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ROYAL AIRCRAFT ESTABLISHMENT

ORBIT DETERMINATION AT THE ROYAL AIRCRAFT ESTABLISHMENT

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21

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SUMMARY

This paper describes the development of a series of computer programs for the orbit determination of earth satellites. The paper is based on the first of two lectures given at the ESRO summer school on Spacecraft Operations held at Gravenbruch, near Frankfurt, West Germany, in August 1970. The second lecture is available as Technical Memorandum Space 157.

1 INTRODUCTION

This, the first of my two lectures, covers three related topics:-

(1) survey of OD (orbit determination) programs at RAE;

(2) general principles of OD;

(3) orbital parameters and their perturbations.

The third topic provides a natural link with my second lecture.

In this lecture I use the words 'orbit' and 'determination' in a special way. I regard an 'orbit' as a set of mathematical parameters - usually not more than ten - which are equivalent to complete knowledge (of limited, but adequate, accuracy) of a satellite's motion during a given period, e.g. a week. (Sometimes I use the word in this sense, but sometimes in the more conventional sense.) By the 'determination' of such an orbit I mean the refinement of initial estimates of the parameters, where these estimates must be good enough for a linear, iterative, 'differential-correction' method of refinement to converge. (If sufficiently accurate initial estimates are not available, then the problem is irretrievably non-linear, and special methods of solution have to be used.) The basis for an OD is the existence of observations of the satellite over the given period - e.g. 100 observations over a week - to which the orbit can be fitted. Since it is essential to have good geographical coverage, if a complete set of parameters is to be refined, observations will normally be required from at least two stations, but there is a special situation when observations from a single station will suffice; this occurs when two distinct arcs of the orbital path of the satellite can be tracked, one when it is near the station's latitude moving north, and the other when it is near the station's latitude moving south - the situation is equivalent to the existence of two separated stations.

2 SURVEY OF OD PROGRAMS AT RAE

Four computer programs have been developed at the RAE. The first two of these were written in machine orders for a particular computer, Pegasus, while the other two have been written in Standard Fortran IV and will work (in principle) on any large computer (with minimum storage of about 700 000 bits).

The first program[1,2] was a particularly simple program, written very quickly, immediately after the launching of the first earth satellite in 1957. It was limited to directional observations from a single transit at a single station. This meant, as already remarked, that a complete orbit could not be fitted to the observations: the size and shape had to be held fixed at initial values and only four orbital parameters were actually refined. The program was in operation for about two years, and was used in particular[1,2] for the determination of the orbit of Sputnik 2. It was from the Sputnik 2 results, which showed that the orbital plane of the satellite was precessing less rapidly than expected, that it first became clear[1,2,3] - as I will explain in my second lecture - that the then accepted value for the earth's flattening was about 0.3% too large.

The second program[4,5,6] was a fully-fledged OD program with most of the facilities of the two subsequent programs. It was begun in 1960 and applied to a number of satellites during the period 1962 to 1968. It was only because of the demise of the Pegasus computer that it was eventually abandoned. One of its limitations was that observations had to be of the directional type, though azimuth/elevation, right-ascension/declination and direction cosines were all permitted. Among the orbits analysed was that for the Anglo-American satellite Ariel 2 - the orbit of Ariel 1 was analysed by NASA - throughout the first year of its lifetime[7,8], definitive orbits being determined at 50-node intervals.

The third program[9], started late in 1965 and planned for indefinite evolution, is the most important of the four RAE programs for OD. Known as PROP (Program for Refinement of Orbital Parameters), it has, apart from programming language, a very similar form to its predecessor; both were designed around an analytic perturbation model[10] or 'orbit generator' - i.e. both were based on 'general perturbations' as it is known in the jargon of celestial mechanics - and the differential-correction method has been the same for both. Perhaps the most important innovation in PROP has been the wide extension of the permitted types of observation, so that range and range rate, in particular, may now be used. Among other satellites, PROP has been used for the definitive orbital analysis of Ariel 3; an orbit was determined[11] every three days, for 27 months from launch (May 1967).

The fourth program[12], known as POD (Program for Orbital Determination), is complementary to PROP in that the orbit generator is a numerical-integration model - i.e. it is a 'special perturbation' model in the language of celestial mechanics. In most other respects it is identical with PROP, and employs many of the same Fortran subprograms. It was started in 1967 with the proposed UK military satellite Skynet specifically in mind. This satellite was launched in November 1969 into a synchronous (and virtually geostationary) orbit, for which the gravitational attractions of the sun and moon are of much more importance than for close-earth satellites - it was for this reason that the integration model was deemed to be necessary. Though POD has so far only been used for the Skynet orbit, it is in no way limited to synchronous orbits. Its main limitations - and this relates to its role of complementing PROP - is that (at least in the present version) there is no provision for including air-drag terms among the perturbing forces.

3 GENERAL PRINCIPLES OF OD

Fig.1 illustrates the input and output for a program such as PROP or POD. Here 'constants' refers to a set of standard values of, for example, the earth's equatorial radius, that are held with the program in a Fortran common block; thus each constant is numerically defined at a single point, no matter in how many statements of how many different subroutines there is a reference to that constant, and hence there is no difficulty in changing standard values when desired. The control card specifies, among other things, the number of parameters in the orbital model, the maximum number of iterations to be allowed before the program gives up hope of convergence and stops, and the minimum number of observations with which it is worthwhile to continue at any stage - the number of observations being used may change from one iteration to the next, as a different subset of observations is rejected. The epoch card specifies the date/time to which the orbit refers. The remainder of the input comprises the initial orbit, station data (reference numbers and coordinates, one card per station) and the observations themselves (two cards for a pair of direction cosines, but a single card for other types of observation). Output consists firstly of the final orbital parameters (assuming that convergence has taken place) which are both printed and punched on cards, and secondly of information derived from the covariance matrix of these parameters - standard deviations and correlations are printed, while the matrix itself is punched on cards.

It is useful, before proceeding with the mathematics of the differential-correction process, to distinguish between 'observations' and 'observed quantities'; an 'observation' consists of data obtained from a given station at a particular time, being composed of one or more 'observed quantities'. Thus directional observations consist of pairs of observed quantities, and a simultaneous observation of range, range rate and direction cosines would comprise four observed quantities.

Let us suppose we have a set of q observations of a satellite, comprising altogether n (usually greater than q) observed quantities, and let the orbit, which is to be corrected using these observations, consist of the N parameters p_i, $i = 1, 2, \ldots, N$. If there were no errors in either the observations or the orbit generator, a typical observed quantity would be related to the true values of the parameters by a model function

$$\theta = \theta(p_i, t) \quad . \tag{1}$$

We continue to ignore errors in the orbit generator, but now take into account the fact that there are errors in the available values of the p_i, and also in the observations. Thus we distinguish between θ, which represents a function of the p_i and t, as above, and θ_{obs}, which denotes the numerical value of an actual observed quantity; the error in θ_{obs} is assumed to be random, normally distributed, and uncorrelated with the error in any other observed quantity. The difference $\theta_{obs} - \theta$ is a residual R, which would be zero if the true values of the p_i were available and if the random error in θ_{obs} was zero. (We assume that the correct time t is known, but in practice errors in time should also be considered.)

Let θ, together with R, correspond to estimates p_i of the orbital parameters at the beginning of an iteration of the differential-correction process. At the end of the iteration the parameters are incremented by Δp_i, and it is our object to obtain expressions for the Δp_i in terms of known quantities.

If

$$p_i' = p_i + \Delta p_i \quad \text{and} \quad \theta' = \theta(p_i', t) \quad ,$$

then

$$\theta' = \theta + \sum_i \frac{\partial\theta}{\partial p_i} \Delta p_i + O\,(\text{terms like } (\Delta p_i)\,(\Delta p_j)) \quad . \tag{2}$$

Thus if the Δp_i are small, the residuals R', which will be obtained during the next iteration, are given by

$$R' = R - \sum_i \frac{\partial\theta}{\partial p_i} \Delta p_i \quad . \tag{3}$$

Ideally, every R' would be zero, so that one iteration would suffice; then equation (3) could be written

$$\sum_i \frac{\partial\theta}{\partial p_i} \Delta p_i = R \quad , \tag{4}$$

and solution of equations of type (4) would be trivial if there were the same number of equations as unknowns, i.e. if n was equal to N. In practice, of course, n is much greater than N; also the (Δp_i) (Δp_j) terms are not negligible, so that one iteration does not suffice.

The fact that $n > N$ is dealt with by the application of the method of least squares. We start by expressing all the equations, typified by (3), in matrix form. If

$$M = \begin{pmatrix} \frac{\partial\theta_1}{\partial p_1} & \frac{\partial\theta_1}{\partial p_2} & \cdots & \frac{\partial\theta_1}{\partial p_N} \\ \vdots & \vdots & & \vdots \\ \frac{\partial\theta_n}{\partial p_1} & \frac{\partial\theta_n}{\partial p_2} & \cdots & \frac{\partial\theta_n}{\partial p_N} \end{pmatrix}, \quad Y = \begin{pmatrix} R_1 \\ \vdots \\ R_n \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} \Delta p_1 \\ \vdots \\ \Delta p_N \end{pmatrix},$$

then the given equations reduce to

$$Y' = Y - M Z , \tag{5}$$

where Y is known (in terms of the current p_i) and it is assumed that M is likewise known. The least-squares refinement to minimize $\sum R'^2$ leads to

$$\frac{\partial(Y'^T Y')}{\partial p_i} = 0 \quad \text{for} \quad i = 1, 2, \ldots, N ,$$

where T denotes transposition, and this reduces fairly easily to

$$M^T M Z = M^T Y . \tag{6}$$

It is worth noting that equation (4) leads to $MZ = Y$, so that the effect of 'rectangularity' is allowed for simply by pre-multiplication by M^T on both sides of this equation.

Thus the solution for the Δp_i in any iteration is given by the simple matrix formula

$$Z = (M^T M)^{-1} M^T Y . \tag{7}$$

Although the sum of squares $\sum R^2$, or Y^TY in matrix notation, is not required in this formula, it is of vital importance that this quantity should be computed. For Y^TY is precisely the quantity which is undergoing minimization, and hence it is used as a criterion for the convergence of the iterative process; if it increases from one iteration to the next, due to non-linearity, the process is diverging; if it decreases, the process is converging, and if it decreases by less than some conventional amount (taken in PROP and POD to be 1% of the current value), the convergence is regarded as complete. It is important to realise that the value of Y^TY which becomes available at the end of an iteration is the value associated with the p_i derived at the end of the previous iteration; this is obvious, since the residuals associated with the new p_i cannot be formed until the next iteration, but since Z (and hence $p_i + \Delta p_i$) and Y^TY become known at the same time it is easy to be misled.

The quantity Y^TY is important, apart from the question of convergence. If the observations have been weighted correctly (where I now assume that the matrices M and Y include appropriate weighting factors, though I am omitting the details, which are straightforward), then the final value of Y^TY at convergence should be approximately equal to the number of degrees of freedom of the process, viz. $n - N$. Putting this another way, if $\mathcal{E}$ denotes statistical expectation and ε is defined by

$$\varepsilon^2 = \frac{Y^TY}{n - N} , \tag{8}$$

then

$$\mathcal{E}(\varepsilon) = 1.0$$

after the completion of convergence. If the final value of ε significantly exceeds 1.0, as is often the case, then, unless this is due to errors in the orbit generator or the statistical assumptions, it must be because the observations are less accurate than had been assumed.

This mathematical section of the lecture is completed by the derivation of the formula for the covariance matrix of the final parameters yielded by the differential-correction process. The derivation should be easily understood if the following notation is introduced. Let $\hat{Y}$ be the true vector of residuals at the start of the last iteration; i.e. $\hat{Y}$ is based on residuals $\hat{\theta} - \theta$, where $\hat{\theta}$ is the observed quantity minus its random error and $\theta = \theta(p_i, t)$ as usual. Let $\hat{Z}$ be the vector of Δp_i that would then result from the computation

$$\hat{Z} = (M^T M)^{-1} M^T \hat{Y} \quad .$$

Thus $\hat{Z}$, assuming linearity, would lead to the true values of the parameters, though in practice $\hat{Z}$ can never be known since the $\hat{\theta}$ can never be known.

Then the covariance matrix of the orbital parameters, considering the population of all possible sets of errors in the θ_{obs}, is given by

$$\begin{aligned} \mathrm{cov}\ (Z) &= \mathcal{E}\,[(Z - \hat{Z})\ (Z - \hat{Z})^T] \\ &= (M^T M)^{-1}\ M^T\ \mathcal{E}\,[(Y - \hat{Y})\ (Y - \hat{Y})^T]\ M\ (M^T M)^{-1} \quad . \end{aligned} \tag{9}$$

But if the observations have been consistently weighted, $\mathcal{E}\,[(Y - \hat{Y})\ (Y - \hat{Y})^T]$, i.e. cov (Y), is a scalar matrix. In fact

$$\mathrm{cov}\ (Y) = \sigma^2\ I \quad ,$$

where I is the unit matrix of order n and σ would be unity if the weighting was correct and not merely consistent. Then one of the $(M^T M)^{-1}$ factors in (9) cancels out and we get

$$\mathrm{cov}\ (Z) = \sigma^2\ (M^T M)^{-1} \quad . \tag{10}$$

The matrix $(M^T M)^{-1}$ is already known of course, and all the statistics are contained in the scalar factor σ^2. An estimate of σ^2 is available at once, viz. ε^2 as given by equation (8). (If the weighting of the observations is altered in a consistent manner, so that vectors Y and M are multiplied by some scalar, then not only Z at each stage, as given by (7), but also cov (Z), as given by (10), is unaltered; for if Y and M are multiplied by λ, say, then $(M^T M)^{-1}$ is multiplied by λ^{-2}, but σ^2, as estimated by ε^2 in (8), is multiplied by λ^2. Thus the results of an orbit determination, other than the final value of ε, are independent of absolute weighting, and this effectively means that if all observations are from instruments of the same accuracy it does not matter what actual accuracy is assumed.)

Fig.2 summarizes the flow for a program such as PROP or POD. After input the observations undergo some preliminary processing before the differential-correction process is started. The most important part of the processing consists of the incorporation of station coordinates, rotated from earth-fixed to standard (geocentric) sidereal axes, into the appropriate 'observation block', but certain small corrections to the observed quantities themselves, for example for refraction, are also made. After the processing of all the observations there are q of these 'blocks', one associated with each observation, and I have already remarked that q is usually less than n, the total number of observed quantities. The quantities in a block are indicated in Fig.2:- here the 'rejection state' word is set to zero, and the way in which this word changes is considered later; 'type' is a code, e.g. 7 for range, right ascension and declination; date/time specifies the MJD (modified Julian day number), held as a Fortran integer quantity, and fraction-of-a-day past the MJD midnight, held as a Fortran real quantity; station position contains 7 items of information (cartesian coordinates, together with sine and cosine of geodetic latitude and longitude); the number of observed quantities depends on the type; and the sigmas are the accuracies (used for weighting) associated with the observed quantities.

Fig.3 summarizes the flow during differential correction. During each iteration the observations are introduced one at a time and the matrices M^TM, M^TY and Y^TY are built up. The matrix M^TM is of order $N \times N$, where N is usually much less than n (e.g. $N = 7$ and $n = 100$) and M itself, of order $n \times N$, never has to be stored. Actually the matrices are accumulated at four different 'levels', according to the size of the weighted residuals. This is done to facilitate rejection. Once all the q observations have been dealt with (in a given iteration), the number of observations associated with each level is considered. A special subroutine is called in and decides how many of the levels, starting at the top level, will be rejected. From the accepted level the basic operations, leading to $(M^TM)^{-1}$ (M^TY) and Y^TY, are carried out and the orbit improved. There is then a convergence test, after which there is a further iteration or else the final output occurs. 'Final output' may consist of the desired results, or of a statement that the process has been discontinued without convergence, for example because it has reached the maximum number of iterations permitted.

Further understanding of PROP can best be obtained by reference to the output of a typical run. Figs.4 to 11 illustrate such output, edited into a more compact form than the standard computer output. (POD output, apart from the printing of the parameters, which are defined quite differently, is very similar.)

Fig.4 illustrates the initial printing which occurs after the control and epoch cards have been read. The satellite was Ariel 3, and this was the first possible orbit determination, with an epoch only six hours after injection.

Fig.5 shows how, for reference, the values of the initial elements are printed immediately after being read in. I shall be explaining the layout of elements later; for the present it is worth noting that the initial estimate of orbital inclination is 80.1903 degrees, changing at the rate of -0.00015 degrees per day. On the principle that information about all input data cards should be included in the printed output, it is usual to obtain a listing (actually before the initial elements) of all the station and observation data, but here this listing has been suppressed (by means of the zero values of the control parameters ISENSR and IOBSNS).

The observations used for the Ariel 3 OD were all from the interferometers of the NASA Minitrack network. Each observation consists of a pair of direction cosines, classified by PROP as Type 3, as seen in Fig.6. The residuals for the first observation, as obtained during the first iteration - i.e. from the initial elements - were 0.00098 and -0.00546, equivalent to weighted values (since the *a priori* sigma is 0.00029) of 3 and -19. PROP always prints both weighted and unweighted residuals, and those for 14 of the 40 observations of the specimen run are given in Fig.6.

Fig.7 illustrates the output which occurs at the end of each PROP iteration. Since, returning to our earlier notation, $N = 7$ and $n = 2q = 80$, there are 73 degrees of freedom available during the first iteration, and at the end of that iteration the inclination has increased by 0.0027 degrees to 80.1930 degrees. I will ignore the footnote, though it is claimed to be 'important', but full details of this (and PROP generally) are available from the program's operating manual[9]. Note, however, that ε, the quantity I have said should ideally decrease to a limit near to unity, has changed from 200.000 (an arbitrary value corresponding to 'iteration zero') to 19.928, which is essentially an rms value of the weighted residuals from Fig.7, and that ε (as I have already remarked) is always an iteration behind, since the residuals of Fig.6 are associated with the elements of Fig.5, not Fig.7.

Fig.8 indicates the behaviour during the next two iterations. The new factor present is that observations are being rejected - 3 on the second iteration and 4 on the third - and that there are therefore fewer degrees of freedom. Referring to the third iteration, it may be seen that there are now observations at three different levels - the no-star, one-star and two-star levels - but as yet none at the highest possible (or three-star) level; corresponding to these three levels, the rejection-state words which I referred to earlier, are set to 1, 2 and 3, respectively. At the end of the iteration PROP, through its special rejection subroutine, decides to accept only the lowest level and hence prints '* REJECT 4 OBS ...'. (Had the next lowest level also been accepted the printing would have been '** REJECT 2 OBS ...'.

Fig.9 shows that convergence is proceeding nicely during iterations 4, 5 and 6. It is clear from the first six iterations, however, that it is often the continual change in the set of observations being accepted, rather than the importance of neglected second-order effects in the process, which causes the program to go on for many iterations.

This run finally converged after two more iterations, with final = 1.864 and no further change in the rejections, but details of these are omitted. The final elements are as shown in Fig.10, where the orbital inclination has converged to 80.1791 degrees.

Finally, Fig.11 gives standard deviations, in the same layout as the elements themselves. (Zeros occur for elements which are not being fitted, i.e. which are not 'parameters of the orbit' - I elaborate on this a little later.) It is seen that the standard deviation of inclination is 0.0015 degrees, and from the matrix of correlations it is also possible to read off, for example, the correlation coefficient between eccentricity and inclination - this is 0.168.

4 ORBITAL PARAMETERS AND PERTURBATIONS

I will consider the parameters and perturbations of the POD model first, both because the model is conceptually simpler and because an account of the PROP model, which will conclude this first lecture, will then provide a natural link with the second lecture.

POD, as I have said, has a numerical-integration model, and the parameters of the orbit are simply the epoch components of position and velocity, denoted by x_o, y_o, z_o, $\dot{x}_o$, $\dot{y}_o$, $\dot{z}_o$; thus there are always exactly six parameters. Epoch has to be chosen as some time instant which occurred prior to all the observations (or after all of them if the integration is specified to proceed backwards) and is normally at a midnight, though it does not have to be. (PROP epochs, in contrast, have to be at midnights, but are usually taken somewhere near the centre of the period spanned by observations.) At the beginning of each iteration of a POD run, a table of values of x, y, z, $\dot{x}$, $\dot{y}$, $\dot{z}$, $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ is set up, by integration of the standard equations of motion in an (almost) inertial cartesian axis system, i.e. the equations expressed in Cowell's form. The integration starts from x_o etc. at epoch, operating initially in a fourth-order Runge-Kutta mode but switching to an eighth-order Gauss-Jackson process as soon as this is possible. The integration proceeds until the period spanned by the observations has been completely covered. The step length (for the Gauss-Jackson process) is typically 30 min for Skynet, but is actually a control parameter of the program. The interval of the table set up by the integration is a multiple of the step length, typically 60 min for Skynet.

The POD orbital model consists simply of interpolation (fifth-order for x, y, z; fourth-order for $\dot{x}$, $\dot{y}$, $\dot{z}$) in the table just described, combined of course with the appropriate geometrical calculations necessary to derive topocentric range, declination, or whatever is required, from geocentric x etc.

At the beginning of certain iterations, simultaneously with the setting up of the table of x etc, a table of partial derivatives is also set up in a similar way. The derivatives involved are those of x, y, z, $\dot{x}$, $\dot{y}$, $\dot{z}$, $\ddot{x}$, $\ddot{y}$, $\ddot{z}$, with respect to the orbital parameters - i.e. they occur in sets of 54 - and they are generated by the method of Goodyear[13]. I have already explained why partial derivatives are required in a differential correction process. Since their accuracy does not have to be as good as that required for x etc., they are stored at a wider time interval and not always re-generated after every iteration.

Perturbations - i.e. effects due to forces on the satellite other than the inverse square law force towards the centre of the earth - are incorporated in POD very simply, by inclusion of the perturbing accelerations directly in the equations to be integrated. The perturbations which are covered in the present version of the program are luni-solar perturbations, using stored ephemerides of the sun and moon, and five of the gravitational harmonics of the earth, viz. J_2, J_3, J_4, $C_{2,2}$ and $S_{2,2}$ (all of which I will be defining in my second lecture); perturbations due to air drag are not covered.

Before describing the parameters of the PROP model I must introduce the six standard elements of an elliptic (unperturbed) orbit. Five of these define the orbit itself and are illustrated in Fig.12; in a standard notation these are a (semi-major axis), e (eccentricity), i (inclination), Ω (right ascension of the node) and ω (argument of perigee). The sixth element permits the distinction of one satellite from all other possible satellites moving in the same orbit; this element is the mean anomaly at the epoch of reference, and is denoted sometimes by σ and sometimes by M_o. For a circular orbit it is simply the angle by which the satellite, at epoch, is ahead of perigee. The significance of the zero-suffix notation is that, though semi-major axis etc. are constant for an elliptic orbit, mean anomaly increases linearly with time and is given by the formula

$$M = M_o + M_1 t \quad ,$$

where t is time from epoch and M_1, the 'mean motion', is related directly to a by (Kepler's third law)

$$M_1^2 a^3 = \text{const.} \quad (= 398\ 602\ \text{km}^2/\text{sec}^2 \text{ for the earth}).$$

For a perturbed orbit it is no longer true that a, e etc. are constant, but 'osculating' values may be defined precisely, at any instant of time, by considering the hypothetical removal of all perturbations; then the osculating a, e etc. are six functions of time, instantaneous position and instantaneous velocity, which change only slowly with time - therein lies their use. The variation of an element with time usually consists of a secular (or monotonic) component, together with periodic components of various frequencies. Fig.13, for example, shows the variation of the eccentricity of Ariel 2. Here the osculating eccentricity does not appear at all, since 'short-periodic' components have been removed; these components have amplitude of order 0.001 and periods about 90 min (the orbital period in fact) and 30 min (1/3 of the orbital period). It is clear that short-periodic terms are unplottable and it is standard practice to define 'mean elements' which are free of such terms.

The continuous curve in Fig.13 is the mean eccentricity, as found at RAE from 210 separate orbit determinations, and it is clear that there are still two major components present. One of these is a 'long-periodic' component of amplitude similar to the short-periodic terms and period about 120 days (the period of the rotation of perigee in fact). The other component is a secular decrease of about 0.006 over a year, due to air drag.

Before leaving Fig.13, I have some further remarks on the long-periodic perturbations. It is clear that the values of eccentricity retained by NASA, shown as circles, have had long-periodic, as well as short-periodic, terms removed. This removal of long-periodic terms, leading to 'mean mean' elements, is common to the orbit determination programs of a number of organizations, but at RAE we think it is not a good idea. There are two reasons for this: one is that the expressions used for the long-periodic perturbations are, unlike those for short-periodic perturbations, unbounded, and become infinite at the so-called 'critical inclination', when the motion of perigee disappears; the other reason is that removal of these terms tends to hide the true evolution of the orbit, while the explicit retention of long-periodic amplitudes may be very useful, as will appear in my second lecture. Thus in Fig.13 it seemed preferable to compare RAE and NASA values of eccentricity by adding long-periodic terms back into the NASA values, rather than by removing them from the RAE values.

The PROP model can now be described. Each of the parameters e, i, Ω, ω and M is represented as a polynomial - the degree, n, may be chosen in the range $0 \leqslant n \leqslant 5$ for e, i, Ω and ω, and in the range $1 \leqslant n \leqslant 5$ for M; however the usual choice is $n = 1$ for e, i, Ω and ω and $n = 2$ or 3 for M. Then $e = e_o + e_1 t$, for example, and the complete set of coefficients e_o, e_1, etc. is considered as a set of 'orbital elements'. The 'orbital parameters' are defined as a subset of the set of elements, such that if M_s is a parameter, for example, then M_j must also be a parameter if $j < s$, but need not be if $j > s$.

The usual parameter model is illustrated by Fig.14, which relates to the printed layout of elements illustrated earlier. Here a_o is of course a derived element, and e_o, i_o, Ω_o and ω_o are parameters, while e_1, i_1, Ω_1 ω_1 are not; the M elements are all parameters. The reason for this is that e_1 etc. represent perturbations that can be calculated - as functions of the parameters - to sufficient accuracy, while M_2 etc. cannot. (M_2 etc. are essentially empirical drag parameters which represent the unknown state of the atmosphere, whereas e_1 etc. represent second-order drag effects, which can be expressed as known functions of M_2, and the effects of zonal harmonics, the values of these harmonics being sufficiently well known.)

Thus the secular perturbations of the basic PROP element are represented through the extended set of elements. Periodic perturbations, representing the non-secular zonal-harmonic effects and further second-order drag effects, are also represented, but this representation is by internal formulae, with no direct evidence available in the printed output provided by the program.

The perturbations explicitly covered by PROP are the following:- zonal harmonics, as many as desired up to J_{16} (by recurrence relations, the only limitation being due to storage); the sectorial harmonics $C_{2,2}$ and $S_{2,2}$; and drag. Thus, in particular, luni-solar perturbations are not covered.

I will end this lecture by remarking briefly on one of the theoretical methods by which the formulae expressing satellite perturbations are developed. In the absence of perturbing forces the potential function (equal to the negative of the satellite's potential energy divided by its mass) is simply μ/r. Let us write the potential in the general case (so long as the force field is still conservative) as $\mu/r + U$. Then the variation of the osculating elements a, e, i, Ω, ω and σ arises purely from U. If U can be expressed as a function of the satellite's elements together with time, instead of its position, then the rates of changes of the elements can be expressed in terms of the partial derivatives of this function. Fig.15 gives the explicit formulae - Lagrange's planetary equations - which are exact. Now the zero-order approximation to a, e etc. is simply that they are all constant. Hence a first-order approximation is obtained by assuming a, e etc. constant on the right-hand sides of the planetary equations and integrating, just quadratures being involved. A second-order approximation can then be obtained by substituting the first-order solutions into the right-hand sides of the equations, etc., but the mathematics rapidly becomes unmanageable. I have followed this technique myself[14] in studying the perturbations due to the zonal harmonics.

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2	D. G. King-Hele R. H. Merson	Satellite orbits in theory and practice. J. Brit. Interplan. Soc., 16 (8), p.446 (1958)
3	R. H. Merson D. G. King-Hele	Use of artificial satellites to explore the earth's gravitational field: results from Sputnik 2 (1957β). Nature, 182, p.640 (1958)
4	R. H. Merson	A Pegasus computer programme for the improvement of the orbital parameters of an earth satellite. RAE Technical Note Space 16 (1962)
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9	R. H. Gooding R. J. Tayler	A PROP3 users' manual. RAE Technical Report 68299 (1968)
10	R. H. Merson	The dynamic model for PROP, a computer program for the refinement of the orbital parameters of an earth satellite. RAE Technical Report 66255 (1966)
11	R. H. Gooding	The orbit of Ariel 3 (1967-42A). RAE Technical Report 69275 (1969)

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No.	Author(s)	Title, etc.
12	R. H. Gooding	A POD1 users' manual. RAE Technical Report (to be issued)
13	W. H. Goodyear	Completely general closed-form solution for coordinates and partial derivatives of the two-body problem. Astron. J., 70 (3), p.189 (1965)
14	R. H. Gooding	Satellite motion in an axi-symmetric field, with an application to luni-solar perturbations. RAE Technical Report 66018 (1966)

Fig.1- 6

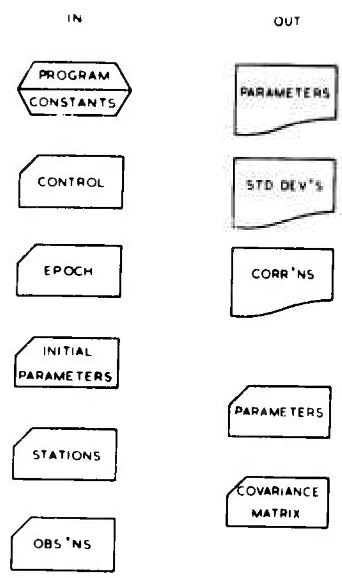

Fig.1

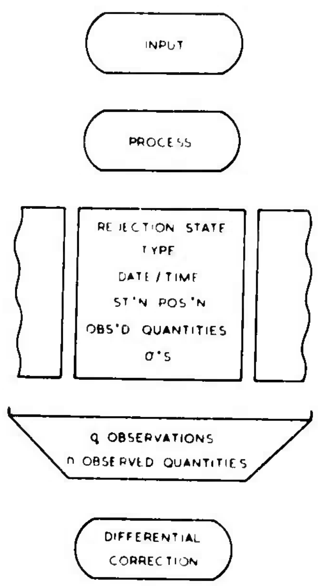

Fig.2

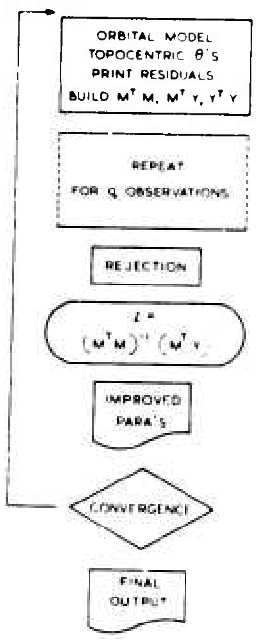

Fig.3

ORBIT DETERMINATION BY PROP

CONTROL PARAMETERS ARE 22223 (KKKKK), 1113 (MMMMM),
0 (MODE), 10 (MAXITN), 20 (MINOBS),
0 (JELTYP), 0 (IENSR), 0 (IOBSNS)

GEOPHYSICAL CONSTANTS ARE 398602 0 (EMU), 6378 163 (ERAD
1 8 (EJ22 * E6), etc etc etc

USE ZONAL HARMONICS UP TO J 9 ONLY

EPOCH IS 1967 MAY 6 0 (MJD 39616),
SATELLITE IDENTIFICATION IS 6704201 (ARIEL 3)

Fig.4

INITIAL ELEMENTS ARE

6930·0964	(DERIVED SEMI-MAJOR AXIS)	
0·007939	-0·0000032	
80·1903	-0·00015	
169·4424	-1·27044	
162 3614	-3·16172	
331·8500	5419·17919	0·017649

Fig.5

FIRST ITERATION ITERATE IN MODE 0

1	TYPE 3	3	-19	0·00098	-0 00546	DCS
2	TYPE 3	2	-16	0 00054	-0 00466	DCS
3	TYPE 3	5	-11	0 00152	0 00323	DCS
4	TYPE 3	6	-6	0 00188	-0 00175	DCS
5	TYPE 3	-42	-39	-0 01220	0 01147	DCS
6	TYPE 3	-45	-29	-0 01306	-0 00837	DCS
	ETC					
33	TYPE 3	2	6	0 00072	0 00186	DCS
34	TYPE 3	5	8	0 00139	0 00247	DCS
35	TYPE 3	14	-2	0 00393	-0 00070	DCS
36	TYPE 3	14	2	0 00399	-0 00045	DCS
37	TYPE 3	-66	-41	-0 01910	-0 01193	DCS
38	TYPE 3	-67	-44	-0 01941	-0 01294	DCS
39	TYPE 3	3	18	-0 00080	0 00530	DCS
40	TYPE 3	-2	19	-0 00046	0 00548	DCS

Fig.6

Fig.1-6

Fig.7-12

```
NO OBSERVATIONS HAVE BEEN REJECTED
THERE ARE 73 DEGREES OF FREEDOM
EPSILON HAS CHANGED FROM 200 000 TO 19 928
ELEMENTS BECOME
      6930 0935   DERIVED SEMI-MAJOR AXIS)
         0 007932      -0 0000032
        80 1930        -0 00015
       169 4395        -1 27010
       162 3445        -3 16202
       331 8453      5419 18275        0 017650
IMPORTANT - ANY OF THE FIRST FOUR SUB-I ELEMENTS
    PRESENT HAS ITS FULL VALUE. THE MAIN SECULAR
    COMPONENT OF WHICH HAS BEEN COMPUTED INTERNALLY.
    THE EXTERNAL COMPONENTS, AND ONLY THESE APPEAR
    IN PUNCHED CARD FORMAT, ARE RESPECTIVELY,
    -0 0000001, -0 00015, 0 00003, 0 00114
```

Fig.7

```
FURTHER ITERATION - NUMBER 2       |      NUMBER 3
                                   |
  1  TYPE 3          8    -6       |          2    -3
  2  TYPE 3          4    -3       |         -3    -3
  3  TYPE 3          6    -1       |          0    -3
  4  TYPE 3          7     0       |          1    -2
  5  TYPE 3    #   -42   -51       |    #   -40   -36
  6  TYPE 3        -43   -42       |    #   -43   -28
     ETC                           |
 33  TYPE 3          3    11       |          2    -8
 34  TYPE 3          5    13       |          4    -7
 35  TYPE 3         18   -14       |          0     7
 36  TYPE 3         18   -13       |          1     6
 37  TYPE 3    #   -60   -33       |   ##  -101   -83
 38  TYPE 3    #   -60   -37       |   ##  -104   -87
 39  TYPE 3        -13    24       |          1    -1
 40  TYPE 3         -1    25       |          1    -1

# REJECT 3 OBSERVATION(S)          | # REJECT 4 OBS
   - 3 FRESH -                     |    - 1 FRESH -
WITH RESIDUAL ABOVE 50             | WITH RES ABOVE 32
THERE ARE 67 DEGREES OF FREEDOM    | THERE ARE 65 D.O.F.
EPSILON HAS CHANGED FROM           | ε FROM 12.762 TO 3.675
19.928 TO 12.762                   | ELEMENTS BECOME ETC
ELEMENTS BECOME ETC                |
```

Fig.8

```
ITERATIONS  4.                    5.                      6.

 1             1    -1              2    -1                 2    -1
 2            -3     0             -3     1                -3     1
 3            -1     0             -1     0                -1     0
 4             0     1              0     1                 0     1
 5 ***       -42   -52     ***    -42   -53      ***      -43   -53
 6 ***       -43   -45     ***    -43   -46      ***      -43   -46

33             2    -7       *      2    -7       **        2    -8
34             3    -5       *      3    -6        *        3    -7
35             2    -1              2     1                 2     1
36             2     0              2     1                 2     2
37 ***       -92   -71     ***    -92   -72      ***      -92   -74
38 ***       -94   -74     ***    -94   -75      ***      -95   -77
39            -1     1             -1     0                -1    -1
40            -1     1             -1     0                -1    -1

* REJECT 4 OBSNS        * REJECT 6 OBSNS        * REJECT 6 OBSNS
  - 0 FRESH -             - 2 FRESH -             - 0 FRESH -
WITH RESDLS ABOVE 9     WITH RESDLS ABOVE 6     WITH RESDLS ABOVE 5
65 D.O.F.               61 D.O.F.               61 D.O.F.
ε FROM 3.675 TO 2.346   ε FROM 2.346 TO 1.998   ε FROM 1.998 TO 1.929
```

Fig.9

```
ELEMENTS BECOME
    6930.0905      (DERIVED SEMI-MAJOR AXIS)
       0.007747       -0.0000049
      80.1791         -0.00016
     169.4548         -1.27187
     163.1191         -3.16049
     331.0528       5419.18576          0.027630

IMPORTANT :- ANY OF THE FIRST FOUR SUB-I ELEMENTS
    PRESENT HAS ITS FULL VALUE, THE MAIN SECULAR
    COMPONENT OF WHICH HAS BEEN COMPUTED INTERNALLY.
    THE EXTERNAL COMPONENTS, AND ONLY THESE APPEAR
    IN PUNCHED CARD FORMAT, ARE RESPECTIVELY,
    -0.0000001, -0.00015, 0.00003, 0.00114
```

Fig.10

```
      0.0039       (S.D. FOR DERIVED SEMI-MAJOR AXIS)
      0.000012      0.0000000
      0.0015        0.00000
      0.0010        0.00000
      0.0783        0.00000
      0.0786        0.00452            0.002060
```

1.000	0.168	-0.150	0.503	-0.500	0.632	-0.653
0.168	1.000	-0.215	0.090	-0.090	0.072	-0.074
-0.150	-0.215	1.000	-0.045	0.043	-0.086	0.087
0.503	0.090	-0.045	1.000	-1.000	0.689	-0.655
-0.500	-0.090	0.043	-1.000	1.000	-0.688	0.653
0.632	0.072	-0.086	0.689	-0.688	1.000	-0.988
-0.653	-0.074	0.087	-0.655	0.653	-0.988	1.000

Fig.11

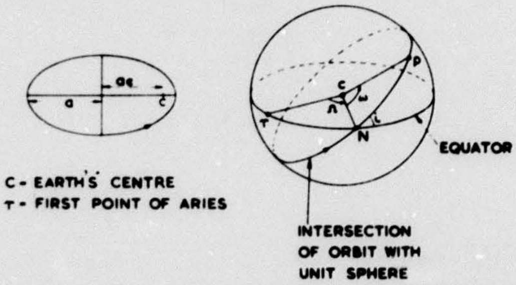

Fig.12

Fig.7-12

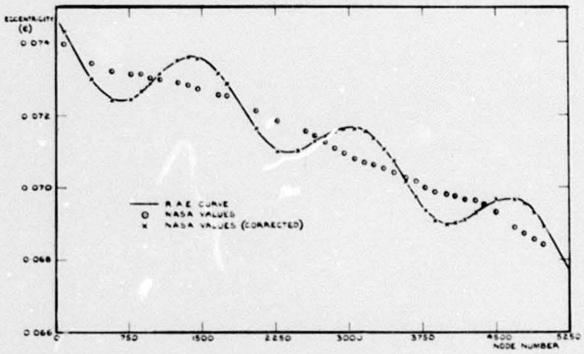

Fig.13

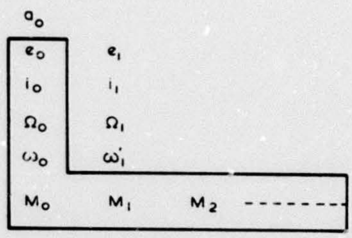

Fig.14

$$\frac{da}{dt} = \frac{2}{na}\frac{dU}{d\sigma}$$

$$\frac{de}{dt} = \frac{1}{na^2 e}\left((1-e^2)\frac{dU}{d\sigma} - \sqrt{1-e^2}\frac{dU}{d\omega}\right)$$

$$\frac{di}{dt} = \frac{\operatorname{cosec} i}{na^2\sqrt{1-e^2}}\left(\cos i\frac{dU}{d\omega} - \frac{dU}{d\Omega}\right)$$

$$\frac{d\Omega}{dt} = \frac{\operatorname{cosec} i}{na^2\sqrt{1-e^2}}\frac{dU}{di}$$

$$\frac{d\omega}{dt} = \frac{1}{na^2}\left(\frac{\sqrt{1-e^2}}{e}\frac{dU}{de} - \frac{\cot i}{\sqrt{1-e^2}}\frac{dU}{di}\right)$$

$$\frac{d\sigma}{dt} = -\frac{1}{na^2}\left(\frac{1-e^2}{e}\frac{dU}{de} + 2a\frac{dU}{da}\right)$$

Fig.15

Fig.13-15