vibrating motion v will satisfy the equation of motion above if

$$v = v_{0_1} e^{i\omega_{n_R} t} + v_{0_2} e^{-i\omega_{n_R} t}.$$

This shows that if $\Omega > \Omega_d = \sqrt{k/m}$ one of the two terms in the solution will diverge. Thus, Ω_d represents the borderline of a semi-infinite region (i.e., from $\Omega = \Omega_d$ to $\Omega = \infty$) of statically divergent instability. While Ω_d is, as mentioned earlier, a "critical" speed, this is *not* what is normally referred to as a critical speed in the literature.

2.3 Apparent Change in Frequency

Before examining a simple example of the phenomena which *are* traditionally thought of as critic. l spe^r ls, it will be helpful to note that if we picture the slotted mass oscillating freely when the shaft is running below the unstable range, i.e., when $\Omega < \Omega_c$, and choosing, arbitrarily, the maximum amplitude v_0 to occur when $t = 0$, then

$$v = v_0 \cos(\omega_{n_R} t).$$

But now the oscillatory displacement v instantaneously causes radial forces analogous to the radial forces of Eq. (1), so that we can write for the forces in the suppcrt bearings the equations

y direction

$$\begin{aligned}(kv)\cos\Omega t &= kv_0(\cos\omega_{n_R} t \ \cos\Omega t) \\ &= \frac{kv_0}{2}\left\{\cos(\omega_{n_R} + \Omega)t + \cos(\omega_{n_R} - \Omega)t\right\}, \end{aligned} \tag{2.4}$$

z direction

$$\begin{aligned}(kv)\sin\Omega t &= kv_0(\cos\omega_{n_R} t \ \sin\Omega t) \\ &= \frac{kv_0}{2}\left\{\sin(\omega_{n_R} + \Omega)t - \sin(\omega_{n_R} - \Omega)t\right\}. \end{aligned}$$

From this it is seen that an oscillatory mass motion of frequency ω_{n_R} in the rotating system causes oscillatory forces at frequencies

$$\omega_{n_R} \pm \Omega$$

in the fixed system. When $\Omega = \Omega_d$, i.e., the speed for borderline stability, ω_{n_R} is equal to zero, so in the fixed system the mass appears to be oscillating at the rotational speed $\Omega = \Omega_d$.

2.4 Critical Speed

Returning to the original disk without a slotted mass but with mass imbalance, suppose that rather than having a rigid shaft on rigid bearings, the rigid shaft is mounted in flexible supports as shown in Fig. 4. Here the *y, z* axes, in addition to being nonrotating, are also fixed at the undeflected bearing position. This system may be viewed in the *y, z* frame of reference as a mass *m* with two (uncoupled) degrees of freedom excited by forces as given in Eq. (2.1). Now suppose

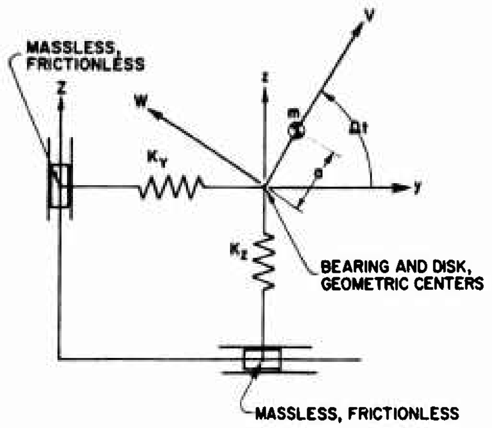

Fig. 4. Disk with eccentric-mass center on an inelastic shaft and flexibly mounted bearings.

$k_z = k_y = k$, and $\Omega = \Omega_c = \sqrt{k/m}$; i.e., the undamped natural frequency as viewed in the fixed system. The forcing functions then drive each degree of freedom at resonance, one 90° from the other. In the absence of damping, the amplitudes will grow without limit; that is, the solutions will be

$$y = \frac{\Omega_c e}{2}(A + t)\sin \Omega_c t + B \cos \Omega_c t$$

and

$$z = \frac{-\Omega_c e}{2}(A' + t)\cos \Omega_c t + B' \sin \Omega_c t, \qquad (2.5)$$

where A, A', B, and B' depend on the displacements and velocities of y and z at time $t = 0$. The sum of the two translational motions is a diverging spiral, so that the postulated $\Omega = \Omega_c$ is also a "critical" speed, and also is numerically equal to an undamped natural frequency. In this case, however, where $\Omega > \Omega_c$, the motion is bounded. Thus, the hazardous speed Ω_c will result in a whirling divergence which in the rotating reference frame is linear with time* and occurs only at the one specific rotational speed $\Omega = \Omega_c$. By comparison, the slotted mass case (Fig. 2) involves a hazardous speed Ω_d *beyond* which there will be a whirling divergence which, in the rotating system, is exponential with time and

*The polar amplitude $\sqrt{y^2 + z^2} = (e\Omega_c)/2(A + t)$, for example, if $A' = A$, $B' = B = 0$. Note that in theory it is possible to have more than one natural frequency with the same value, so that Ω_c could have not only t but t^2 and t^3 divergencies. In practice this is virtually impossible, since manufacturing tolerances preclude the absolute absence of coupling between modes required for two equal natural frequencies.

is more or less rapid depending on how far the unstable range defined by $\Omega \geqslant \Omega_d$ is penetrated. Note that the change in frequency by $\pm\Omega$ which occurs in going from fixed to rotating systems and vice-versa (illustrated by Eq.(2.4)) means that at Ω_c, the motion appears in the rotating system as a static divergence.

The literature is reasonably consistent in calling phenomena such as are associated with Ω_c "critical speeds" and those associated with ranges such as $\Omega \geqslant \Omega_d$ "instabilities." Although these definitions will be adhered to from this point on in the text, the foregoing discussion has attempted to show that they are in fact arbitrary, since both phenomena are associated with rotational speeds which are critical and both result in the kind of divergent motion associated with instability.

2.5 Shaft Flexibility vs Mount Flexibility

To examine more realistic shaft configurations, consider a disk mounted on a massless, elastic shaft constrained to motion in the y, z or V, W plane. The mathematical model now is that shown in Fig. 5b. The bearings in which the shaft runs are now taken to be at the origin, and the shaft is assumed infinitely stiff in torsion. Thus, while the V axis no longer passes through the mass center, it still rotates along with the disk with no loss of angular motion. The transverse (bending) flexibility of the shaft, however, allows translational motion of the disk in the plane of rotation, which may lag the shaft rotation by an angle ϕ, as shown in Fig. 5a; lead it, as shown in 5b; or be coincident with it, as shown in 5c and d. In these figures, of course, the effect of shaft bending is projected into the plane of the disk and represented as the motion of a linear spring. If the shaft bending rigidities are not polar-symmetric, then there would be different elastic restoring forces among Figs. 5a, b, and c, even though the magnitude of the displacements in the direction of the schematic linear spring were all equal. For a shaft infinitely stiff in the direction W, with finite flexibility in the direction V, as shown in Fig. 5c, the system is very much like that of Fig. 3, described by Eqs. (2.3) and (2.4). One might expect from this that shafts with significant differences in stiffness in two directions would be subject to instabilities.

Flexibility effects in the shaft bearings and/or supports, as shown in Fig. 4, and those due to the shaft's elastic properties as shown in Fig. 5, are clearly quite similar. If the shaft of the latter figure has polar-symmetric stiffness, and if the bearing/support spring rates of the former are isotropic, these two-dimensional (planar) descriptions become interchangeable. For example, if the shaft whirled about its bearing centers in a circular motion the spring in Fig. 5 would provide a steady radial force; the springs $k_y = k_z$ in Fig. 4 would provide the same steady resultant radial force. Since springs are, by definition, nondissipative mechanisms, the fact that the latter would be individually oscillating is of no consequence. This suggests, of course, that shafts with flexibility are subject to critical speeds such as described by Eq. (2.5) for the case of rigid shafts on flexible mountings.

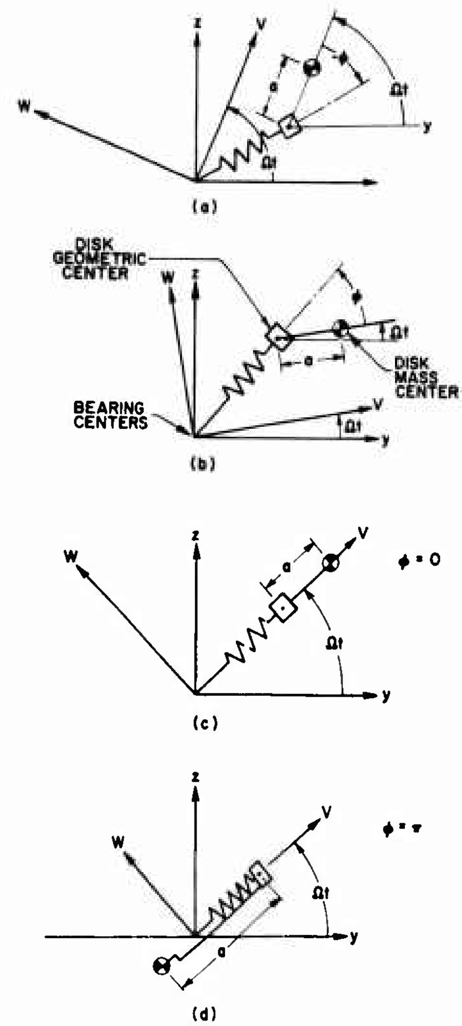

Fig. 5. Possible configurations for steady motion of a disk with eccentric mass on a massless elastic shaft.

2.6 Shaft Damping vs Mount Damping

Using Figs. 4 and 5, it is not difficult for us to imagine the effect of damping. Structural damping in the shaft is representable as a (radial) damper, in parallel with the springs in Figs. 5a through d. The damping forces that might arise from friction elements in parallel with the springs k_y and k_z in Fig. 4, however, whether isotropic or not, act on the shaft in a manner very different from friction in the rotating shaft itself. As has been mentioned in comparing shaft flexibility effects with those due to bearing/support flexibility, a constant-amplitude, circular whirl would cause no oscillation in the shaft spring of Fig. 5c, but would continuously cycle the springs in Fig. 4. Clearly, dampers in parallel with the springs in the latter figure would be a powerful dissipative medium, whereas a damper parallel to the spring in Fig. 5 would do nothing in the postulated motion.

2.7 Effects of Gravity

If the shaft and disk representation of Fig. 5 were turned until the shaft axis were horizontal and the disk balanced, then the force of gravity would become a transverse excitation source, as shown in Fig. 6. While gravity appears as a constant force in the negative z direction, it appears in the rotating system as sinusoidally varying with the frequency of rotation, that is

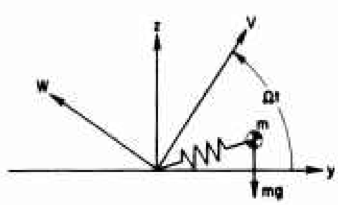

Fig. 6. Balanced disk on a massless, horizontal elastic shaft.

$$-mg \sin \Omega t \quad \text{in the } V \text{ direction,}$$

and (2.6)

$$-mg \cos \Omega t \quad \text{in the } W \text{ direction.}$$

This is analogous to the manner in which centrifugal force, steady in the rotating system, appears as a first harmonic sinusoidal force in the fixed system.

2.8 Progressive-Regressive Whirl

When transverse shaft vibrations occur in two mutually perpendicular planes, the frequency and phasing between such motions at a given point along the shaft length will determine a plane closed curve often called a Lissajous figure. When the frequencies of vibration are the same in the two planes, figures such as those shown in Fig. 7 occur. Such vibrations can occur in either the fixed or the rotating system. If they occur in a rotating frame of reference and with other than zero phase, the shaft center line will appear to rotate, as shown in Figs. 7b and c. When this apparent rotation is in the same direction as the true rotational velocity of the shaft, the shaft is said to be in a forward, advancing, or progressive mode. When the apparent rotation, viewed in the rotating system, is in the opposite direction to that of true shaft rotation, it is said to be vibrating in a reverse, backward, or retrogressive mode.

It is noteworthy that in much of the literature the term *forward whirl* is used to describe the classical shaft critical speed phenomenon, which—it should now be clear—is a limiting case of zero apparent rotation of the shaft center in the rotating system. There are other more complex phenomena, however, in which the forward or backward precession, as viewed in the rotating system, occurs at integer multiples of rotational speeds. If, for example, there is a backward mode

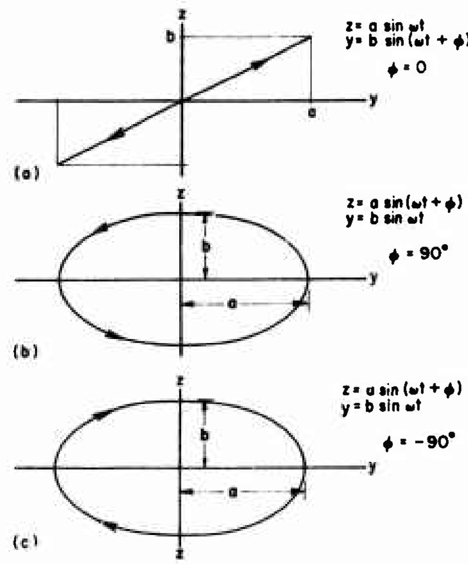

Fig. 7. Lissajous figures traced by a point on a shaft at a given longitudinal station undergoing transverse vibratory motions at the same frequency on two planes.

at twice rotational speed in the rotating system, this would appear as a "backward whirl" in the fixed system at the shaft rotational speed. Such cases, which may be or may only appear to be resonant depending on the source of the exciting force, are discussed further in Chapter 3, Section 3.1.

2.9 Summary

This discussion has attempted to define critical shaft speeds, shaft instabilities, and related effects, and to illustrate some important aspects of the phenomena by examining simplified cases. The change of observed frequency by $\pm\Omega$ when a lateral vibratory phenomenon in nonrotating coordinates is viewed in the rotating system (or vice versa) has been singled out as a potential source of confusion. This effect, of course, includes zero frequency as a degenerate case. Thus, critical speeds which appear as once-per-revolution vibratory resonances in the fixed system are static divergences in the rotating system; thus, strain gages on the rotating shaft in this situation would indicate nonoscillatory stresses. Borderline stabilities, showing the onset of static divergencies in the rotating system, may exhibit the same frequency as resonant vibrations in the fixed system. The important distinctions, then, between the two phenomena are that critical speeds occur only at discrete values of shaft speed and diverge linearly with time; shaft instabilities occur within a range of operating speeds and diverge exponentially with time.

For a rotating shaft there normally exists a number of possible sources of unstable motion. Internal damping, lack of symmetry in the rotating parts, and oil films in journal bearings, each may give rise to an instability. The amplitude of the resulting, self-excited vibration may very often be greater than that due to the resonance vibration of incorrectly balanced rotors. Furthermore, there is the added danger of cyclic stresses in the shaft.

All these matters will be addressed with more mathematical rigor in Chapter 4. Having made the physical introduction of the preceding paragraphs, however, it will be helpful to review some well-accepted results from the cumulative experience of researchers in this field.

CHAPTER 3
BRIEF REVIEW OF EXPERIENCE: LATERAL SHAFT PROBLEMS

In this chapter certain general results of importance from the literature are summarized as regards the phenomena, their classification, and prediction methods. Further summaries of results from the literature, which seem to require the presentation of mathematical foundations as prerequisite to a reasonable discussion, are contained in Chapter 4.

3.1 General

In most shaft systems dissipative (i.e., damping) forces in both fixed and rotating components are small compared to inertial and elastic forces. Furthermore, shaft flexibility is usually polar-symmetric and bearing supports are either relatively rigid or nearly isotropic; thus, the equivalence of Figs. 4 and 5 (discussed in the previous section) makes excitation of the shaft at its natural frequency in the fixed system by the once-per-revolution excitation of mass imbalance the usual case. Viewed in the rotating system, the steady forces due to mass imbalance are "resonant" with the zero natural frequency of the rotating mode of the shaft system.

Alternative terminologies often found in the literature for this kind of phenomena are "critical angular velocity for synchronous precession," "critical speed of forward precession," and "critical whirling speed." Some authors simply refer to "natural frequencies," but risk the possibility of misunderstanding. For example, the natural frequencies of a nonrotating shaft will not always be close to the critical speeds. In general, the natural frequencies of a rotating shaft are a continuous function of the operating speed. It is true that often the dependence is weak, but there are also cases where the natural frequencies of the rotating vs the nonrotating shafts may be very different. A common example occurs when mass moments of inertia about axes transverse to the shaft are large relative to that of the shaft cross section.

The excitation frequencies which cause other resonance phenomena in shafts are usually integral multiples of the operating speed. These cases, which are related to but different from basic shaft critical phenomena, occur when that integral multiple of the operating speed equals a natural frequency or, equivalently, when the operating speed is equal to a natural frequency divided by that integer.

The particular case where gravity acts on a horizontal shaft is, as discussed previously, an example in which the frequency of the exciting force is zero in a stationary coordinate system, but once per revolution in the shaft system. Thus, a resonance could be anticipated when the natural frequency of the rotating shaft as measured in the rotating system is equal to the rotational speed. The critical speed due to gravity is a so-called critical speed of the second order which occurs at an operating speed approximately equal to 1/2 the first normal critical speed. It is interesting to note, however, that a true secondary critical speed will not exist unless the shaft is one with unequal stiffnesses about axes 90° from one another. If the shaft stiffness is isotropic, then, under steady operating conditions, there will be a constant shaft deflection due to gravity, but no resonance.

Another often-mentioned rotating shaft phenomenon is that of reverse precession. This term is used to describe a situation where the shaft center viewed in the fixed system follows a path whose direction of rotation is opposite to that of the true shaft rotation and at the same speed. This can be a resonance if there is an exciting force whose vector rotates with the same frequency as the rotational speed but in the opposite direction. It is generally much more difficult to excite reverse whirl than resonance due to imbalance or gravity. Eubanks and Eshleman [3] mentioned that Lowell [4] was able to excite backward whirl (reverse precession) with pulsating torques. Dimentberg [5] shows that mass imbalance may cause a shaft center line to precess in a direction opposite to that of the shaft rotation, but with the same speed, provided that the support elasticity is different in two directions perpendicular to the shaft. This occurs at operating speeds between two classical, forward-whirl critical speeds and is not a resonance phenomenon.

Certain shaft system characteristics have been alluded to in preceding paragraphs; among them shaft and bearing stiffness and damping, disk eccentricity, and mass moment of inertia. It is well, however, to consider the parameters important to rotating shaft dynamics from a fundamental viewpoint.

There are two classes of effects to be considered: linear and nonlinear. Table 1 tabulates system parameters of consequence according to these two classes.

Table 1. Classification of Effects of System Parameters on Critical Speeds

Linear	*Nonlinear*
Imbalance	Bearing clearance
Axial force and constant torque	Coulomb friction and hydraulic damping
Gyroscopic moments and rotary inertia	Oil film effects in journal bearings
Linear elastic restoring forces, including transverse shear	Nonlinear elastic restoring forces
Viscous damping	Electromagnetic forces
Rotating and nonrotating asymmetry	Faulty mountings, shrink fit, and deformation of ball-bearing races

A linear system typically has a certain set of fixed natural frequencies and vibration modes. The steady response to periodic excitation is unique; at a natural frequency, the system amplitude grows with time to a finite amplitude for nonzero damping, and linear resonance is said to occur.

Nonlinear systems differ in two respects. First, the amplitude-frequency relationships are not in general unique, i.e., single valued. The second and more important effect is that additional resonances can appear, apart from those approximately predicted by the linear analysis.

3.2 Influence of Certain Parameters: Linear Effects

Axial Force and Constant Torque

Southwell and Gough [6] concluded in 1921 that critical speeds are significantly affected by either constant axial force (compression) or constant torque. They found that the effect of both of these parameters is to lower the critical speed. These results were viewed as refuting Greenhill's [7] earlier conclusion that these effects are unimportant.

The subject, however, is not yet closed. N. Willems and S. M. Holzer, in a very recent paper [8], concluded that the effect of torque on the critical speed of a shaft is small for practical ranges of shaft parameters. On the other hand, Eshleman and Eubanks [3] state that constant axial torque does have a significant effect on the critical speeds of slender rotors, and their experimental studies confirm this.

One of the most important experimental results of Eshleman and Eubanks' work was that if an oscillatory torque is superimposed upon a steady torque, the system has at least one region of unstable lateral motion. This result is discussed again in Chapter 4, under instabilities.

Gyroscopic Effects and Rotary Inertia

The influence of gyroscopic effects on the critical speeds of rotating shafts has been studied by many authors. Tondl, Dimentberg, Green, Eubanks, and Eshleman are a few of the most notable contributors. The latter pointed out that the gyroscopic effect may increase or decrease the critical speeds significantly, depending on the operating speed, the size and geometry of the gyroscopic disks, and the location of the disks on the rotating shaft.

It is important to note that the terms *gyroscopic effect* and *effect of rotary inertia* are very often confused in the literature. The gyroscopic moment is proportional to the time rate of change of the shaft's transverse angular deflection and its direction is 90° from that of the transverse angular velocity. Thus, to account for the gyroscopic influence of the mass moment of inertia of shaft cross sections or attached disks, it is necessary to consider the simultaneous bending of the shaft in two planes. It follows from the two-plane aspects that the polar mass moment of inertia about the shaft center line is the parameter of importance to gyroscopic effects.

In contrast, the rotary inertia effect can make itself felt with just transverse angular deflections; i.e., it is a steady phenomenon when viewed in the rotating system. Furthermore, the resulting moment acts in the same plane as the deflection angle. The single-plane character of the rotary inertia effect is associated with the fact that the mass moment of inertia about an axis normal to the shaft center line is the important property of these phenomena. In general one can state that the moments arising from transverse bending angular deflections and centrifugal force tend to stiffen the shaft proportionably to the shaft angular velocity squared. Those resulting from transverse angular acceleration, which for this purpose may be viewed as a variable independent of shaft angular velocity, tend to lower the natural transverse bending frequencies; such terms are proportional to that natural frequency squared. Thus, when the bending natural frequency of a rotating shaft is higher than the rotational speed, increasing the corresponding mass moment of inertia will lower that natural frequency, and reducing the mass moment of inertia will raise that natural frequency. The opposite results are obtained for frequencies lower than the rotational speed.

If one is concerned with forward whirling due to the mass imbalance of an isotropic shaft on, say, rigid supports, then, in a coordinate system fixed in the shaft, the deflections and angles are constant and independent of time. For such motion, the gyroscopic effect cannot influence the classical critical speeds of forward whirling. The true gyroscopic effect can influence the critical speeds of backward whirl or of forward whirl, provided there exist anisotropic supports or some other mechanism for causing the displacement and angles (as measured in the rotating frame) to vary instantaneously with time.

Transverse Shear Deflections

Deflections due to transverse shear become more important in the calculation of critical speeds–just as in any beam problem–when the diameter of a shaft becomes significant relative to its length. According to Eshleman and Eubanks, this effect can be significant for critical speed phenomena when the diameter of a shaft is only as large as one hundredth of the length.

Transverse shear, gyroscopic, and rotary inertia effects also play a rather unusual role in explaining one of the paradoxes associated with critical speeds. A. Tondl [9], in treating the critical speeds of a rotor with uniformly distributed mass, shows that if the rotary inertia and gyroscopic terms are neglected, there is an infinite number of critical speeds in both forward and reverse precession. On the other hand, if the rotary inertia and gyroscopic terms are included, there is an infinite number of critical speeds in reversed precession, but only a finite number in forward precession. Now, if the continuous system is approximated by n disks separated by massless elastic sections, there will be n critical speeds for forward precessions and n for reverse precessions. This must still be true as the number is made arbitrarily large, and one would suppose that in the limit as $n \rightarrow \infty$, the continuous system would be approached. Why, then, does the continuous-system approach not always give an infinite number of critical speeds? This question was answered by Dimentberg [5], who proved that the paradox results from neglecting shear deformations in formulating the bending equations for an elemental shaft section.

Rotating and Nonrotating Asymmetry

The presence of asymmetry in either the rotating or nonrotating frame of reference generally doubles the total number of critical speeds, as compared to the fully symmetric case with the critical speeds occurring in pairs corresponding to each mode. From a mathematical viewpoint, the equations of motion of an idealized shaft system can be in the form of linear differential equations with constant coefficients, when expressed in the frame of reference which possesses the asymmetry. If, in each case, the alternative frame is used, the equations of motion are linear but with periodic coefficients.

With a knowledge of these results one might suppose that, when there is asymmetry in both frames, there might be four critical speeds while the fully symmetric system has one. This supposition turns out to be basically correct and will be discussed at greater length subsequently. One can also conclude, again correctly, that for cases of unsymmetric shafts on unsymmetric supports, there is no possible frame of reference in which the equations of motion have constant coefficients. This is unfortunate because the existing mathematical methods for handling linear differential equations with periodic coefficients are more complex than for constant coefficients and are unfamiliar to many engineers.

Some early contributions to the understanding of critical speeds of unsymmetric shafts operating on unsymmetrical supports were made by Smith [10]. He discussed critical speeds as well as other unstable motions of a flexible shaft operating in flexible bearing supports, and showed that there are four critical speeds in the region where a similar, symmetric shaft/bearing system has only one. In addition, however, he discussed the presence of other minor critical speeds for which periodic motion is possible, with the operating speed in the neighborhood of an integral fraction of the natural frequency of a nonrotating symmetrical system. Periodic motions associated with *odd* integral fractions of natural frequency can be excited by steady forces in the rotating system, such as caused by imbalance, although the amplitudes will usually be small. Periodic motions in the range of *even* integral fractions of a nonrotating natural frequency can be excited by a steady disturbing force in the fixed system, such as gravity. With an operating speed of one-half of a nonrotating natural frequency, for example, the criticals are in the neighborhood of the classical secondary critical speeds associated with a nonsymmetric shaft.

Smith also discussed [10] the additional critical speeds arising from the gyroscopic effects of attached disks, when both the shaft and supports are asymmetrical. The introduction of asymmetry has an effect on the distribution of critical speeds arising from moments of inertia about transverse axes similar to that on the more basic critical speeds. Smith pointed out that those critical speeds associated with lateral moments of inertia corresponding to forward precession may not exist for short rotors. For example, a very thin disk attached to a massless elastic shaft would not introduce an additional forward critical speed, although it may modify the critical speed associated with motion of the center of mass.

About ten years after the Smith paper, Foote, Poritsky, and Slade [11] also considered the combined effects of asymmetry in both the rotating and nonrotating parts of a shaft system. Their concern for these effects was with application to two-pole turbogenerators. In this paper, they claimed that Smith missed one of the unstable intervals in the immediate vicinity of the critical speed of the system with asymmetry set equal to zero. Crandall and Brosens, however, reported in 1961 [12] the results of research on a problem similar to that studied by Foote, Poritsky, and Slade. Their analysis contains the additional complicating factor of gyroscopic coupling, but the essence of their results was not essentially different from that of their predecessors of two and three decades earlier.

Finally, Tondl [9] treated the almost identical problem using a different mathematical approach. His results are also not different from those predicted by Smith. In addition to treating the problem of a shaft having stiffnesses different in two directions, on supports which have different stiffnesses in two directions in the presence of external friction, he accounts for the mass properties of the flexible supporting structure. Additional critical speeds are shown to exist because of the degrees of freedom of the support itself.

3.3 Influence of Certain Parameters: Nonlinear Effects

The mathematical difficulty of nonlinear system analysis has hindered the development of a theory for nonlinear vibrations of all kinds, and the theory of rotating shaft dynamics is no exception. The crucial aspect, of course, is that nonlinear equations eliminate the use of the superposition principle.

In rotating shafts there are various causes of nonlinearity, some of which are weak and others strong. The action of the oil film in journal bearings has a fundamentally nonlinear nature. On the other hand, the elastic restoring forces provided by a deflected shaft or a supporting foundation may usually be considered linear for tolerable deflections.

Other common sources of nonlinearity are clearance in bearings, magnetic force between rotor and stator, and elastic restoring forces due to the deformation of ball-bearing races.

The most fruitful work in the area of nonlinear resonance has been done with some variation of the small-parameter approach. Yamamoto [13], Hayashi [14], and Tondl [15] have made significant contributions to the theory of nonlinear resonance vibrations in rotating shafts, and each has resorted to a small-parameter approach. Tondl concerned himself most with the nonlinearity arising from the action of the oil film, which he simulated by a nonlinear spring support. In his analysis he considered a symmetrically located disk on a massive support, so that there are two degrees of freedom and two classical critical speeds for the linearized system. His theoretical results show that, in addition to main resonances (which degenerate to the critical speeds in the linear case), there may exist subharmonic resonances and what are called internal and combination resonances.

3.4 Prediction Methods

The majority of references dealing with the dynamics of rotating shafts are concerned with qualitative aspects of the problem rather than with analytical or numerical procedures for design purposes. These qualitative studies are facilitated by working either with systems which have only a very few degrees of freedom, or with those which are uniform, simply supported, symmetrical, and straight. In either case, the characteristic equations which define the critical speeds are usually found in a straightforward fashion, and the process of predicting critical speeds presents no real problem.

Real-world problems are usually less than ideal, and it is necessary to have a convenient computational method for predicting critical speeds. It has been mentioned earlier that natural frequencies are generally a function of operating speed. It has also been pointed out that, if the gyroscopic and moment of inertia effects are neglected, the natural frequencies of the system as measured in the fixed frame will be independent of operating speed. It is this fact which accounts for the use of critical speed calculation methods which were originally intended for finding the natural frequencies of nonrotating, beamlike structures.

The methods which have been employed are summarized as follows:

- The Rayleigh method
- The Ritz method
- The Prohl or Myklestad method or transfer-matrix approach
- The force-displacement method or influence coefficient method
- Dunkerley formulas
- Impedance-matching method.

The most classic of these approaches is the Rayleigh method. In this method, one assumes a deflection shape and that the system is oscillating through these deflections sinousoidally at an unknown frequency. The corresponding maximum kinetic and potential energies are then calculated in terms of the unknown frequency. Equating the maximum kinetic and potential energies allows this frequency to be evaluated, and it corresponds to the approximate first natural frequency.

In actuality, there is no true vibration of the rotating shaft for the critical speed usually sought in these calculations, and those using this approach should realize that they are depending on the numerical equivalence of natural frequency and critical speed as discussed in Chapter 2. The difficulties of the Rayleigh method are well known; it is usually good only for the first natural frequency and, therefore, the first critical speed.

The Ritz method [16] improves on the basic Rayleigh method by using more than one deflection shape, each of which satisfies the boundary conditions, and each of which is as close to orthogonal* to the others as possible. A variational principle is employed which ensures that the relative contribution of the assumed mode shapes will be determined so as to satisfy equilibrium (i.e., Lagrange's equations). This method not only gives the first critical, but as many criticals as one assumes mode shapes. Successively better approximations to the lower criticals are obtained by assuming more and more modes. It can be shown that the Ritz and Rayleigh methods always give a first frequency (or critical speed) which is higher than the exact solution for the same physical system.

The Prohl or Myklestad method has been widely used, and in its more modern form is known as a transfer-matrix approach [17]. In this method, the generalized forces and displacements (i.e., forces, moments and torques, and translations and rotations, respectively) at one end of a shaft system are related to those at the other end by means of successive multiplications of matrices, which account for the effects of the stiffness and inertia properties of the various sections of the system. This technique yields as many critical speeds as, and to the degree of accuracy determined by, the number of stations into which the shaft is divided. It is especially suitable for machine computation. This technique has been used to compute both the natural frequencies as a function of operating speed and the forced response due to unbalanced loads.

The force [18] and displacement methods [19] are matrix techniques for determining structural influence coefficients, which in turn can be used in the dynamic matrix iteration method to give all the critical speeds. Generally, but not necessarily, the higher mode accuracy depends on the accuracy with which lower modes are obtained. However, the choice between the transfer-matrix and matrix-iteration approaches is a matter of personal preference, to a large extent.

The Dunkerley [20] formulas for getting approximate values of the first critical speed are in fairly common use. The general idea here is to determine the first critical speed of a system from a knowledge of the critical speeds of smaller, less complex subsystems. Fernlund [21] pointed out that the standard Dunkerley formulas are approximate, even if the gyroscopic effect and the distributed mass

*"Orthogonal" here implies that the inertial forces in one mode will do no net work when they act through the deflections of the other mode.

of the shaft may be neglected, except for a shaft which carries only one concentrated mass. Fernlund has developed an improved Dunkerley formula which gives better results. In either case, however, great care should be exercised in making use of these formulas, since they are generally only first approximations.

Finally, one of the most successful approaches is the use of impedance-matching methods [5, 22]. In this case, natural frequencies or critical speeds are obtained from a knowledge of the impedances of the component parts of a system.

CHAPTER 4
MATHEMATICAL FOUNDATIONS: LATERAL SHAFT PROBLEMS

The purpose of this chapter is to discuss both resonantly forced and unstable lateral motion.

The critical speed of a rotating shaft was defined earlier as a resonant phenomenon, i.e., one in which the applied force reinforces the system's response by occurring at the natural frequency. Mathematically speaking, natural frequencies arise as eigenvalues of the governing equations, with the terms independent of motion (the forcing functions) equal to zero. Physically, they imply free, undamped motion; i.e., frictionless, in vacuo, and with no applied forces. All the natural frequencies of a rotating shaft are more or less dependent on the operating speed of the shaft. Since a forcing function in this case can be the mass imbalance, and since it occurs (viewed in the nonrotating system) at a frequency equal to the rotational frequency, then when a rotating natural frequency (viewed in the fixed system) is also equal to the rotational frequency, that operating condition will be a critical speed.

A simple model of a rotating shaft, which is employed by almost all researchers in this field at one point or another, is a disk located at the center of an elastic, massless shaft. Both ends of the shaft are assumed to be identically supported. Figure 8 illustrates this simple system. It is customary in dealing with this model to assume that the disk remains in one plane, so that gyroscopic effects do not manifest themselves. This requires, of course, that the imbalance (or, for that matter, any disturbance along the shaft system) always occurs symmetrically with respect to the disk.

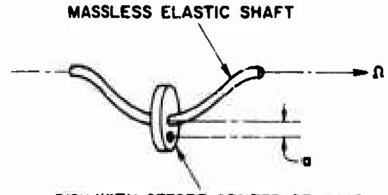

Fig. 8. Simple model of a rotating shaft.

Even with these simplifying assumptions, this model allows an investigation into the effects of many different shaft characteristics on critical speeds and instabilities.

The equations of motion of the system shown in Fig. 8 are easily derived. Here, the disk center of mass is assumed to be offset a distance a from the center of the shaft, and internal damping is included.

It is convenient to introduce a coordinate system which rotates with the shaft at an angular speed Ω (Fig. 9). Letting (v, w) be the coordinates in the rotating frame of the shaft center, the position vector from the origin to the center of mass is given by

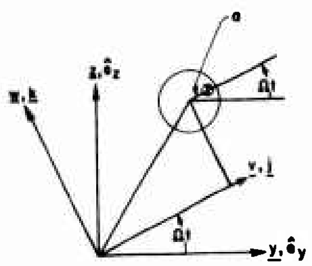

Fig. 9. Fixed and rotating coordinate systems.

$$\bar{\mathbf{r}} = (a + v)\mathbf{j} + w\mathbf{k}. \qquad (4.1)$$

Alternatively, in the fixed system of coordinates,

$$\bar{\mathbf{r}} = (y + a\cos\Omega t)\hat{e}_y + (z + a\sin\Omega t)\hat{e}_z, \qquad (4.2)$$

where (y, z) are the fixed-system coordinates of the center of the shaft.

After properly differentiating the position vector twice, the acceleration of the mass center results:

$$\ddot{\bar{r}} = (\ddot{v} - \Omega^2 v - 2\Omega\dot{w} - \Omega^2 a)\mathbf{j} + (\ddot{w} - \Omega^2 w + 2\Omega\dot{v})\mathbf{k}$$

$$= (\ddot{y} - a\Omega^2\cos\Omega t)\hat{e}_y + (\ddot{z} - a\Omega^2\sin\Omega t)\hat{e}_z. \qquad (4.3)$$

One force acting on the system is the elastic restoring force, given by

$$\bar{f}_s = -\kappa(v\mathbf{j} + w\mathbf{k})$$

$$= -\kappa(y\hat{e}_y + z\hat{e}_z). \qquad (4.4)$$

The internal friction force is given by

$$\bar{f}_i = -c_i(\dot{v}\mathbf{j} + \dot{w}\mathbf{k})$$

$$= -c_i[(\dot{y} + \Omega z)\hat{e}_y + (\dot{z} - \Omega y)\hat{e}_z]. \qquad (4.5)$$

The external friction is given by

$$\bar{f}_e = -c_e[(\dot{v} - \Omega w)\mathbf{j} + (\dot{w} + \Omega v)\mathbf{k}]$$

$$= -c_e(\dot{y}\hat{e}_y + \dot{z}\hat{e}_z). \qquad (4.6)$$

The equations of motion in either frame of reference are now easily written from

$$m\ddot{\bar{r}} = \bar{f}_s + \bar{f}_e + \bar{f}_i. \tag{4.7}$$

Fixed frame

$$m\ddot{y} + (c_i + c_e)\dot{y} + \kappa y + c_i\Omega z = m\Omega^2 a \cos \Omega t,$$

$$m\ddot{z} + (c_i + c_e)\dot{z} + \kappa z - c_i\Omega y = m\Omega^2 a \sin \Omega t. \tag{4.8}$$

Rotating frame

$$m\ddot{v} + (c_i + c_e)\dot{v} + (\kappa - m\Omega^2)v - 2m\Omega\dot{w} - c_e\Omega w = m\Omega^2 a,$$

$$m\ddot{w} + (c_i + c_e)\dot{w} + (\kappa - m\Omega^2)w + 2m\Omega\dot{v} + c_e\Omega v = 0. \tag{4.9}$$

Most authors find it is convenient to deal with a complex representation of the displacements (y, z) and (v, w). Therefore, rather than the Eqs. (4.8) and (4.9), one often finds in the literature equations of the following form:

Stationary frame

$$m\ddot{x} + (c_i + c_e)\dot{x} + \kappa x - jc_i\Omega x = m\Omega^2 a e^{j\Omega t}, \tag{4.10}$$

where

$$x \equiv y + jz.$$

Rotating frame

$$m\ddot{u} + (c_i + c_e)\dot{u} + (\kappa - m\Omega^2)u + 2m\Omega j\dot{u} + c_e\Omega j u = m\Omega^2 a, \tag{4.11}$$

where

$$u \equiv v + jw.$$

It has been mentioned previously that the classical critical speed is excited by the unbalanced mass. With this in mind, the critical speed should be apparent in the steady solution of the equations of motion corresponding to the unbalanced load.

Selecting the equations of motion as expressed in the rotating frame of reference, it is obvious that a particular solution of Eq. (4.11) is in the form of a complex constant. Assume, therefore, that

$$u = u_0 = \text{constant}. \tag{4.12}$$

Then, according to Eq. (4.11),

$$u_0 = \frac{m\Omega^2 a}{(\kappa - m\Omega^2 + c_e\Omega j)}$$

$$= \frac{\Omega^2 a e^{-j \tan^{-1}\{c_e\Omega/[m/(\kappa/m - \Omega^2)]\}}}{\sqrt{\left(\frac{\kappa}{m} - \Omega^2\right)^2 + \left(\frac{c_e\Omega}{m}\right)^2}}. \tag{4.13}$$

If there were no external damping, the expression for u_0 would reduce to

$$u_0 = \frac{\Omega^2 a}{\left(\frac{\kappa}{m} - \Omega^2\right)}. \tag{4.14}$$

The critical speed is the value of Ω such that

$$\Omega = \omega_0 = \sqrt{\frac{\kappa}{m}}. \quad (4.15$$

The presence of external damping limits the amplitude near critical speed. Notice that internal damping does not enter the solution at all, since the shaft is not undergoing any flexure with this motion.

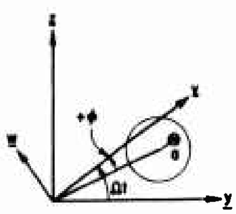

Fig. 10. Shaft motion at a speed below critical.

An illustration of the shaft motion at a speed below critical is shown in Fig. 10. The shaft center line is lagging behind the rotation of the shaft itself by an amount equal to

$$\phi = \tan^{-1}\left\{c_e\Omega\Big/\left[m\left(\frac{\kappa}{m} - \Omega^2\right)\right]\right\}. \tag{4.16}$$

As the speed approaches the critical, the angle ϕ approaches $\pi/2$, so that the configuration is as shown in Fig. 11. Finally, at supercritical speed, the angle ϕ approaches π and the magnitude of u_0 approaches a. See Fig. 12.

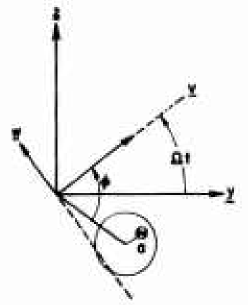

Fig. 11. Shaft motion at a speed approaching critical.

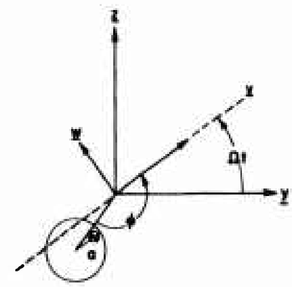

Fig. 12. Shaft motion at supercritical speed.

To consider the undamped natural frequency, we set $c_i = c_c = 0$ and examine the homogeneous solution of Eqs. (4.10) or (4.11). In Eq. (4.10), for example, assume that

$$x = x_0 e^{st}.$$

In order for this to be a solution, the characteristic equation must hold which states that

$$ms^2 + \kappa = 0, \tag{4.17}$$

so that

$$s = \pm j\sqrt{\frac{\kappa}{m}}, \tag{4.18}$$

and free vibrations are of the form

$$x = x_{10} e^{-jt\sqrt{\kappa/m}} + x_{20} e^{jt\sqrt{\kappa/m}}. \tag{4.19}$$

In the rotating frame, the free vibrations are of the form

$$u = u_{10} e^{-jt(\Omega + \sqrt{\kappa/m})} + u_{20} e^{-jt(\Omega - \sqrt{\kappa/m})}. \tag{4.20}$$

The natural frequency in the fixed frame is $\sqrt{\kappa/m}$, but there exist two natural frequencies in the rotating frame given by

$$\Omega \pm \sqrt{\frac{\kappa}{m}}\,.$$

If the simple shaft system is disturbed from its steady rotation described by Eq. (4.13), then oscillatory stresses at frequencies $(\Omega \pm \sqrt{\kappa/m})$ will be recorded by a strain gage attached to the shaft.

In this particular problem, the natural frequencies are a function of operating speed in the rotating system but not in the fixed system. This is not always the case. When rotary inertia or gyroscopic effects are included, the natural frequencies are generally dependent on the operating speed in both frames of reference.

We now consider the question of stability of this simple system and show the relationships of natural frequencies and instabilities. Let us focus on the previous results for natural frequencies in the rotating frame. The natural frequencies correspond to the imaginary part of the eigenvalues for the complex variable s in the free vibration solution of the form

$$u = u_0 e^{st}.$$

In the complex s-plane we may plot the locus of eigenvalues as functions of operating speed (see Fig. 13). As long as the locus of eigenvalues is on the imaginary axis, the implication is that the vibration mode is neutrally stable. That is, it will neither decay nor grow.

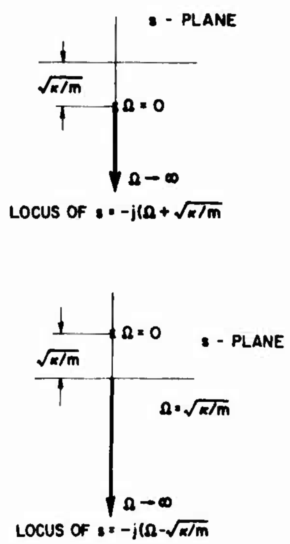

Fig. 13. Locus of eigenvalues in the complex plane (neutral stability case).

To determine the effect of damping on the stability of the system, we again examine the characteristic equation for the case of finite internal and external damping.

It should be pointed out that the only reason we are able to determine stability from the homogeneous solution alone is that the system is linear. If there were any nonlinear elements in the system, the stability would depend on the external forces (in this case, the imbalance).

The characteristic equation, as found from Eq. (4.11), is given as follows:

$$s^2 + \frac{(c_i + c_e)}{m} s + \left(\frac{\kappa}{m} - \Omega^2\right) + 2\Omega j s + \frac{c_e}{m}\Omega j = 0. \quad (4.21)$$

The two eigenvalues are given by

$$s = -j\Omega - \frac{(c_i + c_e)}{2m} \pm \sqrt{\left(\Omega j + \frac{c_i + c_e}{2m}\right)^2 - \left(\frac{\kappa}{m} - \Omega^2 + \frac{c_e}{m}\Omega j\right)}. \quad (4.22)$$

As an approximation, we may consider the case where both c_i and c_e are small enough so that, with only first order terms in either c_i or c_e, the two eigenvalues are

$$s = -j(\Omega - \sqrt{\kappa/m}) - \frac{1}{2m}\left(c_e + c_i - \frac{\Omega c_i}{\sqrt{\kappa/m}}\right) \tag{4.23}$$

and

$$s = -j(\Omega + \sqrt{\kappa/m}) - \frac{1}{2m}\left(c_e + c_i + \frac{\Omega c_i}{\sqrt{\kappa/m}}\right). \tag{4.24}$$

The vibration mode with the frequency $(\Omega + \sqrt{\kappa/m})$ is always positively damped, whereas the mode with frequency $(\Omega - \sqrt{\kappa/m})$ will be amplified exponentially with time, provided that $\Omega > \sqrt{\kappa/m}(1 + c_e/c_i)$. These equations make it clear that, for the model examined, instability will occur only when there is damping in the rotating system; i.e., $c_i \neq 0$.

The locus of roots for each complex eigenvalue, i.e., given by Eqs. (4.23) and (4.24), is plotted in Fig. 14. When the locus of roots crosses over to the right half-plane, the mode becomes unstable. The mode which goes unstable is the one which corresponds to a free vibration in the fixed frame of the form

$$x = x_{20}e^{jt\sqrt{\kappa/m}}, \tag{4.25}$$

which is the forward precessing mode. Thus, if a system characterized by this model were excited by a random disturbance, the oscillatory stress in the shaft with frequency $(\Omega - \sqrt{\kappa/m})$ could be very large.

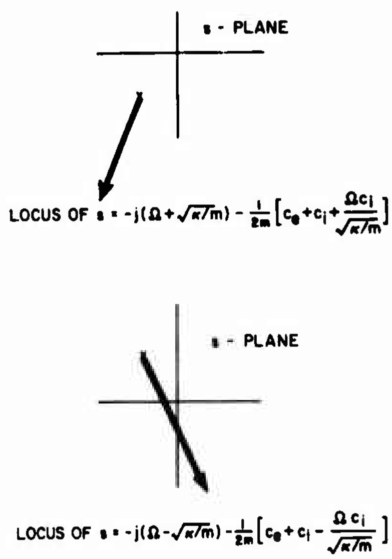

Fig. 14. Locus of eigenvalues in the complex plane (damped case).

4.1 Basic Causes of Unstable Shaft Motions

Internal Friction

Kimball [23] and Newkirk [24] were among the first to recognize that internal friction could cause unstable shaft motion when operating above the first critical speed. Newkirk's contributions were made in the 1920's when he was doing research into the cause of shaft whipping, which had been a source of irritation for high-speed rotor manufacturers. Dr. Newkirk obtained a significant amount of information on the phenomenon through an experimental program. However, his experimental results were in many cases not correlated with theory until much later, and then it was by other workers in the field.

Some of the important results of Newkirk's investigations are worth discussing. The theoretical explanations of many of these efforts are reported by others in the results of recent research. Newkirk found that the onset of unstable motion was not postponed by improving shaft balance. This was true because the instability due to internal friction is a linear phenomenon and thus is independent of amplitude; it is, of course, also independent of the magnitude of the external exciting forces. So long as the system can be considered linear, then, instability will be independent not only of balance but also of orientation with respect to gravity. Newkirk's systems never appeared unstable when operated below the first critical speed. Their stability has since been confirmed by numerous experimental and theoretical researchers.

At the onset of instability, the rotor center line precessed at a speed very close to the first critical speed. An increase in operating speed did not change the precession rate. The precession results from a negatively damped, free vibration of the rotating system at its natural frequency. Only synchronous precession is observed because the reverse precession, while a possible mode, is positively damped by internal friction.

The onset of unstable whirl could vary widely between machines of similar construction. This would probably not be true if the source of internal friction were solely due to, say, hysteresis of the shaft material. The fact of the matter is, however, that losses due to fretting or the rubbing of adjacent parts, such as a disk shrunk on a shaft, account for the greater part of internal friction. This type of action is clearly rather inconsistent because the tightness of a nut or the tolerance of a press fit could easily change the amount of internal friction. Furthermore, the onset of instability is dependent on the ratio of internal to external damping rather than the magnitude of each. This means that in a system with low internal and external damping a small change in either can have a large effect. The fact that most of the internal friction arises from the rubbing of parts also accounts for the fact that Newkirk only observed the whirling phenomenon in built-up rotors.

One result which was especially perplexing to Newkirk was that an increase in system flexibility would delay the onset of instability. He naturally expected that an increase in bearing support flexibility would reduce the critical speed and hence bring about unstable motion earlier. The answer to the apparent paradox was that the support stiffnesses were not equal in two directions perpendicular to the rotor. Kellenberger [25] and, more recently, Gunter [2] present numerous maps and graphs showing the effect of bearing/support asymmetry in narrowing the range of operating speeds for which instabilities arise in the presence of internal and external friction.

Kellenberger reconsidered the stability problem of a disk attached at the center of a rotating, massless, elastically isotropic shaft in the presence of internal and external damping. He introduced the additional influence of anisotropic bearing supports and found that an increasing degree of anisotropy delays the onset of the unstable region to higher operating speeds.

The nature of a realistic law of friction in solids, for the most part, remains a matter of conjecture. With regard to internal friction in rotating shafts, most authors make the assumption that the force of internal friction is proportional to the linear or angular velocity of the shaft center line, measured with respect to a coordinate system which is rotating with the shaft.

It can generally be stated that this simple assumption gives results which are qualitatively correct in many cases, but certainly not in every case. The fact that the instability threshold associated with internal shaft friction is in the vicinity of the first critical speed for low external damping is accurately predicted by linear theory. On the other hand, linear theory does not account for the fact that it is sometimes necessary to strike the shaft of a well-balanced rotor with varying degrees of intensity in order to induce the nonsynchronous whirl which is associated with rotating shaft instability. Nor does the assumption of simple linear friction provide a true picture of either the range of instability or the final amplitude involved.

It is not difficult to see how the nature of the internal friction could be responsible for the dependence of the threshold speed on the degree of shaft imbalance. Consider, for example, the following. First, it is well known that the most important component of internal damping in rotating shafts is that caused by dry friction between contact surfaces of rotor parts during deformations of the rotor. Suppose that the dry friction characteristic is such that there will be no relative motion until the shear force at the interface exceeds some given value. At that point the two contacting surfaces will have a relative velocity proportional to the shear force (as in the usual simplified approach). With such a friction law there will be effectively no internal damping until the shear force (which depends on the degree of imbalance) reaches some threshold value. Furthermore, for a given shaft system, there will generally be more than one source of internal damping, e.g., between the shaft and impeller hub, rotor keyways, and so forth. For a given imbalance, slippage between one set of contact surfaces may not begin at the same operating speed as between another set. On the other hand, an increasing imbalance may initiate more sources of internal friction for a given operating speed. The new result of this kind of friction law is that the onset of unstable motion would depend on the imbalance or the strength of a lateral disturbance applied to the supercritically operating shaft.

Tondl made attempts to account realistically for the effects of internal friction in rotating shafts by investigating both the effect of hysteretic damping in shafts of uniform mass and dry friction between press-fitted parts [9]. He examined the problem of a single disk at the center of a massless shaft subject to internal friction in the form

$$c_i(max \times |\zeta|)\frac{\dot{\zeta}}{|\dot{\zeta}|} ,$$

rather than the more elementary viscous friction form $c_i\dot{\zeta}$. This assumes that the hysteretic damping force is such that it acts in the same direction as the relative velocity $\dot{\zeta}$ but that the magnitude is constant and dependent only on maximum displacement.

Tondl shows that, with a more realistic form of internal damping, the amount of instability due to internal friction is not generally independent of the imbalance or of the orientation of the shaft. For example, a shaft which is stable for supercritical operation while operating in a vertical position will also be stable in a horizontal position, but the converse is not necessarily true. Further, his treatment of internal friction reveals that enough external damping can eliminate the unstable range altogether if the friction is due to hysteresis, whereas there will always be a threshold value of operating speed above which the shaft will be unstable if the source of internal damping is dry friction.

It is noteworthy that in performing experiments dealing with the effects of internal damping, the usual procedure is to use a shaft running in ball bearings in order to reduce the possibility of having unstable motion arising from fluid film bearings. The latter subject is treated in the section on bearing lubricants, p. 37.

Asymmetry of Rotating Parts

Smith [26] was among the earliest investigators of instabilities arising from asymmetry in a rotating shaft system. His observations in this area were remarkably accurate. In fact, some of his observations, such as those dealing with the effect of bearing support asymmetry on the instability due to internal friction, seem to have been rediscovered by at least two different authors two decades or more later.

Other important contributors to this subject include Brosens [12], who considered rotors with unequal principal moments of inertia (he also considered the problem of an unsymmetrical shaft operating in anisotropically flexible supports); and Ariaratnam [27] who concerned himself with the presence of internal and external friction in a horizontal shaft with unequal stiffness about its two transverse axes. The shaft system Ariaratnam treated was assumed to have a constant mass per unit length. In addition, however, it was allowed to have a small initial bend of arbitrary shape and also a continuously varying mass imbalance.

Ariaratnam's approach expands all spatial functions as an infinite series in the normal modes of the nonrotating shaft. His results confirm that instabilities in the absence of damping occur in operating speed ranges which lie between the two natural frequencies associated with each nonrotating mode. External damping narrows the widths of the unstable regions associated with shaft anisotropy to the point where some particular value will eliminate one or more of the unstable zones altogether. If only internal damping is present in the anisotropic shaft, the whole operating range above the first critical speed becomes unstable. If external damping is much greater than internal damping, however, the onset of instability due to internal damping is delayed to a much higher operating speed. The consistency of these results with those of Tondl for shafts with internal hysteresis damping, discussed earlier, is, of course, reassuring.

Ariaratnam also treated the forced response of an anisotropic shaft due to initial bends, mass eccentricity, and gravity. The results show that shaft imbalance and lack of straightness can cause system resonance at a doubly infinite set of critical speeds which bound the unstable regions of free vibration. Resonance due to gravity is shown to occur only when there is asymmetry of the rotor. Furthermore, the amplitude buildup at secondary critical speeds is generally small, unless the stiffnesses of the shaft in the two principal directions are radically different.

Yamamoto *et al.* [28] have published a series of papers in the area of vibrations of unsymmetrical rotating shafts. The only asymmetries they treat are of the type due to a rotor which has unequal principal moments of inertia. The model consists of a massless shaft with an attached disk (not generally located at the center of the shaft). They discuss zones of instability in which the free vibrations grow exponentially with time.

The instability arising from asymmetry in the flexibility of a rotating shaft is most simply illustrated for the case of a disk at the center of a massless elastic shaft. We can write the equations of motion for such a system by modifying the equations developed at the beginning of this chapter to account for the asymmetry in shaft flexibility.

The changes appear in the elastic restoring force. Rather than Eq. (4.4) we have instead

$$\begin{aligned} \bar{f}_s &= -\kappa_j v\mathbf{j} - \kappa_w w\mathbf{k} \\ &= -\bar{\kappa}(v\hat{e}_y + z\hat{e}_z) - \kappa'(v\cos 2\Omega t + z\sin 2\Omega t)\hat{e}_y \\ &\quad - \kappa'(v\sin 2\Omega t - z\cos 2\Omega t)\hat{e}_z\,. \end{aligned} \tag{4.26}$$

where

$$\bar{\kappa} = \left(\frac{\kappa_v + \kappa_w}{2}\right)$$

$$\kappa' = \left(\frac{\kappa_v - \kappa_w}{2}\right). \tag{4.27}$$

The equations of motion are then given by:

Stationary frame

$$m\ddot{y} + (c_i + c_e)\dot{y} + y(\overline{\kappa} + \kappa' \cos 2\Omega t) + \kappa' z \sin 2\Omega t + c_i \Omega z = m\Omega^2 a \cos \Omega t$$

$$m\ddot{z} + (c_i + c_e)\dot{z} + z(\overline{\kappa} - \kappa' \cos 2\Omega t) + \kappa' y \sin 2\Omega t - c_i \Omega y = m\Omega^2 a \sin \Omega t \tag{4.28}$$

Rotating frame

$$m\ddot{v} + (c_e + c_i)\dot{v} + (\kappa_v - m\Omega^2)v - 2m\Omega\dot{w} - c_e\Omega w = m\Omega^2 a$$

$$m\ddot{w} + (c_e + c_i)\dot{w} + (\kappa_w - m\Omega^2)w + 2m\Omega\dot{v} + c_e\Omega v = 0 \tag{4.29}$$

The equations of motion in the stationary frame are linear, but they have periodic coefficients. In the rotating frame, the equations take the form of linear differential equations with constant coefficients. Since solutions to equations with periodic coefficients are less commonplace, the rotating frame is normally used when there is asymmetry only in the rotating parts of the system. The following derivation also proceeds along those lines; it should be noted, however, that much can be learned about systems described by equations with periodic coefficients by studying systems such as this in which a transformation of coordinates eliminates the periodicity in the coefficients.

Employing the complex notation again for (v, w), Eq. (4.29) may be replaced by

$$m\ddot{u} + (c_e + c_i)\dot{u} + (\overline{\kappa} - m\Omega^2)u + \kappa'\overline{u} + 2m\Omega j\dot{u} + c_e\Omega ju = m\Omega^2 a. \tag{4.30}$$

In stability investigations one looks for solutions for v and w such that

$$v = v_0 e^{st}$$

and

$$w = w_0 e^{st},$$

so that

$$u = u_0 e^{st}, \quad \overline{u} = \overline{u}_0 e^{st}, \tag{4.31}$$

and

$$\left\{ms^2 + [(c_e + c_i) + 2m\Omega j]s + [\bar{\kappa} - m\Omega^2 + c_e\Omega j]\right\}u_0 + \kappa'\bar{u}_0 = 0.$$

The conjugate of this equation also holds. That is, we can replace j by $-j$, u_o by $\bar{u}_o$, $\bar{u}_o$ by u_o; thus,

$$\left[\begin{array}{c|c} s^2 + \left(\dfrac{c_e + c_i}{m} + 2\Omega j\right)s + \left(\dfrac{\bar{\kappa}}{m} - \Omega^2 + \dfrac{c_e\Omega j}{m}\right) & \dfrac{\kappa'}{m} \\ \hline \dfrac{\kappa'}{m} & s^2 + \left(\dfrac{c_e + c_i}{m} - 2\Omega j\right)s + \left(\dfrac{\bar{\kappa}}{m} - \Omega^2 - \dfrac{c_e\Omega j}{m}\right) \end{array}\right] \left\{\begin{array}{c} u_o \\ \hline \bar{u}_o \end{array}\right\} = \left\{\begin{array}{c} 0 \\ \hline 0 \end{array}\right\}. \tag{4.32}$$

The characteristic equation is then given by

$$\left[s^2 + \frac{c_e + c_i}{m}s + \left(\frac{\bar{\kappa}}{m} - \Omega^2\right)\right]^2 + \left(2\Omega s + \frac{c_e\Omega}{m}\right)^2 - \left(\frac{\kappa'}{m}\right)^2 = 0. \tag{4.33}$$

It can be shown by means of the Routh criterion that, in addition to an unstable region due to internal damping, there may be an instability in the range of operating speeds between $\sqrt{\kappa_w/m}$ and $\sqrt{\kappa_v/m}$, i.e., between the two critical speeds of the undamped shaft.

The effect of external damping on this latter instability range is to shift its center to slightly lower speeds and to narrow it down; as noted earlier in this section, sufficient external damping may eliminate the instability altogether.

Bearing Lubricants

Probably the most frequent source of unstable shaft motion is the action of an oil-lubricated journal in its bearings. It is also probably one of the most dangerous in that it can easily lead to bearing seizure.

It was Newkirk who first recognized this type of instability. He had been involved in the study of shaft whipping (i.e., synchronous precession due to internal damping), when he observed a violent whipping motion which obviously did not arise from either hysteresis or dry friction. Newkirk referred to this new phenomenon as oil whip, and many subsequent researchers adopted this terminology. Newkirk and Taylor [29] published the first paper that offered an explanation of the cause of this instability. From a simple consideration of the continuity of oil flow about the journal, the journal center was shown to precess at an angular rate equal to one-half the operating speed.

Some of the principal investigators whose analytical work followed the experimental findings of Newkirk are Robertson [30], Hagg [31], Hori [32], Poritsky [33], Pinkus [34], Tondl [9], and many more.

The instability due to the oil film in journal bearings, of course, involves fluid mechanics, and early advances in the theory were mainly due to improvements in the approximate equations governing fluid films. Robertson developed expressions for the forces acting on a journal due to the action of a load-carrying oil film at a very early date (1933). His results are valid for a viscous, incompressible lubricant where there is no side leakage.

The results of many investigations, both theoretical and experimental, were often in disagreement in many respects. In particular, there was no universal agreement on the angular velocity of the self-excited vibrations. Most authors, however, felt that it was at the natural frequency of the rotor.

Stability analyses of journal bearings usually represent the nonlinear bearing forces as perturbations about some equilibrium position. It is then possible to use the Routh-Hurwitz criterion to determine stability boundaries. The resulting linearized bearing forces affect rotor stability in a manner which is not very different from that of the linearized internal friction force.

Effect of Nonlinearities

While Tondl's investigations into the effects of nonlinear friction forces provide one mechanism for harmonic whirling at a finite amplitude (i.e., in a limit cycle), there are other possible sources of nonlinearity which could have similar results. Billett [35], for example, treated the case of a shaft in bearings with symmetrical nonlinear flexibility and internal friction. This type of nonlinearity is cited as perhaps characteristic of ball bearings.

The specific mathematical model used by Billett is, again, the disk mounted at the center of a massless, elastic shaft. Both internal and external frictions are assumed to be of the simple viscous type. The equations of motion of this system are given by

$$m\ddot{x} + m(\mu + \nu)\dot{x} + F(r)(x^2 + y^2)^{-1/2}x + m\nu\Omega y = mr^2 a \cos \Omega t \tag{4.34}$$

and

$$m\ddot{y} + m(\mu + \nu)\dot{y} + F(r)(x^2 + y^2)^{-1/2}y - m\nu\Omega x = m\Omega^2 a \sin \Omega t, \tag{4.35}$$

where $r = \sqrt{x^2 + y^2}$ and $F(r)$ is the radial restoring force in the plane of the rotor.

The stiffness of the support is assumed to increase with deflection so that

$$F(r) > 0,$$

$$F'(r) > \frac{F(r)}{r}. \tag{4.36}$$

It can be seen that Eqs. (4.34) and (4.35) reduce to the linear equations as presented early in this chapter for the case where

$$F'(r) = k \quad \text{or} \quad F = kr.$$

Making use of a complex representation of the deflection in the stationary frame, i.e., $z = x + iy$, the following equation of motion may be written:

$$\ddot{z} + (\mu + \nu)\dot{z} + \frac{F(r)}{m} z(z\bar{z})^{-\frac{1}{2}} - i\Omega\nu z = \Omega^2 a e^{i\Omega t}. \tag{4.37}$$

Billett considers first the case of synchronous whirling in which $z = Re^{i(\Omega t - \phi)}$. It is shown that there will be discrete values for R and ϕ, given a speed Ω. These values, in this case, will depend on the imbalance, since $F(r)$ is constant but dependent on the imbalance. Billett's stability analysis is linearized, in the usual manner, by considering a small perturbation about a mean whirl amplitude: those solutions lead in turn to the existence of nonsynchronous whirl of finite amplitude.

4.2 Methods for Predicting Instabilities

Stability Criteria vs Eigenvalue Locations

Many papers in the literature which treat the instabilities of rotating shafts due to friction, asymmetry, journal bearings, and so forth, use the Routh-Hurwitz criterion to test the stability of the characteristic polynomials.

For nth-order linear differential equations with constant coefficients, an investigation into the free vibrations of the form $e^{i\lambda t}$ always leads to an nth-order characteristic equation of the form

$$(a_0 + ib_0)\lambda^n + (a_1 + ib_1)\lambda^{n-1} + \ldots + (a_n + ib_n) = 0. \tag{4.38}$$

The roots of this equation may be complex whether the coefficients are real or complex.

The stability criterion for such polynomials consists of a requirement that n particular determinants each be of a certain sign. The necessary and sufficient condition for all modes to be stable is that

$$-\begin{vmatrix} a_0 & a_1 \\ b_0 & b_1 \end{vmatrix} > 0 \quad \text{and} \quad \begin{vmatrix} a_0 & a_1 & a_2 & 0 \\ b_0 & b_1 & b_2 & 0 \\ 0 & a_0 & a_1 & a_2 \\ 0 & b_0 & b_1 & b_2 \end{vmatrix} > 0, \tag{4.39}$$

and that in general

$$(-1)^n \begin{vmatrix} a_0 & a_1 & \cdots & a_n & 0 & 0 & \cdots & \\ b_0 & b_1 & \cdots & b_n & 0 & 0 & \cdots & \\ 0 & a_0 & \cdots & a_{n-1} & a_n & 0 & \cdots & \\ 0 & b_0 & \cdots & b_{n-1} & b_n & 0 & \cdots & \vdots \\ 0 & 0 & a_0 & \cdots & & & & \vdots \\ \vdots & 0 & b_0 & \cdots & & & & 0 \\ & & & & & & & 0 \\ & & & & & & & a_n \\ & & & & & & & b_n \end{vmatrix} > 0.$$

If any of the eigenvalues of λ have negative imaginary parts, the solutions are of the form $e^{i\omega t + \nu t}$, which grow exponentially with time and indicate an instability. For example, consider the simple model considered in the first section of this chapter. The characteristic equation was given by Eq. (4.21). Since s is equivalent to $i\lambda$, we see that, for that example,

$$(a_0 + ib_0) = -1,$$

$$(a_1 + ib_1) = i\left(\frac{c_i + c_e}{m}\right) - 2\Omega, \tag{4.40}$$

and

$$(a_2 + ib_2) = \frac{c_e}{m}\Omega i + \left(\frac{\kappa}{m} - \Omega^2\right).$$

Therefore, in order that all the modes be stable the necessary and sufficient condition is that

$$-\begin{vmatrix} -1 & -2\Omega \\ 0 & \dfrac{c_i + c_e}{m} \end{vmatrix} > 0$$

and

$$\begin{vmatrix} -1 & -2\Omega & \left(\dfrac{\kappa}{m} - \Omega^2\right) & 0 \\ 0 & \dfrac{c_i + c_e}{m} & \dfrac{c_e \Omega}{m} & 0 \\ 0 & -1 & -2\Omega & \left(\dfrac{\kappa}{m} - \Omega^2\right) \\ 0 & 0 & \left(\dfrac{c_i + c_e}{m}\right) & \dfrac{c_e \Omega}{m} \end{vmatrix} > 0. \quad (4.41)$$

The first condition is always true so long as the constants and c_e and c_i are positive.

The second condition requires that

$$\frac{\kappa}{m}\left(\frac{c_i + c_e}{m}\right)^2 > \left(\frac{\Omega c_i}{m}\right)^2. \quad (4.42)$$

Equivalently, all modes are stable as long as

$$\Omega < \left(1 + \frac{c_e}{c_i}\right)\sqrt{\frac{\kappa}{m}}. \quad (4.43)$$

This is the identical result obtained at the beginning of this chapter by taking a first approximation for the complex roots.

The advantage of using the Routh-Hurwitz criterion in this particular example is that no approximations need be made in calculating stability boundaries. On the other hand, if only this stability criterion is used, no insight as to the behavior of the motion in the unstable regime is gained. The unstable mode is not identified, and the growth rates will be undetermined. To predict these characteristics, one can solve for the roots of the characteristic equation. In practice, instabilities can probably be investigated best by a combination of techniques.

The foregoing example is really a very special case. For this simple, second-order system, the process of finding the roots exactly is not difficult; the Routh-Hurwitz criterion, therefore, while straightforward, adds no information.

In practical cases many degrees of freedom are involved. Generally, then, there will be many complex roots of the characteristic equation, and the procedure for their solution will require either approximations or iterative techniques. In either case it will probably be worthwhile to apply the Routh-Hurwitz criterion.

Consider a ten-degree-of-freedom system, for which the stability of the system within a certain operating range is of interest. Determining all ten roots will require a trial and error or iterative process for each root at each of the many different values of operating speed in the range of interest. Applying the Routh-Hurwitz criterion, however, will involve evaluating ten different determinants (of order 2, 4, 6, . . ., 20) for the same number of speeds in the range of interest, but no iterations.

An efficient approach would be to first scan the regime of interest with the Routh-Hurwitz criterion. If no instabilities are indicated, there may be no need to know the roots. If an instability is indicated in a certain range, the roots could then be evaluated for those limited operating speeds, to determine the unstable mode or modes and their frequency and growth rate. Such information is likely to suggest design changes to suppress the instability. For example, a knowledge of the growth rate indicates the characteristics of the external damping device needed to stabilize the system.

The analytical techniques used to arrive at the characteristic equation governing the stability of the shaft system are, of course, identical to those which are used to compute the natural frequencies as a function of operating speed. Stability analysis requires only that the frequencies be allowed to be complex. It should be emphasized that all these procedures apply only when the governing equations of motion are linear differential equations with constant coefficients.

When there are asymmetries in the elastic or inertial or damping properties of both the rotating and nonrotating parts of a shaft system, the governing equations will have periodic coefficients, no matter what frame of reference is used. For example, if a slotted shaft, carrying a disk at its center, is operating on supports which have unequal flexibilities in two directions, the governing equations are given by the following expressions:

Stationary frame

$$
\begin{aligned}
m\ddot{y} + (c_i + c_e)\dot{y} + (\bar{\kappa} + k_y + \kappa' \cos 2\Omega t)y & \\
+ (c_i\Omega + \kappa' \sin 2\Omega t)z &= m\Omega^2 a \cos \Omega t, \\
m\ddot{z} + (c_i + c_e)\dot{z} + (\bar{\kappa} + k_z - \kappa' \cos 2\Omega t)z & \\
- (c_i\Omega - \kappa' \sin 2\Omega t)y &= m\Omega^2 a \sin \Omega t, \qquad (4.44)
\end{aligned}
$$

Rotating frame

$$m\ddot{v} + (c_i + c_e)\dot{v} + (\kappa_v + \bar{k} + k' \cos 2\Omega t - m\Omega^2)v$$

$$- 2m\Omega\dot{w} - (c_e\Omega + k' \sin 2\Omega t)w = m\Omega^2 a,$$

and

$$m\ddot{w} + (c_i + c_e)\dot{w} + (\kappa_w + \bar{k} - k' \cos 2\Omega t - m\Omega^2)w$$

$$+ 2m\Omega\dot{v} + (c_e\Omega - k' \sin 2\Omega t)v = 0. \quad (4.45)$$

These equations assume that the imbalance lies along the principal axis of shaft bending. However, this does not enter into stability considerations, since only the left-hand sides of the equations are considered in a linear analysis. The important point here is that the presence of periodic coefficients apparently can not be avoided. There are no known closed-form solutions to coupled equations of this type. Accordingly, most theoretical investigators have avoided problems in which they arise. In practical situations, however, there will always be asymmetries in both rotating and nonrotating parts to some degree. Assuming otherwise involves an approximation.

Direct Numerical Time Integration

The direct numerical time integration approach can be made to work very well for particular design problems but gives little insight into the physics of the predicted behavior. Any number of numerical schemes such as Runge-Kutta can be applied to the time integration.

Generalized Hill's Method

An extension of Hill's procedure to the coupled, linear differential equations with periodic coefficients encountered with an unsymmetrical shaft operating in unsymmetrically flexible supports was first published by Foote *et al.* [11] in 1943. Similar work was published by Brosens and Crandall [36] in 1961 and by Coleman and Feingold [37] in 1958. The Coleman work addressed the mechanical instability of helicopter rotors with hinged blades, but the equations of motion are similar to those of concern here, and their methods of analysis therefore apply. The most recent work known at this writing along the lines of generalizing Hill's technique is due to Crimi [38].

Asymptotic Methods

If the source of asymmetry in rotating shaft systems is relatively small, each periodic coefficient in the governing equation of motion is usually multiplied by some small parameter. For example, in Eq. (4.45) the quantity κ' may be considered very small compared to $\bar{\kappa}$, as may be seen on examination of Eq. (4.27).

Tondl [9] uses a method wherein the solution is assumed in the form of a power series in the small parameter. His approach predicts the instability intervals by a method of successive approximations. Tondl's method seems easy to use and, also, is useful in determining the approximate solution of the stability of linear systems with constant coefficients when there is a large number of degrees of freedom.

Unfortunately, the method is not always applicable. For example, a shaft with a very unsymmetrical cross section, operating on a stand which is flexible in one direction but extremely stiff in the other, allows the relative differences in flexibilities of neither shaft nor supports to be assumed small. However, for many practical systems the asymmetry of at least the rotating parts is small. This technique, therefore, is useful for a broad class of practical problems.

CHAPTER 5
FORCED BENDING RESPONSE AND TRANSITION THROUGH CRITICAL: LATERAL SHAFT PROBLEMS

Analysis of forced response—as, for example, results from mass imbalance—shows that large amplitudes may be encountered in the vicinity of critical speeds. This requires, of course, that the operator not allow the speed to linger in the region of a critical speed. In practice, then, the magnitudes of shaft deflection are strongly dependent on the rate at which the shaft is accelerated or decelerated near or through critical speeds. Nevertheless, much information can be inferred from forced response analyses and stability studies wherein the operating speed is assumed constant.

5.1 Forced Steady State Response

It is usual, in determining the forced steady state response of a linear elastic system in the presence of a periodic external force, to seek a periodic solution having the same period as the forcing function. For the rotating shaft carrying an unsymmetrical rotor, however, this is not the only solution. Yamamoto [39] showed, for example, that a massless shaft carrying a disk with two different principal lateral moments of inertia and rotating with angular velocity Ω, would, when excited by a (fixed system) periodic external force with frequency ω_0, undergo forced vibrations consisting of the sum of two periodic functions. These motions would have frequencies ω_0 and $(2\Omega - \omega_0)$, as detected by a vibration-measuring device on the bearing pedestal. It is emphasized that the amplitude of the response at the latter frequency might be significantly larger than the amplitude at the forcing frequency.

It is important to note that these rather peculiar results manifest themselves because this behavior has been described as it is observed in the stationary coordinate system. If the motion is described as it is observed in the rotating coordinate system, the results seem more conventional. As mentioned above, the response as measured in the stationary frame has frequencies $\Omega \pm (\omega_0 - \Omega)$. Any oscillation or forcing function which in the stationary frame has the form of a constant $xe^{i\omega_0 t}$ would be measured as having the frequency $(\omega_0 - \Omega)$ in a reference frame fixed in a shaft rotating at an angular velocity equal to Ω. Thus, the forcing function has a frequency $(\omega_0 - \Omega)$ when measured in the rotating frame, and the two components of the response in the rotating frame have frequencies $\Omega \pm (\omega_0 - \Omega) - \Omega = \pm(\omega_0 - \Omega)$. Therefore, the frequency magnitudes of the two response components are the same when measured in the rotating system. The difference in sign is associated with the direction of rotation (as measured in the rotating frame).

The exciting frequency ω_0 is general, so that the force of gravity is a possible special case in which the force is constant both in magnitude and direction so that the forcing frequency is zero ($\omega_0 = 0$). Consequently, in the stationary frame, vibrations at zero frequency (i.e., static deflection) and at 2Ω are detected. In the rotating reference frame, one would measure vibrational frequencies $-\Omega$ and $+\Omega$, which are, of course, vibrations at frequencies equal to the angular velocity of the shaft. Since it has been noted that the portion of the response with frequency equal to that of the forcing function may be the smaller of the two, gravity excitation could result in twice-per-revolution response greater than the static deflection. Here both deflections are considered in the nonrotating stationary frame.

5.2 Response During Transition

When the forcing function is due to imbalance, the frequency ω_0 is obviously equal to the operational speed Ω. Then, only a vibrational frequency of value Ω is detected in the stationary system. In the rotating system, the frequency is zero. All this is true, however, only if the rotational speed Ω is constant. In considering the problem of transition through the critical speed, it is necessary to account for the variable speed in order to determine the maximum deflection or stress involved, since response magnitudes are dependent on acceleration or deceleration rates.

A complete and accurate picture of the transition situation can be obtained only by considering the torque and angular velocity relationships of the driving motor. Goloskokov [40, 41] treated the transition problem by assuming that the moment output characteristics of the driving motor are independent of acceleration; this is, of course, only an approximation. Another approach [42], which allows for the time-varying nature of the coefficients, to be seen in the following pages, is to simulate the shaft motion during transition by an analog computer.

There have been, in fact, very few theoretical studies of the transition problem. They have all been restricted to the simplest configurations, and they usually disregard the interaction of torque and angular velocity. This is done by assuming a definite form for the acceleration, e.g., constant. Of course, in order for the angular velocity to increase linearly with time, the applied torque may be required to vary in an unrealistic way. But by specifying the angular-velocity time history, as has been done, the torque itself often remains implicit. The assumption of constant acceleration, of course, has the advantage that it allows a fairly straightforward analysis. The results are likely to be valid as long as the shaft speed range investigated is small enough.

The brief analysis of transition through critical speed which follows is based on Dimentberg's solution [5] for a disk at the center of a massless elastic shaft.

Let the position vector from the line of centers of the supports to the center of mass of the unbalanced disk (Fig. 15) be given by

$$\overline{r} = v\mathbf{j} + w\mathbf{k} + a\mathbf{j} \qquad \text{(rotating frame)}, \qquad (5.1)$$

and

$$\overline{r} = y\hat{e}_j + z\hat{e}_k \qquad \text{(fixed frame)}.$$

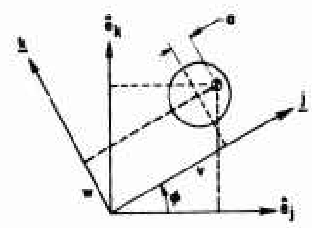

Fig. 15. Fixed and rotating coordinate systems.

The angular velocity of the $\mathbf{i}, \mathbf{j}, \mathbf{k}$ triad is given by $\Omega\mathbf{i}$, where

$$\Omega = \Omega_0 + 2\alpha t. \qquad (5.2)$$

The velocity of the center of mass is equal to the first time derivation of the position vector;

$$\dot{\overline{r}} = \dot{v}\mathbf{j} + \dot{w}\mathbf{k} + \Omega v\mathbf{k} - \Omega w\mathbf{j} + \Omega a\mathbf{k}$$

$$= \dot{y}\hat{e}_j + \dot{z}\hat{e}_k. \qquad (5.3)$$

Differentiating once more, we have the acceleration of the center of mass:

$$\ddot{\overline{r}} = \left[\frac{d}{dt}(\dot{v} - \Omega w) - \Omega(\dot{w} + \Omega v + \Omega a)\right]\mathbf{j}$$

$$+ \left[\frac{d}{dt}(\dot{w} + \Omega v + \Omega a) + \Omega(\dot{v} - \Omega w)\right]\mathbf{k}.$$

$$= \ddot{y}\hat{e}_j + \ddot{z}\hat{e}_k. \qquad (5.4)$$

The elastic restoring force on the shaft is provided by the shaft stiffness:

$$\overline{f}_e = -\kappa(v\mathbf{j} + w\mathbf{k}),$$

$$= -\kappa[(y - a\cos\phi)\hat{e}_j + (z - a\sin\phi)\hat{e}_k]. \qquad (5.5)$$

Newton's second law provides the equation of motion of the system. Having developed the expressions for force and acceleration in both the stationary and moving frames of reference, we are able to write the equations of motion in both frames:

Stationary frame

$$m\ddot{y} + \kappa y = \kappa a \cos\phi,$$

$$m\ddot{z} + \kappa z = \kappa a \sin\phi, \qquad (5.6)$$

where

$$\phi = \Omega_0 t + \alpha t^2;$$

Rotating frame

$$\ddot{v} + \left(\frac{\kappa}{m} - \Omega^2\right)v - 2\Omega\dot{w} - \dot{\Omega}w = \Omega^2 a,$$

$$\ddot{w} + \left(\frac{\kappa}{m} - \Omega^2\right)w + 2\Omega\dot{v} + \dot{\Omega}v = -\Omega^2 a, \qquad (5.7)$$

where

$$\dot{\Omega} = 2\alpha, \qquad \Omega^2 = (\Omega_0 + 2\alpha t)^2.$$

It is clear that Eqs. (5.5) to (5.7) may be solved without a knowledge of the torque if the acceleration α is specified. However, for completeness, we proceed now to develop an expression for the torque.

The angular momentum of the disk about its center of mass is equal to its polar moment of inertia times the angular velocity of the disk itself. That is, $\mathbf{L}_{c.m.} + I_p\Omega\mathbf{i}$.

The total moment of force acting about the center of mass is

$$T\mathbf{i} - a\mathbf{j}x\bar{f}_e = (T + a\kappa w)\mathbf{i}. \qquad (5.8)$$

Equating the total moment and the rate of change of angular momentum, we have

$$T = -\kappa a w + 2\alpha I_p \qquad (5.9)$$

or

$$T = \kappa a(y \sin\phi - z\cos\phi) + 2\alpha I_p. \qquad (5.10)$$

Equations (4.9) and (4.10) will provide the torque time history required to accelerate the shaft as assumed.

The governing equations, (5.6) or (5.7), are to be solved subject to the initial conditions the shaft had at the instant the acceleration was begun. Those conditions are found from the particular solution with $\phi = \Omega_0 t$ and $\Omega = \Omega_0$. Thus,

$$v = \frac{a\Omega_0^2}{(\omega_n^2 - \Omega_0^2)}, \qquad \dot{v} = 0;$$

$$w = 0, \qquad \dot{w} = 0; \tag{5.11}$$

where

$$w_n^2 = \frac{\kappa}{m}.$$

Or, in terms of the stationary coordinates,

$$y = \frac{a \cos \Omega_0 t}{1 - \dfrac{\Omega_0^2}{\Omega_n^2}}, \qquad \dot{y} = \frac{-\Omega_0 a \sin \Omega_0 t}{1 - \dfrac{\Omega_0^2}{\Omega_n^2}};$$

$$z = \frac{a \sin \Omega_0^2}{1 - \dfrac{\Omega_0^2}{\omega_n^2}}, \qquad \dot{z} = \frac{\Omega_0 a \text{ oos } \Omega_0 t}{1 - \dfrac{\Omega_0^2}{\omega_n^2}}. \tag{5.12}$$

Starting at $t = 0$, we have

$$v_0 = \frac{\dfrac{a\Omega_0^2}{\omega_n^2}}{1 - \dfrac{\Omega_0^2}{\omega_n^2}}, \qquad \dot{v}_0 = 0;$$

$$w_0 = 0, \qquad \dot{w}_0 = 0; \tag{5.13}$$

or

$$y_0 = \frac{a}{1 - \dfrac{\Omega_0^2}{\omega_n^2}}, \qquad \dot{y}_0 = 0;$$

$$z_0 = 0, \qquad \dot{z}_0 = \frac{\Omega_0 a}{1 - \dfrac{\Omega_0^2}{\omega_n^2}}. \tag{5.14}$$

Dimentberg approached this problem by working with the fixed coordinate system to the point where a general solution to the differential equations was obtained. Then expressions for the displacements in a rotating coordinate system were found using a coordinate transformation. Inspection of Eq. (5.6) vs Eq. (5.7) reveals the rationale for this approach; it is simpler to solve the equations of motion, in this case, in the fixed system, even though the primary interest is in the displacements and stresses in the rotating system. While both sets are linear differential equations, they have constant coefficients in the fixed frame, as opposed to nonconstant coefficients in the rotating frame.

The formal solution of Eq. (5.6), subject to the initial conditions of Eq. (5.14) is given by:

$$y = a\omega_n \int_0^t \sin \omega_n(t-\tau) \cos (\Omega_0 \tau - \alpha\tau^2) d\tau ,$$

$$z = a\omega_n \int_0^t \sin \omega_n(t-\tau) \sin (\Omega_0 \tau - \alpha\tau^2) d\tau$$

$$+ \left(\frac{a\Omega_0}{\omega_n}\right) \frac{\sin \omega_n t}{1 - \dfrac{\Omega_0^2}{\omega_n^2}} . \qquad (5.15)$$

Through use of a coordinate transformation, and after considerable algebra, the expressions for the deflections in a system rotating with the shaft are obtained from Eqs. (5.15);

$$v = a\omega_n \sqrt{\frac{\pi}{8\alpha}} \Big\{ -\sin \nu [c(\nu) - c(\nu'_0)] + \cos \nu [s(\nu) - s(\nu_0)]$$

$$+ \sin \nu' [c(\nu') - c(\nu'_0)] - \cos \nu' [s(\nu') - s(\nu'_0)] \Big\}$$

$$+ \frac{a}{1 - \dfrac{\Omega_0^2}{\omega_n^2}} \Big[\cos \omega_n t \cos (\Omega_0 t + \alpha t^2)$$

$$+ \frac{\Omega_0}{\omega_n} \sin \omega_n t \sin (\Omega_0 t + \alpha t^2) \Big] , \qquad (5.16)$$

and

$$w = a\omega_n \sqrt{\frac{\pi}{8\alpha}} \left\{ -\cos\nu[c(\nu)-c(\nu_0)] - \sin\nu[s(\nu)-s(\nu_0)] + \cos\nu'[c(\nu')-c(\nu_0')] + \sin\nu'[s(\nu')-s(\nu_0')] \right\} + \frac{a}{1-\frac{\Omega_0^2}{\omega_n^2}} \left[-\cos\omega_n t \sin(\Omega_0 t + \alpha t^2) + \frac{\Omega_0}{\omega_n} \sin\omega_n t \cos(\Omega_0 t + \alpha t^2) \right], \tag{5.17}$$

where

$$c(\nu) \equiv \int_0^{\nu} \frac{\cos\sigma}{\sqrt{2\pi\sigma}}\, d\sigma$$

and

$$s(\nu) \equiv \int_0^{\nu} \frac{\sin\sigma}{\sqrt{2\pi\sigma}}\, d\sigma$$

are tabulated functions known as Fresnel integrals, and

$$\nu = \left(t\sqrt{\alpha} + \frac{\Omega_0 - \omega_n}{2\sqrt{\alpha}}\right)^2, \qquad \nu_0 = \left(\frac{\Omega_0 - \omega_n}{2\sqrt{\alpha}}\right)^2,$$

and

$$\nu' = \left(t\sqrt{\alpha} + \frac{\Omega_0 + \omega_n}{2\sqrt{\alpha}}\right)^2, \qquad \nu_0' = \left(\frac{\Omega_0 + \omega_n}{2\sqrt{\alpha}}\right)^2.$$

The expressions given by Eqs. (5.16) and (5.17) are exact for the model being discussed. They may be plotted against time as the system passes through critical in order to assess the actual deflections encountered in the transition.

Dimentberg presented a plot of (v, w) as the shaft passes through critical for a system having $\omega_n = 105$ rad/sec, $\Omega_0 = 94.5$ rad/sec, $\alpha =$ rad/sec^2. His plot was redrawn and is displayed in Fig. 16. Notice that the amplitudes reach maxima somewhat beyond the point where the instantaneous operating speed is equal to the critical speed ω_n. The maximum amplitude is about seven times greater than the initial amplitude at 0.94 of critical, for this case with zero damping.

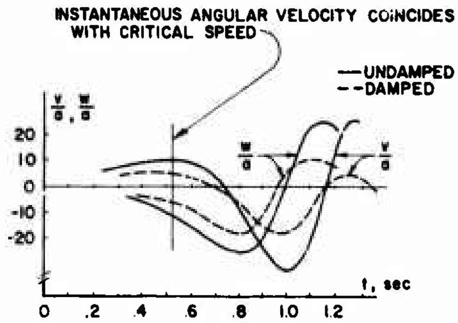

Fig. 16. Deflections as the shaft passes through critical.

A close look at the form of Eqs. (5.16) and (5.17) shows that the transition through critical induces vibrations at frequencies equal to the sum and difference of the instantaneous operating speed and the critical speed. Based on the discussion of Section 5.1, this means that, in the stationary frame, there will be response with a frequency equal to the critical speed.

5.3 Effect of Damping on Transition

If external damping is present in the system, transition deflections will be reduced. Dimentberg also considered an unbalanced disk at the center of a massless shaft subject to external friction. The equations of motion given by Eq. (5.6) are then modified by the addition of viscous damping terms on the left-hand side, i.e., $c\dot{y}$ and $c\dot{z}$, respectively. Dimentberg's analysis showed that the same case cited above with friction added in an amount such that $c/m = 1$ would experience a maximum amplitude only about three times the value at 0.94 of critical. This is less than 50 percent of the maximum amplitude for the undamped case, as shown in Fig. 16 with dotted lines.

Dimentberg also treats the problem of an undamped shaft accelerated through critical when the supports are both flexible and anisotropic. The equations of motion in the final system are no more difficult to solve than before, since equations are uncoupled and the unequal supports merely change the effective value of κ in each equation.

5.4 Transition Through Secondary Critical Speed

Dimentberg also considered a shaft of unequal stiffness operating horizontally and accelerating through its secondary critical speed. The asymmetry of the rotating part introduces nonconstant coefficients into the equations of motion expressed in the fixed system. Dimentberg therefore worked directly in the rotating system, but still found it necessary to ignore certain terms to reach a solution. Essentially, he dropped the second derivatives and solved the resulting system of first order equations by methods of integral calculus.

CHAPTER 6
COUPLED BENDING AND TORSIONAL MOTION

The principal concern in this chapter is with phenomena in which lateral and torsional motion interact.

The rationale commonly used in ignoring the coupling is that the coupling is usually determined by mass imbalance, which is a small quantity. Further, natural frequencies of lateral and torsional motion have, in the past, been so far removed from one another that they made dynamic interaction unlikely. However, with the advent of supercritical-speed shafts, which often have relatively small diameters, the uncoupled torsional frequencies can fall into the same range as the bending frequencies which are above the fundamental but below operating speed. Neglecting bending and torsion coupling in such cases may give inaccurate predictions of the critical speeds. More importantly, fundamental phenomena such as additional instabilities–where they exist–would be missed altogether. The coupling of bending and torsional motion also becomes important when investigating the influence of unsteady torque loading.

Just as pure lateral shaft motion has been the subject of the monograph to this point, it is well to begin the discussion of coupled bending and torsion with an initial review of pure torsional motion.

6.1 Pure Torsional Motion

Pure torsional vibrations can reasonably be considered whenever lateral motion is substantially absent, as for example because the shaft is extremely stiff laterally or because there are sufficiently frequent bearing locations constraining the system–or if coupling effects are negligible. For example, torsional vibrations of engines and geared systems have usually been considered solely torsional problems because of the lateral constraints used.

Pure torsional free vibrations of a rotating shaft do not differ in their dynamics from those of a shaft which is not rotating. This is in contrast to lateral free vibration characteristics, which are generally a function of the shaft operating speed. The reason for this behavior is, of course, that, by assuming a purely torsional motion, one eliminates the possibility of centripetal or Coriolis forces doing work on the motions of interest.

In the pure torsional vibrations of a crankshaft, for example, the rotational speed affects the vibrations only through its influence on the frequency of the periodic torques felt by the shaft. The preponderance of the literature concerning torsional vibrations of shafts is, therefore, devoted to computation of torsional natural frequencies and mode shapes.

The theory of torsional vibrations of rotating shafts with attached disks is generally much simpler than that for lateral vibration. On the other hand, some complications often arise in the torsional case which need seldom be considered for lateral vibrations. For example, it is fairly common to speak of torsional vibrations of branched systems, such as encountered in gear trains, whereas the lateral vibrations of such configurations rarely receive attention. (An exception is the lateral bending of launch rockets for space vehicles in which the former have "clustered" tanks [43, 44].

If a torsional system is composed of n disks on a rotating shaft, then the equation of motion of an individual disk is given by (see Fig. 17)

$$I_i\ddot{\Phi}_i + k_i(\Phi_i - \Phi_{i+1}) + k_{i-1}(\Phi_i - \Phi_{i-1}) = T_i, \qquad (6.1)$$

where

T_i is the torque applied to the ith disk,
Φ_i is the instantaneous angle of rotation of the ith disk,
I_i is the polar moment of inertia about the fixed point at the center of the shaft.

If, instead, we deal in a rotating coordinate system we have

$$\Phi_i = \omega t + \phi_i, \qquad (6.2)$$

and the form of the equation of motion is invariant.

The solution of the equations of motion for the case of free vibration (i.e., with the applied torques set equal to zero) yields the uncoupled torsional natural frequencies of the system.

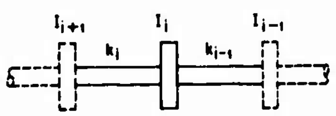

Fig. 17. Model for pure torsional vibrations.

Resonance, of course, results when a periodic applied torque, which has a frequency equal to one of the natural frequencies, either arises within the system or is applied.

In practice, the frequencies of the external torques are usually related to the operating speed of the shaft by some integral multiple. For example, gravity institutes a once-per-revolution pulsating torque on a horizontal shaft if any attached disk has imbalance; i.e., if the center of mass of the disk does not coincide with the center of the shaft. In that case, a torsional critical speed occurs when the operating speed is equal to one of the torsional natural frequencies. On the other hand, the basic period of the external torque in an engine may be twice that of the shaft rotation, so that resonance will occur when the operating speed is equal to twice the natural frequency. This is the case with the usual four-cycle, internal combustion engine.

Reference to the linear instability of a shaft undergoing purely torsional vibrations does not appear in the literature. It would appear that such unstable modes of vibration, which can be predicted with the assumption of pure, uncoupled torsion, have not been encountered. Such problems could perhaps occur in the presence of electromechanical interactions such as occur in modern motor-driven systems. Certainly, unstable torsional motion can arise when there are feedback elements in the system. Moreover, when coupling exists between bending and torsion, unstable torsional motions can occur due to basically bending instabilities, such as those arising from internal shaft friction. Finally, it also is true that instabilities exist, the basic character of which is different from those arising in either uncoupled bending or uncoupled torsion cases. Before considering in more detail such additional instabilities, however, it is well to consider how such phenomena as critical speeds and natural frequencies are influenced by bending-torsion coupling.

6.2 Coupled Critical Speeds

Consider the coupled equations of motion lateral bending–torsion for an unbalanced disk at the center of an elastic shaft, where the shaft is not assumed to be rotating at a constant speed. If it is assumed that the instantaneous angular rotation rate Ω is given by a steady value Ω_0 plus a small perturbation $\dot{\phi}$, then the linearized equations of motion in the coordinate system fixed to the shaft are written as follows:

$$\ddot{v} + \left(\frac{k}{m} - \Omega_0^2\right) v - 2\Omega_0 \dot{w} = \Omega_0^2 a + 2\Omega_0 \dot{\phi} a,$$

$$\ddot{w} + \left(\frac{k}{m} - \Omega_0^2\right) w + 2\Omega_0 \dot{v} = -a\ddot{\phi},$$

$$I_p \ddot{\phi} - kaw + k_\phi \phi = T = 0. \qquad (6.3)$$

Note that, for the moment, the applied torque is taken to be zero in developing these equations.

The particular solution corresponding to the unbalanced force is assumed to consist of constants for v, w, and ϕ. Substitution into Eqs. (6.3) yields (since the first and second time derivatives vanish)

$$\left(\frac{k}{m} - \Omega_0^2\right) v_0 = \Omega_0^2 a,$$

$$w_0 = 0,$$

and

$$\phi_0 = 0. \tag{6.4}$$

This shows that the classical critical speed is unaffected by the inclusion of torsional stiffness. That is to say, v_0 still becomes very large at $\Omega_0 = \sqrt{k/m}$.

It was mentioned earlier that for symmetric shafts the secondary critical speed due to gravity excitation is unimportant since, without asymmetry, there can be no resonance phenomenon at an operating speed near half the critical. Consider now how these conditions are affected by the coupling of bending and torsion. The equations of motion of the centrally located disk on a horizontal shaft (as expressed in the stationary frame this time) are given by

$$m\ddot{y} + ky = ka\cos(\Omega_0 t + \phi) - mg,$$

$$m\ddot{z} + kz = ka\sin(\Omega_0 t + \phi),$$

and

$$I_p\ddot{\phi} + k_\phi\phi + ka(y\sin\phi - z\cos\phi) = 0. \tag{6.5}$$

If there is no imbalance ($a = 0$), the particular solution corresponding to the gravity loading takes the form

$$y_0 = \frac{-mg}{k},$$

$$z_0 = 0,$$

and

$$\phi_0 = 0. \tag{6.6}$$

Once again, if there is no imbalance, the result does not change from the pure bending situation. In any real system, however, there is always some imbalance. If a is not zero, it is necessary to consider its effects simultaneously with those of mg, since a contributes to the homogeneous part of the differential equations.

Taking the gravity loading and imbalance into account simultaneously, the equations of motion in the rotating frame are the same as given by Eqs. (6.3) except for the addition of the terms $mg \cos(\phi + \Omega_0 t)$ and $-\,mg \sin(\phi + \Omega_0 t)$ on the right-hand sides of the first and second equations in the set, respectively.

It is convenient to introduce the complex notation $u = v + jw$, $\bar{u} = v - jw$. Then the equations of motion can be written as

$$\ddot{u} + \left(\frac{k}{m} - \Omega_0^2\right)u + 2\Omega_0 j\dot{u} + 2\Omega_0 a\dot{\phi} + ja\ddot{\phi} =$$

$$\Omega_0^2 a + ge^{-j(\Omega_0 t + \phi)},$$

and

$$I_p\ddot{\phi} - \frac{ka}{2j}(u - \bar{u}) + k_\phi \phi = 0. \tag{6.7}$$

The equations are further simplified by multiplying the first equation in Eqs. (6.7) by $e^{j\phi}$ and dropping second order terms. Then,

$$\ddot{u} + \left(\frac{k}{m} - \Omega_0^2\right)u + 2\Omega_0 j\ddot{u} + 2\Omega_0 a\dot{\phi} + ja\ddot{\phi} - j\Omega_0^2 a\phi =$$

$$\Omega_0^2 a + ge^{-j\Omega_0 t},$$

and

$$I_p\ddot{\phi} - \frac{ka}{2j}(u - \bar{u}) + k_\phi \phi = 0. \tag{6.8}$$

To obtain the steady response due to gravity, it is logical to assume a solution of the form

$$u = Ae^{j\Omega_0 t} + Be^{-j\Omega_0 t},$$

and

$$\phi = jCe^{j\Omega_0 t} + jDe^{-j\Omega_0 t}, \tag{6.9}$$

where A, B, C, and D are real numbers. In order for this to be a solution, it is necessary that

$$\left(\frac{k}{m} - 4\Omega_0^2\right)A = 0,$$

$$\frac{k}{m}B + 4\Omega_0 aD = g,$$

$$\frac{ka}{2}(B-A) - (k_\phi - I_p\Omega_0^2)C = 0,$$

and

$$\frac{ka}{2}(A-B) - (k_\phi - I_p\Omega_0^2)D = 0. \qquad (6.10)$$

As long as Ω_0^2 is not equal to $k/(4m)$, then the quantities A, B, C, and D are uniquely determined.

It results then that

$$u = v + jw = \frac{mg}{k}\,\frac{\left(\frac{k_\phi}{I_p} - \Omega_0^2\right)e^{-j\Omega_0 t}}{\left[\frac{k_\phi}{I_p} - \Omega_0^2\left(1 - \frac{2ma^2}{I_p}\right)\right]}$$

and

$$\phi = \frac{-mga}{I_p}\,\frac{\sin \Omega_0 t}{\left[\frac{k_\phi}{I_p} - \Omega_0^2\left(1 - \frac{2ma^2}{I_p}\right)\right]}\,. \qquad (6.11)$$

The second equation shows that the torsional critical speed is given by $\Omega_0^2 = k_\phi/(I_p - 2ma^2)$. The first shows that the force of gravity can excite a large bending deflection when the operating speed is close to the torsional natural frequency. However, this resonance is very sharp and local due to the similar behavior of the numerator and denominator (except in a very narrow range).

The above solution breaks down if $(4\Omega_0^2 - k/m) = 0$, that is, where $\Omega_0 = \frac{1}{2}\sqrt{k/m}$, since then nothing can be said about the magnitude of A. One circumstance under which the assumed form of the solution would not be valid where $\Omega_0 = \frac{1}{2}\sqrt{k/m}$ is if there is a natural frequency equal to $\frac{1}{2}\sqrt{k/m}$. It is this possibility which leads to the secondary critical speed due to gravity as described by Timoshenko [45]. This is discussed further in the next section.

6.3 Free Coupled Vibrations: Natural Frequencies

The coupled equations of motion without the forcing terms due to imbalance and gravity may be written as

$$\ddot{u} + 2\Omega_0 j\dot{u} + \left(\frac{k}{m} - \Omega_0^2\right)u + a[2\Omega_0\dot{\phi} + j(\ddot{\phi} - \Omega_0^2\phi)] = 0$$

and

$$\frac{-ka}{2j}(u - \bar{u}) + I_p\ddot{\phi} + k_\phi\phi = 0.$$

One can solve for the quantities u, $\bar{u}$, and ϕ with just those two equations, since the complex conjugate of the first equation also holds. That is,

$$\ddot{\bar{u}} - 2\Omega_0 j\dot{\bar{u}} + \left(\frac{k}{m} - \Omega_0^2\right)\bar{u} + a[2\Omega_0\dot{\phi} - j(\ddot{\phi} - \Omega_0^2\phi)] = 0.$$

Solutions to this set are given by the exponential form

$$u = u_0 e^{j\lambda t},$$

$$\bar{u} = \bar{u}_0 e^{j\lambda t},$$

and

$$\phi = \phi_0 e^{j\lambda t},$$

which leads to the characteristic determinant

$$\begin{vmatrix} -\lambda^2 - 2\Omega_0\lambda + \left(\frac{k}{m} - \Omega_0^2\right) & 0 & ja[2\Omega_0\lambda - \lambda^2 - \Omega_0^2] \\ 0 & -\lambda^2 + 2\Omega_0\lambda + \left(\frac{k}{m} - \Omega_0^2\right) & ja[2\Omega_0\lambda + \lambda^2 + \Omega_0^2] \\ \frac{-ka}{2j} & \frac{ka}{2j} & -I_p\lambda^2 + k\phi \end{vmatrix} = 0.$$

Solving for the eigenvalues λ yields all the natural frequencies as a function of the operating speed Ω_0.

Of particular interest is the case where the operating speed is given by

$$\Omega_0 = \frac{1}{2}\sqrt{\frac{k}{m}}\,.$$

It is possible to show that, under these conditions, two of the six characteristic values of λ are given by $\lambda = \pm\Omega_0$, so that vibrations occur with the form

$$u = u_0 e^{\pm j\Omega_0 t} = u_0 e^{\pm j\,t/2\,\sqrt{k/m}}.$$

6.4 Effect of Torque Loading

In a practical shaft system, it is usually very difficult to describe the applied torque or the instantaneous angular operating speed as an explicit function of time. It is normally assumed that the operating speed is constant. In the foregoing analysis it was assumed that the torque was zero. Attempts to improve on these approximations variously assume that the applied torque is proportional to the instantaneous operating speed or that speed, acceleration, and torque are all coupled by the characteristics of the motor.

The effect of a purely pulsating torque is interesting because its presence can be induced by electrical components. To demonstrate its effect on the simple system used as an example in the preceding sections, it is only necessary to look at the particular solution of the set of Eq. (6.3) corresponding to an oscillatory torque loading in the moment equation. The imbalance again serves to couple the system, so that the oscillatory torque excites lateral vibrations.

6.5 Instabilities

Tondl [9] has produced analytical results which show that instabilities can be introduced by the mutual interaction of torsional and flexural vibrations. The particular system examined was a two-mass system composed of a turbine and a generator acted on by a driving moment, viscous drag moments, and a synchronizing torque. Gyroscopic terms were neglected, so that the complete set of governing equations are similar in form to Eqs. (6.3). Naturally, there are equations for each mass, and these are elastically coupled. After performing extensive algebraic manipulations in which series solutions were found in the form of expansions in powers of the imbalance, Tondl came to the conclusion that intervals of instability can be found in the range of operating speeds given by

$$\Omega = |\omega_{n_T} \pm \omega_{n_b}|,\ 2\omega_{n_T},$$

where

ω_{n_T} are the natural frequencies of torsional vibrations
ω_{n_b} are the natural frequencies of flexural vibrations.

If, as in many practical systems, the torsional natural frequencies are very much higher than the bending frequencies, and if operation is limited to the lower critical speeds, then certainly one does not have to worry about the above unstable ranges, since

$$\Omega \simeq \omega_{n_b} \text{ and } \omega_{n_T} >> \omega_{n_b}.$$

On the other hand, designers (of power transmission shafts, for example) strive for higher speeds to reduce the torque required for a fixed amount of power. This, coupled with weight and size limitations, leads to operating speeds in excess of first, second, or even fifth or sixth critical, so that the importance of the mutual interaction of bending and torsional vibration is substantially increasing.

Regimes in which the flexural and torsional natural frequencies are of the same order become especially crucial, since regions of instability may then merge and overlap.

Unlike the instabilities discussed in preceding parts of the monograph, such as those due to asymmetry or internal damping, those due to coupled bending and torsion are affected by the degree and location of imbalance. In fact, as has been pointed out earlier in this chapter, if the equipment is perfeclty balanced there may be no coupling between bending and torsion at all, as long as the assumptions of a linear system are made.

It follows, then, that the widths of the unstable regions decrease continuously to zero as the imbalance is reduced. This fact alone offers good reason for careful balancing when a system must operate in the regime of the torsional natural frequencies.

As in other types of shaft instability, a sufficient amount of external damping also serves to narrow, or eliminate altogether, the unstable regimes associated with the combined flexural and torsional stiffness.

One final word with regard to coupled bending-torsion instabilities. A centrifugal pendulum, mounted on a rotating part and tuned to some particularly troublesome exciting frequency, has been used as a dynamic absorber to reduce vibrations in rotating systems. The motions of such a seismic mass, in a plane normal to the shaft center line, are analogous to torsional motions. If such devices are disposed without polar symmetry about the shaft axis, their response will couple lateral bending and torsion motions. Most important, however, they can give rise to a separate class of linear instabilities which can exist in the absence of imbalance, damping, and asymmetry. The corresponding theory has been well developed (for helicopter rotors) by Coleman [37] and shows that instability is precluded if the natural frequency of the centrifugal pendulum, as measured in the rotating system, is higher than rotational speed.

CHAPTER 7
BALANCING

Operating speed is always an important parameter in deciding the degree to which a rotating system must be balanced. Historically, the procedures for balancing evolved as the need arose for higher and higher speed machines. For high speed machines, of course, even a small geometric imbalance may cause a large oscillatory force to be transmitted to the supporting structure.

The most direct unfavorable effect of imbalance is the initiation of oscillatory loads on supports. If the imbalance is relatively small, the consequences may still be important from other viewpoints; for example, noise transmission. If the imbalance is large enough there is the possibility of fatigue failures in the rotating shaft or premature bearing wear and seizure. Generally speaking, the more accurately balanced the shaft system, the smaller will be the vibrations and dynamic reactions in the supports; hence, the longer the component life of the system.

For machines which operate sufficiently below the first critical speed, shafts may usually be considered rigid. The advent of high speed systems, especially those operating above the first critical, has made it necessary to consider the flexibility of the shafts in the balancing process. For many applications, especially modern gyroscopic instruments and devices, it has become necessary to take into account the gyroscopic effect of the rotating parts.

7.1 Three Classes of Imbalance: Static, Dynamic, and Flexible Shafts

Static imbalance is the term used to describe the offset of the center of gravity of a rotor from the axis of rotation, where the principal axis of inertia of the rotating body is still oriented in the same direction as the axis of rotation. Such would be the case, for example, if a perfectly massless shaft were carrying a disk, mounted perpendicular to the shaft but offset a certain amount (Fig. 18).

Dynamic imbalance is the term used to describe the angular misalignment of the principal axis of inertia of the rotating body with respect to the axis of rotation. An example of purely dynamic imbalance is shown in Fig. 19, where a disk

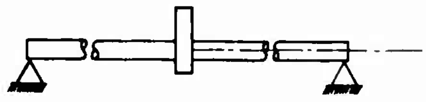

Fig. 18. Static imbalance.

is mounted on a massless shaft so that the plane of the disk is not perpendicular to the shaft axis, but the center of gravity of the disk lies on the shaft axis. The action of centrifugal force is such that a force perpendicular to the rotating shaft results from static imbalance and a moment perpendicular to the rotating shaft from dynamic imbalance. It is clear that both effects cause forces to be applied to the bearings.

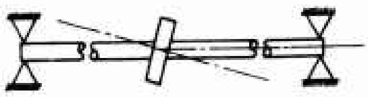

Fig. 19. Dynamic imbalance.

In the general case, both static and dynamic imbalance are always present to some extent due to manufacturing tolerances. Given a rigid shaft which is both statically and dynamically out of balance it is always possible, in principle, to balance the system for all speeds by placing weights in each of two arbitrarily chosen cross-sectional planes. When a rotating shaft is operating at speeds in the vicinity of first critical and higher, it is usually no longer possible to neglect shaft flexibility, and perfect balance then requires an infinite number of shaft stations in the most general case. The following paragraphs attempt to review the procedures used in practice.

7.2 Balancing Methods and Theory

Static Balancing

Pure static imbalance can be detected without rotation of the rotor. If, for example, a statically unbalanced shaft (with concentrically round surfaces) is set across two perfectly level, flat, and parallel blades, the shaft will roll until the center of mass is at the lowest possible point. Mass is added to the shaft 180° away from the imbalance until the shaft will not roll on the blades, regardless of the position from which it is started. The accuracy of this static balancing method is limited by the rolling friction between the blades and the rotor. For small rotors it is sometimes convenient to mount the shafts in holders that have their own ball bearings. The accuracy is then limited by the friction in the ball bearings rather than friction between rotor and rails. In either case, friction effects may be reduced by introducing low-level vibrations to the supporting base.

The theory of pure static balancing is, of course, very simple. Given a rigid rotor, the center of gravity of which is not on the axis of rotation, it is only necessary to add one mass in order to balance the rotor. The unbalanced shaft has the following values for the position of the center of gravity (see Fig. 20):

$$\bar{x} = 0,$$

$$\bar{y} = \epsilon$$

$$\bar{z} = 0.$$

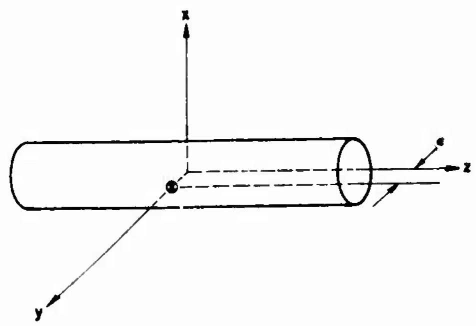

Fig. 20. Static balancing model.

The center of gravity, with an additional mass m added on the surface of the shaft 180° away, is given by

$$\bar{x} = \bar{z} = 0$$

and

$$\bar{y} = \frac{(M\epsilon - mr)}{(m + M)},$$

where M is the original mass. If $mr = M\epsilon$, the shaft will be statically balanced.

Dynamic Balancing

Dynamic balancing is commonly carried out by mounting the rotor on the mobile platform of a balancing machine. When the rotor is rotated, the platform vibrates due to the imbalance. The location and magnitude of the required balancing weights is determined by measuring the amplitude and phase of the platform vibration.

The basic components of a dynamic balancing machine are

- A mechanical platform assembly which permits the necessary degrees of freedom of the rotor,
- A drive system which imparts a definite speed of rotation to the rotor,
- Measuring devices to accurately detect the motion of the platform, and
- An accurate device for adding or removing material at a definite place on freedom of the rotor,

The mechanical resonance of the platform assembly can be used to advantage in amplifying the motion caused by the imbalance. Many measuring devices for detecting the amplitude and phase of the mobile platform are optical.

Three different variations of dynamic balancing machines of the resonance type which were widely used in the 1930's were discussed by Timoshenko [45]. The advantages are simplicity, reliability of operation, and relatively modest cost.

Some disadvantages are low productivity, difficulty in balancing the rotor in its own bearings or housing, and difficulty in automating the balancing process.

More highly productive or automatic machines do not depend on resonance for amplification but, rather, on highly sensitive electrical methods of measurement. In fact, some machines have fixed rather than mobile supports, where measurements are based on the dynamic force reactions.

One way to express dynamic balance theoretically is to state that, for a rigid rotor (rotating about, say, the z axis), the cross products of inertia must equal zero. That is,

$$I_{zx} = I_{yz} = 0. \tag{7.1}$$

If this is true and the x and y coordinates of the center of mass are also zero, that is

$$\bar{x} = \bar{y} = 0,$$

then the rotor is both statically and dynamically balanced.

It should be emphasized that by balancing rotors on machines intended for dynamic balancing, static and dynamic imbalance are removed simultaneously.

The mass products of inertia, I_{zx} and I_{zy}, are given by

$$I_{zx} = \int_V zxdm$$

and

$$I_{zy} = \int_V yzdm, \tag{7.2}$$

where dm is an elemental mass ($dm = \rho d\tau$) of the body, and the integration is carried out over the entire volume of the body.

If the rotating body is made of a perfectly homogeneous material, then ρ is constant over the volume τ. The quantities I_{zx} and I_{zy} then equal zero, provided that the rotor is symmetrical with respect to a plane normal to its axis of rotation.

If the rotor in question is a perfectly straight circular cylinder, such a plane clearly exists, and there is no dynamic imbalance. If the shaft center line were slightly bent into an arc symmetrical about the normal to the shaft midstation, the shaft would be statically out of balance but not dynamically, as can be seen from the symmetry argument. On the other hand, if it were bent so that there was no symmetry about the orthogonal midstation plane there would be dynamic imbalance as well. Any inhomogeneity would also cause dynamic imbalance (even if the shaft were perfectly straight and circular) unless the density distribution were again symmetrical with respect to the dividing plane.

If a rigid rotor is dynamically out of balance, it is possible to balance it by adding two correcting weights in two arbitrarily positioned planes, called correction planes, normal to the axis of rotation. The two correction planes are located at $z = z_1$ and $z = z_2$ (Fig. 21). The correction masses are denoted as m_1 and m_2, and have coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively. If the original unbalanced shaft had mass M and cross products of inertia I_{zx} and I_{zy}, then the new cross products of inertia, with the correction mass added, are

$$I_{xz}^{new} = I_{zx} + m_1x_1z_1 + m_2x_2z_2$$

and

$$I_{zy}^{new} = I_{yz} + m_1y_1z_1 + m_2y_2z_2 . \qquad (7.3)$$

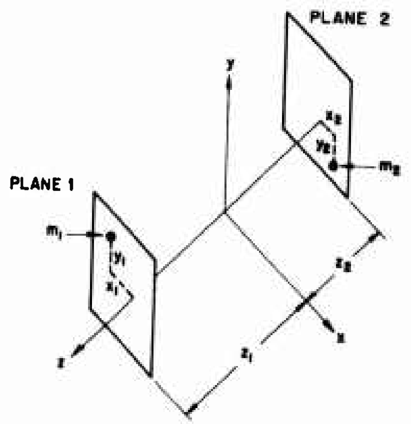

Fig. 21. Correction planes.

The new center of mass is defined by

$$\bar{x}_c^{new} = \frac{M\bar{x}_c + m_1x_1 + m_2x_2}{M + m_1 + m_2}$$

and

$$\bar{y}_c^{new} = \frac{M\bar{y}_c + m_1y_1 + m_2y_2}{M + m_1 + m_2} . \qquad (7.4)$$

To have the rotor both statically and dynamically balanced, it is necessary that

$$m_1x_1 + m_2x_x = -M\bar{x}_c,$$

$$m_1y_1 + m_2y_2 = -M\bar{y}_c, \qquad (7.5)$$

$$z_1(m_1x_1) + z_2(m_2x_2) = -I_{xz},$$

and

$$z_1(m_1y_1) + z_2(m_2y_2) = -I_{yz}.$$

Equation (7.5) constitutes four independent equations for the four unknowns (m_1x_1), (m_2x_2), (m_1y_1), and (m_2y_2). Since only the products of mass and coordinate are important, it is possible to use standard correction masses, with the coordinates determined by the equations. Note, too, that the masses may be negative quantities–obtainable, for example, by removing material from the disks attached to the shaft.

For the removal of an imbalance caused by manufacturing inaccuracies, the above analytical formulas are not directly useful because it is difficult to determine x_c, y_c, I_{xz}, and I_{yz} experimentally (without already having balanced the shaft). The formulas only prove that it is possible to balance the shaft by using two masses.

On the other hand, a rotating shaft may be required to carry cams or gears, which can cause a gross imbalance that is independent of manufacturing accuracy. In that case, the original center of mass and the products of inertia may be calculated in a straightforward way so that all the information is available to balance the system to within normal tolerances.

Flexible Shaft Balancing

When operational requirements demand that shafts be operated at speeds in excess of the lowest natural frequency, it is no longer adequate to regard the shaft as rigid for balancing purposes. The true position of the mass center of a flexible shaft, as well as its instantaneous cross products of inertia, will generally be different for each operating speed. It is thus impossible to be perfectly balanced for all speeds. This largely accounts for the fact that rotors balanced on a balancing machine are often no longer balanced when they are operating in their own bearings and at a different speed than that of the test machine. In fact, by placing masses in two different planes, a flexible rotor can at best be balanced for one speed.

The most widely discussed procedure for balancing flexible rotors is based on the idea of balancing rotors according to their particular vibration modes. In the

United States, Den Hartog [46]; in the Soviet Union, Dimentberg [5] and Kushul' [47]; and in Britain, Bishop [48] and Parkinson [49] have all been strong proponents of this idea, and it has received wide acceptance.

The theory supporting this modal balancing technique is usually developed on the assumption that the rotary inertia effects of shaft cross sections, as well as of attached disks, may be neglected. The immediate consequence of this assumption is that the natural vibration frequencies and mode shapes of the rotating shaft are the same as those of the stopped shaft. This assumption need not be made, however, in view of the power of modern computing methods, and in principle, accounting for rotary inertia effects is straightforward. The change of mode shapes with operating speed, however, probably means that "balance" would have to be sought with some preconception of weighting one operating speed vs others.

The modal balancing theory was first developed by completely symmetrical systems. Bishop and Parkinson have since extended the purely modal analysis to balance shafts which are either axially symmetric operating in asymmetric bearings or nonaxially symmetric operating in symmetrical bearings. However, it is evidently still relatively unknown what effect modal balancing has if gyroscopic mass effects do occur.

The fundamental aspect of the modal balancing technique is the use of natural vibration modes as generalized coordinates. The orthogonality of the modes makes it possible to obtain completely uncoupled equations for the response in each mode due to the imbalance. Correction weights are added in sequence to successively balance out the reactions corresponding to the lower modes; if such weights may be considered small, first order quantities, the modes may be assumed to be not fully coupled by the additional mass.

It has recently been pointed out by Den Hartog [46] and Kushul' [47] that a pure modal balancing technique may not always guarantee quiet operation at the required range of operating speeds, even if gyroscopic moments are completely negligible. Concentrated imbalances are often associated with this difficulty. At its source is the fact that the generalized unbalanced force, which is weighted by each particular modal deflection shape, appears in each modal equation as the (only) driving function in normal mode analysis. If the imbalance distribution has sharp variations or if it is shaped—by coincidence—very much like the deflection shape of one of the higher modes, the generalized forces which excite the higher modes may be large enough so that, unless those modes are also included in the balancing procedure, the dynamic bearing reactions may still be large. In other words, a modal balancing technique using only the lower modes is satisfactory only if the harmonic content of the imbalance distribution is not too great.

Some approaches taken to eliminate difficulties associated with balancing by harmonics are known as combined methods of balancing. One of these is based on the following theorem due to Den Hartog [46]. A massless flexible rotor carrying (or representable by) r concentrated masses, supported on b bearings, with an imbalance, regardless of its distribution, can be completely balanced by weights distributed in $n = r + b$ different planes along the rotor length. *Complete balancing* is defined here as that which results in no dynamic reaction in any bearing at any operating speed. This theorem holds provided that (a) the unbalanced masses are small compared to the total mass of the rotor, and (b) that flexure due to the imbalance is small compared to the eccentricities of the static imbalance.

Another combined approach to the balancing of flexible shafts has been developed by Kushul' [47]. In at least some cases, the approach of Kushul' may give better results than that of Den Hartog. In general, it may be stated that the combined methods of both Den Hartog and Kushul' can be much more effective at reducing the reaction forces at supports than the older methods. This is especially true if there is a practical limitation on the number of correction weights which may be used. The theoretical basis of both methods is presented in Ref. 47. That reference also gives some simple numerical examples that illustrate the differences in methodology and performance of the purely harmonic and combined approaches.

One point deserves emphasis. Even the most exact balancing procedure does not eliminate the fact that a rotor has flexibility, mass, natural frequencies, and possibly instabilities. All balancing is directed at minimizing the forced response due to geometric imperfections or material inhomogeneities. No matter how well a shaft is balanced, it can still experience vibration and transmit oscillatory forces at the supports if the natural vibration modes are excited by a source other than mass imbalance.

The consensus regarding the state of the art and future of flexible rotor balancing seems to be that the combined methods of balancing have given the best results and that continued development of the theory along those lines is justified. In particular, a theory for balancing flexible rotors which have significant gyroscopic moments in the system remains poorly developed.

7.3 New Concepts in Automatic and Self-Balancing

The best balancing procedures for flexible shafts generally are aimed only at operation in a small range of speeds. Furthermore, as the name balancing implies, these approaches are directed at reducing steady deflections in the rotating system arising from intrinsic mass imbalance. Since correction weights are always fixed to the shaft, they hardly can be expected to reduce other than once-per-revolution disturbances as measured in a stationary coordinate system, and certainly not random excitations.

One of the most promising areas of research for balancing is the use of correction weights which are not fixed to the shaft but, rather, are free to rotate with respect to the shaft. Self-balancing devices which are based on this principle have been used in washing machines, food blenders, automobile wheels, and more recently helicopter rotor shafts [50]. In all cases, at least two freely swinging weights or spheres which are free to ride in an annular track are allowed to rotate with the shaft and to take equilibrium azimuthal positions. For operation below critical speed, the added weights aggravate the imbalance but, for supercritical operation, the free weights tend to reduce the shaft deflections to zero.

The principle upon which these self-balancing devices are based is a well-known phenomenon which is evidenced by the simple case of an unbalanced disk attached to the center of an elastic shaft, discussed in Chapter 4. A free mass will always seek the maximum radius. Consequently, below critical speed the free masses will move to a radius which is greater than that of either the shaft center line or the center of mass. Above critical speed, however, the free mass will move to a position diametrically opposite the center of mass, i.e., with respect to the shaft center line. Therefore, above critical speed the combined center of mass will move toward the shaft center line and hence reduce the net imbalance. These conditions are shown in Fig. 22.

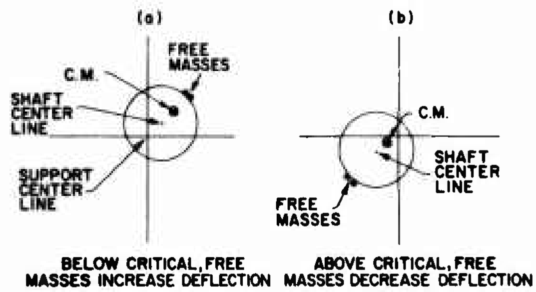

Fig. 22. Self-balancing.

Other configurations are being investigated for helicopter applications [50], wherein the track that the rotating masses ride in is itself geared to rotate at a different speed than the shaft, and even in the opposite direction. In this way it may be possible to reduce vibrations at other harmonics.

It is interesting to note that the idea of such a self-balancing device was conceived as early as 1900 by Leblanc (see Ref. 46).

The success that has been achieved with such self-balancing devices suggests that more research should be undertaken in this area. While progress is being made in developing working models for helicopter applications, there seems to be little corresponding analytical work under way to further the understanding of those systems. Analyses of the passage through critical and response to disturbances are particularly difficult because of their nonlinear nature, but they should nevertheless be pursued, since the real advantage of such devices may lie therein.

CHAPTER 8
CONCLUSIONS

The literature dealing with the subject of rotating shaft dynamics is large, growing, and diverse. This diversity reflects the variety of machines in which rotation is a fundamental mode of operation.

A literature review suggests three trends. For one, methodology is unquestionably becoming more sophisticated. While a natural trend in any field, a driving force here has been the ever-increasing speeds of rotation. Second, the broader, more general treatments of the subject continue to deal with simpler systems and more theoretical analyses. Third, the more complete treatments usually apply to specific configurations; hence they are relatively narrow in application and rely frequently on large-scale digital computers and numerical methods of analysis. This last is hardly unexpected—particularly if one contrasts the differences between, say, high speed gear trains and large, overhung mixers. It is unfortunate, however, and a trend with which researchers in this field should not be content, since understanding is always enhanced by the ability to generalize.

A general assertion might be cautiously attempted regarding the collective characteristics of the many authors who have contributed to the field. Authors outside the United States (who contributed about half our literature) attacked the same problems as those in this country. In doing so, however, they generally seemed to take a more traditional, theoretical approach toward solution, one involving much labor. Authors in the United States, in general, seem to have taken, most often more straightforward, brute-force approaches. This may simply reflect what has been until now the large difference in available digital computing power in the United States as compared to other countries.

Except for people of high stature and experience (like Dimentberg, Tondl, and many others), the literature abounds with indications of confusion among such differing phenomena as natural frequencies, critical speeds, instabilities, and gyroscopic and rotary inertia effects. It is hoped this monograph will help eliminate some of the confusion.

Some conclusions can be stated regarding critical speeds based on linear theory. First, critical speeds may legitimately be classed as resonances since they involve amplitudes which are predicted to increase linearly with time to infinite amplitudes in the absence of damping, at discrete operating speeds; this despite the fact that stresses and motions as sensed in the rotating system may be non-oscillatory. Constant (compressive) axial force and torque generally lower critical speeds. Gyroscopic effects increase or decrease critical speeds, depending on operating speed, size, disk geometry, and disk location. The natural frequencies of rotating shafts are speed-dependent because of rotary inertia; this is also one effect that differentiates critical speeds from nonrotating natural frequencies. Transverse shear is reported to affect critical speed when the diameter is as small as one-hundredth of the length; the influence increases with the ratio of diameter to length. It is noted that this presumes a certain ratio of material shear to bending modulus. If more unusual materials find use in rotating machinery, this threshold may be quite different. The presence of asymmetry, in either the rotating or nonrotating frame, generally doubles the total number of critical speeds compared to the fully symmetric case. Asymmetry in both frames generally quadruples the number of critical speeds. Additional critical speeds exist when degrees of freedom are added, such as degrees of freedom of the supporting structure.

There is a dearth of experimental verification concerning the existence of reverse precession, which is one of the kinds of critical speed theoretically predicted. The embryonic state of nonlinear vibration theory, in general, hinders understanding such phenomena as they occur in rotating shafts. Among the sources of nonlinearities in practical systems are oil films in journal bearings, structural restoring forces, bearing clearances, magnetic forces between rotor and stator, faulty mountings, shrink fits, and deformations of ball-bearing races. In general, nonlinearities introduce additional resonances, such as subharmonics of resonances which degenerate to critical speeds in the linear case.

There are six standard methods for predicting critical speeds. The Rayleigh method is only good for the first natural frequency (rotational dynamics are usually not included). The Ritz method is an improvement in that higher modes are determined. The transfer-matrix approach yields all (linear theory) critical speeds to any desired degree of accuracy and is especially suitable for machine computations. The force and displacement or other methods leading to a matrix of influence coefficients and a dynamic matrix are also good for linear systems. They do require the manipulation of high order matrices, are somewhat more complex in the formulation, and require a larger computer capacity. They will, however, generally require less machine time. The choice between the dynamic matrix and the transfer-matrix formulas are generally only good for first approximations. Impedance-matching methods have been successful.

An instability, as distinct from a resonance, occurs when deflections grow exponentially with time regardless of applied forces; instabilities will usually exist over a range of operating speeds. A number of physical characteristics are important to the stabilities of a rotating system, e.g., internal friction, asymmetry of the rotating parts, bearing lubricants, and nonlinearities. The onset of an instability is dependent on the ratio of internal damping to external damping, rather than the magnitude of each. The assumptions of simple, linear internal friction, which are usually made, will not give a true picture of either the instability range or the final amplitude. The amount of instability due to internal friction is not generally independent of the imbalance or the orientation of the shaft, since internal friction usually is nonlinear with amplitude. Probably the most frequent source of instability is the action of a journal in its bearings.

There are four established methods for mathematically predicting the existence of instabilities: the Routh-Hurwitz criterion, direct numerical time integration, the generalized Hill's method, and asymptotic methods. Each has advantages and disadvantages. The problem is often complicated mathematically by the existence of periodic coefficients in the equations of motion. The Routh-Hurwitz criterion does not provide any knowledge of the behavior of the motion, the deflection shape and its growth rate if the system is unstable, and not much of an idea of the margin from instability if the system is stable. Conversely, direct numerical time integration gives little insight into the physics of the problem. The generalized Hill's method faces periodic coefficients directly but is limited by the need to deal with infinite determinants, which obviously must be approximated in practice. Asymptotic methods sometimes suffer from the need to identify a crucial parameter which may be considered small with respect to others; this may or may not be reasonable in particular cases.

Experimentally observable frequencies of system forced response will depend on the observer's frame of reference (i.e., fixed or rotating frames). The response may well be larger for vibrations at a frequency other than the forcing frequency. For example, with gravity as the forcing function (zero forcing frequency in the fixed frame) the response can be greater at a twice-per-revolution frequency than at the zero forcing frequency (static deflection).

There have been few analytical studies of the transition-through-critical problem. Those have been restricted to the simplest configurations and have usually disregarded torque–angular velocity relations. An accurate picture requires consideration of the latter; constant acceleration assumptions are valid only for adequately small ranges of investigation. Damping has the effect of reducing the maximum amplitudes that arise during transition.

The mutual interaction of lateral and torsional motion becomes important for supercritical speeds, especially when small-diameter shafts are involved, and for investigating the influence of torque loading. The literature is devoid of references treating the linear instabilities of a shaft undergoing purely torsional vibrations. For the case of coupled bending and torsion, one can obtain unstable torsional motion due to basically bending instabilities. Coupled theory predicts additional instabilities that are not predicted by uncoupled theory.

Generally speaking, the more accurately balanced the shaft system, the smaller will be the vibrations and support reaction. For high speed machines (working above first critical), shaft flexibility must be considered in the balancing process. For even higher speeds, it is also necessary to account for gyroscopic effects. The classical idea of static and dynamic balancing is tied to the concept of rigid rotating parts. Both are achieved by machines designed for dynamic balancing: the converse is not true. All balancing is directed at minimizing the forced response due to geometric imperfections or material inhomogeneities. However, no matter how well balanced, a shaft can still experience vibrations and transmit oscillatory forces to the supports if natural vibration modes are excited by any source. So-called combined methods of balancing seem to give the best results. Continued development of the theory is justified; in particular, the theory for balancing flexible rotors that have significant gyroscopic moments remains an area worthy of increased attention. It is also necessary to improve methods for measuring the responses due to imbalance so that full advantage can be taken of the advances in the theory which are certain to come. One of the most promising areas of balancing research is in the use of correction weights which are free to rotate with respect to the shaft. Methods for predicting response while passing through critical and in the presence of transient disturbances are important to realizing the full potential of such devices.

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