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13. ABSTRACT

It has been shown by Birnbaum, Esary and Marshall that the class of survival functions with increasing hazard rate average (IHRA) is closed under the formation of coherent systems. Moreover, this is the smallest class of survival functions which is closed both under the formation of coherent system and limits in distribution, and which contains the exponential survival functions. In this paper a number of other classes are found which are closed under the formation of coherent systems and limits in distribution. Associated subclasses that play a generating role like the exponential class in the IHRA case are exhibited. In addition, several methods are presented for deriving closed classes from closed classes.

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1. Introduction

In the analysis of system reliability, one is often concerned with properties of a system life distribution which can be guaranteed from properties of component life distributions without reference to details of the system structure. We consider here some properties that the life distribution of every coherent system will inherit from component life distributions. A class of survival functions (those which possess a given property) is said to be closed under the formation of coherent systems if the survival function of every coherent system is in the class whenever the component survival functions are all in the class. Birnbaum, Esary and Marshall (1966) have shown that the class of survival functions with increasing hazard rate average (IHRA) is closed under the formation of coherent systems. Moreover, this is the smallest class of survival functions which is closed both under formation of coherent systems and limits in distribution, and which contains the exponential survival functions. Following the methods of Birnbaum, Esary and Marshall (1966), we obtain here a number of other closed classes, together with the associated subclasses that play a generating role like the exponential class in the IHRA case.

Not all of the closed classes obtained have clear interpretations in the context of reliability theory, because some classes consist of survival functions supported by the whole real line or even by the negative axis.

In addition to these results, several methods are presented for deriving closed classes from closed classes.

2. Preliminaries

We present here some required definitions, notations and preliminary remarks. Except for some minor extensions, these ideas are not new; see, e.g., Birnbaum, Esary and Saunders (1961), Barlow and Proschan (1965, p. 202), Birnbaum, Esary and Marshall (1966), Esary and Marshall (1970), and Esary, Marshall and Proschan (1970).

A function ϕ on $\{\underset{\sim}{x} = (x_1, \cdots, x_n) : x_i = 0$ or 1 for all $i\}$ which takes the values 0 or 1 is called a semi-coherent structure function of order n if ϕ is non-decreasing in each of its arguments. If in addition, $\phi(0, \cdots, 0) = 0$ and $\phi(1, \cdots, 1) = 1$, then ϕ is said to be coherent.

The reliability function h of a semi-coherent structure ϕ is a function on $\{\underset{\sim}{p} = (p_1, \ldots, p_n) : 0 \leq p_i \leq 1$ for all $i\}$ which is defined via independent Bernoulli random variables X_i with expectations $EX_i = p_i$, $i = 1, 2, \ldots, n$, by

$$h(p_1, \ldots, p_n) = E\phi(X_1, \ldots, X_n) .$$

We refer to such a function as a coherent reliability function if ϕ is coherent.

A survival function $\bar{F}$ is a function such that $\bar{F} = 1 - F$ for some right-continuous proper distribution function F. This terminology is most appropriate when $F(0) = 0$, but we wish not to imply such a restriction. For any coherent reliability function h and survival functions $\bar{F}_1, \ldots, \bar{F}_n$, it is obvious that

$$\bar{F}(t) = h(\bar{F}_1(t), \ldots, \bar{F}_n(t)) \quad , \quad -\infty < t < \infty \tag{2.1}$$

defines a survival function $\bar{F}$. For any family S of coherent reliability functions and any family $\mathcal{F}$ of survival functions, we denote by $\mathcal{F}^S$

the class of all survival functions $\bar{F}$ which have the form of (2.1) for some $h \in S$ and some $\bar{F}_1,\ldots,\bar{F}_n \in \mathcal{F}$. In particular, when S is the family of all coherent reliability functions, we use the notation $\mathcal{F}^{CS}$ in place of $\mathcal{F}^S$, and call $\mathcal{F}^{CS}$ the <u>closure of $\mathcal{F}$ under the formation of coherent systems</u>.

The formation of $\mathcal{F}^{CS}$ is a bona fide closure operation in that, (i) the closure of $\mathcal{F}$ contains $\mathcal{F}$, (ii) the closure of $\mathcal{F}$ contains the closure of $\mathcal{A}$ whenever $\mathcal{F}$ contains $\mathcal{A}$, (iii) the closure is closed, and (iv) the closure of the empty set is empty. See Birnbaum, Esary and Marshall (1966, p. 820). In general, the formation of $\mathcal{F}^S$ fails to be a closure operation unless

(2.2) $h, h_1, \ldots, h_n \in S$ where h is of order $n \Rightarrow h^* \in S$, where

$$h^*(\underset{\sim}{p}^{(1)},\ldots,\underset{\sim}{p}^{(n)}) = h(h_1(\underset{\sim}{p}^{(1)}),\ldots,h_n(\underset{\sim}{p}^{(n)})),$$

and

(2.3) the reliability function $h(p) = p$ of order 1 is in S.

If (2.2) and (2.3) hold, we call S a <u>closed family of reliability functions</u>, and refer to $\mathcal{F}^S$ as the closure of $\mathcal{F}$ under the formation of S-systems.

The survival function $\bar{F}$ of (2.1) can be interpreted physically as representing the life distribution of a coherent system with structure function ϕ, reliability function h, and mutually independent components with life distributions $F_1,\ldots,F_n$. To see this, let

(2.4)

$$X_i(t) = 1 \text{ for } t < T_i, \text{ and } X_i(t) = 0 \text{ for } t \geq T_i,\ i = 1,2,\ldots,n,$$

$$X(t) = 1 \text{ for } t < T, \text{ and } X(t) = 0 \text{ for } t \geq T,$$

where T_i has distribution F_i and is the failure time of the i^{th} component,

$i = 1,2,\ldots,n$, T has distribution F and is the system failure time. Then,

(2.5) $$X(t) = \phi(X_1(t),\ldots,X_n(t)),$$

$E\,X_i(t) = \bar{F}_i(t)$, $i = 1,2,\ldots,n$, and $E\,X(t) = \bar{F}(t) = h(\bar{F}_1(t),\ldots,\bar{F}_n(t))$.

An interesting generalization is obtained when (2.1) is replaced by

(2.6) $$\bar{F}(t) = h_t(\bar{F}_1(t),\ldots,\bar{F}_n(t)) \ , \quad -\infty < t < \infty \ ;$$

In this case (2.5) is replaced by

(2.7) $$X(t) = \phi_t(X_1(t),\ldots,X_n(t)) \ .$$

where h_t is the reliability function of ϕ_t. We shall call $\{\phi_t, -\infty < t < \infty\}$ a <u>time-degrading</u> <u>coherent</u> <u>structure</u> if ϕ_t, $-\infty < t < \infty$, are semi-coherent structure functions of a common order, say n, satisfying

(2.8) $$\phi_s(\underset{\sim}{x}) \geq \phi_t(\underset{\sim}{x}) \quad \text{for all } \underset{\sim}{x} \text{ and } s \leq t \ ,$$

(2.9) $$\text{for some } s,\ \phi_s(1,\ldots,1) = 1; \quad \text{for some } t,\ \phi_t(0,\ldots,0) = 0,$$

(2.10) $$\phi_t(\underset{\sim}{x}) \text{ is right continuous in } t.$$

Condition (2.8) guarantees that $\bar{F}$ in (2.6) is decreasing; (2.9) guarantees $\lim_{t\to-\infty} \bar{F}(t) = 1$, $\lim_{t\to\infty} \bar{F}(t) = 0$; condition (2.10) guarantees that $\bar{F}$ is right continuous. Closure of a class of survival functions under the formation of time degrading coherent systems is defined in the obvious way.

3. Closure Under Coherent and Time-Degrading Coherent Systems

The following proposition records the fact that closure under formation of coherent systems is often equivalent to closure under formation of time-degrading coherent systems.

3.1 Proposition. If $\mathcal{F}$ is a class of survival functions which is closed under the formation of coherent systems, and if $\mathcal{F}$ contains the degenerate survival functions, then $\mathcal{F}$ is closed under the formation of time-degrading coherent systems.

Proof. We suppose that $\bar{F}(t) = h_t(\bar{F}_1(t),\ldots,\bar{F}_n(t))$ where ϕ_t satisfies (2.8), (2.9), (2.10), and $\bar{F}_1,\ldots,\bar{F}_n \in \mathcal{F}$. We must show that $\bar{F} \in \mathcal{F}$. Because there are only finitely many coherent structures of order n, there exist finitely many points $-\infty = t_0 < t_1 < \ldots < t_m < t_{m+1} = \infty$ such that $h_r(\underset{\sim}{p}) = h_s(\underset{\sim}{p})$ for all $\underset{\sim}{p}$ whenever $t_j \leq r < s < t_{j+1}$ for some j. With an abuse of notation, we write h_0 in place of h_t when $t_0 < t < t_1$, and h_j in place of h_t when $t_j \leq t < t_{j+1}$, $j = 1,2,\ldots,m$.

Consider now the structure function ϕ^* of order m+n defined by

$$\phi^*(\underset{\sim}{x},\underset{\sim}{y}) = 1 - (1 - \phi_m(\underset{\sim}{x}))\prod_{i=0}^{m-1} (1 - y_{i+1}\phi_i(\underset{\sim}{x})) ,$$

where ϕ_i is the semi-coherent structure function corresponding to h_i. This structure function is diagramed in Fig. 3.1.

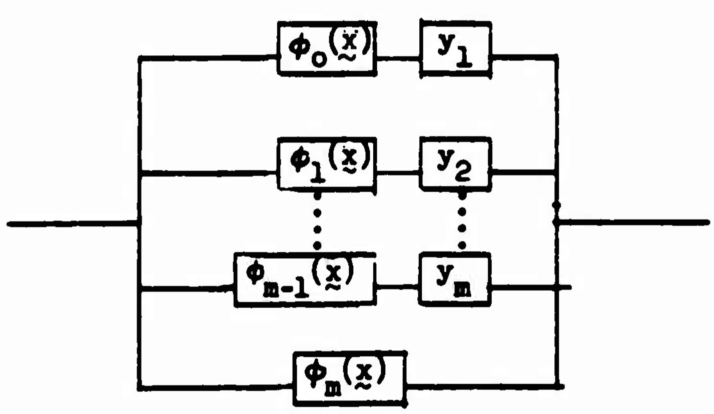

Figure 3.1

From (2.8), it follows that $\phi_0(\underset{\sim}{x}) \geq \cdots \geq \phi_m(\underset{\sim}{x})$ for all $\underset{\sim}{x}$. If $\{Y_i(t),\ -\infty < t < \infty\}$ is the degenerate process

$$Y_i(t) = 1 \quad \text{for} \quad t < t_i, \quad Y_i(t) = 0 \quad \text{for} \quad t \geq t_i, \quad i = 1,2,\ldots,m,$$

and if $\underset{\sim}{X}(t) = (X_1(t),\ldots,X_n(t))$ is defined as in (2.4), then

$$\phi^*(\underset{\sim}{X}(t),\ \underset{\sim}{Y}(t)) \equiv \phi_i(\underset{\sim}{X}(t)) \ , \quad t_i \leq t < t_{i+1} \ , \quad i = 0,1,\ldots,m+1.$$

Thus, in our original notation (with ϕ_t corresponding to h_t),

$$\phi^*(\underset{\sim}{X}(t),\ \underset{\sim}{Y}(t)) = \phi_t(\underset{\sim}{X}(t)) \ , \quad -\infty < t < \infty \ .$$

This means that since $\bar{F}_i(t) = E\,X_i(t)$, $\bar{F}(t) = h_t(\bar{F}_1(t),\ldots,\bar{F}_n(t))$ has the representation

$$\bar{F}(t) = E\phi^*(\underset{\sim}{X}(t),\ \underset{\sim}{Y}(t)) = h^*(\bar{F}_1(t),\ldots,\bar{F}_n(t),\ \bar{G}_1(t),\ldots,\bar{G}_m(t)),$$

where $\bar{F}_1,\ldots,\bar{F}_n \in \mathcal{F}$, and $\bar{G}_i$ (degenerate at t_i) $\in \mathcal{F}$, $i = 1,2,\ldots,m$. Since ϕ^* is coherent and since $\mathcal{F}$ is closed under the formation of coherent systems, $\bar{F} \in \mathcal{F}$. ||

4. Operations Which Generate Closed Classes From Closed Classes

There are a number of operations which generate classes closed under the formation of coherent systems from similarly closed classes. A few such operations are listed in the following theorem:

4.1 Theorem. Let S be a closed family of coherent reliability functions. If $\mathcal{F}$ is closed under the formation of S-systems, i.e., if $\mathcal{F} = \mathcal{F}^S$, then the following classes are also closed in the same sense:

(a) $\mathcal{F}^{LD} = \{\bar{F}: \bar{F}$ is the limit in distribution of some sequence of survival functions in $\mathcal{F}\}$,

(b) $\mathcal{F}^{ac} = \{\bar{F}: \bar{F} \in \mathcal{F}$ and $\bar{F}$ is absolutely continuous$\}$,

(c) $\mathcal{F}^{\xi} = \{\bar{F}:$ for some $\bar{G} \in \mathcal{F}$, $\bar{F}(t) = \bar{G}(\xi(t))$, $-\infty < t < \infty\}$, where ξ is a right continuous non-decreasing function on $(-\infty, \infty)$ such that $\lim_{t \to \infty} \xi(t) \geq b$, $\lim_{t \to -\infty} \xi(t) \leq a$, and $a < b$ satisfy $\bar{F}(a) - \bar{F}(b) = 1$ for $\bar{F} \in \mathcal{F}$,

(d) $\mathcal{F}^{I} = \{\bar{F}:$ for some $\bar{G} \in \mathcal{F}$, $\bar{F}(t) = 1$ for $t < a$, $\bar{F}(t) = \bar{G}(t)$ for $a \leq t < b$, $\bar{F}(t) = 0$ for $t \geq b\}$,

(e) $\mathcal{F}^{CA} = \{\bar{F}:$ for some $\bar{G} \in \mathcal{F}$, $\bar{F}(t) = \bar{G}(t)$ for all $t \in A\}$, where $A \subset R$

(f) $\mathcal{F}^{SG} = \{\bar{F}:$ for some $\bar{G} \in \mathcal{F}$, $\bar{F}(t) \geq \bar{G}(t)$, $-\infty < t < \infty\}$,

$\mathcal{F}^{SL} = \{\bar{F}:$ for some $\bar{G} \in \mathcal{F}$, $\bar{F}(t) \leq \bar{G}(t)$, $-\infty < t < \infty\}$.

These examples can be easily obtained as applications of the following propositions. In these propositions, S need not be closed, except where noted.

4.2 Proposition $\mathcal{F}^{LD,S} \subset \mathcal{F}^{S,LD}$.

For S the family of all reliability functions, this result has been given by Birnbaum, Esary and Marshall (1966). If S is closed, then by putting $\mathcal{F}^S$ in place of $\mathcal{F}$ in this proposition, we obtain that

$$\mathcal{F}^{S,LD,S} = \mathcal{F}^{S,LD} .$$

4.3. Proposition $\mathcal{F}^{ac,S} \subset \mathcal{F}^{S,ac}$:

Proof. $\bar{F} \in \mathcal{F}^{ac,S}$ implies $\bar{F}(t) = h(\bar{F}_1(t),\ldots,\bar{F}_n(t))$ where each $\bar{F}_i \in \mathcal{F}^{ac}$, $h \in S$. This means $\bar{F} \in \mathcal{F}^S$. Moreover $\bar{F}$ is absolutely continuous (see Esary and Marshall, 1970, Application 5.3), so that $\bar{F} \in \mathcal{F}^{S,ac}$. ||

It can be that $\mathcal{F}^{S,ac} \subset \mathcal{F}^{ac,S}$, is false. E.g., suppose that $\mathcal{F} = \{\bar{F}_1, \bar{F}_2\}$ where F_1 is absolutely continuous and has suport $[0,1]$, and F_2 is absolutely continuous except for a discontinuity at 2. Suppose that $S = \{h\}$ consists of but one reliability function, $h(p_1,p_2) = p_1p_2$. Then $\mathcal{F}^{S,ac} = \{\bar{F}_1^{\,2}, \bar{F}_1\bar{F}_2\}$, $\mathcal{F}^{ac,S} = \{\bar{F}_1^{\,2}\}$.

4.4 Proposition $\mathcal{F}^{\xi,S} = \mathcal{F}^{S,\xi}$.

Proof. If $\bar{F} \in \mathcal{F}^{\xi,S}$, there exists a reliability function $h \in S$ (of some order, say n, and $\bar{F}_1,\ldots,\bar{F}_n \in \mathcal{F}^{\xi}$ such that $\bar{F}(t) = h(\bar{F}_1(t),\ldots,\bar{F}_n(t))$. Since $\bar{F}_i \in \mathcal{F}^{\xi}$, $\bar{F}_i(t) = \bar{G}_i(\xi(t))$ for some $\bar{G}_i \in \mathcal{F}$. Morever, $\bar{F}(t) = \bar{G}(\xi(t))$ where $\bar{G}(t) = h(\bar{G}_1(t),\ldots,\bar{G}_n(t))$, i.e., $\bar{F} \in \mathcal{F}^{S,\xi}$. Conversely, if $\bar{F} \in \mathcal{F}^{S,\xi}$, there exists for some n a reliability function $h \in S$ of order n and $\bar{G}_1,\ldots,\bar{G}_n \in \mathcal{F}^S$ such that $\bar{F}(t) = \bar{G}(\xi(t)) = h(\bar{G}_1(\xi(t)),\ldots,$ $\bar{G}_n(\xi(t))) = h(\bar{F}_1(t),\ldots,\bar{F}_n(t))$, where $\bar{F}_i \in \mathcal{F}^{\xi}$, Thus $\bar{F} \in \mathcal{F}^{\xi S}$. ||

In a rather different form, and with S the class of all reliability functions, this proposition has been given by Esary and Marshall (1970, Application 5.1).

We obtain from Proposition 4.4 the following corollary which has an important application in § 5.

4.5 Corollary. If $\mathcal{J}^S = \mathcal{F}$, then $\mathcal{J}^{\xi,S} = \mathcal{F}^{\xi}$; if $\mathcal{J}^{S,LD} = \mathcal{F}$, then $\mathcal{J}^{\xi,S,LD} = \mathcal{F}^{\xi}$.

Proof. To obtain the first assertion, note from Proposition 4.4 that $\mathcal{J}^{\xi,S} = \mathcal{J}^{S,\xi}$; but $\mathcal{J}^{S,\xi} = \mathcal{F}^{\xi}$. The second assertion follows similarly, but requires additonally the fact that $\mathcal{J}^{\xi,LD} = \mathcal{J}^{LD,\xi}$. ||

4.6 <u>Proposition</u> $\mathcal{F}^{I,S} = \mathcal{F}^{S,I}$.

<u>Proof.</u> This follows directly from Proposition 4.4 with $\xi(x) = -\infty$, $x < a$, $\xi(x) = x$, $a \leq x < b$, $\xi(x) = \infty$, $x \geq b$. ||

4.7 <u>Proposition</u> $\mathcal{F}^{CA,S} = \mathcal{F}^{S,CA}$.

<u>Proof.</u> If $\bar{F} \in \mathcal{F}^{CA,S}$, then $\bar{F}(t) = h(\bar{F}_1(t),\ldots,\bar{F}_n(t))$ for some $h \in S$ and some $\bar{F}_1,\ldots,\bar{F}_n \in \mathcal{F}^{CA}$. This implies that $\bar{F}(t) = h(\bar{G}_1(t),\ldots, \bar{G}_n(t))$ for some $\bar{G}_1,\ldots,\bar{G}_n \in \mathcal{F}$, so that $\bar{F} \in \mathcal{F}^{S,CA}$. The proof of the converse is similar. ||

One can, with inclusion only, generalize Propositions 4.6 and 4.7 as follows: Let $(\mathcal{F},\mathcal{H};A)$ be the set of all survival functions that coincide on A with a member of $\mathcal{F}$, and that coincide off A with a member of $\mathcal{H}$.

4.8 <u>Proposition</u> $(\mathcal{F},\mathcal{H};A)^S \subset (\mathcal{F}^S,\mathcal{H}^S;A)$.

The proof is similar to proofs previously given. If $A = [a,b]$ and the survival functions $\mathcal{G}$ have no mass outside A, then this result follows from Proposition 4.6. If $\mathcal{H}$ consists of all survival probabilities, this result follows from Proposition 4.7.

4.9 <u>Proposition</u> $\mathcal{F}^{SG,S} \subset \mathcal{F}^{S,SG}$ and $\mathcal{F}^{SL,S} \subset \mathcal{F}^{S,SL}$.

The proof is again similar to those previously given.

5. Some Classes Closed Under the Function of Coherent Systems

Probably the most important class of survival functions known to be closed under the formation of coherent systems is the class of survival functions with an increasing (i.e., non-decreasing) "hazard rate average". If a distribution function F has a density f, then it has a hazard rate r defined by $r(t) = f(t)/\bar{F}(t)$ for all t such that $\bar{F}(t) > 0$. The condition that $\bar{F}(0) = 1$ and that the hazard rate average $t^{-1}\int_0^t r(x)dx$ is increasing in $t > 0$ is equivalent to the condition that

$$(5.1) \qquad \log \bar{F}(at) \leq - a \log \bar{F}(t) \quad \text{whenever} \quad 0 \leq a \leq 1 \quad \text{and} \quad t \geq 0.$$

Whether or not F has a density, we say that $\bar{F}$ has an increasing hazard rate average if (5.1) is satisfied and we denote the class of such functions by $\{IHRA\}$. The exponential survival functions (i.e., those which for some $\lambda > 0$ have the form $e^{-\lambda t}$ for $t \geq 0$) constitute the subclass, denoted by $\{exp\}$, for which equality holds in (5.1). Birnbaum, Esary and Marshall (1966) show that

$$\{exp\}^{CS, LD} = \{IHRA\}, \quad \text{and that} \quad \{IHRA\}^{CS} = \{IHRA\} .$$

Another class of survival functions $\bar{F}$ which is closed under the formation of coherent systems are those which satisfy $\bar{F}(t) \geq \bar{F}(x+t)/\bar{F}(x)$. Because the right side of this inequality can be interpreted as a conditional survival probability given survival to time x, survival functions which satisfy the inequality are said to be new better than used, and the class is denoted by $\{NBU\}$. The fact that $\{NBU\}^{CS} = \{NBU\}$ is proved by Esary, Marshall and Proschan (1970). No interesting proper subclass $\mathcal{F}$ of $\{NBU\}$ is known which satisfies either $\mathcal{F}^{CS} = \{NBU\}$ or $\mathcal{F}^{CS, LD} = \{NBU\}$. We shall not further discuss the class $\{NBU\}$ or related classes which can be obtained from it using Theorem 4.1.

There are a number of other classes of survival functions which are easily shown to be closed under the formation of coherent systems. For example, those survival functions which are absolutely continuous, those which are discrete, those which are singular, and those which are degenerate, each constitute a closed class.

Consider again the IHRA case. Let $\bar{G}(t) = e^{-t}$, $t \geq 0$, and let

$$\mathcal{F} = \{\bar{F} : \bar{G}^{-1}\,\bar{F}(at) \leq a\,\bar{G}^{-1}\,\bar{F}(t) \text{ for all } a\epsilon(0,1] \text{ and } t \geq 0\}\ ,$$

$$\mathcal{S} = \{\bar{F} : \bar{G}^{-1}\,\bar{F}(at) = a\,\bar{G}^{-1}\,\bar{F}(t) \text{ for all } a\epsilon(0,1] \text{ and } t \geq 0\}\ .$$

Then $\mathcal{F} = \{\text{IHRA}\}$, $\mathcal{S} = \{\exp\}$, and we know that $\mathcal{S}^{CS,LD} = \mathcal{F}$, $\mathcal{F}^{CS} = \mathcal{F}$. It is of interest to ask if other survival functions can play the role of the exponential $\bar{G}(t) = e^{-t}$, $t \geq 0$, in this development.

To answer this question, we recall that Birnbaum, Esary and Marshall (1966) obtain the exponential-IHRA result described above via an inequality for reliability functions. Moreover, only a special case of their inequality is utilized in proving that $\{\exp\}^{CS,LD} = \{\text{IHRA}\}$. In the remainder of this paper, we show that other special cases of the inequality also have potential for proving closure results.

The inequality of Birnbaum, Esary and Marshall (1966) is given in the following lemma.

5.1 <u>Lemma</u>. Let h be a coherent reliability function of order n. If ψ is a function on $[0,1]$ satisfying

$$r\psi(y) + (1-r)\,\psi(x) + (y-x)\,\psi(r) \geq \psi(ry+(1-r)x)$$

(5.2)

for all r,x and y such that $0 \leq r \leq 1$, $0 \leq x \leq y \leq 1$,

and

$$\psi(0) = \psi(1) = 0 ,$$

then

$$\sum_{i=1}^{n} \psi(p_i)\, \partial h(\underset{\sim}{p})/\partial p_i \geq \psi h(\underset{\sim}{p}) . \tag{5.4}$$

The class of functions ψ which satisfy (5.2) and (5.3), and the class which satisfies (5.4) are discussed in some detail in § 6. We remark here that $\psi_1(z) = -z \log z$, $\psi_2(z) = -(1-z)\log(1-z)$ and $\psi_3(z) = z(1-z)$, $0 \leq z \leq 1$ all satisfy (5.2) and (5.3). For the special case $\psi = \psi_3$, (5.4) was obtained by Esary and Proschan (1963) for the still more special case that $\psi = \psi_3$ and $p_1 = \ldots = p_n$, (5.4) was obtained by Moore and Shannon (1956).

The case $\psi = \psi_1$ was used by Birnbaum, Esary and Marshall (1966). In this case, Esary, Marshall and Proschan (1970) have shown that (5.4) can be rewritten in the form

$$\eta(a\underset{\sim}{x}) \leq a\eta(\underset{\sim}{x}) \quad 0 \leq a \leq 1, \quad 0 \leq x_i < \infty, \quad i = 1,2,\ldots,n ,$$

where

$$\eta(\underset{\sim}{x}) = -\log h(e^{-x_1},\ldots,e^{-x_n}) , \quad 0 \leq x_i < \infty, \quad i = 1,2,\ldots,n .$$

They found this form to be particularly convenient in proving that the IHRA distributions are closed under the formation of coherent systems.

The function η is called the <u>hazard transform</u> of the coherent system because it gives the system hazard function R in terms of component hazard functions R_i:

$$R(t) = \eta\,(R_1(t),\ldots,R_n(t))$$

(The <u>hazard function</u> of a distribution F is given by $-\log \bar{F}(t)$, and is the integral $\int_0^t r(x)dx$ of the more familiar hazard rate r).

In the definition of the hazard transform, the exponential function can be replaced by certain other survival functions $\bar{G}$. Suppose that $G(0) = 0$ and that G has a positive derivative g on $(0,\infty)$. Then $\bar{G}$ has an inverse $\bar{G}^{-1}$ on $[0,1]$ which satisfies

$$\bar{G}\bar{G}^{-1}(p) = p \quad \text{whenever} \quad 0 \leq p \leq 1, \quad \text{and} \quad \bar{G}^{-1}\bar{G}(x) = x \quad \text{whenever} \quad 0 \leq x < \infty .$$

In this case, the G-<u>hazard transform</u> η of a semi-coherent structure with reliability function h of order n is given by

$$\eta(\underset{\sim}{x}) = \bar{G}^{-1}h(\bar{G}(x_1),\ldots,\bar{G}(x_n)), \quad 0 \leq x_i < \infty, \quad i = 1,2,\ldots,n .$$

5.2 <u>Lemma</u>. Let G be a distribution function such that $G(0) = 0$ and suppose that G has a positive derivative g on $(0,\infty)$. If ψ, defined by

(5.5) $\quad \psi(z) = \bar{G}^{-1}(z)\cdot g\,\bar{G}^{-1}(z), \quad 0 < z < 1, \quad \text{and} \quad \psi(0) = \psi(1) = 0,$

satisfies (5.2) and if η is the G-hazard transform of a coherent structure of order n, then

(5.6) $\quad \eta(a\underset{\sim}{x}) \leq a\eta(\underset{\sim}{x}), \quad 0 \leq a \leq 1, \quad 0 \leq x_i < \infty, \quad i = 1,2,\ldots,n .$

We defer the proofs of these lemmas to § 6. However, we remark that (5.6) can be reformulated as a monotonicity condition, and that (5.4) is the corresponding condition that a derivative be non-negative.

Let $\mathcal{A}$ be the class of all survival functions $\bar{G}$ for which

(i) $G(0) = 0$ and G has a positive derivative on $(0,\infty)$, and

(ii) the G-hazard transform η of every coherent structure satisfies (5.6). Lemma 5.2 provides a sufficient condition for $\bar{G} \in \mathcal{A}$.

Consider now the utility of (5.6) for obtaining closed classes.

5.3 <u>Theorem</u>. If $\bar{G}\epsilon\mathcal{a}$ and if

$$\mathcal{F}_G = \{\bar{F} : \bar{G}^{-1}\bar{F}(at) \leq a\bar{G}^{-1}\bar{F}(t) \text{ for all } a\epsilon[0,1] \text{ and all } t \geq 0\},$$

$$\mathcal{B}_G = \{\bar{F} : \bar{G}^{-1}\bar{F}(at) = a\bar{G}^{-1}\bar{F}(t) \text{ for all } a\epsilon[0,1] \text{ and all } t \geq 0\},$$

then

$$\mathcal{F}_G^{CS} = \mathcal{F}_G, \quad \text{and} \quad \mathcal{B}_G^{CS,LD} = \mathcal{F}_G .$$

<u>Proof</u>. Let $\bar{F}$ be the survival function of a coherent system with reliability function h, G-hazard transform η, and component survival functions $\bar{F}_1,\ldots,\bar{F}_n \,\epsilon\, \mathcal{F}_G$. If $0 \leq a \leq 1$ and $t \geq 0$, then

$$\bar{G}^{-1}\bar{F}(at) = \bar{G}^{-1}h(\bar{F}_1(at),\ldots,\bar{F}_n(at)) = \eta(\bar{G}^{-1}\bar{F}_1(at),\ldots,\bar{G}^{-1}\bar{F}_n(at))$$

$$\leq \eta(a\bar{G}^{-1}\bar{F}_1(t),\ldots,a\bar{G}^{-1}\bar{F}_n(t)) \leq a\eta(\bar{G}^{-1}\bar{F}_1(t),\ldots,\bar{G}^{-1}\bar{F}_n(t)) = a\bar{F}(t)$$

The first inequality follows from the fact that η is increasing and each $\bar{F}_i \,\epsilon\, \mathcal{F}_G$. The second inequality follows because $\bar{G} \,\epsilon\, \mathcal{a}$ so that (5.6) holds. This means that $\mathcal{F}_G^{CS} \subset \mathcal{F}$, and hence $\mathcal{F}_G^{CS} = \mathcal{F}$.

Clearly $\mathcal{B}_G^{CS,LD} \subset \mathcal{F}_G$. The proof that $\mathcal{F}_G \subset \mathcal{B}_G^{CS,LD}$ is virtually the same as the proof given by Birnbaum, Esary and Marshall (1966) for the case that $\bar{G}(t) = e^{-t}$, $t \geq 0$. ||

Note that if $\bar{F} \,\epsilon\, \mathcal{B}_G$, then for some $c > 0$,

$$\bar{F}(t) = \bar{G}(ct).$$

5.4 <u>Corollary</u>. If $\bar{G} \,\epsilon\, \mathcal{a}$ and if ξ is an increasing function satisfying

$$\lim_{t\to-\infty} \xi(t) \leq 0, \quad \lim_{t\to\infty} \xi(t) = \infty ,$$

and if

$$\mathcal{F}_G^{\xi} = \{\bar{F} : \bar{G}^{-1}\bar{F}(t)/\xi(t) \text{ is non-decreasing in } t \text{ for which } \xi(t) > 0\}$$

$$\mathcal{S}_G^{\xi} = \{\bar{F} : \bar{G}^{-1}\bar{F}(t)/\xi(t) \text{ is constant in } t \text{ for which } \xi(t) > 0\},$$

then

$$\mathcal{F}_G^{\xi,CS} = \mathcal{F}_G^{\xi}, \text{ and } \mathcal{S}_G^{\xi,CS,LD} = \mathcal{F}_G^{\xi}.$$

Proof. This result follows from Theorem 5.3 and Corollary 4.5.//

Here it is trivial that $\mathcal{S}_G^{\xi}$ is the set of all survival functions which, for some $c > 0$, have the form $\bar{F}(t) = \bar{G}(c\xi(t))$, t such that $\xi(t) > 0$.

Observe that if ξ_1 and ξ_2 satisfy the condtions of Corollary 5.4, if $\xi_2(t) > 0$ implies $\xi_1(t) > 0$ and if $\xi_1(t)/\xi_2(t)$ is increasing in t for which $\xi_2(t) > 0$, then

$$\mathcal{F}_G^{\xi_1} \subset \mathcal{F}_G^{\xi_2}. \tag{5.7}$$

5.5 Corollary. Let ψ be a function satisfying (5.4) for all coherent reliability functions h, and let $\theta(t) \geq 0$ for all t. Then

$$\mathcal{F} = \{\bar{F} : \bar{F} \text{ has a density } f \text{ and } f(t)/\psi\bar{F}(t) \geq \theta(t) \text{ for all } t\}$$

is closed under the formation of coherent systems.

A proof of Corollary 5.5 can be given by utilizing (5.4) and slightly modifying the proof of Theorem 4.1, Birnbaum, Esary and Marshall (1966). The corollary can also be obtained from Theorem 4.1(b) and Corollary 5.4 with $\xi(t) > 0$ and $\theta(t) = \xi'(t)/\xi(t)$.

Remark. We have assumed that if $\bar{G} \in \mathcal{Q}$, then $G(0) = 0$ and G has a positive derivative on $(0,\infty)$. Most of the above development can be modified to eliminate these conditions, although one must still be able to define $\bar{G}^{-1}$. We have avoided the extra complexities this entails, because the only known $\bar{G}$ for which (5.6) holds do in fact satisfy our assumed conditions.

Although the class $\mathcal{A}$ and corresponding functions ψ defined by (5.5) are discussed in § 6, we consider some important examples here.

5.6 Example. From the viewpoint of reliability theory, the most important results are obtained with

(5.8) $\bar{G}(t) = e^{-t}$, $t \geq 0$, i.e., $\psi(z) = -z \log z$, $0 \leq z \leq 1$.

Here,

$$\mathcal{F}_G^{\xi} = \{\bar{F} : [-\log \bar{F}(t)]/\xi(t) \text{ is increasing in } t \text{ for which } \xi(t) > 0\},$$

and

$$\mathcal{H}_G^{\xi} = \{\bar{F} : \bar{F}(t) = e^{-c\xi(t)},\ c > 0,\ t \text{ such that } \xi(t) > 0\}.$$

With $\xi(t) = t$, $\mathcal{F}_G^{\xi} = \{\text{IHRA}\}$ and $\mathcal{H}_G^{\xi} = \{\exp\}$. With quite general ξ, the class $\mathcal{F}_G^{\xi}$ was introduced by Saunders (1968).

With $\xi(t) = t^{\alpha}$, $\alpha > 0$, $t \geq 0$, $\mathcal{H}_G^{\xi}$ consists of the Weibull distributions with shape parameter α. Use of this fact has been made by Barlow and Gupta (1969). From (5.7) or directly, we see that the closed classes

$$\mathcal{F}_G^{\xi} = \{\bar{F} : [-\log \bar{F}(t)]/t^{\alpha} \text{ is increasing in } t > 0\}$$

are nested: $\mathcal{F}_G^{t^{\alpha}} \subset \mathcal{F}_G^{t^{\beta}}$ when $\alpha > \beta$.

Of course, Weibull distributions are extreme value distributions for minimums. Other extreme value distributions arise in our context in a similar fashion. If $\xi(t) = e^{\alpha t}$, $\alpha > 0$, $-\infty < t < \infty$, then $\bar{F} \in \mathcal{H}_G^{\xi}$ has the form

$$\bar{F}(t) = \exp[-ce^{\alpha t}],\quad -\infty < t < \infty,\quad c > 0,$$

and if $\xi(t) = (-t)^{-\alpha}$, $t < 0$, $\alpha > 0$, then $\bar{F} \in \mathcal{H}_G^{\xi}$ has the form

$$\bar{F}(t) = \exp[-c(-t)^{\alpha}],\quad t \leq 0,\quad c > 0.$$

Another interesting case is $\xi(t) = e^{\alpha t} - 1$, $\alpha > 0$, $t \geq 0$. Here $\bar{F} \in \mathscr{A}_G^{\xi}$ has the form of a Gompertz distribution

$$\bar{F}(t) = \exp\{-c(e^{\alpha t}-1)\}, \quad t \geq 0, \quad c > 0.$$

More generally, if $\xi(t) = -\log \bar{H}(t) = R_H(t)$ is the hazard function of the survival function $\bar{H}$, then it is easily checked that

$$\mathscr{A}_G^{\xi} = \{\bar{F} : \bar{F}(t) = [\bar{H}(t)]^c \text{ for all } t, \text{ some } c > 0\}$$

5.7 <u>Example</u>. Let

(5.9) $\qquad \bar{G}(t) = 1 - e^{-1/t}, \quad t > 0, \quad$ i.e., $\psi(z) = -(1-z)\log(1-z)$

Closed classes obtained here have the form

$$\mathscr{F}_G^{\xi} = \{\bar{F} : -\xi(t)\log F(t) \text{ is decreasing in } t \text{ for which } \xi(t) > 0\}$$

and the corresponding generating class is

$$\mathscr{A}_G^{\xi} = \{\bar{F} : F(t) = \exp[-b/\xi(t)], \ b > 0, \ t \text{ for which } \xi(t) > 0\}.$$

Here, there are choices for ξ such that $\mathscr{A}_G^{\xi}$ consists of extreme value distributions for maximums:

(i) If $\xi(t) = (-t)^{-\alpha}$, $\alpha > 0$, $t \geq 0$, then

$$\mathscr{A}_G^{\xi} = \{\bar{F} : F(t) = \exp[-b(-t)^{\alpha}], \ t \leq 0, \ b > 0\}.$$

(ii) If $\xi(t) = e^t$, $-\infty < t < \infty$, then

$$\mathscr{A}_G^{\xi} = \{\bar{F} : F(t) = \exp[-be^{-t}], \ -\infty < t < \infty, \ b > 0\}.$$

(iii) If $\xi(t) = t^{\alpha}$, $\alpha > 0$, $t \geq 0$, then

$$\mathcal{M}_G^{\xi} = \{\bar{F} : F(t) = \exp[-bt^{-\alpha}], \quad t \geq 0, \quad b > 0\}.$$

With $\xi(t) = -1/\log H(t)$,

$$\mathcal{M}_G^{\xi} = \{\bar{F} : F(t) = [H(t)]^b \text{ for all } t, \text{ some } b > 0\}.$$

5.8 __Example.__ Let

(5.10) $\bar{G}(t) = 1/(1+t)$, $t \geq 0$, i.e., $\psi(z) = z(1-z)$.

Here, closed classes $\mathcal{F}_G^{\xi}$ have the form

$$\mathcal{F}_G^{\xi} = \{\bar{F} : F(t)/\bar{F}(t)\xi(t) \text{ is increasing in } t \text{ such that } \xi(t) > 0\},$$

and the corresponding generating class is

$$\mathcal{M}_G^{\xi} = \{\bar{F} : \bar{F}(t) = 1/[1+c\xi(t)], \quad c > 0\}.$$

An interesting special case is $\xi(t) = e^t$, $-\infty < t < \infty$, in which case $\mathcal{M}_G^{\xi}$ is a family of logistic distributions. If $\xi(t) = t^{\alpha}$, then $\bar{F}$ in $\mathcal{M}_G^{\xi}$ is of the form $\bar{F}(t) = 1/[1+ct^{\alpha}]$; this is the survival function of a ratio of two variables which have Weibull distributions, each with shape parameter α.

It is not difficult to verify that the closed class $\mathcal{F}_G^{\xi}$ with $\psi(x) = x(1-x)$ contains both $\mathcal{F}_G^{\xi}$ with $\psi(x) = -x \log x$ and $\mathcal{F}_G^{\xi}$ with $\psi(x) = -(1-x)\log(1-x)$.

Early in this section, we asked what survival functions $\bar{G}$ can replace the exponential e^{-t}, $t \geq 0$ in the result $\{\exp\}^{CS,LD} = \{IHRA\}$. We have given a sufficient condition ($\bar{G} \in \mathcal{Q}$) and some examples; it may also be worth showing by example that some conditions are indeed necessary.

5.9 <u>Example</u>. Let $\bar{G}(t) = \sqrt{1-t}$, $0 \leq t \leq \frac{1}{2}$, $\bar{G}(t) = \exp[-2 \log \sqrt{2}\ t]$, $t \geq \frac{1}{2}$. Then $\bar{G}^{-1}(u) = 1-u^2$, $1/\sqrt{2} \leq u \leq 1$, $\bar{G}^{-1}(u) = -\log u/2 \log \sqrt{2}$, $0 \leq u \leq 1/\sqrt{2}$. Let

$$\mathcal{M}_G = \{\bar{F} : \bar{G}^{-1}\bar{F}(at) = a\bar{G}^{-1}\bar{F}(t),\ 0 \leq a \leq 1 \text{ and } t \geq 0\},$$

$$\mathcal{F}_G = \{\bar{F} : \bar{G}^{-1}\bar{F}(at) \leq a\bar{G}^{-1}\bar{F}(t),\ 0 \leq a \leq 1 \text{ and } t \geq 0\}.$$

Then, with the reliability function $h(p_1, p_2) = p_1 p_2$, we see that $\bar{F}(t) = \bar{G}^2(t) \in \mathcal{M}_G^{CS}$. But for $t < \frac{1}{2}$, $t^{-1}\bar{G}^{-1}\bar{G}^2(t) = 2 - t^2$ is decreasing in t, so that $\bar{G}^2 \notin \mathcal{F}_G$, and thus $\mathcal{M}_G^{CS,LD} \not\subset \mathcal{F}_G$.

6. Further Details

In § 5, we introduced the class $\mathcal{Q}$ of survival functions $\bar{G}$ for which the G-hazard transform η of all coherent systems satisfies

$$\eta(ax) \leq a\eta(x) .$$

In this section, we investigate some properties of the class $\mathcal{C}$ of functions ψ which satisfy the basic inequality (5.4) for all reliability functions h. An implication of Lemma 5.2 is that $\bar{G} \in \mathcal{Q}$ if $\psi(z) = \bar{G}^{-1}(z)\, g\bar{G}^{-1}(z) \in \mathcal{C}$.

Proof of Lemma 5.1

We prove (5.4) for all semi-coherent reliability functions h of order n by induction on n. If n=1, then h(p) = 0, h(p) = 1 or h(p) = p; in each case, (5.4) is trivial in view of (5.3). Suppose then that (5.4) holds for n = m-1 and that h is a reliability function of order m. If $h(\underset{\sim}{p}, 0_m) = h(p_1, \ldots, p_{m-1}, 0)$ and $h(\underset{\sim}{p}, 1_m) = h(p_1, \ldots, p_{m-1}, 1)$ then $h(\underset{\sim}{p}) = p_m\, h(\underset{\sim}{p}, 1_m) + (1-p_m)\, h(\underset{\sim}{p}, 0_m)$. Thus

$$\sum_{i=1}^{m} \psi(p_i)\, \partial h(\underset{\sim}{p})/\partial p_i = \sum_{i=1}^{m-1} \psi(p_i)\, \partial h(\underset{\sim}{p})/\partial p_i + \psi(p_m)\, \partial h(\underset{\sim}{p})/\partial p_i$$

$$= p_m \sum_{i=1}^{m-1} \psi(p_i)\, \partial h(\underset{\sim}{p}, 1_m)/\partial p_i + (1-p_m) \sum_{i=1}^{m-1} \psi(p_i)\, \partial h(\underset{\sim}{p}, 0_m)/\partial p_i + \psi(p_m)\partial h(\underset{\sim}{p})/\partial p_m$$

$$\geq p_m \psi h(\underset{\sim}{p}, 1_m) + (1-p_m)\, \psi h(\underset{\sim}{p}, 0_m) + \psi(p_m)[h(\underset{\sim}{p}, 1_m) - h(\underset{\sim}{p}, 0_m)]$$

$$\geq \psi(p_m h(\underset{\sim}{p}, 1_m) + (1 - p_m)\, h(\underset{\sim}{p}, 0_m)) = \psi h(\underset{\sim}{p}) .$$

The first inequality is from the induction hypothesis; the second follows from (5.2). ||

Proof of Lemma 5.2

Observe first that (5.6) is equivalent to $\eta(a\underset{\sim}{x})/a$ increasing in $a \epsilon (0,1]$ whenever each $x_i \geq 0$. Thus,

$$\sum z_i \frac{\partial \eta(\underset{\sim}{z})}{\partial z_i} \geq \eta(\underset{\sim}{z}), \quad \text{if each } z_i > 0 \ . \tag{6.1}$$

We compute

$$\frac{\partial \eta(z)}{\partial z_i} = -\bar{G}^{-1'}\ h(\bar{G}(z_1),\ldots\bar{G}(z_n)) \frac{\partial h(\underset{\sim}{u})}{\partial u_i}\bigg|_{\underset{\sim}{u}=(\bar{G}(z_1),\ldots,\bar{G}(z_n))} g(z_i) \ .$$

Since $-\bar{G}^{-1'}(p) = 1/[g\bar{G}^{-1}(p)] > 0$, (6.1) is equivalent to

$$\sum z_i g(z_i) \frac{\partial h(\underset{\sim}{u})}{\partial u_i}\bigg|_{\underset{\sim}{u}=(\bar{G}(z_1),\ldots,\bar{G}(z_n))} \geq \bar{G}^{-1'} h(\underset{\sim}{u}) \cdot g\bar{G}^{-1} h(\underset{\sim}{u})\bigg|_{\underset{\sim}{u}=(\bar{G}(z_1),\ldots,\bar{G}(z_n))} \ .$$

With $\bar{G}(z_i) = p_i$ and $\psi(p) = \bar{G}^{-1}(p) \cdot g\bar{G}^{-1}(p)$, this inequality becomes (5.4).||

6.1 Proposition. If ψ is continuous, then equality holds in (5.4) identically in $\underset{\sim}{p}$ if and only if one of the following conditions holds:

(i) $\psi(u) = -\, c u \log u$ for some $c > 0$ and

$$h(\underset{\sim}{p}) = \prod_{i \epsilon E} p_i \ ;$$

(ii) $\psi(u) = -\, c(1-u)\log(1-u)$ for some $c > 0$ and

$$h(\underset{\sim}{p}) = 1 - \prod_{i \epsilon E}(1-p_i);$$

(iii) $\psi(u) = 0, \quad 0 \leq u \leq 1$

(iv) $h(\underset{\sim}{p}) = 1$ for all $\underset{\sim}{p}$, or $h(\underset{\sim}{p}) = 0$ for all $\underset{\sim}{p}$;

(v) $n = 1$.

The set E indexes the essential components of the structure, i.e., the components upon which $\phi(\underset{\sim}{x})$ and $h(\underset{\sim}{p})$ truly depend. From a practical viewpoint there is no loss in assuming that all components are essential, and with this assumption, the statement of Proposition 6.1 is somewhat simplified.

A result related to Proposition 6.1 has been given by Esary, Marshall and Proschan (1970, Theorem 4.1). There, it is assumed that $\psi(p) = -p\log p$, and that equality holds in (5.6) for some triplet a_1, a_2 and $a_1 + a_2$; the conclusion is that h has the form (i) of Proposition 6.1.

Proof of Proposition 6.1. It is easily checked that (5.4) becomes an equality whenever one of these conditions is satisfied (case (iv) requires the result $\psi \in \mathcal{C}$ implies $\psi(0) = \psi(1) = 0$ that we prove below in Proposition 6.6).

Suppose that equality holds in (5.4) for all $\underset{\sim}{p}$, and that none of the conditions (iii), (iv) or (v) hold. In order to prove that this implies (i) or (ii), we suppose in addition that h is the reliability function of a coherent system with a minimal path set P and a minimal cut set K both of size at least 2, and show that this leads to a contradiction. (Minimal path and cut sets are discussed by Birnbaum, Esary and Saunders (1961), and by Esary and Proschan (1963)).

If P is a minimal path set,

$$h(\underset{\sim}{p}) = \prod_{i\epsilon P} p_i \quad \text{whenever} \quad p_i = 0, \quad i \notin P.$$

Equality in (5.4) then yields

$$\Sigma_{i\epsilon P}\psi(p_i)\prod_{j\epsilon P, j\neq i} p_j = \psi(\prod_{i\epsilon P} p_i) \quad \text{if} \quad p_i = 0, \quad i \notin P .$$

If P contains at least two elements, the transformation $\theta(u) = \psi(e^u)/e^u$ converts this functional equation to Cauchy's equation

$$\Sigma_{i\epsilon P}\, \theta(u_i) = \theta(\Sigma_{i\epsilon P}\, u_i), \quad u_i \leq 0, \quad i\epsilon P.$$

Since we have assumed that ψ is continuous, it follows that θ is continuous so that $\theta(u) = au$, i.e., $\psi(u) = -cu\log u$. Since we have assumed (iii) does not hold, and since $-cu\log u$ fails to satisfy (5.3) unless $c \geq 0$, we conclude that $c > 0$.

If K is a minimal cut set,

$$h(\underset{\sim}{p}) = 1 - \prod_{i\epsilon K}(1-p_i) \quad \text{whenever} \quad p_i = 1, \quad i \notin K,$$

and equality in (5.4) yields

$$\sum_{i\in K}\psi(p_i)\prod_{j\in K, j\neq i}(1-p_j) = (1-\prod_{i\in K}(1-p_i)) \quad \text{if} \quad p_i = 1, \; i\notin K.$$

This functional equation can also be transformed to Cauchy's equation and we can conclude that $\psi(u) = -c(1-u)\log(1-u)$, $c > 0$.

These two definitions of ψ cannot be reconciled, and we conclude that either all minimal path sets are of size 1 ($h(\underset{\sim}{p}) = 1 - \prod_{i\in E}(1-p_i)$), or all minimal cut sets are of size 1 ($h(\underset{\sim}{p}) = \prod_{i\in E} p_i$). ||

There are several interesting properties of the class $\mathcal{C}$ of functions ψ satisfying (5.4), which we enumerate in the following propositions.

6.2 Proposition. If $\psi \in \mathcal{C}$, then $\psi(0) = \psi(1) = 0$.

Proof. From (5.4) with $h(p) = p_1p_2$ and $h(p) = 1 - (1-p_1)(1-p_2)$ we have $p_1\psi(p_2) + p_2\psi(p_1) \geq \psi(p_1p_2)$ and $(1-p_2)\psi(p_1) + (1-p_1)\psi(p_2) \geq \psi(p_1 + p_2 - p_1p_2)$. With $p_1 = p_2 = 0$, it follows that $0 \geq \psi(0)$ and $2\psi(0) \geq 0$. Thus $\psi(0) = 0$, and $\psi^D(0) = \psi(1) = 0$. ||

6.3 Proposition. $\mathcal{C}$ is a pointed (proper) convex cone.

Proof. It is easily seen that $\mathcal{C}$ is a convex cone, i.e., ψ_1 and $\psi_2 \in \mathcal{C}$ implies $c_1\psi_1 + c_2\psi_2 \in \mathcal{C}$ whenever $c_1 \geq 0$, $c_2 \geq 0$. That $\mathcal{C}$ is pointed (i.e., $\psi \in \mathcal{C}$, $-\psi \in \mathcal{C}$ implies $\psi = 0$) follows from Proposition 6.1, since $\psi \in \mathcal{C}$ and $-\psi \in \mathcal{C}$ implies that equality holds in (5.4) for all reliability functions h, and this implies $\psi(u) = 0$. ||

6.4 Proposition. $\mathcal{C}$ is closed under the formation of maximums.

Proof. If $\psi_a \in \mathcal{C}$, $a \in A$, then since $\partial h(\underset{\sim}{p})/\partial p_i \geq 0$,

$$\sum_{i=1}^{n} \max_{a\in A} \psi_a(p_i)\, \partial h(\underset{\sim}{p})/\partial p_i \geq \sum_{i=1}^{n} \psi_a(p_i)\, \partial h(\underset{\sim}{p})/\partial p_i \geq \psi_a h(\underset{\sim}{p})$$

for all $a \epsilon A$. Hence

$$\sum_{i=1}^{n} \max_{a \epsilon A} \psi_a(p_i)\, \partial h(\underset{\sim}{p})/\partial p_i \geq \max_{a \epsilon A} \psi_a h(\underset{\sim}{p}). \quad \|$$

6.5 <u>Proposition.</u> If $\psi \epsilon \mathcal{C}$, then $\psi^D \epsilon \mathcal{C}$, where $\psi^D(u) = \psi(1-u)$.

<u>Proof.</u> If h is the reliability function of a coherent structure ϕ and $h^D(\underset{\sim}{p}) = 1 - h(1-p_1,\ldots,1-p_n)$, then h^D is the reliability function of the coherent structure ϕ^D defined by $\phi^D(\underset{\sim}{x}) = 1 - \phi(1-x_1,\ldots,1-x_n)$. Since $\partial h^D(\underset{\sim}{p})/\partial p_i = \partial h(1-p_1,\ldots,1-p_n)/\partial p_i$, we have $\sum_{i=1}^{n} \psi^D(p_i)\partial h^D(\underset{\sim}{p})/\partial p_i =$
$= \sum_{i=1}^{n} \psi(1-p_i)\, \partial h(1-p_1,\ldots,1-p_n)/\partial p_i \geq \psi h(1-p_1,\ldots,1-p_n) = \psi^D h^D(\underset{\sim}{p})$. $\|$

It is easily seen that Propositions 6.3, 6.4 and 6.5 remain true if $\mathcal{C}$ is replaced by the class of continuous functions satisfying (5.4), for all reliability functions h. Also, these propositions are true if $\mathcal{C}$ is replaced by the class of functions satisfying (5.2), or (5.2) and (5.3). The question of whether there exist discontinuous functions in $\mathcal{C}$ has not been resolved.

6.6 <u>Proposition.</u> If $\psi \epsilon \mathcal{C}$ is continuous on (0,1), then $\psi(u) \geq 0$, $0 \leq u \leq 1$.

<u>Proof.</u> A sequence of "k out of n" systems ($\phi_{k,n}(\underset{\sim}{x}) = 1$ if $S_n(\underset{\sim}{x}) = \sum_{i=1}^{n} x_i \geq k$, $\phi_{k,n}(\underset{\sim}{x}) = 0$ if $S_n(\underset{\sim}{x}) < k$) chosen so that $n \to \infty$ while $k/n \to \theta$, $0 \leq \theta \leq 1$ has the property that

$$\begin{aligned} h_{k,n}(p,\ldots,p) = P\{S_n(X_1,\ldots,X_n)/n \geq k/n\} &\to 1 \text{ if } p \geq \theta \\ &\to 0 \text{ if } p < \theta , \end{aligned}$$

where X_i are independent random variables such that $P\{X_i = 1\} = p$, $P\{X_i = 0\} = 1 - p$. Thus, the set of points $\theta \epsilon (0,1)$ such that

$$h_{k,n}(\theta,\ldots,\theta) = \theta \text{ for some } k,n$$

is dense in [0,1]. Moreover for such a point θ,

$$\sum_{j=1}^{n} \partial h_{k,n}(p_1,\ldots,p_n)/\partial p_j \Big|_{p_1=\ldots=p_n=\theta} = dh_{k,n}(\theta,\ldots,\theta)/d\theta > 1 ,$$

so that $\psi(\theta) < 0$ means

$$\sum_{j=1}^{n} \psi(\theta) \frac{\partial h(\underset{\vee}{p})}{\partial p_j}\Bigg|_{p_1=\ldots=p_n=\theta} < \psi(\theta) = \psi h(\theta,\ldots,\theta) .$$

This contradicts (5.4). Consequently the set of points θ such that $\psi(\theta) \geq 0$ is dense in [0,1], so the proof is complete by continuity of ψ. ||

We wish to correct an error in an earlier paper (Birnbaum, Esary and Marshall (1966)). There, the statement is made (p. 820) that $\psi \in \mathcal{C}$ implies ψ is concave. That this is false can be seen by taking $\psi = \max(\psi_1, \psi_2)$, where $\psi_1(u) = -u\log u$ and $\psi_2(u) = \psi_1^D(u)$; $\psi \in \mathcal{C}$ by Proposition 6.3, but ψ is <u>not</u> concave. It is true, however, that $\psi \in \mathcal{C}$ implies $\psi(p_1) - \psi(p_1p_2) \geq \psi(p_1+p_2-p_1p_2) - \psi(p_2)$. This is a concavity-like property, since $p_1 \leq p_1+p_2-p_1p_2$ and $p_1-p_1p_2 = (p_1+p_2-p_1p_2) - p_2$.

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It has been shown by Birnbaum, Esary and Marshall that the class of survival functions with increasing hazard rate average (IHRA) is closed under the formation of coherent systems. Moreover, this is the smallest class of survival functions which is closed both under the formation of coherent system and limits in distribution, and which contains the exponential survival functions. In this paper a number of other classes are found which are closed under the formation of coherent systems and limits in distribution. Associated subclasses that play a generating role like the exponential class in the IHRA case are exhibited. In addition, several methods are presented for deriving closed classes from closed classes.

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