AN INTRODUCTION TO GREEN'S FUNCTIONS

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Preface

These notes are intended for students with no previous experience with Green's functions. A knowledge of applied mathematics at an advanced undergraduate level is assumed; for example, it is assumed that the reader will be familiar with contour integration and with expansions of functions of several variables in the usual sets of orthogonal functions. A knowledge of integral transform methods would also be useful. However, a brief introduction to the Fourier and Laplace transforms is given in one of the appendices. Since a wide variety of conventions is in use this also serves to define the conventions and notations used in the text.

This work was sponsored by the Institute of Ocean Science and Engineering at The Catholic University of America for use by workers in acoustics. This interest is reflected in the chapter on the Helmholtz equation (Chapter 5) where the applications are all chosen from acoustics. For readers with an interest in this field but with no previous knowledge of Green's functions it is suggested that the notes be read from the beginning with the possible exception of the chapter on the diffusion equation (Chapter 3).

The material in its present form is considered to be a preliminary presentation. It was felt that it would be desirable to make this available fairly quickly and to write a second edition after some feedback had been obtained. Consequently, the author will be especially grateful to any readers who will take the time to offer criticisms and corrections.

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CONTENTS

I. INTRODUCTION

We will introduce Green's function by means of a simple example, and in later chapters discuss some particular equations in detail. To explain our choice we first review some general properties of second order linear partial differential equations.

1.1 Classification of partial differential equations

Any discussion of second order equations must begin with their classification as elliptic, parabolic, or hyperbolic, and the prototypes of these categories, the Laplace, diffusion, and wave equations:

$$\nabla^2 \psi = 0$$

$$\nabla^2 \psi - \frac{1}{\alpha^2} \frac{\partial \psi}{\partial t} = 0$$

$$\nabla^2 \psi - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = 0$$

Mathematically, the classification arises from questions of existence, the problem being to determine boundaries and boundary conditions such that existence theorems can be established. We review only the general outlines of this scheme here, and leave more precise statements for later chapters.

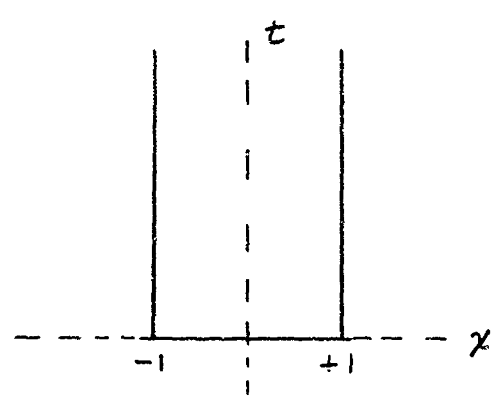
A condition which specifies the values of a function on a boundary is called a Dirichlet condition. If the derivative ($\frac{\partial}{\partial n} \equiv \vec{n} \cdot \nabla$) of the function in a direction $\vec{n}$ normal to the boundary is given the condition is called a Neumann condition. A linear combination of the function and its normal derivative is called a mixed condition, while specifying these independently is called a Cauchy condition.

For an elliptic equation Dirichlet, Neumann, or mixed conditions on a closed boundary* are appropriate. For example, the steady state temperature distribution in a solid satisfies Laplace's equation. A Dirichlet condition would give the temperature on the surface, while a Neumann condition would give the heat flux through the surface. (By Fourier's law the heat flux is given by $\vec{H} = k\nabla T$, k constant. Therefore, the normal component of $\vec{H}$ is $H_n = k\vec{n}\cdot\nabla T = k\ \partial T/\partial n$.)

If a Neumann condition is given it is evident that an additional condition must be imposed, for if the heat flux were everywhere into the body it is clear that a steady state could not be reached and the boundary value problem would have no solution. Therefore, we impose the additional condition that the net heat flux through the surface vanish, i.e.,

$$\int ds \frac{\partial T}{\partial n} = 0$$

For a hyperbolic equation an open boundary is needed. Cauchy conditions are usually appropriate over at least part of the boundary, while Dirichlet, Neumann, or mixed conditions may be given over the remainder. For example, the displacement of a vibrating string, ψ say, satisfies the wave equation. If the string is clamped at $x = \pm 1$ then the boundary is an open rectangle in the x-t plane with Dirichlet conditions ($\psi = 0$) given on $x = \pm 1$, $t > 0$ and Cauchy conditions (the initial values of ψ, $\frac{\partial \psi}{\partial t}$) given on $t = 0$, $-1 \leq x \leq 1$



*A region with a closed boundary is completely enclosed. The boundary may be closed at infinity. If the solution is required to vanish at infinity this is equivalent to imposing a Dirichlet condition on a sphere and taking the limit as its radius becomes infinite. Closed boundary should not be confused with the term closed region, by which we mean that the region includes its boundary.

Parabolic equations represent an intermediate case between the elliptic and the hyperbolic. A Dirichlet condition over at least part of an open boundary is usually appropriate, with Dirichlet, Neumann, or mixed conditions on the remainder. For example, the transient temperature distribution in a rod is governed by the diffusion equation

$$\frac{\partial^2 T}{\partial x^2} - \frac{1}{\alpha^2}\frac{\partial T}{\partial t} = 0, \qquad \alpha \text{ constant.}$$

If the rod lies between $x = \pm 1$, the boundary in the x-t plane is an open rectangle with a Dirichlet condition (initial temperature) given on $t = 0$, $-1 \leq x \leq 1$. On the remainder of the boundary $x = \pm 1$, $t > 0$ a Dirichlet condition (temperature at ends), a Neumann condition (heat flux through ends), or a combination of these may be given.

In addition to existence there are two other requirements for a well posed problem: uniqueness and stability. By stability we mean that the solution depends continuously on the boundary conditions, i.e., a small change in the boundary conditions implies a small change in the solution. The above rough sketch of appropriate boundary conditions has taken these requirements into account. For a discussion of the concept of a well posed problem see Courant and Hilbert (1962, pp. 226-232).

Equations may also be characterized by their effect on the boundary conditions. The elliptic operator is a smoothing operator: any discontinuities in the boundary condition, as well as its general form, are smoothed when continued away from the boundary. In contrast with this type of behavior, the hyperbolic operator has propagation properties: any discontinuities and the general form of the boundary conditions persist, are propagated, when continued away from the boundary. Again, the parabolic equation is an intermediate case. In the example given above the initial condition is smoothed while the boundary conditions are propagated.

Physically, elliptic equations usually describe static phenomena. For example, electrostatics, magnetostatics, steady state temperatures are governed by Laplace's equation. Hyperbolic equations describe dynamic, time dependent, phenomena, e.g., electromagnetic and acoustic radiation. Parabolic equations are often very useful as an approximation, e.g., circuit theory, but their appearance in physics is always suspect. In the example given changes in the boundary conditions are propagated instantaneously, which is decidedly unphysical.

1.2 Introductory example

To introduce the Green's function associated with a second order partial differential equation we begin with the simplest case, Poisson's equation

$$\nabla^2\phi = -4\pi\rho$$

which is simply Laplace's equation with an inhomogeneous, or source, term. A convenient physical model to have in mind is the electrostatic potential $\phi(x,y,z)$ which arises from a source $\rho(x,y,z)$ which is a volume distribution of charge (charge per unit volume). We imagine that a point charge of unit magnitude (unit source) is at a point $\vec{r}_o(x_o,y_o,z_o)$ (source point) and that the potential is measured at a point $\vec{r}(x,y,z)$ (field point). From Coulomb's law the potential is just the reciprocal distance between the two points (Gaussian units are being used). Written as a function of $\vec{r}$ and $\vec{r}_o$ we call this potential the Green's function

$$G(\vec{r},\vec{r}_o) = \frac{1}{|\vec{r}-\vec{r}_o|}$$

In general, a Green's function is just the response or effect due to a unit point source. We also note the symmetry property (reciprocity relation)

$$G(\vec{r},\vec{r}_o) = G(\vec{r}_o,\vec{r})$$

Suppose that there is a charge distribution $\rho(\vec{r})$ in a certain region R of space. The potential at a point $\vec{r}$ inside or outside of R can be written

$$\Phi(\vec{r}) = \int_R d\tau_o \frac{\rho(\vec{r}_o)}{|\vec{r}-\vec{r}_o|} = \int_R d\tau_o\, \rho(\vec{r}_o)\, G(\vec{r},\vec{r}_o)$$

($d\tau_o \equiv dx_o\, dy_o\, dz_o$) This results from simply treating each element of charge $\rho(\vec{r}_o)\, d\tau_o$ as a point charge and taking the sum of the contributions from all elements. The integrand is singular if $\vec{r}$ is in R, but the integral is convergent. Thus, we have obtained a solution of Poisson's equation by means of G.

So far this is intuitively obvious and, therefore, unremarkable. But what comes next is quite remarkable. We now imagine that in addition to the source ρ we also have present certain boundaries on which boundary conditions are specified. Since Poisson's equation is elliptic, we know that, for example, a Dirichlet condition is appropriate. Therefore, let us suppose that the equation holds in a closed regio. R and that the value of Φ is specified as a continuous but otherwise arbitrary function on the surface σ of R. We assume a continuous boundary condition for simplicity in this introductory example; later we will be interested in relaxing this condition.

We anticipate two very surprising and pleasing results. The first is that the definition of G ($\vec{r},\vec{r}_o$), given above in the boundary-free case, can be extended simply and used to obtain a solution of the boundary value problem; the second is that the reciprocity relation G ($\vec{r},\vec{r}_o$) = G ($\vec{r}_o,\vec{r}$) continues to hold for the Green's function so defined.

We have defined G in the boundary-free case as the response to a unit point source. This implies that the equation

$$\nabla^2 G(\vec{r},\vec{r}_o) = -4\pi\, \delta(\vec{r}-\vec{r}_o) \qquad (1)$$

is satisfied, and we assume that it continues to be satisfied when boundaries are present. This will be part of the definition of G. The source term, our unit point source, is a delta function

$$\delta(\vec{r}-\vec{r}_o) \equiv \delta(x-x_o)\delta(y-y_o)\delta(z-z_o)$$

In our electrostatic model $\delta(\vec{r}-\vec{r}_o)$ is the volume distribution $\rho(\vec{r})$ of a point charge at $\vec{r}_o$. We shall use delta functions freely as the simplest formal device for manipulating Green's functions. An introduction to the formalism of delta functions is given in Appendix I. We recall the formal rules

$$\delta(x-x_o) = 0 \qquad x \neq x_o$$

$$\int_{-\infty}^{\infty} f(x)\ \delta(x-x_o)dx = f(x_o)$$

where f (x) is a continuous function. The latter is the so-called "sifting property". Thus, we may say loosely that $\delta(\vec{r}-\vec{r}_o)$ is zero except at $\vec{r} = \vec{r}_o$ where it is infinite in such a way that

$$\int d\vec{r}_o\ \delta(\vec{r}-\vec{r}_o) = 1$$

and hence, represents a unit source.

We have emphasized the word formal because within the context of the classical theory of real variables there is no function with these properties. A function is defined only if it has a definite value for each point in its range (infinity is not a definite value). Therefore, to make the use of these functions rigorous a broader mathematical context is needed. The best known of these is the theory of distributions. An admirable introduction to this theory is the book of Lighthill (1958).

We shall not be concerned with justifying the use of delta functions because all of our results can be obtained more laboriously without their use.* Moreover, we regard the delta function not only as a convenient shortcut, but also as supplying useful insight because of its interpretation as a unit point source.

Of central importance in our work will be Green's theorem

$$\int_R d\tau \,\{U\nabla^2 V - V\nabla^2 U\} = \int_\sigma ds \,\{U \frac{\partial V}{\partial n} - V \frac{\partial U}{\partial n}\}$$

which is valid if U and V are continuous with continuous partial derivatives up to the second order in the closed region R. $\vec{n}$ is the unit outward normal to R and $\frac{\partial}{\partial n} \equiv \vec{n}\cdot\nabla$ is the derivative in this direction. This follows from the divergence theorem (Gauss' theorem)

$$\int_R d\tau \;\; \mathrm{div}\, \vec{F} = \int_\sigma \vec{n}\cdot\vec{F}\, ds$$

by substituting

$$\vec{F} = U\nabla V$$

and expanding

$$\mathrm{div}\,(U\nabla V) = \nabla U\cdot\nabla V + U\nabla^2 V$$

by the well known vector identity, interchanging U and V, and subtracting.

Returning now to our boundary value problem we write

$$\nabla_o^2 \phi = -4\pi\, \rho(\vec{r}_o) \tag{2}$$

$$\nabla_o^2\, G(\vec{r},\vec{r}_o) = -4\pi\, \delta(\vec{r}-\vec{r}_o) \tag{3}$$

where

$$\nabla_o^2 \equiv \frac{\partial^2}{\partial x_o^2} + \frac{\partial^2}{\partial y_o^2} + \frac{\partial^2}{\partial z_o^2}$$

*As an example of this approach we cite the book of Sommerfeld (1949); concise, rigorous proofs of the theorems needed are given by Titchmarsh (1958, chap. 21).

In the first equation we have simply replaced $\vec{r}$ by $\vec{r}_o$. The second follows from eq. (1), the symmetry of the Green's function, and the fact that the delta function is even,

$$\delta(x-x_o) = \delta(x_o-x)$$

However, since we have not fully defined, let alone proved the symmetry, of Green's function, we might do better to simply interchange the roles of $\vec{r}$ and $\vec{r}_o$ in our interpretation of the G in eq. (3) and regard it as the response at $\vec{r}_o$ to a source at $\vec{r}$, Really, an interpretation at this point is not necessary, since what follows is simply formal manipulation. We could just as easily take eq. (3) as part of the definition of G and later prove the reciprocity relation and derive eq. (1).

Multiplying eq. (2) by G, eq. (3) by Φ, integrating and subtracting we obtain

$$\int_R d\tau_o \, \{G\nabla_o^2\Phi - \Phi\nabla_o^2 G + 4\pi \, G\rho(\vec{r}_o) - 4\pi\Phi \, \delta(\vec{r}-\vec{r}_o)\} = 0$$

By Green's theorem

$$\int_R d\tau_o \, \{G\nabla_o^2\Phi - \Phi\nabla_o^2 G\} = \int_\sigma ds_o \, \{G \frac{\partial\Phi}{\partial n_o} - \Phi \frac{\partial G}{\partial n_o}\}$$

and by the sifting property

$$\int d\tau_o \, \Phi(\vec{r}_o) \, \delta(\vec{r}-\vec{r}_o) = \Phi(\vec{r})$$

Therefore, we have

$$\Phi(\vec{r}) = \int d\tau_o \, \rho(\vec{r}_o) G(\vec{r},\vec{r}_o) + \frac{1}{4\pi} \int ds_o \, \{G \frac{\partial\Phi}{\partial n_o} - \Phi \frac{\partial G}{\partial n_o}\} \quad (4)$$

which expresses Φ in terms of its values and the values of its normal derivative on the surface. This is not a solution of the boundary value problem because both Φ and $\partial\Phi/\partial n$ are not given on the surface. However, if we can impose a further condition on G, viz., that it vanish on σ, we can eliminate the term involving $\partial\Phi/\partial n_o$. This additional condition completes the definition of G. We would then have

$$\Phi(\vec{r}) = \int_R d\tau_o\ \rho(\vec{r}_o)G(\vec{r},\vec{r}_o) - \frac{1}{4\pi}\int_\sigma ds_o\ \Phi(\vec{r}_o)\frac{\partial G}{\partial n_o} \qquad (5)$$

which expresses Φ in terms of its boundary values. We have obtained this relation assuming that Φ is a solution of Poisson's equation. We may now ask if we substitute for Φ in the surface integral an arbitrary continuous function does this relation give a solution of the boundary value problem? It does if a solution exists, for if it exists it must satisfy this relation. It is important for us to be aware of the appropriate existence theorems. We may conclude that eq. (5) gives a solution of the Dirichlet problem because there is an existence theorem (cf.chap. 2), but eq. (4) does not give a solution of the Cauchy problem for Poisson's equation because, in general, a solution does not exist.

We have reduced the original problem to the simpler problem of finding the Green's function, i.e., of finding the potential of a point change subject to the condition of zero potential on the boundaries (grounded boundaries). There are two general methods of constructing Green's functions, which we will explore in the next chapter. They are the method of expansion in eigenfunctions and the method of reflection or imaging. The latter method works only for some rather special geometries, but when it is available it provides a simple analytical expression for the solution, with obvious advantages over an infinite series.

Having interpreted G as the potential of a point charge in the presence of grounded conductors, we can illustrate the reciprocity relation $G(\vec{r},\vec{r}_0) = G(\vec{r}_0,\vec{r})$ as follows. Imagine that a point charge is placed at $\vec{r}_0$ near a grounded conductor and that the potential is measured at $\vec{r}$. This potential is due to the point charge and a surface distribution of charge on the conductor which is brought up from the earth and distributed so as to maintain the potential of the conductor at zero. If now the point charge is moved to $\vec{r}$ and the potential measured at $\vec{r}_0$, the surface charge on the conductor will be rearranged so that the two measurements are identical.

To summarize, the solution of Dirichlet's problem for Poisson's equation is given by

$$\Phi(\vec{r}) = \int_R d\tau_0 \, \rho(\vec{r}_0) G(\vec{r},\vec{r}_0) - \frac{1}{4\pi} \int_\sigma ds_0 \, \Phi(\vec{r}_0) \frac{\partial G}{\partial n_0}$$

where the Green's function is defined by

$$\nabla^2 G = -4\pi \, \delta(\vec{r}-\vec{r}_0)$$

with the condition that G vanish on the boundary. We emphasize that a Green's function is defined not only with respect to an equation and its boundary conditions, but also with respect to a particular region.

II. LAPLACE'S EQUATION

The theory of Laplace's equation is usually called potential theory. The classic work is Kellogg's "Foundations of Potential Theory", where rigorous proofs of the following existence theorems may be found (chap. 11).

2.1 Existence theorems

By an interior problem we shall mean the problem of finding a solution of Laplace's equation, $\nabla^2\Phi = 0$, within a closed finite region R (the region includes its surface σ) which satisfies boundary conditions on σ. By an exterior problem we mean the problem of finding a solution in the infinite region outside of a surface subject to the additional condition that Φ vanish at least as fast as r^{-1} as $r \to \infty$. This condition is needed for uniqueness. Physically, it means that at sufficiently large distances the source distribution should look like a point source. We have the following theorems.

There are unique solutions of the interior and exterior Dirichlet problems and of the exterior Neumann problem for continuous boundary values.

There is a solution of the interior Neumann problem, unique to within an additive constant, for continuous boundary values provided

$$\int_\sigma ds \frac{\partial}{\partial n}\Phi = 0 \quad .$$

To give rigorous proofs of these theorems one must impose certain conditions on the shape of the region R. For a thorough discussion Kellogg's book may be consulted. The conditions are very weak from the point of view of physical applications and we may assume that physical boundaries are always sufficiently regular.

The theorems apply to Poisson's equation if the source term is piecewise differentiable. (This is not the weakest possible condition, but it is a convenient one.)

2.2 Method of reflection

We consider the Dirichlet problem first. We have seen that there is a solution

$$\phi = \int_R d\tau_o\, \rho(\vec{r}_o) G(\vec{r},\vec{r}_o) - \frac{1}{4\pi}\int_\sigma ds_o\, \phi(\vec{r}_o)\frac{\partial G}{\partial n_o}$$

of Poisson's equation provided we can find a Green's function which satisfies

$$\nabla^2 G = -4\pi\, \delta(\vec{r}-\vec{r}_o)$$

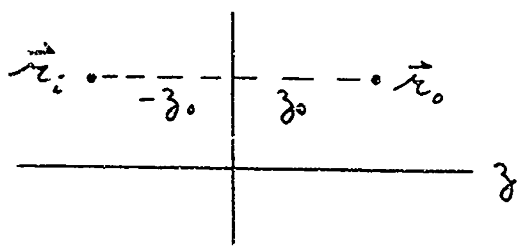
and vanishes on the boundary. For exterior problems we add the condition that G vanish as r^{-1} for large r in order to satisfy the conditions of the existence theorem. A particular solution is $|\vec{r}-\vec{r}_o|^{-1}$ to which we may add any solution of the homogeneous equation, F say,

$$G = \frac{1}{|\vec{r}-\vec{r}_o|} + F(\vec{r})$$

and hope to choose F so that the boundary condition is satisfied.

The method of reflection consists of choosing for F solutions of Poisson's equation corresponding to point sources at positions outside of the region of interest R. Within R these are solutions of the homogeneous equation, $\nabla^2 G = 0$.

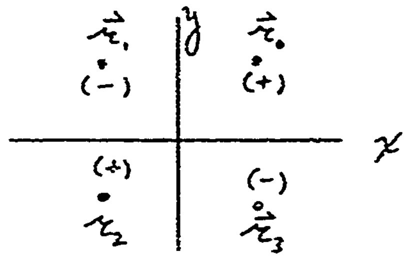
For example, suppose that Φ is given on the plane z = o and we wish a solution for the half-space z >o. G can be written as the potential of a positive point change at $\vec{r}_o$ (x_o,y_o,z_o) plus the potential of a negative point change at an "image point" $\vec{r}_1$ $(x_o,y_o,-z_o)$ obtained by reflecting $\vec{r}_o$ in the plane z = o.



$$G(\vec{r},\vec{r}_o) = \frac{1}{|\vec{r}-\vec{r}_o|} - \frac{1}{|\vec{r}-\vec{r}_1|}$$
$$= \{(x-x_o)^2 + (y-y_o)^2 + (z-z_o)^2\}^{-1/2}$$
$$- \{(x-x_o)^2 + (y-y_o)^2 + (z+z_o)^2\}^{-1/2}$$

The boundary condition is obviously satisfied on $z = o$.

This trick is well known in electrostatics and can be used for other regions bounded by planes. For example, consider the quarter-space $x > 0$, $y > 0$. We place image sources outside this region at



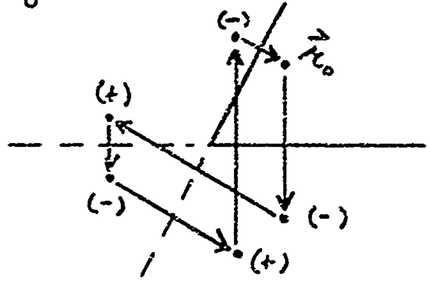
(−) $\vec{r}_1(-x_o, y_o, z_o)$,

(+) $\vec{r}_2(-x_o, -y_o, z_o)$, and

(−) $\vec{r}_3(x_o, -y_o, z_o)$.

The signs of the sources are indicated. It is easily verified that the boundary condition is satisfied on both of the boundary planes.

We can think of this as a process of successive imaging, each reflection involving a sign change, until we return to $\vec{r}_o$. As a further example (Sommerfeld, p.80) consider a 60° wedge. There are five images, formed by successive reflections as shown in the figure. Obviously, the method is useful only if there is a finite number of images, i.e., if we do eventually return to the starting point $\vec{r}_o$. For a wedge this will be so if the wedge angle is π times a rational number.



2.3 Neumann's problem

An obvious starting point is eq. (1.4)

$$\phi(\vec{r}) = \int d\tau_o\, \rho(\vec{r}_o)G(\vec{r},\vec{r}_o) + \frac{1}{4\pi}\int ds_o\, \{G\frac{\partial\phi}{\partial n_o} - \phi\frac{\partial G}{\partial n_o}\}$$

Our first thought would be to proceed as we did for Dirichlet's problem and impose a condition on G, this time requiring that $\partial G/\partial n = 0$ on the boundary. However, for the interior Neumann problem this is not possible. We can see this as follows.

In Green's theorem let $U = 1$, $V = G$.

$$\int ds \frac{\partial G}{\partial n} = \int d\tau\, \nabla^2 G = -4\pi \int d\tau\, \delta(\vec{r}-\vec{r}_0) = -4\pi$$

Obviously, $\partial G/\partial n$ cannot vanish everywhere on the surface. However, we can set

$$\partial G/\partial n = \text{const.} = -4\pi/\Sigma$$

on the surface, where Σ is the total surface area. Then from eq. (1.4) we have the solution

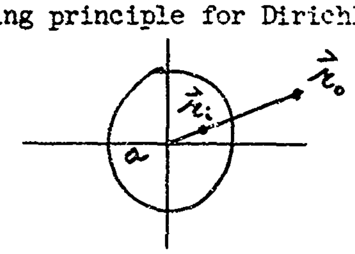
$$\phi(\vec{r}) = \int d\tau_0\, \rho(\vec{r}_0) G(\vec{r},\vec{r}_0) + \frac{1}{4\pi} \int ds_0\, G \frac{\partial \phi}{\partial n_0} + \text{const.}$$

which reflects the fact that uniqueness is proved only to within an additive constant for the interior Neumann problem.

The method of reflection in a plane works for Neumann's problem if it works for Dirichlet's problem. The only difference is that there is no sign change under reflection.

2.4 <u>Reflection in a sphere</u>

For a spherical boundary there is an imaging principle for Dirichlet's problem but not for Neumann's. The image $\vec{r}_1$ is on the same radius as the unit source $\vec{r}_0$. Let $r_1 r_0 = a^2$



where a is the radius of the sphere and let the strength of the image source be $-a/r_0$.

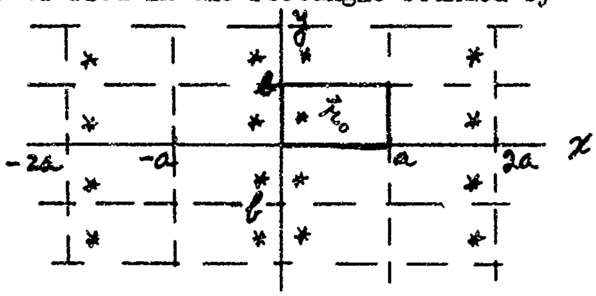
Then

$$G = \frac{1}{|\vec{r}-\vec{r}_0|} - \frac{a/r_0}{|\vec{r} - (\frac{a}{r_0})^2 \vec{r}_0|}$$

which is valid inside or outside of the sphere.

2.5 Expansion in orthogonal functions

The method of reflection may produce an infinite number of images. As an example, suppose Laplace's equation is satisfied in the rectangle bounded by $x = o,a$ and $y = 0,b$.



To satisfy the condition on $x=o$ we reflect in this line. A reflection in $x=a$ satisfies the condition on this line but now the condition on $x=o$ is no longer satisfied so the second image is reflected in $x=o$, etc. In this way we completely fill the plane with images. For detailed examples of this approach see Courant and Hilbert (1953, pp. 378-386).

A more reasonable approach is to satisfy the boundary conditions at the outset by attempting an expansion in orthogonal functions each of which satisfies the conditions. These are obtained by the separation of variables method. For the present example they are sines and the expansion

$$G = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

is a double Fourier sine series.

Substituting into the equation

$$\nabla^2 G = -2\pi\, \delta(x-x_o)\delta(y-y_o)$$

which we adopt for G in two dimensions, and using the orthogonality relation

$$\int_0^a \sin \frac{m\pi x}{a} \sin \frac{m'\pi x}{a}\, dx = \frac{a}{2}\, \delta_{mm'}$$

and the sifting property to determine the coefficients Amn we obtain

$$G = \frac{8}{\pi ab} \sum_{m,n} \frac{\sin m\pi x/a \sin m\pi x_o/a \sin n\pi y/b \sin n\pi y_o/b}{(m/a)^2 + (n/b)^2} \tag{1}$$

Since we know that G is singular at $x = x_o$, $y = y_o$, we are inclined to ask in what sense it can be represented by an ordinary Fourier series. Justification can be found in the theory of distributions or generalized functions (Lighthill, 1958). We only remark that as the delta function has a well known representation as a Fourier integral

$$\delta(x-x_o) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ik(x-x_o)}$$

so a row of delta functions (an infinite array of sources) can be represented by a Fourier series, e.g.,

$$\sum_{m=-\infty}^{\infty} (-1)^m \delta(x-x_o-ma) = \frac{2}{a} \sum_{n=0}^{\infty} \sin \frac{n\pi x_o}{a} \sin \frac{n\pi x}{a} \tag{2}$$

(cf. Lighthill, chap. 5)

It may also occur to the reader to wonder how the expanded G when substituted into*

$$\Phi = -\frac{1}{2\pi} \oint ds_o\, \Phi \frac{\partial G}{\partial n_o} \tag{3}$$

can give a solution which satisfies the boundary condition when each term in the expansion vanishes on the boundary.

*We have adopted the convention of writing Poisson's equation in two dimensions in the form $\nabla^2\Phi = -2\pi\rho$. Therefore, in this formula we have a factor of 2π instead of 4π and a line integral in place of the surface integral.

The answer is that the boundary must be approached from inside the region. To take a simple example, suppose the potential is f (x) on y = o $0 \leq x \leq a$ and zero on the rest of the rectangle. Then eq. (3) becomes

$$\Phi = \frac{1}{2\pi} \int_0^a dx_o \ f(x_o) \ [\frac{\partial G}{\partial y_o}]_{y_o=0} \tag{4}$$

(the outward normal is in the - y_o direction).

From eq. (1),

$$[\frac{\partial G}{\partial y_o}]_{y_o=0} = \frac{8}{a} \sum_m \sin \frac{m\pi x}{a} \sin \frac{m\pi x_o}{a} \sum_n \frac{n \sin n\pi y/b}{n^2 + (\frac{nb}{a})^2} \tag{5}$$

It can be verified that the expansion

$$\frac{\pi}{2} \frac{\sinh k\pi(1-y/b)}{\sinh k\pi} = \sum_n \frac{n \sin n\pi y/b}{n^2 + k^2}$$

is valid for $0 < y < b$. Hence, as $y \to 0+$ the inner sum in eq. (5) approaches $\pi/2$. Therefore,

$$\begin{aligned} [\frac{\partial G}{\partial y_o}]_{\substack{y_o=0 \\ y\to 0+}} &= \frac{4\pi}{a} \sum_m \sin \frac{m\pi x}{a} \sin \frac{m\pi x_o}{a} \\ &= 2\pi \ \delta(x-x_o) \end{aligned}$$

from eq. (2). Substituting into eq. (4) we see that the boundary condition is satisfied.

$$\Phi(x,y\to 0+) = \int dx_o \ f(x_o)\delta(x-x_o) = f(x)$$

2.6 Discontinuous boundary conditions

Our formulas have been derived under the assumption that Φ is a solution in a closed region R (the region includes its boundary). This implies continuous values of Φ on the boundary and this condition also appears in the existence theorems quoted. However, we notice that the surface integrals which occur in the formulas may exist even when the boundary conditions are discontinuous, and so it is natural to ask if they can, in some sense, represent a solution.

It is clear in this case that there cannot be a solution in the closed region. However, we can re-define the boundary value problem so that we seek a solution in the open region which approaches the (possibly discontinuous) boundary values as the surface is approached, and an existence theorem can be proved for piecewise continuous boundary values (Tychonov and Samarski, p. 261 ff.).

This may actually be a more reasonable way to state the problem for certain physical applications. For example, across a surface charge distribution ω there is a jump condition on the normal derivative of the potential

$$\left(\frac{\partial\Phi}{\partial n}\right)_1 - \left(\frac{\partial\Phi}{\partial n}\right)_2 = 4\pi\omega \quad ,$$

the subscripts indicating that the surface is approached from one side or the other. $\partial\Phi/\partial n$ is not defined on the surface itself.

To justify our derivations, the conditions imposed on surface values of functions for the validity of Green's theorem (or Gauss' theorem, on which it is based) must be weakened. This can be done; see Kellogg (p. 119).

We now give an example to show how the boundary condition is approached at a point of discontinuity. Suppose that Laplace's equation is satisfied in the upper half-plane $y > o$ and that on $y = o$ we impose

$$\Phi(x,0) = \begin{cases} 0 & -\infty < x < 0 \\ 1 & 0 < x < 1 \\ 0 & 1 < x < \infty \end{cases}$$

We first construct Green's function for this region by the method of reflection. Since this is a two dimensional problem, the potential of a [illegible] change is $-\log|\vec{r}-\vec{r}_o|$.

Therefore,

$$G = -\log\sqrt{(x-x_o)^2+(y-y_o)^2} + \log\sqrt{(x-x_o)^2+(y+y_o)^2}$$

$$\left[\frac{\partial G}{\partial n_o}\right]_{boundary} = -\left[\frac{\partial G}{\partial y_o}\right]_{y_o=0} = \frac{-2y}{(x-x_o)^2+y^2}$$

(n_o is the outward normal)

$$\phi = \frac{-1}{2\pi}\int dx_o\,\phi\,\frac{\partial G}{\partial n_o} = \frac{1}{\pi}\int_0^1 dx_o\,\frac{y}{(x-x_o)^2+y^2}$$

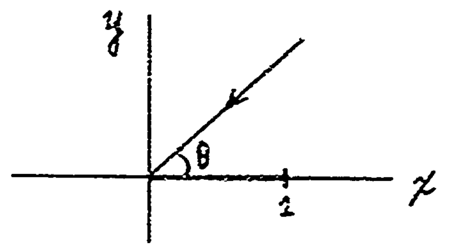
Let us consider the approach to the discontinuity at the origin along

ray $y = \alpha x$

making an angle θ

with the x - axis,

$\alpha = \tan\theta$.



The integral can be performed giving

$$\phi = \frac{1}{\pi}\tan^{-1}\frac{x_o-x}{\alpha x}\Bigg|_0^1$$

Taking the limit $x \to o$

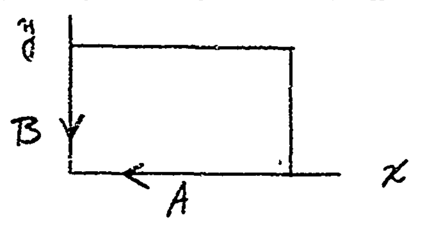
$$\phi \to \frac{1}{\pi}\left\{\frac{\pi}{2}+\tan^{-1}\cot\theta\right\} = \frac{\pi-\theta}{\pi}$$

We see that the discontinuity is not continued away from the boundary, but is immediately smoothed, and that all values in the range of the jump are approached as θ varies over 180°.

The approach of this and the previous section can be used to justify a method which is given in many textbooks. Suppose we wish a solution of Laplace's equation in a rectangle which satisfies a Dirichlet condition on the boundary. We break the problem up into four parts in each of which a condition is given on one side of the rectangle with zero on the other sides. The separate problems can be solved by the separation of variables method and it is claimed that the solution of the original problem is just the sum of the separate solutions.

However, the separate solutions are given as infinite series whose uniform convergence can only be established if the boundary condition is continuous, and, therefore, zero at the corners. But we know from the existence theorems that the boundary condition for the original problem need not vanish at the corners.

We can justify a discontinuous boundary condition at the corners for the separate problems by means of the expanded Green's function of §2.5. If $\phi(x,0) = f(x)\ \ 0 \leq x \leq a$, for example, with $\phi = 0$ on the rest of the boundary, it can be shown that at the origin



$\phi \to f(0)$ along A

$\phi \to 0$ along B

and that ϕ approaches intermediate values continuously along rays between A and B.

The method can be used for other regions with corners, such as boxes and cylinders.

2.7 Exercises

1. Prove the reciprocity relation for Dirichlet's problem. Hint: Multiply the equation for $G(\vec{r},\vec{r}_1)$ by $G(\vec{r},\vec{r}_2)$ and vice versa. Subtract the equations, integrate, and use Green's theorem.

2. Show that for Neumann's problem the reciprocity relation may always be imposed as a separate condition.

3. Find the solution of Laplace's equation, $\phi_{xx} + \phi_{yy} = 0$, in the half-space $y \geq 0$ which satisfies $\phi_y(x,0) = g(x)$. (Subscripts indicate partial derivatives.)

4. Replace the boundary condition of #3 by $\phi(x,0) = f(x)$.

5. Find the solution in the quarter-space $x \geq 0$, $y \geq 0$ which satisfies $\phi(x,0) = 0$ $\phi_x(0,y) = g(y)$.

6. Replace the boundary conditions of #5 by

$$\phi(x,0) = f(x) \qquad \phi_x(0,y) = 0 .$$

7. Find the solution of Laplace's equation, $\phi_{xx} + \phi_{yy} + \phi_{zz} = 0$, in the half-space $z \geq 0$ which satisfies $\phi(x,y,0) = f(x,y)$.

8. Replace the boundary condition of #7 by $\phi_z(x,y,0) = g(x,y)$.

9. Find the solution in the quarter-space $x \geq 0$, $y \geq 0$, $-\infty < z < \infty$ which satisfies $\phi(0,y,z) = 0$ $z \geq 0$ $\phi(x,0,z) = f(x,z)$ $x \geq 0$.

10. Replace the boundary conditions of #9 by

$$\phi_x(0,y,z) = g(y,z) \quad y \geq 0 \quad \phi(x,0,z) = f(x,z) \quad x \geq 0$$

11. An infinite plane has a hemispherical boss of radius a. Find Green's function for the half-space for which this surface is convex (see diagram). Assume that Dirichlet conditions are given on the surface.

Find Green's Function for this region.

12. Assuming that Dirichlet conditions are given find Green's function for the region between concentric spheres of radii b > a. Hint: Write

$$G = \frac{1}{|\vec{r}-\vec{r}_o|} + F(\vec{r})$$

Expand F in spherical harmonics and use the well known expansion

$$\frac{1}{|\vec{r}-\vec{r}_o|} = \sum \frac{(\ell-|m|)!}{(\ell+|m|)!} \frac{r_<^{\ell}}{r_>^{\ell+1}} Y_{m\ell}(\theta,\phi)\, Y^*_{m\ell}(\theta_o,\phi_o)$$

where $r_<$ ($r_>$) is the smaller (larger) of r, r_o and $Y_{m\ell}(\theta,\phi) = P^m_\ell(\cos\theta)e^{im\phi}$.

III. THE DIFFUSION EQUATION

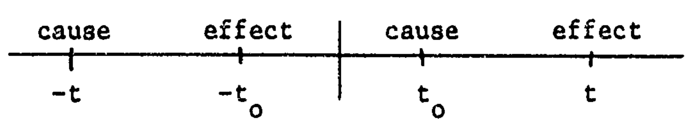
We must now have a unit point source in time, as well as in space. Therefore, we write

$$\{\nabla^2 - \frac{1}{\alpha^2}\frac{\partial}{\partial t}\}\, G(\vec{r},\vec{r}_o;t,t_o) = -4\pi\, \delta(\vec{r}-\vec{r}_o)\delta(t-t_o)$$

We shall see that $G = 0$ $t < t_o$ holds as well as the reciprocity relation:

$$G(\vec{r},\vec{r}_o;t,t_o) = G(\vec{r}_o,\vec{r};-t_o,-t)$$

We emphasize that these are not imposed conditions, but are derived. They are usually described by the word causality, from the interpretation of G as the response or effect due to a cause represented by the source term (a pulse at $t = t_o$). Thus, the first relation says that the effect cannot precede the cause, and the second shows the following symmetry between cause and effect with respect to the time scale



It is quite reasonable to make this interpretation because time's arrow is built into the diffusion equation: it is not invariant under time reversal. This can also be understood in terms of stability, which we mentioned in the introduction as one of the requirements for a well posed problem. The problem of continuing an initial condition into the future is stable, whereas continuation into the past is unstable. We shall see that the causal interpretation is somewhat more arbitrary when applied to the wave equation, which is invariant under time reversal.

3.1 The Boundary value problem

We begin by writing

$$\nabla_o^2\psi - \frac{1}{\alpha^2}\frac{\partial\psi}{\partial t_o} = -4\pi\,\rho(\vec{r}_o,t_o) \qquad \Big| G$$

$$\nabla_o^2 G + \frac{1}{\alpha^2}\frac{\partial G}{\partial t_o} = -4\pi\,\delta(\vec{r}-\vec{r}_o)\delta(t-t_o) \qquad \Big| \psi$$

The second equation follows from the reciprocity relation. The operator on the left hand side is the adjoint of the diffusion operator. (The Laplace and wave equations are self adjoint.) We multiply the first equation by G, the second by ψ, integrate and subtract, giving

$$\int_0^{t+} dt_o \int d\tau_o \left\{G\nabla_o^2\psi - \psi\nabla_o^2 G - \frac{1}{\alpha^2}\left(\psi\frac{\partial G}{\partial t_o} + G\frac{\partial\psi}{\partial t_o}\right) + 4\pi\rho\, G\right\} = 4\pi\,\psi(\vec{r},t)$$

where t + indicates the limit as $\varepsilon \to 0+$ of $t+\varepsilon$. We write

$$\psi\frac{\partial G}{\partial t_o} + G\frac{\partial\psi}{\partial t_o} = \frac{\partial}{\partial t_o}(\psi G)$$

perform the t_o integration for this term,

$$[G\psi]_o^{t+} = -[G\psi]_{t_o=0} \qquad \text{since } G(t,t+) = 0$$

and using Green's theorem obtain

$$\psi(\vec{r},t) = \int_0^{t+} dt_o \int d\tau_o \, \rho(\vec{r}_o,t_o) G$$

$$+ \frac{1}{4\pi} \int_0^{t+} dt_o \int ds_o \, \{G \frac{\partial\psi}{\partial n_o} - \psi \frac{\partial G}{\partial n_o}\}$$

$$+ \frac{1}{4\pi\alpha^2} \int d\tau_o \, [G\psi]_{t_o=0}$$

We impose the vanishing of G on the boundary if Dirichlet conditions are given, and of $\partial G/\partial n$ for Neumann conditions. Thus, the first term represents the contribution from the source, the second from the boundary conditions, and the third from the initial condition.

3.2 Boundary-free case

We consider first the case in which there are no boundaries and no initial condition. Then, having obtained this solution, we can construct other Green's functions using the method of reflection. For Laplace's equation we were able to write this solution immediately be appealing to Coulomb's law, but here the answer is not so obvious. Let us begin with the one dimensional equation

$$\frac{\partial^2 G}{\partial x^2} - \frac{1}{\alpha^2}\frac{\partial G}{\partial t} = -4\pi \, \delta(x-x_o)\delta(t-t_o)$$

Taking a Fourier k-transform on x and a Fourier ω-transform on t, the equation becomes (see Appendix II)

$$(-k^2 + \frac{i\omega}{\alpha^2})G_{k\omega} = -4\pi \, e^{ikx_o} \, e^{i\omega t_o}$$

$G_{k\omega}$, which is a function of the transform variables k and ω, is the transform of G, i.e.,

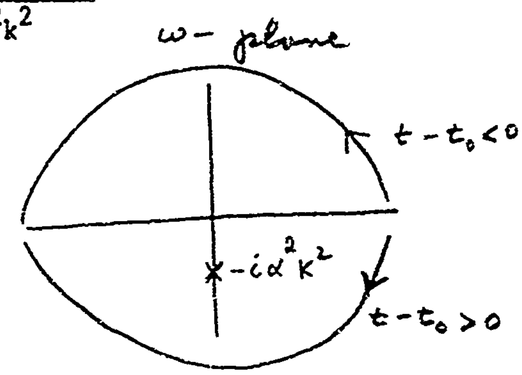
$$G_{k\omega} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dt \, G(x,x_o;t,t_o) \, e^{ikx} \, e^{i\omega t}$$

Solving the transformed equation

$$G_{k\omega} = 4\pi \frac{e^{ikx_0} e^{i\omega t_0}}{\frac{i\omega}{\alpha^2} - k^2} = 4\pi\alpha^2 i \frac{e^{ikx_0} e^{i\omega t_0}}{\omega + i\alpha^2 k^2}$$

Inverting the ω-transform

$$G_k(t,t_0) = \frac{4\pi\alpha^2 i}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{e^{ikx_0} e^{-i\omega(t-t_0)}}{\omega + i\alpha^2 k^2}$$



The integranal has a simple pole in the lower half of the complex ω-plane. We may complete the contour in the upper (lower) half-plane for $t-t_0 \lessgtr 0$.
The integral is zero for $t-t_0 < 0$ since no singularity is enclosed. For $t-t_0 > 0$ we have $-2\pi i$ times the residue at the pole (the path of integration is in the negative sense):

$$G_k = 4\pi\alpha^2 e^{ikx_0} e^{-\alpha^2 k^2 (t-t_0)} \qquad (t > t_0)$$

Finally, inverting the k-transform

$$G = \frac{4\pi\alpha^2}{2\pi} \int_{-\infty}^{\infty} dk\, e^{-ik(x-x_0)} e^{-\alpha^2 k^2 (t-t_0)}$$

$$= 2\alpha^2 \int_{-\infty}^{\infty} dk \cos k(x-x_0) e^{-\alpha^2 k^2 (t-t_0)}$$

$$= \frac{2\alpha\sqrt{\pi}}{\sqrt{t-t_0}} \exp - (x-x_0)^2/4\alpha^2(t-t_0) \qquad (t > t_0)$$

We see that at any instant after t_o the effect of the pulse is felt everywhere, propagation is instantaneous.

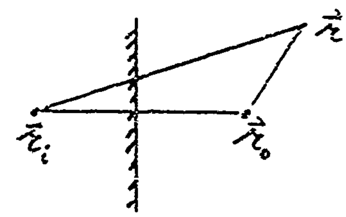
The method is trivially extended to n dimensions. We simply take Fourier transforms on all variables and obtain

$$G = 4\pi\alpha^2 \left(2\alpha\sqrt{\pi(t-t_o)}\right)^{-n} U(t-t_o) \exp - |\vec{r}-\vec{r}_o|^2/4\alpha^2(t-t_o)$$

The unit function U, which is equal to one (zero) for positive (negative) argument, appears because G vanishes for $t < t_o$. In the older literature these are called the heat pole solutions.

3.3 Method of reflection

We are now in a position to use the method of reflection just as we did for Laplace's equation. For example, consider the three dimensional problem in which Dirichlet conditions are given on a plane. We reflect the source at $\vec{r}_o$ in the plane (with a sign change) and write



$$G = \frac{(t-t_o)^{-3/2}}{2\alpha\sqrt{\pi}} U(t-t_o) \{e^{-|\vec{r}-\vec{r}_o|^2/4\alpha^2(t-t_o)} - e^{-|\vec{r}-\vec{r}_i|^2/4\alpha^2(t-t_o)}\}$$

Unfortunately, there is no principle of reflection in a sphere for the diffusion and wave equations.

3.4 Expansion in eigenfunctions

Let us take a Fourier ω-transform on t and transform the equation

$$\nabla^2 G - \frac{1}{\alpha^2}\frac{\partial G}{\partial t} = -4\pi\, \delta(\vec{r}-\vec{r}_o)\delta(t-t_o)$$

to

$$\nabla^2 G_\omega + \frac{i\omega}{\alpha^2} G_\omega = -4\pi\, \delta(\vec{r}-\vec{r}_o) e^{i\omega t_o}$$

Assume that $\psi_n(\vec{r})$ are orthonormal eigenfunctions, i.e.,

$$\int d\tau \, \psi^*_{n'} \, \psi_n = \delta_{nn'}$$

corresponding to eigenvalues k_n of the problem $\nabla^2\psi + k^2\psi = 0$ plus whatever boundary conditions we wish to impose on G (any weighting function can be absorbed by a suitable definition of ψ_n). The subscript n is intended to be a generic symbol which stands for whatever indices may be present.

We substitute the expansion

$$G_\omega = \sum a_n \psi_n(\vec{r})$$

into the equation for G_ω :

$$\sum a_n \{-k_n^2 + \frac{i\omega}{\alpha^2}\} \psi_n(\vec{r}) = -4\pi \, \delta(\vec{r}-\vec{r}_o) e^{i\omega t_o}$$

From the orthogonality

$$a_n = -4\pi \frac{\psi^*_n(\vec{r}_o) e^{i\omega t_o}}{\frac{i\omega}{\alpha^2} - k_n^2}$$

Inverting the transform

$$G = 2i\alpha^2 \sum \psi_n(\vec{r}) \psi^*_n(\vec{r}_o) \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega(t-t_o)}}{\omega + i\alpha^2 k_n^2}$$

The integrand has a simple pole in the lower half of the ω-plane. We may complete the contour in the upper (lower) half-plane for $t-t_o \lessgtr 0$. Thus, G vanishes for $t-t_o < 0$. We obtain

$$G = 4\pi\alpha^2 \, U(t-t_o) \sum \psi_n(\vec{r}) \psi^*_n(\vec{r}_o) \, e^{-\alpha^2 k_n^2 (t-t_o)}$$

In a rectangle, for example, with Dirichlet conditions the eigenfunctions are

$$\psi_{mn} = \frac{2}{\sqrt{ab}} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

with

$$k^2_{mn} = (\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2 \quad .$$

3.5 Exercises

1. Find the solution of the one dimensional diffusion equation for $t \geq 0$ which satisfies the initial condition

$$\psi(x,0) = f(x) \qquad -\infty < x < \infty$$

2. Find the solution for $x \geq 0$, $t \geq 0$ which satisfies $\psi(x,0) = f(x)$ $\psi(0,t) = g(t)$.

3. Replace the boundary conditions of #2 by $\psi(x,0) = f(x)$ $\psi_x(0,t) = g(t)$

4. Find the solution of the two dimensional diffusion equation for $t \geq 0$ which satisfies the initial condition $\psi(x,y,0) = f(x,y)$ $-\infty < x < \infty$ $-\infty < y < \infty$

5. Find the solution of the two dimensional equation for $x \geq 0$ which satisfies

$$\psi(x,y,0) = f(x,y) \qquad \psi(0,y,t) = g(y,t)$$

6. Replace the boundary conditions of #5 by $\psi(x,y,0) = f(x,y)$ $\psi_x(0,y,t) = g(y,t)$.

7. Find the solution of the three dimensional diffusion equation for $t \geq 0$ which satisfies the initial condition $\psi(x,y,z,0) = f(x,y,z)$ $-\infty < x < \infty$ $-\infty < y < \infty$ $-\infty < z < \infty$.

8. Find the solution of the three dimensional equation for $x \geq 0$ which satisfies

$$\psi(x,y,z,0) = f(x,y,z) \qquad \psi(0,y,z,t) = g(y,z,t)$$

9. Replace the boundary conditions of #8 by $\psi(x,y,z,0) = f(x,y,z)$ $\psi_x(0,y,z,t) = g(y,z,t)$.

10. Find the solution for

$$x \geq 0, \qquad y \geq 0, \qquad -\infty < z < \infty, \qquad t \geq 0$$

which satisfies

$$\psi(x,y,z,0) = f(x,y,z)$$
$$\psi(0,y,z,t) = g(y,z,t)$$
$$\psi(x,0,z,t) = 0$$

11. Find the solution of the two dimensional diffusion equation in a rectangle of sides a and b which satisfies

$$\psi(x,y,0) = f(x,y) \qquad \psi = 0 \quad \text{on boundary.}$$

12. Replace the boundary conditions of #11 by

$$\psi(x,y,0) = \psi(0,y,t) = \psi(a,y,t) = \psi(x,b,t) = 0$$

$$\psi(x,0,t) = f(x,t)$$

13. Find the solution inside a sphere of radius a which satisfies

$$\psi(a,\theta,\phi,t) = f(\theta,\phi,t)$$

$$\psi(r,\theta,\phi,0) = 0$$

IV. THE WAVE EQUATION

Green's function for the wave equation satisfies

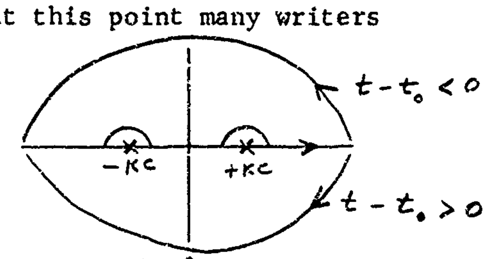
$$\{\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\}\, G(\vec{r},\vec{r}_o,t,t_o) = -4\pi\, \delta(\vec{r}-\vec{r}_o)\delta(t-t_o) \qquad (1)$$

Let us try to obtain a solution in the boundary-free case as we did for the diffusion equation. We take Fourier k_1, k_2, k_3 transforms on x, y, z and a Fourier ω - transform on t obtaining $\{-k^2 + \frac{\omega^2}{c^2}\}G_{k\omega} = -4\pi\, e^{i\vec{k}\cdot\vec{r}_o}\, e^{i\omega t_o}$ where, as before, $\vec{k} = (k_1,k_2,k_3)$ and the subscript k indicates a function of k_1, k_2, and k_3.

However, we find when we try to invert, say, the ω-transform,

$$G_k = \frac{-4\pi c^2}{2\pi}\int d\omega\, \frac{e^{-i\omega(t-t_o)}}{\omega^2-k^2c^2}\, e^{i\vec{k}\cdot\vec{r}_o} \qquad (2)$$

The integral does not exist since there are poles at $\omega = \pm kc$ on the path of integration (the real axis of the ω - plane). At this point many writers circumvent the difficulty by saying that the path of integration must detour around the singularities in such a



way as to satisfy the physical principle of causality. If the δ - function in eq. (1) represents a pulse at $\vec{r}_o$, t_o then we must have $G = 0 \quad t < t_o$ since the effect cannot precede the cause. Therefore, the path is chosen as shown in the diagram and completed by a large semi-circle in the upper (lower) half-plane for $t - t_o \lessgtr 0$. The Green's function vanishes for $t < t_o$ since in this case the path encloses no singularities.

There are two criticisms we can make of this approach. First, it is not clear by what authority one is allowed to push the path of integration off the real axis. Secondly, since the problem is purely mathematical (finding a solution of a differential equation), the resolution of the difficulty should

...ical! It is not necessary to introduce a physical principle, ... is misleading to do so since, as we shall see, this really has ... causality. While it is often helpful to think of the ... term in a differential equation as a source (the "cause" of the ...eld), this is not at all a necessary interpretation.*

... as easily think of the δ - function in eq. (1) as a ... which the fields are building up throughout ... after which the fields vanish.

... reen's function

... rier transforms has failed because of a singularity on the ... integral. Let us, therefore, try a Laplace transform ... difficulty cannot possibly arise since the inversion contour ... of all singularities in the complex plane.

... this: the inversion integral of the Laplace transform ... the function only for positive values of its argument, ... zero for the range $(- \infty,0 \quad)$

$$\mathcal{L}^{-1}\mathcal{L}\{f(x)\} = \begin{cases} f(x) & x > 0 \\ 0 & x < 0 \end{cases}$$

---

* ... is more reasonable for Poisson's equation, for it can be ... homogeneous equation (Laplace's equation) is satisfied ... the only solution is the trivial one (a constant). There are ... sources". But this is not so for the wave equation. There ... solutions, e.g., plane waves, even when the homogeneous ... fied everywhere - effects without causes! This is an indication ... theory described by hyperbolic equations the fields, e.g. a ... r all of the properties of, and are largely independent of, (matter).

Therefore, since G may be taken to be a function of $t - t_0$, we must assume that $G\ (t - t_0)$ vanishes for negative values of its argument. This additional condition is, of course, just the causality condition again:

$$G = 0 \qquad t < t_0 \tag{3}$$

As we shall see in the next section, the Green's function so-defined is suitable for a particular type of initial value problem, viz., one in which the initial conditions are to be projected into the future. Other Green's functions can be defined for other problems, e.g., projecting the initial conditions into the past, in which case the causality condition is not appropriate.

Since G is a function of $t - t_0$ which vanishes for negative values of this argument, we may assume $t > t_0 \geq 0$. Taking Fourier k_1, k_2, k_3 transforms on x, y, z and a Laplace s - transform on t, we obtain from eq. (1)

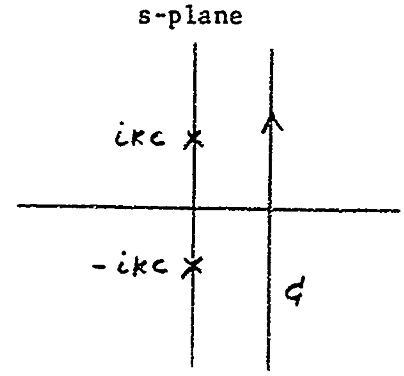
$$\{-k^2 - s^2/c^2\}\ G_{ks} = -4\pi\ e^{i\vec{k}\cdot\vec{r}_0}\ e^{-st_0}$$

since $G = G_t = 0$, $t = 0$, $t_0 > 0$ by condition (3).

Inverting the s - transform,

$$G_k = \frac{4\pi c^2}{2\pi i}\ e^{i\vec{k}\cdot\vec{r}_0} \int_C ds\ \frac{e^{s(t-t_0)}}{\omega^2 + k^2c^2}$$

s-plane



The integrand has simple poles at $\omega = \pm\ ikc$. By evaluating the residues at these poles we obtain:

$$G_k = \begin{cases} 4\pi c\ \dfrac{\sin kc(t-t_0)}{k}\ e^{i\vec{k}\cdot\vec{r}_0} & t > t_0 \\ 0 & t < t_0 \end{cases}$$

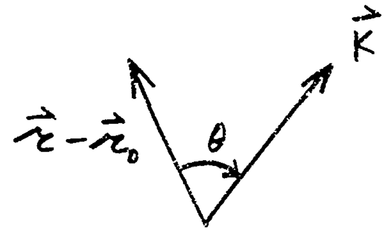
Now inverting the k_1, k_2, k_3 transforms we have, for $t > t_0$:

$$G = \frac{4\pi c}{(2\pi)^3} \iiint dk_1\, dk_2\, dk_3\, \frac{\sin kc(t-t_o)}{k}\, e^{-i\vec{k}\cdot(\vec{r}-\vec{r}_o)}$$

The integral is evaluated by means of the following trick. We introduce spherical coordinates in the k_1, k_2, k_3 space so that

$$dk_1\, dk_2\, dk_3 = k^2 \sin\theta\, d\theta\, d\phi\, dk$$

and we orient the coordinate system so that θ is measured from the direction of $\vec{r}-\vec{r}_o$. The φ integration can be performed immediately, giving



$$G = \frac{4\pi c}{(2\pi)^2} \int_0^\infty kdk\, \sin kc\,(t-t_o) \int_0^\pi d\theta\, \sin\theta\, e^{-ik|\vec{r}-\vec{r}_o|\cos\theta}$$

The θ integration is also readily performed:

$$\int_0^\pi d\theta\, \sin\theta\, e^{-ik|\vec{r}-\vec{r}_o|\cos\theta} = \frac{2 \sin k\, |\vec{r}-\vec{r}_o|}{k\, |\vec{r}-\vec{r}_o|}$$

We express the sines as exponentrals, and since the integrand is even we may write

$$G = \frac{-c/4\pi}{|\vec{r}-\vec{r}_o|} \int_{-\infty}^{\infty} dk\, \{e^{ikc(t-t_o)} - e^{-ikc(t-t_o)}\} \cdot \{e^{ik|\vec{r}-\vec{r}_o|} - e^{-ik|\vec{r}-\vec{r}_o|}\}$$

Recalling the wellknown representation of the delta function

$$\delta(p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipx}\, dx$$

we have

$$G = \frac{-c/2}{|\vec{r}-\vec{r}_o|} \{\delta[\,|\vec{r}-\vec{r}_o| + c(t-t_o)]$$

$$+ \delta[-|\vec{r}-\vec{r}_o'| - c(t-t_o)]$$

$$- \delta[\,|\vec{r}-\vec{r}_o| - c(t-t_o)]$$

$$- \delta[-|\vec{r}-\vec{r}_o| + c(t-t_o)]\,\}$$

Since we have assumed $t > t_o > 0$ the arguments of the first two delta functions can never be zero and, therefore, these functions vanish. The other delta functions are identical because the function is even. Making use of the identity

$$\delta(ax) = \frac{1}{|a|}\,\delta(x)$$

we have

$$G = \frac{1}{|\vec{r}-\vec{r}_o|}\,\delta[t-t_o - \frac{|\vec{r}-\vec{r}_o|}{c}] \qquad (3a)$$

We have obtained this expression assuming $t > t_o > 0$. However, we now notice that the expression automatically gives zero for $t < t_o$ and, therefore, is correct for all t, t_o.

If we think of G as the response to a pulse at $\vec{r}_o, t_o$, then we see that this response is itself a (spherical) pulse expanding with a velocity c and damped by the reciprocal of its radius.

In the boundary-free case with no initial conditions (i.e., assuming the sources vanish sufficiently far in the past) a solution of the inhomogeneous wave equation

$$\{\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\}\psi = -4\pi\, f(\vec{r},t)$$

May be written

$$\psi(\vec{r},t) = \int d\tau_o \int dt_o\ f(\vec{r}_o,t_o)\ G(\vec{r},\vec{r}_o,t,t_o)$$

and using the sifting property of the delta function to perform the t_o integration we have

$$\psi = \int d\tau_o \frac{f(\vec{r}_o, t-|\vec{r}-\vec{r}_o|/c)}{|\vec{r}-\vec{r}_o|}$$

This solution is sometimes called the retarded solution. It is an integration over the sources evaluated not at t but at a certain time before t, the retarded time $t - |\vec{r}-\vec{r}_o|/c$. The dependence on the retarded time reflects the finite velocity of propagation. Any change in the source at $\vec{r}_o$ is not felt at $\vec{r}$ until a time $|\vec{r}-\vec{r}_o|/c$ later.

Let us again emphasize that the above is an interpretation of the purely mathematical problem: find a solution which vanishes as $t \to -\infty$ of the wave equation with an inhomogeneous term which is non-zero in a finite region of space-time. The interpretation consists of identifying the inhomogeneous term as the source, or cause, of the solution. We can also find a solution which vanishes as $t \to \infty$ by using a different Green's function, the so-called advanced Green's function (see §4.3).

4.2 The initial value problem

We now assume that we seek a solution for $t > 0$ which satisfies initial conditions at $t = 0$ as well as boundary conditions on whatever surfaces may be specified. The wave equation is symmetric with respect to past and future, but the statement of the initial value problem introduces asymmetry: we wish to continue the initial conditions into the future rather than into the past. The asymmetry is, for this problem, appropriately introduced into the Green's function by imposing the causality condition (3).

We first establish the reciprocity relation

$$G(\vec{r},\vec{r}_o,t,t_o) = G(\vec{r}_o,\vec{r},-t_o,-t)$$

The relation between t and t_o would hold for any function of $t - t_o$, while the relation between $\vec{r}$ and $\vec{r}_o$ is stronger, expressing the equivalence of source and field points. We first write

$$\{\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\} G(\vec{r},\vec{r}_o,t,t_o) = -4\pi\ \delta(\vec{r}-\vec{r}_o)\delta(t-t_o)$$

$$\{\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\} G(\vec{r},\vec{r}_1,-t,-t_1) = -4\pi\ \delta(\vec{r}-\vec{r}_1)\delta(t-t_1)$$

We multiply the first equation by $G(\vec{r},\vec{r}_1,-t,-t_1)$, the second by $G(\vec{r},\vec{r}_o,t,t_o)$, subtract and integrate over t and whatever volume is of interest.

$$\int_{-\infty}^{\infty} dt \int d\tau\ \{G(\vec{r},\vec{r}_1,-t,-t_1)\nabla^2\ G(\vec{r},\vec{r}_o,t,t_o)$$

$$- G(\vec{r},\vec{r}_o,t,t_o)\nabla^2\ G(\vec{r},\vec{r}_1,-t,-t_1)\}$$

$$- \frac{1}{c^2}\int_{-\infty}^{\infty} dt \int d\tau\ \frac{\partial}{\partial t}\ \{G(\vec{r},\vec{r}_1,-t,-t_1)\ \frac{\partial}{\partial t}\ G(\vec{r},\vec{r}_o,t,t_o)$$

$$- G(\vec{r},\vec{r}_o,t,t_o)\ \frac{\partial}{\partial t}\ G(\vec{r},\vec{r}_1,-t,-t_1)\}$$

$$= 4\pi\ \{G(\vec{r}_1,\vec{r}_o,t_1,t_o) - G(\vec{r}_o,\vec{r}_1,-t_o,-t_1)\}$$

By Green's theorem the first integral is

$$\int_{-\infty}^{\infty} dt \int ds \left\{ G(\vec{r},\vec{r}_1,-t,-t_1) \frac{\partial}{\partial n} G(\vec{r},\vec{r}_o,t,t_o) - G(\vec{r},\vec{r}_o,t,t_o) \frac{\partial}{\partial n} G(\vec{r},\vec{r}_1,-t,-t_1) \right\}$$

We now impose the condition that G satisfies homogeneous boundary conditions of the type given on the surface. For example, if Dirichlet conditions are given G must vanish on the surface; if Neumann conditions are given $\partial G/\partial n$ must vanish on the surface. In the latter case there is no arbitrary additive constant as there was for Laplace's equation because any such constant would be fixed by the initial conditions. Having imposed this additional condition the surface integral vanishes. The second integral

$$\int d\tau \ [G(\vec{r},\vec{r}_1,-t,-t_1) \frac{\partial}{\partial t} G(\vec{r},\vec{r}_o,t,t_o) - G(\vec{r},\vec{r}_o,t,t_o) \frac{\partial}{\partial t} G(\vec{r},\vec{r}_1,-t,-t_1)]_{-\infty}^{\infty} \qquad (4)$$

vanishes because of the causality condition and, therefore, the theorem is proved.

To solve the initial value problem we write

$$\{\nabla_o^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t_o^2}\} \ \psi(\vec{r}_o,t_o) = -4\pi \ f(\vec{r}_o,t_o)$$

$$\{\nabla_o^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t_o^2}\} \ G(\vec{r},\vec{r}_o,t,t_o) = -4\pi \ \delta(\vec{r}-\vec{r}_o)\delta(t-t_o)$$

The second equation follows from eq. (1) by the reciprocity relation. We multiply the first equation by G, the second by $\psi(\vec{r}_o,t_o)$, subtract and integrate over the volume of interest and over $0 \leq t < \infty$. Now,

$$\int_0^{\infty} dt_o \ \{\psi \frac{\partial^2 G}{\partial t_o^2} - G \frac{\partial^2 \psi}{\partial t_o^2}\} = [\psi \frac{\partial G}{\partial t_o} - G \frac{\partial \psi}{\partial t_o}]_0^{\infty}$$

which vanishes at the upper limit because of the causality condition. Using Green's theorem

$$\psi(\vec{r},t) = \int_0^{t+} dt_o \int d\tau_o\, f(\vec{r}_o,t_o)\, G(\vec{r},\vec{r}_o,t,t_o)$$

$$+ \frac{1}{4\pi} \int_0^{t+} dt_o \int ds_o \left\{G \frac{\partial\psi}{\partial n_o} - \psi \frac{\partial G}{\partial n_o}\right\}$$

$$+ \frac{1}{4\pi c^2} \int d\tau_o \left[G \frac{\partial\psi}{\partial t_o} - \psi \frac{\partial G}{\partial t_o}\right]_{t_o=0} \qquad (5)$$

Because of the causality condition the integrals over t have reduced to integrals over [o, t +] where, as before, t + indicates $t + \varepsilon$ in the limit $\varepsilon \to 0+$.

Eq. (5) is a solution of the boundary value problem. The boundary conditions enter through the surface integral in the second term. We recall that we have assumed that G staisfies homogeneous boundary conditions of the type imposed on ψ . If Dirichlet conditions are given then G vanishes on the surface and the integrand is $\psi\ \partial G/\partial n_o$.
If Neumann conditions are given the integrand is $G\ \partial\psi/\partial n_o$. If mixed conditions specifying $\psi + \alpha\ \partial\psi/\partial n$ on the surface are imposed then G satisfies

$$G + \alpha \frac{\partial G}{\partial n} = 0$$

on the surface and the integrand may be written

$$G \frac{\partial\psi}{\partial n_o} - \psi \frac{\partial G}{\partial n_o} = \frac{1}{\alpha} G \left\{\psi + \alpha \frac{\partial\psi}{\partial n_o}\right\}$$

The effect of the initial (Cauchy) conditions which specify ψ and its time derivative at $t = 0$ is contained in the third integral in eq. (5).

4.3 Other Green's functions

The Green's function of the last two sections is sometimes called the retarded Green's function, $G_{ret.}$ If, given the initial conditions, we wish to find the solution at some time in the past $G_{ret.}$ is no longer appropriate. Instead we must use the advanced Green's function, $G_{adv.}$, which is obtained by imposing the condition

$$G = 0 \qquad t > t_o \tag{6}$$

which we might call the "anti-causality condition". We see that auxiliary conditions such as (3) and (6) are only devices used to obtain solutions of certain types of problems, and need not be given a causal interpretation. Indeed, in the case of eq. (6) such an interpretation would say that the effect precedes the cause!

In the boundary-free case $G_{adv.}$, can be calculated by the methods of § 4.1 giving

$$G_{adv.} = \frac{1}{|\vec{r}-\vec{r}_o|} \delta[t-t_o + \frac{|\vec{r}-\vec{r}_o|}{c}]$$

The initial value problem is solved as in § 4.2, integrating t_o over $[-\infty, 0]$. Finally, we note that $G_{adv.}$ satisfies the same reciprocity relation as $G_{ret.}$, for condition (6) also causes the integral (4) to vanish.

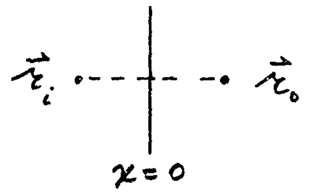
A Green's function which is symmetric in the time can also be constructed. We have noted the difficulty associated with the integral (2) due to the simple poles on the path of integration. By using the theory of distributions it can be shown that it is correct to take the principal value of this integral (Lighthill, 1958, p.31). It is readily shown that the Green's function obtained in this way, $\bar{G}$ say, is

$$\bar{G} = \frac{1}{2}\{G_{ret.} + G_{adv.}\}$$

This Green's function is useful in problems which are symmetric in the time, for example, in Dirac's theory of the electron and in certain problems in quantum field theory. See also § 5.3.

4.4 Method of reflection

We turn now to the problem of constructing Green's functions which satisfy homogeneous boundary conditions on surfaces. For the wave equation the method of reflection is unfortunately limited to conditions on planes. For example, suppose we seek a solution in the half-space $x > 0$ with Dirichlet (Neumann) conditions on the plane $x = 0$. We must construct a Green's function which satisfies $G = 0$ ($\partial G/\partial n = 0$) on $x = 0$.



This can be accomplished by placing an image at $\vec{r}_1 = (-x_0, y_0, z_0)$ which is the reflection of $\vec{r}_0$ in the plane.

$$G = \frac{1}{|\vec{r}-\vec{r}_0|}\,\delta\!\left(t-t_0-\frac{|\vec{r}-\vec{r}_0|}{c}\right) \mp \frac{1}{|\vec{r}-\vec{r}_1|}\,\delta\!\left(t-t_0-\frac{|\vec{r}-\vec{r}_0|}{c}\right)$$

The upper sign is used for Dirichlet conditions, the lower for Neumann.

If we think of the first term as a spherical wave spreading out from the point $\vec{r}_0$, then we may think of the second term as a wave reflected from the plane.

4.5 Method of eigenfunction expansions

This method is more general and more complicated than the method of reflection. We operate on eq. (1) with a Laplace s-transform on t obtaining

$$(\nabla^2 + k^2)G_s = -4\pi\ \delta(\vec{r}-\vec{r}_o)e^{-st_o} \qquad t_o > 0 \qquad (7)$$

where $k = i\ s/c$. Let $\psi_n(\vec{r})$ be ortho-normal eigenfunctions of the homogeneous equation

$$(\nabla^2 + k_n^2)\psi_n = 0$$

(the Helmholtz equation) which satisfy the boundary conditions imposed on G. For the eigenvalues k_n and the eigenfunctions ψ_n the subscript n is a generic symbol which stands for whatever indices may be present.

For example, for the interior of a unit sphere the (unnormalized) eigenfunctions which vanish on the surface are

$$\psi_{n\ell m}(r,\theta,\phi) = J_{\ell+\frac{1}{2}}(k_{\ell n}r)P_\ell^m(\cos\theta)\ e^{im\phi}$$

where the eigenvalues k_{ln} are the roots of

$$J_{\ell+\frac{1}{2}}(k) = 0$$

which can be shown to be real. Here l and m are integers and the subscript n is used to number the infinitely many roots of the l^{th} equation.

We substitute the expansion

$$G_s = \Sigma\ a_n\ \psi_n$$

into eq. (7) obtaining

$$\Sigma\, a_n(-k_n^2 + k^2)\psi_n(\vec{r}) = -4\pi\, \delta(\vec{r}-\vec{r}_o)e^{-st_o}$$

and since by our assumption of ortho-normalized eigenfunctions

$$\int d\vec{r}\ \psi_n(\vec{r})\psi^*_{n'}(\vec{r}) = \delta_{nn'}$$

we obtain

$$a_n = -4\pi \frac{\psi^*_n(\vec{r}_o)e^{-st_o}}{k^2-k_n^2} = 4\pi c^2 \frac{\psi^*_n(r_o)e^{-st_o}}{s^2 + k_n^2c^2}$$

Inverting the s-transform

$$G = \frac{1}{2\pi i}\int_c G_s\, e^{st}\, ds$$

$$= \frac{4\pi c^2}{2\pi i}\ \Sigma\, \psi_n(\vec{r})\psi^*_n(\vec{r}_o)\int_c \frac{e^{s(t-t_o)}}{s^2 + k_n^2c^2}\, ds$$

where the contour c is to the right of all singularities (in our example of the sphere, these are on the imaginary axis of the s-plane).

Evaluating the residues,

$$G = 4\pi c^2\ \Sigma\, \psi_n(\vec{r})\psi_n(\vec{r}_o)\ \frac{e^{ik_nc(t-t_o)} - e^{-ik_nc(t-t_o)}}{2i\, k_n\, c}$$

$$= 4\pi c\ \Sigma\ \frac{\sin k_n\, c(t-t_o)}{k_n}\ \psi^*_n(\vec{r}_o)\psi_n(\vec{r})$$

4.6 Exercises

1. Obtain the boundary-free retarded and advanced Green's functions for the one dimensional wave equation:

$1 - U[|x-x_o| \mp c(t-t_o)]\}$

function

$$(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

btain D' Alembert's solution of the initial

:

$$= \frac{1}{2}\left\{ (x-ct,0) + (x+ct,0) + \frac{1}{c}\int_{x-ct}^{x+ct} \psi_t(\xi,0)d\xi \right\}$$

valid for both positive and negative t Hint:

unit function is a delta function, dU/dx =

ne dimensional wave equation for x > 0, t > 0

$f(x) \quad \psi_t(x,0) = g(x) \qquad \psi(0,t) = h(t)$

tions of #3 by

$f(x) \quad \psi_t(x,0) = g(x) \qquad \psi_x(0,t) = h(t)$

ded boundary-free Green's function for the two

equation is given by

$$\frac{(t-t_o - |\vec{r}-\vec{r}_o|/c)}{( )^2 - |\vec{r}-\vec{r}_o|^2}$$

Laplace transforms lead to a difficult

point source in two dimensions is

urce in three dimensions. Therefore,

G can be obtained from the three dimensional Green's function by integration over z_0.

6. Solve the initial value problem in two dimensions assuming that no boundaries are present.

7. Find a solution of the three dimensional wave equation for $z > 0$ which satisfies

$$u(x,y,0,t) = f(x,y,t) \qquad -\infty < t < \infty$$

8. Eigenfunctions such as those given in §4.5 for the interior of a sphere are not always available. For example, for the exterior of a sphere we require functions which give only outgoing waves at infinity. This is the so-called radiation condition; it is equivalent to the causality condition. Therefore, instead of the Bessel function we would have to use $H^{(1)}_{\ell+\frac{1}{2}}(kr)$. But $H^{(1)}$ has no real zeros and, therefore, cannot satisfy the boundary condition on the sphere.

When an expansion of the fundamental solution (such as that used in problem 2.12) is known, the following approach may be used. Let $G_k = G_s \exp st_0$. Show that in the boundary-free case

$$G_k = \frac{1}{R} e^{ikR} \qquad \text{where} \qquad R = |\vec{r}-\vec{r}_0|$$

so that generally G_k behaves as $(\exp i k R)/R$ near the singularity at $\vec{r}_0$.

Therefore, write

$$G_k = \frac{e^{ikR}}{R} + F(\vec{r})$$

where $F(\vec{r})$ is regular in the region of interest. By expanding F in appropriate eigenfunctions and making use of the expansion (Morse and Ingard, p. 352)

$$\frac{e^{ikR}}{R} = ik \sum (2\ell+1) \frac{(\ell-|m|)!}{(\ell+|m|)!} j_\ell(kr_<)h_\ell(kr_>)Y_{m\ell}(\theta,\phi)Y^*_{m\ell}(\theta_o,\phi_o)$$

construct a G_k for the exterior of a sphere of radius a which vanishes on the surface. In the above expansion j_ℓ and h_ℓ are spherical Bessel and Hankel functions, $r_>$ ($r_<$) is the larger (smaller) of r, r_o, and $Y_{m\ell}(\theta,\phi) = P_\ell^m(\cos\theta)e^{im\phi}$.

V. ACOUSTIC RADIATION AND SCATTERING

Our purpose in this chapter is not to give a general introduction to acoustics, but simply to give some examples of the use of the Green's function for the scalar wave equation. Readers seeking a general background in acoustics or examples with more complicated physics than those presented here are refered to Morse and Ingard (1968).

5.1 Basic Equations of Acoustics

For simplicity we assume a homogeneous, inviscid, compressible fluid. The conservation of mass is expressed by the continuity equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho\vec{\mu} = q$$

where ρ, $\vec{\mu}$, and q are the density, velocity, and source strength (rate at which fluid is "created" (introduced into the flow field from outside) per unit volume. The force equation (Newton's second law) for fluids is usually called Euler's equation:

$$\rho \frac{\partial \vec{\mu}}{\partial t} + \rho\vec{\mu}\cdot\nabla\vec{\mu} = -\nabla p + \vec{F}$$

where p is the pressure and $\vec{F}$ the external force per unit volume. The field variables (unknowns) are ρ, p, and $\vec{\mu}$. Therefore, another relation is needed to complete the above equations. For this we assume adiabatic motion (which is quite realistic for sound) so that

$$p = \text{const. } \rho^{\gamma}$$

where $\gamma = c_p/c_v$, the ratio of specific heats at constant pressure and constant volume.

The basic equations of acoustics are obtained by linearization. We write

$$\rho = \rho_o + \rho' \qquad p = p_o + p' \qquad \vec{F} = \vec{F}_o + \vec{F}'$$

$$\vec{\mu} = \vec{\mu}_o + \vec{\mu}' \qquad q = q_o + q'$$

where the quantities with subscripts are mean or equilibrium values, which satisfy the equations separately, and the primed quantities are fluctuations assumed small with respect to mean values. We assume that there is no mean flow so that $\vec{\mu}_o = 0$ and substitute these relations into the above equations neglecting quadratic terms in the primed variables. From the adiabatic law we obtain

$$p' = c^2 \rho'$$

where $c = (\gamma p_o/\rho_o)^{1/2}$ is the velocity of sound. Using this relation to eliminate ρ', we obtain by linearization of the first two equations the basic field equations of acoustics:

$$\frac{1}{c^2}\frac{\partial p}{\partial t} + \rho_o \operatorname{div} \vec{\mu} = q \tag{1}$$

$$\rho_o \frac{\partial \vec{\mu}}{\partial t} + \nabla p = \vec{F} \tag{2}$$

where for convenience the primes have been dropped from perturbation quantities.

We now note two consequences of these equations. By multiplying the first by p, the second by $\vec{\mu}\cdot$ and adding we obtain the conservation of energy

$$\frac{\partial w}{\partial t} + \operatorname{div} \vec{s} = pq/\rho_o + \vec{F}\cdot\vec{\mu}$$

where w is the energy density

$$w = \frac{1}{2}\left(p^2/\rho_o c^2 + \rho_o \mu^2\right)$$

and $\vec{s}$ (the Poynting vector) is the energy flux density (energy/cm^2 sec)

$$\vec{s} = p\vec{\mu} \tag{3}$$

The terms on the right hand side represent dissipation.

By eliminating $\vec{\mu}$ between eqns. (1-2) we obtain a wave equation for the pressure:

$$\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = \nabla\cdot\vec{F} - \frac{\partial q}{\partial t} \tag{4}$$

5.2 Multipole Fields

We will not be concerned with the mechanism of radiation in this section, but will simply assume that there is a finite source region, i.e., a region in which energy is somehow introduced into the acoustic field. The mechanism, for example, could be a vibrating surface or turbulence. We further assume a harmonic time dependence. A typical time dependent quantity is then written

$$p(\vec{r},t) = p_\omega(\vec{r})e^{-i\omega t}$$

This assumption is, of course, equivalent to taking a Fourier transform.

We write the source term in the wave equation as a function

$$f(\vec{r},t) = f_\omega(\vec{r})e^{-i\omega t}$$

Then the wave equation (4) for p becomes the Helmholtz equation for p_ω:

$$\nabla^2 p_\omega + k^2 p_\omega = -4\pi f_\omega \qquad k = \omega/c$$

Solving the wave equation by means of the boundary-free Green's function (4.3a) gives

$$p = p_\omega e^{-i\omega t} = \int d\tau_o\, dt_o\, f_\omega\, e^{-i\omega t_o}\, G$$

$$= e^{-i\omega t} \int d\tau_o \frac{e^{ik|\vec{r}-\vec{r}_o|}}{|\vec{r}-\vec{r}_o|} f_\omega(\vec{r}_o)$$

or

$$p_\omega = \int d\tau_o\, G_\omega(\vec{r},\vec{r}_o)\, f_\omega(\vec{r}_o)$$

where G_ω is the boundary-free Green's function for the Helmholtz equation.

$$G_\omega(\vec{r},\vec{r}_o) = \frac{\exp ik|\vec{r}-\vec{r}_o|}{|\vec{r}-\vec{r}_o|} \qquad (5)$$

We assume that the origin of our coordinate system is somewhere within the souce region and that we wish to calculate the field far from this region so that r is large compared with the values of r_o within its range of integration. We expand $|\vec{r}-\vec{r}_o|$ in a Taylor series in $\vec{r}_o$ about the origin

$$|\vec{r}-\vec{r}_o| = r - \frac{\vec{r}\cdot\vec{r}_o}{r} + \ldots$$

Then the expansion

$$e^{ik|\vec{r}-\vec{r}_o|} = e^{ikr} \exp ik\vec{r}_o \cdot \{\frac{\vec{r}}{r} + \ldots\}$$

$$= e^{ikr} \{1 + ik \frac{\vec{r}_o \cdot \vec{r}}{r} + \ldots\}$$

is valid provided we make the further assumption $kr_o << 1$. This means that we assume the dimensions of the source small compared with the wavelength λ $(k = 2\pi/\lambda)$. With this assumption we may expand G_ω in a Taylor series in $\vec{r}_o$,

$$G_\omega = \frac{e^{ikr}}{r} + \sum_i \left(\frac{\partial}{\partial x_{oi}} \frac{e^{ik|\vec{r}-\vec{r}_o|}}{|\vec{r}-\vec{r}_o|}\right)_{\vec{r}_o=0} x_{oi} + \ldots$$

Substituting into the integral

$$p_\omega = \int d\tau_o \, f_\omega(\vec{r}_o) G_\omega$$

we obtain

$$p_\omega = S \frac{e^{ikr}}{r} - \vec{D} \cdot \nabla \frac{e^{ikr}}{r} + \ldots$$

where we might call S the monopole strength

$$S = \int d\tau_o \, f_\omega(\vec{r}_o)$$

since, as we see from the Green's function, the first term is the field of a point source at the origin.

In writing the second term we have used the fact that $\partial/\partial x_{oi} = -\partial/\partial x_i$ when these operate on functions of $|\vec{r}-\vec{r}_o|$ only. The second term is a dipole field with dipole moment

$$D_i = \int d\tau_o \, x_{oi} \, f_\omega(\vec{r}_o)$$

Higher order terms in the expansion represent quadrupoles, octupoles etc. These terms reflect the symmetries of the source. For example, a sphere vibrating purely radially has only a monopole term, which is the only spherically symmetric multipole. The dipole term reflects symmetry about an axis and the higher order multipoles are correspondingly more complicated.

5.3 Radiation Calculations

We will discuss two methods of calculating the energy radiated by sources. The first and best known of these we call the far field method. It consists of integrating the Poynting vector $\vec{s}$ (3) over a sphere whose radius is arbitrarily large. Thus, it represents only energy which is ultimately lost by the system, i.e., radiated away. Since the element of surface area in spherical coordinates is $r^2 \sin\theta \, d\theta \, d\phi$ where r is the radius of the sphere, and since $\vec{s}$ is quadratic in the fields, only terms of order $1/r$ will contribute to this integral in the limit $r \to \infty$. These terms are called the radiation fields.

For example, the field of a dipole along the z - direction is

$$\frac{\partial}{\partial z} \frac{e^{i(kr-\omega t)}}{r} = (ik \frac{z}{r^2} - \frac{z}{r^3}) e^{i(kr-\omega t)}$$

The first term gives the radiation field since it is the only term of order $1/r$.

$$p_{rad.} = \frac{ik \cos\theta}{r} e^{i(kr-\omega t)}$$

The velocity u in the radiation field is obtained from eq. (2)

$$i \omega \rho_o \mu_{rad.} = p_{rad.} (ikz/r)$$

and the time-averaged Poynting vector is

$$s_{rad.} = \langle \mathrm{Re}\, p_{rad.} \, \mathrm{Re}\, \mu_{rad.} \rangle$$
$$= \frac{1}{2} \frac{k^2}{\rho_o c} \frac{\cos^2\theta}{r^2}$$

in the radial direction. This shows the directional properties of the radiation. The total power may be obtained by integration.

In addition to the radiant energy in the field there is also reactive energy, energy which fluctuates between the source and field. The second method, the near field method, consists of integrating the energy flux density over the source. This has the advantage of giving the reactive as well as the radiant energy, and, therefore, the mechanical impedance.

For example, assume the normal velocity $\mu(\vec{r})e^{-i\omega t}$ is specified on a vibrating surface. By eq. (2) this is equivalent to giving the normal derivative of the pressure (we are still assuming harmonic time dependence). Assuming the (retarded) Green's function is known we may substitute

$$p = \frac{1}{4\pi}\int \frac{\partial p}{\partial n} G_{ret.}\, ds_o\, dt_o = \frac{i\omega\rho_o}{4\pi}\int \mu(\vec{r}_o)e^{-i\omega t}\, G_{ret.}\, dt_o\, ds_o$$

into the expression for the total power

$$P = \int p\,\mu\, ds \tag{6}$$

We may now write

$$G_{ret.} = \bar{G} + H$$
$$\bar{G} = \frac{1}{2}\{G_{ret.} + G_{adv.}\}$$
$$H = \frac{1}{2}\{G_{ret.} - G_{adv.}\}$$

where $\bar{G}$ is the time symmetric Green's function of §4.3. H is not a Green's function since it is a solution of the homogeneous wave equation, as we can see by subtracting the equations for $G_{ret.}$ and $G_{adv.}$.

After substituting into eq. (6) we find that if the sign of t is changed the part involving $\bar{G}$ changes sign and the part involving H is unchanged Therefore, the former represents reactive power and the latter resistive or radiant power. This elegant method was perhaps first used by Schwinger (1949) in connection with electromagnetic radiation.

As an example consider an infinite plane $z = 0$ on which $\mu = 0$ except for a small portion σ which vibrates. The Green's functions are easily obtained by the method of reflection with an image ppint $\vec{r}_i = (x_o, y_o, -z_o)$. The effect of the harmonic time dependence is to replace $G_{ret.,adv.}$ by

$$G_{\omega ret.,adv.} = \frac{e^{\pm ik|\vec{r}-\vec{r}_o|}}{|\vec{r}-\vec{r}_o|} + \frac{e^{\pm ik|\vec{r}-\vec{r}_i|}}{|\vec{r}-\vec{r}_i|}$$

On the plane $|\vec{r}-\vec{r}_o| = |\vec{r}-\vec{r}_i|$. We have for the time-averaged radiative power

$$P_{rad.} = \int_\sigma \mu(\vec{r})\, ds \int_\sigma \frac{\omega\rho_o}{2\pi} \mu(\vec{r}_o) \frac{\sin k|\vec{r}-\vec{r}_o|}{|\vec{r}-\vec{r}_o|} ds_o \tag{7}$$

and for the reactive power

$$P_x = -\frac{\omega\rho_o}{2\pi} \int_\sigma ds \int_\sigma ds_o\, \mu(\vec{r})\, \mu(\vec{r}_o) \frac{\cos k|\vec{r}-\vec{r}_o|}{|\vec{r}-\vec{r}_o|}$$

The integrand in the last equation is singular, but by a well known theorem for improper integrals (Kellogg, p. 149) the integral is convergent.

Detailed calculations are available for the case in which σ is circular. If, further, μ is constant (circular piston) then the coefficients of μ^2 in the expressions for $P_{rad.}$ and P_x give the resistive and reactive parts of the complex mechanical impedance. For further details see Bouwkamp (1946) and Morse and Ingard (p. 381 ff.).

If directional properties of the radiation are desired we can make use of the relation (§ 4.1)

$$\frac{\sin k|\vec{r}-\vec{r}_o|}{|\vec{r}-\vec{r}_o|} = \frac{k}{4\pi} \int e^{ik\vec{n}\cdot(\vec{r}-\vec{r}_o)}\, d\Omega$$

where $d\Omega$ is the element of solid angle about the unit vector $\vec{n}$. Then from eq. (7)

$$p_{rad.} = \int d\Omega \; p(\theta,\phi)$$

where

$$p(\theta,\phi) = \frac{\omega\rho_o k}{8\pi^2} \{ \int ds_o \; \mu(\vec{r}_o) e^{-ik\vec{n}\cdot\vec{r}_o} \}^2$$

is the power radiated into $d\Omega$. If θ,ϕ are spherical angles with the normal to the plane as the z - direction and ϕ_o is the azimuthal angle in the plane, then

$$\begin{aligned} \vec{n}\cdot\vec{r}_o &= r_o \cos(\vec{n},\vec{r}_o) \\ &= r_o \sin\theta \cos(\phi-\phi_o) \end{aligned}$$

in the above formula.

5.4 Mechanisms of Radiation

Let us now take a closer look at the source terms in eq. (4). We shall add to these a term from the stress tensor which is of the order neglected in the derivation. Lighthill (1952,1954) has shown that this term may not be of second order, at least for turbulence in air. Therefore, we write

$$\nabla^2 p_\omega + k^2 p_\omega = i\omega q_\omega + \nabla\cdot\vec{F}_\omega - \sum_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \rho_o \mu_{\omega i}\mu_{\omega j}$$

Solving by means of the Green's function (5) we have

$$p_\omega = \int d\tau_o \; G_\omega(\vec{r},\vec{r}_o) \{ i\omega q_\omega + \nabla_o\cdot\vec{F}_\omega - \sum_{ij} \frac{\partial^2}{\partial x_{oi}\partial x_{oj}} \rho_o \mu_{\omega i}\mu_{\omega j} \} \qquad (8)$$

where $\nabla_{oi} = \partial/\partial x_{oi}$.

Let us look at the terms in this integral separately. If we expand the first integral, the one containing q_ω , as in § 5.2 the first term in the expansion will be a monopole field. For this reason q_ω (or $\partial q/\partial t$) is called a monopole source. It represents the rate at which fluid is added or withdrawn. A vibrating surface is a source of this type, for its effect is as if fluid were alternately injected and withdrawn. An example is the cavitation behind a propeller. A collapsing bubble is cushioned by vapor inside it causing a bouncing effect. Of course, we should keep in mind that the field is not a pure monopole field, but rather the monopole term is the dominant term in a multipole expansion.

The second integral may be transformed through the identity

$$\int d\vec{r}_o\ \nabla_o \cdot G_\omega \vec{F}_\omega = \int d\vec{r}_o\ G_\omega \nabla_o \cdot F_\omega + \int d\vec{r}_o\ \vec{F}_\omega \cdot \nabla_o\ G_\omega$$

We have assumed that the sources are non-zero in a finite region. The left hand side may be written as a surface integral by Grauss' theorem, but since the volume integral is over all space the limit must be taken as the surface goes to infinity. As the limit is taken the surface will eventually lie in a region where $\vec{F}_\omega$ vanishes. Hence, the left hand side is zero and the contribution to the pressure is

$$\int d\tau_o\ \vec{F}_\omega \cdot \nabla_o\ G_\omega$$

The leading term of a multipole expansion of this integral will contain derivatives of G_ω : it is a dipole term and, therefore, $\nabla \cdot \vec{F}$ is called a dipole source. An example is the sound field due to the force exerted on a fluid by a propeller blade.

The third integral in eq. (8) can be transformed by two integrations by parts giving

$$\int d\tau_o \sum_{ij} \rho_o \mu_{\omega i} \mu_{\omega j} \frac{\partial^2}{\partial x_{oi} \partial x_{oj}} G_\omega$$

The leading term in a multipole expansion contains second derivatives of G_ω; consequently, it is a quadrupole term. Thus, for example, a turbulent region is a source of quadrupole radiation.

5.5 Scattering and Diffraction

Scattering and diffraction refer to the same phenomenon, the interaction of a wave with an obstacle. The word diffraction is usually used when the dimensions of the obstacle are large compared with the wavelength, as, for example, when sound passes through an aperture in an infinite screen. The obstacle need not be a solid body. It may, for example, be a region of turbulence or inhomogeneity.

Green's function is used in these problems to transform the boundary value problem into an integral equation. Let us consider the case of a solid body. For harmonic time dependance and with no sources present a solution is given by the surface integral

$$p_\omega = \frac{1}{4\pi} \int ds_o \left\{\frac{\partial p_\omega}{\partial n_o} G_\omega - p_\omega \frac{\partial G_\omega}{\partial n_o}\right\} \tag{9}$$

This is not a solution of the boundary value problem since it is not proper to specify both p_ω and its normal derivative on the surface. However, instead of now imposing a boundary condition on G as we have in previous problems, we may instead regard eq. (9) as an integral equation for p_ω. The solution is not unique since we have not yet imposed any boundary conditions. We may add to it any solution of the homogeneous wave equation. Therefore, we write

$$p = p_i + \frac{1}{4\pi} \int ds_o \left\{ \frac{\partial p_\omega}{\partial n_o} G_\omega - p_\omega \frac{\partial G_\omega}{\partial n_o} \right\}$$

where p_i is a given incident wave. The integral represents the scattered wave if we use the retarded Green's function

$$G_\omega = \frac{e^{ik|\vec{r}-\vec{r}_o|}}{|\vec{r}-\vec{r}_o|}$$

which gives outgoing waves.* The boundary condition may now be imposed. For a rigid body we would set the normal derivative of the pressure equal to zero on the boundary. For a compliant body a linear combination of p_ω and its normal derivative would be specified.

For simple geometries the integral equation can often be solved by expanding p_ω, p_i, and G_ω in appropriate orthogonal functions, fixing undetermined coefficients by evaluating p_ω on the surface. Such problems can be solved by direct methods, so that nothing is really gained by the transformation to an integral equation. When an exact solution is not possible, however, the integral equation suggests an approximation scheme based on iteration (Born approximation). In general, this should succeed if the scattered wave is small, as, for example, when the wavelength

* With our choice for the time dependence, $(\exp \pm ikr - i\omega t)/r$ represents outgoing (incoming) spherical waves.

is large compared with the dimensions of the scatterer.

As a simple example we calculate the radiation field of the scattered wave from a rigid sphere in this approximation with the assumption $ka \ll 1$, a being the radius of the sphere. After one iteration the scattered wave is given approximately by

$$p_s \approx \frac{a^2}{4\pi} \iint d\Omega_o \left[p_i(\vec{r}_o) \frac{\partial}{\partial r_o} \frac{e^{ik|\vec{r}-\vec{r}_o|}}{|\vec{r}-\vec{r}_o|}\right]_{r_o=a}$$

where p_i is the incident wave, which we assume to be plane

$$p_i = A\, e^{i\vec{k}\cdot\vec{r}}$$

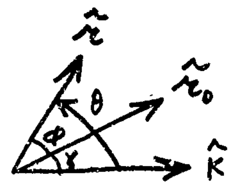
There are two approximations. We keep only the $\frac{1}{r}$ terms (the radiation field) and we expand in powers of ka keeping only the first non-zero contribution.

$$p_s = \frac{a^2}{4\pi} \iint d\Omega_o\, A\, e^{i\vec{k}\cdot\vec{r}_o} \frac{\partial}{\partial r_o} \left.\frac{\exp ik(\vec{r}-\hat{r}\cdot\vec{r}_o + \ldots)}{r + \ldots}\right|_{r_o=a}$$

$$= \frac{Aa}{4\pi} \frac{e^{ikr}}{r} \iint d\Omega_o \{1 + ika\cos\gamma + \ldots\}\{-ika\cos\phi - (ka)^2\cos^2\phi + \ldots\}$$

where

$$\cos\phi = \cos(\hat{r},\hat{r}_o) \qquad \cos\gamma = \cos(\hat{k},\hat{r}_o)$$



Carets indicate unit vectors. To perform the integral we substitute $\gamma = \theta - \phi$, where θ is the angle of $\hat{r}$ measured from the direction of the incident wave, $\hat{k}$. The first non-vanishing integral gives

$$p_s \approx \frac{Aa}{3} \frac{e^{ikr}}{r} (ka)^2 \{1-\cos\theta\}$$

A typical diffraction problem is that of a wave passing through an aperture in an infinite screen. Eq. (9) is usually used for the field behind the screen together with the Kirchhoff approximation which assumes that the values of p_ω and $\partial p_\omega/\partial n$ on the aperture are just what would be given by the incident wave if the screen were not present. Then the integral equation is well-defined. Methods of solution are discussed in most books on acoustics, optics, and electromagnetic theory.

Appendix I : The Delta Function

The delta function is sometimes "defined" by the relations

$$\delta(x) = 0 \qquad x \neq 0$$

$$\int_{-a}^{b} \delta(x)\, dx = 1 \qquad a,b > 0 \tag{1}$$

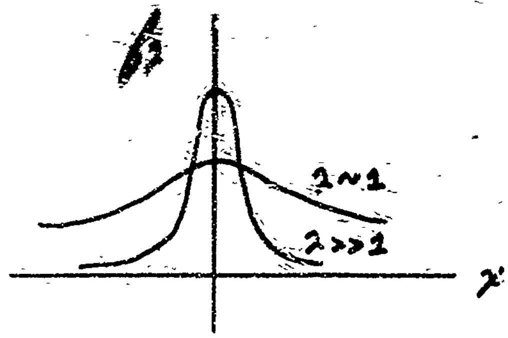
from which immediately follows the so-called sifting property:

$$\int_{a}^{b} f(x)\delta(x-x_o)dx = f(x_o) \qquad a < x < b$$

Intuitively, it is a function which is zero everywhere except at $x = 0$ where it is infinite in such a way that its integral is unity. It is clear that within the context of ordinary real variable theory there is no such function, for a function is defined only if it is assigned a definite value for every point within its range (infinity is not a definite value). Moreover, the area under a point is always zero.

However, the behavior described by eqns. (1) can be approximated with arbitrary accuracy by ordinary functions. For example, consider the sequence

$$f_\lambda(x) = \sqrt{\frac{\lambda}{\pi}}\, e^{-\lambda x^2}$$



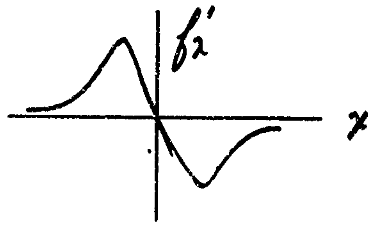
The area under f_λ is always unity and the maximum at $x = 0$ becomes arbitrarily sharp as λ is made large.

Therefore, we let $\delta(x)$ represent the limiting behavior of such a sequence and write, symbolically

$$\delta(x) = \lim_{\lambda\to\infty} f_\lambda(x)$$

Identities involving derivatives and integrals of $\delta(x)$, e.g., the sifting property, can be proved provided that the interchange of these operations with the limit can be justified. Justification can only be found within a broader mathematical context than the ordinary theory of real variables. Several such contexts are available, of which the best known is probably the theory of distributions as presented by Lighthill (1958), whose book can be read with a knowledge of calculus only. For an intuitive approach we also recommend the book of van der Pol and Bremmer (1964, chap. 5). Our purpose in this appendix is not to supply rigor, but only to discuss the formal manipulation of delta functions.

The derivative of f_λ is a function having two sharp and arbitrarily close extrema as λ is made large. It too has a sifting property.



It picks the values of a function at two points which are arbitrarily close and, therefore, produces the derivative. By a formal integration by parts we have

$$\int f(x)\, \delta'(x-x_o)\, dx = -f'(x_o)$$

and, generally,

$$\int f(x)\, \delta^{(n)}(x-x_o)\, dx = (-1)^n f^{(n)}(x_o)$$

The following are useful identities;

$$\delta(-x) = \delta(x)$$

$$\delta(ax) = \frac{1}{|a|}\,\delta(x)$$

$$x\,\delta(x) = 0$$

$$x\,\delta'(x) = -\delta(x)$$

$$\delta(x^2-a^2) = \frac{1}{2a}\{\delta(x-a) + \delta(x+a)\}$$

$$\frac{dU}{dx} = \delta(x) \qquad U(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

By the equality in these relations we mean that the right and left hand sides give the same result when substituted into an integral. Thus, to "prove" the second relation we form the integral of $\delta(ax)$ with an arbitrary continuous function, $f(x)$ say, and show, by a simple change of the variable of integration, that the result is $|a|^{-1} f(0)$, which is what is given by the right hand side of the identitiy.

This interpretation of the equality is sufficient for our purposes, since our delta functions always ultimately disappear by being substituted into integrals and, thus, do not appear in the final results of our calculations. Their use can be regarded as an intermediate shortcut. For example, eq. (1.4) can be derived without the use of the delta function as follows. Let G be a harmonic function in a region v except for a singularity of the type $|\vec{r}-\vec{r}_o|^{-1}$ at one point. Green's theorem can be applied to the region $v - v_o$ which excludes from v a small sphere v_o of radius R_o centered at the singular point $\vec{r}$.

$$\int_{v-v_o} d\tau_o \, \{G\nabla_o^2\phi - \phi\nabla_o^2 G\} = \int_{\sigma+\sigma_o} ds_o \, \{G \frac{\partial\phi}{\partial n_o} - \phi \frac{\partial G}{\partial n_o}\} .$$

In the limit $R_o \to 0$ we have

$$\lim_{R_o\to 0} \int_{\sigma_o} R_o^2 \, d\Omega_o \, \{G \frac{\partial\phi}{\partial r_o} - \phi \frac{\partial G}{\partial r_o}\} = -4\pi \, \phi(\vec{r})$$

where only the second term has contributed, and the result is established by substituting $\nabla_o^2\phi = -4\pi \, \rho(\vec{r}_o)$, $\nabla_o^2 G = 0$ in the volume integral over $v-v_o$.

Appendix II: Fourier and Laplace Transforms

The Fourier integral theorem

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dt \; f(t) \; e^{-ip(x-t)} \qquad (1)$$

may be written in two steps:

$$F(p) = \int_{-\infty}^{\infty} f(t) \; e^{ipt} \; dt \qquad (2)$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(p) \; e^{-ipx} \; dp \qquad (3)$$

These form a transform pair. The function $F(p)$ is called the Fourier transform of $f(x)$, and the operator in eq. (3) through which the original function is recovered is called the inverse Fourier transform. The following notation is also used:

$$F(p) \equiv F\{f\} \equiv f_p$$

$$F^{-1}\{f_p\} = f(x)$$

In forming the transform pair the position of the factor of 2π and of the plus and minus signs in the exponentials is arbitrary. Since all possible combinations are in use in the literature, definitions should be carefully checked when using tables of Fourier transforms.

The classical theorem for the validity of eq. (1) is the Plancherel theorem which states, essentially, that a sufficient condition is the existence of $\int_{-\infty}^{\infty} dx |f(x)|^2$, or, more precisely, that $f(x)$ is Lebesgue square integrable $(f \in L^2(-\infty,\infty))$. However, this theorem is too restrictive if we wish to admit generalized functions, such as the delta function and its derivatives. Therefore, we shall use, instead, a sufficient condition given by Lighthill (1958, p.21), viz., the existence of $\int_{-\infty}^{\infty} dx \frac{|f(x)|}{(1+x^2)^N}$ for some integer N. This allows functions bounded by polynomials as $|x| \to \infty$.

For example,

$$F^{-1}\{\delta(p)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ipx}\, \delta(p)dp = \frac{1}{2\pi}$$

or

$$F\{1\} = 2\pi\, \delta(p)$$

Integrating by parts we have

$$F^{-1}\{\delta'(p)\} = ix/2\pi$$

or

$$F\{x\} = -2\pi i\, \delta'(p)$$

and, in general,

$$F\{x^n\} = 2\pi(-i)^n\, \delta^{(n)}(p) \qquad n = \text{integer} \geq 0$$

Perhaps the most important property of the Fourier transform for our purposes is the way it operates on derivatives. Operating on eq. (3) with the derivative we have

$$f' = F^{-1}\{-ipf_p\}$$

or

$$F\{f'\} = -ipf_p$$

and by repetition

$$F\{f''\} = -p^2 f_p \qquad \text{etc.}$$

Thus, a differential equation is transformed to an algebraic one.

These and some additional properties are summarized in the following table.

$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dp\, e^{-ipx} f_p$		$f_p = \int_{-\infty}^{\infty} dx\, e^{ipx} f(x)$
x^n	$n =$ integer ≥ 0	$2\pi(-i)^n \delta^{(n)}(p)$
	$\frac{d}{dx} f(x)$	$-ip\, f_p$
	$ix\, f(x)$	$\frac{d}{dp} f_p$
	$e^{ixp_o} f(x)$	f_{p+p_o}
	$f(x+x_o)$	$e^{-ipx_o} f_p$
	$f(ax+b)$	$\frac{1}{\lvert a \rvert} e^{-ipb/a} f_{p/a}$
	$f(x)\, g(x)$	$\frac{1}{2\pi} f_p * g_p$
	$f(x) * g(x)$	$f_p\, g_p$

The star indicates the convolution of two function, which is defined by

$$f(x)*g(x) = \int_{-\infty}^{\infty} f(\xi)\, g(x-\xi)\, d\xi = g * f$$

Our sufficient condition does not admit functions which are exponentially large as $x \to \infty$. We can allow such functions, and in the process define the Laplace transform, if we are willing to throw away half of the function and assume it to vanish for $x < 0$. Thus, we assume

$$f(x) = 0 \qquad x < 0$$

$$|f(x)| < e^{ax} \qquad x > x_o > 0 \qquad a > 0$$

Since the Fourier transform of f (x) may not exist, we define a new function

$$\phi(x) = f(x)e^{-bx} \qquad \text{where} \qquad b > a$$

which is exponentially small as $x \to \infty$. If we first re-define the Fourier transform pair by interchanging the signs in the exponentials of eqs. (2, 3) we may write

$$F\{\phi\} = \int_{-\infty}^{\infty} e^{-ipt}\, \phi(t)dt = \int_{0}^{\infty} e^{-(b+ip)t}\, f(t)dt$$

Let $s = b + ip$. As a function of p $F\{\phi\}$ is the Fourier transform of ϕ. As a function of s we define it to be the Laplace transform of f:

$$\mathcal{L}\{f(x)\} = \int_{0}^{\infty} e^{-st}\, f(t)dt$$

We obtain the inversion theorem through the inversion theorem for the Fourier transform according to which

$$\phi(x) = f(x)e^{-bx} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dp\; e^{ipx}\, \phi_p$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dp\; e^{(b+ip)x}\phi_p = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} ds\; e^{sx}\, \mathcal{L}\{f\}$$

There was originally no upper limit for b so that the contour for the inversion integral, C say, which is a straight line parallel to the imaginary axis of the s-plane, is arbitrarily far to the right of the origin, but by Cauchy's theorem is equivalent to any straight line to the right of all singularities. For $x > 0$ the integral can be evaluated in terms of contributions from the poles, branch cuts, etc. to the left of C. For $x < 0$ the inversion integral automatically gives zero since the integrand vanishes as C is moved to the right.

Integration by parts gives the effect of the Laplace transform on derivatives, which we include in the following table.

$f(x)$	$\mathcal{L}\{f\} \equiv f_s$
$\frac{d}{dx} f(x)$	$sf_s - f(0)$
$\frac{d^2}{dx^2} f(x)$	$s^2 f_s - sf(0) - f'(0)$
$x^n f(x)$ $n =$ integer ≥ 0	$(-1)^n \frac{d^n}{ds^n} f_s$
$\frac{1}{x} f(x)$	$\int_s^\infty f_s \, ds$
$\int_0^x f(\xi) d\xi$	$\frac{1}{s} f_s$
$f(x-b)$ where $f(x) = 0$ $x < 0$	$e^{-bs} f_s$
$\frac{1}{a} e^{bx/a} f(x/a)$	f_{as-b}
$f * g$	$f_s g_s$

For the Laplace transform convolution is defined by

$$f * g = \int_0^x d\xi\, f(\xi)\, g(x-\xi) = g * f$$

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SOLUTIONS

Chapter II

2. Suppose $\int \{G(\vec{r},\vec{r}_1) - G(\vec{r},\vec{r}_o)\}ds \neq 0$. Since G is determined to within an additive constant, let $F(\vec{r}_1) = \int ds\, G(\vec{r},\vec{r}_1)$ and replace $G(\vec{r},\vec{r}_1)$ by $G(\vec{r},\vec{r}_1) - F(\vec{r}_1)/s$ where s is the total area.

3. $$\phi = -\frac{1}{2\pi}\int_{-\infty}^{\infty} dx_o\, g(x_o) \log \{(x-x_o)^2 + y^2\}$$

4. $$\phi = \frac{1}{\pi}\int_{-\infty}^{\infty} dx_o\, f(x_o) \frac{y}{(x-x_o)^2 + y^2}$$

5. $$\phi = \frac{1}{2\pi}\int_{0}^{\infty} dy_o\, g(y_o) \log \frac{x^2 + (y+y_o)^2}{x^2 + (y-y_o)^2}$$

6. $$\phi = \frac{1}{\pi}\int_{0}^{\infty} dx_o\, f(x_o) \left\{\frac{y}{y^2 + (x-x_o)^2} + \frac{y}{y^2 + (x+x_o)^2}\right\}$$

7. $$\phi = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx_o \int_{-\infty}^{\infty} dy_o\, f(x_o,y_o) \frac{z}{[(x-x_o)^2 + (y-y_o)^2 + z^2]^{3/2}}$$

8. $$\phi = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx_o \int_{-\infty}^{\infty} dy_o\, g(x_o,y_o)\, [(x-x_o)^2 + (y-y_o)^2 + z^2]^{-1/2}$$

9. $$\phi = \frac{1}{2\pi}\iint dx_o dz_o\, f(x_o,z_o)\{y[(x-x_o)^2 + y^2 + (z-z_o)^2]^{-3/2}$$
$$- y[(x+x_o)^2 + y^2 + (z-z_o)^2]^{-3/2}\}$$

10. $$\phi = \frac{1}{2\pi}\iint dy_o dz_o\, g(y_o,z_o)\, \{[x^2 + (y-y_o)^2 + (z-z_o)^2]^{-1/2}\}$$
$$- [x^2 + (y+y_o)^2 + (z-z_o)^2]^{-1/2}$$
$$+ \frac{1}{2\pi}\iint dx_o dz_o\, f(x_o,z_o)\, y\{[(x-x_o)^2 + y^2 + (z-z_o)^2]^{-3/2}$$
$$+ [(x+x_o)^2 + y^2 + (z-z_o)^2]^{-3/2}\}$$

11. $$G(\vec{r},\vec{r}_o) = \frac{1}{|\vec{r}-\vec{r}_o|} - \frac{1}{|\vec{r}-\vec{r}_3|} + \frac{a}{r_3}\frac{1}{|\vec{r}-\vec{r}_4|} - \frac{a}{r_o}\frac{1}{|\vec{r}-\vec{r}_2|}$$

$$\vec{r}_3 = (-x_o,y_o,z_o) \qquad \vec{r}_2 = a^2\vec{r}_o/r_o^2 \qquad \vec{r}_4 = a^2\,\vec{r}_3/r_3^2$$

12. $$G(\vec{r},\vec{r}_o) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{(\ell-|m|)!}{(\ell+|m|)!}\,\frac{Y_{m\ell}(\theta,\phi)Y^*_{m\ell}(\theta_o,\phi_o)}{1-(a/b)^{2\ell+1}}$$

$$(r_<^{\ell} - \frac{a^{2\ell+1}}{r_<^{\ell+1}})(\frac{1}{r_>^{\ell+1}} - \frac{r_>^{\ell}}{b^{2\ell+1}})$$

Chapter III

1. $$\psi = \frac{1}{2\alpha\sqrt{\pi t}}\int_{-\infty}^{\infty} dx_o\; f(x_o)\;\exp - (x-x_o)^2/4\alpha^2 t$$

2. $$\psi = \frac{1}{2\alpha\sqrt{\pi t}}\int_{0}^{\infty} dx_o\; f(x_o)\;\{e^{-\frac{(x-x_o)^2}{4\alpha^2 t}} - e^{-\frac{(x+x_o)^2}{4\alpha^2 t}}\}$$

$$+ \frac{x}{2\alpha\sqrt{\pi}}\int_0^t dt_o\; g(t_o)(t-t_o)^{-3/2}\;\exp - x^2/4\alpha^2\;(t-t_o)$$

3. $$\psi = \frac{1}{2\alpha\sqrt{\pi t}}\int_{0}^{\infty} dx_o\; f(x_o)\;\{e^{-\frac{(x-x_o)^2}{4\alpha^2 t}} + e^{-\frac{(x+x_o)^2}{4\alpha^2 t}}\}$$

$$+ \frac{\alpha}{\sqrt{\pi}}\int_0^t dt_o\; g(t_o)(t-t_o)^{-1/2}\;\exp -x^2/4\alpha^2\;(t-t_o)$$

4. $$\psi = \frac{1}{4\alpha^2\pi t}\iint_{-\infty}^{\infty} dx_o dy_o\; f(x_o,y_o)e^{-\frac{(x-x_o)^2+(y-y_o)^2}{4\alpha^2 t}}$$

$$5. \quad \psi = \frac{1}{4\alpha^2\pi t}\iint dx_o dy_o\ f(x_o,y_o)\ \{e^{-\frac{(x-x_o)^2+(y-y_o)^2}{4\alpha^2 t}} - e^{-\frac{(x+x_o)^2+(y-y_o)^2}{4\alpha^2 t}}\}$$

$$+ \frac{x}{4\pi\alpha^2}\int_0^t dt_o \int dy_o\ g(y_o,t_o)(t-t_o)^{-2}\ e^{-\frac{x^2+(y-y_o)^2}{4\alpha^2(t-t_o)}}$$

$$6. \quad \psi = \frac{1}{4\alpha^2\pi t}\iint dx_o dy_o\ f(x_o,y_o)\ \{e^{-\frac{(x-x_o)^2+(y-y_o)^2}{4\alpha^2 t}} + e^{-\frac{(x+x_o)^2+(y-y_o)^2}{4\alpha^2 t}}\}$$

$$+ \frac{1}{2\pi}\int_0^t dt_o \int dy_o\ g(y_o,t_o)(t-t_o)^{-1}\ e^{-\frac{x^2+(y-y_o)^2}{4\alpha^2(t-t_o)}}$$

$$7. \quad \psi = \frac{1}{8\alpha^3(\pi t)^{3/2}}\iiint dx_o dy_o dz_o\ f(x_o,y_o,z_o) e^{-\frac{|\vec{r}-\vec{r}_o|^2}{4\alpha^2 t}}$$

where $|\vec{r}-\vec{r}_o|^2 = (x-x_o)^2 + (y-y_o)^2 + (z-z_o)^2$

$$8. \quad \psi = \frac{1}{8\alpha^3(\pi t)^{3/2}}\iiint d_o\ f(\vec{r}_o)\{e^{-\frac{|\vec{r}-\vec{r}_o|^2}{4\alpha^2 t}} - e^{-\frac{|\vec{r}-\vec{r}_i|^2}{4\alpha^2 t}}\}$$

$$+ \frac{x}{8\alpha^3\pi^{3/2}}\int_0^t dt_o \iint dy_o dz_o\ g(y_o,z_o,t_o)(t-t_o)^{-5/2}\ e^{-\rho^2/4\alpha^2(t-t_o)}$$

where $|\vec{r}-\vec{r}_i|^2 = (x+x_o)^2 + (y-y_o)^2 + (z-z_o)^2$

$\rho^2 = x^2 + (y-y_o)^2 + (z-z_o)^2$

9. $$\psi = \frac{1}{8\alpha^3(\pi t)^{3/2}} \iiint d\tau_o \; f(\vec{r}_o) \; e^{-\frac{|\vec{r}-\vec{r}_o|^2}{4\alpha^2 t}} + e^{-\frac{|\vec{r}-\vec{r}_1|^2}{4\alpha^2 t}} \}$$

$$+ \frac{1}{4\alpha\pi^{3/2}} \int_0^t dt_o \iint dy_o dz_o \; g(y_o, z_o, t_o)(t-t_o)^{-3/2} \; e^{-\rho^2/4\alpha^2(t-t_o)}$$

10. $$\psi = \frac{x}{8\alpha^3\pi^{3/2}} \int_0^t dt_o \iint dy_o dz_o \; g(y_o, z_o, t_o)(t-t_o)^{-5/2}$$

$$\{\exp - \rho_o^2/4\alpha^2(t-t_o) - \exp - \rho_1^2/4\alpha^2(t-t_o)\}$$

$$+ \frac{1}{8\alpha^3(\pi t)^{3/2}} \iiint d\tau_o \; f(\vec{r}_o) \sum_{i=0}^{3} (-1)^i \exp - |\vec{r}-\vec{r}_i|^2/4\alpha^2 t$$

$$\rho_o^2 = x^2 + (y-y_o)^2 + (z-z_o)^2 \qquad \rho_1^2 = x^2 + (y+y_o)^2 + (z-z_o)^2$$

$$x_1 = x_2 = -x_3 = -x_o \quad y_1 = -y_2 = -y_3 = y_o \quad z_1 = z_2 = z_3 = z_o$$

11. $$\psi = \frac{4}{ab} \sum_{m,n} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \; e^{-\alpha^2 k_{mn}^2 t} \; U(t)$$

$$\cdot \int_0^a dx_o \int_0^b dy_o \; f(x_o, y_o) \sin\frac{m\pi x_o}{a} \sin\frac{n\pi y_o}{b}$$

where $k_{mn}^2 = (\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2$

12. $$\psi = \frac{4\alpha^2\pi}{ab^2}\sum_{m,n} n \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$\cdot \int_0^t dt_o \int_0^a dx_o\, f(x_o,t_o)\sin\frac{m\pi x_o}{a}\, e^{-\alpha^2 k_{mn}^2(t-t_o)}$$

13. $$\psi = -\alpha^2 \sum_{\ell,m,n} \frac{\psi_{\ell mn}(\vec{r})}{\sqrt{c_{\ell mn}}} \int_0^t dt_o \iint a^2 \sin\theta_o\, d\theta_o\, d\phi_o\, f(\theta_o,\phi_o,t_o)$$

$$\cdot \frac{\partial}{\partial r_o}\{\frac{1}{\sqrt{r_o}} J_{\ell+\frac{1}{2}}(k_{\ell n}r_o)\} Y^*_{\ell m}(\theta_o,\phi_o)\ \exp - \alpha^2 k^2_{\ell n}(t-t_o)$$

$$\psi_{\ell mn}(\vec{r}) = \frac{1}{\sqrt{r}} J_{\ell+\frac{1}{2}}(k_{\ell n}r)\ Y_{\ell m}(\theta,\phi)/\sqrt{c_{\ell mn}}$$

$$c_{\ell mn} = \int_0^a r^2 dr \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi\, \frac{1}{r} J^2_{\ell+\frac{1}{2}}(k_{\ell n}r)\,|Y_{\ell m}(\theta,\phi)|^2$$

$k_{\ell n}$ are the roots of $J_{\ell+\frac{1}{2}}(k_{\ell n}a) = 0$

Chapter IV

3. $$\psi = \frac{1}{2}\{f(x+ct) + f(x-ct)\} + \frac{1}{2c}\int_{|x-ct|}^{x+ct} d\xi\, g(\xi) + h(t-x/c)U(t-x/c)$$

where f(x) is defined for negative argument by $f(-x) = -f(x)$

4. $$\psi = \frac{1}{2}\{f(x+ct) + f(x-ct)\} - cU(t-x/c)\int_0^{t-x/c} h(\xi)d\xi$$

$$+ \frac{1}{c}U(ct-x)\int_0^{ct-x} d\xi\, g(\xi) + \frac{1}{2c}\int_{|x-ct|}^{x+ct} d\xi\, g(\xi)$$

where f(x) is defined for negative argument by $f(-x) = f(x)$

6. $$\psi(x,y,t) = \frac{1}{2\pi c}\iint \frac{\psi_t(x_o,y_o,0)dx_o dy_o}{\sqrt{c^2t^2 - (x-x_o)^2 - (y-y_o)^2}} + \frac{1}{2\pi c}\frac{\partial}{\partial t}\iint \frac{\psi(x_o,y_o,0)dx_o dy_o}{\sqrt{c^2t^2 - (x-x_o)^2 - (y-y_o)^2}}$$

where the integration is over a circle of radius ct centered at (x,y).

Hint: $\partial G/\partial t_o = -\partial G/\partial t$.

7. $$\psi = \frac{1}{2\pi}\iint dx_o dy_o \left\{\frac{z}{R_o^3} f(x_o,y_o,t - R_o/c) - \frac{z}{cR_o^2} f_t(x_o,y_o,t - R_o/c)\right\}$$

where $R_o^2 = (x-x_o)^2 + (y-y_o)^2 + z^2$

8. $$F(r) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} A_{m\ell}\, h_\ell(kr) Y_{m\ell}(\theta,\phi)$$

$$A_{m\ell} = -ik(2\ell+1)\frac{(\ell-|m|)!}{(\ell+|m|)!}\frac{j_\ell(ka)}{h_\ell(ka)}\, h_\ell(kr_o) Y^*_{m\ell}(\theta_o,\phi_o)$$

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13. ABSTRACT

This report is an introduction to Green's functions intended for workers in acoustics but also suitable for general purposes. The first chapter reviews the properties of the various types of second order linear partial differential equations and discusses a simple Green's function as an introductory example. Subsequent chapters discuss the Green's functions associated with the Laplace, diffusion, and wave equations, and the final chapter deals with the Helmholtz equation through some applications from acoustics. Appendix 1 discusses the formal manipulation of delta functions, and Appendix 2 is a brief introduction to Fourier and Laplace transforms. A bibliography, exercises, and solutions are also included.

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