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Optimal Measurement Strategies
for Linear Stochastic Systems

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Lexington, Massachusetts



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MASSACHUSETTS INSTITUTE OF TECHNOLOGY
LINCOLN LABORATORY

OPTIMAL MEASUREMENT STRATEGIES
FOR LINEAR STOCHASTIC SYSTEMS

M. ATHANS

Consultant, Group 32

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ABSTRACT

This note presents the formulation of a class of optimization problems dealing with selecting, at each instant of time, one measurement provided by one out of many sensors. Each measurement has an associated measurement cost. The basic problem is then to select an optimal measurement policy, during a specified observation time interval, so that a weighted combination of "prediction accuracy" and accumulated "observation cost" is minimized. The current analysis is limited to the class of linear stochastic dynamic systems and measurement subsystems. The problem of selecting the optimal measurement strategy can be transformed into a deterministic optimal control problem. An iterative digital computer algorithm is suggested for obtaining numerical results. It is shown that the optimal measurement policy and the associated "matched" Kalman-type filter can be precomputed, i. e. specified before the measurements actually occur.

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1. INTRODUCTION

There are many engineering situations in which a variety of possible measurements can be carried out on a physical system or process. However, there are also physical constraints that impose the requirement that, at each instant of time, one is able to use only one out of a possible total of M sensors. In such cases, one has to make a decision: Which measurement to make at present, and when to make an alternate measurement. It may also happen that one can associate with each type of measurement a per unit-of-time "measurement cost" reflecting the fact that some measurements are more costly or difficult to make than others, although they may contain more useful or reliable information.

This type of problem arises in the following types of systems:

Telemetry-Data Aerospace Systems. Consider a space vehicle and suppose that it contains M sensors each of which measure a different signal (i. e. yaw, roll, pitch, or perhaps their rates). Suppose that one has a single bandlimited telemetry link so that at each instant of time one can communicate to the ground the data signal from only one of the sensors. Furthermore, suppose that the ground-control can command which sensor output is to be communicated. How should the ground command the telemetry system to sequence the data? What are the criteria that the ground-control can use in order to make such decisions? We should like to stress that this is not the multiplexing problem. Each sensor can provide a group of measurements which can be multiplexed for transmission to the ground. However, we assume that not all measurements can be

transmitted to the ground at once. Hence, we can transmit a group of measurements now, some other group later on, and so on.

Tracking and Discrimination: Consider the observation of a target by a radar. Suppose that the radar has the capability to transmit over a specified interval at each instant of time one out of M possible waveforms and that each one can be used for making a different type of measurement (e. g. tracking, Doppler, wake measurements, etc.). There may be different radar power requirements for each signal; furthermore, there may be different computational overhead and real-time requirements associated with the data processing of each type of return. If the defense is interested in discrimination and impact-point prediction, how does one **decide which waveform is to be transmitted at each instant of time?**

Socioeconomic Problems: In many socioeconomic problems one can make measurements by assigning a group of people to collect data or carry out polls or look up statistics etc. How does the manager of an information-gathering subdepartment allocate his data-collecting resources at each instant of time so that an accurate forecasting of future trends of the entire company or government agency can be carried out?

The above situations (and many others) represent typical situations in which the allocation of limited measurement resources is important.

They can be abstracted in the following context:

- 1) A stochastic dynamic system (satellite, target vehicle, socioeconomic system) is involved.
- 2) One cannot measure all of the significant variables (state variables) of this system.

- 3) At each instant of time one has the choice of making only one out of many possible measurements
- 4) Each type of measurement may have an associated measurement cost per unit time
- 5) Each measurement is unreliable (noisy).
- 6) Usually a prediction (forecasting) is involved, whose accuracy will depend upon the judicious choice of a measurement policy.
- 7) There exist tradeoffs between total cost of measurements vs prediction accuracy.

The purpose of this note is to formulate this class of problems for the simplest possible class of problems; the assumptions we make are:

- a) the dynamics of the process are linear.
- b) each signal available for measurement is a linear combination of the process state variables.
- c) each measurement is corrupted by white noise.

Somewhat related problems have been studied before, although the availability of results has been relatively scarce. Athans and Schweppe [1], [2] have studied the problem of the timing of a measurement, under constraints on the available measurement energy. Specific applications were given by Schweppe [3] and Schweppe and Gray [4]. Control problems with costly observations and pertaining to the timing of observations have also been studied by Kushner [7], Vandelinde and Lavi [8] and Sano and Terao [10]. The optimal control of systems with observation constraints has been studied by Meier, Peschon, and Dressler [9]. The studies reported in the literature differ from the problem under consideration in this report due to the fact that alternate observation policies dealing with allocation of costly resources have not been explicitly considered.

The structure of this note is as follows. In Section 2 we define precisely the plant under consideration, the assumed statistics, the sensor constraints, and the measurement constraints. An optimization problem is defined in Section 2.7 which involves the selection of the optimal observation policy and the associated prediction algorithm so that a weighted combination of prediction error and measurement cost is minimized. In Section 3 it is shown how such problems can be attacked using the Kalman Bucy [5] filtering and prediction framework. This leads to an alternate formulation of the basic optimization problem in Section 4; the new optimization problem involves the minimization of a deterministic cost functional with matrix differential equation constraints. In Section 5, the matrix minimum principle [6] is used to deduce the necessary conditions for optimality. Section 6 suggests a computational algorithm to solve the two point boundary value problem. Section 7 contains a discussion of the off-line and on-line computational requirements.

2. PROBLEM FORMULATION

In this section we shall summarize the basic definitions, notation, and assumptions which relate to the problem under consideration.

2.1 Plant Dynamics

Consider a linear, possibly time-varying, plant described by the stochastic differential equation

$$\dot{\underline{x}}(t) = \underline{A}(t)\underline{x}(t) + \underline{B}(t)\underline{u}(t) + \underline{\xi}(t) \quad ; \quad \underline{x}(t_0) = \underline{x}_0 \quad (2.1)$$

We assume that:

- The plant state vector $\underline{x}(t)$ is an n-dimensional column vector
- The plant control $\underline{u}(t)$ is an r-dimensional column vector; we assume that $\underline{u}(t)$ is deterministic and known for all $t \geq t_0$.
- $\underline{A}(t)$ is a known deterministic nxn time-varying matrix.
- $\underline{B}(t)$ is a known deterministic nxr time-varying matrix.
- $\dot{\underline{x}}(t) \stackrel{\Delta}{=} \frac{d}{dt} \underline{x}(t)$
- The plant driving noise, $\underline{\xi}(t)$, is an n-dimensional column vector. We assume that $\underline{\xi}(t)$ is a white noise process with zero mean for all t, i. e.

$$E\{\underline{\xi}(t)\} = \underline{0} \quad \text{for all } t \quad (2.2)$$

and known covariance matrix:

$$\text{cov}[\underline{\xi}(t); \underline{\xi}(\tau)] = E\{\underline{\xi}(t)\underline{\xi}'(\tau)\} = \underline{\Xi}(t) \delta(t-\tau) \quad (2.3)$$

The nxn matrix $\underline{\Xi}(t)$ is assumed known for all $t \geq t_0$. It is a symmetric positive semidefinite matrix i. e.

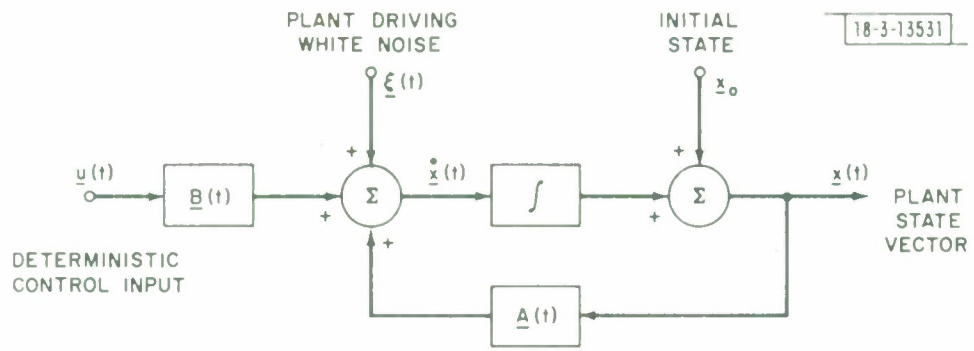


Fig. 1. Block diagram of the linear stochastic plant.

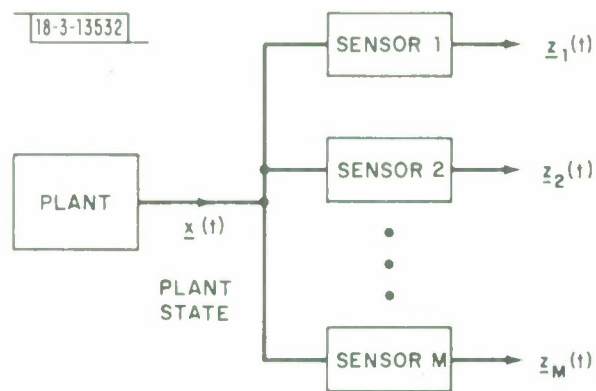


Fig. 2. Visualization of the multiple sensors and of the possible measurements.

$$\underline{\Xi}(t) = \underline{\Xi}'(t) \geq \underline{0} \quad (2.4)$$

The initial plant state \underline{x}_0 is not known. It is modelled as a vector-valued random variable with known mean $\bar{\underline{x}}_0$, i. e.

$$E\{\underline{x}_0\} = \bar{\underline{x}}_0 \quad (2.5)$$

and known covariance matrix $\underline{\Sigma}_0$ (symmetric, positive semidefinite)

$$\text{cov}[\underline{x}_0; \underline{x}_0] = E\{(\underline{x}_0 - \bar{\underline{x}}_0)(\underline{x}_0 - \bar{\underline{x}}_0)'\} = \underline{\Sigma}_0 \quad (2.6)$$

$$\underline{\Sigma}_0 = \underline{\Sigma}_0' \geq \underline{0} \quad (2.7)$$

\underline{x}_0 is independent of $\underline{\xi}(t)$ for all $t \geq t_0$, i. e.

$$\text{cov}[\underline{x}_0; \underline{\xi}(t)] = \underline{0} \text{ for all } t \geq t_0.$$

Figure 1 illustrates a vector block-diagram of the plant under consideration.

2.2 Sensor Constraints

Let us suppose that we have available M sensors which can carry out, not necessarily independent, measurements: This situation is illustrated in Figure 2. We shall let $\underline{z}_j(t)$ denote the measurement vector (set of signals) obtained from the j -th sensor at time t .

We shall assume that the sensor measurement vector $\underline{z}_j(t)$ is an m_j -dimensional vector given by

$$\underline{z}_j(t) = \underline{y}_j(t) + \underline{\theta}_j(t) = \underline{C}_j(t) \underline{x}(t) + \underline{\theta}_j(t), \quad j=1, 2, \dots, M \quad (2.9)$$

where $\underline{y}_j(t)$ is the output signal defined by

$$\underline{y}_j(t) = \underline{C}_j(t) \underline{x}(t) \quad (2.10)$$

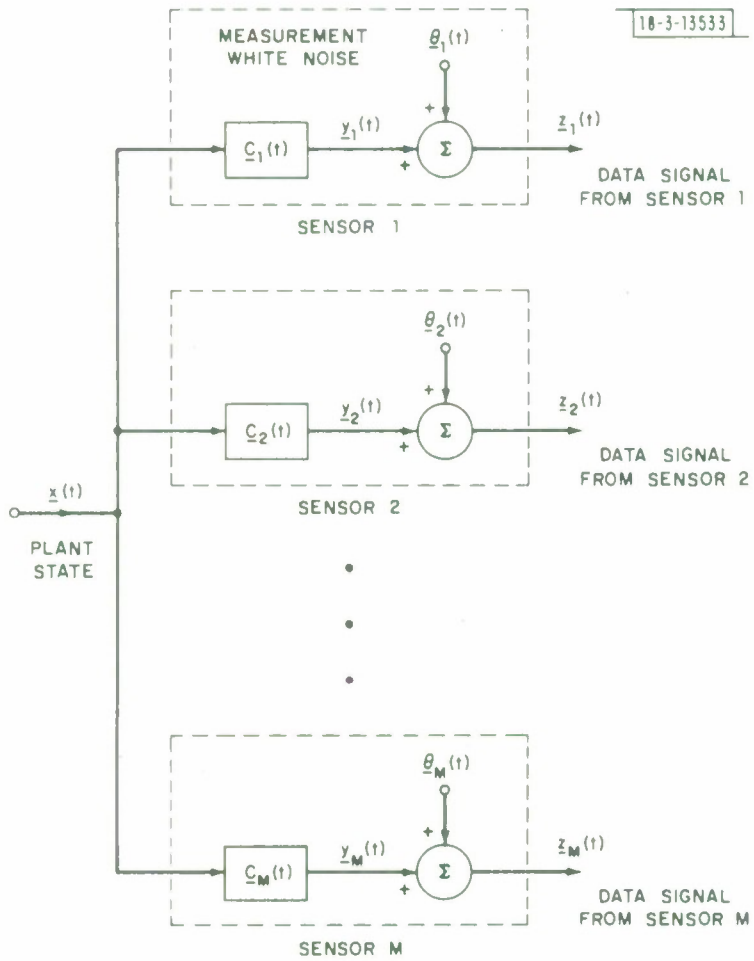


Fig. 3. Modelling of the sensor structure and sensor noise.

and $\underline{C}_j(t)$ is a known $m_j \times n$ time-varying matrix, for each j .

In Equation (2.9) $\underline{\theta}_j(t)$ is the measurement noise associated with the j -th sensor. We assume that each $\underline{A}_j(t)$ is a white noise process with zero mean

$$E\{\underline{\theta}_j(t)\} = \underline{0}; \quad j=1, 2, \dots, M; \quad t \geq t_0 \quad (2.11)$$

and known (symmetric, positive-definite) covariance matrix

$$\text{cov}[\underline{\theta}_j(t); \underline{\theta}_j(\tau)] = E\{\underline{\theta}_j(t)\underline{\theta}_j'(\tau)\} = \underline{\Theta}_j(t)\delta(t-\tau) \quad (2.12)$$

$$\underline{\Theta}_j(t) = \underline{\Theta}_j'(t) > \underline{0} \quad (2.13)$$

We assume that each noise process is independent of \underline{x}_0 and $\underline{\xi}(t)$, i. e.

$$\text{cov}[\underline{x}_0; \underline{\theta}_j(t)] = \underline{0}, \quad \text{for all } t \geq t_0, \quad \text{all } j=1, 2, \dots, M \quad (2.14)$$

$$\text{cov}[\underline{\xi}(t); \underline{\theta}_j(\tau)] = \underline{0}, \quad \text{for all } t, \tau, \quad \text{all } j=1, 2, \dots, M \quad (2.15)$$

The measurement noises $\underline{\theta}_k(t)$ and $\underline{\theta}_j(t)$ may be dependent.

Figure 3 indicates, in block diagram form, our assumptions regarding the structure of each sensor.

2.3 Discussion

It is important to realize that our assumptions imply that each sensor can provide a group of noisy measurements. For example, suppose that we deal with the attitude of a satellite whose state variables are

φ = roll angle

ψ = yaw angle

θ = pitch angle

$\dot{\varphi}$ = roll-rate

$\dot{\psi}$ = yaw-rate

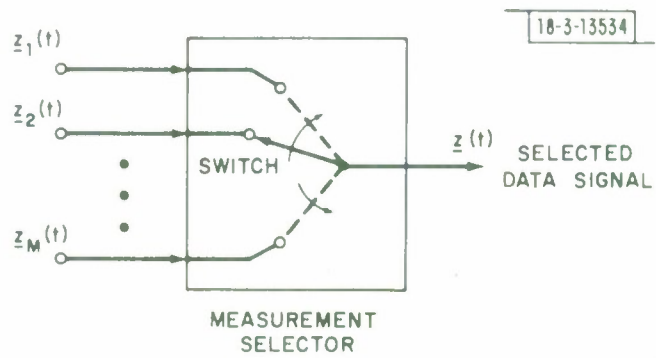


Fig. 4. Visualization of the selected signal $z(t)$.

$\hat{\theta}$ = pitch-rate

Then sensor #1 may yield the measurement (scalar)

roll angle + noise

while sensor #2 may yield the measurement (vector)

roll angle + noise

yaw rate + noise

and sensor #3 may yield the measurement (vector)

roll angle + noise

pitch angle + noise

roll rate + noise

yaw rate + noise.

2.4 Measurement Constraints

As we have indicated in the introduction, we shall assume that at each instant of time, t , we are constrained in looking at only one of the data signals available from the sensors. We are free, of course, to switch from one sensor to another.

To motivate our definitions one can imagine that we have a measurement selector box whose output $\underline{z}(t)$ can be connected directly to either $\underline{z}_1(t)$ or $\underline{z}_2(t), \dots$ or $\underline{z}_M(t)$. This is illustrated in Figure 4.

A convenient way of modelling this simple switching task of the measurement selector is to define M time functions

$$v_1(t), v_2(t), \dots, v_M(t) \quad (2.16)$$

with the following properties

- a) at each instant of time $v_j(t)$ ($j=1, 2, \dots, M$) can attain either the value 0 or the value 1

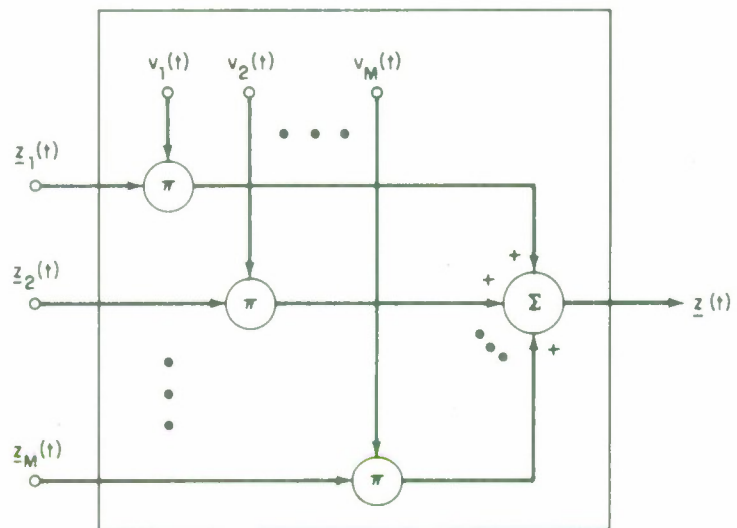


Fig. 5. Alternate modelling of the signal $\underline{z}(t)$.

b) If $v_j(t) = 1$, then $v_k(t) = 0$, for $k = 1, 2, \dots, j-1, j+1, \dots, M$

Mathematically, we can define the switching vector $\underline{v}(t)$ whose components are

$$\underline{v}(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \\ \dots \\ v_M(t) \end{bmatrix} \quad (2.17)$$

and

$$\begin{cases} v_j(t) \in \{0, 1\} \\ \sum_{j=1}^M v_j(t) = 1 \end{cases} \quad (2.18)$$

$$(2.19)$$

The selected data signal $\underline{z}(t)$ can then be written as

$$\underline{z}(t) = v_1(t)\underline{z}_1(t) + v_2(t)\underline{z}_2(t) + \dots + v_M(t)\underline{z}_M(t) \quad (2.20)$$

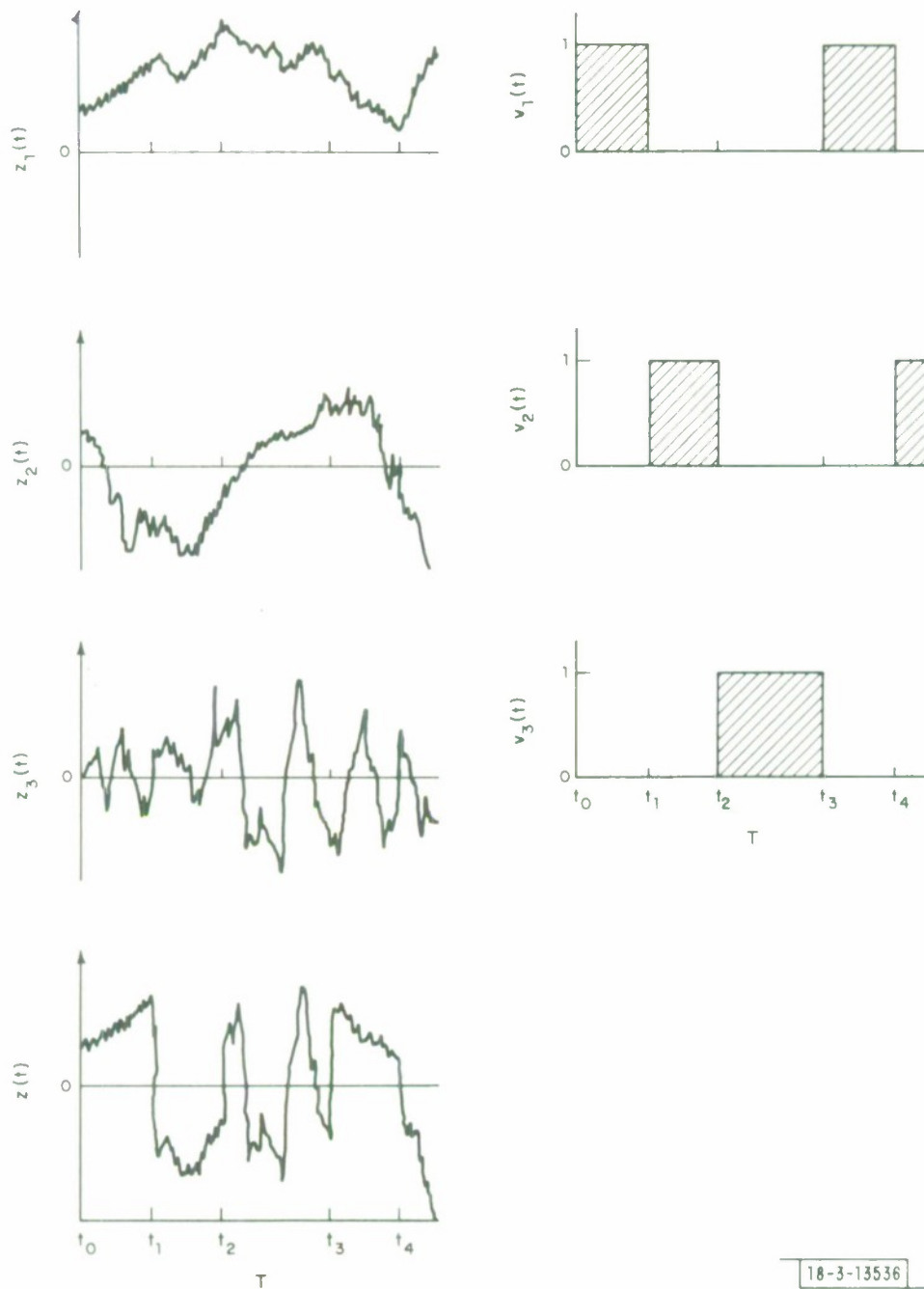
with (dim meaning dimension of column vector)

$$\dim \underline{z}(t) = \dim \underline{z}_j(t) \text{ when } v_j(t) = 1 \quad (2.21)$$

As shown in Figure 5, we can then model the measurement selector by multiplying each measurement $\underline{z}_j(t)$ by $v_j(t)$ and "adding" the results. Figure 6 illustrates how $\underline{z}(t)$ is formed from individual possible measurements.

2.5 Cost of Observations

As we have indicated in the introduction one can associate an observation cost to each one of the M possible observations. Such a cost can be used to reflect that



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Fig. 6. Illustration of the relation of the signal $z(t)$ to three signals $z_1(t)$, $z_2(t)$, $z_3(t)$ as a function of the observation variables $v_1(t)$, $v_2(t)$, $v_3(t)$.

- a) Special resources or instruments may be required to carry out a specific observation.
- b) Special computational algorithms (different in programming overhead and/or real-time requirements) may be required to process each special observation.

For this reason, we shall assume that there is an inherent cost that must be taken into account in order to arrive at an optimal observation policy.

We shall denote by $q_j(t)$ the per-unit-of time cost of making the observation $\underline{z}_j(t)$ at time t . We assume that

$$0 \leq q_j(t); \quad j=1, 2, \dots, M \quad (2.22)$$

Since one is limited to a specific observation at each instant of time, then one can associate with each observation policy $\underline{v}(t)$ a total cost, denoted by $q(\underline{v})$, by

$$q(\underline{v}) = \int_{t_0}^T \left[\sum_{j=1}^M q_j(t)v_j(t) \right] dt \quad (2.23)$$

$q(\underline{v})$ then represents the total observation cost associated with the use of the observation strategy $\underline{v}(t)$ in the time interval $t_0 \leq t \leq T$.

2.6 Prediction Requirements

The definition of an optimal observation policy during a time interval $t_0 \leq t \leq T$ cannot be made on the basis of the observation cost alone. Usually one makes observations upon a physical plant or process in order to predict its future response. (This in turn may be used for control purposes, if it turns out that the future response is in some sense unsatisfactory). Intuitively, one would expect that the accuracy of any prediction or fore-

casting policy will depend on the information content and accuracy of the observations that have been already made. Hence, an optimal observation policy must depend (in addition to the cost of observation) upon the accuracy of the prediction for which observations are made.

In this note, we shall assume that the purpose of observations during a time-interval $t_0 \leq t \leq T$ is to predict the value of a vector $\underline{w}(t)$, associated with the plant variables, at some value of time $t = T_p$, $T_p \geq T$, where $T_p - T$ is the length of the prediction interval.

For example, consider the radar measurements carried out on a ballistic target. Suppose that diverse measurements are carried out during the time-interval $[t_0, T]$. Then one may be interested in predicting at some time T_p , $T_p > T$,

- a) the position vector of the target
- b) its mass

to aid in decisions involving its potential threat to a defense site and/or potential interception.

To be specific, we assume that $\underline{w}(t)$ is an k -dimensional vector, whose components summarize important plant characteristics. We shall call $\underline{w}(t)$ the important plant vector. We shall assume that $\underline{w}(t)$ is linearly related to the plant state vector $\underline{x}(t)$ by the equation.

$$\underline{w}(t) = \underline{D}(t)\underline{x}(t) \quad (2.24)$$

where $\underline{D}(t)$ is a known, possibly time-varying, $k \times n$ matrix.

We can now state in a precise manner the prediction requirements of our problem. Let $\hat{\underline{w}}(t)$ denote an estimate of $\underline{w}(t)$. Then, the accuracy of our prediction scheme hinges on

- a) whether or not $E\{\underline{w}(T_p) - \hat{\underline{w}}(T_p)\} = \underline{0}$ i. e. whether or not at the prediction time T_p the estimation error vector $\underline{w}(T_p) - \hat{\underline{w}}(T_p)$ has zero mean.
- b) the value of the mean square error

$$\hat{J}(T_p) = E\{(\underline{w}(T_p) - \hat{\underline{w}}(T_p))'(\underline{w}(T_p) - \hat{\underline{w}}(T_p))\} \quad (2.25)$$

(the smaller $\hat{J}(T_p)$, the more accurate the prediction).

2.7 Statement of Optimization Problem

We are now ready to formulate in precise terms the optimization problem under consideration, whose solution will specify the optimal observation program defined by the observation vector $\underline{v}(t)$. Given the plant

$$\dot{\underline{x}}(t) = \underline{A}(t)\underline{x}(t) + \underline{B}(t)\underline{u}(t) + \underline{\xi}(t); \quad \underline{x}(t_0) = \underline{x}_0 \quad (2.26)$$

and the important plant vector

$$\underline{w}(t) = \underline{D}(t)\underline{x}(t) \quad (2.27)$$

Given the sensor signals

$$\underline{z}_j(t) = \underline{C}_j(t)\underline{x}(t) + \underline{\theta}_j(t); \quad j=1, 2, \dots, M \quad (2.28)$$

Let $t_0 \leq t \leq T$ be a fixed time interval and let T_p , $T_p \leq T$ be a fixed prediction time. Determine the scalar variables

$$v_1(t), v_2(t), \dots, v_m(t), \quad t \in [t_0, T] \quad (2.29)$$

subject to the constraints

$$v_j(t) \in \{0, 1\}; \quad \sum_{j=1}^M v_j(t) = 1 \quad (2.30)$$

and a prediction algorithm, such that if $\hat{w}(T_p)$ denotes an estimate of $w(T_p)$, given an observation program and observations during $t_0 \leq t \leq T$, then

$$E\{w(T_p) - \hat{w}(T_p)\} = 0 \quad (2.31)$$

and the scalar cost functional

$$\begin{aligned} J &\triangleq \alpha \int_{t_0}^T \left[\sum_{j=1}^M q_j(t) v_j(t) \right] dt + E\{(w(T_p) - \hat{w}(T_p))' (w(T_p) - \hat{w}(T_p))\} \\ &= \alpha q(\underline{v}) + \hat{J}(T_p) \end{aligned} \quad (2.32)$$

with $\alpha \geq 0$ (2.33)

is minimized.

We remark that α is a weighting constant that reflects the relative importance of the total observation cost

$$q(\underline{v}) = \int_{t_0}^T \left[\sum_{j=1}^M q_j(t) v_j(t) \right] dt \quad (2.34)$$

with respect to the "mean square error"

$$\hat{J}(T_p) = E\{(w(T_p) - \hat{w}(T_p))' (w(T_p) - \hat{w}(T_p))\} \quad (2.35)$$

in the overall cost functional J .

3. PREDICTION ACCURACY FOR ANY GIVEN OBSERVATION POLICY

Let $\underline{v}(t), t \in [t_0, T]$ denote any fixed observation policy during the time-interval $t_0 \leq t \leq T_p$. For any such observation policy one can determine a Kalman-Bucy filter, [5], which is "matched" to the observation policy.

3.1 State Estimation

Denote by $\hat{\underline{x}}_v(t)$ the estimate of the plant state $\underline{x}(t)$, given an observation program $\underline{v}(\tau), t_0 \leq \tau \leq t$ and the subsequent observation $\underline{z}(\tau), t_0 \leq \tau \leq t$, as generated by the Kalman-Bucy filter. The subscript v is used to stress the dependence of the estimate upon the observation policy used.

Let $\underline{\Sigma}_v(t)$ denote the state estimate error covariance matrix, i.e. the covariance matrix of the state estimation error $\underline{x}(t) - \hat{\underline{x}}_v(t)$,

$$\underline{\Sigma}_v(t) = E\{(\underline{x}(t) - \hat{\underline{x}}_v(t))(\underline{x}(t) - \hat{\underline{x}}_v(t))'\} \quad (3.1)$$

It can be shown* that the estimate $\hat{\underline{x}}_v(t)$ is generated by the solution of the stochastic differential equation (Kalman-type filter)

$$\begin{aligned} \frac{d}{dt} \hat{\underline{x}}_v(t) = & \underline{A}(t)\hat{\underline{x}}_v(t) + \underline{\Sigma}_v(t) \left[\sum_{j=1}^M v_j(t) \underline{C}'_j(t) \underline{\Theta}_j^{-1}(t) \right] \\ & \cdot \left[\underline{z}(t) - \left(\sum_{j=1}^M v_j(t) \underline{C}_j(t) \right) \hat{\underline{x}}_v(t) \right] + \underline{B}(t)\underline{u}(t); \quad \hat{\underline{x}}_v(t_0) = \bar{\underline{x}}_0 \end{aligned} \quad (3.2)$$

where $\underline{z}(t)$ is the actual observation signal obtained from the policy $\underline{v}(t)$

$$\underline{z}(t) = \sum_{j=1}^M v_j(t) \underline{z}_j(t) \quad (3.3)$$

The error covariance matrix $\underline{\Sigma}_v(t)$ is the solution of the matrix Riccati differential equation

* because the essential linearity of the equations is not affected from the use of a specific measurement policy.

$$\begin{aligned} \frac{d}{dt} \underline{\Sigma}_v(t) &= \underline{A}(t)\underline{\Sigma}_v(t) + \underline{\Sigma}_v(t)\underline{A}'(t) + \underline{\Xi}(t) \\ &- \underline{\Sigma}_v(t) \left[\sum_{j=1}^M v_j(t) \underline{C}_j'(t) \underline{\Theta}_j^{-1}(t) \underline{C}_j(t) \right] \underline{\Sigma}_v(t); \underline{\Sigma}_v(t_0) = \underline{\Sigma}_0 \end{aligned} \quad (3.4)$$

We remark that, for any given $\underline{v}(t)$, the state estimation error has zero mean:

$$E\{\underline{x}(t) - \hat{\underline{x}}_v(t)\} = \underline{0} \quad \text{for all } t \in [t_0, T] \quad (3.5)$$

3.2 State Prediction

The predicted estimate $\hat{\underline{x}}(T_p)$ of the state $\underline{x}(T_p)$ can be computed from the state estimate $\hat{\underline{x}}_v(T)$ by

$$\hat{\underline{x}}_v(T_p) = \underline{\Phi}(T_p, T) \hat{\underline{x}}_v(T) + \int_{T_p}^T \underline{\Phi}(T_p, \tau) \underline{B}(\tau) \underline{u}(\tau) d\tau \quad (3.6)$$

where $\underline{\Phi}(t, \tau)$ is the transition matrix defined by $\underline{A}(t)$, i. e.

$$\frac{d}{dt} \underline{\Phi}(t, \tau) = \underline{A}(t) \underline{\Phi}(t, \tau); \quad \underline{\Phi}(\tau, \tau) = \underline{I} \quad (3.7)$$

3.3 Important Parameter Prediction

The predicted estimate $\hat{\underline{w}}_v(T_p)$ of $\underline{w}(T_p)$ is generated by

$$\hat{\underline{w}}_v(T_p) = \underline{D}(T_p) \hat{\underline{x}}(T_p) \quad (3.8)$$

It can be shown that this leads to a zero mean prediction error, i. e.

$$E\{\underline{w}(T_p) - \hat{\underline{w}}_v(T_p)\} = \underline{0} \quad (3.9)$$

Let $\underline{S}_v(T_p)$ denote the covariance matrix of the prediction error $\underline{w}(T_p) - \hat{\underline{w}}_v(T_p)$, i. e.

$$\underline{S}_v(T_p) = E\{(\underline{w}(T_p) - \hat{\underline{w}}_v(T_p))(\underline{w}(T_p) - \hat{\underline{w}}_v(T_p))'\} \quad (3.10)$$

Then the error covariance matrices $\underline{S}_v(T_p)$ and $\underline{\Sigma}(T_p)$ are related by

$$\underline{S}_v(T_p) = \underline{D}(T_p) \underline{\Sigma}_v(T_p) \underline{D}'(T_p) \quad (3.11)$$

But the predicted state error covariance matrix $\underline{\Sigma}_v(T_p)$ is related to the state error covariance matrix $\underline{\Sigma}_v(T)$, at the end time T of the observation interval, by

$$\underline{\Sigma}_v(T_p) = \underline{\Phi}(T_p, T) \underline{\Sigma}_v(T) \underline{\Phi}'(T_p, T) \quad (3.12)$$

where $\underline{\Phi}(t, \tau)$ is the transition matrix defined by Equation (3.7).

Hence, from Equations (3.11) and (3.12) we conclude that

$$\underline{S}_v(T_p) = \underline{D}(T_p) \underline{\Phi}(T_p, T) \underline{\Sigma}_v(T) \underline{\Phi}'(T_p, T) \underline{D}'(T_p) \quad (3.13)$$

Let us now recall that our prediction accuracy was measured by the "mean square error" (See Equation (2.35))

$$\hat{J}(T_p) = E\{(\underline{w}(T_p) - \hat{\underline{w}}_v(T_p))'(\underline{w}(T_p) - \hat{\underline{w}}_v(T_p))\} \quad (3.14)$$

By a matrix identity, $\hat{J}(T_p)$ can also be written as

$$\begin{aligned} \hat{J}(T_p) &= E\left\{ \text{tr} \left[(\underline{w}(T_p) - \hat{\underline{w}}_v(T_p))(\underline{w}(T_p) - \hat{\underline{w}}_v(T_p))' \right] \right\} \\ &= \text{tr} \left[E\{ (\underline{w}(T_p) - \hat{\underline{w}}_v(T_p))(\underline{w}(T_p) - \hat{\underline{w}}_v(T_p))' \} \right] \\ &= \text{tr} \left[\underline{S}_v(T_p) \right] \end{aligned} \quad (3.15)$$

and so, in view of Equation (3.13), $\hat{J}(T_p)$ is given by

$$\hat{J}(T_p) = \text{tr} \left[\underline{D}(T_p) \underline{\Phi}(T_p, T) \underline{\Sigma}_v(T) \underline{\Phi}'(T_p, T) \underline{D}'(T_p) \right] \quad (3.16)$$

We can now see that any given observation policy $\underline{v}(t)$, $t \in [t_0, T]$, defines a state error covariance matrix $\underline{\Sigma}_v(T)$, by the solution of the matrix Riccati differential equation (3.4) and, hence, a value of $\hat{J}(T_p)$ from Equation (3.16).

4. REFORMULATION OF THE OPTIMIZATION PROBLEM

The above discussion points out that the optimization problem stated in Section 2.7 can be reformulated as follows:

Given the matrix Riccati differential equation

$$\frac{d}{dt} \underline{\Sigma}(t) = \underline{A}(t)\underline{\Sigma}(t) + \underline{\Sigma}(t)\underline{A}'(t) + \underline{\Xi}(t) - \underline{\Sigma}(t) \left(\sum_{j=1}^M v_j(t) \underline{C}'_j(t) \underline{\Theta}_j^{-1}(t) \underline{C}_j(t) \right) \underline{\Sigma}(t);$$

$$\underline{\Sigma}(t_0) = \underline{\Sigma}_0 \quad (4.1)$$

(The elements of $\underline{\Sigma}(t)$ are viewed as the state variables and the $v_j(t)$ as the control variables)

Given the constraints on the $v_j(t)$, $j=1, 2, \dots, M.$

$$v_j(t) \in \{0, 1\}, \quad \text{for all } t \in [t_0, T] \quad (4.2)$$

$$\sum_{j=1}^M v_j(t) = 1, \quad \text{for all } t \in [t_0, T] \quad (4.3)$$

Find the optimal $v_j^*(t)$ such that the cost functional, with t_0, T fixed

$$J = \alpha \int_{t_0}^T \left[\sum_{j=1}^M q_j(t) v_j(t) \right] dt + \text{tr} \left[\underline{D}(T_p) \underline{\Phi}(T_p, T) \underline{\Sigma}(T) \underline{\Phi}'(T_p, T) \underline{D}'(T_p) \right]$$

(4.4)

is minimized.

We remark that this is a deterministic optimal control problem. Since the dynamic constraints (4.1) are naturally expressed via a matrix differential equation, one can obtain the solution through the use of the matrix minimum principle (Athans, [6]).

5. APPLICATION OF THE MATRIX MINIMUM PRINCIPLE

Let $\underline{P}(t)$ denote an $n \times n$ costate matrix associated with the covariance matrix $\underline{\Sigma}(t)$.

Define the scalar Hamiltonian function for the posed optimization problem as follows:

$$\begin{aligned} H &= H(\underline{\Sigma}(t), \underline{P}(t), v_j(t), t). \\ &= \alpha \sum_{j=1}^M q_j(t) v_j(t) + \text{tr}[\underline{\Sigma}(t) \underline{P}'(t)] \end{aligned} \quad (5.1)$$

or

$$\begin{aligned} H &= \alpha \sum_{j=1}^M q_j(t) v_j(t) + \text{tr} [\underline{A}(t) \underline{\Sigma}(t) \underline{P}'(t)] \\ &+ \text{tr} [\underline{\Sigma}(t) \underline{A}'(t) \underline{P}'(t)] + \text{tr} [\underline{\Xi}(t) \underline{P}'(t)] \\ &+ \text{tr} \left[\underline{\Sigma}(t) \left(\sum_{j=1}^M v_j(t) \underline{C}_j'(t) \underline{\Theta}_j^{-1}(t) \underline{C}_j(t) \right) \underline{\Sigma}(t) \underline{P}'(t) \right] \end{aligned} \quad (5.2)$$

5.1 Conditions for Optimality

Let $v_j^*(t)$ characterize the optimal observation policy, $\underline{\Sigma}^*(t)$ the resultant state error covariance matrix, and $\underline{P}^*(t)$ the corresponding costate matrix. Then the following properties are true.

Canonical equations:

$$\begin{aligned} \frac{d}{dt} \underline{\Sigma}^*(t) &= \left. \frac{\partial H}{\partial \underline{P}(t)} \right|_* = \underline{A}(t) \underline{\Sigma}^*(t) + \underline{\Sigma}^*(t) \underline{A}'(t) + \underline{\Xi}(t) \\ &= \underline{\Sigma}^*(t) \left[\sum_{j=1}^M v_j^*(t) \underline{C}_j'(t) \underline{\Theta}_j^{-1}(t) \underline{C}_j(t) \right] \underline{\Sigma}^*(t) \end{aligned} \quad (5.3)$$

$$\begin{aligned}
\frac{d}{dt} \underline{P}^*(t) = - \frac{\partial H}{\partial \underline{\Sigma}(t)} \Big|_* &= - \underline{P}^*(t) \underline{A}(t) - \underline{A}'(t) \underline{P}^*(t) \\
&+ \underline{P}^*(t) \underline{\Sigma}^*(t) \left(\sum_{j=1}^M v_j^*(t) \underline{C}_j'(t) \underline{\Theta}_j^{-1}(t) \underline{C}_j(t) \right) \\
&+ \left(\sum_{j=1}^M v_j^*(t) \underline{C}_j'(t) \underline{\Theta}_j^{-1}(t) \underline{C}_j(t) \right)' \underline{\Sigma}^{*'}(t) \underline{P}^*(t)
\end{aligned} \tag{5.4}$$

Boundary Conditions

At $t = t_0$

$$\underline{\Sigma}^*(t_0) = \underline{\Sigma}_0 \tag{5.5}$$

At $t = T$

$$\begin{aligned}
\underline{P}^*(T) &= \frac{\partial}{\partial \underline{\Sigma}(T)} \text{tr} \left[\underline{D}(T_p) \underline{\Phi}(T_p, T) \underline{\Sigma}(T) \underline{\Phi}'(T_p, T) \underline{D}'(T_p) \right] \Big|_* \\
&= \underline{\Phi}'(T_p, T) \underline{D}'(T_p) \underline{D}(T_p) \underline{\Phi}(T_p, T)
\end{aligned} \tag{5.6}$$

Hamiltonian Minimization

The inequality

$$H(\underline{\Sigma}^*(t), \underline{P}^*(t), v_j^*(t), t) \leq H(\underline{\Sigma}^*(t), \underline{P}^*(t), v_j(t), t) \tag{5.7}$$

or (see equation (5.2)).

$$\begin{aligned}
&\alpha \sum_{j=1}^M q_j(t) v_j^*(t) - \text{tr} \left[\underline{\Sigma}^*(t) \left(\sum_{j=1}^M v_j^*(t) \underline{C}_j'(t) \underline{\Theta}_j^{-1}(t) \underline{C}_j(t) \right) \underline{\Sigma}^*(t) \underline{P}^{*'}(t) \right] \\
&\leq \alpha \sum_{j=1}^M q_j(t) v_j(t) - \text{tr} \left[\underline{\Sigma}^*(t) \left(\sum_{j=1}^M v_j(t) \underline{C}_j'(t) \underline{\Theta}_j^{-1}(t) \underline{C}_j(t) \right) \underline{\Sigma}^*(t) \underline{P}^{*'}(t) \right]
\end{aligned} \tag{5.8}$$

must hold at each $t \in [t_0, T]$ and for all $v_j(t) \in \{0, 1\}$, $\sum_{j=1}^M v_j(t) = 1$.

5.2 Implications of Necessary Conditions

The properties of the trace function can be used to simplify the inequality (5.7). Since

$$\begin{aligned} & \text{tr} \left[\underline{\Sigma}^*(t) \left(\sum_{j=1}^M v_j(t) \underline{C}_j'(t) \underline{\Theta}_j^{-1}(t) \underline{C}_j(t) \right) \underline{\Sigma}^*(t) \underline{P}^{*'}(t) \right] \\ &= \left[\left(\sum_{j=1}^M v_j(t) \underline{C}_j'(t) \underline{\Theta}_j^{-1}(t) \underline{C}_j(t) \right) \underline{\Sigma}^*(t) \underline{P}^{*'}(t) \underline{\Sigma}^*(t) \right] \\ &= \sum_{j=1}^M v_j(t) \text{tr} [\underline{C}_j'(t) \underline{\Theta}_j^{-1}(t) \underline{C}_j(t) \underline{\Sigma}^*(t) \underline{P}^{*'}(t) \underline{\Sigma}^*(t)] \end{aligned} \quad (5.9)$$

define, for notational simplicity, the (symmetric at least positive semi-definite) matrices $\underline{L}_j(t)$ by *

$$\underline{L}_j(t) \triangleq \underline{C}_j'(t) \underline{\Theta}_j^{-1}(t) \underline{C}_j(t) \quad (5.10)$$

Using the above, the inequality (5.7) can be written as

$$\begin{aligned} & \sum_{j=1}^M v_j^*(t) \left[\alpha q_j(t) - \text{tr} [\underline{L}_j(t) \underline{\Sigma}^*(t) \underline{P}^{*'}(t) \underline{\Sigma}^*(t)] \right] \\ & \leq \sum_{j=1}^M v_j(t) \left[\alpha q_j(t) - \text{tr} [\underline{L}_j(t) \underline{\Sigma}^*(t) \underline{P}^{*'}(t) \underline{\Sigma}^*(t)] \right] \end{aligned} \quad (5.11)$$

* One can think of the $\underline{L}_j(t)$ as the matrices that are related to the "signal-to-noise" ratio of the \underline{L}_j j -th possible observation at time t .

Define the switching functions $s_j^*(t)$, $j=1, 2, \dots, M$

$$s_j^*(t) = \alpha q_j(t) - \text{tr}[\underline{L}_j(t)\underline{\Sigma}^*(t)\underline{P}^{*'}(t)\underline{\Sigma}^*(t)] \quad (5.12)$$

In view of the constraints on the $v_j(t)$ we can conclude that

$$\begin{cases} v_j^*(t) = 1 & \text{if } s_j^*(t) \leq s_k^*(t) \text{ for all } k = 1, 2, \dots, M, k \neq j \\ v_j^*(t) = 0 & \text{otherwise} \end{cases} \quad (5.13)$$

6. NUMERICAL SOLUTION OF THE TWO POINT BOUNDARY VALUE PROBLEM

The equations that define the properties of the optimal observation policy, derived by the matrix minimum principle, and stated in Section 5 represent a nonlinear two point boundary value problem. Since we deal with nonlinear matrix differential equations, techniques such as Newton's method are difficult to apply since they involve the computation and inversion of a fourth order tensor quantity. On the other hand, standard gradient techniques cannot be used due to the "on-off" nature of the observation variables $v_j(t)$.

A technique which can be **used, in a** relatively straight forward manner, is the H-minimal technique suggested by Kelley [11]. For the sake of completeness the algorithm is summarized in the flow chart of Figures 7.1 to 7.4. In the construction of this algorithm the following properties (which are easy to verify) have been used

- . The costate matrix $\underline{P}^*(t)$ is symmetric and at least positive semidefinite
- . The covariance matrix $\underline{\Sigma}^*(t)$ is symmetric and at least positive semidefinite

There is no guarantee of convergence of the H-minimal algorithm in general. Also, the nonlinear nature of the matrix differential equations involved precludes any a priori knowledge of existence of locally optimal observation strategies in addition to the globally optimal one. Additional research is currently underway to determine convergence properties and the use of alternate computational algorithms for the solution of the 2-point boundary value problem.

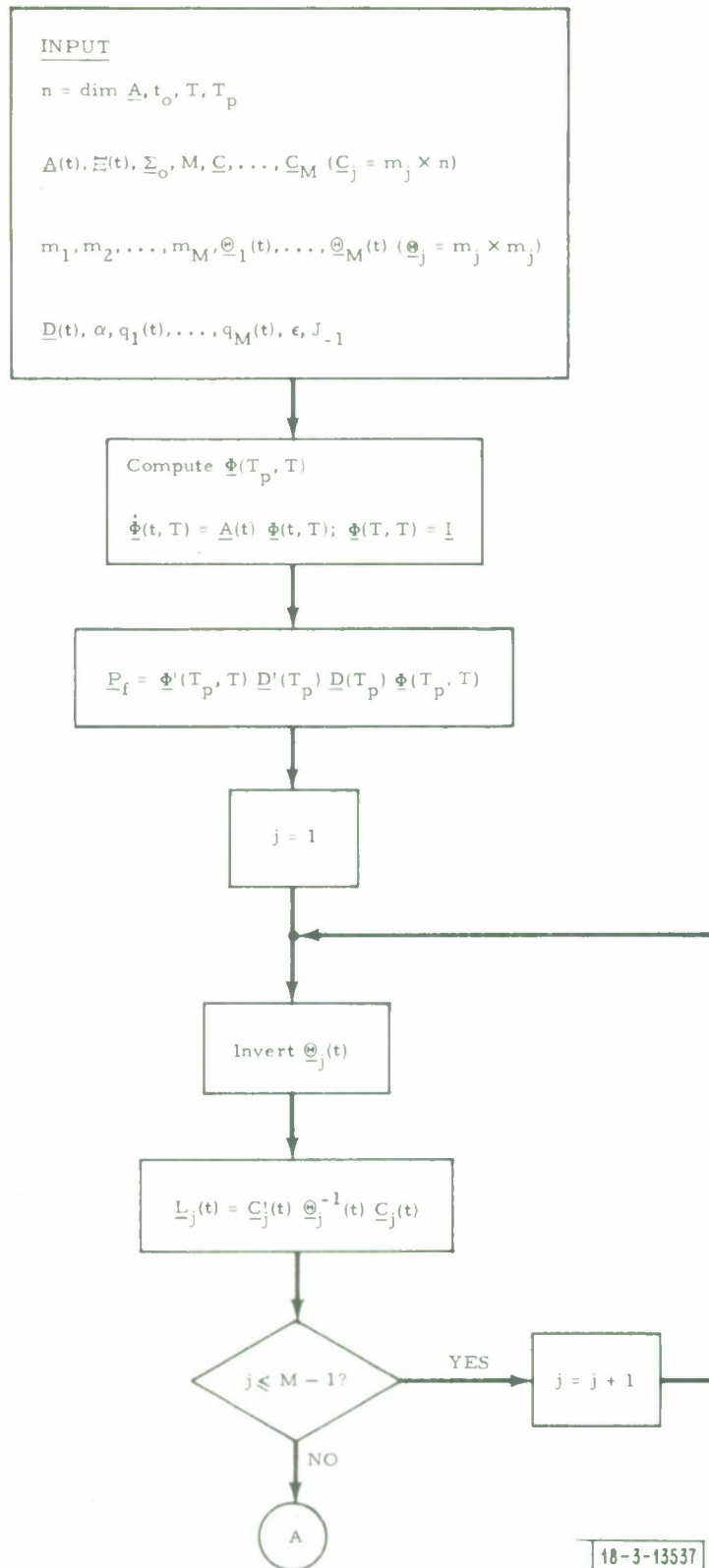


Fig. 7. Structure of the digital computer algorithm for the determination of the optimal observation policy.

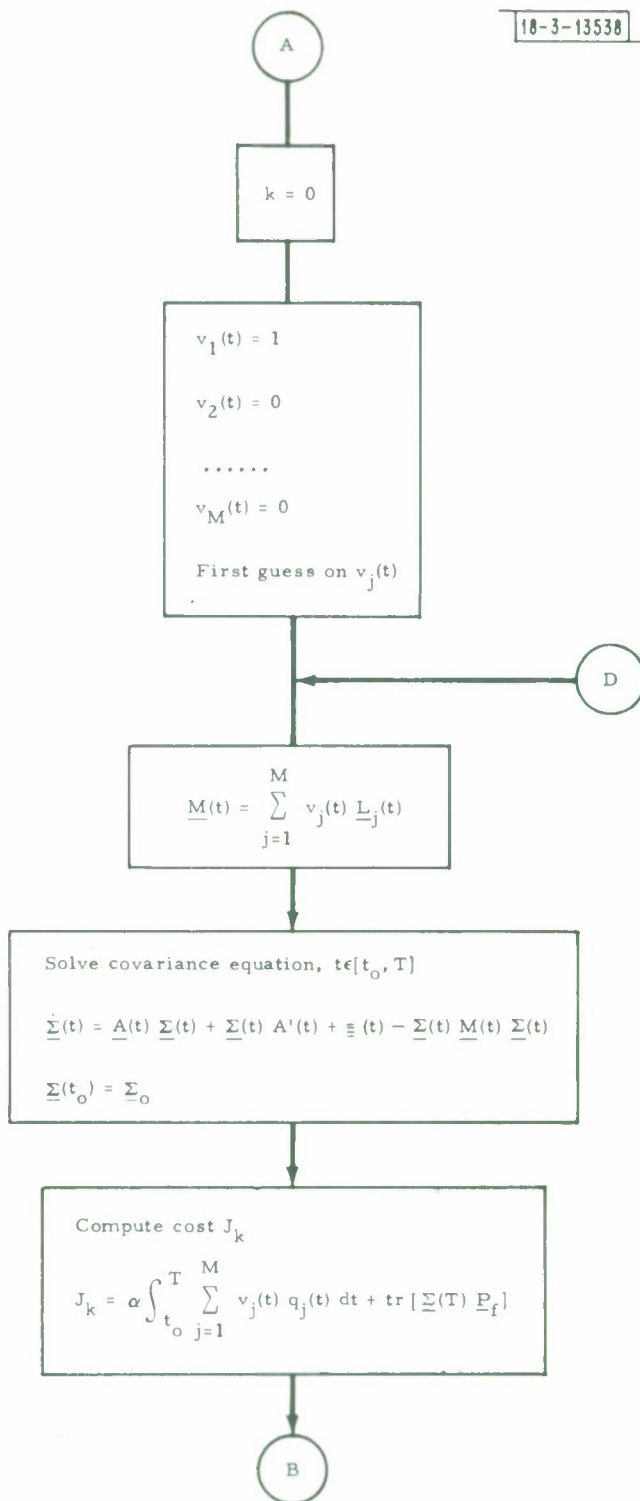


Fig. 7. Continued.

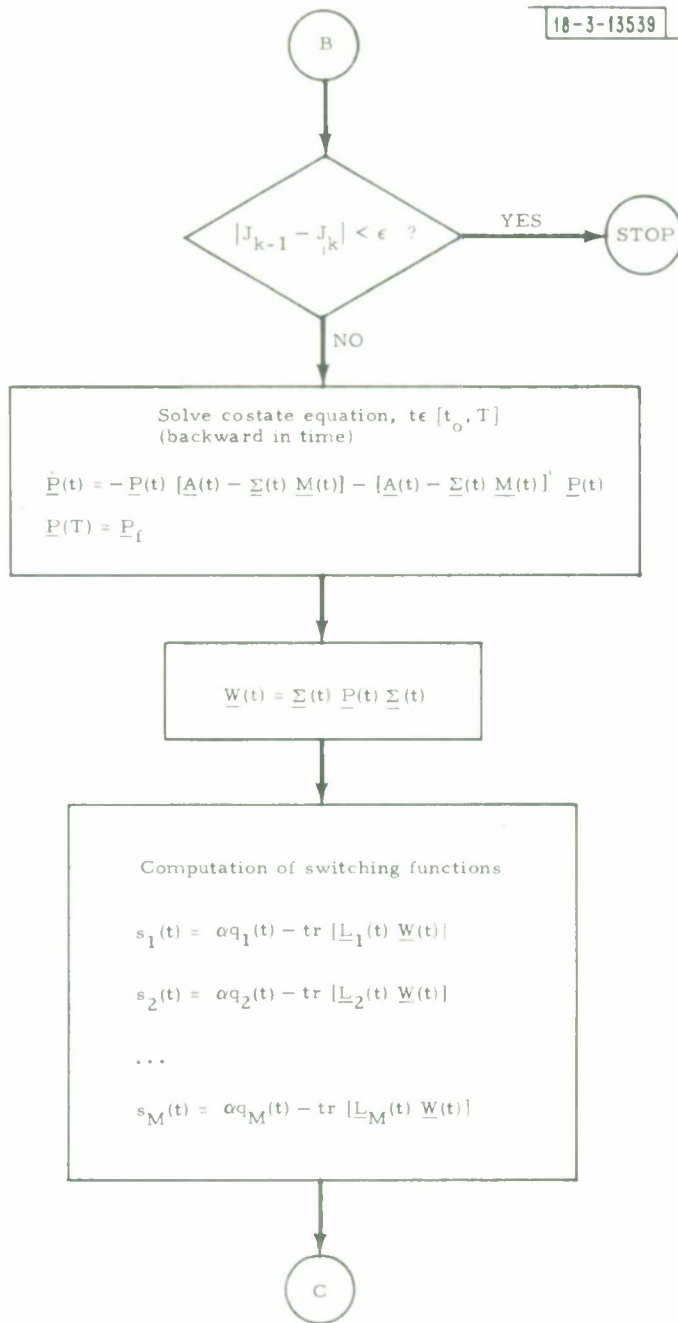


Fig. 7. Continued.

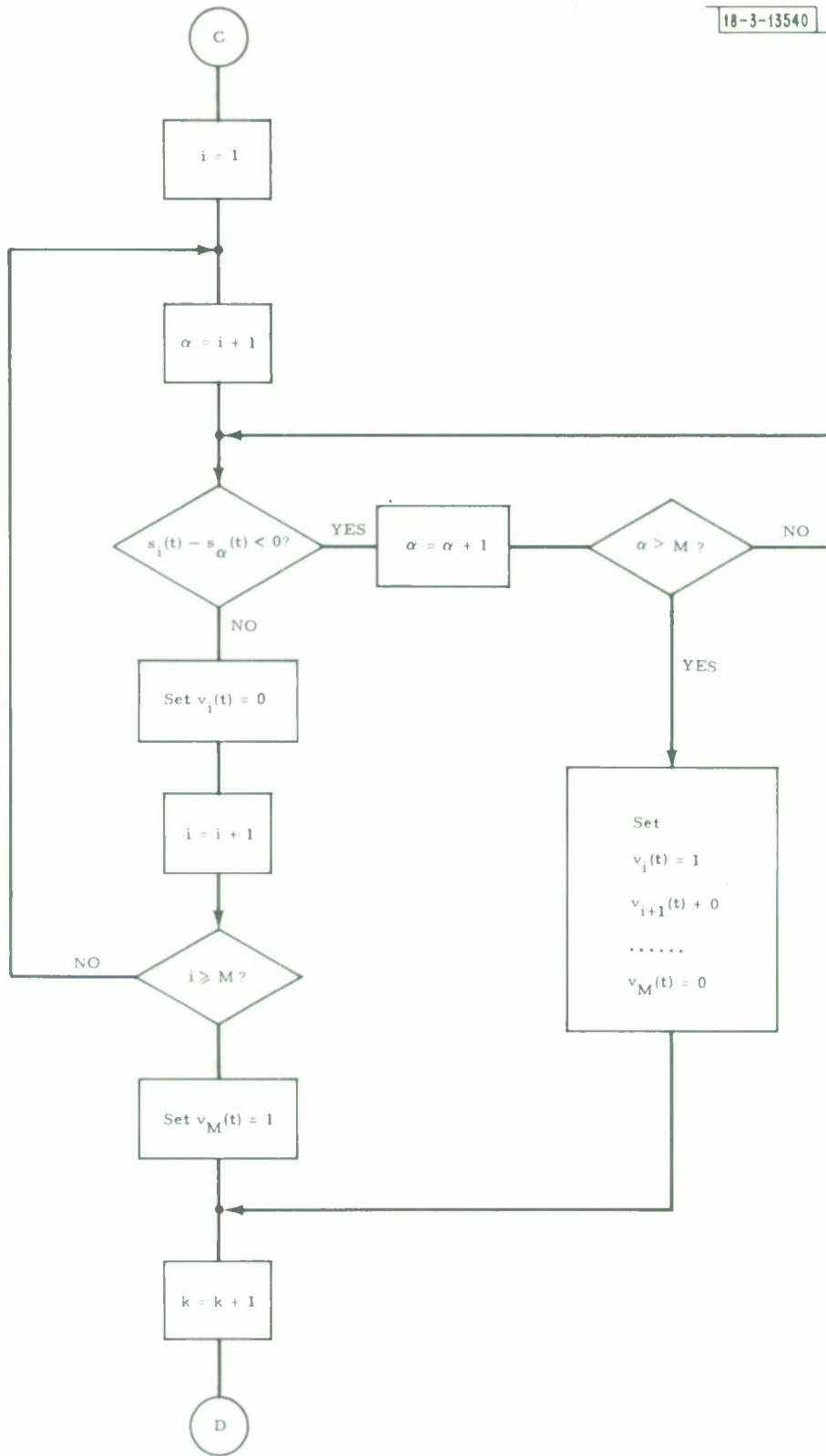


Fig. 7. Continued.

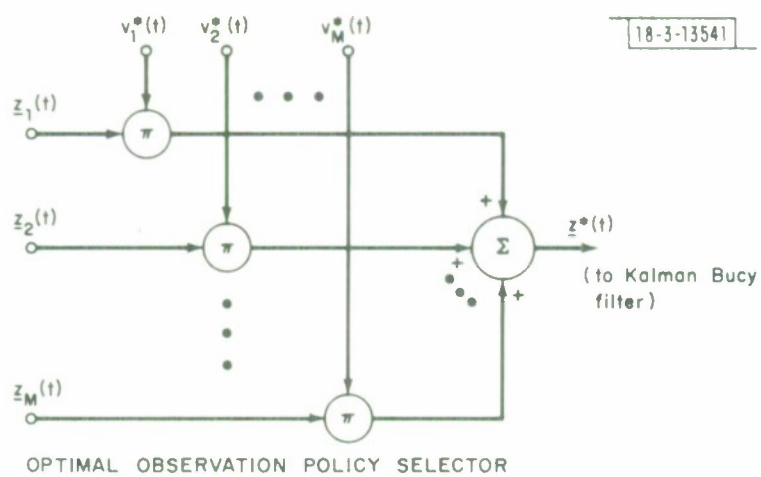


Fig. 8. Generation of optimal measurement $\underline{z}^*(t)$.

7. IMPLEMENTATION

It is important to recognize that the determination of the optimal observation policy $\underline{v}^*(t)$, $t \in [t_0, T]$, is an off-line problem. That is, the solution of the two-point-boundary value problem does not have to be done while measurements are being made. The reason is that the actual measurements $\underline{z}_j(t)$ do not enter in the equations of the optimization problem whose solution determines the optimal observation policy; rather, it is only the statistics of the problem and the plant dynamics that are relevant, rather than the measurements themselves.

Once the optimal observation policies $v_1^*(t)$, $v_2^*(t), \dots, v_M^*(t)$ has been computed (off-line!), then one can implement the "matched" Kalman Bucy filter and predictor which operates upon the actual measurements to generate the optimal estimate $\hat{\underline{w}}^*(t)$ of the important plant parameter vector $\underline{w}(t)$ at any instant of time and at the prespecified prediction time T_p .

Figure 8 shows the generation of the actual signal $\underline{z}^*(t)$ that drives the Kalman filter once $\underline{v}^*(t)$ has been obtained (compare with Figure 5) from the naturally available measurements $\underline{z}_1(t), \dots, \underline{z}_M(t)$.

$$\underline{z}^*(t) = \sum_{j=1}^M v_j^*(t) \underline{z}_j(t) \quad (7.1)$$

The optimal state error covariance matrix $\underline{\Sigma}^*(t)$

$$\underline{\Sigma}^*(t) \triangleq \underline{\Sigma}_{\underline{v}^*}^*(t) \quad (7.2)$$

can be computed off-line, once $\underline{v}^*(t)$ has been obtained by solving the matrix Riccati equation (compare with Equation (3.4)).

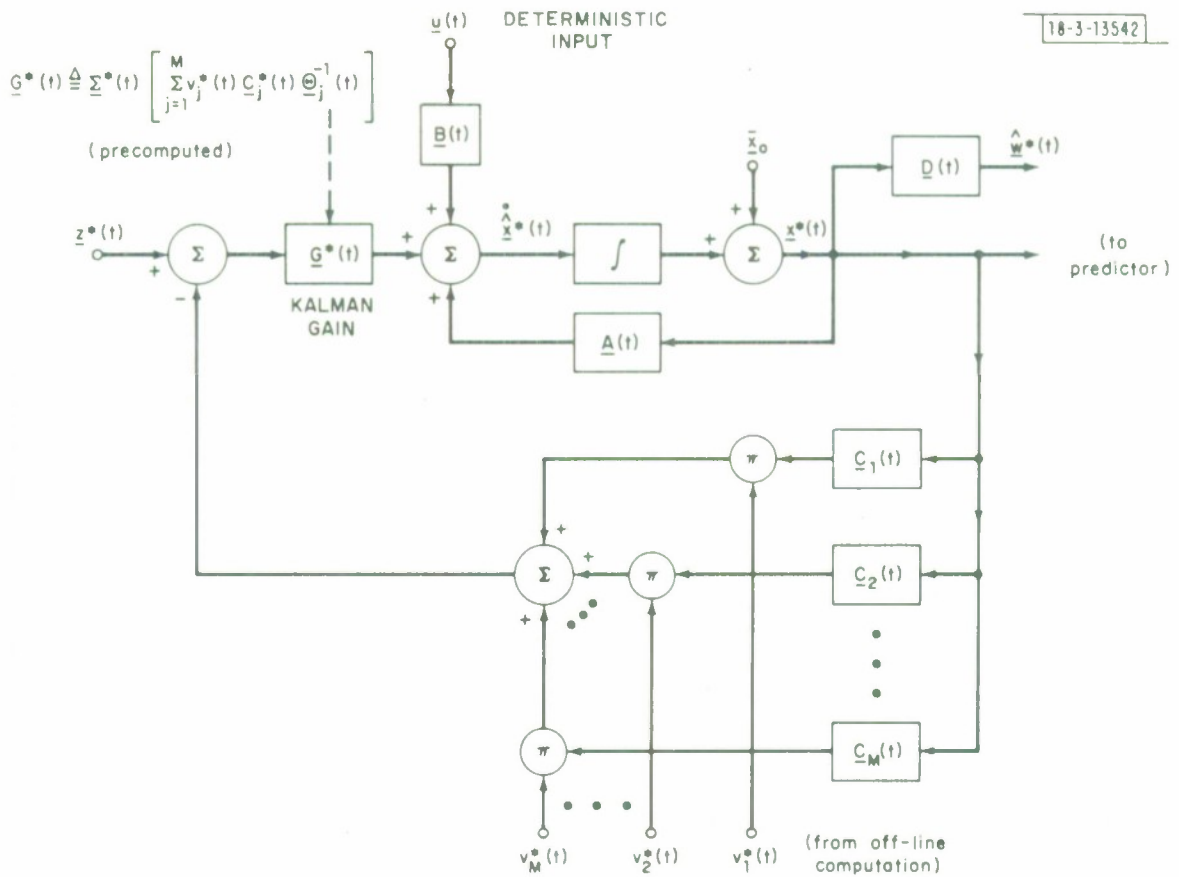


Fig. 9. Structure of the Kalman-type filter which is "matched" to the optimal observation policy.

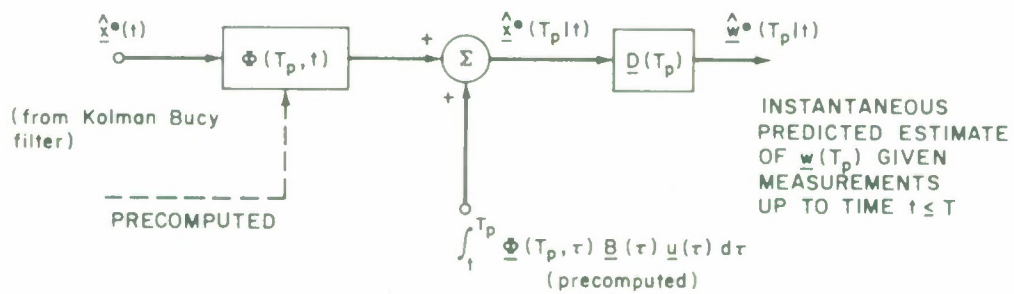


Fig. 10. Structure of the predictor.

$$\frac{d}{dt} \underline{\Sigma}^*(t) = \underline{A}(t)\underline{\Sigma}^*(t) + \underline{\Sigma}^*(t)\underline{A}'(t) + \underline{\Xi}(t) \quad (7.3)$$

$$- \underline{\Sigma}^*(t) \sum_{j=1}^M v_j^*(t) \underline{C}_j'(t) \underline{Q}_j^{-1}(t) \underline{C}_j(t) \underline{\Sigma}^*(t); \quad \underline{\Sigma}^*(t_0) = \underline{\Sigma}_0$$

The structure of the Kalman filter that generates the state estimate $\hat{\underline{x}}^*(t)$

$$\hat{\underline{x}}^*(t) \stackrel{\Delta}{=} \hat{\underline{x}}_{\underline{v}^*}^*(t) \quad (7.4)$$

is shown in Figure 9. The Kalman gain matrix $\underline{G}^*(t)$

$$\underline{G}^*(t) = \underline{\Sigma}^*(t) \sum_{j=1}^M v_j^*(t) \underline{C}_j(t) \underline{Q}_j^{-1}(t) \quad (7.5)$$

can be computed off-line once $\underline{v}^*(t)$ and $\underline{\Sigma}^*(t)$ have been found.

The diagram of Figure 9 helps to visualize how the optimal observation variables $v_1^*(t), \dots, v_M^*(t)$ determine the signal to be subtracted from $\underline{z}^*(t)$, generated by the selector of Figure 8. One can obtain the instantaneous estimate $\hat{\underline{w}}^*(t)$ by simply multiplying the state $\hat{\underline{x}}^*(t)$ of the Kalman Bucy filter by the known matrix $\underline{D}(t)$.

As the actual observations are being made, one can compute the predicted estimate $\hat{\underline{w}}^*(T_p | t)$ of $\underline{w}(T_p)$ given observations only up to time t ($t \leq T$). Figure 10 illustrates the on-line computations required to generate this predicted estimate.

8. CONCLUDING REMARKS

A digital computer program is currently being developed to solve the two point boundary value problem discussed in Section 6. Its performance as well as numerical examples will be reported in the future.

Extensions of the basic ideas to the nonlinear dynamics case are currently under investigation. The approach consists of matching an extended Kalman filter to a particular observation program and then attempting to optimize the observation program. However, in the nonlinear case the situation is much more complex, since in the extended Kalman filter the (pseudo) error covariance matrix cannot be accurately precomputed and, in fact, it is coupled to the estimation equation. For this reason, the optimal observation program has to be computed and updated on line. This may present excessive on-line computational requirements. The projected research effort will be focused on techniques that have less severe on-line computational requirements; however, these may yield suboptimal measurement strategies. Hence, trade-off studies will be necessary in order to establish concrete results in this important class of problems.

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