FINAL REPORT

STUDIES IN ELECTROMAGNETIC SCATTERING FROM TURBULENT WAKES

Drexel University

DISCLAIMER NOTICE

THIS DOCUMENT IS THE BEST QUALITY AVAILABLE.

COPY FURNISHED CONTAINED A SIGNIFICANT NUMBER OF PAGES WHICH DO NOT REPRODUCE LEGIBLY.

FINAL REPORT

STUDIES IN ELECTROMAGNETIC SCATTERING FROM TURBULENT WAKES

Principal Investigator

John Jarem

Graduate Research Assistants

Joseph Chawki Elias Hassab
Richard J. Wasneski

Prepared For:

Army Research Office
Box CM, Duke Station
Durham, North Carolina 27706

Sponsored By:

Contract No. DAH-CO-4-67C-0072
Continuation Oct 1969 - Oct 1970
ARPA Order 1009
Drexel University Project No. 349

TABLE OF CONTENTS

Abstract Report 1 Scalar Scattering from Turbulent Plasmas by a Modified Method of Iteration

The scattering of scalar waves from a turbulent plasma is investigated by a modification of the iteration procedure used in deriving the so-called first order smoothing equation. Electron density fluctuations are modeled by locally homogeneous turbulence.

A wave number space representation is used for the random scattering integral equation which is decomposed into coupled integral equations for coherent and fluctuating scattering amplitudes. These equations are solved by invoking the "large scatterer approximation". This density independent approximation is used to express the coherent field in terms of the fluctuating field. Successive iteration then yields a solution for the fluctuating field. The incoherent radar cross section is calculated to second order for an axially symmetric wake section illuminated nose-on. The turbulence is modeled by an isotropic exponential correlation function and a turbulent intensity which is axially uniform with a Gaussian fall-off radially. The present model predicts a cross section saturation with increasing turbulent fluctuations.

Abstract Report 2 Vector Scattering from a Turbulent Plasma by a Modified Method of Iteration

The analysis of Report 1 is generalized to the scattering of electromagnetic waves. For vector scattering we must solve two coupled vector integral equations for the coherent and fluctuating fields. The "large-scatterer approximation" (scatter dimensions large compared to correlation

length) leads to a vector integral equation for the coherent field in terms of the fluctuating field involving a non-symmetric tensor whose elements depend on scatterer characteristics and the dyadic free space Green's function. This is used to obtain an integral equation for the fluctuating field which is solved by iteration to second order in the density. Using the quasi-normal hypothesis a general expression is derived for the incoherent bistatic scattering cross section of an axi-symmetric wake section illuminated at non-normal incidence and arbitrary polarization.

Abstract Report 3 Application of the Modified Method of Iteration for Scattering to SRI Experiment

The general expression for the incoherent scattering cross section derived in Report 2 is specialized to compute the backscatter cross section for direct and cross polarized directions. The SRI turbulent flame is illuminated at normal incidence with direct polarization along the flame axis. The turbulent intensity is modeled by a Gaussian radial and axially uniform distribution with a correlation function of exponential form. These models are close approximations to the measured SRI scatterer characteristics. Direct and cross polarized backscatter cross sections are calculated for a wide range of RMS electron density values including underdense and near critical density regions. Agreement between experimental and theoretical absolute cross sections fall well within the 3 db SRI measurement uncertainty. For the very underdense case the theoretical model yields results in agreement with first Born theory for direct polarization and second Born theory (Ruffive-DeWolfe) for cross polarization. As critical density is approached the model predicts cross section saturation observed experimentally (SRI, RCA Montreal, Bell Labs).

Abstract Report 4 Wave Scattering from a Unidimensional Layer with Strong Random Irregularities by the Method of Smoothing

Under study is the multiple scattering of a plane wave normally incident on a layer filled with a strongly fluctuating plasma. The electron density fluctuations are locally homogeneous, and the mean dielectric constant of the layer may be appreciably different from the imbedding medium. The formulation of the problem is based on the Dyson-Smoothing equations in the bilocal approximation. The approximate treatment of the mean dielectric constant as homogeneous throughout is shown to be generally inconsistent. Exact treatment of the problem is developed in the limit of small-scale fluctuations; the dependence of the scattered power on a broad range of plasma parameters is displayed and discussed for typical casses of physical interest.

In the case of a mean dielectric constant throughout, a new mathematical formulation is presented. The resulting integral equation is singular and belongs to the standard form studied in the literature. This formulation is closer to reality since it avoids the assumption of sharp boundaries for the fluctuations.

Abstract Report 5 Scattering Coefficient of One-dimensional Plasmas of Epstein-type Profiles with Random Irregularities

The problem of electromagnetic scattering by inhomogeneous one-dimensional plasmas with electron density of Epstein-type profiles, superimposed with random irregularities is studied by means of a perturbation method. An exact Green's function is derived for the problem, and the resulting integral equation is solved approximately by the Neumann

iterative technique. Coherent and incoherent power reflection and transmission coefficients are obtained to a consistent second order accuracy, and an expression is given to define the parameter validity region of the solution. In addition, the incoherent reflection coefficient is graphically displayed for an interesting parameter range.

Where meaningful, the special case of the halfspace of plasma is compared to an exact Monte Carlo computer experiment for a plasma slab. It is shown that for the given region of validity, the solution compares favorably with experiment.

REPORT 1

Scalar Scattering from Turbulent Plasmas by a Modified Method of Iteration

1. Introduction

In this report we investigate the scattering of electromagnetic waves from a turbulent plasma by a modification of the iteration procedure in the method of smoothing. The integral equation for the scattered field is transformed into an integral equation for the scattering amplitude. Coupled integral equations for the coherent and fluctuating amplitudes are formed and solved by a method which invokes the "large scatterer approximation" during the iteration.

2. Analysis

We consider the scalar scattering problem defined by the integral equation,

$$\psi(\bar{r}) = e^{-i\bar{k}\cdot\bar{r}} + \psi_s(\bar{r}) = e^{-i\bar{k}\cdot\bar{r}} + k^2\Omega_{pl}^2 \int_V d\bar{r}_1 \frac{e^{-ik|\bar{r}-\bar{r}_1|}}{-4\pi|\bar{r}-\bar{r}_1|}\, n(\bar{r}_1)\, \psi(\bar{r}_1) \qquad (1)$$

where

$\psi(\bar{r})$ is the total field, $\psi_s(\bar{r})$ is the scattered field

$\Omega_{pl}^2 = \dfrac{n_{RMS}e^2}{m\varepsilon_o\omega^2}$, the normalized plasma frequency based on a reference RMS electron density

$n(\bar{r}_1) = \dfrac{\delta n(\bar{r}_1,t)}{n_{RMS}}$ electron density fluctuation normalized on n_{RMS}.

V is the scattering volume.

In the far field Eq. (1) for the scattered field is

$$\psi_s(\bar{r}) = \frac{e^{-ikr}}{r}\left[\frac{k^2\Omega_{pl}^2}{-4\pi}\int_V d\bar{r}_1\, e^{ik\hat{r}\cdot\bar{r}_1}\, n(\bar{r}_1)\, \psi(\bar{r}_1)\right] \qquad (2)$$

We identify the bracket term as the scattering amplitude in the $\hat{r}$ direction, $F(k\hat{r}) = F(\bar{\kappa})$. We derive an integral equation for this quantity by multiplying (1) thru by $\eta(\bar{r})\exp(i\bar{\kappa}\cdot\bar{r})$ and integrating over V.

$$\int_V \psi(\bar{r})\,\eta(\bar{r})e^{i\bar{\kappa}\cdot\bar{r}}\,d\bar{r} = \int_V e^{-i\bar{k}\cdot\bar{r}}\,\eta(\bar{r})e^{i\bar{\kappa}\cdot\bar{r}}\,d\bar{r} \tag{3}$$

$$+ k^2\Omega_{pl}^2 \int_V \eta(\bar{r})e^{i\bar{\kappa}\cdot\bar{r}}\,d\bar{r} \int_V d\bar{r}_1 \int_{-\infty}^{\infty} \frac{e^{i\bar{p}\cdot(\bar{r}_1-\bar{r})}}{(2\pi)^3(k^2-p^2)}\,d\bar{p}\;\eta(\bar{r}_1)\;\psi(\bar{r}_1)$$

where we have used the spectral representation

$$\frac{e^{-ikR}}{-4\pi R} = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} \frac{e^{i\bar{p}\cdot\bar{R}}\,d\bar{p}}{k^2 - p^2} \quad , \quad \bar{R} = \bar{r}_1 - \bar{R} \tag{4}$$

with

$$I(\bar{\kappa}) = \int_V \psi(\bar{r})\,\eta(\bar{r})e^{i\bar{\kappa}\cdot\bar{r}}\,d\bar{r} \tag{5}$$

$$\eta(\bar{\kappa}) = \int_V \eta(\bar{r})e^{i\bar{\kappa}\cdot\bar{r}}\,d\bar{r} \tag{6}$$

equation (3) becomes

$$I(\bar{\kappa}) = \eta(\bar{\kappa} - \bar{k}) + \frac{k^2\Omega_{pl}^2}{(2\pi)^3}\int_{-\infty}^{\infty} \frac{d\bar{p}}{k^2 - p^2}\;\eta(\bar{\kappa} - \bar{p})\;I(\bar{p}) \tag{7}$$

This is an integral equation for the scattering function $I(\bar{\kappa})$ and according to (2) the scattering amplitude is

$$F(\bar{\kappa}) = \frac{k^2\Omega_{pl}^2}{-4\pi}\,I(\bar{\kappa}) \tag{8}$$

The quantity $\eta(\bar{\kappa})$ is a spectral representation for the electron density fluctuation. We assume that scatterer dimensions are large compared to the fluctuation scale (correlation length ℓ). The scatterer is modeled by an electron density fluctuation super-imposed on a mean electron density background. The fluctuations are described by a locally homogeneous turbulence

whose intensity is proportional to the square of the mean density. At the outset (eq. (1)) we have assumed that the mean background contribution to the scattering is negligible. If in Eq. (7) we measure wavelength in units of correlation length then all quantities can be interpreted as non-dimensional and we indentify

$$\varepsilon = \frac{(2\pi \frac{\ell}{\lambda} \Omega_{p1})^2}{(2\pi)^3} \tag{9}$$

as the natural expansion parameter for the problem.

Next we form the equations for the coherent ($I_o(\bar{\kappa})$) and fluctuating ($I_1(\bar{\kappa})$) scattering functions by applying the ensemble averaging operator P to Eq. (7).

$$PI(\kappa) = I_o(\kappa) = P\eta(\kappa-k) + \varepsilon\, G(p)\, P\eta(\kappa-p)\, I(p)$$

$$I_o(\kappa) = \varepsilon\, G(p)\, P\eta(\kappa-p)\, I_1(p) \tag{10}$$

where we used $P\eta = 0$, $I = I_o + I_1$, $PI_1 = 0$,

$G(p) \equiv \int_{-\infty}^{\infty} \frac{d\bar{p}}{k^2-p^2}$, and vector arrows were suppressed.

If Eq. (10) is subtracted from (7) we obtain

$$I_1(\kappa) = \eta(\kappa-k) + \varepsilon\, G(p)\, \eta(\kappa-p)\, I_o(p) + \varepsilon\, G(p)\, \eta(\kappa-p)\, I_1(p) - \varepsilon\, G(p)\, P\, \eta(\kappa-p)\, I_1(p) \tag{11}$$

We shall now solve pair (10) and (11) by iteration with ε as an expansion parameter. First use (11) to form $P\,\eta(\kappa-p)\, I_1(p)$ for use in (10).

$$P\, I_1(p)\, \eta(\kappa-p) = P\, \eta(p-k)\eta(\kappa-p) + \varepsilon\, G(p)\, P\, \eta(p-p_1)\eta(\kappa-p)\, I_o(p_1) + \varepsilon\, G(p)\, P\, \eta(p-p_1)\eta(\kappa-p)\, I_1(p) \tag{12}$$

In a previous study[1] it has been shown that

$$p\, \eta(p_1)\, \eta(p_2) = A(p_1+p_2)\, \Phi(\frac{p_1-p_2}{2})$$

$$p\, \eta(p_1)\, \eta^*(p_2) = A(p_1-p_2)\, \Phi\, (\frac{p_1+p_2}{2}) \tag{13}$$

---

1 J.Jarem, "Studies in Electromagnetic Scattering from Turbulent Wakes" Drexel Univ., Proj. 349 Report, Sept. 1969, Contract DAH-CO-4-67C-0072, ARPA Order 1009.

where $A(p)$ and $\Phi(p)$ are the Fourier transforms of the turbulent intensity and correlation functions respectively. The scale over which the intensity changes is large compared to the correlation length. In the limit of a large scatterer $A(p)$ approaches a delta function. We shall use this property in evaluating certain integrals in the iteration. When the averages in (12) are expressed according to (13) we obtain

$$PI_1(p)n(\kappa-p) = A(\kappa-k)\Phi(\frac{\kappa+k}{2} - p)$$
$$+ \varepsilon\, G(p_1)I_o(p_1)A(\kappa-p_1)\Phi(\frac{\kappa+p_1}{2} - p) \qquad (14)$$
$$+ \varepsilon\, G(p_1)Pn(\kappa-p)n(p-p_1)I_1(p_1)$$

Viewing $A(\kappa-p_1)$ as a delta function centered about $p_1 = \kappa$ the second RHS term is integrated approximately to

$$\varepsilon\, I_o(\kappa)\, \Phi(\kappa-p)\, G(p_1)\, A(\kappa-p_1) \qquad (15)$$

When the above results are used in (10) we obtain

$$I_o(\kappa) = \varepsilon\, G(p)\, A(\kappa-k)\, \Phi(\frac{\kappa+k}{2} - p)$$
$$+ \varepsilon^2\, I_o(\kappa)\, G(p)\, \Phi(\kappa-p)\, G(p_1)\, A(\kappa-p_1)$$
$$+ \varepsilon^2\, G(p)\, G(p_1)\, P\, n(\kappa-p)n(p-p_1)I_1(p_1) \qquad (16)$$

Letting $g_1(\kappa) = \int_{-\infty}^{\infty} \frac{d\bar{p}\ \Phi(\kappa-p)}{p^2 - k^2}$ (17)

$$g_2(\kappa) = \int_{-\infty}^{\infty} \frac{d\bar{p}_1\ A(\kappa- p_1)}{p_1^2 - k^2} \qquad (18)$$

we can solve for $I_o(\kappa)$ to obtain

$$I_o(\kappa) = \frac{1}{1-\varepsilon^2 g_1(\kappa)g_2(\kappa)} \{ \varepsilon\, A(\kappa-k)\, G(p)\, \Phi\, (\frac{\kappa+k}{2} - p)$$
$$+ \varepsilon^2\, G(p)\, G(p_1)\, P\, n(\kappa-p)n(p-p_1)\, I_1(p_1)\} \qquad (19)$$

Next we use (19) to eliminate the coherent scattering function I_o from (11).

$$I_1(\kappa) = \eta(\kappa-k) + \varepsilon^2\, G(p)\, \eta(\kappa-p)\, \frac{A(p-k)\, G(p_1)\, \Phi\, (\frac{p+k}{2} - p_1)}{1 - \varepsilon^2\, g_1(p)\, g_2(p)}$$

$$+ \varepsilon^3\, G(p)\, \eta(\kappa-p)\, \frac{G(p_1)\, G(p_2)\, P\, \eta(p-p_1)\, \eta(p_1-p_2)\, I_1(p_2)}{1 - \varepsilon^2\, g_1(p)\, g_2(p)}$$

$$+ \varepsilon\, G(p)\, \eta(\kappa-p)\, I_1(p) - \varepsilon\, G(p)\, P\, \eta(\kappa-p)\, I_1(p) \qquad (20)$$

The first and second RHS terms can be combined after approximate integration to

$$\eta(\kappa-h) + \varepsilon^2\, \frac{\eta(\kappa-k)\, G(p)}{1-\varepsilon^2 g_1(k) g_2(k)}\, A(p-k)\, G(p_1)\, \Phi(k-p_1) = \eta(\kappa-k) \frac{1}{1 - \varepsilon^2 g_1(k) g_2(k)} \qquad (21)$$

and (20) can be written

$$I_1(\kappa) = \frac{\eta(\kappa-k)}{1-\varepsilon^2 g_1(k) g_2(k)} + \varepsilon\, G(p)\, \eta(\kappa-p)\, I_1(p) - \varepsilon\, G(p)\, P\, \eta(\kappa-p)\, I_1(p)$$

$$+ \varepsilon^3 G(p)\, \eta(\kappa-p)\, \frac{G(p_1)\, G(p_2) P\, \eta(p-p_1)\, \eta(p_1-p_2)\, I_1(p_2)}{1 - \varepsilon^2\, g_1(p)\, g_2(p)} \qquad (22)$$

To order ε we can neglect the last term in (22) which contributes at order ε^3. A single iteration of the remaining terms yields

$$I_1(\kappa) = \frac{1}{1-\varepsilon^2 g_1(k) g_2(k)}\, [\eta(\kappa-k) + \varepsilon\, G(p)\eta(\kappa-p)\eta(p-k) - \varepsilon G(p) P \eta(\kappa-p)\eta(p-k)] \qquad (23)$$

The incoherent differential scattering cross section is related to the scattering amplitude[1] by

$$\sigma(\kappa) = P\, F_1(\kappa)\, F_1^*(\kappa) = P\, (\frac{k^2\Omega^2 p1}{-4\pi})^2\, I_1(\kappa)\, I_1^*(\kappa) = \frac{\sigma_{RC}(\kappa)}{4\pi} \qquad (24)$$

Making use of (23) and $P\eta = P\eta^3 = 0$ the cross section can be written

$$\sigma(\kappa) = \frac{k^4 \Omega_{p1}^4}{(4\pi)^2 |1-\epsilon^2 g_1(k) g_2(k)|^2} [p\, \eta(\kappa-k)\eta^*(\kappa-k) + \epsilon^2 G(p)G^*(p_1)P\eta(\kappa-p)\eta(p-k)\eta^*(\kappa-p_1)\eta^*(p_1-k) - \epsilon^2 G(p)\, P\eta(\kappa-p)\eta(p-k)G^*(p_1)P\eta^*(\kappa-p_1)\eta^*(p_1-k)] \quad (25)$$

For quasi-normal turbulence the fourth order moment in (25) can be expressed as the following sum:

$$P\eta(\kappa-p)\eta(p-k)\, P\eta^*(\kappa-p_1)\eta^*(p_1-k) + P\eta(\kappa-p)\eta^*(\kappa-p_1)P\eta(p-k)\eta^*(p_1-k) + P\eta(\kappa-p)\eta^*(p_1-k)\, P\eta(p-k)\eta^*(\kappa-p_1) \quad (26)$$

We note that the first term in (26) when used in (25) cancels the last term and using (13) to reduce the remaining terms leads to

$$\sigma(\kappa) = \frac{k^4 \Omega_{p1}^4}{(4\pi)^2 |1-\epsilon^2 g_1 g_2|^2} [A(0)\, \phi(\kappa-k) + \epsilon^2\, G(p)G^*(p_1)A(p_1-p)\phi(\kappa-\frac{p_1+p}{2})A(p-p_1)\phi(\frac{p+p_1}{2}-k) + \epsilon^2\, G(p)G^*(p_1)A(\kappa+k-p-p_1)\phi(\frac{\kappa-k+p_1-p}{2})A(p+p_1-k-\kappa)\phi(\frac{\kappa-k+p-p_1}{2})] \quad (27)$$

If A is again viewed as a delta function for the p_1 integration we shall finally obtain, with A(p) and $\phi(p)$ even and real,

$$\sigma(\kappa) = \frac{A(0)\, k^4 \Omega_{p1}^4}{(4\pi)^2 |1-\epsilon^2 g_1 g_2|^2} [\phi(\kappa-k) + \epsilon^2\, G(p)\, \phi(p-\kappa)\, \phi(p-k)\, G^*(p_1)\, A(p_1-p) + \epsilon^2\, G(p)\, \phi(p-\kappa)\, \phi(p-k)\, G^*(p_1)\, A(p_1-(\kappa+k-p))] \quad (28)$$

3. Backscattering Example

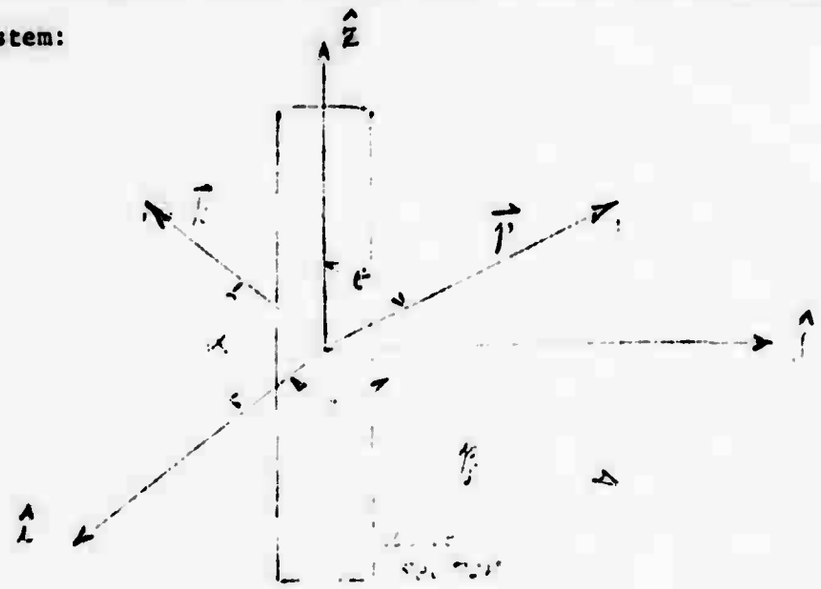
For our first example we shall compute the backscatter cross section of an axially symmetric wake section illuminated nose-on. The turbulence is modeled by an isotropic exponential correlation function $\exp(-\frac{R}{\ell})$ with corresponding spectrum

$$\Phi(\bar{p}) = \int_{-\infty}^{\infty} d\bar{R}\, e^{i\bar{p}\cdot\bar{R}}\, e^{-\frac{R}{\ell}} = \frac{\Phi_o}{[1 + (p\ell)^2]^2}, \quad \Phi_o = 8\pi\ell^3 \tag{29}$$

and a turbulent intensity which is uniform axially over a length 2b and has a Gaussian fall-off radially with an e-folding radius $a/\sqrt{2}$. For such a model[1]

$$A(\bar{p}) = A_o\, e^{-\frac{1}{2}(\frac{ap_\rho}{2})^2}\, \frac{\sin bp_3}{bp_3}, \quad A_o = C_T \pi a^2 b \tag{30}$$

To carry out the integrals called for in (28) we choose the following corrdinate system:



The (x,z) plane is defined by vectors $(\hat{k},\hat{z})$ with $\hat{k} = \hat{i}\cos\alpha + \hat{z}\sin\alpha$ where α = angle of incidence and $\hat{z}$ falls along the wake axis. $(\hat{i},\hat{j},\hat{z})$ form a rectangular base. For nose illumination $\alpha = 90°$ and $\hat{k} = \hat{z}$. For backscatter $\bar{\kappa} = k\hat{r} = k(-\hat{k}) = -k\hat{z}$. From (17)

$$g_1(-k\hat{z}) = \int_{-\infty}^{\infty} \frac{d\bar{p}\ \Phi(-k\hat{z}-\bar{p})}{p^2-k^2+i\varepsilon} = \int_{-\infty}^{\infty} \frac{d\bar{p}\ \Phi(p)}{p^2-2\bar{k}\cdot\bar{p}+i\varepsilon} \tag{31}$$

In spherical corrdinates (p,θ,ϕ) with $\bar{k}\cdot\bar{p} = kp\cos\theta$ we have for isotropic turbulence

$$g_1 = 2\pi \int_0^\infty p^2\, dp\, \phi(p) \int_0^\pi \frac{\sin\theta\, d\theta}{p^2 - 2kp\cos\theta + i\varepsilon}$$

$$= 2\pi \int_0^\infty p^2\, dp\, \phi(p) \int_0^1 \frac{du}{p^2 - 2kpu + i\varepsilon} \tag{32}$$

The integration over p can be expressed as

$$\int_{-\infty}^{\infty} \frac{p^2\, dp\, \phi(p)}{p^2 - 2kup + i\varepsilon} = \frac{1}{2ku} \int_{-\infty}^{\infty} \frac{p^2\, dp\, \phi(p)}{p - (2ku - i\varepsilon)} - \frac{1}{2ku} \int_{-\infty}^{\infty} \frac{p^2\, dp\, \phi(p)}{p - i\varepsilon} \tag{33}$$

$$\lim_{\varepsilon \to o} \int_{-\infty}^{\infty} \frac{p^2\, dp\, \phi(p)}{p - i\varepsilon} = i\pi\, p^2\, \phi(p)]_{p=o} + P \int_{-\infty}^{\infty} p\, \phi(p)\, dp = 0 \tag{34}$$

Using model (29) in the first RHS term in (31) we find by contour integration

$$\frac{1}{2ku} \int_{-\infty}^{\infty} \frac{p^2\, dp}{(p - 2ku + i\varepsilon)} \frac{\phi_o}{[1 + (p\ell)^2]^2} = \frac{-i\, 2\pi\phi_o}{k^3\, \ell^4} \frac{u + \frac{k\ell}{4i}((2u)^2 - \frac{1}{k^2\ell^2})}{((2u)^2 + \frac{1}{k^2\ell^2})^2} \tag{35}$$

When this result is used in (30) we obtain

$$g_1(k) = (2\pi)^3\, \ell^2\, \frac{(1 - i\, 2k\ell)}{1 + (2k\ell)^2} \tag{36}$$

Next we compute $g_2(-k\hat{z})$ according to (18).

$$g_2(k) = \int_{-\infty}^{\infty} \frac{d\bar{p}\, A(\bar{p})}{\bar{p}\cdot(\bar{p} + 2\bar{k}) + i\varepsilon} = \int_{-\infty}^{\infty} \frac{d\bar{p}\, A(\bar{p})}{\bar{p}\cdot 2\bar{k} + i\varepsilon} = \int_{-\infty}^{\infty} \frac{d\bar{p}\, A(\bar{p})}{2kp_3 + i\varepsilon} \tag{37}$$

With model (30) for $A(\bar{p})$ we write

$$g_2(k) = \int_0^\infty p_\rho dp_\rho\, A_o e^{-\frac{1}{8}(ap_\rho)^2} \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dp_3\, \frac{\sin bp_3}{bp_3}\, \frac{1}{2kp_3 + i\varepsilon} \tag{38}$$

Plemelj's formula yields

$$\lim_{\varepsilon \to o} \int_{-\infty}^{\infty} dp_3 \, \frac{\sin bp_3}{bp_3} \, \frac{1}{2kp_3 + i\varepsilon} = -\frac{i\pi}{2k} \tag{39}$$

When this result is used in (38) we obtain after integration

$$g_2(k) = \frac{-i_{\backslash}(2\pi)^3 \, C_T \, b}{2k} \tag{40}$$

We complete the evaluation of (28) by first combining the last two terms.

For backscatter ($\bar{\kappa} = -\bar{k}$) we have, calling this sum C_2,

$$C_2 = 2\varepsilon^2 G(\bar{p}) \, \phi(\bar{p}+\bar{k}) \, \phi(\bar{p}-\bar{k}) \, G^*(\bar{p}_1) \, A(\bar{p}_1-\bar{p}) \tag{41}$$

For model (32) the above integration reduces to[1]

$$C_2 = 2\pi^3 \varepsilon^2 \, \frac{\sqrt{2\pi} \, A_o}{ab} \int_o^{2\pi} d\phi \int_o^{\pi} d\theta \, \phi(\hat{pk}+\bar{k}) \, \phi(\hat{pk}-\bar{k}) \tag{42}$$

For the present turbulent model the product spectrum is written

$$\frac{\phi_o}{(1+k^2\ell^2|\hat{p}+\hat{k}|^2)^2} \, \frac{\phi_o}{(1+k^2\ell^2|\hat{p}-\hat{k}|^2)^2} = \frac{\phi_o^2}{[1+(2k\ell)^2]^4 [1-(\frac{(2k\ell)^2}{1+(2k\ell)^2})^2 \cos^2\theta]^2} \tag{43}$$

If this is used in (42) we will obtain

$$C_2 = 2^7 \pi^8 \varepsilon^2 \sqrt{2\pi} \, C_T \, a \, \ell^6 \, \frac{2 + 4(2k\ell)^2 + (2k\ell)^4}{[1 + (2k\ell)^2]^3 \, [1 + 2(2k\ell)^2]^{3/2}} \tag{44}$$

If (44), (40) and (36) are used in (25) we finally obtain

$$\sigma_{RC} = \frac{\xi^2 \frac{\pi}{4} C_T a^2(kb) \frac{\beta^3}{(1+\beta^2)^2} \{1+\xi^2 \frac{\pi\sqrt{\pi}}{32} C_T(ka) \frac{\beta^3(2+4\beta^2+\beta^4)}{(1+\beta^3)(1+2\beta^2)^{3/2}}\}}{|1 + \xi^2 \frac{C_T(kb)}{8} \frac{\beta^3}{1+\beta^2} + i \, \xi^2 \frac{C_T(kb)}{8} \frac{\beta^2}{1+\beta^2}|^2}$$

where

$$\xi = \Omega_{pl}^2 = \frac{n_{RMS}}{n_{crit.}}$$

$\beta = 2k\ell$, ℓ = correlation length

$C_T \sim \frac{1}{2}$, constant of proportionality relating m.s. fluctuation to square of mean electron density.

$2b$ = length of wake section

$\frac{a}{\sqrt{2}}$ = effective radius of wake, e folding distance in Gaussian radial fall-off.

σ_{RC} = Radar cross section

REPORT 2

Vector Scattering from a Turbulent Plasma by a Modified Method of Iteration

1. Introduction

In Report 1 we investigated the scalar scattering of electromagnetic waves from a turbulent plasma by a modification of the usual iteration procedure used to derive the integral equation for the coherent field in the so called method of smoothing. In this report we generalize the modification to include vector scattering for the purpose of investigating polarization effects.

2. Analysis

We start with the vector integral equations for the electric field[1]

$$\bar{E}(\bar{r}) = \bar{E}_o(\bar{r}) + k^2\Omega_{pl}^2 \int_V d\bar{r}_1 \overleftrightarrow{G}(\bar{r}-\bar{r}_1)\cdot\bar{E}(\bar{r}_1)\ \eta(\bar{r}_1,t) \tag{1}$$

where $\bar{E}$ is the total field; $\bar{E}_o = \hat{q}\, e^{-i\bar{k}\cdot\bar{r}}$ is the incident field; Ω_{pl} is the plasma frequency based on a reference RMS electron density; $\overleftrightarrow{G}(\bar{r},\bar{r}_1)$ dyadic Green's function for free space; $\eta(\bar{r},t)$ electron density fluctuation normalized on n_{RMS}. Integral Eq. (1) is transformed to an integral equation for the scattering function $\bar{I}(\bar{\kappa})$[1]; $\bar{\kappa} = \hat{k}\bar{r}$,

$$\bar{I}(\bar{\kappa}) = \hat{q}\,\eta(\bar{\kappa}-\bar{k}) + \frac{k^2\Omega_{pl}^2}{(2\pi)^3}\int_{-\infty}^{\infty} d\bar{p}\ \overleftrightarrow{G}(\bar{p})\cdot\bar{I}(\bar{p})\ \eta(\bar{\kappa}-\bar{p}) \tag{2}$$

where

$$\bar{I}(\bar{p}) = \int_V d\bar{r}\, e^{i\bar{p}\cdot\bar{r}}\ \eta(\bar{r})\ \bar{E}(\bar{r}) \tag{3}$$

$$\eta(\bar{\kappa}) = \int_V d\bar{r}\, e^{i\bar{p}\cdot\bar{r}}\ \eta(\bar{r}) \tag{4}$$

---

1 J. Jarem, "Studies in Electromagnetic Scattering from Turbulent Wakes". Drexel Univ. Project 349 Report, Sept. 1969, Contract DAH-CO-4-67C-0072, ARPA Order 1009.

$$\overleftrightarrow{G}(\bar{p}) = \int d\bar{R}\, e^{i\bar{p}\cdot\bar{R}}\, \overleftrightarrow{G}(\bar{R}) = \frac{\bar{p}\bar{p}k^{-2} - \overleftrightarrow{\delta}}{p^2 - k^2 + i\varepsilon} + \frac{\overleftrightarrow{\delta}}{3k^2} \tag{5}$$

$$\overleftrightarrow{G}(\bar{R}) = (\overleftrightarrow{\delta} + \frac{\nabla\nabla}{k^2})\frac{e^{-ikR}}{-4\pi R} + \frac{\overleftrightarrow{\delta}\,\delta(R)}{3k^2}\,, \quad \bar{R} = \bar{r} - \bar{r}_1, \tag{6}$$

If the integral equation can be solved for $\bar{I}(\bar{\kappa})$ then the scattering amplitude $\bar{F}(\bar{\kappa})$ is given by

$$\bar{F}(\bar{\kappa}) = -\frac{k^2\Omega^2_{pl}}{4\pi}(\overleftrightarrow{\delta} - \hat{r}\hat{r})\cdot\bar{I}(\bar{\kappa}) \tag{7}$$

and the radar and differential scattering cross sections are respectively[17]

$$\sigma_{RC}(\bar{\kappa}) = 4\pi\,\sigma(\bar{\kappa}) = 4\pi\,\langle\bar{F}(\bar{\kappa})\cdot\bar{F}^*(\bar{\kappa})\rangle \tag{8}$$

To solve (2) we consider $\varepsilon = \frac{k^2\Omega^2_{pl}}{(2\pi)^3}$ as an expansion parameter and we form the coupled integral equations for the coherent ($\bar{I}_o(\bar{\kappa})$) and fluctuating ($\bar{I}_1(\bar{\kappa})$) scattering functions in the usual way to obtain:

$$\bar{I}_o(\bar{\kappa}) = \varepsilon\int_{-\infty}^{\infty} d\bar{p}\,\overleftrightarrow{G}(\bar{p})\cdot P\,\bar{I}_1(\bar{p})\,\eta(\bar{\kappa}-\bar{p}) \tag{9}$$

$$\bar{I}_1(\bar{\kappa}) = \hat{q}\,\eta(\bar{\kappa}-\bar{k}) + \varepsilon\int_{-\infty}^{\infty} d\bar{p}\,\overleftrightarrow{G}(\bar{p})\cdot\bar{I}_o(\bar{p})\,\eta(\bar{\kappa}-\bar{p}) \tag{10}$$

$$+ \varepsilon\int_{-\infty}^{\infty} d\bar{p}\,\overleftrightarrow{G}(\bar{p})\cdot(\bar{I}_1(\bar{p}) - P\,\bar{I}_1(\bar{p}))\,\eta(\bar{\kappa}-\bar{p})$$

We have used $P\,\bar{I}(\bar{\kappa}) = \langle\bar{I}(\bar{\kappa})\rangle = \bar{I}_o(\bar{\kappa})$, $P\,\bar{I}_1(\bar{\kappa}) = 0$, and $P\,\eta(\bar{\kappa}) = 0$. If we let

$$\overleftrightarrow{G}(\bar{p}) = \int_{-\infty}^{\infty} d\bar{p}\,\overleftrightarrow{G}(\bar{p})$$

then (9) and (10) can be written, suppressing arrows over vector arguments,

$$\bar{I}_o(\kappa) = \varepsilon\,\overleftrightarrow{G}(p)\cdot P\,\bar{I}_1(p)\,\eta(\kappa-p) \tag{11}$$

---

17 P.M. Morse & H. Feshbach, Methods of Theoretical Physics, Part II, McGraw-Hill, 1953.

$$\bar{I}_1(\kappa) = \hat{q}\,\eta(\kappa-k) + \varepsilon\,\overleftrightarrow{G}(p)\cdot\bar{I}_o(p)\,\eta(\kappa-p) + \varepsilon\,\overleftrightarrow{G}(p)(I-P)\cdot\bar{I}_1(p)\,\eta(\kappa-p) \qquad (12)$$

We use expression (12) for $\bar{I}_1$ in Eq. (11) to evaluate $P\,\bar{I}_1(p)\,\eta(\kappa-p)$. Eq. (11) becomes

$$\bar{I}_o(\kappa) = \varepsilon\,\overleftrightarrow{G}(p)\cdot\hat{q}\,P\,\eta(\kappa-p)\,\eta(p-k)$$

$$+ \varepsilon^2\,\overleftrightarrow{G}(p)\cdot\overleftrightarrow{G}(p_1)\cdot\bar{I}_o(p_1)\,P\,\eta(\kappa-p)\,\eta(p-p_1)$$

$$+ \varepsilon^2\,\overleftrightarrow{G}(p)\cdot\overleftrightarrow{G}(p_1)\cdot P\,\eta(\kappa-p)\,(I-P)\,\bar{I}_1(p_1)\,\eta(p-p_1) \qquad (13)$$

The correlations can be expressed[17] as

$$P\,\eta(\kappa-p)\,\eta(p-k) = A(\kappa-k)\,\phi\left(\frac{\kappa+k}{2} - p\right)$$

$$P\,\eta(\kappa-p)\,\eta(p-p_1) = A(\kappa-p_1)\,\phi\left(\frac{\kappa+p_1}{2} - p\right)$$

where $A(p)$ and $\phi(p)$ are the Fourier transforms of the turbulent intensity and correlation functions respectively. For large scatterer $A(p)$ approaches a delta function and using this property (13) becomes

$$\bar{I}_o(\kappa) = \varepsilon\,A(\kappa-k)\,\overleftrightarrow{G}(p)\cdot\hat{q}\,\phi\left(\frac{\kappa+k}{2} - p\right)$$

$$+ \varepsilon^2\,\overleftrightarrow{G}(p)\cdot\overleftrightarrow{G}(p_1)\cdot\bar{I}_o(\kappa)\,A(\kappa-p_1)\,\phi(\kappa-p)$$

$$+ \varepsilon^2\,\overleftrightarrow{G}(p)\cdot\overleftrightarrow{G}(p_1)\cdot P\,\eta(\kappa-p)\,\eta(p-p_1)\,\bar{I}_1(p_1) \qquad (14)$$

The symmetry of the dyadic operator $\overleftrightarrow{G}(p)$ enables us to write the second RHS term in (14) as

$$\varepsilon^2\,\bar{I}_o(\kappa)\cdot\overleftrightarrow{G}(p_1)\,A(\kappa-p_1)\cdot\overleftrightarrow{G}(p)\,\phi(\kappa-p)$$

and the equation for the coherent scattering function becomes

$$\bar{I}_o(\kappa)\cdot(\delta-\varepsilon^2\,\overleftrightarrow{G}(p_1)\,A(\kappa-p_1)\cdot\overleftrightarrow{G}(p)\,\phi(p-\kappa)) = \bar{I}_o(\kappa)\cdot\overleftrightarrow{T}(\kappa)$$

$$= \varepsilon\,A(\kappa-k)\,\overleftrightarrow{G}(p)\cdot\hat{q}\,\phi\left(\frac{\kappa+k}{2}-p\right)$$

$$+ \varepsilon^2\,\overleftrightarrow{G}(p)\cdot\overleftrightarrow{G}(p_1)\cdot P\,\bar{I}_1(p_1)\,\eta(\kappa-p)\,\eta(p-p_1) \qquad (15)$$

where the $\overleftrightarrow{T}(\kappa)$ tensor defined in (15) is not symmetric in general.

Introduce the tensor $\overleftrightarrow{S}(\kappa)$ such that

$$\overleftrightarrow{T}(\kappa)\cdot\overleftrightarrow{S}(\kappa) = \overleftrightarrow{\delta}$$

In terms of $\overleftrightarrow{T}(\kappa)$ we have

$$\overleftrightarrow{S}(\kappa) = \frac{1}{T}\begin{bmatrix} T_{22}T_{33}-T_{32}T_{23} & T_{32}T_{13}-T_{12}T_{33} & T_{12}T_{23}-T_{22}T_{13} \\ T_{31}T_{23}-T_{21}T_{33} & T_{11}T_{33}-T_{31}T_{13} & T_{21}T_{13}-T_{11}T_{23} \\ T_{21}T_{32}-T_{22}T_{31} & T_{31}T_{12}-T_{11}T_{32} & T_{11}T_{22}-T_{21}T_{12} \end{bmatrix} \tag{16}$$

where $T = \det(\overleftrightarrow{T}(\kappa))$.

Multiply each side of (15) from the right by $\overleftrightarrow{S}(\kappa)$ to obtain

$$\bar{I}_o(\kappa) = \varepsilon\, A(\kappa-k)\, \overleftrightarrow{G}(p)\cdot\hat{q}\, \phi(\frac{\kappa+k}{2} - p)\cdot\vec{S}(\kappa)$$

$$+ \varepsilon^2\, \vec{G}(p)\cdot\vec{G}(p_1)\cdot P\, \bar{I}_1(p_1)\, n(\kappa-p)\, n(p-p_1)\cdot\vec{S}(\kappa) \tag{17}$$

Use this expression to eliminate $\bar{I}_o(p)$ in (12)

$$\bar{I}_1(\kappa) = \hat{q}\, n(\kappa-k) + \varepsilon^2\vec{G}(p)\cdot n(\kappa-p)\, A(p-k)\, \vec{G}(p_1)\cdot\hat{q}\, \phi(\frac{p+k}{2} - p_1)\cdot\overleftrightarrow{S}(p)$$

$$+ \varepsilon^3\vec{G}(p)\cdot n(\kappa-p)\, \overleftrightarrow{G}(p_1)\cdot\vec{C}(p_2)\cdot P\, \bar{I}_1(p_2)\, n(p-p_1)\, n(p_1-p_2)\cdot\overleftrightarrow{S}(p)$$

$$+ \varepsilon\, \vec{G}(p)\, (I-P)\cdot\bar{I}_1(p)\, n(\kappa-p) \tag{18}$$

We again make use of the delta function like property of A(p-k) to simplify the second RHS term in (18). This term is approximately

$$\varepsilon^2\, n(\kappa-k)\, \vec{G}(p)\cdot A(p-k)\, \overleftrightarrow{G}(p_1)\cdot\hat{q}\, \phi(k-p_1)\cdot\overleftrightarrow{S}(k)$$

Combining this term with the first RHS term and letting

$$\vec{Q}(k) = \hat{q} + \varepsilon^2 \overleftrightarrow{G}(p)\, A(p-k)\cdot\hat{q}\cdot\overleftrightarrow{G}(p_1)\Phi(p_1-k)\cdot\overleftrightarrow{S}(k) \tag{19}$$

we can write (18) as

$$\begin{aligned}\vec{I}_1(\kappa) = {} & \vec{Q}n(\kappa-k) + \varepsilon\, \overleftrightarrow{G}(p)\cdot(\delta-P)\vec{I}_1(p)n(\kappa-p) \\ & + \varepsilon^3\, \overleftrightarrow{G}(p)\cdot\overleftrightarrow{G}(p_1)\cdot\overleftrightarrow{G}(p_2)\cdot n(\kappa-p)Pn(p-p_1)n(p_1-p_2)\vec{I}_1(p_2)\cdot\overleftrightarrow{S}(p)\end{aligned} \tag{20}$$

We now use an iteration technique to solve (20). Let

$$\vec{I}_1(\kappa) = \vec{J}_0(\kappa) + \varepsilon\, \vec{J}_1(\kappa) + \varepsilon^2\vec{J}_2(\kappa) + \varepsilon^3\vec{J}_3(\kappa) + \theta(\varepsilon^4) \tag{21}$$

Using this perturbation expansion in (20) and comparing terms of same order in ε we obtain the following iterative series:

$$\begin{aligned}\vec{J}_0(\kappa) &= \vec{Q}n(\kappa-k) \\ \vec{J}_1(\kappa) &= \overleftrightarrow{G}(p)\cdot(\delta-P)n(\kappa-p)\vec{J}_0(p) \\ \vec{J}_2(\kappa) &= \overleftrightarrow{G}(p)\cdot(\delta-P)n(\kappa-p)\vec{J}_1(p) \\ \vec{J}_3(\kappa) &= \overleftrightarrow{G}(p)\cdot(\delta-P)n(\kappa-p)\vec{J}_2(p) + \overleftrightarrow{G}(p)\cdot\overleftrightarrow{G}(p_1)\cdot\overleftrightarrow{G}(p_2)n(\kappa-p)Pn(p-p_1)n(p_1-p_2) \\ & \quad \cdot\vec{J}_0(p_2)\cdot\overleftrightarrow{S}(p)\end{aligned} \tag{22}$$

The scattering amplitude $\vec{F}_1(\kappa)$ to order ε^3 is

$$\vec{F}_1(\kappa) = -\frac{(k\Omega_{p_1})^2}{4\pi}(\overleftrightarrow{\delta}-\hat{r}\hat{r})\cdot(\vec{J}_0(\kappa) + \varepsilon\vec{J}_1(\kappa) + \varepsilon^2\vec{J}_2(\kappa) + \varepsilon^3\vec{J}_3(\kappa)) \tag{23}$$

In the direct ($\hat{q}$) polarization direction the scattering amplitude is given by

$$F(\kappa) = \hat{q}\cdot\vec{F}_1(\kappa) = -\frac{(k\Omega_{p_1})^2}{4\pi}(\hat{q}-\hat{q}\cdot\hat{r}\hat{r})\cdot\vec{I}_1(\kappa) \equiv \vec{d}\cdot\vec{I}_1(\kappa) \tag{24}$$

In the cross polarized direction ($\hat{q}_\perp = \hat{q} \times \hat{k}$) the scattering amplitude is written

$$F_\perp(\kappa) = \hat{q}_\perp\cdot\vec{F}_1(\kappa) = -\frac{(k\Omega_{p_1})^2}{4\pi}(\hat{q}_\perp-\hat{q}_\perp\cdot\hat{r}\hat{r})\cdot\vec{I}_1(\kappa) \equiv \vec{c}\cdot\vec{I}_1(\kappa) \tag{25}$$

To form the cross section in the direct polarization direction we square and average (24) and use the following expressions for the second and fourth order moments: (odd moments are assumed to vanish)

$$\begin{aligned}
&Pn(p_1)n(p_2) = A(p_1+p_2)\phi\left(\frac{p_1-p_2}{2}\right)\\
&Pn^*(p_1)n(p_2) = A(p_1-p_2)\phi\left(\frac{p_1+p_2}{2}\right)\\
&Pn(p_1)n(p_2)n(p_3)n(p_4) = Pn(p_1)n(p_2)Pn(p_3)n(p_4)\\
&\qquad + Pn(p_1)n(p_3)Pn(p_2)n(p_4)\\
&\qquad + Pn(p_1)n(p_4)Pn(p_2)n(p_3)
\end{aligned} \tag{26}$$

$$\sigma(\kappa) = PF(\kappa)F^*(\kappa) = P\,|\,\vec{d}\cdot\vec{I}_1(\kappa)\,|^2$$

$$= P\,|\,\vec{d}\cdot\vec{Q}n(\kappa-k) + \varepsilon\vec{d}\cdot\overleftrightarrow{G}(p)\cdot\vec{Q}(n(\kappa-p)n(p-k)-A(\kappa-k)\phi(\tfrac{\kappa+k}{2}-p))$$

$$+ \varepsilon^2\vec{d}\cdot\overleftrightarrow{\ }(p)\cdot\overleftrightarrow{G}(p_1)\cdot\vec{Q}(n(\kappa-p)n(p-p_1)n(p_1-k)-n(\kappa-p)A(p-k)\phi(\tfrac{p+k}{2}-p_1))$$

$$+ \varepsilon^3 \vec{d}\cdot\overleftrightarrow{}(p)\cdot\overleftrightarrow{G}(p_1)\cdot\overleftrightarrow{G}(p_2)\cdot\vec{Q}(n(\kappa-p)n(p-p_1)n(p_1-p_2)n(p_2-k)$$

$$- n(\kappa-p)n(p-p_1)A(p_1-k)\phi(\frac{p_1+k}{2}-p_2) \qquad (27)$$

$$- Pn(\kappa-p)n(p-p_1)n(p_1-p_2)n(p_2-k)$$

$$+ Pn(\kappa-p)n(p-p_1)A(p_1-k)\phi(\frac{p_1+k}{2}-p_2))\ |^2$$

After considerable reduction we finally obtain

$$\sigma(\kappa) = A_0\phi(\kappa-k)|\ \vec{d}\cdot\vec{Q}\ |^2$$

$$+ \varepsilon^2 2\phi(-k)\text{Real}(\vec{d}^*\cdot\vec{Q}^*)(\vec{d}\cdot\overleftrightarrow{G}(p)\cdot\overleftrightarrow{G}(p_1)\cdot\vec{Q})$$

$$X\phi(\kappa-p)[A^2(p_1-\kappa) + A^2(p_1-p-k+\kappa)] \qquad (28)$$

$$+ \varepsilon^2(\vec{d}\cdot\overleftrightarrow{G}(p)\cdot\vec{Q})(\vec{d}\cdot\overleftrightarrow{G}(^*q)\cdot\vec{Q}^*)$$

$$X\phi(\kappa-p)\phi(p-k)[A^2(q-p) + A^2(q-\kappa-k+p)]$$

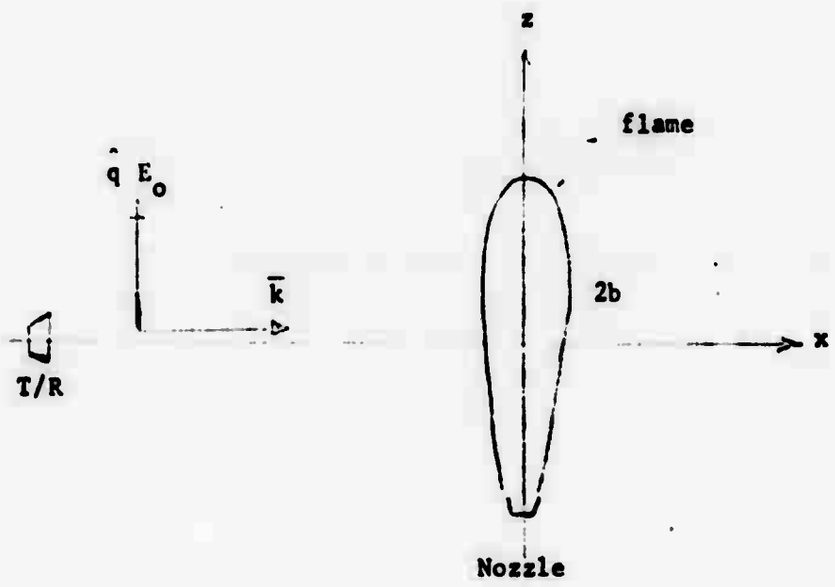
Equation (28) is the direct polarization differential cross section in the scattering direction $\bar{\kappa} = k\hat{r}$. The result neglects terms of order ε^4 or higher. The corss polarized cross section is readily obtained by replacing $\bar{d}$ by $\bar{c}$ in (28).

REPORT 3

APPLICATION OF THE MODIFIED METHOD OF ITERATION FOR SCATTERING TO SRI EXPERIMENT

I. Introduction

In Report 2 the modified method of smoothing was applied to calculate the direct and cross polarization cross sections for scattering from a turbulent plasma. In this report we specialize those results to compute the backscatter cross sections for both polarizations for the SRI flame scattering experiment at 9.4 GHz. The scattering geometry is shown below:



The transmitter-receiver is located about 50 cm above the nozzle and the flame is illuminated broadside with direct polarization ($\hat{q}$) along the z axis. The SRI jet has a non-homogeneous electron density distribution in

which the turbulent electron density fluctuations are characterized by a locally homogeneous turbulence of the form

$$\langle \delta n(\bar{r}_1,t)\ \delta n(\bar{r}_2,t)\rangle = (n_o')^2\ A\left(\frac{\bar{r}_1+\bar{r}_2}{2}\right)\ B(\bar{r}_1-\bar{r}_2) \qquad (1)$$

where n_o' is the rms value of electron density at the reference point in the flame ($z=0$, $\rho=0$), A is the normalized turbulence intensity function, and $B(\bar{r}_1-\bar{r}_2)$ is the normalized turbulence correlation function. The turbulence intensity is modeled by a Gaussian radial and axially uniform distribution

$$A(\bar{R}) = e^{-\frac{2\rho^2}{a^2}}\ Z(z) \qquad (2)$$

where $\bar{R} = \hat{\rho}\,\rho + \hat{Z}\,z = \dfrac{\bar{r}_1+\bar{r}_2}{2}$

$a \sim 3$ to 5 cm

$Z(z) = 1$, $-b<Z<b$, $b \sim 5$ to 10 cm

$= 0$ otherwise

For the correlation function we assume the analytically simple isotropic exponential function

$$B(\bar{r}_1 - \bar{r}_2) = e^{-\frac{|\bar{r}_1-\bar{r}_2|}{\ell}} \qquad (3)$$

where the correlation length $\ell \sim 1.75$ to 2.5 cm. Measurements indicate that the radial turbulence scale is about 1.75 cm and the longitudinal scale is 2.5 cm. The spectrum representing $B(\bar{r})$ is

$$\Phi(\kappa) = \frac{\Phi_o}{[1+(\kappa\ell)^2]^2} \quad \text{where } \Phi_o = 8\pi\ell^3 \qquad (4)$$

We will also need the spectrum of the turbulence intensity distribution

$$A(\bar{p}) = \int_{-\infty}^{\infty} d\bar{R}\, A(\bar{R}) e^{i\bar{p}\cdot\bar{R}} = A_o e^{-\frac{1}{8}(p_\rho a)^2} \frac{\sin p_3 b}{p_3 b}, \quad A_o = \pi a^2 b \tag{5}$$

2. Cross Section Functions for the SRI Model

In this section we derive the explicit form of the functions necessary to compute the scattering cross sections for the SRI model. In (R2.17) we need the vector

$$\bar{Q}(\bar{k}) = \hat{q} + \epsilon^2 G(\bar{p})\, A(\bar{p}-\bar{k})\cdot\hat{q}\cdot\overleftrightarrow{G}(\bar{p}_1)\, \phi(\bar{p}_1-\bar{k})\cdot\overleftrightarrow{S}(\bar{k}) \tag{6}$$

where $\overleftrightarrow{S}(\bar{k})$ is defined in terms of the elements of tensor $\overleftrightarrow{T}(\bar{k})$. The latter is given by (R2.15)

$$\overleftrightarrow{T}(\bar{k}) = \delta - \epsilon^2 \overleftrightarrow{G}(\bar{p}_1)\, A(\bar{k}-\bar{p}_1)\cdot\overleftrightarrow{G}(\bar{p})\, \phi(\bar{p}-\bar{k}) \tag{7}$$

Introducing the tensors

$$\overleftrightarrow{g}(\bar{k}) = \overleftrightarrow{G}(\bar{p})\, \phi(\bar{p}-\bar{k}) = \int_{-\infty}^{\infty} d\bar{p}\, \phi(\bar{p}-\bar{k})\, \left(\frac{\bar{p}\bar{p}\bar{k} - \overleftrightarrow{\delta}}{p^2-k^2+i\varepsilon} + \frac{\overleftrightarrow{\delta}}{3k^2}\right) \tag{8}$$

and

$$\overleftrightarrow{f}(\bar{k}) = \overleftrightarrow{G}(\bar{p})\, A(\bar{p}-\bar{k}) \tag{9}$$

We can write (7) as

$$\overleftrightarrow{T}(\bar{k}) = \delta - \epsilon^2 \overleftrightarrow{f}(\bar{k})\cdot\overleftrightarrow{g}(\bar{k}) \tag{10}$$

We proceed to the evaluation of $\overleftrightarrow{g}$ and $\overleftrightarrow{f}$ for models (4) and (5). Let

$$a_{ij} = k^2 \int_{-\infty}^{\infty} \frac{d\bar{p}\, \phi(\bar{p}-\bar{k})\, p_i p_j}{p^2-k^2-i\varepsilon} \tag{11}$$

In spherical coordinates with

$$\bar{p} = \hat{i}\, p \sin\theta \cos\phi + \hat{j}\, p \sin\theta \sin\phi + \hat{z} \cos\theta \tag{12}$$

$$a_{11} = k^2 \int_0^\infty dp\, p^4 \int_0^{2\pi} d\phi \int_0^\pi \sin\theta\, d\theta\, \phi(|\bar{p}-\bar{k}|)\, \frac{\cos^2\theta \cos^2\phi}{p^2-k^2-i\varepsilon} \tag{13}$$

Taking the polar axis along $\bar{k}$ we have for model (4)

$$a_{11} = \frac{\pi\phi_o}{k^2\ell^4} \int_0^\infty dp\, p^4 \int_0^\pi \frac{d\theta \sin\theta \cos^2\theta}{(p^2-k^2-i\varepsilon)(p^2+k^2+\tau^2-2kp\cos\theta)^2} \tag{14}$$

$$= \frac{\pi\phi_o}{2k^2\ell^4} \int_{-\infty}^\infty dp\, p^4 \int_o^\pi \frac{d\theta \sin\theta \cos^2\theta}{(p^2-k^2-i\varepsilon)(p^2-2kp\cos\theta + k^2+\tau^2)^2} \tag{15}$$

where $\tau^2 = \ell^{-2}$

Letting $u = \cos\theta$ we can write (15) as

$$a_{11} = \frac{\pi\phi_o}{2k^2\ell^4} \int_{-1}^1 u^2\, du \int_{-\infty}^\infty \frac{dp\, p^4}{(p^2-k^2-i\varepsilon)(p^2-2kup+k^2+\tau^2)^2} \tag{16}$$

Calling the integral over p, I_1, we write

$$I_1 = \lim_{\varepsilon\to o} \int_{-\infty}^\infty dp \left[\frac{A_1}{p-(k+i\varepsilon)} + \frac{B_1}{p-(-k-i\varepsilon)} + \frac{C_1}{[p-(\alpha+i\beta)]^2} + \frac{C_2}{p-(\alpha+i\beta)} \right.$$

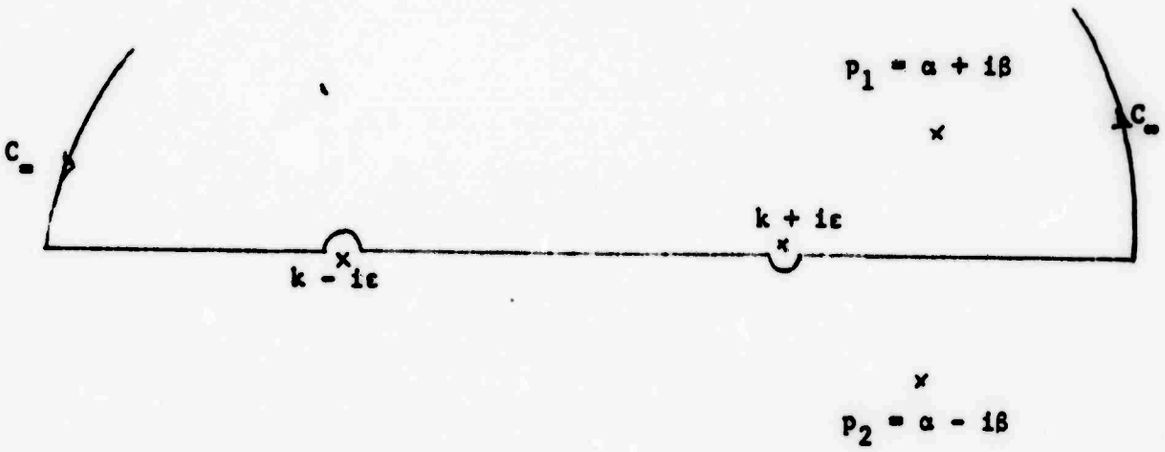
$$\left. + \frac{D_1}{[p-(\alpha-i\beta)]^2} + \frac{D_2}{p-(\alpha-i\beta)} \right] \tag{17}$$

where $\alpha = ku$, $\beta = \sqrt{\tau^2 + k^2(1-u^2)} > o$ and

$$A_1 = \frac{k^4}{2k(k-p_1)^2(k-p_2)^2}, \quad C_2 = \frac{-2p_1^3}{(p_1^2-k^2)(p_1-p_2)^2} \left(\frac{k^2}{p_1^2-k^2} + \frac{p_2}{p_1-p_2}\right), \tag{18}$$

$p_1 = \alpha + i\beta$, $p_2 = \alpha - i\beta$.

Integral (17) can be evaluated with the contour $C = \Gamma + C_\infty$ show below:



$$\oint_C dp \frac{p^4}{(\ )(\ )^2} = \int_\Gamma dp\ (\ ) + \int_{C_\infty} dp\ (\ ) = 2\pi i(A_1 + C_2) \tag{19}$$

We have since the contribution over C_∞ vanishes

$I_1 = 2\pi i(A_1 + C_2)$ and

$$a_{11} = \frac{\pi\phi_o}{2k^2\ell^4}\int_{-1}^{1} u^2\, du\, 2\pi i \left[\frac{k^3}{2(k-p_1)^2(k-p_2)^2} - \frac{2p_1^3}{(p_1^2-k^2)(p_1-p_2)^2}\left(\frac{k^2}{p_1^2-k^2} + \frac{p_2}{p_1-p_2}\right)\right] \tag{20}$$

Calling the contribution from A_1, I_{11}, we have

$$I_{11} = \frac{i\pi^2\,\phi_o\,k}{2\ell^4}\int_{-1}^{1} \frac{u^2\,du}{(k-\alpha-i\beta)^2(k-\alpha+i\beta)^2} \tag{21}$$

Carrying out this integration and letting $\ell_o = k\ell$ we obtain

$$I_{11} = \frac{-i\pi\,\phi_o k}{8\,\ell_o^6}\left[(1+2\ell_o^2)\ln(1+4\ell_o^2) - 2\ell_o^2\left(1 + \frac{(1+2\ell_o^2)^2}{1+4\ell_o^2}\right) \right] \tag{22}$$

With I_{12} the contribution over integral C_2 we write

$$I_{12} = \frac{-\pi^2\phi_o\, 2i}{k^2\ell^4}\int_{-1}^{1} \frac{u^2\,du\,p_1^3}{(p_1^2-k^2)(p_1-p_2)^2}\left(\frac{k^2}{p_1^2-k^2} + \frac{p_2}{p_1-p_2}\right) \tag{23}$$

If we let $x = p_1 = \alpha + i\beta = ku + i\sqrt{k^2+\tau^2-k^2u^2}$

then integral (23) is transformed to

$$I_{12} = \frac{-\pi^2\phi_o}{4h^5\ell^4}\int_{-k+i\tau}^{k+i\tau} \frac{x\,dx\,(x^2+k^2+\tau^2)^2}{(x^2-k^2)(x^2-k^2-\tau^2)}\left(\frac{k^2}{x^2-k^2} + \frac{k^2+\tau^2}{x^2-k^2-\tau^2}\right) \tag{24}$$

and integration by partial fractions yields

$$I_{12} = \frac{-i\pi^2 k\phi_o}{8\ell_o^6}\left[(1+2\ell_o^2)\ \ln\frac{1-i2\ell_o}{1+i2\ell_o} + i4\ell_o(1+\ell_o^2 + \frac{\ell_o^4}{1+4\ell_o^2}) \right] \tag{25}$$

Combining (22) and (25) we obtain

$$a_{11} = \frac{-i\pi^2\phi_o k}{8\ell_o^6}\left[(1+2\ell_o^2)\ \ell n(1+4\ell_o^2) - 2\ell_o^2\left(1 + \frac{(1+2\ell_o^2)^2}{1+4\ell_o^2}\right)\right.$$

$$\left. + (1+2\ell_o^2)\ \ell n\left(\frac{1-i2\ell_o}{1+i2\ell_o}\right) + i4\ell_o\left(1 + \ell_o^2 + \frac{\ell_o^4}{1+4\ell_o^2}\right)\right] \tag{26}$$

Next we calculate a_{22} according to

$$a_{22} = k^{-2} \int_0^\infty dp\ p^2 \int_0^{2\pi} d\phi \int_0^\pi d\theta\ \sin\theta\ \phi(|\bar{p}-\bar{k}|)\ \frac{p^2 \sin^2\theta \sin^2\phi}{p^2-k^2-i\varepsilon}$$

$$= \frac{\pi\phi_o}{2k^2\ell^4} \int_{-\infty}^{\infty} dp\ p^4 \int_0^\pi \frac{d\theta \sin\theta \sin^2\theta}{(p^2-k^2-i\varepsilon)(p^2-2kp\cos\theta + \tau^2+k^2)^2} \qquad (27)$$

with $u = \cos\theta$ equation (27) becomes

$$a_{22} = \frac{\pi\phi_o}{2k^2\ell^4} \int_{-1}^{1} du(1-u^2) \int_{-\infty}^{\infty} \frac{dp\ p^4}{(p^2-k^2-i\varepsilon)(p^2-2kup+k^2+\tau^2)^2} \qquad (26)$$

The above integral can be broken up into the following integrals:

$$a_{22} = -a_{11} + \frac{i\pi^2\phi_o k}{2\ell^4} \int_{-1}^{1} \frac{du}{(k-p_1)^2(k-p_2)^2}$$

$$- \frac{2i\pi^2\phi_o}{k^2\ell^4} \int_{-1}^{1} \frac{du\ p_1^3}{(p_1^2-k^2)(p_1-p_2)^2} \left(\frac{k^2}{p_1^2-k^2} + \frac{p_2}{p_1-p_2}\right) \qquad (29)$$

Calling the middle term I_{21} we have

$$I_{21} = \frac{i\pi^2 k\phi_o}{2\ell^4} \int_{-1}^{1} \frac{du}{(2k^2+\tau^2-2k^2u)^2} = \frac{i\pi^2 k\phi_o}{2\ell^4} \frac{2}{\tau^2(\tau^2+4k^2)} \qquad (30)$$

with $\ell_o = k\ell$ this becomes

$$I_{21} = \frac{i\pi^2\ k\ \phi_o}{1+4\ell_o^2} \qquad (31)$$

Calling the last term in (29) I_{22} and letting $x = p_1$

$$I_{22} = \frac{-2i\pi^2\phi_o}{k^2\ell^4} \frac{1}{2k} \int_{-k+i\tau}^{k+i\tau} \frac{dx\ x^3}{(x^2-k^2)(x^2-k^2-\tau^2)} \left(\frac{k^2}{x^2-k^2} + \frac{k^2+\tau^2}{x^2-k^2-\tau^2}\right) \qquad (32)$$

Carrying out this integration we finally obtain

$$I_{22} = \frac{-i2\pi^2\phi_o}{h^2\ell^4} \; \frac{i}{4\tau} \; \frac{\tau^2+5k^2}{\tau^2+4k^2}$$

$$I_{22} = \frac{\pi^2 k\phi_o}{2\ell_o^3} \; \frac{1+5\ell_o^2}{1+4\ell_o^2} \tag{33}$$

Using (31) and (33) in (29) there results

$$a_{22} = -a_{11} + \frac{i\pi^2 k\phi_o}{1+4\ell_o^2} + \frac{\pi^2 k\phi_o}{2\ell_o^3} \; \frac{1+5\ell_o^2}{1+4\ell_o^2} \tag{34}$$

Combining terms in (26) a_{11} simplified to

$$a_{11} = \frac{-i\pi^2 k\phi_o}{4\ell_o^6} \left[(1+2\ell_o^2)\, \ell n(1-i2\ell_o) - 2\ell_o \, \frac{\ell_o(1+4\ell_o^2+2\ell_o^4)-i(1+5\ell_o^2+5\ell_o^4)}{1+4\ell_o^2} \right] \tag{35}$$

When this is used in (29) there results

$$a_{22} = \frac{i\pi^2 k\phi_o}{4\ell_o^6} \left[(1+2\ell_o^2) \;\; n(1-i2\ell_o) - 2\ell_o(\ell_o-i) \right] \tag{36}$$

Next we evaluate a_{33} according to

$$a_{33} = k^{-2} \int_o^\infty dp\; p^2 \int_o^{2\pi} d\phi \int_o^\pi \sin\theta \; d\theta \; \phi(|\bar{p}-\bar{k}|) \; \frac{p^2 \cos^2\theta}{p^2-k^2-i\varepsilon} \tag{37}$$

Comparing this integral with (27) we observe after integration over ϕ that

$$a_{33} = 2\, a_{11} \tag{38}$$

Since $a_{ij}(i\neq j)$ involve $\sin\phi$ or $\cos\phi$ integrations over $(o,2\pi)$ these components vanish, or,

$$a_{ij} = o, \; i \neq j \tag{39}$$

The next type of integral to evaluate $\overleftrightarrow{g}$ in (8) is

$$b_{ij} = -\delta_{ij} \int_{-\infty}^{\infty} d\bar{p} \; \frac{(|\bar{p}-\bar{k}|)}{p^2-k^2-i\varepsilon} = \frac{\overleftrightarrow{-\delta}\phi_o}{\ell^4} \int_{-\infty}^{\infty} \frac{d\bar{p}}{(p^2-k^2-i\varepsilon)(\tau^2+|p-k|^2)^2} \tag{40}$$

This integral is evaluated in Morse and Feshbach (pp. 1083).

$$b_{ij} = \frac{-\pi^2 k\phi_o(1 + i2\ell_o)}{\ell_o(1 + 4\ell_o^2)} \quad \delta_{ij} \tag{41}$$

The last integral in (8), call it c_{ij}, is

$$c_{ij} = \frac{\overleftrightarrow{\delta}}{3k^2} \int_{\infty}^{\infty} d\bar{p}\ \Phi(|\bar{p}-\bar{k}|) = \frac{4\pi\overleftrightarrow{\delta}}{3k^2} \int dp\ p^2\ \Phi(|\bar{p}|) \tag{42}$$

$$c_{ij} = \frac{4\pi\ \Phi_o\overleftrightarrow{\delta}}{3k^2\ell^4} \int_o^{\infty} \frac{x^2\,dx}{(x^2+\tau^2)^2} = \frac{\pi^2 k\ \Phi_o}{3\ell_o^3}\ \delta_{ij} \tag{43}$$

Combining (35), (41) and (43) we obtain g_{11}.

$$g_{11} = a^*_{11} + b^*_{11} + c^*_{11}$$

$$\frac{g_{11}}{\pi^2 k\phi_o} = \frac{i}{4\ell_o^6}\left[(1+2\ell_o^2)\ \ell n(1+i2\ell_o) - \frac{2\ell_o}{1+4\ell_o^2}(\ell_o[1+4\ell_o^2+2\ell_o^4]+i[1+5\ell_o^2+5\ell_o^4])\right]$$

$$+ \frac{1+\ell_o^2+i6\ell_o^3}{3\ell_o^3(1+4\ell_o^2)} \tag{44}$$

similarily (36), (41) and (43) yield

$$\frac{g_{22}}{\pi^2 k\phi_o} = a^*_{22} + b^*_{22} + c^*_{22} = \frac{-i(1+2\ell_o^2)\ \ell_n(1+i2\ell_o)}{4\ell_o^6} - \frac{(1-i\ell_o)}{2\ell_o^5} + \frac{1+\ell_o^2+i6\ell_o^3}{3\ell_o^2(1+4\ell_o^2)} \tag{45}$$

Finally (38), (41) and (43) yield

$$\frac{g_{33}}{\pi^2 k\phi_o} = \frac{i}{2\ell_o^6}\left[(1+2\ell_o^2)\,\ell n(1+2\ell_o) - \frac{2\ell_o}{1+4\ell_o^2}\,(\ell_o[1+4\ell_o^2+2\ell_o^4] + i[1+5\ell_o^2+5\ell_o^4])\right] + \frac{1+\ell_o^2+i6\ell_o^3}{3\ell_o^3(1+4\ell_o^2)} \tag{46}$$

Next we evaluate $f_{ij}(\bar{k})$ according to (9).

$$f_{ij}^* = \int_{-\infty}^{\infty} d\bar{p}\; A(\bar{p}-\bar{k})\left[\frac{k^{-2}p_i\,p_j-\delta_{ij}}{p^2-k^2-i\varepsilon} + \frac{\delta_{ij}}{3k^2}\right] \tag{47}$$

For model (5) the last integral in (47) is

$$\frac{\delta_{ij}}{3k^2}\int_{-\infty}^{\infty} d\bar{p}\; A(\bar{p}-\bar{k}) = \frac{\delta_{ij}}{3k^2}\int_0^{\infty} p_\rho\, dp_\rho \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dp_3\, A_o e^{-\frac{p_\rho^2 a^2}{8}}\,\frac{\sin bp_3}{bp_3} = \frac{(2\pi)^3\,\delta_{ij}}{3k^2} \tag{48}$$

The first integral in (47) is evaluated approximately treating $A(\bar{p}-\bar{k})$ as a quasi delta function.

$$\int \frac{d\bar{p}\; A(\bar{p}-\bar{k})\;[p_i p_j\, k^{-2}-\delta_{ij}]}{p^2-k^2-i\varepsilon} \simeq (\hat{k}_i\hat{k}_j-\delta_{ij}) \int_{-\infty}^{\infty} \frac{d\bar{p}\; A(\bar{p}-\bar{k})}{2\bar{k}\cdot(\bar{p}-\bar{k})-i\varepsilon} \tag{49}$$

For model (5) the last integral in (49) is

$$\int \frac{d\bar{p}\, A(\bar{p})}{2\bar{k}\cdot\bar{p}-i\varepsilon} = \int_{-\infty}^{\infty} \frac{dp_1 dp_2 dp_3 A_0 e^{-\frac{a^2}{8}(p_1^2+p_2^2)}}{2kp_1\cos\alpha + 2kp_3\sin\alpha - i\varepsilon} \frac{\sin bp_3}{bp_3} \tag{50}$$

Integrating over p_3 and recognizing that the principal value contribution vanishes we obtain

$$\int \frac{d\bar{p}\, A(\bar{q})}{2\bar{k}\cdot\bar{p}-i\varepsilon} = \frac{i\pi A_0}{2k\sin\alpha} \int dp_2 \int dp_1 e^{-\frac{a^2}{8}(p_1^2+p_2^2)} \frac{\sin(bp_1\cot\alpha)}{bp_1\cot\alpha} \tag{51}$$

$$\text{with } \int_{-\infty}^{\infty} dp_2 e^{-\frac{a^2}{8}p_2^2} = \frac{2\sqrt{2\pi}}{a} \tag{52}$$

equation (51) reduces to, $x = bp_1\cot\alpha$, $q = \frac{a}{\sqrt{8}b}\tan\alpha$

$$= \frac{i\pi\sqrt{2\pi}\, A_0}{ka\sin\alpha} \frac{2\tan\alpha}{b} \int_0^{\infty} dx \frac{\sin x}{x} e^{-q^2x^2} \tag{53}$$

But it can be shown that

$$\int_0^{\infty} dx \frac{\sin x}{x} e^{-q^2x^2} = \frac{\pi}{2}\,\mathrm{erf}\left(\frac{1}{2q}\right) \tag{54}$$

Using this result in (53) and recalling that $A_0 = \pi a^2 b$

$$= \frac{i\sqrt{\pi}(2\pi)^3 kb}{4k^2\sin\alpha} \frac{\mathrm{erf}\left(\frac{\sqrt{2}\,b}{a\tan\alpha}\right)}{\left(\frac{\sqrt{2}\,b}{a\tan\alpha}\right)} \tag{55}$$

We note that $\frac{\mathrm{erf}(Z)}{Z} \underset{Z\to 0}{\Longrightarrow} \frac{2}{\sqrt{\pi}}$ and $\mathrm{erf}(Z) \underset{Z\to\infty}{\Longrightarrow} 1$.

Using (48), (49), and (55) in (47) we obtain

$$\overleftrightarrow{f}(\vec{k}) = \frac{(2\pi)^3}{k^2}\left\{\frac{\overleftrightarrow{\delta}}{3} + (\overleftrightarrow{\delta}-\hat{k}\hat{k})f(\alpha)\right\} \tag{56}$$

where we wrote

$$f(\alpha) = \frac{i\sqrt{\pi}\ kb}{4\ \sin\alpha}\ \frac{\mathrm{erf}\left(\frac{\sqrt{2}\ b}{a\ \tan\alpha}\right)}{\left(\frac{\sqrt{2}\ b}{a\ \tan\alpha}\right)} \tag{57}$$

Expanding (56) we write

$$\overleftrightarrow{f}(\vec{k}) = \frac{(2\pi)^3}{k^2}\begin{bmatrix} \frac{1}{3} + \sin^2\alpha f(\alpha) & 0 & -\sin\alpha\cos\alpha f(\alpha) \\ 0 & \frac{1}{3} + f(\alpha) & 0 \\ -\sin\alpha\cos\alpha f(\alpha) & 0 & \frac{1}{3} + \cos^2\alpha f(\alpha) \end{bmatrix} \tag{58}$$

In terms of $\overleftrightarrow{g}$ and $\overleftrightarrow{f}$ the tensor (7) becomes

$$\overleftrightarrow{T}(\vec{k}) = \begin{bmatrix} 1 - \varepsilon^2 f_{11}g_{11} & 0 & f_{13}g_{33} \\ 0 & 1 - \varepsilon^2 f_{22}g_{22} & 0 \\ f_{13}g_{11} & 0 & 1 - \varepsilon^2 f_{33}g_{33} \end{bmatrix} \tag{59}$$

$$T = \det(\overleftrightarrow{T}) = T_{11}T_{22}T_{33} - T_{13}T_{22}T_{31}$$

$$= (1-\epsilon^2 f_{22}g_{22}) \quad (1-\epsilon^2 f_{11}g_{11})(1-\epsilon^2 f_{33}g_{33}) - g_{11}g_{33}f_{13} \tag{60}$$

The elements of $\overleftrightarrow{S}$ (Report 2, eq. (16)) are

$$\overleftrightarrow{S} = \frac{1}{T}\begin{bmatrix} (1-\epsilon^2 f_{22}g_{22})(1-\epsilon^2 f_{33}g_{33}) & 0 & -f_{13}g_{33}(1-\epsilon^2 f_{22}g_{22}) \\ 0 & (1-\epsilon^2 f_{11}g_{11})(1-\epsilon^2 f_{33}g_{33})-f_{13}^2 g_{11}g_{33} & 0 \\ -f_{13}g_{11}(1-\epsilon^2 f_{33}g_{22}) & 0 & (1-\epsilon^2 f_{22}g_{22})(1-\epsilon^2 f_{11}g_{11}) \end{bmatrix} \tag{61}$$

For the STI experiment the flame is illuminated broadsides so that $\alpha = 90°$ and $f_{13}(\alpha = 0) = 0$. Also, from (57)

$$f(o) = \frac{i\sqrt{\pi}a_0}{4\sqrt{2}}$$

so that

$$\overleftrightarrow{f} = \frac{(2\pi)^3}{k^2}\begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} + \frac{i\sqrt{\pi}a_0}{4\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{3} + \frac{i\sqrt{\pi}a_0}{4\sqrt{2}} \end{bmatrix} \tag{62}$$

$$f_{ij} = \frac{(2\pi)^3}{k^2} (\hat{i}\hat{i} + \hat{j}\hat{j})(\frac{1}{3} + \frac{ikb}{2}) + \hat{z}\hat{z}\,\frac{1}{3}) \tag{62}$$

The elements of S_{ij} simplify to

$$S_{ij} = \begin{bmatrix} \frac{1}{1 - \varepsilon^2 f_{11} g_{11}} & 0 & 0 \\ 0 & \frac{1}{1 - \varepsilon^2 f_{22} g_{22}} & 0 \\ 0 & 0 & \frac{1}{1 - \varepsilon^2 f_{33} g_{33}} \end{bmatrix}$$

$$\overleftrightarrow{S} = \frac{\hat{i}\hat{i}}{1-\varepsilon^2 f_{11} g_{11}} + \frac{\hat{j}\hat{j}}{1-\varepsilon^2 f_{22} g_{22}} + \frac{\hat{z}\hat{z}}{1-\varepsilon^2 f_{33} g_{33}} \tag{63}$$

When (62) and (63) are used in (6) we obtain for $\overline{Q}$ the following result for $\hat{q} = \hat{z}$:

$$\overline{Q}(\overline{k}) = \hat{q} + \varepsilon^2(\hat{i}\hat{i}\, f_{11} + \hat{j}\hat{j}\, f_{22} + \hat{z}\hat{z}\, f_{33})\cdot\hat{q}(\hat{i}\hat{i}\, g_{11} + \hat{j}\hat{j}\, g_{22} + \hat{z}\hat{z}\, g_{33})\cdot\overleftrightarrow{S}$$

$$= \frac{\hat{z}}{1 - \varepsilon^2 f_{33} g_{33}} = \hat{z}\, Q \tag{64}$$

3. Direct Polarization Backscatter Cross Section

In this section we use the general result of Report 2 (Eq. 27) to calculate the backscatter differential cross section for the direct polarization case. We recall that $\hat{q} = \hat{z}$ (direct polarization direction), α=0 (normal illumination), $\hat{k} = \hat{i}$ (direction of incident illumination), $\overline{\kappa} = k\hat{r} = -\overline{k}$ (backscattering). For backscattering and direct polarization we have

$$\bar{d} = -\frac{k^2\Omega^2}{4\pi} p_1 \hat{z}$$

$$\bar{Q} = \hat{z}\, Q = \frac{\hat{z}}{1-\varepsilon^2 f_{33} g_{33}}$$

$$\bar{d}\cdot\bar{Q} = -\frac{k^2\Omega^2}{4\pi} p_1 Q$$

$$\bar{d}\cdot\overleftrightarrow{G}(p)\cdot\overleftrightarrow{G}(p_1)\cdot\bar{Q} = -\frac{k^2\Omega^2}{9k^4 4\pi} p_1 Q \int \frac{d\bar{p}[3(\hat{z}\cdot\bar{p})\bar{p}+\hat{z}(p^2-4k^2)]}{p^2-k^2+i\varepsilon} \cdot \int \frac{d\bar{p}_1[3(\hat{z}\cdot\bar{p}_1)\bar{p}_1+\hat{z}(p_1^2-4k^2)]}{p_1^2-k^2+i\varepsilon}$$

$$\bar{d}\cdot\overleftrightarrow{G}(p)\cdot\bar{Q} = -\frac{k^2\Omega^2}{3k^2 4\pi} p_1 Q \int \frac{d\bar{p}[3(\hat{z}\cdot\bar{p})^2 + p^2-4k^2]}{p^2-k^2+i\varepsilon}$$

When these results are used in (28) the direct polarization backscatter cross section becomes

$$\sigma(-\bar{k}) = \frac{k^4\Omega^4 p_1 A_0 \phi(2k)|Q|^2}{(4\pi)^2} \Big\{1 + \frac{2\varepsilon^2}{9k^4 A_0} \text{Real} \int \frac{d\bar{p}\, h_1(\bar{p})\phi(\bar{p}+\bar{k})}{p^2-k^2+i\varepsilon} \int \frac{d\bar{q}\, A^2(\bar{q}+\bar{k})}{q^2-k^2+i\varepsilon}$$

$$+ \frac{2\varepsilon^2}{9k^4 A_0} \text{Real} \int \frac{d\bar{p}\, h_2(\bar{p})\phi(\bar{p}+\bar{k})}{p^2-k^2+i\varepsilon} \int \frac{d\bar{q}\, A^2(\bar{q}-\bar{p}-2\bar{k})}{q^2-k^2+i\varepsilon} \qquad (65)$$

$$+ \frac{2\varepsilon^2}{9k^4 A_0} \phi^{-1}(2k) \int \frac{d\bar{p}\, h_3(\bar{p})\phi(\bar{p}+\bar{k})\phi(\bar{p}-\bar{k})}{p^2-k^2+i\varepsilon} \int \frac{d\bar{q}\, A^2(\bar{q}-\bar{p})}{q^2-k^2-i\varepsilon} \Big\}$$

Call the second, third and fourth terms in the brackets of (65) σ_1, σ_2, and σ_3, respectively, and where the polarization dependent functions are defined by

$$h_1(\bar{p}) = [3(\hat{z}\cdot\bar{p})\bar{p}+\hat{z}(p^2-4k^2)]\cdot[3(\hat{z}\cdot\bar{k})\bar{k}+\hat{z}(k^2-4k^2)]$$
$$= 3k^2(4k^2-p^2-3(\hat{z}\cdot\bar{p})^2) \tag{66}$$

$$h_2(\bar{p}) = [3(\hat{z}\cdot\bar{p})\bar{p}+\hat{z}(p^2-4k^2)]\cdot[3\,\hat{z}\cdot(\bar{p}+2\bar{k})(\bar{p}+2\bar{k})+\hat{z}(\bar{p}+2\bar{k})^2-24k^2]$$
$$= (p^2-4k^2)(p^2+4\bar{k}\cdot\bar{p})+3(\hat{z}\cdot\bar{p})^2(5p^2+10\bar{k}\cdot\bar{p}-4k^2) \tag{67}$$

$$h_3(\bar{p}) = (3(\hat{z}\cdot\bar{p})^2+p^2-4k^2)^2 \tag{68}$$

Next we proceed to the evaluation of σ_1.

$$\sigma_1 = \frac{2\epsilon^2}{9k^4A_0}\,\text{Real}\int\frac{d\bar{p}\;3k^2(-p^2-3(\hat{z}\cdot\bar{p})+4k^2)}{p^2-k^2+i\epsilon}\int\frac{d\bar{q}\;A^2(\bar{q})}{(\bar{q}-\bar{k})^2-k^2+i\epsilon} \tag{69}$$

We evaluate the integral over $\bar{q}$ (call it σ_{12}) approximately.

$$\sigma_{12} = \int\frac{d\bar{q}\;A^2(\bar{q})}{q^2-2\vec{k}\cdot\vec{q}+i\epsilon} \approx \int\frac{dq\;A^2(q)}{-2\vec{k}\cdot\vec{q}+i\epsilon} = A_0^2\int\frac{dq_1 e^{-\frac{a^2}{4}q_1^2}}{-2kq_1+i\epsilon}\int dq_2 e^{-\frac{a^2q_2^2}{4}}\int dq_3\left[\frac{\sin bq_3}{bq_3}\right]^2 \tag{70}$$

$$= -\frac{i\pi}{2k}\frac{\sqrt{4\pi}}{a}\frac{\pi}{b}(\pi a^2 b)^2 = -\frac{i\pi^3\sqrt{\pi}}{k}A_0$$

The integral over $\bar{p}(\sigma_{11})$ is evaluated next. Take $\hat{i}$ as the polar axis with $\hat{k}\cdot\bar{p} = p\cos\theta$, $\hat{Z}\cdot\bar{p} = p\sin\theta\sin\phi$, $u = \cos\theta$.

$$\sigma_{11} = \int_0^\infty \frac{dp\ p^2}{p^2-k^2+i\varepsilon} \int_0^\pi d\theta\sin\theta \int_0^{2\pi} d\phi\ 3k^2\ (-3p^2\sin^2\theta\sin^2\phi - p^2 + 4k^2) \times \frac{\Phi_0}{[1 + \ell^2(p^2+2kp\cos\theta+k^2)]^2} \tag{71}$$

$$\sigma_{11} = -\frac{\pi\Phi_0 3k^2}{2} \int_{-1}^{1} du \int_{-\infty}^{\infty} dp\ p\ \frac{(3p^2(1-u^2) + 2(p^2-4k^2))}{[1 + \ell^2(p^2+2kup+k^2)]^2} \tag{72}$$

$$= -\frac{\pi\Phi_0 3k^2}{2} \int_{-1}^{1} du \left\{-i\pi k\ \frac{(3k^2(1-u^2)-6k^2)}{[1 + 2\ell^2k^2(1+u)]^2} + P\int \frac{dp\,p}{p-k}\right\}$$

Using results (70) and (72) in (69) we obtain, with $\Phi_0 = 8\pi\ell^3$, $A_0 = \pi a^2 b$, $a_0 = ka$, $\ell_0 = k\ell$, $b_0 = kb$, $\varepsilon = \frac{k^2\Omega^2 p_1}{(2\pi)^3}$:

$$\sigma_1 = -\Omega^4_{p_1}\ \frac{\sqrt{\pi}}{8}\ a_0\ell_0^3 \int_{-1}^{1} \frac{du\ (u^2+1)}{[1 + 2\ell_0^2(1+u)]^2} = -\Omega^4 p_1\ \frac{\sqrt{\pi}}{8}\ a_0\ell_0^3 m_1(\ell_0) \tag{73}$$

The last integral is evaluated to yield

$$m_1(\ell_0) = \int_0^2 \frac{1 + (x-1)^2}{(1 + 2\ell_0^2 x)^2}\,dx = \frac{4\ell_0^4 + 4\ell_0^2 + 1}{\ell_0^6(1 + 4\ell_0^2)} - \frac{H2\ell_0^2}{4\ell_0^6} \ln\ (H4\ell_0^2) \tag{74}$$

Next we evaluate σ_2 approximately.

$$\sigma_2 = \frac{2\epsilon^2}{9k^4A_0} \text{ Real} \int \frac{d\bar{p}\, h_2(\bar{p})\Phi(p+k)}{p^2-k^2+i\epsilon} \int \frac{d\bar{q}\, A^2(\bar{q})}{(\vec{q}+\vec{p}+2\vec{k})^2-k^2+i\epsilon} \tag{75}$$

Evaluate the integral over $\bar{q}(\sigma_{22})$ first. Neglecting small $\bar{q}$ terms in the denominator we have approximately

$$\sigma_{22} = \int \frac{d\bar{q}\, A^2(\bar{q})}{4\vec{k}\cdot\vec{q}+p^2+4\vec{k}\cdot\vec{p}+3k^2+i\epsilon} \tag{76}$$

With $4\bar{k}\cdot\bar{q} = 4kq_1$ and $p_0 = \frac{p^2+4\bar{k}\cdot\bar{p}+3k^2}{4k}$ we write

$$\sigma_{22} = \frac{A_0^2}{4k} \int \frac{dq_1 e^{-\frac{a^2}{4}q_1^2}}{q_1+p_0+i\epsilon} \int dq_2 e^{-\frac{a^2q_2^2}{4}} \int dq_3 \frac{\sin bq_3}{bq_3}$$

$$= \frac{\pi^2 a\sqrt{\pi}A_0}{2k} \left\{ -i\pi e^{-\frac{a^2p_0^2}{4}} + P\int_{-\infty}^{\infty} \frac{dq_1 e^{-\frac{a^2}{4}q_1^2}}{q_1+p_0} \right\} \tag{76}$$

But it can be shown that the principal value integral is expressible in terms of the Dawson integral

$$P\int_{-\infty}^{\infty} \frac{dq_1 e^{-\frac{a^2}{4}q_1^2}}{q_1+p_0} = 2\sqrt{\pi}e^{-(\frac{a}{2}p_0)^2} \int_0^{\frac{ap_0}{2}} dx e^{x^2} = 2\sqrt{\pi}D(\tfrac{ap_0}{2}) \tag{77}$$

$$\sigma_{22} = \frac{\pi^2 a\sqrt{\pi}A_0}{2k} \left\{ -i\pi e^{-(\frac{ap_0}{2})^2} + 2\sqrt{\pi}D(\tfrac{ap_0}{2}) \right\} \tag{78}$$

When this result is used into (75) and we introduce spherical coordinates we have after p integrations

$$\sigma_2 = \frac{2\varepsilon^2}{9k^4A_0}\ \text{Real}\ \frac{1}{2}\int_{-\infty}^{\infty} dp\ \frac{P}{p-k+i\varepsilon}\int_{-1}^{1} du[2\pi(p^2+4kpu)(p^2-4k^2)+\pi p^2(1-u^2)(15p^2+30kpu-12k^2)]$$

$$\times \frac{\Phi_0}{[1+\ell^2(p^2+2kpu+k^2)]^2}\ \frac{\pi^2 a\sqrt{\pi}A_0^2}{2k}\left\{-i\pi e^{-(\frac{ap_0}{2})^2} + 2\sqrt{\pi}D(\frac{ap_0}{2})\right\}$$

Carrying out the p integration and letting $x = p\ell$, $\ell_0 = k\ell$, $a_0 = ka$, we obtain,

$$\sigma_2 = \frac{2\varepsilon^2}{9k^4}\left\{\frac{\pi^5\Phi_0\sqrt{\pi}a^3k^4}{4}\ m_{21}(a_0,\ell_0) + \frac{\pi^4\Phi_0\sqrt{2}a}{2k\ell^5}\ m_{22}(a_0,\ell_0)\right\} \qquad (80)$$

where

$$m_{21}(a_0,\ell_0) = \int_{-1}^{1} du\ \frac{10u^3+u^2-2u+1}{[1+2\ell_0^2(1+u)]^2}\ \exp\left(-\frac{a_0^2(1+u)^2}{4}\right) \qquad (81)$$

$$m_{22}(a_0,\ell_0) = \int_{-1}^{1} du\ P\int_{-\infty}^{\infty}\frac{dx\ x}{x-\ell_0}\ D\left(\frac{a_0[x^2+4\ell_0ux+3\ell_0^2]}{8\ell_0^2}\right)$$

$$\frac{x^2(1-u^2)(15x^2+30\ell_0xu-12\ell_0^2)+2(x^2-4\ell_0^2)(x^2+4\ell_0xu)}{(1+x^2+2\ell_0xu+\ell_0^2)^2} \qquad (82)$$

For the SRI experiment $a_0 = 7.8$ and $\ell_0 = 4.4$. Numerical integration yields $m_{21}(7.8,4.4) = -0.12$, $m_{22}(7.8,4.4) = 203$, and $m_1(4.4) = 0.0475$.

The fourth term in (65), σ_3, is now evaluated. Treating $A(\bar{p}-\bar{q})$ as a quasi delta function and writing

$$\frac{1}{(p^2-k^2+i\varepsilon)(q^2-k^2-i\varepsilon)} = \frac{1}{(\bar{q}-\bar{p})\cdot(\bar{q}+\bar{p})-i\varepsilon}\left\{\frac{1}{p^2-k^2+i\varepsilon} - \frac{1}{q^2-k^2-i\varepsilon}\right\}$$

we obtain approximately

$$\sigma_3 = \frac{2\varepsilon^2\psi^{-1}(2k)}{9k^4\ A_0}\int d\bar{p}\ h_3(\bar{p})\,\psi(\bar{p}+\bar{k})\,\psi(\bar{p}-\bar{k})\left(\frac{1}{p^2-k^2+i\varepsilon} - \frac{1}{p^2-k^2-i\varepsilon}\right)\int\frac{d\bar{q}\ A^2(\bar{q}-\bar{p})}{2\vec{p}\cdot(\vec{q}-\vec{p})-i\varepsilon} \qquad (83)$$

Carrying out the σ_3 integration and recognizing that the principal value integral vanishes, $A(\bar{q}) = A(-\bar{q})$, we find the $\bar{q}$ integration reduces to

$$\int\frac{d\bar{q}\ A^2(\bar{p}-\bar{q})}{2\vec{p}\cdot(\vec{q}-\vec{p})-i\varepsilon} = \frac{i\pi}{2p_3}\int dq_1dq_2A^2\left(q_1,q_2,-\frac{p_1q_1-p_2q_2}{p_3}\right) \qquad (84)$$

But

$$\frac{1}{p^2-k^2+i\varepsilon} - \frac{1}{p^2-k^2-i\varepsilon} = -\frac{i\pi}{k}\left\{\delta(p-k)+\delta(p+k)\right\} \qquad (85)$$

and model (5) for $A(\bar{p})$ in (84) yields

$$\int\frac{d\bar{q}\,A^2(\bar{q})}{2p\cdot q-i\varepsilon} = \frac{i\pi^2\sqrt{\pi}A_0^2}{ab\ p_0} \qquad (86)$$

When this result is used into (75) and we introduce spherical coordinates we have after ϕ integrations

$$\sigma_2 = \frac{2\epsilon^2}{9k^4A_0} \text{ Real } \frac{1}{2}\int_{-\infty}^{\infty} dp \frac{p}{p-k+i\varepsilon} \int_{-1}^{1} du[2\pi(p^2+4kpu)(p^2-4k^2)+\pi p^2(1-u^2)(15p^2+30kpu-12k^2)]$$

$$\times \frac{\Phi_0}{[1+\ell^2(p^2+2kpu+k^2)]^2} \frac{\pi^2 a\sqrt{\pi}\ell_0^2}{2k}\left\{-i\pi e^{-(\frac{ap_0}{2})^2} + 2\sqrt{\pi}D(\frac{ap_0}{2})\right\}$$

Carrying out the p integration and letting $x = p\ell$, $\ell_0 = k\ell$, $a_0 = ka$, we obtain,

$$\sigma_2 = \frac{2\epsilon^2}{9k^4}\left\{\frac{\pi^5\Phi_0\sqrt{\pi}a3k^4}{4} m_{21}(a_0,\ell_0) + \frac{\pi^4\Phi_0\sqrt{2}a}{2k\ell^5} m_{22}(a_0,\ell_0)\right\} \tag{80}$$

where

$$m_{21}(a_0,\ell_0) = \int_{-1}^{1} du \frac{10u^3+u^2-2u+1}{[1+2\ell_0^2(1+u)]^2} \exp\left(-\frac{a_0^2(1+u)^2}{4}\right) \tag{81}$$

$$m_{22}(a_0,\ell_0) = \int_{-1}^{1} du \; P\int_{-\infty}^{\infty} \frac{dx\, x}{x-\ell_0} D(\frac{a_0[x^2+4\ell_0 ux+3\ell_0^2]}{8\ell_0^2}) \tag{82}$$

$$\frac{x^2(1-u^2)(15x^2+30\ell_0 xu-12\ell_0^2)+2(x^2-4\ell_0^2)(x^2+4\ell_0 xu)}{(1+x^2+2\ell_0 xu+\ell_0^2)^2}$$

For the SRI experiment $a_0 = 7.8$ and $\ell_0 = 4.4$. Numerical integration yields $m_{21}(7.8,4.4) = -0.12$, $m_{22}(7.8,4.4) = 203$, and $m_1(4.4) = 0.0475$.

The fourth term in (65), σ_3, is now evaluated. Treating $A(\bar{p}-\bar{q})$ as a quasi delta function and writing

$$\frac{1}{(p^2-k^2+i\varepsilon)(q^2-k^2-i\varepsilon)} = \frac{1}{(\bar{q}-\bar{p})\cdot(\bar{q}+\bar{p})-i\varepsilon}\left\{\frac{1}{p^2-k^2+i\varepsilon} - \frac{1}{q^2-k^2-i\varepsilon}\right\}$$

we obtain approximately

$$\sigma_3 = \frac{2\varepsilon^2\phi^{-1}(2k)}{9k^4\ A_0}\int d\bar{p}\ h_3(\bar{p})\phi(\bar{p}+\bar{k})\phi(\bar{p}-\bar{k})\left(\frac{1}{p^2-k^2+i\varepsilon} - \frac{1}{p^2-k^2-i\varepsilon}\right)\int\frac{d\bar{q}\ A^2(\bar{q}-\bar{p})}{2\vec{p}\cdot(\vec{q}-\vec{p})-i\varepsilon} \qquad (83)$$

Carrying out the q_3 integration and recognizing that the principal value integral vanishes, $A(\bar{q}) = A(-\bar{q})$, we find the $\bar{q}$ integration reduces to

$$\int\frac{d\bar{q}\ A^2(\bar{p}-\bar{q})}{2\vec{p}\cdot(\vec{q}-\vec{p})-i\varepsilon} = \frac{i\pi}{2p_3}\int dq_1dq_2A^2\left(q_1,q_2,-\frac{p_1q_1-p_2q_2}{p_3}\right) \qquad (84)$$

But

$$\frac{1}{p^2-k^2+i\varepsilon} - \frac{1}{p^2-k^2-i\varepsilon} = -\frac{i\pi}{k}\left\{\delta(p-k)+\delta(p+k)\right\} \qquad (85)$$

and model (5) for $A(\bar{p})$ in (84) yields

$$\int\frac{d\bar{q}A^2(\bar{q})}{2p\cdot q-i\varepsilon} = \frac{i\pi^2\sqrt{\pi}A_0^2}{ab\ p_\rho} \qquad (86)$$

When (85) and (86) are used in (83) we obtain with model (5) for $\phi(\bar{p})$ and

$$A_0 = \pi a^2 b,\ \Phi_0 = 8\pi \ell^3,\ \varepsilon = \frac{k^2\Omega^2 p_1}{(2\pi)^3},\ a_0 = ka,\ \ell_0 = k\ell,\ \beta = \left(\frac{2\ell_0^2}{1+2\ell_0^2}\right)^2 \tag{87}$$

$$\sigma_3 = \Omega^4 p_1 \frac{\sqrt{\pi}}{8} a_0 \ell_0^3 \frac{16 J(\beta)(1+4\ell_0^2)}{\pi(1+2\ell_0^2)^4} \tag{88}$$

where

$$J(\beta) = \int_0^{\pi/2} d\theta \sin^4\theta \int_0^{\pi/2} d\phi (1-\beta\sin^2\theta\cos^2\phi)^{-2} \tag{89}$$

Making use of the substitution $x = 1-\beta\sin^2\theta\cos^2\phi$ we can carry out the double integration in terms of complete elliptic integrals:

$$J(\beta) = \frac{\pi}{12\beta^2(1-\beta)} \left\{(2\beta^2-3\beta+4)E(\beta) + (\beta-4)(1-\beta)K(\beta)\right\} \tag{90}$$

Recognizing that

$$E(\beta) = \frac{\pi}{2}\left(1 - \frac{\beta}{4} - \frac{1}{3}\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\beta^2 - \cdots\right) \underset{\beta\to 0}{\rightarrow} \frac{\pi}{2}$$

$$K(\beta) = \frac{\pi}{2}\left(1 + \left(\frac{1}{2}\right)^2\beta + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\beta^2 + \cdots\right) \underset{\beta\to 0}{\rightarrow} \frac{\pi}{2}$$

$$\lim_{\beta\to 1} E(\beta) = 1,\ \lim_{\beta\to 1} K(\beta) = \ln\left(\frac{16}{1-\beta}\right)^{1/2}$$

we obtain the following limiting values:

$$J(0) = \frac{3\pi^2}{32} \quad \text{and} \quad \lim_{\beta\to 1} J(\beta) = \frac{\pi}{4(1-\beta)}$$

A curve of $J(\beta)$ is shown in Fig. 1. We note that

$$\lim_{\beta\to 1} \frac{16J(\beta)(1+4\ell_0^2)^2}{\pi(1+2\ell_0^2)^4} = \frac{4(1+4\ell_0^2)}{(1+2\ell_0^2)^2}$$

Next we use results (73), (80) and (88) in equation (65) for the direct polarization cross section. With the definitions in (87) and writing the first Born radar cross section

$$\sigma_B = \frac{4\pi k^4 \Omega^4 p_1 A_0 \Phi(2k)}{(4\pi)^2} = \frac{4\pi\Omega^4 p_1 a_0^2 b_0 \ell_0^3}{2k^2(1+4\ell_0^2)^2} \tag{91}$$

We obtain

$$\sigma(-\vec{k}) = \sigma_B |Q|^2 [1 + \Omega^4 p_1 \frac{\sqrt{\pi}}{8} a_0 \ell_0^3 (-m_1 + \frac{m_{21}}{6} + \frac{m_{22}}{9\pi^{3/2}\ell_0^5} + \frac{16J(\beta)(1+4\ell_0^2)^2}{\pi(1+2\ell_0^2)^4}] \tag{92}$$

According to (64), (62) and (46)

$$Q = \frac{1}{1 - \varepsilon^2 f_{3383}} = \frac{1}{1 - \varepsilon^2 \frac{(2\pi)^3}{k^2} (\frac{1}{3} + \frac{i\sqrt{\pi} a_0}{4\sqrt{2}}) \pi^2 k \Phi_0 g(\ell_0)} \tag{93}$$

where

$$g(\ell_0) = \frac{i}{2\ell_0^6}\left\{(1+2\ell_0^2)\ln(1+i2\ell_0) - \frac{2\ell_0}{1+4\ell_0^2}(\ell_0[1+4\ell_0^2+2\ell_0^4]+i[1+5\ell_0^2+5\ell_0^4])\right\}$$

$$+ \frac{1+\ell_0^2+i6\ell_0^3}{3\ell_0^3(1+4\ell_0^2)} \qquad (94)$$

or separating into real and imaginary components

$$\ell_0^3 g(\ell_0) = -\frac{1}{\ell_0^2}\left\{\frac{16\ell_0^4+16\ell_0^2+3}{3(1+4\ell_0^2)} - \frac{1+2\ell_0^2}{2\ell_0}\tan^{-1}(2\ell_0)\right\}$$

$$+ -\frac{i}{\ell_0}\left\{-1 + \frac{0.575(1+2\ell_0^2)}{\ell_0^2}\log_{10}(1+4\ell_0^2)\right\} \qquad (95)$$

When definitions (87) are used in (93) we obtain

$$Q = \frac{1}{1 - \Omega^4 p_1 \left(\frac{1}{3} + i\frac{\sqrt{\pi}a_0}{4\sqrt{2}}\right)\ell_0^3 g(\ell_0)} \qquad (96)$$

For $a_0 = 7.85$, $b_0 = 15.7$, $\ell_0 = 4.4$, $\ell_0^3 g(\ell_0) = 1.05 + i\,0.282$,

$$|Q|^2 = \frac{1}{|1 - \Omega^4 p_1(-0.34 + i\,2.66)|^2} = \frac{1}{(1 + 0.34\Omega^4 p_1)^2 + 7.1\Omega^8 p_1} \qquad (97)$$

$$\sigma_B = \Omega^4 p_1 \frac{4\pi(7.85)^2\ 15.7\ (4.4)^3\ X\ 10^{-4}}{2(1.96)^2\ (1+78.5)^2} = 2.14 \times 10^{-3}\ \Omega^4 p_1\ m^2 \tag{98}$$

$$\beta = \left|\frac{2\ell_0^2}{1+2\ell_0^2}\right|^2 = \left|\frac{38.8}{39.8}\right|^2 = 0.95,\ J(0.95) = 15.5$$

Applying the numerical values to (92) we obtain for the direct polarization radar cross section

$$\sigma_{(RC)} = \frac{2.14 \times 10^{-3}\Omega^4 p_1}{(1+0.34\Omega^4 p_1)^2 + 7.1\Omega^8 p_1} \left\{1+\Omega^4 p_1 0.222(7.85)(4.4)^3(-0.0475 - \frac{0.12}{6}\right.$$

$$\left. + \frac{203}{9\pi^3\ 2(4.4)^5} + \frac{16(15.5)(79.5)^2}{\pi(39.8)^4}\right)\Bigg\}$$

$$\sigma_{(RC)} = \frac{2.14 \times 10^{-3}\Omega^4 p_1 (1 + 19.6\Omega^4 p_1)}{(1+0.34\Omega^4 p_1)^2+7.1\Omega^8 p_1}\ m^2 \tag{99}$$

4. Cross polarization backscatter cross section

Next we specialize the general cross section formula (Report 2, eq. 28) to determine the cross polarization backscatter cross section. Setting $\bar{\kappa} = -\bar{k}$, letting $\bar{d} \to \bar{c} = -\frac{k^2\Omega^2 p_1}{4\pi}\hat{j}$ and recalling that $\vec{Q} = \hat{Z}Q$ we observe

that the first two terms in (P2.28) vanish and we have for the cross polarization cross section ($\sigma_\perp$)

$$\sigma_\perp(-\vec{k}) = 2\varepsilon^2(\vec{c}\cdot\overleftrightarrow{G}(p)\cdot\vec{Q})(\vec{c}\cdot\overleftrightarrow{G}^*(q)\cdot\vec{Q}^*)\phi(p+k)\phi(p-k)A^2(q-p)$$

$$= 2\varepsilon^2\frac{\Omega^4 P_1}{(4\pi)^2}|Q|^2\int d\vec{p}\; h_\perp^2(\vec{p})\phi(\vec{p}-\vec{k})\phi(\vec{p}+\vec{k})\int\frac{d\vec{q}\;A^2(\vec{q}-\vec{p})}{(p^2-k^2+i\varepsilon)(q^2-k^2-i\varepsilon)} \tag{100}$$

where $h_\perp(\vec{p}) = (\hat{j}\cdot\vec{p})(\hat{z}\cdot\vec{p})$

For model (5) the integral terms in (100) reduce to

$$\int d\vec{p}(\;)\int d\vec{q}(\;) = \frac{\pi^3\sqrt{\pi}A_0^2}{ab}\int_0^\pi d\theta\int_0^{2\pi} d\phi\; h_\perp^2(\hat{pk})\phi(\hat{pk}-\bar{k})\phi(\hat{pk}+\bar{k}) \tag{101}$$

With $\hat{p}$ expressed in spherical coordinates

$$h_\perp^2(\hat{pk}) = k^4\sin^2\theta\cos^2\theta\sin^2\phi \tag{102}$$

When turbulence model (4) and (102) are used in (101) and the result is used in (100) there results

$$\sigma_\perp(-\vec{k}) = \sigma_B|Q|^2\Omega^4 P_1\frac{2}{\sqrt{\pi}}a_0\ell_0^3 J_\perp(\beta)\frac{(1+4\ell_0^2)^2}{(1+2\ell_0^2)^4} \tag{103}$$

where

$$J_{\perp}(\beta) = \int_0^{\pi/2} d\theta \sin^2\theta\cos^2\theta \int_0^{\pi/2} d\phi \frac{\sin^2\phi}{(1-\beta \sin^2\theta\cos^2\phi)^2}$$

$$= \frac{\pi}{12\beta^2}\left\{(2-\beta)E(\beta) - 2(1-\beta)K(\beta)\right\} \tag{104}$$

Limiting values of $J_{\perp}(\beta)$ are

$$J_{\perp}(0) = \frac{\pi^2}{64} \text{ and } J_{\perp}(1) = \frac{\pi}{12}$$

$J_{\perp}(\beta)$ is shown in Fig. 1 for $0.7 < \beta < 1$. For the numerical values used in the previous section the cross polarized radar cross section is, with $J_{\perp}(0.95) = 0.235$,

$$\sigma_{\perp(RC)} = \frac{2.14 \times 10^{-3}\Omega^4 p_1}{(1+0.34\Omega^4 p_1)^2+7.1\Omega^8 p_1} \Omega^4 p_1 \frac{2}{\sqrt{\pi}}(7.85)(4.4)^3(0.235)\frac{(78.5)^2}{(39.8)^4}$$

$$\sigma_{\perp(RC)} = \frac{0.935 \times 10^{-3}\Omega^8 p_1}{(1+0.34\Omega^4 p_1)^2 + 7.1\Omega^8 p_1} \text{ m}^2 \tag{105}$$

Plots of equations (99) and (105) are shown in Fig. 2. These results compare favorably with SRI experimental cross sections when the measurement's uncertainty is approximately 3 db.

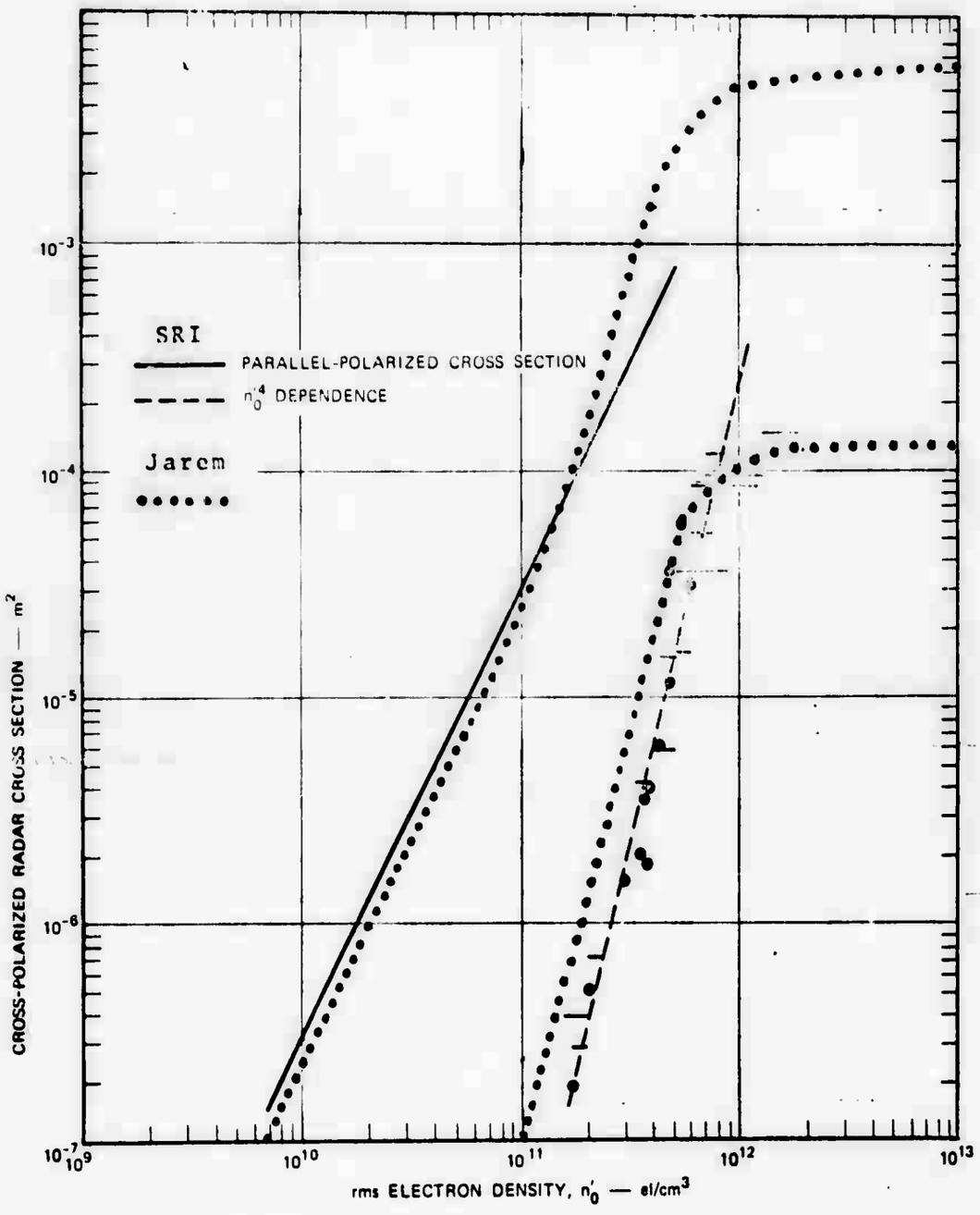

SRI
PARALLEL-POLARIZED CROSS SECTION
$n_0'^4$ DEPENDENCE
Jarem
CROSS-POLARIZED RADAR CROSS SECTION — m^2
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{9}
10^{10}
10^{11}
10^{12}
10^{13}
rms ELECTRON DENSITY, n_0' — el/cm^3

Figure 2

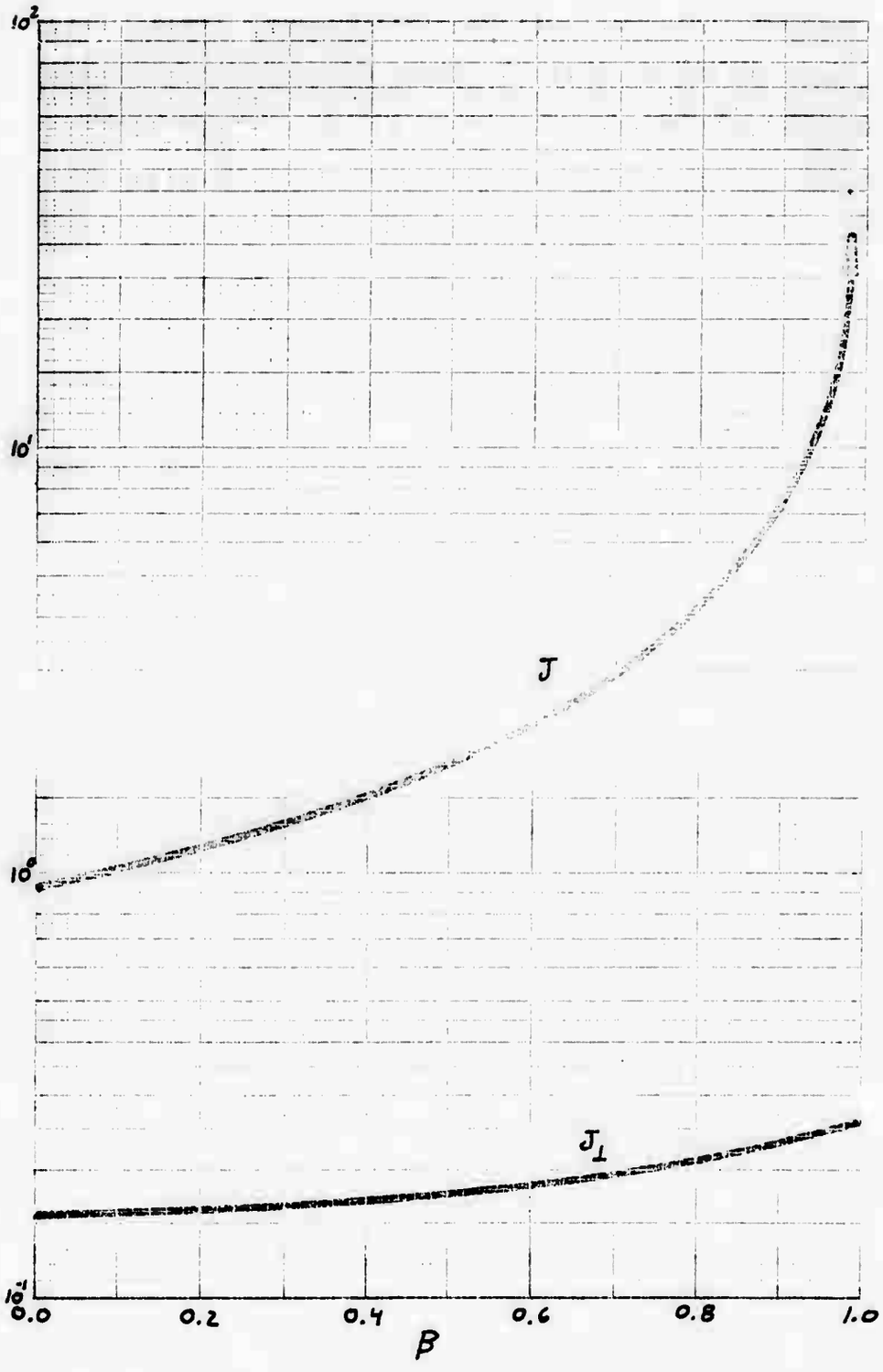


Figure 3

Report 4

Wave Scattering from a Unidimensional Layer with Strong Random Irregularities by the Method of Smoothing

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

PREFACE

In recent years, an increasing amount of interest has been focused upon the problem of wave propagation and scattering in random media, both from the viewpoint of theory and measurement. The underlying reasons for this interest are the numerous instances of wave propagation through turbulent regions such as: light or short wave communication in the atmosphere, under water scattering, and scattering from reentry objects.

To investigate these problems analytically, one has to utilize a mathematical model. Unfortunately, the model which presents the best physical description is mathematically intractable. This is due to the complicated dependence of the medium upon its physical parameters, i.e. temperature, pressure, humidity, electron density, velocity, etc..

In Chapter I, the choice of a simpler model which may describe the prominent characteristics of the wave medium interaction is adopted. In Section B., the medium is mathematically defined; the mean, rms, and correlation

functions are presented as adequate measures of the randomness in the medium. In Section C., the problem explored in this thesis is mathematically formulated for a layer of infinite extent in the yz direction and of thickness x_o in the x direction. This layer is illuminated by a plane wave propagating normally to the fluctuations in the index of refraction. This layer is also characterized by a collision frequency, a homogeneous mean and a strong small-scale fluctuation in the electron density. The resulting random differential equation of the field is formalized and its difficulties are explained. Since a straight Neumann series solution fails under the present circumstances, selective summation techniques are used.

Chapter II. develops these techniques (diagram and smoothing) with the appropriate approximation to account for large fluctuations and multiple scattering. Though both techniques lead to the same integro-differential equation, the derivations are completely different; consequently, these derivations shed light on different aspects of the problem and on the interpretation of the results.

The preceding techniques have been used in the study of wave propagation in infinite media. In 1962, Bourret (5), (6), derived the first order Dyson equation (15) through the hypothesis of local independence; he applied

the resulting equation to wave propagation in an infinite homogeneous random medium. In a series of papers, Keller re-derived (26), (28), the same results and avoided the above assumption. In 1963 Tatarski and Gercenstein (47) derived Bourret's results using a variant of the smoothing method. The resulting integro-differential equation was applied to scalar wave propagation in an infinite homogeneous medium with strong small-scale fluctuations. In 1964 Tatarski (45) studied the electromagnetic wave propagation in an infinite medium with strong dielectric- constant fluctuation. Macrakis (32) (1965) gave a more straightforward derivation of some of Tatarski's results. Brown (8) (1967) applied Tatarski's formulation to the propagation problem; he interpreted the results in terms of coherent and incoherent scattering, and the theory of dielectric. Frisch (16) and later Bassinini (2) (1967) obtained the known solution of the classical random oscillator through the smoothing method.

The application of the Dyson or smoothing formulations to a bounded or inhomogeneous random medium has received little attention. Bassinini et al. (1) (1967) used the infinite medium effective wave number parameter given by Tatarski to study scattering from bounded media. Recent papers (Rosenbaum, 1969; Kupiec et al., 1969; Collin, 1970)

studied some aspects of interface effect on coherent wave motion. Rosenbaum (41) obtained approximately the coherent wave scattered by a random half space whose mean dielectric constant is different from the uniform nonrandom medium in the conjugate half space. For a homogeneous background throughout, Kupiec et al. (30) solved the scalar Dyson equation for a normally incident wave on a random medium occupying a half space and slab geometries. Collin considered an obliquely incident wave on a random half space superimposed on a uniform medium throughout. It is difficult to assess the results obtained since the randomness was assumed to be homogeneous near the boundary

A complete investigation of the bounded random or inhomogeneous random problems is difficult. In this thesis some aspects of these problems are considered and applied to a randomly fluctuating plasma layer whose uniform background is generally different from the outside medium. In Chapter III, the conjecture made by Bassinini et al. about the effective wave number is tested through application to the layer problem. The reflected coherent and incoherent powers are derived. The comparison with the successive approximation method is also made. In Chapter IV, the exact dependence of the scatter power on large electron density fluctuations is investigated in the

limit of small-scale fluctuations. A locally homogeneous randomness, which appropriately describes the medium near the boundary, is used. Typical cases are studied for a wide range of plasma parameters. In Chapter V, a new formulation of the scattering problem is presented for a uniform background throughout with a locally homogeneous randomness, whose mean characteristics change smoothly. This property is advantageous in the Fourier transform space, and a singular integral equation of standard form results. This description appears to be closer to reality since it avoids the presence of a sharp boundary between the stochastic and nonstochastic medium.

I. FORMULATION OF THE PROBLEM

A. Introduction

In this chapter, the medium and its interaction with an incident wave is mathematically described. This description leads to linear differential equations whose coefficients are dependent upon the macroscopic inhomogeneities of the medium.

B. Description of the Medium

Two types of inhomogeneities exist, deterministic and random. It is deterministic if an identical measurement is performed many times and the results obtained are always alike. If, however, all conditions under the control of the experimenter remained the same, the results continually differ from each other, the medium is said to be random.

For a "temperate" and isotropic plasma with electron density fluctuations, the complex dielectric constant is(19):

$$K(x,\mu) = 1 - \frac{\Omega_p^2 (1 + j\Omega_c)}{(1 + \Omega_c^2)} \eta(x,\mu) \qquad (1)$$

where Ω_p , Ω_c are respectively the normalized plasma and collision frequencies; the quantity η is the electron

density distribution normalized on the peak or mean value. Here, the mean value of η is unity and the fluctuation η_1 is random.

In the applications of interest to the present investigation, the fluctuations in electron density are assumed to be related to turbulent mixing (48), (50). The mean and second moments of the fluctuations usually suffice for most scattering calculations. The mean indicates a steady value of the electron density over the period of observation. The second moment measures the correlation between electron density fluctuations at neighboring points. Ensemble averaging over a set of realizations is assumed throughout.

For homogeneous turbulence the second moments are given by

$$\langle \eta(x_1)\,\eta(x_2)\rangle = \sigma_\eta^2(x_1)\,B(x_1 - x_2) \tag{2}$$

For locally homogeneous turbulence, we have

$$\langle \eta(x_1)\,\eta(x_2)\rangle = \sigma_\eta^2\left(\frac{x_1 + x_2}{2}\right) B(x_1 - x_2) \tag{3}$$

The exponential and Gaussian correlative functions for isotropic turbulence,

$$B(x_1 - x_2) = e^{-\frac{|x_1 - x_2|}{l}} \tag{4}$$

$$B(x_1 - x_2) = e^{-\frac{|x_1 - x_2|^2}{l}} \tag{5}$$

are useful in many applications.

C. Derivation of the Wave Equation

Consider a plane wave of the form $\phi_0 e^{-j(\omega t - kx)}$ normally incident on a plasma layer (fig. 1). The slab is characterized by a complex dielectric constant which is a function of space. The medium is assumed to be linear and isotropic. The electron density possesses a homogeneous mean and a strong small-scale fluctuation.

Maxwell's equations for all space can be written as

$$\nabla \times \bar{E} = -j\omega\mu_0 \bar{H} \tag{6}$$

$$\nabla \times \bar{H} = j\omega\epsilon_0 K(x) \bar{E} \tag{7}$$

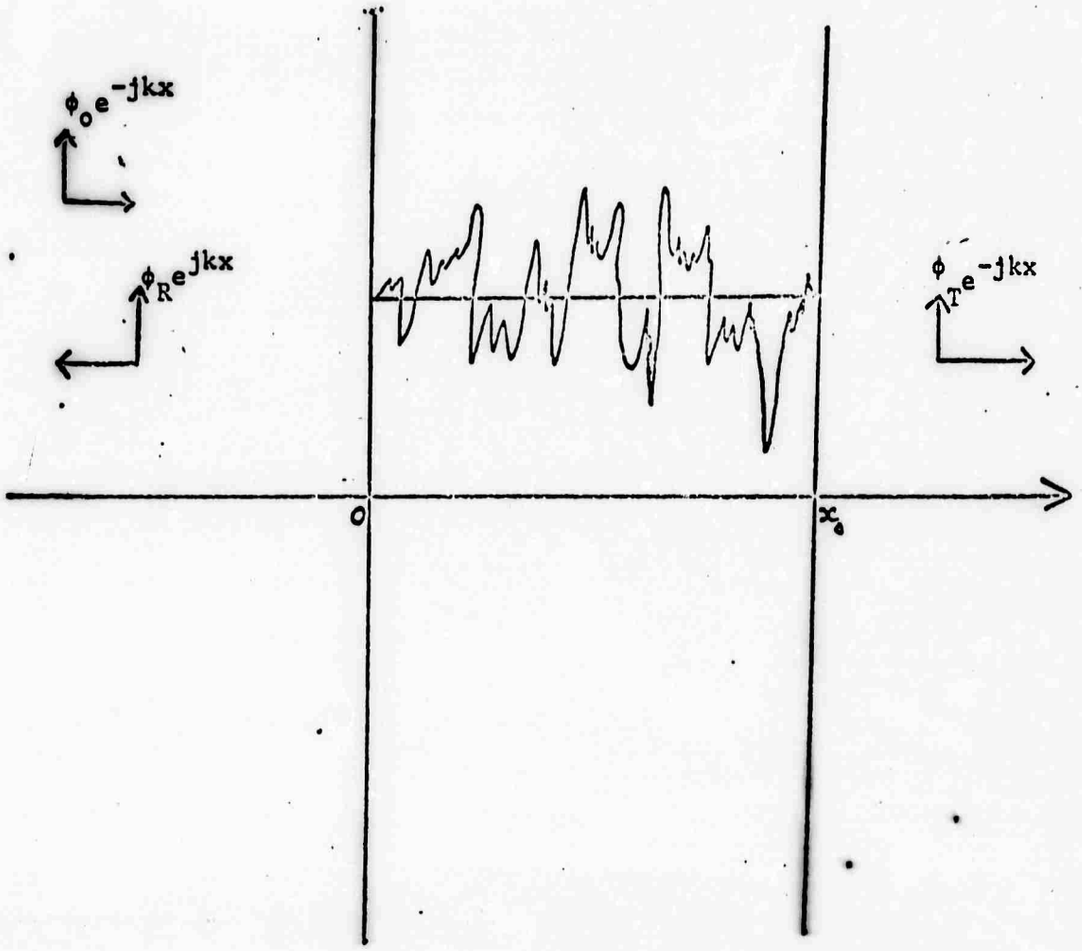


Fig. 1 Random fluctuations of electron density.

where $K(\underline{x})$ is the "effective dielectric constant", equal to one outside the scatterer and varying with $\underline{x}$ inside the scatterer.

The wave equation for $\bar{E}$ is found by vector manipulation of (6) and (7). Let $k^2 = \omega^2 \mu_0 \epsilon_0$, then,

$$(\nabla^2 + k^2 K(\underline{x}))\bar{E} = -\nabla\left(\frac{\bar{E} \cdot \nabla K(\underline{x})}{K(\underline{x})}\right) \tag{8}$$

In cases where the electric field vector is normal to the variation in the refractive index the cross-coupling term in (8) vanishes ; let ϕ be a component of the electric field vector, then

$$(\nabla^2 + k^2 K(\underline{x}))\phi = 0 \tag{9}$$

Complete mathematical solution of this differential equation with random coefficients is still lacking. Here, the study is confined to the investigation of the statistical properties of the solution, $\phi(\underline{x})$, of the random equation. To facilitate the analysis, equation (9) may be put in operational form:

$$L\phi = 0 \tag{10}$$

where

$$L = L_0 + L_1$$

$$L_0 = \frac{d^2}{dx^2} + k_1^2$$

$$L_1 = k^2 N \eta_1(x,t) \tag{11}$$

$$k_1^2 = k^2 K_0 = k^2 (1 - N)$$

$$N = \frac{\Omega_{p_0}^2 (1 + j\Omega_c)}{(1 + \Omega_c^2)}$$

together with the continuity conditions at the boundaries, and radiation conditions at $x = \pm\infty$.

Equation 10 is the reduced wave equation which occurs in classical and quantum fields (§6), (17).

The solution of the reduced wave equation must be found approximately because calculation of L involves a nonuniform component in L_0 and the random component L_1 .

Fortunately, physical interest is focused not on the functional dependence of $\phi(L_i)$, but on the statistical moments of the wave solution. The moments are in fact the observable features of the physical phenomena; it will be our purpose to investigate the coherent field $\langle\phi(*)\rangle$ and the fluctuating field, $\delta\phi$. Specifically, this thesis is concerned with the dependence of the scattered field on the large fluctuations of the refractive index.

The procedure employed consists in the calculation of the inverse operator, L^{-1}. This formulation will be the subject of the next chapter, and reduces to finding the effective characteristics of the density fluctuations, and subsequent determination of the coherent and incoherent scattered power.

II. THE METHODS OF PARTIAL SUMMATION

A. Introduction

The validity of the widely used successive approximation method is restricted to bounded media and small strength random perturbations (Appendix I). The problem under investigation does not satisfy these conditions; and in addition the summation of all terms up to second or third order does not work because the series converges too slowly to be practical. In these cases, two approximation methods (smoothing and diagram methods) are available; they will be described in this chapter. The diagram method (12) involves a summation over a selected class of repeated diagrams. The method of smoothing (29) involves the decoupling of the mean and fluctuating field through the solution of the latter field by formal iteration; an equation for the mean field is then obtained. Though both methods yield the same results (15) when applied to linear random equations, their methods of derivation shed light on different aspects of the problem. In Section B, the development leading to the Dyson equation (15) is carried out. In Section C, the smoothing equation is derived.

B. The Diagram Method

Let the field $\phi(x)$ in a random medium with index of refraction, $k_1(x,\mu)$, be related to its source J through

$$\phi(x) = \int G(x,x')\, J(x')\, dx' \qquad (1)$$

The Green's function, $G(x,x')$, satisfies the random equation in integral form :

$$G(x,x') = G^{(0)}(x,x') - \int G^{(0)}(x,x')\, L_1(x_1',\omega)\, G(x,x')\, dx_1 \qquad (2)$$

when $G^{(0)}(x,x')$ is the free space Green's function (37),

$$G^{(0)}(x,x') = \frac{1}{-2k_j}\, e^{-jk|x-x'|} \qquad (3)$$

The solution of (2) by iteration gives

$$G(x,x') = G^{(0)}(x,x') - \int G^{(0)}(x,1)\, L_1(1)\, G^{(0)}(1,x')\, d1 + \cdots + \qquad (4)$$

$$+ \int G^{(0)}(x,n)\, L_1(n) \cdots L_1(1)\, G^{(0)}(1,x')\, dn \cdots d1$$

An obvious physical interpretation of the n^{th} term is: a wave propagates unhindered from x' to 1, where it is scattered by the random inhomogeneity, the resultant propagates unhindered from 1 to 2, is scattered at 2, etc..

If the index of refraction is normally distributed, the average of the products of the random functions in

the n^{th} term is (8)

$$\langle L_0(1) L_0(2) \cdots L_0(n) \rangle = \sum \langle L_1(1) L_1(2) \rangle \cdots \frac{n}{2} \text{ terms } \langle L_1(n-1) L_1(n) \rangle \qquad (5)$$

or

$$\langle L_0(1) L_0(2) \cdots L_0(n) \rangle = \sum \Gamma_0(1,2) \cdots \Gamma_0(n-1,n) \qquad n \text{ even} \qquad (6)$$

$$0 \qquad n \text{ odd} \qquad (7)$$

and

$$\Gamma_0(1,2) = \langle L_1(1) L_1(2) \rangle \qquad (8)$$

$\Gamma_0(1,2)$, the correlation of two points, tends to zero for separations large compared to the correlation length. The sum in (5) is constructed from $n!/(\frac{n}{2})!\,2^{n/2}$ possible permutation of the indices (8). This summation is over all the combinations of the points 1, 2, ...n into groups of two

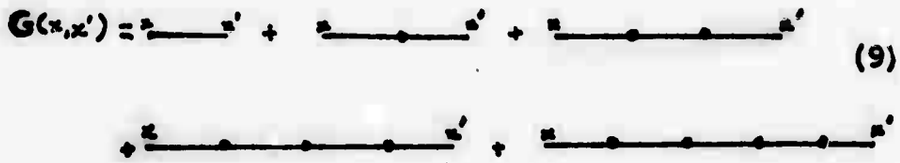
$$\Gamma_0(1,2), \ldots, \Gamma(n-1,n)$$

To facilitate the writing of the functions

$\langle G(x,x') \rangle$ and $\langle G(x,x') G^*(x,x') \rangle$

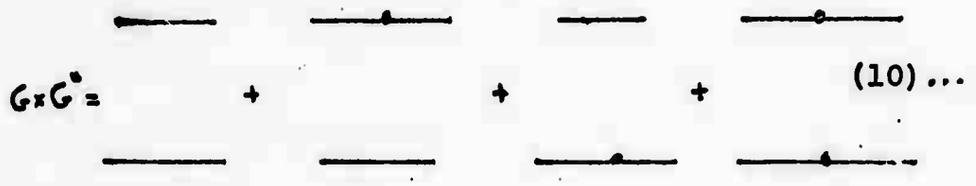
equivalent diagrams of each term in the mathematical series is drawn. A dictionary of the fundamental symbols may be constructed as follows: We represent each $G_{ij}^{(0)}$ by a horizontal line segment; if we have two $G_{ij}^{(0)}$ we join the two lines and put a dot at the junction. The dot indicates the coefficient of the term times the integration over the random inhomogeneity. A two point correlation is identified by a dashed line connecting the two random inhomogeneities. The multiplication of two diagrams is indicated by placing one on top of the other.

The Neumann series of the Green's function $G(x,x')$ may be diagrammed as follows:



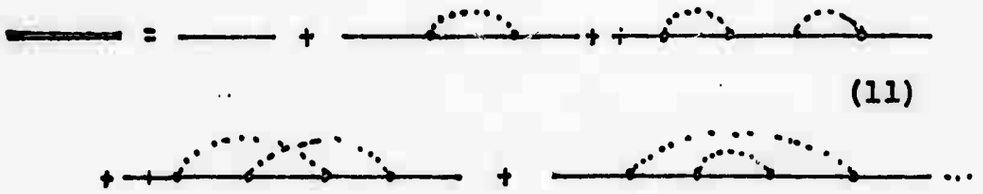
(9)

Its product with itself will be



(10)...

Since only L_1 indicates randomness, the means of (9) and (10) are:



(11)

(12)...

where

$\langle G(x,x') \rangle =$

$\langle G \times G^* \rangle =$ 

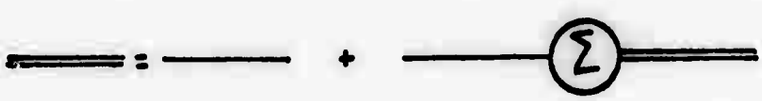
It is proven (Appendix A) that the Green's function series is convergent for a sufficiently small perturbation; the terms in the expansion are then successively smaller

with increasing order. In this case we can estimate the accuracy of the approximation from the highest order term. For large perturbations, the above process fails. In this case, the terms that appear to give the largest contribution are chosen and summed to an equivalent deterministic integral equation. Physical arguments at times enable us to identify the terms to be chosen. Though the diagram technique is lacking, rigorous mathematical justification, it is useful since it yields results which compare favorably in some cases, with exact methods (2).

To perform the summation, each term in the diagram equation of (11) is dismantled into noncorrelated parts. For instance

(13)

Factorization of noncorrelated parts in the diagrams lead to the Dyson (12) and Bethe-Salpeter (43) equations of field theoretic methods:

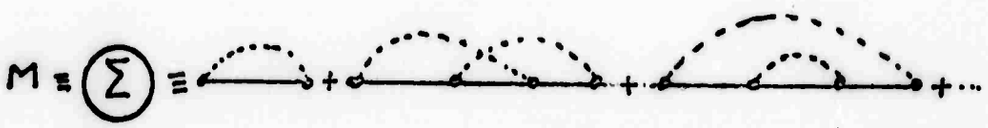


(14)

In operational form, we have

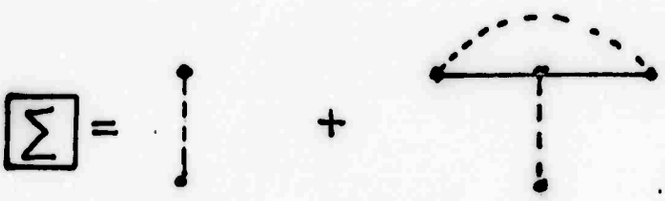
$$\langle G \rangle = G^{(o)} + G^{(o)} M G$$

where



(15)

where



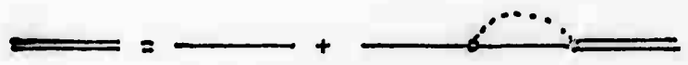
The "effective wave number operator", $\textcircled{\Sigma}$, and "intensity operator", $\boxed{\Sigma}$, are the sum of the diagrams that cannot be further dismantled without breaking dotted lines.

To construct the first approximation to the Dyson equation, the property

$$\Gamma_2(|1-2|) = 0 \qquad |1-2| > \ell \qquad (16)$$

reveals that the correlation between neighboring points

gives the greatest contribution when $|1-2|<\ell$; the diagrams with overlaping dotted lines are then neglected. This amounts to the retention of the first term in the mass operator



(17)

which is the summation of

(18)

It is to be noted that the solution of this equation should not be performed by iteration as this process leads back to a divergent series. This is expected as the Dyson equation is a selective restructuring of the Neumann series.

C. The Method of Smoothing

The smoothing method (15) to be discussed below is an alternative development to the diagram method. For random equations, the two methods lead to the same results. Some ad hoc procedures are also discussed in Appendix B.

To facilitate the analysis, the operational form of the wave equation is adopted:

$$\phi = G_{op}^{(\omega)} J - G_{op}^{(\omega)} \eta \phi \tag{19}$$

where

$$G_{op}^{(\omega)} = L_o^{-1}$$

$$\phi = P\phi + \delta\phi$$

$$P = \langle \rangle \tag{20}$$

$$\delta\phi = (I - P)\,\phi$$

By applying successively the projection, P , and fluctuation, $\delta\phi$, operators to equation (19) we obtain

$$\langle \phi \rangle = G_{op}^{(\omega)} J - G_{op}^{(\omega)} P \eta\, \delta\phi$$

$$\delta\phi = - G_{op}^{(\omega)} (I - P) L_1 (\langle \phi \rangle + \delta\phi) \tag{21}$$

Solution of (20) by formal iteration yields

$$\delta\phi = \sum_{n=1}^{\infty} (-G_{op}^{(\omega)} (I - P) L_1)^n \langle \phi \rangle \tag{22}$$

If we replace $\delta\phi$ in(20) by its value in(22) ,the coherent field becomes:

$$\langle\phi\rangle = G_T^{(0)} J + G_{0p}^{(0)} M \langle\phi\rangle \tag{23}$$

with the mass operator

$$\begin{aligned} M &= \sum_{n=1}^{\infty} P L_1 (- G_{0p}^{(0)} (I - P) L_1)^n P \\ &= - \sum_{n=1}^{\infty} P L_1 (- G_{0p}^{(0)} L_1)^n P \end{aligned} \tag{24}$$

Equation (23) represents the Dyson equation for the coherent field. Above order one,the Dyson equation is usually intractable because of the increasing complexity of the mass operator. In what follows, the analysis and discussion are concerned only with that first approximation; for n=1, equations (21) and (23) reduce to

$$L_0 \langle\phi\rangle + \langle L_1 L_0' L_1 \rangle \langle\phi\rangle = 0 \tag{25}$$

$$\delta\phi = L_0' L_1 \langle\phi\rangle \tag{26}$$

The solution of Equation (25) has been shown (47) to be asymptotic to the exact solution of the classical random oscillator for the condition of strong small-scale fluctuations.

D. Some Observations on the Partial Summation Method

The methods of partial summations transformed the stochastic integral equation (12) into a nonstochastic integro-differential equation, (25), which possesses the mathematical form

$$\frac{d^2}{dx^2}\langle\phi\rangle + k^2 K_0 \langle\phi\rangle - \int K(x,x')\langle\phi(x')\rangle\,dx' = 0 \qquad (27)$$

where $K(x,x')$ is the kernel and $\langle\phi\rangle$, the unknown coherent field.

A general solution of Equation (27) is not known. If a straightforward iteration procedure is used, this leads to Equation (18), which is known to diverge in the case of large fluctuations. When the kernel is of the convolution type(44)

$$K(x,x') = K(x-x')$$

the application of Fourier transform leads to a solution of the wave equation. Also if the integral is of the Wiener-Hopf type(44)

$$\frac{d^2}{dx^2}\langle\phi\rangle + k^2 K_0 \langle\phi\rangle - \int_{-\infty}^{\infty} K(x-x')\langle\phi(x')\rangle\,dx' = 0 \qquad (27)$$

$$-\infty < x < \infty$$

a technique is known for its solution. Both solutions have been discussed in the literature. Physically, both cases required, among other things, the assumption of random homogeneity; this seems hardly justifiable near the boundaries of bounded scatterers. In addition, the appropriate Green's function that satisfies the radiation condition at $x \to \pm\infty$ and the boundary conditions of the scatterer, is not of the convolution type. These physical modifications add to the complexity of the scattering problem considered in this thesis.

E. Conclusion

Further work on the mathematical foundations of the Dyson and smoothing equations are needed (33). At present, the capacity and limitations of the first order Dyson or smoothing equations are best gauged by the examination of the results of their application to prototype problems. The resulting integro-differential equation is difficult to investigate in the case of random, inhomogeneous or bounded media.

III. WAVE SCATTERING FROM A RANDOM UNIDIMENSIONAL LAYER WITH AN APPROXIMATE EFFECTIVE INDEX OF REFRACTION

A. Introduction

Recent papers (5), (32), (27), considered the propagation in an infinite random medium with strong small-scale fluctuations. For this case, the resulting integro-differential equation of the first smoothing equation is of the convolution type; its solution is readily obtained through Fourier transformation. However, examination of a bounded or an inhomogeneous medium requires extreme simplification of the stochastic problem. In a recent paper, Bassanini et al. (1) asserted that the interface effects on the coherent effective wave number parameter are negligible in the limit of small-scale fluctuations; they concluded that the development made by Tatarski (47) in the evaluation of an effective parameter for an infinite random medium with uniform background is applicable to bounded media. Since the mathematical foundation of this procedure is not well set, its results are subject to doubt. This chapter may be viewed as a numerical testing experiment to see if the above assertion has not produced a violation of the principle of conservation of electromagnetic energy. The reflected coherent and incoherent powers are analytically obtained as a function of the strength of the fluctuation. Comparison to the successive iteration method is also made.

B. Infinite Medium Effective Index Parameter

In this section we reduce the original stochastic problem in its coherent part to an equivalent deterministic boundary value problem through an estimate of the effective characteristics of an infinite medium.

The Green's function is taken as

$$G(x,x') = \frac{1}{-2jk\sqrt{\epsilon_0}} e^{(-jk\sqrt{\epsilon_0}|x-x'|)} \qquad (1)$$

The randomness is assumed to be homogeneous and the correlation is exponential. The field within the plasma is governed by an integro-differential equation of the convolution type; its solution has been carried out through the application of Fourier transformation (47) or through the assumption of a wave solution (25) of the form

$$\langle \phi(x) \rangle \sim e^{-j k_e x}$$

Following the second alternative, one obtains, after some manipulations

$$k_e = k(\alpha_e - j\beta_e) \qquad (2)$$

where

$$\alpha_2 = [(a + (a^2 + b^2)^{1/2})/2]^{1/2}$$

$$\beta_2 = [(-a + (a^2 + b^2)^{1/2})/2]^{1/2}$$

$$a = a_1 + (p\beta_1 + q\alpha_1)/[(a_1^2 + a_2^2)^{1/2} - pka_2]$$

$$b = a_2 + (p\alpha_1 - q\beta_1)/[(a_1^2 + a_2^2) + qka_2]$$

$$p = k\Omega_p^4 \sigma^2 a_2 (1 - \Omega_2^2)/(1 + \Omega_2^2)^2$$

$$q = 2k\Omega_p^4 \sigma^2 a_2 \Omega_2/(1 + \Omega_2^2)^2 \qquad (3)$$

$$\alpha_1 = [(a_1 + (a_1^2 + a_2^2)^{1/2})/2]^{1/2}$$

$$\beta_1 = [(-a_1 + (a_1^2 + a_2^2)^{1/2})/2]^{1/2}$$

$$a_1 = 1 - \Omega_p^2/(1 + \Omega_2^2)$$

$$a_2 = \Omega_p^2 \Omega_2/(1 + \Omega_2^2)$$

For $\sigma = 0$, Equation (2) reduces to the deterministic plasma wave number; for $\sigma \neq 0$, the turbulence attenuates the wave; it transforms the coherent energy of the incident wave into incoherent energy; this effect is more prominent as the normalized plasma frequency Ω_p nears the over-dense regime.

C. The Coherent and Incoherent Reflected Powers for a Slab

1. Coherent Reflected Field

The analysis is confined to the simple bounded slab filled with a randomly fluctuating plasma and described in Chapter I (fig.1). Assuming that Bassanini's work is correct, the coherent field equation, within the slab, may be written as

$$\frac{d^2\langle\phi\rangle}{dx^2} + k_c^2\langle\phi\rangle = 0 \qquad (4)$$

The general solution of the coherent field equation is

$$\langle\phi\rangle = A e^{-\gamma_c x} + B e^{\gamma_c x} \qquad (5)$$

where

$$\gamma_c = k(\beta_c + j\alpha_c) \qquad (6)$$

The boundary conditions are

$$1 + \langle\phi_R\rangle = \langle\phi(0)\rangle$$

$$1 - \langle\phi_R\rangle = -\frac{1}{jk}\frac{d\langle\phi(0)\rangle}{dx} \qquad (7)$$

$$\langle \phi_r \rangle e^{-jkx_0} = \langle \phi(x_0) \rangle$$

$$\langle \phi_r \rangle e^{-jkx_0} = -\frac{1}{jk} \frac{d}{dx} \langle \phi(x_0) \rangle \qquad (8)$$

Solution of the preceding algebraic equations will determine the four unknowns, A, B, $\langle \phi_R \rangle$, $\langle \phi_T \rangle$:

$$A = 2 \frac{b_1}{b_3}$$

$$B = -2 \frac{b_2}{b_3} e^{-2\gamma_c x_0}$$

$$\langle \phi_R \rangle = \frac{b_1 b_2}{b_3} (1 - e^{-2\gamma_c x_0}) \qquad (9)$$

$$\langle \phi_T \rangle = \frac{4}{b_3} (\alpha_c - j\beta_c) e^{-(\gamma_c - jk)x_0}$$

where

$$b_1 = 1 + \gamma_c / jk$$

$$b_2 = 1 - \gamma_c / jk \qquad (10)$$

$$b_3 = b_1^2 - b_2^2 e^{-2\gamma_c x_0}$$

2. Incoherent Reflected Field

The incoherent field equation may be written as

$$\frac{d^2}{dx^2}\delta\phi + k_1^2\,\delta\phi = L_1\langle\phi\rangle \tag{11}$$

The solution of Equation (11) by the "variation of parameters" (11) is

$$\delta\phi = A\,\delta\phi_1(x) + B\,\delta\phi_2(x) + \delta\phi_2(x)\int\frac{L_1\langle\phi\rangle\,\delta\phi_1\,dx}{W(\delta\phi_1,\delta\phi_2)} - \delta\phi_1\int\frac{L_1\langle\phi\rangle\,\delta\phi_2\,dx}{W(\delta\phi_1,\delta\phi_2)} \tag{12}$$

where $\delta\phi_1 = e^{\gamma x}$, $\delta\phi_2 = e^{-\gamma x}$ are the homogeneous solutions of (11) , W is the Wronskian.

Applying continuity conditions to (12), the incoherent field, $\delta\phi$, within the plasma becomes

$$\delta\phi(x) = k^2\Omega_p^2(1+j\Omega_c)\Big\{k_1^3\,\eta_{qp}(x_0,\gamma_2-\gamma)\,e^{\gamma x} + k_1^2k_2\left[\eta_{qp}(x_0,\gamma_2-\gamma)\,e^{-2\gamma_2 x_0} + \eta_{qp}(x_0,\gamma-\gamma_2)\,e^{-2\gamma x_0}\right]e^{\gamma x}$$

$$- k_1k_2^2\,e^{-2(\gamma+\gamma_2)x_0}\,\eta_{qp}(x_0,\gamma+\gamma_2)\,e^{\gamma x} + k_2^3\,\eta_{qp}(x,\gamma_2-\gamma)\,e^{-\gamma x}$$

$$+ b_1 b_2^* \left[\eta_{op}(x_0, \gamma_2 - \gamma)\, e^{-2\gamma_2 x_0} + \eta_{op}(x_0, \gamma - \gamma_2)\, e^{-2\gamma x_0} \right] e^{-\gamma x}$$

$$+ b_2^3\, e^{-2(\gamma+\gamma_2)x_0}\, \eta_{op}(x_0, \gamma+\gamma_2)\, e^{-\gamma x} + b_1 b_3\, e^{\gamma_2 x}\, \eta_{op}(x_1, -\gamma-\gamma_2)$$

$$b_2 b_3\, e^{\gamma x}\, \eta_{op}(x_1, -\gamma-\gamma_2) - b_2 b_3\, e^{-2\gamma_2 x_0} e^{\gamma x}\, \eta_{op}(x_1, \gamma_2 - \gamma) \qquad (13)$$

$$b_1 b_3\, e^{-\gamma x}\, \eta_{op}(x_1, \gamma - \gamma_2) + b_2 b_3\, \eta_{op}(x_1, \gamma+\gamma_2)\, e^{-\gamma x} e^{-2\gamma_2 x_0} \Big\}$$

where

$$\eta_{op}(x, q) = \int_0^x \eta(x_1)\, e^{q x_1}\, dx_1$$

3. Reflection Coefficients

The reflected power is now found as a coherent and an incoherent contribution. The coherent reflected power is easily calculated

$$|\langle \phi_R \rangle|^2 = \left| \frac{b_1 b_2 (1 - e^{-2\gamma_2 x_0})}{b_3} \right|^2 \qquad (14)$$

The incoherent reflected power may be written as

$$\langle |\delta \phi_R|^2 \rangle = \left| k^2 \Omega_p^2 (1 + j\Omega_c)(b_1 + b_2) \right| \langle \Big[\Big| - b_1^2\, \eta_{op}(x_0, -\gamma_2 - \gamma)$$

$$h_1 h_2 \eta_{o_T}(x_o, r_c - r) e^{-2\gamma_c x_o} + h_1 h_2 \eta_{o_T}(x_o, r - \xi) e^{-2\gamma x_o}$$

$$- h_2^2 \eta_{o_T}(x_o, r + r_c) e^{-2(\gamma + \gamma_c) x_o} \Big|^2 \Big] \rangle \qquad (15)$$

For the exponential correlation function, statistical averaging reduces (15) to

$$\langle |\delta\phi_R|^2 \rangle = \left| \frac{2 k \gamma_c \Omega_p^2 (1 + j\Omega_c)\sigma}{\gamma h_3^2 (1 + \Omega_c^2)} \right|^2 \Big\{ |h_1|^4 h(-\gamma_c - r, -\gamma_c'' - r'')$$

$$- h_1^2 h_1''^2 h_2''^2 e^{-2\gamma_c'' x_o} h(\gamma_c'' - r'', -\gamma_c - r) - [h_1^2 h_1''^2 h_2''^2 e^{-2\gamma'' x_o}$$

$$h(r'' - \gamma_c'', \gamma_c - r)] + h_1^2 (h_2'')^2 e^{-2(\gamma'' + \gamma_c'') x_o} h(\gamma'' + \gamma_c'', \gamma - r)$$

$$- h_1 h_2 (h_1'')^2 e^{-2\gamma_c x_o} h(\gamma_c - r, -\gamma_c'' - r'') + [|h_1 h_2|^2 e^{-2(\gamma_c + \gamma_c'') x_o}$$

$$h(\gamma_c - r, \gamma_c'' - r'')] + |h_1 h_2|^2 e^{-2(\gamma_c + r'') x_o} h(\gamma_c - r, r'' - \gamma_c'')$$

$$- h_1 h_2 (h_2'')^2 e^{-2(\gamma'' + \gamma_c'' + \gamma_c) x_o} h(\gamma'' + \gamma_c'' + \gamma_c - r)$$

$$- h_1 h_2 (h_1'')^2 e^{-2\gamma x_o} h(-\gamma_c'' - r'', r - \gamma_c) + [|h_1 h_2|^2 e^{-2(\gamma_c'' + r) x_o}$$

$$h(\gamma_c'' - r'', r - \gamma_c)] + |h_1 h_2|^2 e^{-2(r + r'') x_o} h(r'' - \gamma_c'', r - \gamma_c)$$

$$- b_1 b_2 (b_2^*)^2 e^{-2(\gamma^* + \gamma_e^* + \gamma)x_0} h(\gamma^* + \gamma_e^* + \gamma - \gamma_e)$$

$$b_2^2 b_1^* b_2^* e^{-2(\gamma_e^* + \gamma + \gamma_e)x_0} h(\gamma_e^* - \gamma^*, \gamma + \gamma_e)$$

$$b_2^2 b_1^* b_2^* e^{-2(\gamma^* + \gamma + \gamma_e)x_0} h(\gamma^* - \gamma_e^*, \gamma + \gamma_e)$$

$$|b_2|^4 e^{-2(\gamma + \gamma_e + \gamma^* + \gamma_e^*)x_0} h(\gamma^* + \gamma_e^*, \gamma + \gamma_e) \Big\} \qquad (16)$$

where

$$h(c_1, c_2) = -\frac{1}{c_1 + c_2}\left[\frac{1}{\ell - c_1} + \frac{1}{\ell - c_2}\right] + \frac{e^{(c_1 + c_2)x_0}}{c_1 + c_2}\left[\frac{1}{\ell + c_1}\right. \qquad (17)$$

$$\left. + \frac{1}{\ell + c_2}\right] + \frac{e^{(c_1 - \ell)x_0}}{(\ell - c_1)(\ell + c_2)} + \frac{e^{c_2 x_0}}{(\ell + c_1)(\ell - c_2)}$$

For a nonstochastic medium, $\sigma = 0$, the incoherent reflected power zeroes and the coherent reflected power reduces to the well known deterministic result.

If $\gamma = \gamma_e$ the reflected coherent and incoherent powers in Equation (14) and (16) reduce to those obtained by Jarem (22) using the successive approximation method; this condition is approached in practice for very small random variations. When the difference between the mean

values of the slab and the surrounding medium is not too small, the conjecture made by Bassinini et al. becomes unreliable since it is inconsistent with the continuity conditions at the boundaries. In fact, the principle of conservation of electromagnetic energy is violated (fig. 2).

D. Conclusion

The basic conclusion drawn from the analysis of the obtained results points out that the neglect of the interface effect on the coherent field is not generally close to reality: 1. Some observed phenomena, such as the saturation of the scattered power before the critical density is reached , are not displayed. 2. The conservation of electromagnetic energy, near the transition regime, is violated.

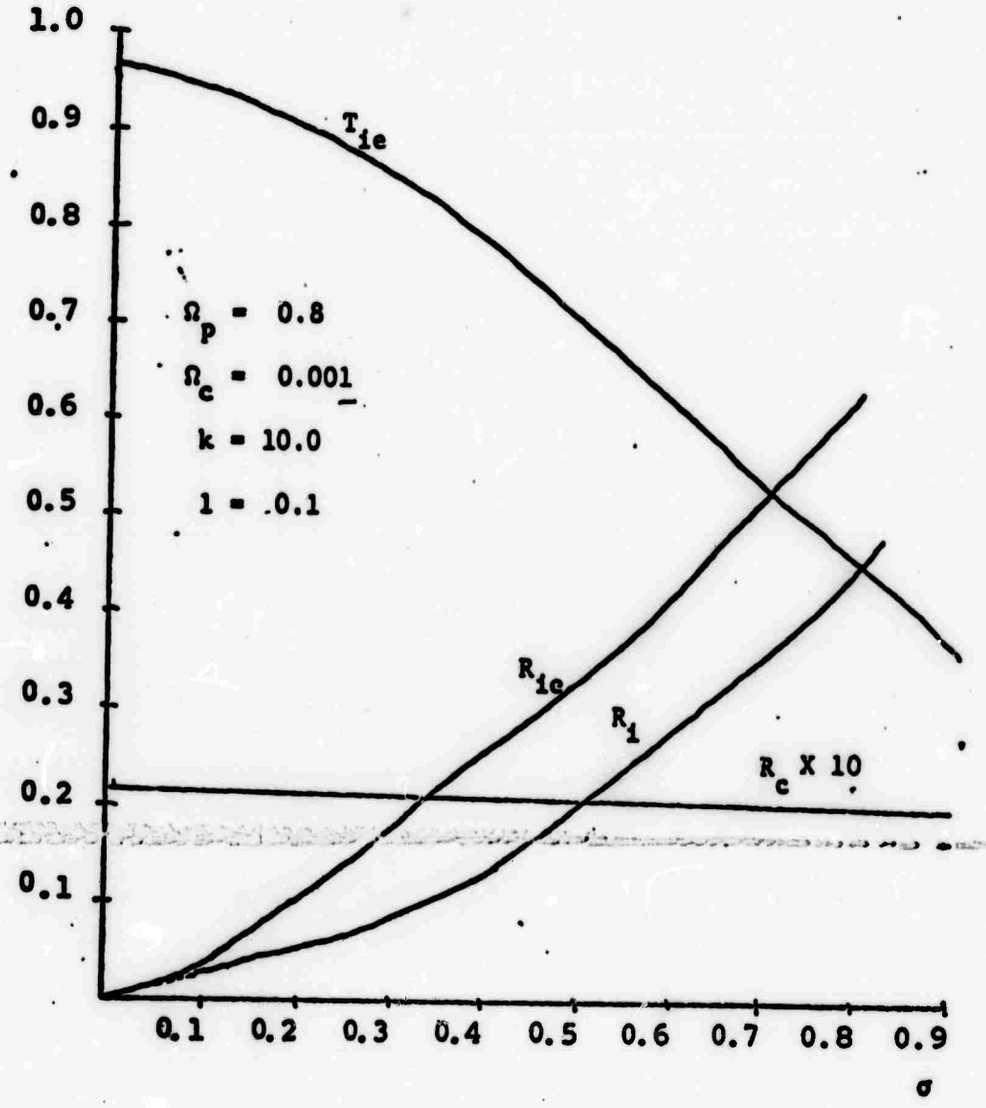


Fig. 2 Violation of conservation of energy principle in approximate smoothing usage.

IV. WAVE SCATTERING FROM A RANDOM UNIDIMENSIONAL LAYER WITH AN APPROPRIATE EFFECTIVE INDEX OF REFRACTION

A. Introduction

In the previous chapter, use of the infinite medium Green's function was shown to be improper for a slab, where the difference between the mean value of its dielectric constant and imbedding medium is not small. In this chapter, exact solution of the problem is obtained in the limit of small-scale fluctuations. We consider a uniform plasma slab with superimposed turbulence whose intensity and short scale behavior follow a Gaussian form.

Numerical evaluations of typical cases are displayed as a function of the mean square fluctuations.

B. Exact Effective Index of Refraction

The coherent field, within the slab, is governed by

$$\frac{d^2}{dx^2}\langle\phi(x)\rangle + k^2(1-N)\langle\phi\rangle - k^4N^2\int G(x,x')\,B(x,x')\langle\phi(x')\rangle\,dx' \tag{1}$$

The Green's function, $G(x,x')$ is that of the background medium wave equation

$$\frac{d^2}{dx^2}\psi(x) + k^2(1-N)\,\psi(x) = 0$$

$$x_0 < x < 0$$

and

$$\frac{d^2}{dx^2}\psi(x) + k^2\psi(x) = 0 \qquad \begin{matrix} x > 0 \\ x < x_0 \end{matrix} \qquad (2)$$

The general solution of Equation (2) can be written in terms of known functions; we let ψ_1 and ψ_2 be the solutions of Equation (2) which satisfy the radiation conditions at $x \to \pm\infty$ and the boundary conditions at $x = 0, x_0$. The Green's function may then be constructed (14):(35), (44):

$$G(x, x') = \frac{\psi_1(x_<)\,\psi_2(x_>)}{W(\psi_1, \psi_2)} \qquad (3)$$

W is the Wronskian; $x_>$ is the greater of the two numbers x and x', and $x_<$ is the lesser of x and x'. Let us determine the functions ψ_1 and ψ_2 for the turbulence free uniform slab (fig.2). The determination follows from the standard solution of the slab problem, together with the boundary conditions and the properties of the Green's function; this leads to

$$\psi_1(x_<) = \begin{cases} e^{jkx} + R_{01}e^{-jkx} & x > 0 \\ A_2 e^{jk_2 x} + B_2 e^{-jk_2 x} & x_0 < x < 0 \\ T_{01}e^{jkx} & x < x_0 \end{cases} \qquad (4)$$

$$\psi_2(x) = \begin{cases} T_{0_2}\, e^{-jkx} & x > 0 \\ A_1\, e^{-jk_p x} + B_1\, e^{jk_p x} & x_0 < x < 0 \\ e^{-jkx} + R_{0_1}\, e^{jkx} & x < x_0 \end{cases} \qquad (5)$$

where

$$A_2 = \frac{2\, e^{j(k_p - k_0)x_0}(1 + k_p/k)}{\Delta}$$

$$B_2 = \frac{-2(1 - k_p/k)\, e^{2jk_p x_0}}{\Delta}$$

$$T_{0_2} = \frac{4(k_p/k)\, e^{j(k_p - k)x_0}}{\Delta}$$

(6)

$$A_1 = \frac{-2e^{-jkx_0}(1 + k_p/k)}{\Delta}$$

$$B_1 = \frac{-2e^{j(k_p - k)x_0}(1 - k_p/k)}{\Delta}$$

$$\Delta = (1 + k_p/k)^2 - (1 - k_p/k)^2 e^{2jk_p x_0}$$

To pursue the analysis of Equation (1) we consider the mean value of the field that has a scale of inhomogeneity

much larger than the correlation scale of the perturbation. The correlation function manifests itself then as a local effect (13), (47)

$$B(x,x') = \delta(x-x') \int B(x,w)\, dw \qquad (7)$$

This means that we are interested in the variation of the coherent field on a scale that is large compared to the correlation length. In this case, the first order smoothing equation becomes for a locally homogeneous fluctuation whose intensity or strength follow a Gaussian form:

$$\frac{d^2\langle\phi\rangle}{dx^2} + k^2 M(x) \langle\phi\rangle = 0 \qquad (8)$$

where

$$M(x) = (1-N) - k^2 N^2 \sigma^2(x)\, G(x,x)$$

This equation reduces to that obtained elsewhere (30) for the case of an infinite uniform background and homogeneous fluctuations for which an analytical solution can be obtained in terms of known functions. However, the solution of Equation (8) for a finite slab thickness is not known analytically and must be integrated numerically for typical cases. No difficulty is encountered as any

singularity of the "effective wave number" is shifted off the real axis by the presence of the randomness. This shift, which appears as a collisional effect, describes the transformation of the energy in the propagating wave into fluctuation energy. The transmitted coherent field at the boundary is chosen to be of unit magnitude and zero phase. The Runge-Kutta method (21) is used to carry the integration to the input boundary, and the reflection and transmission coefficient are then automatically evaluated.

C. Scattered Power

1. Coherent Scattered Power

Consider the following scattering geometry (fig. 2). The change in the positive reference direction is made for computational convenience.

The boundary conditions are:

$x=x_0$

$$\langle \phi(x_0) \rangle = \langle \phi_1 \rangle e^{jkx_0} \tag{9a}$$

$$\frac{1}{jk}\frac{d}{dx}\langle \phi(x_0) \rangle = \langle \phi_1 \rangle e^{jkx_0}$$

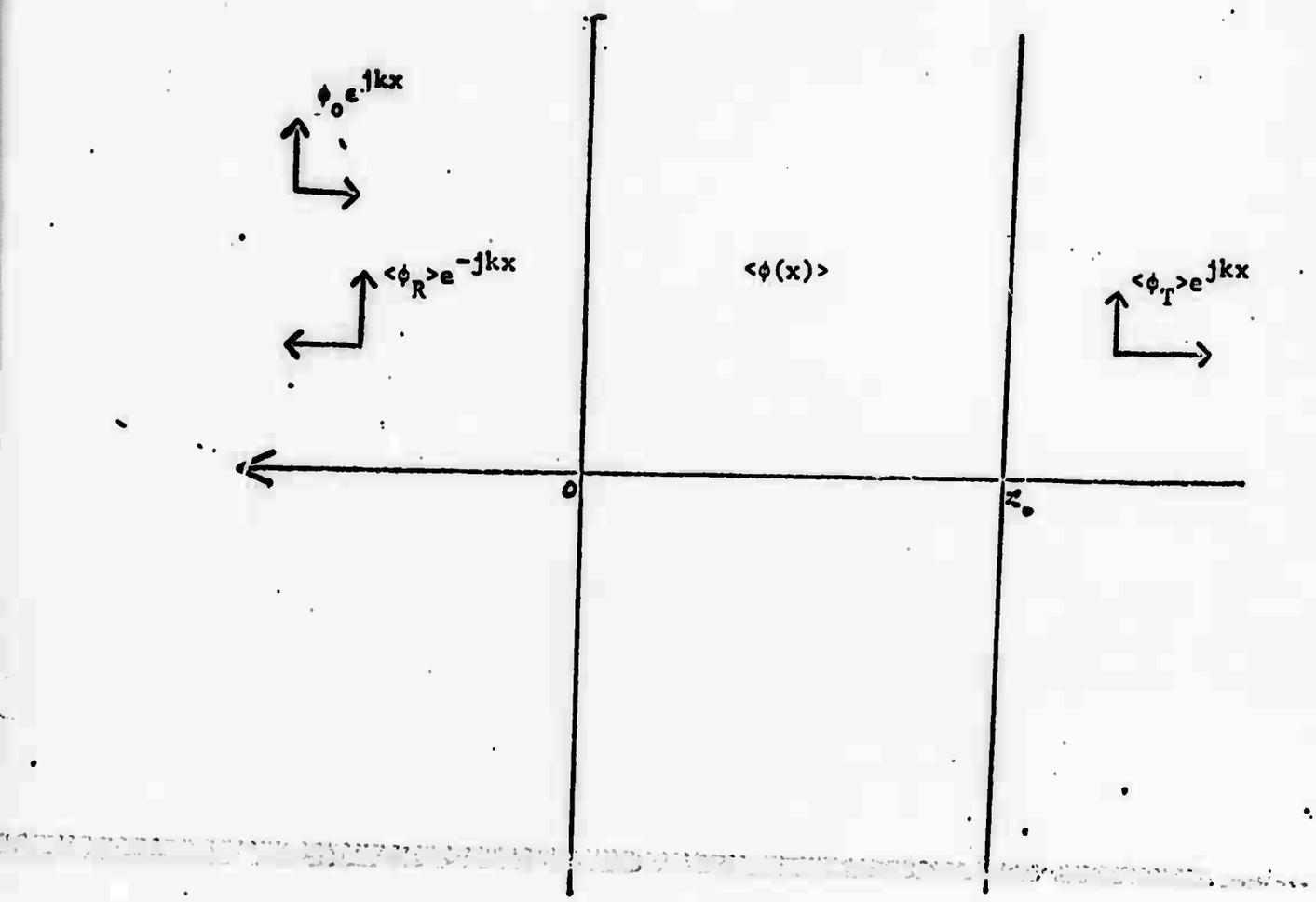


Fig. 3 Geometry of the plasma slab.

$x=0_+$

$$\phi_0 + \langle \phi_x \rangle = \langle \phi(x) \rangle$$

$$\phi_0 - \langle \phi_x \rangle = \frac{1}{jk} \frac{d}{dx} \langle \phi(x) \rangle \tag{9b}$$

Let the complex coherent field be represented in terms of real and imaginary parts as

[illegible] (10)

Let

$$M(x) = M_r(x) + j M_i(x) \tag{11}$$

Equation (8) may be decomposed into a coupled set of second order differential equations with variable coefficients

$$\frac{d^2}{dx^2}\langle \phi_r(x)\rangle + k^2\left[M_r(x)\langle \phi_r(x)\rangle - M_i(x)\langle \phi_i(x)\rangle\right] = 0$$

$$\frac{d^2}{dx^2}\langle \phi_i(x)\rangle + k^2\left[M_r(x)\langle \phi_i(x)\rangle + M_i(x)\langle \phi_r(x)\rangle\right] = 0 \qquad (12)$$

Let

$$\frac{d}{dx}\langle \phi_r(x)\rangle = P_r \quad \text{and} \quad \frac{d}{dx}\langle \phi_i(x)\rangle = P_i \qquad (13)$$

Equation (12) takes the form (11):

$$\begin{bmatrix} P_r' \\ P_i' \\ \langle \phi_r' \rangle \\ \langle \phi_i' \rangle \end{bmatrix} = \begin{bmatrix} 0 & 0 & -k^2 M_r & k^2 M_i \\ 0 & 0 & -k^2 M_i & -k^2 M_r \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_r \\ P_i \\ \langle \phi_r \rangle \\ \langle \phi_i' \rangle \end{bmatrix} \qquad (14)$$

Initial values may be obtained if we take the wave in the slab at $x=x_0$ as the reference; for convenience let $\langle \phi(x_0)\rangle = 1$. there results from the boundary conditions, Equation (9)

$$- \langle \phi_2(x_0) \rangle = 1$$

$$- \langle \phi_1(x_0) \rangle = 0$$

$$- P_2(x_0) = 0 \qquad (15)$$

$$- P_1(x_0) = k$$

We solve for the incident reflected and transmitted coherent field as a function of the field within the plasma at x=0; we obtain

$$\phi_0 = \frac{1}{2}\left(\langle \phi(0) \rangle - j \left\langle \frac{\phi'(0)}{k} \right\rangle \right)$$

$$\langle \phi_2 \rangle = \frac{1}{2}\left(\langle \phi(0) \rangle + j \left\langle \frac{\phi'(0)}{k} \right\rangle \right) \qquad (16)$$

$$- \langle \phi_T \rangle = \langle \phi(x_0) \rangle \, e^{-jkx_0}$$

The coherent reflection and transmission coefficients are

$$|\langle \phi_R \rangle|^2 = \frac{(k\langle \phi_1(0) \rangle - \langle \phi_2'(0) \rangle)^2 + (k\langle \phi_2(0) \rangle - \langle \phi_1'(0) \rangle)^2}{(\langle \phi_2'(0) \rangle - k\langle \phi_1(0) \rangle)^2 + (k\langle \phi_2(0) \rangle + \langle \phi_1'(0) \rangle)^2}$$

$$|\langle\phi_T\rangle|^2 = \frac{4k^2}{(\langle\phi_R'(0)\rangle - k\langle\phi_I(0)\rangle)^2 + (k\langle\phi_R(0)\rangle + \langle\phi_I'(0)\rangle)^2} \tag{17}$$

The respective phase shifts are

$$\angle\langle\phi_R\rangle = \tan^{-1}\frac{k\langle\phi_I(0)\rangle + \langle\phi_R'(0)\rangle}{k\langle\phi_R(0)\rangle - \langle\phi_I'(0)\rangle} - \tan^{-1}\frac{k\langle\phi_I(0)\rangle - \langle\phi_R'(0)\rangle}{k\langle\phi_R(0)\rangle + \langle\phi_I'(0)\rangle}$$

$$\angle\langle\phi_T\rangle = k x_0 - \tan^{-1}\frac{k\langle\phi_I(0)\rangle - \langle\phi_R'(0)\rangle}{k\langle\phi_R(0)\rangle + \langle\phi_I'(0)\rangle} \tag{18}$$

2. Incoherent Scattered Power

The incoherent reflected, $R_i = \langle|\delta\phi_R|^2\rangle$, and transmitted, $T_i = \langle|\delta\phi_T|^2\rangle$, powers are given formally by (II,26). In the one-dimensional case this leads to

$$\langle|\delta\phi_R|^2\rangle = \frac{|\epsilon|^2|\psi(0)|^2}{|W(\psi_1,\psi_2)|^2}\Big[\int dx'\,\psi_2(x')\langle\phi(x')\rangle \tag{19}$$

$$\int dx''\,\psi_2^*(x'')\langle\phi(x'')\,\sigma^2\left(\tfrac{x'+x''}{2}\right)\beta(x'-x'')$$

$$\langle|\delta\phi_1|^2\rangle = \frac{|\epsilon|^2\,\psi_0(x_0)}{|W(\psi_1,\psi_0)|^2}\Big[\int dx'\;\psi_1(x')\langle\phi(x')\rangle$$

$$\int dx''\;\psi_1^*(x'')\langle\phi(x'')\rangle\,\sigma^2\!\left(\frac{x'+x''}{2}\right)B(x'-x'')$$

(20)

Manipulation of (7), (19) and (20) give

$$\langle|\delta\phi_2|^2\rangle = \frac{k^2\Omega_p^*}{4|\phi_0|^2}\int_{-1}^{0}\Big(|A_2|^2+|B_2|^2+A_2^*B_2\,e^{-2jk_1x'}$$

$$+A_2B_2^*\,e^{2jk_1x'}\Big)\,\sigma^2 e^{-s\left(x'+\frac{1}{2}\right)^2}\langle\phi(x')\rangle\,dx'$$

(21)

$$\langle|\delta\phi_1|^2\rangle = \frac{k^2\Omega_p^*}{4|\phi_0|^2}\int_{-1}^{0}\Big(|A_1|^2+|B_1|^2+B_1A_1^*\,e^{2jk_1x'}$$

$$+A_1B_1^*\,e^{-2jk_1x'}\Big)\,\sigma^2 e^{-s\left(x'+\frac{1}{2}\right)^2}\langle\phi(x')\rangle\,dx'$$

(22)

D. Discussion of the Results

The first Born approximation is widely used in the literature (46); this approximation assumes that the incident wave propagates to and from its scattering center unhindered by the rest of the slab. Subject to this assumption the analytical results show a linear relationship between the scattered incoherent powers and the mean square fluctuations, σ^2 . On the other hand, experimental observations display a saturating effect as σ^2 increases; this relays a breakdown of single scattering

theory due to the stronger interaction between the incident wave and the electron fluctuations. The present analysis brings in the effects of multiple scattering of the incident wave and weighs their effects. The results show a linear relationship between the scattered incoherent powers for small mean square fluctuations, σ^2 ; as the value of σ^2 increases, the scattered power begins to saturate (fig.4 ,5). This saturation is accentuated as the normalized plasma frequency, Ω_p , nears the transition regime (fig. 6,7). This is expected since the mean index of refraction appears in the denominator of the coherent equation's effective index of refraction. The saturation eventually levels out, tending generally to some limiting value; this qualitative behavior is supported by experimental observations (40).

If a slight damping is introduced, weak and subdued incoherent scattering results (fig.8 , 9,10), since its source, the coherent field, suffers attenuation as it illuminates the layer. In addition, no appreciable departure from the power scattering law $\langle|\delta R|^2\rangle \sim \sigma^2$ with increasing σ^2 is noticed. However, it should not be surmised that the single scattering theory gives a physical description of the interaction.

A comparison of the homogeneous fluctuational effect with the inhomogeneous one show a relatively higher incoherent scattering in the former case throughout the

variation of the mean square fluctuations (fig. 11); a less definite dependence appears in the coherent scattering case (fig.12,13). For a locally homogeneous randomness, the Gaussian correlation shows a larger scattering power than the exponential one for the cases considered (fig.14).

Typical examples for the dependence of the coherent reflected and transmitted powers on the mean square fluctuations are shown in Figure15,16. The results show a definite dependence on large σ^2 . This correlates with the theoretical formulations since σ^2 appears as an attenuation effect on the coherent field; this attenuation describes analytically the energy transfer from the coherent to the incoherent field. For slight damping the variations in the coherent scattered power as a function of σ^2 are subdued in comparison to zero collision; for $\Omega_c = .1$ the coherent power is comparatively insensitive to the value of the mean square fluctuations (fig 17.).

The results seem to indicate that a nonrandom approach should be sufficient to describe the coherent return with an accuracy of at least 3 db .

It is to be noted that the present formulation, contrary to the preceding one in Chapter III, obeys the principle of conservation of energy as is shown in the varied cases considered.

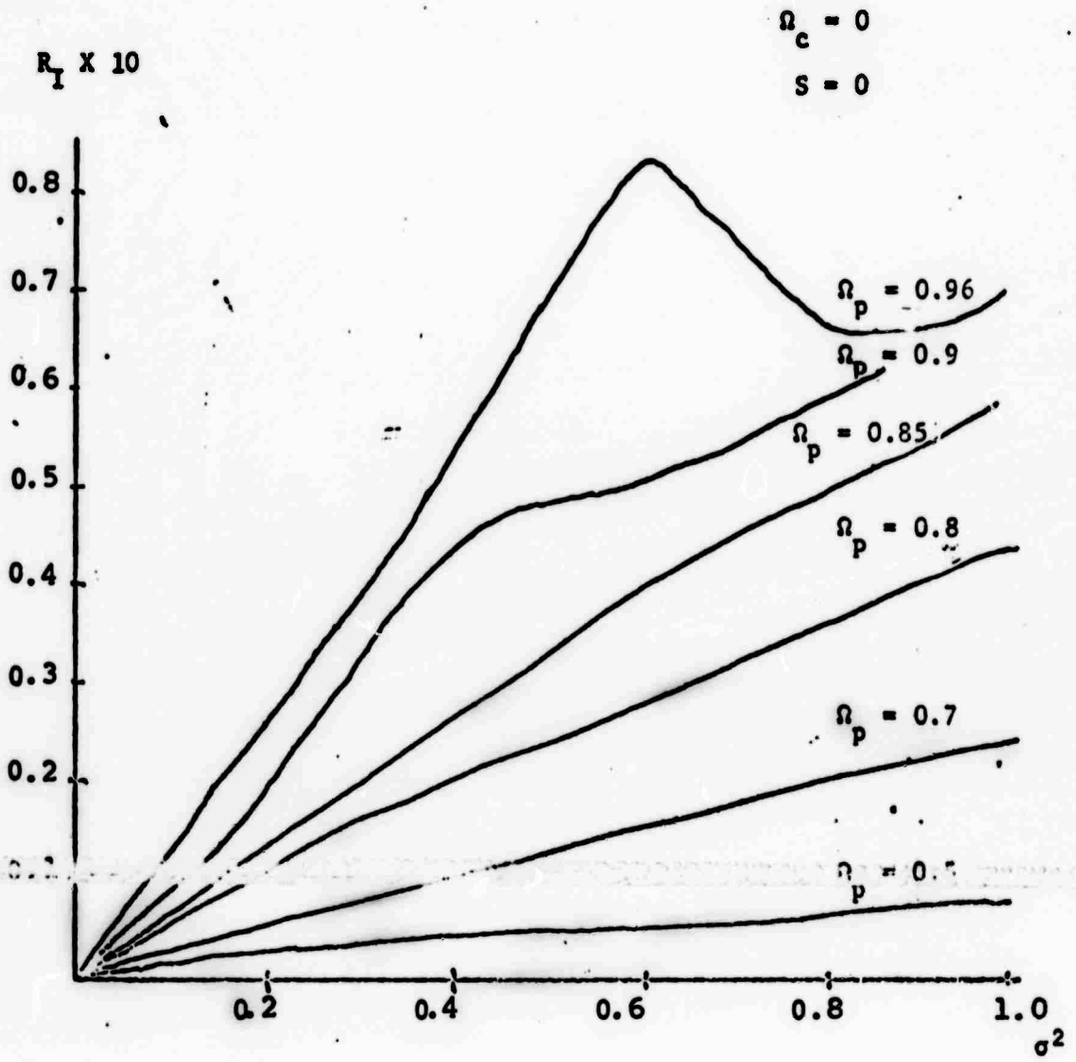


Fig. 4 Incoherent reflected power as a function of the mean square fluctuations with normalized plasma frequency as a parameter.

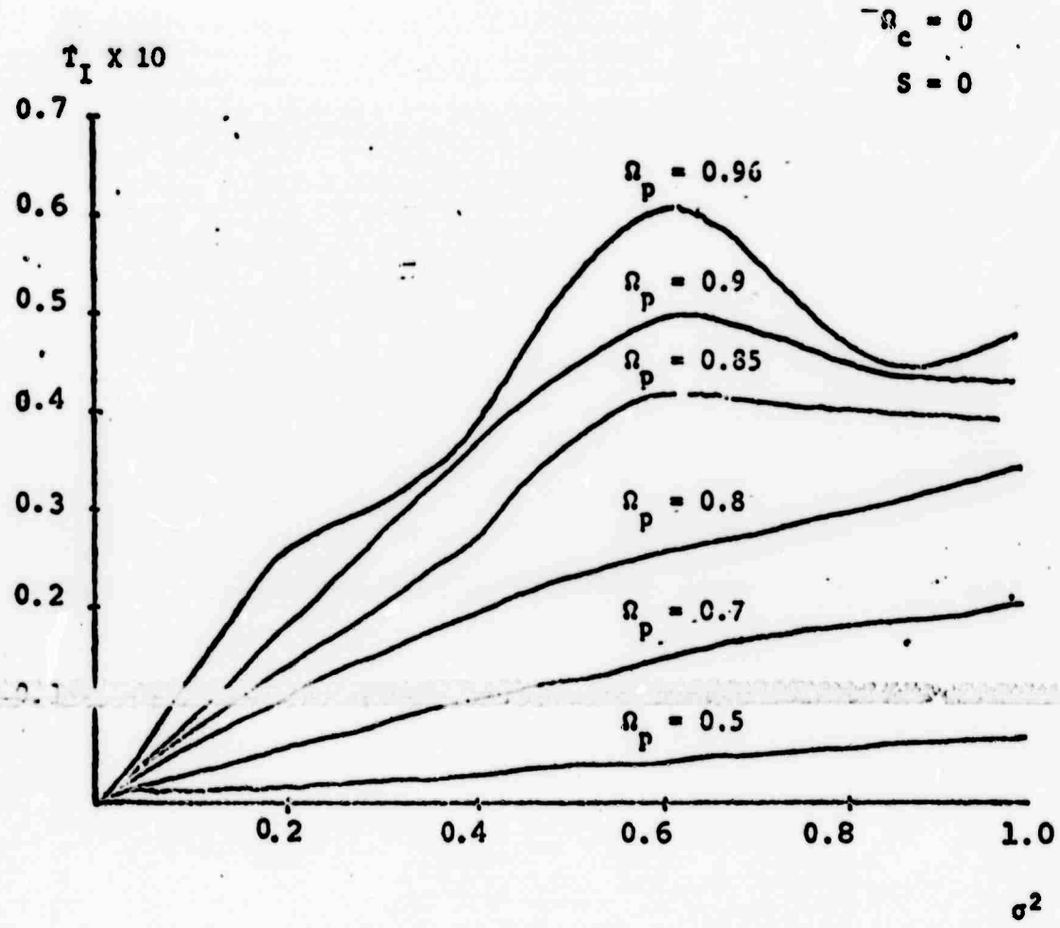


Fig. 5 Incoherent transmitted power as a function of the mean square fluctuations with normalized plasma frequency as a parameter.

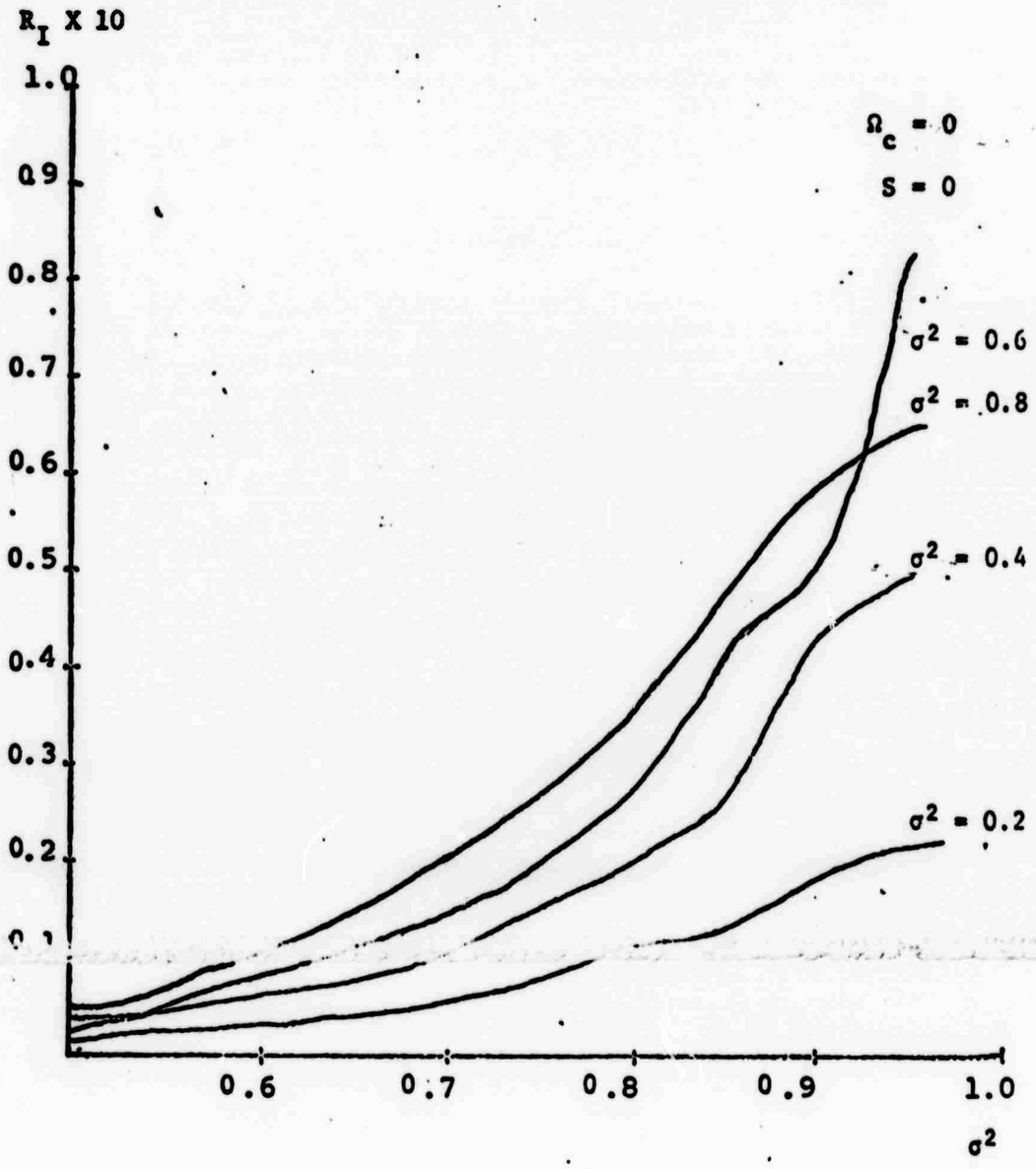


Fig. 6 Effect of plasma frequency on the incoherent reflected power.

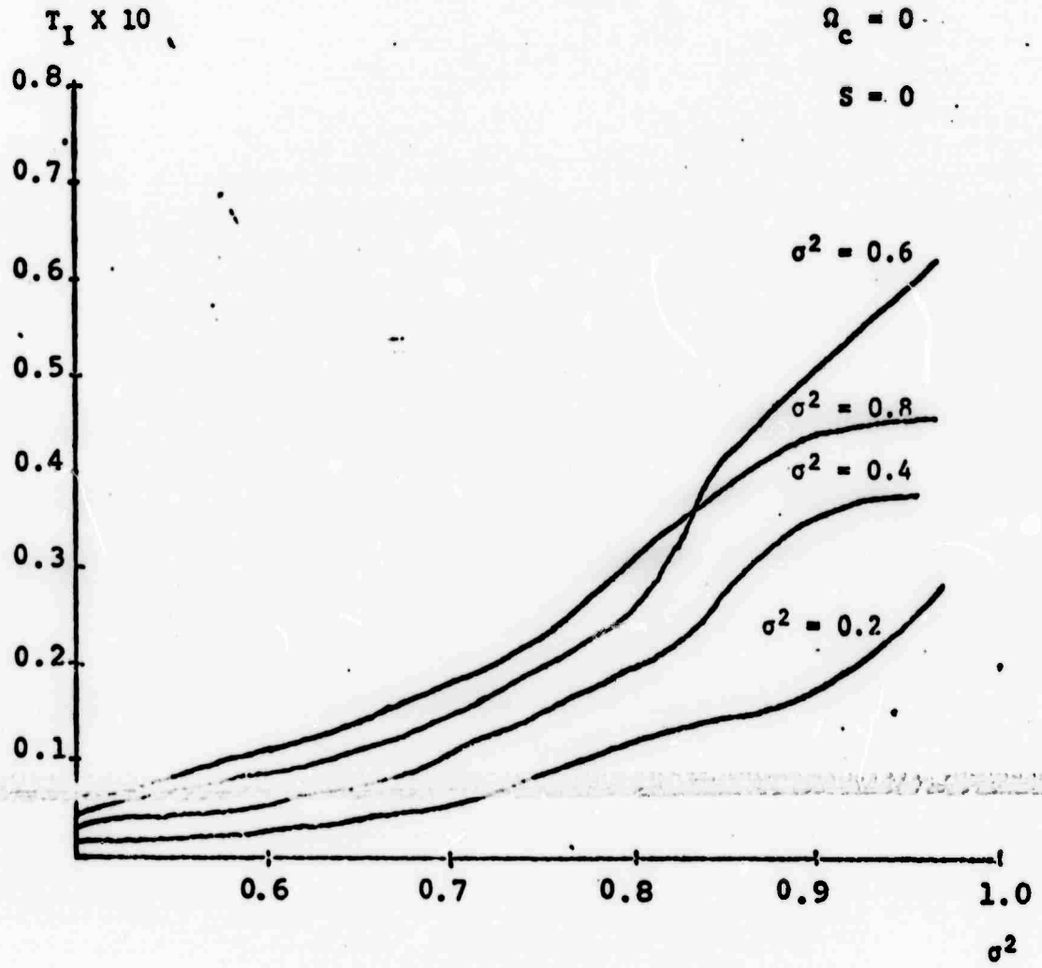


Fig. 7 Effect of plasma frequency on the incoherent transmitted power.

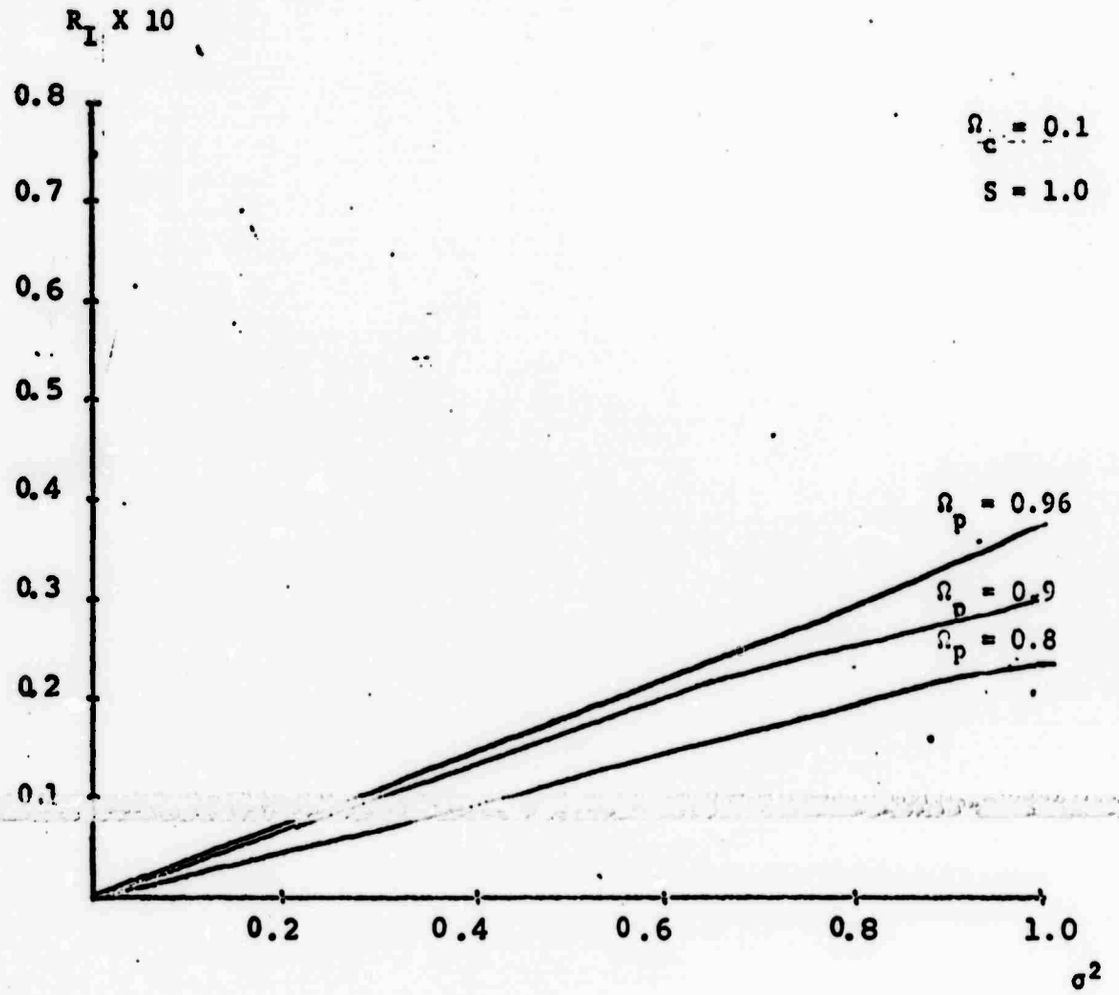


Fig. 8 The incoherent reflected power near the transition regime.

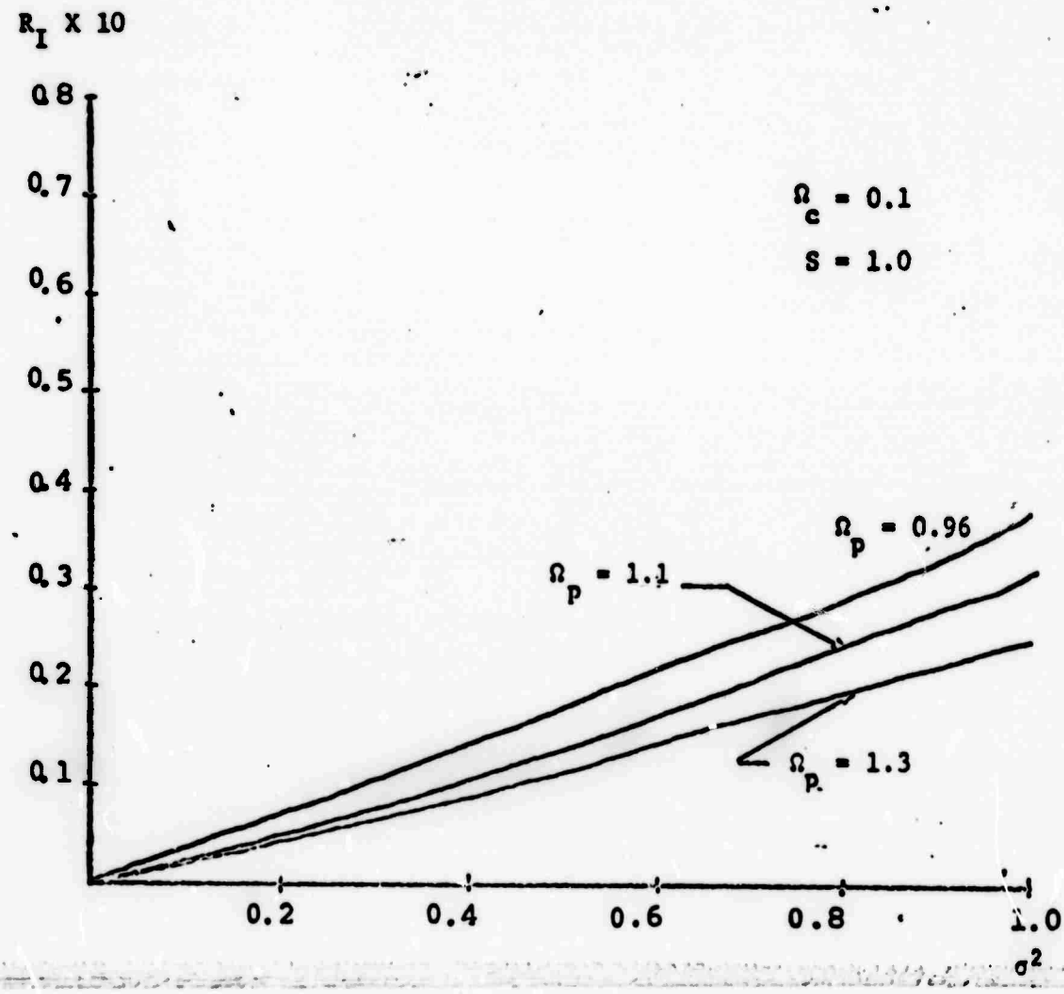


Fig. 9 The incoherent reflected power in the underdense-overdense regime.

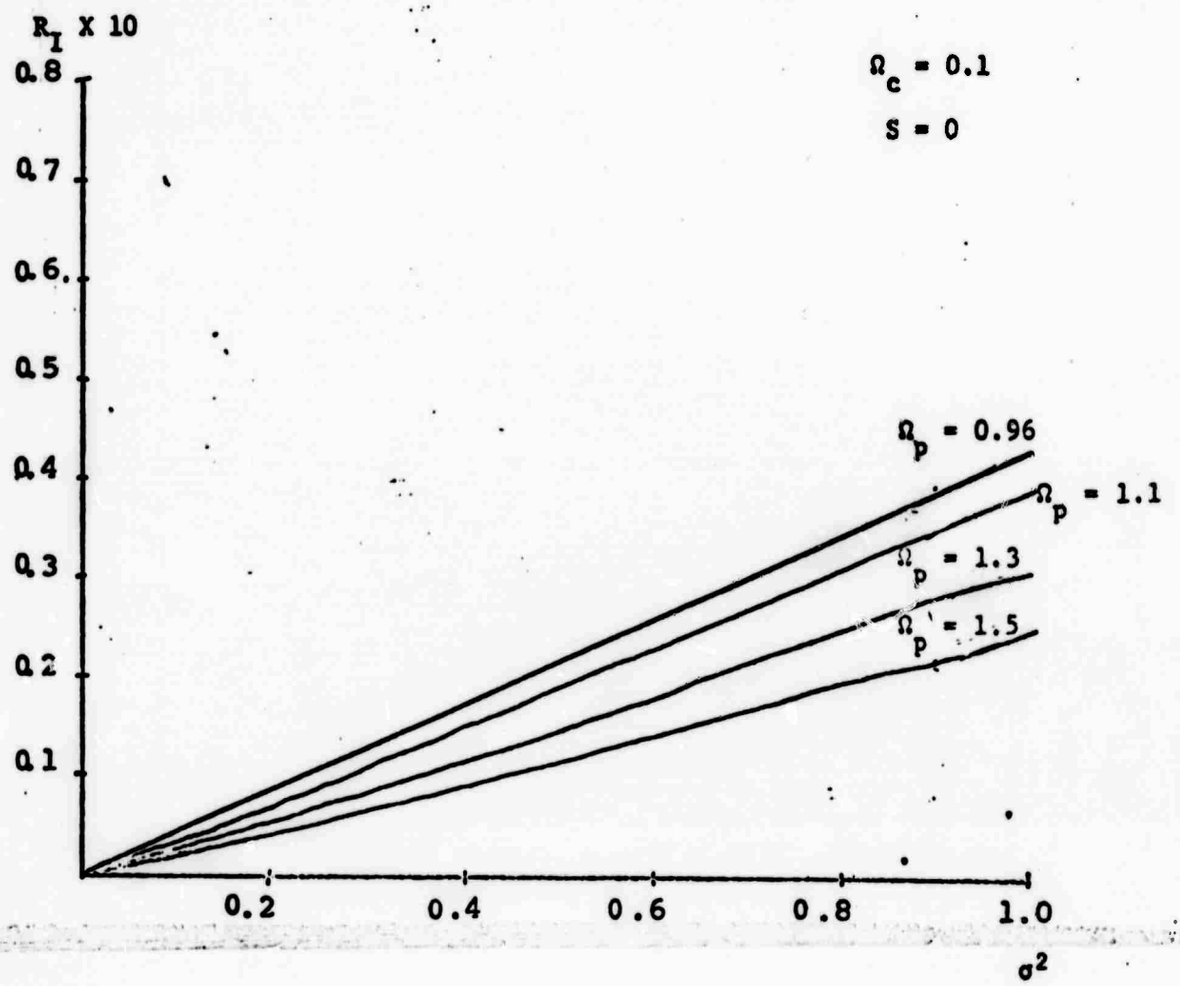


Fig. 10 The incoherent reflected power in underdense-overdense regime.

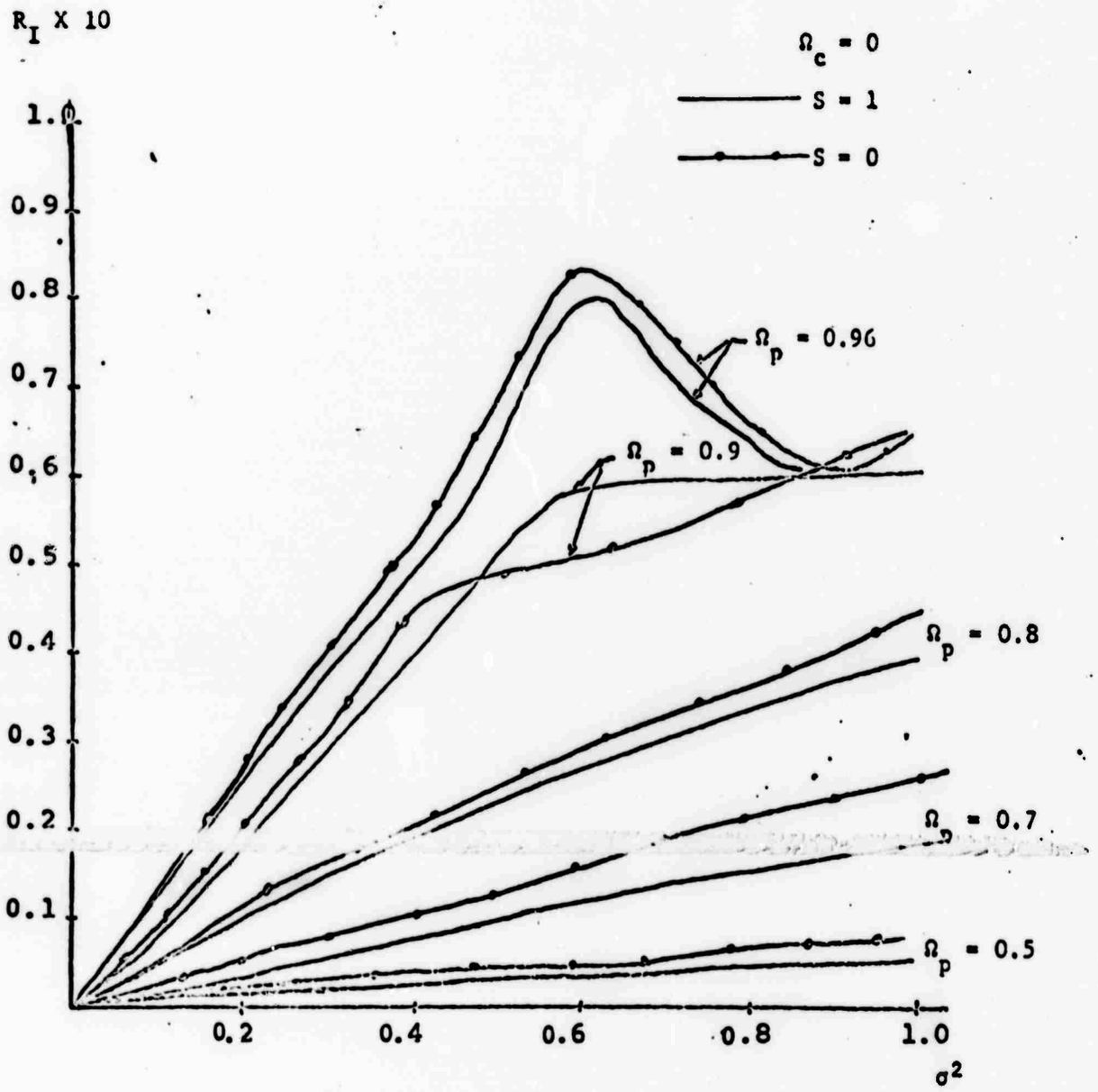


Fig. 11 Comparative effect of turbulent intensity on the incoherent power.

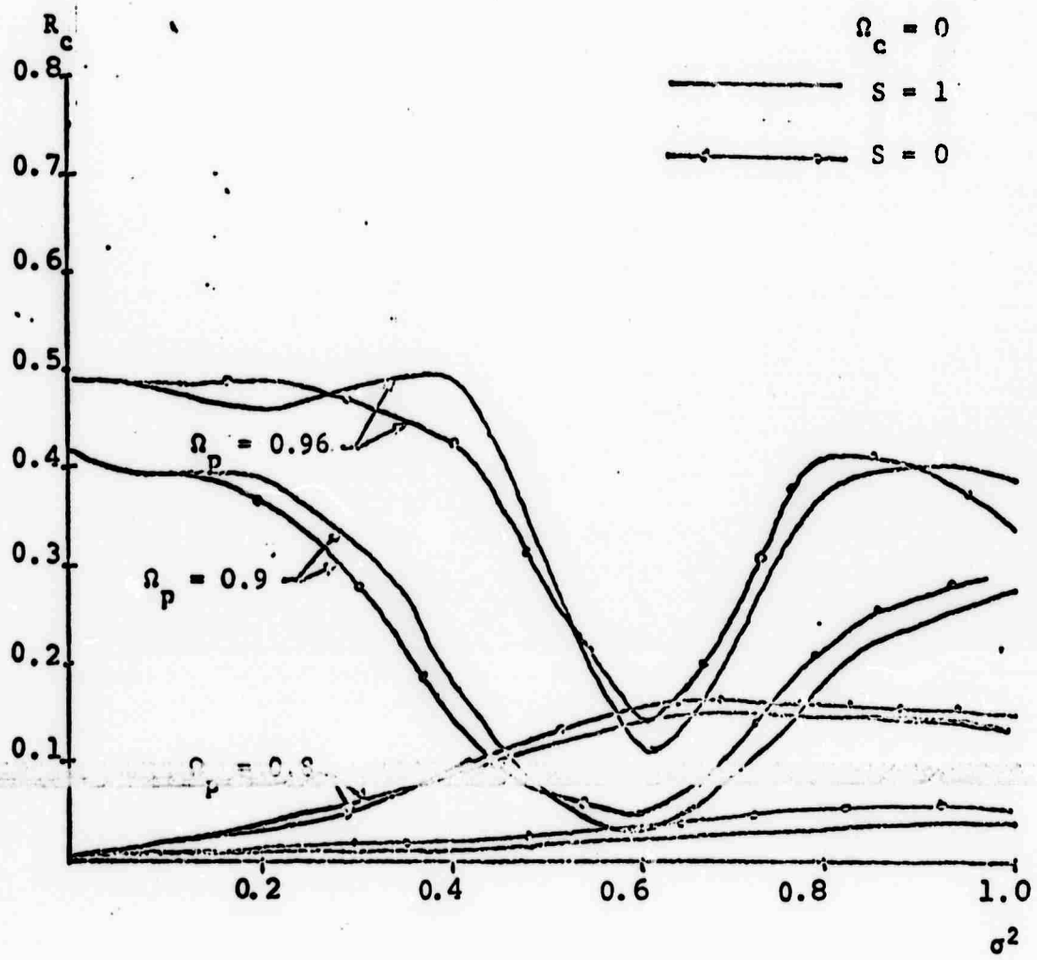


Fig. 12 Comparative effect of turbulent intensity on the coherent power.

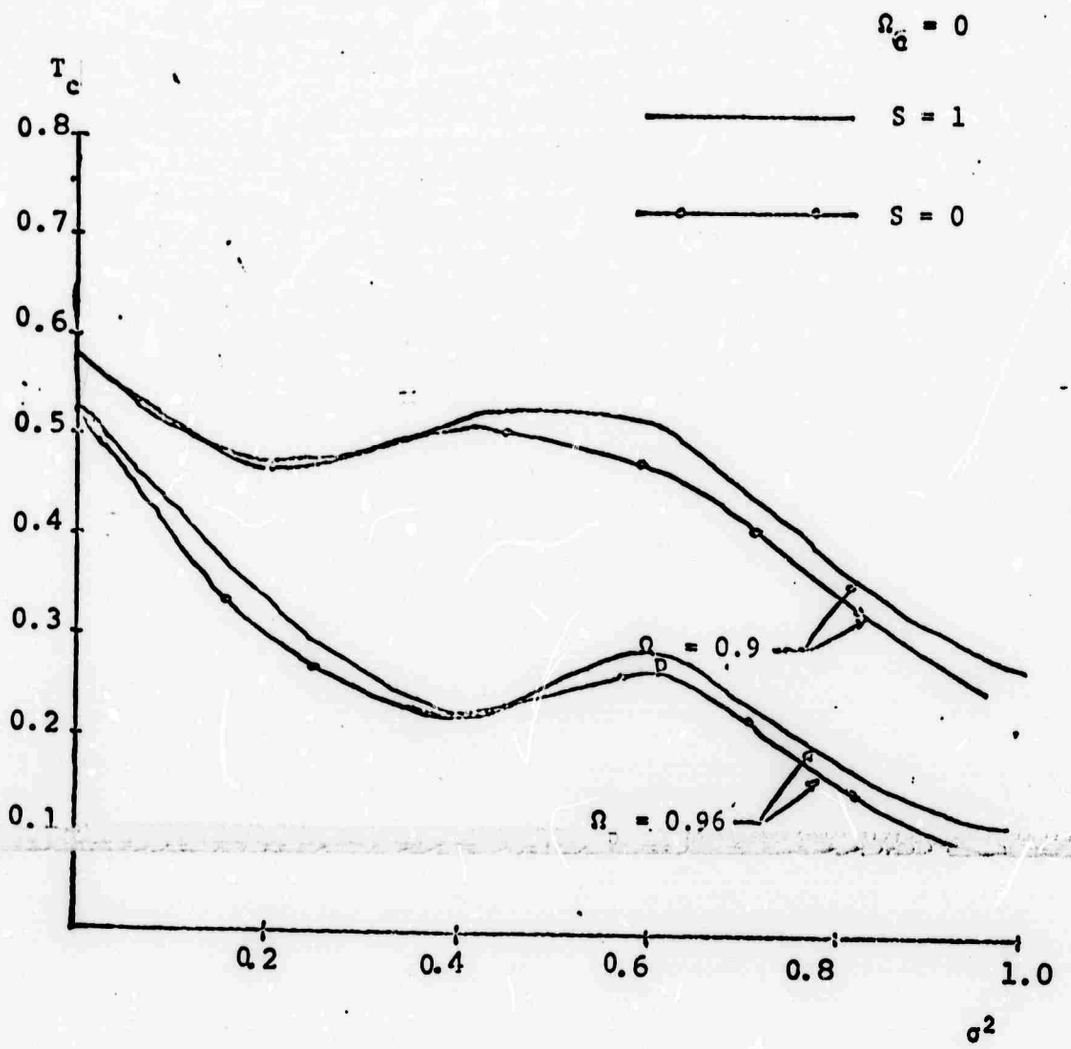


Fig. 13 Comparative effect of turbulent intensity on the incoherent power.

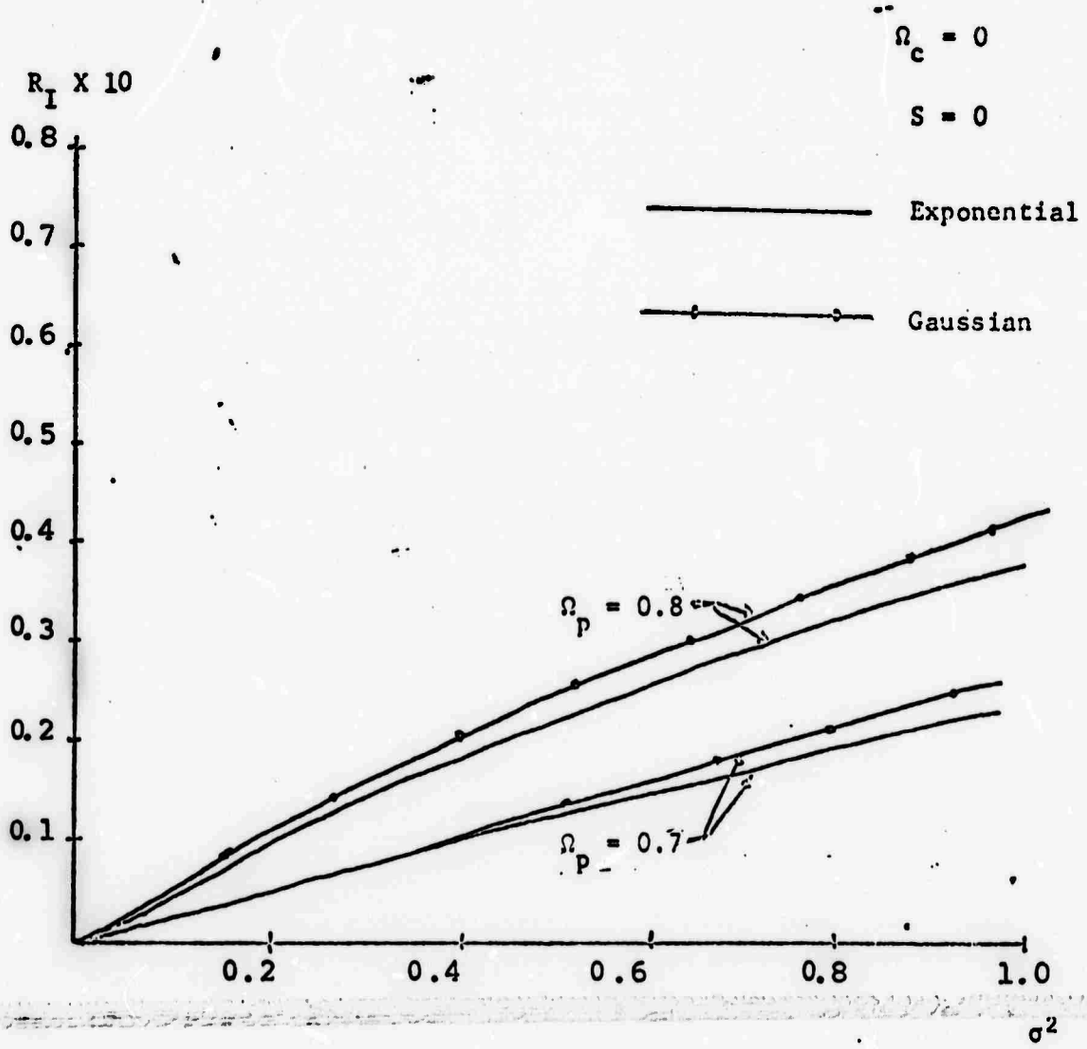


Fig. 14 Comparative effect of correlation function on the reflected power.

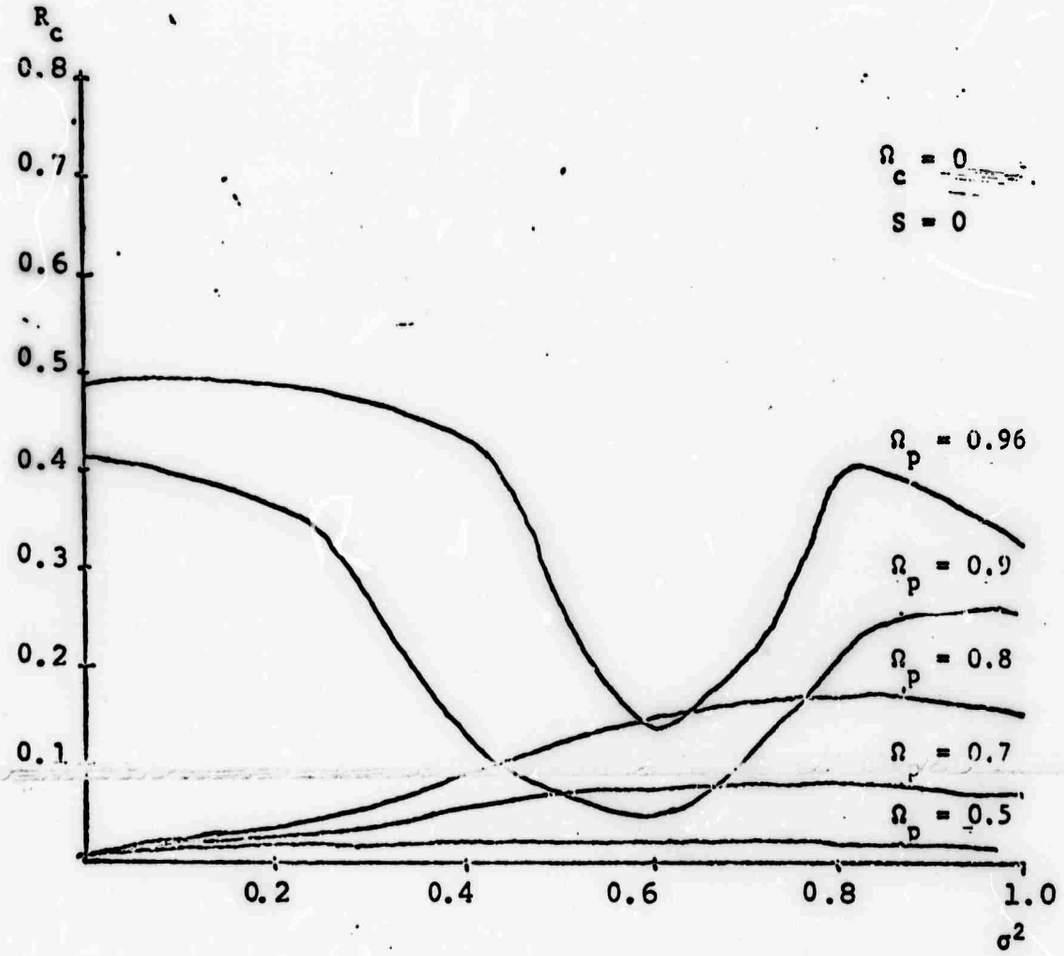


Fig. 15 Coherent reflected power as a function of the mean square fluctuations with normalized plasma frequency as a parameter.

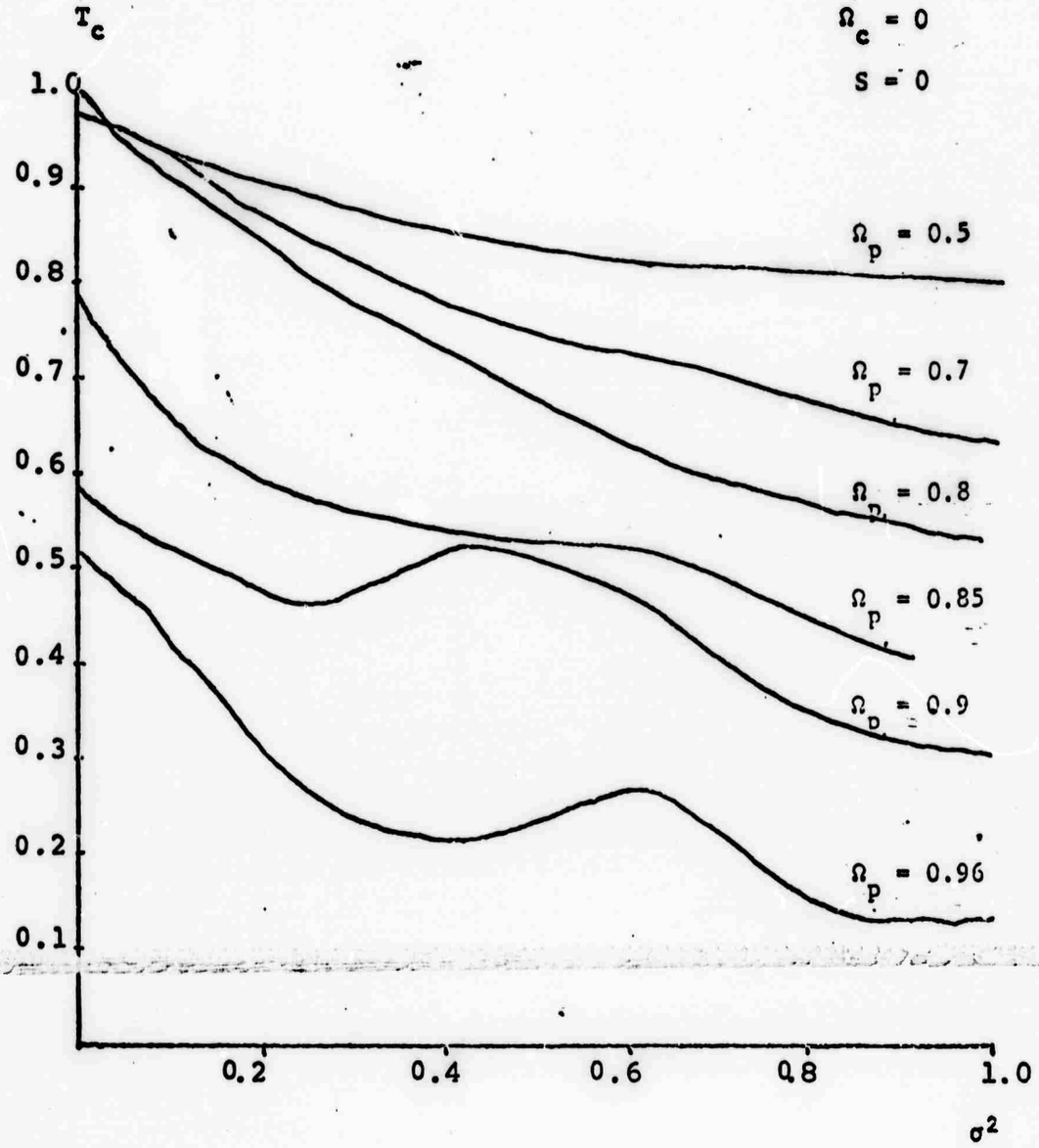


Fig. 16 Coherent transmitted power as a function of the mean square fluctuations with normalized plasma frequency as a parameter.

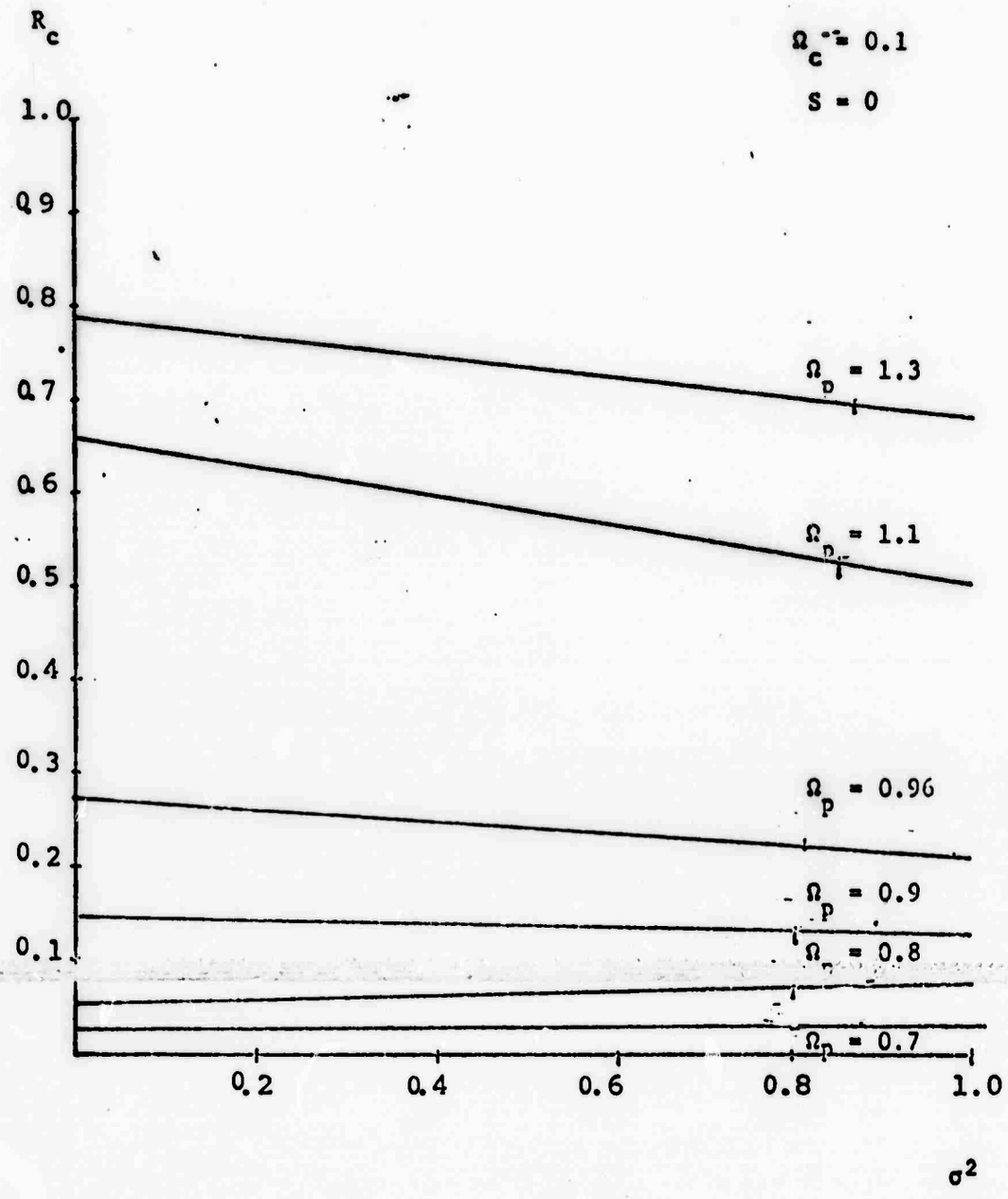


Fig. 17 Damping effect on the coherent reflected power.

V. TRANSFORM METHODS AND THE SMOOTHING EQUATION

A. Introduction

In the preceding chapter we examined the scattering problem in the limit of small scale fluctuations. In this chapter the problem will be formulated as a transform-smoothing equation in order to achieve a more general description of the fluctuational's scale. A singular integral equation of the standard form is obtained. Formal solutions of this equation are produced and discussed. Direct numerical simulation is also investigated.

B. Analysis

For a plane incident wave, the integral formulation of Equation (II,1) takes the form:

$$\phi(x) = e^{-jkx} + \epsilon \int G^{(0)}(x, x')\, \eta(x')\, \phi(x')\, dx' \qquad (1)$$

This equation may be transformed into wave number-space upon multiplying it by $\eta(x)\, e^{jKx}$ and integrating over the whole space:

$$\phi(K) = \eta(K-k) + \epsilon \int dp\, G(p)\, \eta(K-p)\, \phi(p) \qquad (2)$$

where

$$\epsilon = -\frac{k^2 \Omega_p^2 (1 + j\Omega_c)}{2\pi (1 + \Omega_c^2)}$$

(3)

$$G(p) = \frac{1}{p^2 - k^2 + j\theta}$$

$$= \frac{1}{2k}\left[\frac{1}{p - (k - j\theta)} - \frac{1}{p - (-k + j\theta)}\right]$$

Equation (2) may be put in operational form

$$\phi(\kappa) = \eta(\kappa - k) + \epsilon G_{op}(p)\, \eta(\kappa - p)\, \Phi(p) \qquad (4)$$

The Fourier transform of the correlation function (I,2) may be written as (23)

$$\langle \eta(p_1)\, \eta(p_2) \rangle = A(p_1 + p_2)\, \Phi\left(\frac{p_1 - p_2}{2}\right) \qquad (5)$$

where $A(p_1+p_2)$ and $\Phi(\frac{p_1-p_2}{2})$ are respectively the Fourier transform of the intensity and the small scale behavior of the plasma. The first order smoothing equation (4) for the coherent field gives then

$$\langle \phi(k) \rangle = f(k,k) + \epsilon\, G_{op}(p_1)\, f(k,p) \langle \phi(p) \rangle \tag{6}$$

where

$$f(k,p) = \epsilon\, A(k-p)\, g(k+\tfrac{p}{2})$$

$$g(k) = G_{op}(p)\, \Phi(k-p) \tag{7}$$

Equation (7) may be written explicitly as

$$\langle \phi(k) \rangle = f(k,k) + \frac{\epsilon}{2k} \int \frac{dp}{p-k+j\theta} \left[f(k,p) \langle \phi(p) \rangle + f(k,-p) \langle \phi(-p) \rangle \right] \tag{8}$$

Replacing k by $+k$ in equation (8), we obtain the forward scattering amplitude:

$$\langle \phi(k) \rangle = f(k,k) + \frac{\epsilon}{2k} \int \frac{dp}{p-k+j\theta} \left[f(k,p) \langle \phi(p) \rangle + f(k,-p) \langle \phi(p) \rangle \right] \tag{9}$$

The back scattering amplitude is

$$\langle \phi(-k) \rangle = f(-k,k) + \frac{\epsilon}{2k} \int \frac{dp}{p-k+j\theta} \left[f(-k,p) \langle \phi(p) \rangle + f(-k,p) \langle \phi(-p) \rangle \right] \tag{10}$$

The sum and difference of (9) and (10) are respectively

$$\psi_{1,2}(k) \equiv \langle \phi(k) \rangle \pm \langle \phi(-k) \rangle = F_{1,2}(k,k) + \frac{\epsilon}{2k} \int \frac{dp\, F_{1,2}(k,p)\, \psi_{1,2}(p)}{p-k+j\theta} \tag{11}$$

where

$$F_{1,2}(k,k) = f(k,k) \pm f(-k,k) \tag{12}$$

C. Singular Integral Equations

The application of Plemmelj formulae

$$\int_{-\infty}^{\infty} \frac{dp\, F(p)}{p-(k \mp j\theta)} = \mp \pi j F(k) + PV \int \frac{dp\, F(p)}{p-k} \tag{13}$$

to equation (11) leads to:

$$\Psi_{1,2}(k) = \frac{\epsilon^2 \pi j}{2k} f_{1,2}(k,k) - \frac{\epsilon^2 \pi j}{2k} f_{1,2}(k,k)\,\psi_{1,2}(k) + \frac{\epsilon^2}{2k} PV \int \frac{dp\, f_{1,2}(k,p)}{p-k} \Psi_{1,2}(p) \tag{14}$$

where

$$f_{1,2}(k,p) = [A(p-k)\, g\left(\frac{p+k}{2}\right) \pm A(p+k) g\left(\frac{p-k}{2}\right)] \tag{15}$$

Equation (14) may be set in normal form; for example, the kernel $f_1(k,p)$ may be expanded in a Taylor's series; then we have

$$\frac{f_1(k,p)}{p-k} = \frac{f_1(k,k)}{p-k} + K^*(k,p) \tag{16}$$

where

$$K(k,p) = f_1'(k,k) + \frac{1}{2}(p-k) f_1''(k,k) + \ldots \tag{17}$$

manipulation of Equation (14) and (17) leads to

$$a(k)\,\psi(k) + \frac{b(k)}{\pi i}\int \frac{\psi(p)\,dp}{p-k} = f(k,k) - \int k^{*}(p,k)\,\psi(p)\,dp \tag{18}$$

or symbolically

$$K^{\circ}\psi = f - K^{*}\psi \tag{19}$$

where

$$a(k) = \left(1 + \frac{\epsilon^{2}\pi j\, f(k,k)}{2k}\right)^{\frac{1}{\epsilon}}$$

$$b(k) = \epsilon\,\pi j\, f(k,k)/2k \tag{20}$$

$$K^{*}(p,k) = \epsilon\, K(k,p)/2k$$

The expression

$$K^{\circ}\psi = a(k)\,\psi(k) + \frac{b(k)}{\pi i}\int \frac{dp\;\psi(p)}{p-k} \tag{21}$$

is said to be the regular part of the equation.

D. Solution

A general method of solution of the singular integral

equation has been advanced by Carleman and developed by Vekua (38). The method (18) consists in the separation of the characteristic part from the equation and in its solution. This reduces the singular integral equation to a weakly singular one. We shall now consider the characteristic equation. Its solution is well known, (38). We have

$$\Psi(k) = a(k) f(k,k) - \frac{b(k)\, z(k)}{\pi j} \int \frac{dp\, f(p)}{(p-k)\, z(p)} + b(k)\, z(k)\, P_{n-1}(k) \tag{22}$$

where

$$z(k) = \frac{e^{\Gamma(k)}}{[t^{\eta}\, \pi(k)]^{1/2}}$$

$$\Gamma(k) = \frac{1}{2\pi j} \int \frac{dp\, \ln\left[p^{\eta} \pi(p) \frac{a(p) - b(p)}{a(p) + b(p)}\right]}{(p-k)}$$

$$\pi(k) = \prod_{k=1}^{n} (p - z_k)^{n_k} \tag{23}$$

$$\eta = \frac{1}{2\pi j} \int d \ln \frac{a+b}{a-b}$$

If the coefficients of Equation (18) do not satisfy the condition

$$a^2(k) - b^2(k) = 1 \tag{24}$$

we must divide both sides of (18) by

$$[a^2(k) - b^2(k)]^{1/2} \neq 1$$

Applying Equation (22) to (19), we obtain, after some manipulation,

$$\psi(k) + \int K^{**}(p,k)\, \psi(p)\, dp = f^*(k) \tag{25}$$

where the Fredholm kernel is defined by

$$K^{**}(p,k) = a(k)\, K^*(p,k) - \frac{b(k)\, z(k)}{\pi j} \int dp \frac{K^*(p,p)}{z(p)(p-k)} \tag{26}$$

and where the free term has the form

$$f^*(k) = a(k)\, f(k) - \frac{b(k)\, z(k)}{\pi j} \int \frac{dp\, f(p)}{z(p)(p-k)} + b(k) P_{n-1}(k) \tag{27}$$

The Carleman-Vekua method forms a basis of the theory of singular integral equations. In the case the kernel of Equation (10) is degenerate $K^*(k,p) = \sum_{i=1}^{n} \alpha_i(k)\,\beta_i(p)$ the solution of Equation (18) is given in closed form.

$$\psi(k) = K^{*-1} f - \sum K^{*-1} \alpha_i(k) \int \beta_i(p)\, \psi(p)\, dp \qquad (28)$$

This is a Fredholm equation with degenerate kernel. It may be reduced to an algebraic system of equations (34). In fact, a singular integral equation of general type may be reduced to an equation with degenerate kernel. By virtue of Weiertrass' theorem, $K^*(k,p)$ can be approximated by a degenerate kernel

$$K^*(k,p) = \sum \alpha_i(k)\,\beta_i(p) + \gamma(k,p) \qquad (29)$$

Where the norm $\|\gamma(k,p)\|$ in the space L_2 may be made as small as we please.

In the case where the random intensity varies slowly in real space with respect to the coherent field, a simplification of equation (6) results; the coherent field under the integral sign is considered to be sampled at the wave number K ; equation (6) reduces to

$$\langle \phi(k) \rangle = \frac{\epsilon A(k-k)\, g(k+k/2)}{1-\epsilon^2 G_{op}(p_1) A(k-p_1) g(k+p_1/2)} \tag{30}$$

The incoherent field is then

$$\delta\phi = \eta(k-k) + \epsilon G_{op}(p_1) \langle \phi(p_1) \rangle \eta(k-p_1) \tag{31}$$

We have concentrated so far on the analytical methods of solutions. Direct numerical simulation of Equation (11) should not be discarded; the apparent difficulty is due to the singularity in the kernel; it may be avoided through the introduction of an appropriate average value for the integral in the vicinity of p=k or the rewriting of the integral equation so that the singular contribution cancels out.

E. Conclusion

The present development appears to have some definite potentials. A survey of the methods for the approximate solution of the resulting equation is given. For slowly changing characteristics of the random intensity, the scattered field is easily deduced. In addition, the formulation is close to reality since it includes the effect of inhomogeneous turbulent intensity.

BIBLIOGRAPHY

VI. CONCLUSIONS AND RECOMMENDATIONS

A. Conclusions

This thesis treats the case of a random slab whose mean refractive index is different from the deterministic refractive index in the exterior. Previous studies neglected the effect of the boundaries in the computation of the mean wave; this procedure is shown to be energetically inconsistent. The present treatment is consistent and evaluates the coherent and incoherent scattered powers. The principal conclusions of the analysis--that the incoherent scattered powers experience saturation with a further increase in the mean square fluctuations, and that the coherent scattered power may be adequately predicted by a nonrandom approach for moderate collision frequency--have been illustrated graphically for some typical cases.

B. Recommendations

1. The present work has concentrated on the limit of small scale fluctuations; a measure of the sensitivity of the power scattered as a function of the correlation scale is desirable. A proper formulation has been set in Chapter V. of this thesis and can be used to investigate this point.

2. A comparable analysis to the present work is

needed for three dimensional fluctuations. One would still assume that the correlation function varies much more rapidly than the Green's function, and it may be viewed as an impulse function except in the case where divergent integrals are obtained if the unsmeared function is used.

3. The present method of solution is not restricted to a homogeneous background layer with sharp boundaries. It may be used in the case of layers-- Epstein, linear, parabolic profiles-- which have inhomogeneous backgrounds and diffuse boundaries. It would be interesting to study the transitional effect on the scattered power.

BIBLIOGRAPHY

1. Bassanini, P., et al., "Scattering of Waves by a Medium with Strong Fluctuations of Refractive Index," Rad. Sc., 2:1, 1967.

2. Bassanini, P., "Wave Propagation in a One-dimensional Random Medium," Rad. Sci., 2:4, 1967.

3. Blanc-Lapierre, A., Modeles Statistiques pour L'etude de Phènomenes de Fluctuations, Paris: Masson and Cie., 1963.

4. Blanc-Lapierre, A., and Fortet, R., Theorie des Fonctions Aleatoires, Paris: Masson and Cie., 1953.

5. Bourret, R. C., "Propagation of Randomly Perturbed Fields," Canad. J. Phys., 40: , 1962, pp. 782-90.

6. Bourret, R.C., "Stochastically Perturbed Fields with Applications to Wave Propagation in Random Media," Nuovo Cimento, 26: , 1962, pp. 1-31.

7. Brekhovskikh, L., Waves in Layered Media, New York: Academic Press, 1960.

8. Brown, W. P., "Propagation in Random Media--Cumulative Effect of Weak Inhomogeneities," IEEE A-P, 15:1, 1967, pp. 81-89.

9. Chernov, Lev A., Wave Propagation in a Random Medium, New York: McGraw Hill Book Co., Inc., 1960.

10. Collin, R. E., "Coherent Wave Reflection from a Random Half-space," IEEE Trans. A-P, 18:2, 1970, pp. 299-301.

11. Derusso, Paul M., et al., State Variables for Engineers, New York: John Wiley and Sons, Inc., 1966.

12. Dyson, F. J., "The Radiation Theories of Tomonaga, Schwinger, and Feynman, Phys. Rev., 75: , 1949, p. 486.

13. Finkelberg, V. M., "Mean Field Strength in an Inhomogeneous Medium," JETP, 19:2, 1963, p. 494.

14. Friedman, B., Principles and Techniques of Applied Mathematics, New York: John Wiley and Sons, Inc., 1956.

15. Frisch, U.,Wave Propagation in Random Media in Probabilistic Methods in Applied Mathematics, (A. J. Bharucha-Reid, Ed.), New York: Adademic Press, 1968, pp. 76-198.

16. Frisch U.,"Propagation d' ondes dans un Milieu Aleatoire Unidimensional," C. R., 261: , 1965, pp. 55-57.

17. Halperin, B. I., "Green's Function for a Particle in a One Dimensional Random Potential," Phys. Rev., 139: , 1965, pp. 104-117.

18. Hap, F. W., "Approximate Solution of Singular Integral Equations," U. S. S. R. Computational Math. and Phys., 5:2, 1965, pp. 171-84.

19. Hassab, J. C., Scattering from a Lossy Plasma Slab with Strong Sinusoidal Electron Density Variations, Phila., Pa.: Drexel U., M. S. Thesis, June,1968.

20. Hoffman, W. C., "Wave Propagation in a General Random Continuous Medium," Proceedings of Symposia in Applied Mathematics, 16: , 1967, p. 117.

21. James, M. L., Smith, G. M., and Wolford, J. C., Applied Numerical Methods for Digital Computation, Scranton, Pa.: International Textbook Co., 1967.

22. Jarem, John, et al., Topics in Electromagnetic Scattering from a Random Plasma Slab, Research Paper 441, Institute for Defense Analysis, 1969.

23. Jarem, J., "Studies in Electromagnetic Scattering from Turbulent Media," Drexel University Report No. 349, ARPA Order 1009, Contract No. DAH-C0-4-67C-0072.

24. Kampé de Fériet, J., "Random Integrals of Differential Equations," Lectures on Modern Mathematics, (T. L. Saati, Ed.), Vol. III, New York: John Wiley and Sons, Inc., 1965, pp. 277-321.

25. Karal, F. C., and Keller, J. B., "Elastic Electromagnetic and Other Waves in a Random Medium," J. of Math. Phys., 5:4, 1964.

26. Keller, J. B., "Stochastic Equations and Wave Propagation in Random Media," Proc. Symp. Appl. Math., Providence, Rhode Island: American Mathematics Society, 1964, pp. 145-70.

27. Keller, J. B., "Wave Propagation in Random Media," Proc. Symp. Appl. Math., Providence, Rhode Island: American Mathematics Soc., 1962, pp. 247.

28. Keller, J. B., and Karal, F. C., "Effective Dielectric Constant, Permiability and Conductivity of a Random Medium and the Velocity and Attenuation of Coherent Waves," J. Math. Phys., 7: , 1966, pp. 661-70.

29. Krylov, N., and Bogoliulov, N. N., Introduction to Nonlinear Mechanics, Princeton, New Jersey: Princeton University Press, 1948.

30. Kupiec, I., et al., "Reflection and Transmission by a Random Medium," Rad. Sci., 2:1, 1967, pp. 1-18.

31. Lin, S. C., "Radio Echoes from a Manned Satellite during Re-entry," J. Geophys. Res., 67:10, 19, pp. 3851-70.

32. Macrakis, M. S., "Scattering from Large Fluctuations," J. Geophys. Res., 70:19, 1965, pp. 4987-89.

33. Mattuck, Richard D., A Guide to Feynman Diagrams in the Many-body Problem, New York: McGraw-Hill Co., 1967.

34. Mikhlin, S., Linear Integral Equations, New York: Macmillan Co., 1964.

35. Mittleman, M., "Application of the Distorted Wave Born Approximation to the Calculation of Radar Scattering from a Random Plane Parallel Plasma Slab," Research Paper 441, Institute for Defense Analysis, 1969.

36. Mintzer, David, "Wave Propagation in a Randomly Inhomogeneous Medium", J. Acoust. Soc. Am., 25:5, 1953.

37. Morse, P. M., and Feshback, H., Methods of Theoretical Physics, Vol. 2, New York: McGraw Hill Book Co., 1953.

38. Muskelishvili, N. I., Singular Integral Equations, Groningen, Holland: P. Noordhoff Ltd., 19 .

39. Newton, R. G., Scattering Theory of Waves and Particles, New York: McGraw Hill Book Co., 19 .

40. Richard, C., et al., Experimental Check of the Applicability of Single Born Scattering Theory up to Critical Electron Density Fluctuations, Research Report 96716-19, Advanced Research Projects Agency, 1970.

41. Rosenbaum, S., "On the Coherent Wave Motion in Bounded Randomly Fluctuating Regions," Rad. Sci., 4: , 1969, pp. 709-19.

42. Ryzhov, A., et al., "Spatial Dispersion of Inhomogeneous Media," Soviet Physics JETP, 21:2, 1965, pp.433-38.

43. Salpeter, E. E., and Bethe, H. A. "A Relative Equation for Bound-state Problems," Phys. Rev., 84: , 1951, p. 1232.

44. Smirnov, V. I., A Course of Higher Mathematics, Vol. I-IV, New York: Pergamon Press, 1964.

45. Tatarskii, V. I., "Propagation of Electromagnetic Waves in a Medium with Strong Dielectric Constant Fluctuations," Soviet Phys. JETP, 19:4, 1964, pp. 946-53.

46. Tatarskii, V. I., Wave Propagation in Turbulent Medium, New York: McGraw Hill Book Co., 1961.

47. Tatarskii, V. I., and Gertenshtein, M. E., "Propagation of Waves in a Medium with Strong Fluctuation of the Refractive Index," Soviet Phys. JETP, 17:2, 1966, pp. 458-63.

48. Villar, F., and Weisskopf, V. F., "The Scattering of Electromagnetic Waves in Turbulent Atmospheric Fluctuations," Phys. Rev., 94: , 1954, pp. 232-40.

49. Wu, and Ohmura, Quantum Theory of Scattering, New York: Prentice Hall Publishing Co., 19 .

50. Yen, K. T., Effect of Turbulence Intermittency on the Scattering of Electromagnetic Waves by Underdense Plasmas, Research Report R64SD72, General Electric Space Science Laboratory, 1964.

APPENDICES

APPENDIX A

ON THE VALIDITY OF THE BORN APPROXIMATION IN RANDOM WAVE SCATTERING

A. Introduction

The scattering of a plane wave incident upon an appropriate random medium has been widely treated in the first order Born approximation(40). Evidently, there are two points of important consideration: 1. For a given scattering problem, does the Born-Neumann series converge? 2. If this series converges, how large and error is incurred by truncating it after the first term? In this chapter, a simple quantitative criterion is derived that insures convergence and gives an upper bound for the truncation error. This bound is expressed in terms of relevant parameters which characterize the convergence condition, the scattering volume, and the structure of the medium. This result is established using the method of successive approximation, and the inequalities appropriate to the condition of quadratic summability of the kernel in the basic mean square.

B. Statement of the Problem and Solution

A scattering problem may be represented by the following integral equation (37).

$$\phi(\bar{r}) = \phi_0(\bar{r}) + \epsilon \int G(\bar{r}, \bar{r}')\, \eta(\bar{r}')\, \phi(\bar{r}')\, d\bar{r}' \qquad (1)$$

A straight forward approach to solving the integral equation is successive approximations. We begin with the zero order approximation

$$\phi(\bar{r}) \sim \phi_0(\bar{r}) \qquad (2)$$

Let us substitute the zero order approximation into the original equation under the integral sign to obtain a first order approximation, and the process is then repeated. The resulting series for the scattered field is

$$\phi_s(\bar{r}) = \sum_{m=1}^{\infty} \phi_{m_s}(\bar{r}) \qquad (3)$$

where

$$\phi_{m_s}(\bar{r}) = \epsilon^m \int \tilde{K}_m(\bar{r}, \bar{r}')\, \phi_0(\bar{r}')\, d\bar{r}'$$

$$\tilde{K}_m(\bar{r}, \bar{r}') = \int \tilde{K}_2(\bar{r}, \bar{r}_1')\, \tilde{K}_{m-1}(\bar{r}_1', \bar{r}')\, d\bar{r}_1' \qquad (4)$$

$$\tilde{K}_1(\bar{r},\bar{r}') = \tilde{K}(\bar{r},\bar{r}') = G(\bar{r},\bar{r}')\,\gamma(\bar{r}')$$

and $\tilde{K}_m(\bar{r},\bar{r}')$ is called the n^{th} iterated kernel in relation to the given kernel. The first term in the series solution may be regarded as contribution due to the incident wave; the second term, the contribution due to a single scattering of the wave from each inhomogeneity in the scatterer; the third term, the contribution caused by the double scattering of the wave, etc. The formal integration of each term in the series, though random, should amount to the integration of functions when a particular sample function of the ensemble is considered(24). To each sample function corresponds a solution $\phi(\bar{r},\mu)$ and a condition for each validity. The value of the solution is generally different for each sample function, and the solution thus takes a whole set of values. The ensemble averages of these values are of interest in the applications. The method consists of the following steps: The explicit solution(3)of (1) is first determined formally for each member of the ensemble. Then, the mean and the correlation function of $\phi_s(\bar{r},\mu)$ are obtained. As the integration becomes exceedingly complicated, calculations have been limited to the first term. A bound on the truncation error of the series and the condition of convergence of that series

are investigated.

C. Convergence and Error Bounds

The general term in the series solution for the mean square value of the scattered field may be written as

$$\langle |\phi_m(r)|^2 \rangle = \left\langle \left| \int d\bar{r}_m \, K_m(\bar{r}, \bar{r}_m) \, \phi_0(\bar{r}_m) \, \eta(\bar{r}_1) \cdots \eta(\bar{r}_m) \right|^2 \right\rangle \tag{7}$$

The Bunyakovski inequality states that

$$|(\phi_1, \phi_2)| \leq \|\phi_1\| \cdot \|\phi_2\| \tag{8}$$

Using the above relation, we get for the second order moment of a probability density $p(u, v)$

$$\left| \iint uv \, p(u, v) \, du \, dv \right|^2 \leq \langle u^2 \rangle \langle v^2 \rangle \tag{9}$$

We assume that the random variable η follows a Gaussian process with mean zero and rms. Applying the relations in (8) and (9) to the mean square value of (7), we obtain:

$$\langle |\phi_m|^2 \rangle \leq \binom{2m}{2} \sigma^{2m} \int |K_m(\bar{r}, \bar{r}')|^2 \, d\bar{r}' \int |\phi_0(\bar{r}')|^2 \, d\bar{r}' \tag{10}$$

Let

$$C_m = \operatorname{Sup} \int_V |K_m(\bar{r}, \bar{r}')|^2 \, d\bar{r}' \tag{11}$$

and

$$K_m(\bar{r}, \bar{r}') = \int K_{m-1}(\bar{r}, \bar{r}_1) \, K(\bar{r}_1, \bar{r}') \, d\bar{r}_1 \tag{12}$$

Then

$$C_m = \operatorname{Sup} \int |K_m(\bar{r}, \bar{r}')|^2 \, d\bar{r}'$$

$$\leq \int \left[\int |K_{m-1}(\bar{r}, \bar{r}_1)|^2 \, d\bar{r}_1 \int |K(\bar{r}_1, \bar{r}')|^2 d\bar{r}_1 \right] d\bar{r}'$$

$$\leq C_{m-1} B^2 \tag{13}$$

where

$$B^2 = \iint |K(\bar{r}_1, \bar{r}')|^2 \, d\bar{r}_1 \, d\bar{r}' \tag{14}$$

Using the recurrence relation in (13), there results

$$C_m \leq C_1 B^{2(m-1)} \tag{15}$$

Since the scatterer has a finite volume, V, a relation exists between B and C_1 where

$$B^2 \leq C_1 V \tag{16}$$

and (15) becomes

$$C_m \leq C_1^m V^{m-1} \tag{17}$$

Replacing (17) in (10), there results

$$\langle |\Phi_{ms}|^2 \rangle \leq \binom{2m}{2} |A_0|^2 D^m \tag{18}$$

where

$$D = C_1 \sigma^2 V \tag{19}$$

The series is uniquely convergent for D less than one.

Now we establish an error bound on the truncation of the series after the first term, thus estimating the importance of multiple scattering. In view of the elementary inequality

$$|\phi_1 + \phi_2|^2 \leq 2\left(|\phi_1|^2 + |\phi_2|^2\right) \tag{20}$$

the scattered power is:

$$\langle |\phi_2(r)|^2 \rangle \leq \sum 2|A_0|^2 \binom{2m}{2} D^m \tag{21}$$

A comparison of the first Born approximation with (20) gives a measure of the relevance of multiple scattering. Admittedly, this measure is somewhat "loose"; on the other hand, a very stringent bound would not satisfy many practical cases.

APPENDIX B

AD HOC DERIVATIONS OF FIRST ORDER SMOOTHING EQUATION

A. Formulation

For wave propagation in random media, the first order smoothing equation has been derived in different ways (5) (16), (26). The scattering equation in operational form is

$$(L_0 + L_1)\,\phi = 0 \tag{1}$$

The mean is

$$L_0 \langle \phi \rangle + \langle L_1 \delta\phi \rangle = 0 \tag{2}$$

The difference between 1 and 2

$$L_0\,\delta\phi = L_1 \langle \phi \rangle + \widetilde{\delta\phi} \tag{3}$$

where

$$\widetilde{\delta\phi} = L_1\,\delta\phi - \langle L_1\,\delta\phi \rangle$$

Equations 2 and 3 form a set of coupled equations that describe the problem.

B. Local Independence

With the hypothesis of local independence, the wave, ϕ, and the perturbation, L_1, are considered statistically independent, i.e.

$$\langle L_1 L_0^{-1} L_1 \phi \rangle = \langle L_1 L_0^{-1} L_1 \rangle \langle \phi \rangle \qquad (4)$$

Let us multiply (1) by L_1 and average. The resulting equation may be further simplified by applying the local independence hypothesis; there results

$$L_0 \langle \phi \rangle - \langle L_1 L_0^{-1} L_1 \rangle \langle \phi \rangle = 0 \qquad (5)$$

The above technique leads to the same formulation as the first smoothing equation. A physical measure to the assumption of local independence may be given. It is known that sharp changes in the refractive index makes a wave dependent upon the medium; then, a necessary condition for the hypothesis to hold requires that the dependence or correlation between the medium's index of refraction and the wave be zero. In this case, there are reasons to believe that the functions are statistically

independent, or at least the stochastic dependence between them is so weak as to be negligible.

C. "Distorted" Wave Born Approximation

In the so called "distorted" wave Born approximation, the first smoothing equations are derived by neglecting the $\tilde{\delta\phi}$ term in (2), with no apparent justification. Manipulation of (2) and (3) result in

$$L_0 \langle \phi \rangle - \langle L_1 L_0^{-1} L_1 \rangle \langle \phi \rangle = 0 \tag{6}$$

$$\delta\phi = - L_0^{-1} L_1 \langle \phi \rangle \tag{7}$$

which are exactly the same as the first smoothing equation. For higher orders, however, the results of both methods generally differ. This is expected since the neglected term $\tilde{\delta\phi}$ generates terms comparable in size to the remaining ones in (2).

D. Modified Neumann Method

The first smoothing equations are here derived directly from the Neumann series. Let the field ϕ be replaced in (1) by the incident field ϕ_0 and scattered field ϕ_s:

$$L_0 \phi_s = - L_1 (\phi_0 + \phi_s) \tag{8}$$

Solving for ϕ_s , we get

$$\phi_s = M L_0^{-1} L_1 \phi_0 \tag{9}$$

Where

$$M = (1 + L_0^{-1} L_1)^{-1} = 1 + L_0^{-1} L_1 + (L_0^{-1} L_1)^2 + \ldots \tag{10}$$

To first order approximation we get for the mean field

$$L_0 \langle \phi_s \rangle - \langle L_1 L_0^{-1} L_1 \rangle \phi_0 = 0 \tag{11}$$

ϕ_0 , in front of the operator, is the first order iteration of ϕ_s .

$$\phi_0 \sim \phi_s$$

$$L_0 \langle \phi_s \rangle - \langle L_1 L_0^{-1} L_1 \rangle \langle \phi_s \rangle = 0 \tag{12}$$

The fluctuation field is easily solved for, as :

$$\delta \phi_s = - L_0^{-1} L_1 \langle \phi_s \rangle \tag{13}$$

Report 5

Scattering Coefficients of One-dimensional Plasmas of Epstein-type Profiles with Small Random Irregularities

TABLE OF CONTENTS

NOMENCLATURE

A	constant of (72); also integration parameter ($A = 1+\alpha+\beta-\gamma-\delta$)
$\bar{A}$	general vector
a	correlation length; also general parameter
$\bar{a}$	general parameter
a_1, a_2	constants of (73)
B	constant of (72); also integration parameter of (B5)
B_1-B_4	constants defined by (54)
$\bar{B}$	magnetic induction vector ($\bar{B} = \mu_0 \bar{H}$)
b	reciprocal correlation length ($b = 1/a$); also general parameter
$\bar{b}$	general parameter
b_1, b_2	constants of (73)
C_1, C_2	constants defined by (52)
c	general parameter
$\bar{c}$	general parameter
$\bar{D}$	electric displacement vector ($\bar{D} = \epsilon \bar{E}$)
d	general parameter
$\bar{d}$	general parameter
$\bar{E}$	electric field intensity vector
E	$\lvert \bar{E} \rvert$

e	electron charge ($e = 1.602 \times 10^{-19}$ coulombs); also general parameter
$\bar{e}$	general parameter
F	hypergeometric function defined by (50)
F_1, F_2	independent solutions of hypergeometric equation (35)
f	source function; also general parameter
$\bar{f}$	general parameter
G	Green's function; also Green's function operator defined by (5)
G_1, G_2	functions defined by (B27) and (B28); also symmetric components of Green's function
g	profile distribution function defined by (57)
H_1, H_2	functions defined by (B29) and (B30)
$\bar{H}$	magnetic field intensity vector
I	integral
$\underline{I}$	integral given by (B9)
I_g	integrand of (A1)
I_1-I_6	generic integral terms of (B3)
I^0_ξ	general *Type I* integral defined in *Appendix B.1*
i	$(-1)^{\frac{1}{2}}$; also summation index
J	$\lvert J \rvert$; also integral given by (B17)
$\bar{J}$	vector current density
$J^{\xi'}_{-\infty}$	general *Type II* integral defined in *Appendix B.2*

j	summation index
K	constant of (B8) and (B17)
K_1-K_3	parameters defined by (43)
k	wave number defined by (24); also summation index
k_0	free space wave number
k_1	wave number at ∞
l	summation index
M	plasma parameter defined by (46)
m	electronic mass ($m = 9.11 \times 10^{-31}$ kg); also layer parameter defined by (44); also summation index
M	plasma parameter defined by (46)
n	electron density; also summation index
n_0	mean component of electron density
n_1	random component of electron density
P	independent variable transformation function defined by (36) and (44)
Q	function defined by (B31)
R	power reflection coefficient; also $\|\pm\|$
$R_0, R_1, R_2, \cdots$	successive contributions to field reflection coefficient
R_v	field reflection coefficient
$\langle R' \rangle$	reflection coefficient due to turbulence defined by (94)
r	dependent variable transformation function defined by (36) and (40)

r_0	integration constant of (40)
S	layer thickness parameter defined by (47)
s	integration parameter of (B21)
T	power transmission coefficient
t	time; also dummy variable
t'	dummy variable
U_1-U_4	parameters of (B51)
u	integration parameter of (B11)
V_I	incoherent transmission coefficient
W_C	coherent reflection coefficient defined by (91)
W_I	incoherent reflection coefficient defined by (92)
W	Wronskian determinant defined by (78)
X	$+(1-n)^{\frac{1}{2}}$
x	rectangular coordinate; also dummy variable
x'	dummy variable
Z	dependent variable of scalar wave equation (41)
Z_{01-}, Z_{02-}	Epstein profile wave equation solutions in the negative half-space
Z_{01+}, Z_{02+}	Epstein profile wave equation solutions in the positive half-space
Z_1, Z_2	Epstein profile wave equation solutions
z	rectangular coordinate in propagation direction; also dummy variable
z'	Dummy variable

GREEK

α	plasma distribution parameter defined by (48)
α'	Re $[1-\mathcal{N}]$
β	plasma distribution parameter defined by (48)
β'	Im $[1-\mathcal{N}]$
γ	plasma distribution parameter defined by (48)
$\Gamma(z)$	gamma function of argument z
$\delta(z-z')$	impulse function acting at $z = z'$
δ	distribution variable ($\delta = 1$: transition; $\delta = 2$: symmetric)
$\delta\kappa$	related to random part of wave number
ε	effective permittivity (dielectric constant), also measure of turbulent strength
ε_0	permittivity of free space ($\varepsilon_0 = 8.85 \times 10^{-12}$ farad/meter)
ζ	turbulent intensity defined by (60)
η	normalized electron density fluctuation defined by (61)
θ	arg(t)
κ	related to sure part of wave number
κ_0	related to constant part of wave number
κ_1	related to sure spatial varying part of wave number
μ	integration parameter of (B11)

μ_0	permeability of free space ($\mu_0 = 4\pi \times 10^{-7}$ henry/meter)
ν	effective electron-neutral collision frequency; also integration parameter of (B11)
ξ	independent variable defined by (36) and (44); also dummy variable
ξ'	dummy variable
ρ	charge density
σ	conductivity; also $-k_0^2 \delta\kappa$
ϕ	scalar field component; also arg(z)
ϕ_i	solution to homogeneous wave equation
ϕ_0	coherent field component
ϕ_1	fluctuating field component
ϕ_n	n^{th} term of Neumann series
Ω_c	normalized collision frequency defined by (34)
Ω_p	normalized plasma frequency defined by (34)
Ω_{p0}	maximum value of plasma frequency distribution
ω	radian electromagnetic frequency
ω_p	plasma frequency defined by (33)

OPERATORS

∇	gradient
$\times$	cross product
$\cdot$	dot product

$\langle\ \rangle$	ensemble average
$*$	conjugate
I	identity
P	averaging
$\|\ \|$	magnitude
arg	argument
$\dot{x}$	time derivative of x
Z'	total derivative of Z
Re	real part
Im	immaginary part
+	refers to positive half-space
-	refers to negative half-space
$\{\leftrightarrow\}$	interchange operator defined in *Appendix B*

OTHERS

B1-B12	parameters defined by (B32)
1-8	parameters defined by (B32)

1. INTRODUCTION

In contemporary communication processes (e.g., ionospheric and tropospheric scatter links; stellar observations; re-entry wake analysis), electromagnetic scattering plays an important role and has received interest since the very earliest formulation of electromagnetic radiation theory. The influence of matter upon propagating electromagnetic signals can be classified into separate topics, such as, scattering, absorption, depolarization, dispersion [e.g., *Jones*, 1964]. In this work, we investigate the scattering of electromagnetic waves by a non-uniform plasma with random irregularities. Consideration will be restricted to monochromatic waves, which will be allowed to interact with a plasma medium characterized by the Lorentz model [*Heald and Wharton*, 1965]. Such a model has been shown to be valid for radio and radar propagation in plasmas associated with rocket exhausts, re-entry wakes, and the ionosphere, in the absence of a magnetic field [*Jarem*, 1969]. The Lorentz plasma model leads to an effective dielectric constant for the plasma which will be taken to be a random continuum, i.e., the particle character of the media

(electrons, ions, neutral molecules) will be ignored and the random deviation of the dielectric constant from the mean (due to turbulence) will be continuous, varying smoothly in time and linear extent. Complete information about the media will never be available and thus a complete description of the interaction with the electromagnetic waves must be made in terms of a statistical interpretation [e.g., *deWolf*, 1968]. Such media are mathematical models of real-life situations, namely, turbulent atmospheric layers, rocket exhausts, and re-entry wakes. A good deal of interest in these media comes from the irregular disturbances of radio and radar communication signals when propagating through such physical media. Such disturbed signals may also be used as remote sensors for basic information on turbulent flows, and as a diagnostic tool for re-entry wake analysis.

Solutions to the problem defined above are already known in certain limiting cases (e.g., low mean electron density, and small turbulent fluctuation levels; particularly simple geometries). It is the aim of this paper to study the formulated problem for the one-dimensional Epstein profiles [*Epstein*, 1930] for an arbitrary peak level of electron density. Electron density fluctuation levels will not be arbitrary but will be restricted by the perturbation method here employed. Where applicable, the

results will be compared to the statistical computer experiments of *Hochstim and Martens* [1969; 1970], who have examined the case of a randomly fluctuating plasma slab.

2. BACKGROUND

2.1 *BORN APPROXIMATION*

Scattering from turbulent media has drawn increasing interest since the publication of work on tropospheric scattering by *Booker and Gordon* [1950]. Booker and Gordon made use of the perturbational Born approximation [e.g., *Dicke and Wittke*, 1960], and since that time, a number of papers have been written to extend the accuracy of that approximation, many of them attempting to treat the full three-dimensional electromagnetic problem [e.g., *Tatarskii*, 1961; *Salpeter and Treiman*, 1964; *Menkes*, 1964; *Ruffine and deWolf*, 1965]. Because of the perturbational nature of the Born method, solution above the lowest order yields only slight additional information with a great increase in complexity.*

---

*An exception to this statement might be made for the second Born approximation, which yields cross polarized information predicted to be zero by the first Born approximation [*Ruffine and deWolf*, 1965; *Jarem*, 1969b]. This information still comes with great complexity.

To apply the Born approximation we start with a wave equation for the propagation in a turbulent medium (This equation will be derived for a special geometry in Section 5.2.).

$$\{d^2/dz^2 + k_0^2 [\kappa_0 + \kappa_1(z) + \delta\kappa(z)]\}\phi(z) = 0 \qquad (1)$$

where the wave number has been separated into a constant part κ_0, an inhomogeneous sure part κ_1 and a random inhomogeneous part $\delta\kappa$. We then write (1) with the homogeneous term on the left

$$\{d^2/dz^2 + k_0^2\kappa_0\}\phi = -k_0^2[\kappa_1 + \delta\kappa]\phi \qquad (2)$$

Both inhomogeneous terms are thus considered as source terms. A *free space* Green's function [*Friedman*, 1956] can thus be utilized in the integral equation corresponding to (2)

$$\phi(z) = \phi_i(z) + \int_{-\infty}^{\infty} f(z')G(z,z')dz', \qquad (3)$$

where $f(z')$ represents the right side of (2) and $\phi_i(z)$, the solution of (2) with $f = 0$. An approximate solution to the integral equation (3) can then be obtained by iteration of (3) one or more times.* The accuracy of such a solution depends on the size of the entire perturbation $\kappa_1+\delta\kappa$.

Thus, even for *small* random deviations, one is restricted by the variation of κ_1 from zero. For a plasma medium, this restriction limits solution to the underdense regime (See Section 4.). Even when κ_1 is constant, one is restricted to a finite size plasma†, because the formulation of the Born approximation does not allow for attenuation of the incident wave§.

2.2 *THE DISTORTED WAVE BORN APPROXIMATION*

Various formulations categorized as distorted wave Born approximations have been devised to circumvent some limitations of the Born approximation [*Kreea*, 1968]. The easiest to apply involves a *local* solution of the wave

---

*The method becomes very unwieldy for more than two iterations.

†A propagation problem is thus excluded.

§Thus there are an infinite number of scattering events of equal importance, with a corresponding blow-up in the scattering coefficients.

equation written as follows

$$[d^2/dz^2 + k_0^2[\kappa_0 + \kappa_1(z)]]\phi = -k_0^2\delta\kappa\phi. \qquad (4)$$

The Green's function is taken to be of the free space form, but the nonrandom portion of the wave number is allowed to vary in accordance with its actual distribution. This formulation fails when the plasma wave number has rapid variation within the distance of a wavelength, for a local solution does not include the effects of the rate of variation of the inhomogeneity.

An exact formulation of the problem is obtained through the use of the correct Green's function for (4). Such a Green's function is available for only simple plasma profiles such as a slab or halfspace (or as we later show, the Epstein profiles). The resulting integral equation is the same as (3) except that the Green's function, G is correct for the particular profile at hand, and f is now the right side of (4). Thus the random inhomogeneity represents a perturbation about the mean profile. Since this perturbation is often small, a meaningful solution to (4) can be obtained even when the plasma varies greatly from free space conditions (The Green's function represents

an exact solution when there is no random term.). The difficulty introduced by the formulation, is that the Green's function is no longer of the convolution form as was the case of the free space Green's function. Thus, Fourier transformation (which is sometimes useful in formulations using a free space Green's function*) does not offer any apparent benefits (for a non-convolution form Green's function problem).

2.3 *SELECTIVE SUMMATION TECHNIQUES*

A number of techniques have been devised to overcome the limitations of the successive type approximations, including smoothing [*Tatarskii and Gertsenshtein*, 1963], renormalization [*Karal and Keller*, 1964] and diagram methods [*Bourret*, 1962]. To the lowest solvable order, all these techniques are equivalent [*Jarem*, 1970]. We shall outline the smoothing method, and we refer the reader to the discussion given in a summary paper [e.g., *Frisch*, 1968; *Kresa*, 1968] for further details of these solutions.

In the method of smoothing, we iterate in a manner to obtain the fluctuating field in terms of the

---

*A number of applications of Fourier transformation have been given by *Jarem* [1968, 1969b].

mean field, and an equation for the mean field. We may write the scattering integral equation (3) in operator format as

$$\phi = \phi_i + G\sigma\phi, \tag{5}$$

where σ represents $-k_0^2\delta\kappa$. By separately applying the averaging operator P and the fluctuation operator $I-P$ (I being the identity operator) to (5), we obtain a pair of coupled integral equations for the coherent (ϕ_0) and the fluctuating (ϕ_1) fields

$$\begin{aligned} P\phi \equiv \phi_0 &= \phi_i + GP\sigma\phi_1 && \text{(A)} \\ (I-P)\phi \equiv \phi_1 &= G(I-P)\sigma(\phi_0 + \phi_1). && \text{(B)} \end{aligned} \tag{6}$$

Formal iteration of (6B) yields

$$\phi_1 = \sum_{n=1}^{\infty} [G(I-P)\sigma]^n \phi_0. \tag{7}$$

Using this expression in (6A), we obtain a master equation for the coherent field

$$\phi_0 = \phi_i + \sum_{n=1}^{\infty} GP\sigma[G(I-P)\sigma]^n\phi_c. \tag{8}$$

By retaining only the first term of the summation in (8), we obtain the first order smoothing equation

$$\phi_0 = \phi_i + GP\sigma G\sigma\phi_0, \tag{9}$$

which by iteration is equivalent to

$$\phi_0 = \phi_i + \sum_{n=1}^{\infty} [GP\sigma G\sigma]^n\phi_i. \tag{10}$$

Therefore the first order smoothing solution represents a summation of terms which are indicative of multiple scattering and are thus included in ϕ_0 obtained by solving the integral equation (9). The fluctuating field ϕ_1 can now be obtained to first smoothing order by using (7), with $n = 1$. The first order smoothing equation has been determined by *Frisch* [1968] to be valid for $(k_0 a \Omega_p)^2 << 1$,

where a is the correlation length (defined in Section 5.2), and Ω_p is the normalized plasma frequency (defined in Section 4). The integro-differential equation obtained by applying a second order differential operator to (9) has been solved only approximately, even for facile geometries [e.g., *Kupiec et al.*, 1969]. The most simplifying approximation to apply is the assumption of a delta function correlation. This assumption immediately casts the equation for the coherent field into an ordinary differential equation.

2.4 *OTHER SOLUTION METHODS*

In addition to the two categories of solution techniques described above some work has been carried out by energy transport theory and also by information theory concepts [*Kresa*, 1968]. The energy transport method [*Watson*, 1969] considers scattering by individual scatterers, and results in a set of coupled multiple scattering equations.

In the information theory formulation [*Kresa*, 1968], an attempt is made to obtain closure of the stochastically nonlinear wave propagation equation without use of a perturbation expansion or invoking any closure assumptions. This method has been useful in statistical

mechanics, but has not yet been significantly explored for scattering problems.

3. STATEMENT OF PROBLEM

In recent years, research work in random scattering has been carried out for one-dimensional plasmas, not necessarily because there is a correspondence to a physical problem, but because insight is sought for the more realistic three-dimensional geometry. It is in *this* spirit that we shall confine our investigation to a one-dimensional problem, even though there is a direct correspondence to the physical situation of reflection of vertically directed radio waves from a turbulent ionosphere. The analysis may in fact, be easily extended to oblique incidence, but will not be considered here, as it would only introduce one additional variable into a result whose analysis is already burdened by a large number of parameters. Propagation will thus take place in a layered-inhomogeneous medium whose properties only vary along one axis of a rectangular coordinate system

We will consider both wave reflection and transmission by the random inhomogeneous layers, and to obtain a rigorous solution of this problem, we solve the one-dimensional scalar Helmholtz equation, subject to the

appropriate boundary conditions. For a wave of unit magnitude incident from the left these are [*Stratton*, 1941, Chap.. 9]

$$z \to -\infty, \quad \phi(z) \to e^{ik_0 z} + Re^{-ik_0 z} \tag{11}$$

$$z \to \infty, \quad \phi(z) \to Te^{ik_1 z}, \tag{12}$$

where R and T are the reflection and transmission coefficients, respectively, which are to be determined.

Because of the difficulties involved in using the smoothing equation, we have chosen to initially investigate the problem of a very inhomogeneous and random plasma by the perturbation method with an exact Green's function. In addition to providing results for a restricted parameter range, such a technique should yield insight into solving one of the wider range formulations. We will determine the scattering coefficients for a particular distribution of mean layer properties (discussed in Section 5), and for particular values of the relevant stochastic parameters.

As a means of providing confidence in our result, we will compare our work to the computer experiments of *Hochstim and Martens* [1969, 1970] for an appropriate selection of parameters.

Although we have chosen to examine the case of a plasma medium, the analysis may be easily applied to the propagation of acoustic waves in layered turbulent media, or the propagation of electromagnetic waves in a randomly fluctuating real dielectric, such as might represent the troposphere.

4. ELECTROMAGNETIC WAVE PROPAGATION IN PLASMAS

The governing equations for electromagnetic wave propagation are the Maxwell equations, [e.g., *Stratton*, 1941, Chap. 1; *Ramo et al.*, 1965]

$$\begin{aligned} \nabla \times \bar{E} &= -\partial \bar{B}/\partial t \quad (A) & \nabla \cdot \bar{D} &= \rho \quad (C) \\ \nabla \times \bar{H} &= \bar{J} + \partial \bar{D}/\partial t \quad (B) & \nabla \cdot \bar{B} &= 0. \quad (D) \end{aligned} \tag{13}$$

By simple manipulation, the Maxwell equations can be reduced to a vector Helmholtz equation for the electric field. First, by assuming a time dependence of the form $e^{-i\omega t}$, and taking the curl of (13A), we obtain

$$\begin{aligned} \nabla \times \nabla \times \bar{E} &= i\omega\mu_0(\sigma - i\omega\epsilon_0)\bar{E} \\ &= \omega^2\mu_0(\epsilon_0 + i\sigma/\omega)\bar{E}. \end{aligned} \tag{14}$$

An *effective* dielectric constant [*Papas*, 1965] can now be defined by

$$\epsilon = \epsilon_0 + i\sigma/\omega. \tag{15}$$

Equation (14) now becomes

$$\nabla \times \nabla \times \bar{E} = \omega^2 \mu_0 \epsilon \bar{E}. \tag{16}$$

From the standard vector identity [e.g., *Brand*, 1957]

$$\nabla \times \nabla \times \bar{A} = \nabla(\nabla \cdot \bar{A}) - \nabla^2 \bar{A}. \tag{17}$$

Equation (16) becomes

$$\nabla(\nabla \cdot \bar{E}) - \nabla^2 \bar{E} = \omega^2 \mu_0 \epsilon \bar{E}. \tag{18}$$

The field vectors $\bar{D}$ and $\bar{E}$ are related by the constitutive relation [*Stratton*, 1941, Chap. 1]

$$\bar{D} = \epsilon \bar{E}. \tag{19}$$

Equation (13C) now becomes, for a charge free region*

$$\nabla \cdot \epsilon \bar{E} = 0, \tag{20}$$

---

*All effects of plasma free charges are contained in the conductivity and, subsequently, in the effective dielectric constant.

or

$$\nabla\epsilon \cdot \bar{E} + \epsilon\nabla \cdot \bar{E} = 0. \tag{21}$$

Thus we obtain

$$\nabla \cdot \bar{E} = -\nabla\epsilon \cdot \bar{E}/\epsilon. \tag{22}$$

Equation (18) is now written

$$\nabla^2\bar{E} + \omega^2\mu_0\epsilon\bar{E} = -\nabla[\nabla\epsilon \cdot \bar{E}/\epsilon]. \tag{23}$$

The electromagnetic wave number k is defined by,

$$k^2 = \omega^2\mu_0\epsilon, \tag{24}$$

so that (23) now becomes

$$(\nabla^2 + k^2)\bar{E} = -\nabla[\nabla k \cdot \bar{E}/k]. \tag{25}$$

Equation (25) is the inhomogeneous vector Helmholtz equation. For plane wave propagation perpendicular to the refractive index gradient

$$\nabla k \cdot \bar{E} = 0, \tag{26}$$

and any component of $\bar{E}$ may be represented by the *scalar* Helmholtz equation

$$(\nabla^2 + k^2)\phi = 0. \tag{27}$$

The Lorentz model of the plasma conductivity may be derived by considering the motion of a charged particle in an electric field* [*Heald and Wharton*, 1965]. A one-dimensional equation for the forces on such a particle can be written as

$$m\ddot{x} = -eE - \nu m\dot{x}. \tag{28}$$

---

*A more exact derivation of the plasma conductivity has been given by *Jaeger* [1969], and includes a discussion of the validity of the Lorentz model for radar applications.

The second term on the right hand side corresponds to viscous damping by the electron-neutral particle collisions. The steady state solution to (28) for oscillatory fields is given by [e.g., *Rainville*, 1964, Chap. 8]

$$\dot{x} = eE/m\omega(\omega + i\nu). \tag{29}$$

The current density, J, is defined by [*Stratton*, 1941, Chap. 1]

$$J = \sigma E = -ne\dot{x} \tag{30}$$

The conductivity, σ, is then

$$\sigma = ne^2/m(\nu - i\omega) = (ne^2/m)[(\nu + i\omega)/(\nu^2 + \omega^2)]. \tag{31}$$

The effective dielectric constant, defined by (15), becomes

$$\varepsilon = \varepsilon_0 \left[1 - \frac{(\omega_p/\omega)^2}{1 + (\nu/\omega)^2} + i \frac{(\omega_p/\omega)^2(\nu/\omega)}{1 + (\nu/\omega)^2} \right], \tag{32}$$

where the angular plasma frequency, ω_p, is defined by [*Kelso*, 1964]

$$\omega_p^2 = ne^2/m\varepsilon_0. \tag{33}$$

A normalized plasma frequency Ω_p, and collision frequency Ω_c are defined as follows

$$\Omega_p = \omega_p/\omega \; ; \quad \Omega_c = \nu/\omega , \tag{34}$$

where ω is the angular wave frequency. Plasmas for which $\Omega_p < 1$ are usually referred to as being *underdense*, while those for which $\Omega_p > 1$ are called *overdense*.

5. THE EPSTEIN PROFILE

5.1 *THE DETERMINISTIC LAYER*

In this section, we shall basically follow the analysis outlined by *Brekhovskikh* [1960]. The aim of this derivation is to transform the wave equation (27), into an equation whose solution is already known. As will be seen, when the inhomogeneous distribution is that which was studied by *Epstein* [1930], this transformed equation is the hypergeometric equation [*Whittaker and Watson*, 1965]. The logic of our derivation, however, will be deductive, since we will start with the hypergeometric equation, and show how this transforms into the wave equation.

The hypergeometric equation is given by

$$\frac{d^2F}{d\xi^2} - \frac{(\alpha+\beta+1)\xi - \gamma}{\xi(1-\xi)} \frac{dF}{d\xi} - \frac{\alpha\beta}{\xi(1-\xi)} F = 0. \tag{55}$$

Let transformations of the independent, and dependent variables be given by

$$\begin{aligned} F &= r(z)S(z) \\ \xi &= P(z). \end{aligned} \tag{56}$$

The derivatives of F occurring in (35) are then

$$\frac{dF}{d\xi} = \frac{dr}{d\xi} Z + r \frac{dZ}{dz} \frac{dz}{d\xi}$$

$$\frac{d^2F}{d\xi^2} = \frac{d^2r}{d\xi^2} Z + 2 \frac{dr}{d\xi} \frac{dZ}{dz} \frac{dz}{d\xi} + r \frac{d^2Z}{dz^2} \left(\frac{dz}{d\xi}\right)^2 + r \frac{dZ}{dz} \frac{d^2z}{d\xi^2} . \tag{37}$$

Substituting these into (35), we obtain

$$\begin{aligned} &\frac{d^2Z}{dz^2} \left[r\left(\frac{dz}{d\xi}\right)^2\right] \\ &+ \frac{dZ}{dz} \left[2\frac{dr}{d\xi} \frac{dz}{d\xi} + r\frac{d^2z}{d\xi^2} - \frac{(\alpha+\beta+1)r}{(1-\xi)} \frac{dz}{d\xi} + \frac{\gamma r}{\xi(1-\xi)} \frac{dz}{d\xi}\right] \\ &+ Z\left[\frac{d^2r}{d\xi^2} - \frac{(\alpha+\beta+1)}{(1-\xi)} \frac{dr}{d\xi} + \frac{\gamma}{\xi(1-\xi)} \frac{dr}{d\xi} - \frac{\alpha\beta r}{\xi(1-\xi)}\right] = 0. \end{aligned} \tag{38}$$

In order that (35) be equivalent to the wave equation, the coefficient of $dF/d\xi$ must vanish, or

$$2 \frac{dr}{d\xi} \frac{dz}{d\xi} + r\frac{d^2z}{d\xi^2} - \frac{(\alpha+\beta+1)r}{(1-\xi)} \frac{dz}{d\xi} + \frac{\gamma r}{\xi(1-\xi)} \frac{dz}{d\xi} = 0. \tag{39}$$

Equation (39) is a separable differential equation of the variables r, and ξ, and is readily integrated to obtain the transformation

$$r = r_0 (dz/dx)^{\frac{1}{2}} (1-\xi)^{(\gamma-\alpha-\beta-1)/2} \xi^{-\gamma/2}. \tag{40}$$

The function $P(z)$ has not yet been specified, and is, in fact, arbitrary. The wave equation is then written

$$d^2Z/dz^2 + k^2(z)Z = 0, \tag{41}$$

and now the function $k^2(z)$ is given by

$$k^2(z) = \left\{\frac{1}{2}\frac{d^2}{dz^2}\left[\ln\frac{dP}{dz}\right] - \frac{1}{4}\left[\frac{d}{dz}\left[\ln\frac{dP}{dz}\right]\right]^2\right\} - \left[\frac{d}{dz}\ln P\right]^2\left\{K_1 + K_2\frac{P}{1-P} + K_3\frac{P}{(1-P)^2}\right\}, \tag{42}$$

where

$$\begin{aligned} 4K_1 &= \gamma(\gamma-2) \\ 4K_2 &= 1-(\alpha-\beta)^2 + \gamma(\gamma-2) \\ 4K_3 &= (\alpha+\beta-\gamma)^2 - 1. \end{aligned} \tag{43}$$

In order to make the analysis tractable, *Epstein* [1930] chose for the transformation of the independent variable, the simple function

$$P(z) = -e^{mz}. \tag{44}$$

The dielectric distribution in (42) then becomes

$$k^2(x) = k_0^2\left[1 - M\frac{e^{mx}}{(1+e^{mx})} - 4N\frac{e^{mx}}{(1+e^{mx})^2}\right], \tag{45}$$

where the three new constants k_0, M, N have been defined in terms of K_1, K_2, K_3 as

$$K_1 = -(k_0^2/m^2 + 1/4), \quad K_2 = -(k_0/m)^2 M, \quad K_3 = -4(k_0/m)^2 N. \tag{46}$$

In order to further examine the dielectric distribution, a layer thickness parameter S will be defined as

$$S = 2k_0/m. \tag{47}$$

To determine the parameters α, β, γ, the equations (43) are first solved for these in terms of the constants K_1, K_2, K_3. The parameters α, β, γ are then related to the inhomogeneous distribution through (46) and (47) to obtain

$$\begin{aligned}
\alpha &= \{1 + [1 - 4S^2N]^{\frac{1}{2}} + iS[1 - (1-N)^{\frac{1}{2}}]\}/2 \\
\beta &= \{1 + [1 - 4S^2N]^{\frac{1}{2}} + iS[1 + (1-N)^{\frac{1}{2}}]\}/2 \\
\gamma &= 1 + iS.
\end{aligned} \tag{48}$$

The distribution of (45) is most useful when either N or M is taken to be zero. In the former case, the distribution is called a symmetric distribution since it represents a space filled with dielectric symmetrically distributed about $z = 0$, with free space condition as $|z| \to \infty$.

The case $M = 0$, represents a transition layer going from free space conditions at minus infinity to an arbitrary value of dielectric constant at plus infinity. The parameter S [defined by (47)] is representative of the rapidity of this variation, or in the case of the symmetric layer, the width of the layer. Both symmetric and transition layers are displayed in Fig. (1).

The transformation (36) maps the entire real axis $z(-\infty,\infty)$ into the negative real axis $\xi(0,-\infty)$, with $z = 0$ corresponding to $\xi = -1$.

The hypergeometric equation has 24 solutions. In the region $|\xi|<1$ $(z<0)$, a linearly independent pair of solutions is given by* [*Erdélyi et al.*, 1953]

---

*That F_2 is a solution of (35) may be verified by direct substitution. By virtue of the fact that the solutions (49) are two power series which begin at different powers of ξ, they cannot be proportional to one another. F_1 and F_2 are thus linearly independent.

$$F_1 = F(\alpha, \beta; \gamma; \xi)$$
$$F_2 = \xi^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1; 2-\gamma; \xi), \tag{49}$$

where $F(\alpha, \beta; \gamma; \xi)$ is the hypergeometric function defined by

$$F(\alpha,\beta;\gamma;\xi) = 1 + \frac{\alpha\beta}{\gamma} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)\ 1\cdot 2}\xi^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)\ 1\cdot 2\cdot 3}\xi^3 + \cdots . \tag{50}$$

In order to obtain the corresponding solution in the region $|\xi|>1$ ($z>0$), the hypergeometric function must be considered as the representation in the region $|\xi|<1$ of an entire function. In order to find these solutions of (35) corresponding to those in $|\xi|<1$, it is necessary to determine the representation of this entire function in the region $|\xi|>1$. This procedure is illustrated in *Appendix A* and it results in the following analytic continuation formula for the hypergeometric function in the region $|\xi|>1$ ($z>0$)

$$F(a,b;c;z) = C_1(-z)^{-a}F(a,1-c+a;1-b+a;\ 1/z) + C_2(-z)^{-b}F(b,1-c+b;1-a+b;\ 1/z), \tag{51}$$

where

$$C_1 = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}\ ; \qquad C_2 = \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}\ . \tag{52}$$

The solution to the reduced wave equation with an Epstein profile can thus be written down as

$z<0$

$$Z_{01-}(\xi) = m^{-\frac{1}{2}}\xi^{(\gamma-1)/2}(1-\xi)^{(1+\alpha+\beta-\gamma)/2}F(\alpha,\beta;\gamma;\xi)$$

$$Z_{02-}(\xi) = m^{-\frac{1}{2}}\xi^{-(\gamma-1)/2}(1-\xi)^{(1+\alpha+\beta-\gamma)/2} \times F(\alpha-\gamma+1,\beta-\gamma+1;2-\gamma;\xi)$$

$z>0$

$$Z_{01+}(\xi) = m^{-\frac{1}{2}}\xi^{(\gamma-1)/2}(1-\xi)^{(1+\alpha+\beta-\gamma)/2}\{ B_1(-\xi)^{-\alpha}F(\alpha,1-\gamma+\alpha;1-\beta+\alpha;1/\xi) + B_2(-\xi)^{-\beta}F(\beta,1-\gamma+\beta;1-\alpha+\beta;1/\xi)\} \tag{53}$$

$$Z_{02+}(\xi) = m^{-\frac{1}{2}}\xi^{-(\gamma-1)/2}(1-\xi)^{(1+\alpha+\beta-\gamma)/2}\{ B_3(-\xi)^{\gamma-\alpha-1}F(\alpha-\gamma+1,\alpha;\alpha-\beta+1;1/\xi) + B_4(-\xi)^{\gamma-\beta-1}F(\beta-\gamma+1,\beta,\beta-\alpha+1;1/\xi)\},$$

where

$$B_1 = \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)} \quad ; \quad B_2 = \frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}$$

$$B_3 = \frac{\Gamma(2-\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta-\gamma+1)\Gamma(1-\alpha)} \quad ; \quad B_4 = \frac{\Gamma(2-\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha-\gamma+1)\Gamma(1-\beta)} \quad . \tag{54}$$

The asymptotic forms of the solutions as $|z|\to\infty$ can be easily evaluated to be

$$\begin{aligned}
&z\to-\infty \\
&\qquad z_{01-}(z)\to m^{-\frac{1}{2}}(-1)^{iS/2}e^{ik_0 z} \\
&\qquad z_{02-}(z)\to m^{-\frac{1}{2}}(-1)^{-iS/2}e^{-ik_0 z} \\
&z\to\infty \\
&\qquad z_{01+}(z)\to m^{-\frac{1}{2}}(-1)^{-iS/2}\{B_1 e^{ik_1 z} + B_2 e^{-ik_1 z}\} \\
&\qquad z_{02+}(z)\to m^{-\frac{1}{2}}(-1)^{-iS/2}\{B_3 e^{ik_1 z} + B_4 e^{ik_1 z}\} \quad .
\end{aligned} \tag{55}$$

Note that as $z\to-\infty$ each solution is in the form of a traveling wave, one traveling to the right, the other traveling to the left. As $z\to\infty$, the solutions are made up of linear combinations of traveling waves.*

---

*The pair of solutions (49) picked as independent were done so for convenience, as they are not the only independent pair. The imposition of boundary conditions would force the equivalence of all the arbitrary pairs of independent solutions.

A comparison of (24), (32) and (45) will show that for the distribution profiles of electron density the parameters M and N are given by

$$M = N = \Omega_{pc}^2[(1-i\Omega_c)/(1+\Omega_c^2)], \tag{56}$$

where Ω_{p_0} is the maximum value of plasma frequency for the distribution. For both types of distribution profiles, the collision frequency will be considered to be constant, and for convenience in considering both distributions, our equations will only contain N, knowing that $M = N$.

Also, the z-variation in mean properties will be written

$$e^{mz}/(1+e^{mz})^{\delta} \equiv g(z) \tag{57}$$

where $\delta = 1$ for the transition layer, and $\delta = 2$ for the symmetric case.

5.2 *THE TURBULENT LAYER*

In considering the propagation through a turbulent plasma layer, we assume that the distribution of electron density can be regarded as a continuous fluid and that

[illegible] are smooth and slow (though random) functions of space and time [e.g., *deWolf*, 1968]. By this, we mean that the time scale of turbulence [*Hinze*, 1959] is significantly longer than the time necessary for propagation across a dimension characteristic of the layer thickness. We may then write down the equation for a particular realization of the ti[illegible] *Beckmann*, 1967] so formed. We also assume that the ensemble is stationary, i.e., the statistical properties of the medium are not functions of time [*Beckmann*, 1967]. A second, spatial, ensemble is formed by the electron density as a function of location at a particular time. Si[illegible] changes of electron density are governed by hydrodynamic turbulence, we assume that the time and space ensembles [illegible] both stationary and equivalent [*deWolf*, 1967].

The distribution of electron density n, will be [illegible] rts as follows

$$n = n_0(z) + n_1(z,t) \tag{58}$$

where

$$\langle n \rangle = n_0; \ \langle n_1 \rangle = 0. \tag{59}$$

The brackets < >, represent an ensemble averaging process [*Beckmann*, 1967]. In describing the turbulent medium, we shall have use for the following parameters: the turbulent intensity ζ, where

$$\zeta = \langle n_1^2 \rangle / n_0, \tag{60}$$

and the normalized electron density fluctuation η, where

$$\eta = n_1 / \langle n_1^2 \rangle^{\frac{1}{2}}. \tag{61}$$

The fluctuating component of electron density is then written

$$n_1(z) = n_0(z) \zeta \eta(z). \tag{62}$$

We shall restrict this analysis to homogeneous turbulence [*Hinze*, 1958] by taking ζ to be constant throughout the plasma region.

For notational convenience in this section, we will write the squared wave number k^2, as being composed of a mean non-fluctuating component and a random component, so

that we write

$$k^2 = k_0^2[\kappa + \delta\kappa]. \tag{63}$$

The exact term correspondence of this expression may be obtained by a comparison of (33), (34), (45), (56) and (58). The wave equation (41) may thus be written in the form

$$\{d^2/dz^2 + k_0^2[\kappa + \delta\kappa]\}\phi = 0. \tag{64}$$

We take the $k_0^2\delta\kappa\phi$ term to be a source term, so that the technique of Green's function may be applied, i.e., after writing

$$\{d^2/dz^2 + k_0^2\kappa\}\phi = -k_0^2\delta\kappa\phi, \tag{65}$$

the left side of the equation is just that of an Epstein profile.

Let us now construct a Green's function $G(z,z')$ for (65) which is continuous and corresponds to an impulsive force at $z = z'$. We subject the Green's function to the condition of outgoing waves as $|z| \to \infty$. Thus G satisfies

the equation

$$\{d^2/dz^2 + k_0^2\kappa(z)\}G(z,z') = \delta(z-z'), \tag{66}$$

with

$$\begin{aligned} G(-\infty,z') &\propto e^{-ik_0 z} \\ G(\infty,z') &\propto e^{ik_1 z}. \end{aligned} \tag{67}$$

By integrating (66) with respect to z, we may determine that $G(z,z')|_{z=z'}$ is discontinuous by the value 1. The properties of the Green's function can then be summarized as

1. G is continuous at $z = z'$: $G_{z'_+} - G_{z'_-} = 0$
2. G' is discontinuous at $z = z'$: $G'_{z'_+} - G'_{z'_-} = 1.$ (68)

We are seeking the solution to the problem

$$\{d^2/dz^2 + k_0^2\kappa(z)\}\phi = f(z), \tag{69}$$

with an incident and reflected wave as $z\to-\infty$, and a transmitted wave as $z\to\infty$. These boundary conditions are expressed as

$$\begin{aligned} \psi(-\infty) &= e^{ik_0 z} + Re^{-ik_0 z} \\ \psi(\infty) &= Te^{ik_1 z}, \end{aligned} \tag{70}$$

where k_1 may either be equal to k_0 (symmetric layer) or different, as in the transition layer. Let Z_1 and Z_2 be a pair of independent solutions of the homogeneous equation

$$\{d^2/dz^2 + k_0^2 \kappa(z)\} z = 0, \tag{71}$$

with the following asymptotic forms*

$$\begin{aligned} Z_1(-\infty) &= Ae^{ik_0 z} \qquad Z_2(-\infty) = Be^{-ik_0 z} \\ Z_1(\infty) &= A\{F_1 e^{ik_1 z} + F_2 e^{-ik_1 z}\} \\ Z_2(\infty) &= B\{F_3 e^{ik_1 z} + F_4 e^{-ik_1 z}\}. \end{aligned} \tag{72}$$

The form of the Green's function can then be immediately written down as

$$\begin{aligned} &z<z' \\ &\qquad G(z,z') = a_1 Z_1(z) + a_2 Z_2(z) \\ &z>z' \\ &\qquad G(z,z') = b_1 Z_1(z) + b_2 Z_2(z). \end{aligned} \tag{73}$$

---

*These are the asymptotic forms of the solutions to the reduced wave equation as given in (55).

By applying the outgoing wave property of G,

$$\begin{aligned} G_1(-\infty,z') &= a_1 Z_1(z) + a_2 Z_2(z) \propto e^{-ik_0 z} \\ G_2(\infty, z') &= b_1 Z_1(z) + b_2 Z_2(z) \propto e^{ik_1 z} \\ &= e^{ik_1 z}[b_1 AB_1 + b_2 BB_3] \\ &\quad + e^{-ik_1 z}[b_1 AB_2 + b_2 BB_4], \end{aligned} \tag{74}$$

and then

$$a_1 = 0; \quad b_2 = -AB_2 b_1 / BB_4. \tag{75}$$

The continuity properties of G, (68), lead to the following set of equations

$$\begin{aligned} b_1[Z_1(z') - \frac{AB_2}{BB_4} Z_2(z')] - a_2 Z_2(z') &= 0 \\ b_1[Z_1'(z') - \frac{AB_2}{BB_4} Z_2'(z')] - a_2 Z_2'(z') &= 1. \end{aligned} \tag{76}$$

These yield

$$b_1 = -Z_2(z')/W; \; a_2 = [-Z_1(z') + (AB_2/BB_4)Z_2(z')]/W, \tag{77}$$

where W is the Wronskian determinant [*Rainville*, 1964, Chap. 7]

$$W = [Z_1(z)Z_2'(z) - Z_1'(z)Z_2(z)]_{z=z'}. \tag{78}$$

Since the Wronskian of the wave equation is constant*, it may be evaluated at any convenient point. The Green's function is then given by

$$z<z' \qquad G_1(z,z') = (2ik_0B)^{-1}[Z_1(z')Z_2(z)/A - Z_2(z')Z_2(z)B_2/B_4]$$

$$z>z' \qquad G_2(z,z') = (2ik_0B)^{-1}[Z_2(z')Z_1(z)/A - Z_2(z')Z_2(z)B_2/BB_4]. \tag{79}$$

To make use of the Green's function, we multiply (69) by G, and (66) by ϕ to obtain

$$\begin{aligned} G\phi'' + k_0^2\kappa(z)G\phi &= Gf(z) \\ \phi G'' + k_0^2\kappa(z)G\phi &= \phi\delta(z-z'). \end{aligned} \tag{80}$$

Subtracting the two equations and integrating over z from $-\infty$ to ∞, the left hand side may be rewritten as

---

*This is a special case of Abel's identity [*Ince*, 1956].

$$\int_{-\infty}^{\infty} \frac{d}{dz}[G\phi' - \phi G']dz, \tag{81}$$

while the integral containing the delta function may be immediately evaluated. Therefore

$$[G\phi' - \phi G']_{-\infty}^{\infty} = \int_{-\infty}^{\infty} f(z)G(z,z')dz - \phi(z'). \tag{82}$$

By evaluating the bracketed term, using the asymptotic forms of G and ϕ, and interchanging the variables z and z', then

$$\phi(z) = Z_1(z)/A - B_2Z_2(z)/BB_4 + \int_{-\infty}^{\infty} f(z')G(z',z)dz', \tag{83}$$

where

$$\begin{aligned} z<z' \qquad & G(z',z) = G_2(z',z) = (2ik_0B)^{-1} \\ & \times \{Z_2(z)Z_1(z')/A - B_2Z_2(z)Z_2(z')/BB_4\} \\ z>z' \qquad & G(z',z) = G_1(z',z) = (2ik_0B)^{-1} \\ & \times \{Z_1(z)Z_2(z')/A - Z_2(z)Z_2(z')B_2/BB_4\}. \end{aligned} \tag{84}$$

Comparison of (41) and (64), allows us to write down the constants A and B, and the exact formal solution for the case of the turbulent media.* Note that the

---

*For our situation, (83) does not represent a solution, but represents an integral equation in the desired quantity.

proper representation of the homogeneous solutions Z_1 and Z_2 depends upon whether their argument is greater or less than 0. The formal "solution" is thus an integral equation for the field ϕ. To evaluate the reflection or transmission coefficient of the layer, the field is evaluated at $-\infty$ or ∞ respectively.

To solve (83) we shall use the Neumann iteration technique [*Green*, 1969] which generates a series for the solution of the reduced wave equation at $-\infty$ of the form

$$\phi(-\infty) = e^{ik_0 z} + [R_0 + R_1 + R_2 + \ldots]e^{-ik_0 z}. \qquad (85)$$

The terms within the parenthesis represent the successive contributions to the field reflection coefficients given by succeeding iterations of the integral equation (83) (These are physically equivalent to multiple reflections.). If the series representation of the reflection coefficient is convergent, we may attach to each term a measure of turbulent strength ε, so that we may write the field reflection coefficient as

$$R_v = \varepsilon^0 R_0 + \varepsilon^1 R_1 + \varepsilon^2 R_2 + \cdots \quad . \qquad (86)$$

The most convenient measurable parameter of interest is the average power reflection coefficient defined by

$$R = \langle R_v R_v^* \rangle . \tag{87}$$

Using (86) and retaining terms to order ε^2 in smallness, we have

$$R = \langle |R_0|^2 \rangle + \langle R_0^* R_1 \rangle + \langle R_1^* R_0 \rangle + \langle |R_1|^2 \rangle + \langle R_0^* R_2 \rangle + \langle R_0 R_2^* \rangle . \tag{88}$$

Each higher order of the reflection coefficient R_v is proportional to an integral over a higher power of the random quantity η, i.e.,

$$\begin{aligned} R_0 &\sim \text{constant} \\ R_1 &\sim \int f_1(z')\eta(z')dz' \\ R_2 &\sim \int f_2(z')\eta(z')dz \int f_3(z'')\eta(z'')dz'' \\ &\vdots \end{aligned} \tag{89}$$

The medium properties were split into mean and fluctuating components by (58) so that $\langle \eta \rangle = 0$. Thus $\langle R_0 \rangle = R_0$ and $\langle R_1 \rangle = 0$. The reflection coefficient to second order is then

$$R = |R_0|^2 + 2\mathrm{Re}(R_0^* \langle R_2 \rangle) + \langle |R_1|^2 \rangle . \tag{90}$$

By convention, the reflection coefficient is broken up into coherent and incoherent components, where the coherent reflection coefficient is defined as

$$W_c = |\langle R_v \rangle|^2, \tag{91}$$

while the incoherent coefficient is given by

$$\begin{aligned} W_I &= \langle |R_v - \langle R_v \rangle|^2 \rangle \\ &= \langle |R_v|^2 \rangle - |\langle R_v \rangle|^2. \end{aligned} \tag{92}$$

To second order in ϵ,

$$\begin{aligned} W_c &= |R_0|^2 + 2\mathrm{Re}\{R_0^* \langle R_2 \rangle\} \\ W_I &= \langle |R_1|^2 \rangle. \end{aligned} \tag{93}$$

We shall also make use of the special reflection coefficient $\langle R' \rangle$ used by *Hochstim and Martens* [1969; 1970] defined as

$$\langle R' \rangle = R - |R_0|^2. \tag{94}$$

To second order then,

$$\langle R' \rangle = \langle |R_1|^2 \rangle + 2\mathrm{Re}\{R_0^* \langle R_2 \rangle\}. \tag{95}$$

Two iterations of the integral equation (83) (which corresponds to $R_0 + R_1 + R_2$ in the Neumann series) then yield

$$\begin{aligned}\phi(z) = m^{\frac{1}{4}}(-1)^{-iS/2}\{&[Z_{01}(z)-(-1)^{iS}Z_{02}(z)/B_4] \\ &+ 2k_0^2 N\zeta \int_{-\infty}^{\infty} g(z')n(z')[Z_{01}(z')-(-1)^{iS}B_2 Z_{02}(z')/B_4]G(z',z)dz' \\ &+ (k_0^2 N\zeta)^2 \int_{-\infty}^{\infty} g(z')n(z')G(z',z)dz' \\ &\times \int_{-\infty}^{\infty} g(z')n(z'')[Z_{01}(z'')-(-1)^{iS}B_2 Z_{02}(z'')/B_4]G(z'',z')dz''\}\end{aligned} \tag{96}$$

where $G(z',z)$ is given by (84).

The incoherent power reflection coefficient is then given by

$$W_I = (mk_0/2)^2 |N|^2 \zeta^2 \{$$

$$\int_{-\infty}^{\infty} g(z')Z_{01}^{*2}(z')dz' \int_{-\infty}^{\infty} \langle n(z')n(z'')\rangle g(z'')Z_{01}^2(z'')dz''$$

+

$$
\begin{aligned}
&4|B_2/B_4|^2 \int_{-\infty}^{\infty} g(z')Z_{01}^{*}(z')Z_{02}^{*}(z')dz' \\
&\qquad \times \int_{-\infty}^{\infty} \langle n(z')n(z'')\rangle g(z'')Z_{01}(z'')Z_{02}(z'')dz'' \\
&+ |B_2/B_4|^2 \int_{-\infty}^{\infty} g(z')Z_{02}^{*2}(z')dz' \\
&\qquad \times \int_{-\infty}^{\infty} \langle n(z')n(z'')\rangle g(z'')Z_{02}^{2}(z'')dz'' \\
&+ 2\mathrm{Re}[-2(-1)^{iS}(B_2/B_4) \int_{-\infty}^{\infty} g(z')Z_{01}^{*2}(z')dz' \\
&\qquad \times \int_{-\infty}^{\infty} \langle n(z')n(z'')\rangle g(z'')Z_{01}(z'')Z_{02}(z'')dz'' \\
&+ (-1)^{i2S}(B_2/B_4)^2 \int_{-\infty}^{\infty} g(z')Z_{01}^{*2}(z')dz' \\
&\qquad \times \int_{-\infty}^{\infty} \langle n(z')n(z'')\rangle g(z'')Z_{02}^{2}(z'')dz'' \\
&- 2(-1)^{iS}|B_2/B_4|^2(B_2/B_4) \int_{-\infty}^{\infty} g(z')Z_{01}^{*}(z')Z_{02}^{*}(z')dz' \\
&\qquad \int_{-\infty}^{\infty} \langle n(z')n(z'')\rangle g(z'')Z_{02}^{2}(z'')dz'']\}.
\end{aligned}
\tag{97}
$$

The terms containing $\langle n \rangle$ have vanished. Similarly, the average incoherent transmission coefficient is given by

$$
\begin{aligned}
I = & \int_{-\infty}^{\infty} Z_0^*(z') Z_{02}^*(z') dz' \\
& \times \int_{-\infty}^{\infty} \langle n(z') n(z'') \rangle g(z'') Z_{01}(z'') Z_{02}(z'') dz'' \\
& \times \int_{-\infty}^{\infty} g(z') Z_{02}^{*2}(z') dz' \\
& \times \int_{\infty}^{\infty} \langle n(z') n(z'') \rangle g(z'') Z_{02}^2(z'') dz'' \\
& + 2\mathrm{Re} \int_{\infty}^{\infty} g(z') Z_{01}^*(z') Z_{02}^*(z') dz' \\
& \times \int_{-\infty}^{\infty} \langle n(z') n(z'') \rangle g(z'') Z_{02}^2(z'') dz'' \Big] \Big\}.
\end{aligned}
\tag{98}
$$

A typical term of (98) or (99) has the form

$$\int_{-\infty}^{\infty} g(x')Z_{01}^{*}(x')Z_{02}^{*}(x')dx' \int_{-\infty}^{\infty} \langle n(x')n(x)\rangle g(x)Z_{01}(x)Z_{02}(x)dx. \tag{99}$$

Expressions involving similar terms are obtained for the coherent fields through the use of (93) and (96).

The correlation $\langle n(x)n(x')\rangle$ will be assumed to be of the exponential form*

$$\langle n(x')n(x)\rangle = e^{-b|x-x'|}. \tag{100}$$

Because of the varying format of the functions Z_{01} and Z_{02}, the integration of a term like (99) must be broken into six steps as follows

---

*Such a correlation function is analytically convenient and yields a turbulence spectrum which is consistent with experimental results [*Guthart et al.*, 1966].

$$\int_{-\infty}^{0} g(x')Z^{*}_{01-}(x')Z^{*}_{02-}(x')e^{-bx'}dx'$$

$$\times \int_{-\infty}^{x'} e^{bx}g(x)Z_{01-}(x)Z_{02-}(x)dx$$

$$+ \int_{-\infty}^{0} g(x')Z^{*}_{01-}(x')Z^{*}_{02-}(x')e^{bx'}dx'$$

$$\times \int_{x'}^{0} e^{-bx}g(x)Z_{01-}(x)Z_{02-}(x)dx$$

$$+ \int_{-\infty}^{0} g(x')Z^{*}_{01-}(x')Z^{*}_{02-}(x')e^{bx'}dx'$$

$$\times \int_{0}^{\infty} e^{-bx}g(x)Z_{01+}(x)Z_{02+}(x)dx$$

$$+ \int_{0}^{\infty} g(x')Z^{*}_{01+}(x')Z^{*}_{02+}(x')e^{-bx'}dx' \qquad (101)$$

$$\times \int_{-\infty}^{0} e^{bx}g(x)Z_{01-}(x)Z_{02-}(x)dx$$

+

$$\int_0^{\infty} g(x')Z^{*}_{01+}(x')Z^{*}_{02+}(x')e^{-bx'}dx'$$

$$\times \int_0^{x'} e^{bx} g(x)Z_{01+}(x)Z_{02+}(x)dx$$

$$+ \int_0^{\infty} g(x')Z^{*}_{01+}(x')Z^{*}_{02}(x')e^{bx'}dx'$$

$$\times \int_{x'}^{\infty} e^{-bx} g(x)Z_{01+}(x)Z_{02+}(x)dx.$$

We carry out the integration of such a typical term in *Appendix B*. As a result of the algebraic complexity and length of that integration, we defer the final expressions to *Appendix C*, where the results are given in terms of certain functions defined in *Appendix B*. These functions are in general, sets of infinite series which must be computer evaluated.

The relative simplicity found in the reflection and transmission coefficients of the undisturbed plasma ($\zeta = 0$) does not carry over to the turbulent case, even for the restrictive situation of small perturbations from the smooth distribution. To compute the turbulence free

reflection coefficient it is necessary to know the field only at $-\infty$, where the asymptotic forms are simple exponentials. In the disturbed case however, the zeroth order field must actually be known at every point in space in order to sum up the scattering introduced by the random inhomogeneities throughout this space. The exception to this complication is the case for $S \to 0$ in the transition layer. This distribution then becomes that of free space on the left of zero and a half space of plasma of constant mean electron density on the right of zero. The results of this case are of sufficient interest to promote a separate discussion in Section 6.1. The actual results of the computer evaluation of the final expressions are discussed in Section 6.2.

6. DISCUSSION OF RESULTS

6.1 *THE TURBULENT HALF-SPACE*

As discussed in Section 5.2, if we examine the case of the transition layer when we allow $S \to 0$, we arrive at the problem of a half-space distribution. The half-space distribution has already been studied separately by several workers [*Rosenbaum*, 1969; *Kupiec et al.*, 1969], who have made use of smoothing equation techniques. It is not, however, our goal to compare approximate methods of solution, but to compare our solution method with the exact experimental treatment of the slab geometry [*Hochstim and Martens*, 1969; 1970] (where applicable), and to show that, in fact, the solution restrictions which we derive in this section, convey proper information about the validity of our method of solution.

6.1.1 Comparison with the Hochstim and Martens Computer Experiment

Since the only experimental solutions available are for a slab geometry, comparison with a half-space may

be made only for a restricted number of experiments. These are, those cases in which the attenuation of a wave across the width of the slab is so great, that there is effectively no contribution to the reflection coefficient made by the second jump in medium properties (i.e., the rear of the slab). We have arbitrarily chosen the criteria that an experimental slab can be compared to the half-space distribution (all other medium properties being the same), if a wave suffers 10 dB of attenuation in traveling across the width of the slab. This restricts the comparison to the experimental cases which have $1.5 > \Omega_p > 0.8$, $\Omega_c > 0.01$.

As $S \to 0$, the constants occurring in the Epstein profile solution B_1-B_4 may be evaluated by using the following expansion for the gamma function when z is small [*Abromowitz and Stegun*, 1964]

$$\Gamma(z) \approx 1/z. \tag{102}$$

These constants then become

$$\begin{aligned} B_1 &= B_4 \to (1+X)/2X \\ B_2 &= B_3 \to (X-1)/2X, \end{aligned} \tag{103}$$

where $X = +[1-N]^{\frac{1}{2}}$. The homogeneous solution for the Epstein transition profile can then be easily evaluated to be

$$
\begin{aligned}
&Z_{01} \to m^{-\frac{1}{2}}(-1)^{iS/2} e^{ik_0 z} \\
&Z_{02} \to m^{-\frac{1}{2}}(-1)^{-iS/2} e^{-ik_0 z} \\
&Z_{01} \to m^{-\frac{1}{2}}(-1)^{iS/2}[(X+1)e^{ik_1 z} + (X-1)e^{-ik_1 z}]/2X \\
&Z_{02} \to m^{-\frac{1}{2}}(-1)^{-iS/2}[(X-1)e^{ik_1 z} + (X-1)e^{-ik_1 z}]/2X,
\end{aligned}
\tag{104}
$$

The exact integral equation is then

$$
\begin{aligned}
z<0 \qquad \phi(z) &= \zeta k_0^2[\Omega_p^2(1-i\Omega_c)/(1+\Omega_c^2)] \int_0^\infty n(z')\phi(z')G_-(z',z)dz' \\
&\quad + e^{ik_0 z} + [(1-X)/(1+X)]e^{-ik_0 z} \\
z>0 \qquad \phi(z) &= \zeta k_0^2[\Omega_p^2(1-i\Omega_c)/(1+\Omega_c^2)] \int_0^\infty n(z')\phi(z')G_+(z',z)dz' \\
&\quad + [2/(1+X)]e^{ik_0 X z},
\end{aligned}
\tag{105}
$$

with the Green's function given by

$$
\begin{aligned}
&z'<z,\ z'<0 \quad G_-(z',z) = (2ik_0)^{-1}\left[e^{ik_0 z} + \left(\frac{1-X}{1+X}\right)e^{-ik_0 z}\right]e^{-ik_0 z'}\\
&z'>z,\ z'<0 \quad G_-(z',z) = (2ik_0)^{-1}\left[e^{ik_0 z'} + \left(\frac{1-X}{1+X}\right)e^{-ik_0 z'}\right]e^{-ik_0 z}\\
&z'>z,\ z'>0 \quad G_-(z',z) = (2ik_0)^{-1}\left[\left(\frac{2}{1+X}\right)e^{ik_0 X z'}\right]e^{-ik_0 z}\\
&z'<z,\ z'<0 \quad G_+(z',z) = (2ik_0)^{-1}\left[\left(\frac{2}{1+X}\right)e^{-ik_0 X z}\right]e^{-ik_0 z'}\\
&z'<z,\ z'>0 \quad G_+(z',z) = (2ik_0)^{-1}\left[\left(\frac{X-1}{X+1}\right)e^{ik_0 X z} + e^{-ik_0 X z}\right]e^{ik_0 X z'}\\
&z'>z,\ z'>0 \quad G_+(z',z) = (2ik_0)^{-1}\left[\left(\frac{X-1}{X+1}\right)e^{ik_0 X z'} + e^{-ik_0 X z'}\right]e^{ik_0 X z}.
\end{aligned}
\tag{106}
$$

The expression is just that obtained by considering only the half-space geometry [c.f., *Rosenbaum*, 1969]. By applying the asymptotic forms as $|z|\to\infty$ to the expressions for the scattering coefficients, we may straight-forwardly obtain the results for the half-space which are displayed in *Appendix C*.

6.1.2 Validity of the Solution Method

Following the method of analysis given by *Frisch* [1968] for the validity of the Born series, we may derive

a similar validity for the half-space geometry with exact Green's function. Denoting the successive terms of the Neumann series by ϕ_n, it follows from the integral form of the Minkowski inequality that

$$|\phi_{n+1}|^2 \leq k_0^2 \zeta \frac{\Omega_p^2(1-i\Omega_c)}{(1+\Omega_c^2)} \left|\int_V G(z',z)\phi_n(z')dz'\right|^2$$

$$\leq k_0^2 \zeta \frac{\Omega_p^2(1-i\Omega_c)}{(1+\Omega_c^2)} |\phi_n|^2 \int_V |G(z',z)|^2 dz'. \tag{107}$$

Expression (107) may also be obtained from a convergence criteria for the Neumann series derived by *Jarem* [1969]. The expression (107) implies a finite volume of plasma, whereas a half-space of plasma extends to infinity. If we, however, allow collisional damping, any propagating wave is continuously attenuated and an effective volume is created. The final form of the convergence criteria will show that when there are no collisions, our method of solution does not yield even an approximate solution.

From (106),

$$G(z',z) = e^{-ik_0 z'} e^{ik_0 X z} / ik_0(1+X), \tag{108}$$

and

$$|G(z',z)| = k_0^{-1}|1+\chi|^{-1}e^{-k_0\beta' z}. \tag{109}$$

The integral in (107) is then given by

$$I = \{k_0^2\beta'|1+\chi|\}^{-1}, \tag{110}$$

so that

$$|\phi_{n+1}|^2 \leq \zeta\frac{\Omega_p^2}{(1+\Omega_c^2)^{\frac{1}{2}}\beta'}|1+\chi|^{-1}|\phi_n|^2. \tag{111}$$

With a complete function space, the convergence of $\sum_{n=1}^{\infty}\phi_n$ follows from the convergence of $\sum_{n=1}^{\infty}\phi_n{}^2$. Therefore, if

$$\zeta\Omega_p^2/[(1+\Omega_c^2)^{\frac{1}{2}}\beta'|1+\chi|] < 1, \tag{112}$$

the rapidly convergent Neumann series may be used as an approximate solution to the reduced wave equation. Expressing the bound as a limit upon the allowable turbulence level, we have the restriction

$$\varepsilon < \beta'(1+\Omega_e^2)^{\frac{1}{2}}[(1+\alpha')^2 + (\beta')^2]^{\frac{1}{2}}/\Omega_p^2. \qquad (113)$$

The limiting cases of inequality (113) are displayed in Fig. (2). It can be seen there, that the restriction (113) is consistent with the physical interpretation of the scattering integral equations. Since our iteration method considers only a finite number of scattering events (one event for the incoherent coefficient; two events for the coherent coefficients), our solution will not be valid when the medium is such as to be very multiple scattering. Such a medium would be one which is not very absorbing and of large physical extent, so that a wave would not diminish greatly in magnitude following a scattering event, and, in fact, could be scattered again without appreciable attenuation. Obviously, a theory which allows for at most two scattering events cannot be expected to faithfully predict the scattering coefficients for such

a media. In a plasma, the attenuation coefficient goes to zero when $\Omega_p < 1$ and $\Omega_c \to 0$. As can be seen in Fig. (2), for this situation, the turbulence level must satisfy

$$\zeta < (\Omega_c/2)^{\frac{1}{2}}. \tag{114}$$

Or for a non-attenuating medium,

$$\zeta < 0, \tag{115}$$

i.e., the Neumann series cannot give a solution for any level of turbulence, no matter how small.

In Figs. (3)-(7) we have plotted the results of the reflection coefficients for incoherent reflection, and the special reflection coefficient $\langle R' \rangle$ for the experimental conditions of the Hochstim-Martens experiment.* Superimposed are their experimental results for parameter values where the slab and half space may be compared according to the attenuation criteria previously discussed (Section 6.1.1). It can be seen that for the largest value of collision

---

*The only parameter not given explicitly in these figures is the correlation length. In the computer experiment this was taken to be variable according to the form $ka = \pi/2\Omega_p$. This same form has been used in our calculations.

frequency (Ω_c=1.0), the two solutions are in exact agreement even for large values of turbulence level, ζ. As the plasma becomes less attenuating for Ω_c<1 (by decreasing the collision frequency), our result begins to differ from the computer experiment at the higher levels of turbulence*, and in fact for Ω_c=0.01 physically unrealizable incoherent reflection coefficient are obtained (i.e., greater than one). Such a divergence can be allowable only if the plasma parameters are such, as to render the inequality (113) not valid.

As an example, we have examined the convergence situation for Ω_p=1.0, the value of plasma frequency for which there is the greatest disparity between solution and experiment. In Fig. (8), we have plotted the ratio of the perturbation solution incoherent reflection coefficients to those of the computer experiment, as a function of turbulence level, ζ and collision frequency, Ω_c. In addition, we have plotted the bound given by (113). It can be seen that, if we do, in fact, restrict the region of validity to that given by (113), then the reflection coefficient will

---

*This variance is because of our solution method, not because the half space and slab cannot be compared. The 10 dB attenuation criteria is still met.

differ by no more than 1.5 dB from the exact solution.

Speculation has arisen over the reason why the reflection coefficient $\langle R' \rangle$ takes on negative values. *Hochstim and Martens* [1969] report that this is due to negative fluctuations which near $\Omega_p = 1$ decrease the mean reflection coefficient more than positive fluctuations are able to increase it. Additional speculation has attributed these negative values to the integral wavelength slab thickness chosen by Hochstim and Martens. We do however see in Figs. (6) and (7), that the same phenomenon is observed for the half-space, negating this last presumption.

It is our opinion that this negative value is due to the average additional path length traveled by a wave which has undergone two random events, compared to the wave which has been scattered by the mean distribution. By examining (90), we see that since $\langle R_1 \rangle = 0$, a double-scattering term $2\,Re\{R_0^* \langle R_2 \rangle\}$ is the lowest order term which can yield a negative value. Since this term is of the same order as the incoherent term, $\langle |R_1|^2 \rangle$, it is not surprising that their sum, $\langle R' \rangle$ takes on negative values for some region of parameter space.

The following *necessary* condition for the validity of a perturbation technique for the slab geometry has also

been derived by Jarem [*Hochstim and Martens*, 1970]

$$\zeta^2 << [(1-\Omega_p^2)^2 + \Omega_c^2]/\Omega_p^4. \qquad (116)$$

Such a *necessary* condition is of limited use, however, since expression (116) can only tell us that the technique *might* be good. As an example of the confusion such an expression raises, let us examine (116) for $\Omega_c<<1$ and $\Omega_p<<1$. Inequality (116) can be seen not to restrict the volume size in any manner as the plasma becomes nonattenuating, a case for which the perturbation solution obviously breaks down with a large enough volume.

6.1.3 Calculations

In Figs. (9)-(11), we have mapped the incoherent reflection coefficient as a function of plasma frequency, with collision frequency and correlation length as parameters. The plots are all similar in form, although there is a shift in the value of plasma frequency for maximum reflection (This value decreases to $\Omega_p=1$ as $\Omega_c\to 0$.). The shift may be related to the onset of surface

scattering* with the increasing opacity of the plasma, as the plasma frequency increases. A psuedo-surface is formed which transmits a decreasing amount of illumination as the plasma frequency is increased, because of absorption and the shielding of the inner plasma from illumination (and subsequent scattering).

For values of plasma frequency greater than that of maximum incoherent reflection, the incoherent reflection coefficient is relatively insensitive to changes in correlation length. Where this does occur, the skin depth† of the plasma is shorter than the correlation length, and the fluctuations in electron density are relatively uncorrelated (because of the large absorption over the distance of a correlation length). We thus expect a saturation with increasing correlation lengths. As can be seen from Figs. (9)-(11), at a given value of plasma frequency, departure from the saturation values of incoherent reflection coefficient first occurs for the

---

*Surface scattering is scattering primarily from a thin layer of plasma, with very little transmission through the layer.

†Skin depth is the distance to which a wave will travel before its amplitude decreases to $1/e$ of its original magnitude [*Corson and Lorrain*, 1962].

lowest value of correlation length, which is consistent with the above explanation.

6.2 *The Turbulent Transition and Symmetric Layers*

In Figs. (12) and (13), we give the incoherent transition layer reflection coefficient for two values of collision frequency (Ω_c = 1.0, 0.1) with the layer thickness S as a parameter. The correlation length is taken to be ka = 1.0. From Fig. (13) for Ω_c = 0.1, it can be seen that the effect of increasing the transition layer thickness is to decrease the reflection coefficient in the underdense regime, while shifting the plasma frequency of maximum reflection to higher values. Fig. (12) also displays the decrease in underdense reflection with increasing layer thickness, and from a comparison with Fig. (10), it can be assumed that at higher values of plasma frequency, the reflection coefficient also peaks and drops off for a collision frequency of Ω_c = 1.0. The decrease in incoherent reflection with increasing layer thickness can be attributed to a decrease in the overall electron density distribution as the layer thickness is increased [c.f., Fig. (1)]. The shift of peak reflection may be due to the incident wave propagating farther into the plasma (because of the reduced level of electron density at a given point), and effectively

seeing a larger volume of plasma before encountering surface scattering.

In examining the symmetric layer in Figs. (14) and (15), we see that the incoherent reflection coefficient increases with increasing layer thickness, both for the underdense and overdense plasma. This variation is consistent with the observation that the reflection coefficient should approach zero as $S \to 0$, for then the plasma distribution occurs at only one spatial point.

Although we have presented results only for one value of correlation length, we feel that Figs. (9)-(11) for the halfspace, may be used in conjunction with Figs. (12)-(15) to obtain a *feel* for the results in the more general case.

6.2.1 Validity of the Solution Method

In Section 6.1.2, we presented a validity condition for the plasma half-space. Since that case represents the largest inhomogeneity in electron density (the finite step), it seems reasonable that the restriction (113) should insure a valid transition layer solution when $S > 0$, although it may then represent a conservative estimate. For the symmetric layer, (113) represents an even more

conservative bound, for in the symmetric case, we are dealing with a volume which is *almost* finite.

7. CONCLUSIONS AND RECOMMENDATIONS

In this research paper, we have attempted to advance the field of turbulent scattering into the area of generalized plasma distributions. The price paid was the necessity of dealing with a Green's function which was not of the convolution form, and because of the complexity of this function, we restricted our analysis to the most obvious perturbation solution method. We have shown that even such a simple method applied to the problem of a randomly inhomogeneous medium yields results which must be computer evaluated for examination. Over the parameter ranges which we considered, we obtained no results which could not be plausibly explained on a physical basis, and where comparison with experiment was possible, it was much better than one might casually anticipate in applying a perturbation technique.

Because of the complexity of the lowest order perturbation solution, we do not recommend further investigation using such a procedure. Although additional computer calculations might be made, we believe that the section of parameter space for which we have presented calculations

includes those regions where unusual effects might be observed, *and* for which our solution is valid.

We do however, believe that the Epstein profile offers sufficient advantages in studying inhomogeneous and random media to demand further research effort. The relatively small parameter range allowed by our bounding inequality does not permit a valid solution for the useful case of a high altitude re-entry wake (in which $\Omega_c=0$). The smoothing equation discussed in Section 2.3 has been successfully applied to a number of collisionless plasmas, while a modification of the smoothing technique has shown exceptional promise in comparison with experimental data [*Jarem*, 1970]. Although these techniques are rather involved for even simple Green's functions (e.g., the plasma slab), we feel that a firm understanding of randomly inhomogeneous scattering processes dictates that one of these methods be applied to the Epstein profiles. As a possible start, investigation could be made into determining a transformation for the Epstein profile equations which would have benefits equivalent to those of the convolution forms in Fourier transformation.

Numerical techniques which have been developed to solve the lowest order smoothing equation [*Hassab*, 1971],

could be applied for special cases to the Epstein profile. Such a technique would directly make use of the Green's function obtained in Section 5.2.

An additional topic which could be usefully studied with the assistance of the Epstein profiles is surface scattering and its associated effects. Such an investigation would be valuable even *without* including the effects of turbulence. The choice of a proper *effective* surface for scattering formulations based on surface scattering [e.g., *Jarem*, 1964; *Wasneski*, 1968] has been carried out rather arbitrarily (One method employs a skin depth criteria [*Pergament et al.*, 1967].), and we thus recommend that a more comprehensive investigation of nonturbulent surface scattering be carried out using the Epstein profiles. A subsequent study might investigate turbulent effects.

Finally, we conclude that the primary value of this research paper lies in its action as a *probation* to determine the difficulty of the considered problem. We are hopeful that these results may give one an insight into the next appropriate approximation to apply (e.g., by considering the asymptotic functional dependencies), which might eventually produce a more general and efficient solution.

LIST OF REFERENCES

LIST OF REFERENCES

Abramowitz, M. and Stegun, I. A., eds., (1964) *Handbook of Mathematical Functions* (National Bureau of Standards, Washington) Chap. 6.

Beckmann, P. (1967), *Probability in Communication Engineering* (Harcourt, Brace & World, Inc., New York) Chap. 6.

Booker, H. G. and Gordon, W. E. (1950), "A Theory of Radio Scattering in the Troposphere," *Proc. IRE 38*, 401.

Bourret, R. C. (1962), "Propagation of Randomly Perturbed Fields," *Can. J. Phys. 40*, 782.

Brand, L. (1957), *Vector Analysis* (John Wiley & Sons, Inc., New York) Chap. 4.

Brekhovskikh, L. M. (1960), *Waves in Layered Media* (Academic Press, New York) Chap. 3.

Corson, D. R. and Lorrain, P. (1962), *Introduction to Electromagnetic Fields and Waves* (W. H. Freeman and Company, San Francisco) Chap. 10.

deWolf, D. A. (1968), *Wave Propagation and Multiple Scattering in a Random Continuum* (Grere Offset NV, Eindhoven).

Dicke, R. H. and Wittke, J. P. (1960), *Introduction to Quantum Mechanics* (Addison-Wesley Publishing Company, Inc., Reading, Mass.) Chap. 16.

Dwight, H. B. (1961), *Tables of Integrals and Other Mathematical Data* (The Macmillan Company, New York) Chap. 12.

Epstein, P. (1930), "Reflection of Waves in an Inhomogeneous Absorbing Medium," *Proc. Nat. Acad. Sci. U.S.A. 16*, 627.

Erdélyi, A., Magnus, W., Oberhettinger, F., and Tricomi, F. G. (1953), *Higher Transcendental Functions, Vol. I* (McGraw-Hill Book Company, Inc., New York) Chap. 2.

Friedman, B. (1956), *Principles and Techniques of Applied Mathematics* (John Wiley & Sons, Inc., New York) Chap. 3.

Frisch, V. (1968), "Wave Propagation in Random Media," *Probabilistic Methods in Applied Mathematics*, ed. Bharucha-Reid, A. T. (Academic Press, New York).

Gradshteyn, I. S. and Ryzhik, I. M. (1965), *Table of Integrals, Series, and Products* (Academic Press, New York) Chap. 3.

Green, C. D. (1969), *Integral Equation Methods* (Barnes & Noble, Inc., New York) Chaps. 2 and 4.

Guthart, H., Weissman, D. E., and Morita, T. (1966), "Microwave Scattering from an Underdense Turbulent Plasma," *Radio Sci. 1*, 1253.

Hassab, J. C. (1971), *Multiple Scattering by a Layer with Strong Small-Scale Fluctuations* (Ph.D. Thesis, Drexel University, Philadelphia).

Heald, M. H. and Wharton, C. B. (1965), *Plasma Diagnostics with Microwaves* (John Wiley & Sons Inc., New York) Chap. 1.

Hinze, J. O. (1959), *Turbulence* (McGraw-Hill Book Company, Inc., New York).

Hochstim, A. R. and Martens, C. P. (1969), "Electromagnetic Scattering from Underdense Random Plasma Slabs," *Turbulence of Fluids and Plasmas*, ed. Fox, J. (Polytechnic Press of the Polytechnic Institute of Brooklyn, New York) p. 187.

Hochstim, A. R. and Martens, C. P. (1970), "Electromagnetic Backscattering from a Random Plasma Slab," *Phys. Fluids 13*, 402.

Ince, E. L. (1956), *Ordinary Differential Equations* (reprinted Dover Publication, Inc., New York) Chap. 3.

Jarem, J. (1964), "Scattering from a Turbulent Overdense Surface," RCA Re-Entry Analysis Technical Monograph No. 64-08, AD 456 699.

Jarem, J. (1968), "Bistatic Doppler Information from Underdense Wakes" *Studies in Electromagnetic Scattering from Turbulent Wakes*, ed. Jarem, J. (Drexel University, Philadelphia) Rpt. A.

Jarem, J. (1969a), "Radar Reflectivity of Turbulent Rocket Exhaust Plumes - Derivation of Equations," Tech, Rpt. AFRPL-TR-67-251 (Air Force Rocket Propulsion Laboratory, Edwards, California).

Jarem, J., ed. (1969b), *Studies in Electromagnetic Scattering from Turbulent Wakes* (Drexel University, Philadelphia).

Jarem, J., ed. (1970), *Studies in Electromagnetic Scattering from Turbulent Wakes* (to be published, Drexel University, Philadelphia, Pa.).

Jones, D. S. (1964), *The Theory of Electromagnetism* (The Macmillan Company, New York).

Karal, F. C. and Keller, J. B. (1964), "Elastic, Electromagnetic, and Other Waves in a Random Medium," *J. Math. Phys.* 5, 537.

Kelso, J. M. (1964), *Radio Ray Propagation in the Ionosphere* (McGraw-Hill Book Company, Inc., New York) Chap. 4.

Kresa, K., ed. (1968), *Proceeding of the Technical Workshop on Radar Scattering from Random Media* (Advanced Research Projects Agency, Washington).

Kupiec, I., Felsen, L. B., Rosenbaum, S., Keller, J. B., and Chow, P. (1969), "Reflection and Transmission by a Random Medium," *Radio Sci.* 4, 1067.

Menkes, J. (1964), "Scattering of Radar Waves by an Underdense Turbulent Plasma," *AIAA J.* 2, 1154.

Morse, P. M. and Feshbach, H. (1953), *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York) Chap. 5.

Papas, C. H. (1965), *Theory of Electromagnetic Wave Propagation* (McGraw-Hill Book Company, Inc., New York) Chap. 6.

Pergament, H. S., Birnbaum, S., and Nikatarian, R. R. (1967), "Radar Reflectivity of Turbulent Rocket Exhaust Plumes" (U), Tech. Rpt. AFRPL-TR-67-251 (Air Force Rocket Propulsion Laboratory, Edwards, California) (CONFIDENTIAL).

Rainville, E. D. (1964), *Elementary Differential Equations*, 3rd ed. (The Macmillan Company, New York) Chap. 8.

Ramo, S., Whinnery, J. R., and Van Duzer, T. (1965), *Fields and Waves in Communication Electronics* (John Wiley & Sons, Inc., New York).

Rosenbaum, S. (1969), "On the Coherent Wave Motion in Bounded, Randomly Fluctuating Regions," *Radio Sci. 4*, 709.

Ruffine, R. S. and deWolf, D. A. (1965), "Cross-Polarized Electromagnetic Backscatter from Turbulent Plasmas," *J. Geophys. Res. 70*, 4313.

Salpeter, E. E. and Treiman, S. B. (1964), "Backscatter of Electromagnetic Radiation from a Turbulent Plasma," *J. Geophys. Res. 69*, 869.

Stratton, J. A. (1941), *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York).

Tatarskii, V. I. (1961), *Wave Propagation in a Turbulent Medium* (McGraw-Hill Book Company, Inc., New York).

Tatarskii, V. I. and Gertsenshtein, M. E. (1963), "Propagation of Waves in a Medium with Strong Fluctuation of the Refractive Index," *Sov. Phys. JETP 17*, 458.

Wasneski, R. J. (1968), *Scattering from a Turbulent Cylindrical Overdense Surface* (M.S. Thesis, Drexel University, Philadelphia).

Watson, K. M. (1969), "Multiple Scattering of Electromagnetic Waves in an Underdense Plasma," *J. Math. Phys. 10*, 688.

Whittaker, E. T. and Watson, G. N. (1965), *A Course of Modern Analysis*, 4th ed. (reprinted Cambridge at the University Press) Chap. 14.

APPENDICES

APPENDIX A

REPRESENTATION OF THE HYPERGEOMETRIC FUNCTION

Let us examine the integral [*Feshbach*, 1953]

$$I = (2\pi i)^{-1} \oint \frac{\Gamma(a+t)\Gamma(b+t)}{\Gamma(c+t)} \Gamma(-t \quad (A1)$$

where the path of integration is chosen [illegible] one leg of the integration path extends over the imaginary axis $(-i\infty, i\infty)$ and the poles of the functions $\Gamma(a+t)$ and $\Gamma($ [illegible] to the left of the path of integration and the poles [illegible] function $\Gamma(-t)$ lie to the right of the path. To [illegible] the remainder of the integration path let us [illegible] the integrand for $|t| \to \infty$.

From *Abromowitz and Stegun* [1[illegible] asymptotic form of the gamma function is given by

$$\Gamma(t+1) \xrightarrow[t\to\infty]{} (2\pi)^{\frac{1}{2}} e^{-t-1} t^{t+\frac{1}{2}} \quad (A2)$$

Using the relation [*Dwight*, 1961]

$$\Gamma(a+1) = a\Gamma(a), \tag{A3}$$

then

$$\Gamma(a+t) = (a+t)^{-1}\Gamma(a+t+1) \tag{A4}$$

and

$$\Gamma(a+t) \xrightarrow[t\to\infty]{} (2\pi)^{\frac{1}{2}} e^{-(a+t+1)} t^{a+t-\frac{1}{2}}. \tag{A5}$$

Let

$$t = Re^{i\theta};\ z = re^{i\phi}, \tag{A6}$$

then

$$\frac{\Gamma(a+t)\Gamma(b+t)}{\Gamma(c+t)\Gamma(1+t)} \longrightarrow e^{1+c-a-b} R^{a+b-c-1} e^{i(a+b-c-1)\theta}, \tag{A7}$$

and

$$(-z)^{t} = \exp\{R[\cos\theta \ln r-(\phi-\pi)\sin\theta] + iR[\sin\theta\ln r + (\phi-\pi)\cos\theta]\}, \tag{A8}$$

$$(\sin\pi t)^{-1} = \sin\pi t^{*}|\sin\pi t|^{-2} = (2i)^{-1}\frac{e^{\pi R\sin\theta}e^{i\pi R\cos\theta} + e^{-\pi R\sin\theta}e^{-i\pi R\cos\theta}}{\sin^2(\pi R\cos\theta)+\sinh^2(\pi R\sin\theta)}. \tag{A9}$$

By letting $R\to\infty$, then

$$(\sin\pi t)^{-1}\longrightarrow\begin{cases} -2i\, e^{-\pi R\sin\theta}e^{i\pi R\cos\theta} & \pi>\theta>0 \\ -2i\, e^{\pi R\sin\theta}e^{-i\pi R\cos\theta} & -\pi<\theta<0. \end{cases} \tag{A10}$$

Thus, denoting the integrand of (A1) by I_g

$$|I_g| \to 2(R/e)^{a+b-c-1}e^{R\ln r\cos\theta}\begin{cases} e^{-R\phi\sin\theta} & 0<\theta<\pi \\ e^{(2\pi-\phi)R\sin\theta} & -\pi<\theta<0. \end{cases} \tag{A11}$$

Therefore, for $\ln r < 0$ ($|z| < 1$), and $-\pi/2 < \theta < \pi/2$, closure of the integral in (A1) in the right half plane does not contribute to the value of the integral. Thus

$$(2\pi i)^{-1} \oint I_g = (2\pi i)^{-1} \int_{-i\infty}^{i\infty} I_g, \tag{A12}$$

and evaluating the integral of (A1) by summing the residues at the poles enclosed by the integration path of (A1), we obtain

$$I = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(1+n)} (-1)^n (-z)^n. \tag{A13}$$

By writing out several terms of the series in (A13), we can recognize that the term within the brackets is just the hypergeometric function,

$$I = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} \{1 + \frac{ab}{c \cdot 1} z + \frac{a(a+1)b(b+1)}{c(c+1)\ 2 \cdot 1} z^2 + \cdots\}, \tag{A14}$$

or we then have the representation

$$I = [\Gamma(a)\Gamma(b)/\Gamma(c)]F(a,b;c;z). \tag{A15}$$

Similarly, closure takes place in the left hand plane when $|z|>1$. In this case evaluating the integral by residues gives the following result

$$\begin{aligned} I = {} & \frac{\Gamma(b-a)\Gamma(a)}{\Gamma(c-a)}\,(-z)^{-a}F(a,c-a-1;b-a-1;1/z) \\ & + \frac{\Gamma(a-b)\Gamma(b)}{\Gamma(c-b)}\,(-z)^{-b}F(b,c-b-1;a-b-1;1/z). \end{aligned} \tag{A16}$$

The two series, (A15) and (A16) are then representations in different regions of the integral function in (A1). Equivalently, we may consider the two series of (A15) and (A16) to be the analytic continuations of one another, and we may write the formula for this analytic continuation as

$$\begin{aligned} F(a,b;c;z) = {} & \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}\,(-z)^{-a}F(a,c-a-1;b-a-1;1/z) \\ & + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}\,(-z)^{-b}F(b,c-b-1;a-b-1;1/z) \end{aligned} \tag{A17}$$

$$\int_0^{-\infty} \frac{(\xi')^{b/m}}{(1-\xi')^{\delta}} z_{01}^{\dagger}(\xi') z_{02}^{\dagger}(\xi') d\xi'$$

$$\times \int_{\xi'}^{-\infty} \frac{\xi^{-b/m}}{(1-\xi)^{\delta}} z_{01}(\xi) z_{02}(\xi) d\xi \Bigg\}.$$

The homogeneous solutions z_{01}, z_{02} are given in (53). Because of the products of these solutions occur in the integrals of (B3), the number of individual terms is increased considerably above the six of (53). Hence, only a generic term will be evaluated for each of the six terms in (B3), and will serve as an example of the integration technique. These six terms are given by

$$I_1 = \int_0^{-\infty} (\xi')^{-b/m} (1-\xi')^{A^*} F^*(\alpha,\beta;\gamma;\xi') F^*(\alpha-\gamma+1;\beta-\gamma+1;2-\gamma;\xi') d\xi'$$

$$\times \int_0^{\xi'} \xi^{b/m} (1-\xi)^{A} F(\alpha,\beta;\gamma;\xi) F(\alpha-\gamma+1,\beta-\gamma+1;2-\gamma;\xi) d\xi$$

$$I_2 = \int_0^{-\infty} (\xi')^{b/m} (1-\xi')^{A^*} F^*(\alpha,\beta;\gamma;\xi') F^*(\alpha-\gamma+1,\beta-\gamma+1;2-\gamma;\xi') d\xi'$$

$$\times \int_{\xi'}^{-1} \xi^{-b/m} (1-\xi)^{A} F(\alpha,\beta;\gamma;\xi) F(\alpha-\gamma+1,\beta-\gamma+1;2-\gamma;\xi) d\xi$$

APPENDIX B

EVALUATION OF THE SCATTERING INTEGRALS

In this appendix, we shall evaluate a typical term which occurs in the expressions for the reflection and transmission coefficients, namely

$$\int_{-\infty}^{\infty} g(x')Z_{01}^{*}(x')Z_{02}^{*}(x')dx' \int_{-\infty}^{\infty} e^{-b|x-x'|}g(x)Z_{01}(x)Z_{02}(x)dx. \quad \text{(B1)}$$

This term must be broken down into the six terms given in (102). In that equation, we make the substitution

$$\xi = -e^{mx}, \quad \text{(B2)}$$

so that (B1) becomes

$$m^{-2}\left\{\int_{0}^{-\infty} \frac{(\xi')^{-b/m}}{(1-\xi')^{\delta}} Z_{01}^{*}(\xi')Z_{02}^{*}(\xi')d\xi' \right.$$

$$\left. \times \int_{0}^{\xi} \frac{\xi^{b/m}}{(1-\xi)^{\delta}} Z_{01}(\xi)Z_{02}(\xi)d\xi \right.$$

(B3)

$$I_3 = \int_0^{-\infty} (\xi')^{b/m}(1-\xi')^{A^*} F^*(\alpha,\beta;\gamma;\xi')F^*(\alpha-\gamma+1,\beta-\gamma+1;2-\gamma;\xi')d\xi'$$

$$\times \int_{-1}^{-\infty} \xi^{\gamma-2\alpha-1-b/m}(1-\xi)^{A} F(\alpha,\alpha-\gamma+1;\alpha-\beta+1;1/\xi) \qquad \text{(B4)}$$

$$\times F(\alpha,\alpha-\gamma+1;\alpha-\beta+1;1/\xi)d\xi$$

$$I_4 = \int_0^{\infty} (\xi')^{(\gamma-2\alpha-1-b/m)^*}(1-\xi')^{A^*} F^*(\alpha,\alpha-\gamma+1,\alpha-\beta+1;1/\xi')$$

$$\times F^*(\alpha,\alpha-\gamma+1;\alpha-\beta+1;1/\xi')d\xi'$$

$$\times \int_0^{-1} \xi^{b/m}(1-\xi)^{A} F(\alpha,\beta;\gamma;\xi)F(\alpha-\gamma+1,\beta-\gamma+1;2-\gamma;\xi)d\xi$$

$$I_5 = \int_0^{\infty} (\xi')^{(\gamma-2\alpha-1-b/m)^*}(1-\xi')^{A^*} F^*(\alpha,\alpha-\gamma+1;\alpha-\beta+1;1/\xi')$$

$$F^*(\alpha,\alpha-\gamma+1;\alpha-\beta+1;1/\xi')d\xi'$$

$$\times \int_{-1}^{\xi'} \xi^{\gamma-2\alpha-1+b/m}(1-\xi)^{A} F(\alpha,\alpha-\gamma+1;\alpha-\beta+1;1/\xi)$$

$$\times F(\alpha,\alpha-\gamma+1;\alpha-\beta+1;1/\xi)d\xi$$

$$I_6 = \int_0^{\infty} (\xi')^{(\gamma-2\alpha-1+b/m)^*}(1-\xi')^{A^*} F^*(\alpha,\alpha-\gamma+1;\alpha-\beta+1;1/\xi')$$

$$\times F^*(\alpha,\alpha-\gamma+1;\alpha-\beta+1;1/\xi')d\xi'$$

$$\times \int_{\xi'}^{-\infty} \xi^{\gamma-2\alpha-1-b/m}(1-\xi)^{A} F(\alpha,\alpha-\gamma+1;\alpha-\beta+1;1/\xi)$$

$$\times F(\alpha,\alpha-\gamma+1;\alpha-\beta+1;1/\xi)d\xi.$$

The first integration (over ξ) can be categorized into two types:

$$\text{Type I} \quad \int_{\xi'}^{0} ; \int_{-1}^{0} ; \int_{-1}^{\xi'} \xi^{B}(1-\xi)^{A} F(\alpha,\beta;\gamma;\xi) F(a,b;c;\xi)\, d\xi$$

$$\text{Type II} \quad \int_{-\infty}^{-1} ; \int_{\xi'}^{-1} ; \int_{-\infty}^{\xi'} \xi^{B}(1-\xi)^{A} F(\alpha,\beta;\gamma;1/\xi) F(a,b;c;1/\xi)\, d\xi . \tag{B5}$$

B.1 *THE TYPE I INTEGRAL*

From *Erdélyi* [1953],

$$F(a,b;c;z) = (1-z)^{-b} F(b,c-a;c;[-z/(1-z)]). \tag{B6}$$

The integral representation for the hypergeometric function discussed in *Appendix A* will be used to represent the two hypergeometric functions occurring in (B5), i.e.,

$$F(a,b;c;z) = (2\pi i)^{-1} \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \times \int_{-i\infty}^{i\infty} \frac{\Gamma(a+t)\Gamma(b+t)}{\Gamma(c+t)} \Gamma(-t)(-z)^{t} dt, \tag{B7}$$

where the contour of integration is taken so that one leg of the integration path extends over the interval $(-i\infty, i\infty)$ and the poles of the functions $\Gamma(a+t)$ and $\Gamma(b+t)$ lie to the left of the path of integration and the poles of the function $\Gamma(-t)$ lie to the right of the path. Employing a liberal exchange of integration order, we may write (B5) as

$$I = K(2\pi i)^{-2} \int_{-i\infty}^{i\infty} \frac{\Gamma(b+t')\Gamma(c-a+t')}{\Gamma(c+t')} \Gamma(-t')dt' \times \int_{-i\infty}^{i\infty} \frac{\Gamma(\beta+t)\Gamma(\gamma-a+t)}{\Gamma(\gamma+t)} \Gamma(-t)dt \times \int \xi^{B+t+t'}(1-\xi)^{A-\beta-b-t-t'}d\xi. \tag{B8}$$

The integration over ξ is then given by the integral

$$\underline{I} = \int \xi^{B+t+t'}(1-\xi)^{A-\beta-b-t-t'}d\xi. \tag{B9}$$

From *Gradshtyen and Ryzhik* [1965],

$$\int_u^0 x^{\mu-1}(1-x)^{-\nu}dx = -u^{\mu}\mu^{-1}F(\nu,\mu;1+\mu;u), \tag{B10}$$

and by using (B6), this becomes

$$\int_{w}^{0} x^{\mu-1}(1-x)^{-\nu}dx = -\mu^{-1}[w/(1-w)]^{\mu} \times F(\mu,1+\mu-\nu,1+\mu,[-w/(1-w)]). \tag{B11}$$

The integral over the definite limits $\int_{-1}^{0}$ is just a special case of (B10), while the integral over the limits $\int_{-1}^{\xi}$ can be broken into two parts as follows

$$\int_{-1}^{\xi} = \int_{-1}^{0} - \int_{\xi}^{0}. \tag{B12}$$

All three integrals of *Type I* can thus be treated in the same manner. Our *Type I* integral thus reduces to the following double integral

$$I_{\xi}^{0} = -K(2\pi i)^{-2}[\xi'/(1-\xi')]^{B+1} \times \int_{-i\infty}^{i\infty} \frac{\Gamma(b+t')\Gamma(c-a+t')}{\Gamma(c+t')}\Gamma(-t')[\xi'/(1-\xi')]^{t'}dt' \times \int_{-i\infty}^{i\infty} \frac{\Gamma(\beta+t)\Gamma(\gamma-\alpha+t)}{\Gamma(\gamma+t)}\Gamma(-t)[\xi'/(1-\xi')]^{t} \times (B+1+t+t')^{-1}F(B+1+t+t',A+B+2-b-\beta;B+2+t+t';[-\xi'/(1-\xi')])dt. \tag{B13}$$

Evaluation of these integrations can be carried out in the manner of *Appendix A*. Following the first integration, we have

$$I^0_\xi = -K(2\pi i)^{-1}[\xi'/(1-\xi')]^{B+1} \times \int_{-i\infty}^{i\infty} \frac{\Gamma(b+t')\Gamma(c-a+t')}{\Gamma(c+t')} \Gamma(-t')[\xi'/(1-\xi')]^{t'} \times \sum_{l=0}^{\infty} \frac{\Gamma(\beta+l)\Gamma(\gamma-a+l)}{(\gamma+l)} [-\xi'/(1-\xi']^l \qquad \text{(B14)}$$

$$\times (B+1+t'+l)^{-1}F(B+1+t'+l, A+B+2-b-\beta; B+2+t'+l; [-\xi/(1-\xi)])dt .$$

By integrating again after freely interchanging operation orders, we have the result

$$I^0_\xi = -K[\xi'/(1-\xi')]^{B+1} \times \sum_{l=0}^{\infty} \frac{\Gamma(\beta+l)\Gamma(\gamma-a+l)}{\Gamma(\gamma+l)\Gamma(1+l)} [-\xi/(1-\xi)]^l \times \sum_{n=0}^{\infty} \frac{\Gamma(b+n)\Gamma(c-a+n)}{\Gamma(c+n)\Gamma(1+n)} [-\xi/(1-\xi)]^n . \qquad \text{(B15)}$$

B.2 *THE TYPE II INTEGRAL*

Using the formula of (B6), we may write

$$F(a,b;c;1/\xi) = [-\xi/(1-\xi)]F[b,c-a;c;(1-\xi)^{-1}]. \qquad \text{(B16)}$$

Again using the integral representation of the hypergeometric function, and following a change in integration order, the *Type II* integral becomes

$$J = (-1)^{b+\beta} K (2\pi i)^{-2} \int_{-i\infty}^{i\infty} \frac{\Gamma(b+t')\Gamma(c-a+t')}{\Gamma(c+t')} \Gamma(-t')(-1)^{t'} dt'$$

$$\times \int_{-i\infty}^{i\infty} \frac{\Gamma(\beta+t)\Gamma(\gamma-a+t)}{\Gamma(\gamma+t)} \Gamma(-t)(-1)^{t} dt \qquad \text{(B17)}$$

$$\times \int \xi^{B+b+\beta} (1-\xi)^{A-b-\beta-t-t'} d\xi .$$

To integrate over ξ, it is necessary to transform the integral of (B10).

Let

$$x = -1/t, \qquad \text{(B18)}$$

and we obtain

$$\int_{-\infty}^{-1/u} t^{\nu-\mu-1}(1-t)^{-\nu} dt = (-1)^{1-\nu-\mu} u^{\mu} \mu^{-1} F(\nu,\mu;1+\mu;-u). \qquad \text{(B19)}$$

Now let

$$S = -1/u, \tag{B20}$$

to obtain

$$\int_{-\infty}^{S} t^{\nu-\mu-1}(1-t)^{-\nu}dt = (-1)^{1-\nu}S^{-\mu}\mu^{-1}F(\nu,\mu;1+\mu;1/S). \tag{B21}$$

Again using (B6), we obtain the result

$$\int_{-\infty}^{S} t^{\nu-\mu-1}(1-t)^{-\nu}dt = (-1)^{1+\mu-\nu}\mu^{-1}(1-S)^{-\mu} \times F[\mu,1+\mu-\nu;1+\mu;(1-S)^{-1}]. \tag{B22}$$

The remainder of the integration proceeds as in the *Type I* evaluation with the result

$$J_{-\infty}^{\xi'} = (-1)^{-B}(1-\xi')^{A+B+1} \sum_{l=0}^{\infty} \frac{(\beta)_l(\gamma-\alpha)_l}{(\gamma)_l(1)_l}(1-\xi')^{-l} \times \sum_{n=0}^{\infty} \frac{(b)_n(c-a)_n}{(c)_n(1)_n}(1-\xi')^{-n} \times (l+n-A-B-1)^{-1}F[l+n-A-B-1,-B-b-\beta;l+n-A-B;(1-\xi)^{-1}]. \tag{B23}$$

The integral $\int_{-\infty}^{-1}$ can be broken up into the two parts

$$\int_{\xi'}^{-1} = \int_{-\infty}^{-1} - \int_{-\infty}^{\xi'} . \tag{B24}$$

The integration which remains (over ξ') can now be evaluated exactly as has been done for the ξ integration. The results are as follows

$$I = (-1)^{-(B+B^*)}(2)^{-(B+B^*+2)} \sum_{l=0}^{\infty} \frac{(\beta)_l(\gamma-\alpha)_l}{(\gamma)_l(1)_l} 2^{-l} \sum_{n=0}^{\infty} \frac{(b)_n(\alpha-a)_n}{(\alpha)_n(1)_n} 2^{-n}$$

$$\times \sum_{k=0}^{\infty} \frac{(A+B+2-b-\beta)_k}{(B+1+l+n+k)(1)_k} 2^{-k} \sum_{i=0}^{\infty} \frac{(\beta^*)_i(\gamma^*-\alpha^*)_i}{(\gamma^*)_i(1)_i} 2^{-i} \tag{B25}$$

$$\times \sum_{j=0}^{\infty} \frac{(b^*)_j(\alpha^*-a^*)_j}{(\alpha^*)_j(1)_j} 2^{-j} \sum_{m=0}^{\infty} \frac{(A^*+B^*+2-b^*-\beta^*)_m}{(B+B^*+2+l+n+k+i+j+m)(1)_m} 2^{-m}$$

$$J = (-1)^{-(B+B^*)} 2^{A+A^*+B+B^*+2} \sum_{l=0}^{\infty} \frac{(\beta)_l(\gamma-\alpha)_l}{(\gamma)_l(1)_l} 2^{-l} \sum_{n=0}^{\infty} \frac{(b)_n(\alpha-a)_n}{(\alpha)_n(1)_n} 2^{-n}$$

$$\times \sum_{k=0}^{\infty} \frac{[-(B+b+\beta)]_k}{(l+n+k-A-B-1)(1)_k} 2^{-k} \sum_{i=0}^{\infty} \frac{(\beta^*)_i(\gamma^*-\alpha^*)_i}{(\gamma^*)_i(1)_i} 2^{-i} \tag{B26}$$

$$\times \sum_{j=0}^{\infty} \frac{(b^*)_j(\alpha^*-a^*)_j}{(\alpha^*)_j(1)_j} 2^{-j} \sum_{m=0}^{\infty} \frac{[-(B^*+b^*+\beta^*)]_m}{(i+j+l+n+k+m-A^*-A-B^*-B-2)(1)_m} 2^{-m} .$$

In order to make the expression for the scattering coefficients as compact as possible, the results of these integrations have been defined in terms of certain functions as follows

$$G_1(\alpha,\beta,\gamma,a,b,c,B) = (-1)^{-B}2^{-(B+2)} \sum_{l=0}^{\infty} \frac{(\beta)_l(\gamma-\alpha)_l}{(\gamma)_l(1)_l} 2^{-l}$$

$$\times \sum_{n=0}^{\infty} \frac{(b)_n(c-a)_n}{(c)_n(1)_n} 2^{-n} \sum_{k=0}^{\infty} \frac{(B+A+2-b-\beta)_k}{(B+1+l+n+k)(1)_k} 2^{-k} \tag{B27}$$

$$G_2(\alpha,\beta,\gamma,a,b,c,B) = (-1)^{-B}2^{A+B+2} \sum_{l=0}^{\infty} \frac{(\beta)_l(\gamma-\alpha)_l}{(\gamma)_l(1)_l} 2^{-l}$$

$$\times \sum_{n=0}^{\infty} \frac{(b)_n(c-a)_n}{(c)_n(1)_n} 2^{-n} \sum_{k=0}^{\infty} \frac{[-(B+b+\beta)]_k}{(l+n+k-A-B-1)(1)_k} 2^{-k} \tag{B28}$$

$$H_1[\alpha,\beta,\gamma,a,b,c,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{a},\bar{b},\bar{c},B,\bar{B}] = (-1)^{-(B+\bar{B})}2^{-(B+\bar{B}+2)}$$

$$\times \sum_{l=0}^{\infty} \frac{(\beta)_l(\gamma-\alpha)_l}{(\gamma)_l(1)_l} 2^{-l} \sum_{n=0}^{\infty} \frac{(b)_n(c-a)_n}{(c)_n(1)_n} 2^{-n}$$

$$\times \sum_{k=0}^{\infty} \frac{(B+A+2-b-\beta)_k}{(B+1+l+n+k)(1)_k} 2^{-k} \tag{B29}$$

$$\times \sum_{i=0}^{\infty} \frac{(\bar{\beta}^*)_i(\bar{\gamma}^*-\bar{\alpha}^*)_i}{(\bar{\gamma}^*)_i(1)_i} 2^{-i} \sum_{j=0}^{\infty} \frac{(\bar{b}^*)_j(\bar{c}^*-\bar{a}^*)_j}{(\bar{c}^*)_j(1)_j} 2^{-j}$$

$$\times \sum_{m=0}^{\infty} \frac{(\bar{A}^*+\bar{B}^*+2-\bar{b}^*-\bar{\beta}^*)_m}{(B+\bar{B}^*+2+l+n+k+i+j+m)(1)_m} 2^{-m}$$

$$F_2[\alpha,\beta,\gamma,a,b,c,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{a},\bar{b},\bar{c},B,\bar{B}] = (-1)^{-(B+\bar{B})}2^{A+\bar{A}+B+\bar{B}+2}$$

$$\times \sum_{l=0}^{\infty} \frac{(\beta)_l(\gamma-a)_l}{(\gamma)_l(1)_l}\,2^{-l} \sum_{n=0}^{\infty} \frac{(b)_n(c-a)_n}{(c)_n(1)_n}\,2^{-n}$$

$$\times \sum_{k=0}^{\infty} \frac{[-(B+b+\beta)]_k}{(l+n+k-A-B-1)(1)_k}\,2^{-k} \qquad \text{(B30)}$$

$$\times \sum_{i=0}^{\infty} \frac{(\bar{\beta}^*)_i(\bar{\gamma}^*-\bar{\alpha}^*)_i}{(\bar{\gamma}^*)_i(1)_i}\,2^{-i} \sum_{j=0}^{\infty} \frac{(\bar{b}^*)_j(\bar{c}^*-\bar{a}^*)_j}{(\bar{c}^*)_j(1)_j}\,2^{-j}$$

$$\times \sum_{m=0}^{\infty} \frac{[-(\bar{B}^*+\bar{b}^*+\bar{\beta}^*)]_m}{(i+j+k+l+m+n-\bar{A}^*-A-\bar{B}^*-B-2)(1)_m}\,2^{-m}$$

$$
\begin{aligned}
&Q\{a,b,c,d,e,f,\bar{a},\bar{b},\bar{c},\bar{d},\bar{e},\bar{f},U_1,U_2,U_3,U_4\}\\
&= H_1\{a,b,c,d,e,f,(\bar{a},\bar{b},\bar{c},\bar{d},\bar{e},\bar{f})^*,d-\alpha+\gamma-1+b/m,(d-\alpha+\gamma-1-b/m)^*\}\\
&+ G_1[a,b,c,d,e,f,d-\alpha+\gamma-1]G_1^*[\bar{a},\bar{b},\bar{c},\bar{d},\bar{e},\bar{f},d-\alpha+\gamma-1+b/m]\\
&- H_1[a,b,c,d,e,f,(\bar{a},\bar{b},\bar{c},\bar{d},\bar{e},\bar{f})^*,(d-\alpha+\gamma-1-b/m)^*,d-\alpha+\gamma-1+b/m]\\
&+ 2\mathrm{Re}\,[U_1U_3(-1)^{d-\alpha}G_2(1,4,5,1,4,5,B3)\\
&\qquad + U_1U_4(-1)^{-\beta-d}G_2(1,4,5,2,6,7,B4)\\
&\qquad + U_2U_3(-1)^{-\beta-d}G_2(2,6,7,1,4,5,B4)\\
&\qquad + U_2U_4(-1)^{\alpha-d-2\beta}G_2(2,6,7,2,6,7,B5)]\\
&\qquad\qquad \times G_1^*(\bar{a},\bar{b},\bar{c},\bar{d},\bar{e},\bar{f},d-\alpha+\gamma-1+b/m)\}\\
&+ \{U_1U_3(-1)^{-d-\alpha}G_2(1,4,5,1,4,5,B6)\\
&\quad + U_1U_4(-1)^{-d-\beta}G_2(1,4,5,2,6,7,B7) \qquad\qquad (B31)\\
&\quad + U_2U_3(-1)^{-d-\beta}G_2(2,6,7,1,4,5,B7)\\
&\quad + U_2U_4(-1)^{-d-\alpha-2\beta}G_2(2,6,7,2,6,7,B8)\}\\
&\qquad\qquad \times \{U_1U_3(-1)^{-\alpha-\bar{d}}G_2(1,4,5,1,4,5,B3)\\
&\qquad\qquad + U_1U_4(-1)^{-\beta-\bar{d}}G_2(1,4,5,2,6,7,B4)\\
&\qquad\qquad + U_2U_3(-1)^{-\beta-\bar{d}}G_2(2,6,7,1,4,5,B4)\\
&\qquad\qquad + U_2U_4(-1)^{-\alpha-\bar{d}-2\beta}G_2(2,6,7,2,6,7,B5)\}^*\\
&- \{|U_1|^2|U_3|^2(-1)^{-2\mathrm{Re}(d+\alpha)}\\
&\qquad\qquad \times H_2[1,4,5,1,4,5,(1,4,5,1,4,5)^*,B6,B3^*]\\
&+ |U_1|^2U_3U_4^*(-1)^{-d-d^*-\alpha-\beta^*}\\
&\qquad\qquad \times H_2[1,4,5,1,4,5,(1,4,5,2,6,7)^*,B6,B4^*]\\
&+ U_1U_2^*|U_3|^2(-1)^{-d-d^*-\alpha-\beta^*}\\
&\qquad\qquad \times H_2[1,4,5,1,4,5,(2,6,7,2,6,7)^*,B6,B4^*]\\
&+ U_1U_3U_2^*U_4^*(-1)^{-2\beta^*-2\mathrm{Re}(\alpha+d)}\\
&\qquad\qquad \times H_2[1,4,5,1,4,5,(2,6,7,2,6,7)^*,B6,B5^*]
\end{aligned}
$$

$$+ |U_1|^2 U_4 U_3^* (-1)^{-\alpha^*-\beta-d-d^*}$$

$$\times H_2[1,4,5,2,6,7,(1,4,5,1,4,5)^*,B7,B3^*]$$

$$+ |U_1|^2 |U_4|^2 (-1)^{-2\mathrm{Re}(\beta+d)}$$

$$\times H_2[1,4,5,2,6,7,(1,4,5,2,6,7)^*,B7,B4^*]$$

$$+ U_1 U_4 U_2^* U_3^* (-1)^{-2\mathrm{Re}(\beta+d)}$$

$$\times H_2[1,4,5,2,6,7,(2,6,7,1,4,5)^*,B7,B4^*]$$

$$+ U_1 U_2^* |U_4|^2 (-1)^{\alpha^*-d-d^*-\beta-2\beta^*}$$

$$\times H_2[1,4,5,2,6,7,(2,6,7,2,6,7)^*,B7,B5^*]$$

$$+ U_2 U_1^* |U_3|^2 (-1)^{-\alpha^*-\beta-d-d^*}$$

$$\times H_2[2,6,7,1,4,5,(1,4,5,1,4,5)^*,B7,B3^*]$$

$$+ U_2 U_3 U_1^* U_4^* (-1)^{-2\mathrm{Re}(d+\beta)}$$

$$\times H_2[2,6,7,1,4,5,(1,4,5,2,6,7)^*,B7,B4^*]$$

$$+ |U_2|^2 |U_3|^2 (-1)^{-2\mathrm{Re}(d+\beta)}$$

$$\times H_2[2,6,7,1,4,5,(2,6,7,1,4,5)^*,B7,B4^*]$$

$$+ |U_2|^2 U_3 U_4^* (-1)^{-d-d^*+\alpha^*-\beta-2\beta^*}$$

$$\times H_2[2,6,7,1,4,5,(2,6,7,2,6,7)^*,B7,B5^*]$$

$$+ U_2 U_4 U_1^* U_3^* (-1)^{\alpha-\alpha^*-d-d^*-2\beta}$$

$$\times H_2[2,6,7,2,6,7,(1,4,5,1,4,5)^*,B8,B3^*]$$

$$+ U_2 U_1^* |U_4|^2 (-1)^{\alpha-d-d^*-2\beta-\beta^*}$$

$$\times H_2[2,6,7,2,6,7,(1,4,5,2,6,7)^*,B8,B4^*]$$

$$+ |U_2|^2 U_4 U_3^* (-1)^{\alpha-d-d^*-2\beta-\beta^*}$$

$$\times H_2[2,6,7,2,6,7,(2,6,7,1,4,5)^*,B8,B4^*]$$

$$+ |U_2|^2 |U_4|^2 (-1)^{2\mathrm{Re}(\alpha-d-\beta)}$$

$$\times H_2[2,6,7,2,6,7,(2,6,7,2,6,7)^*,B8,B5^*]\}$$

$$+ \left\{ \begin{array}{l} B6 \leftrightarrow B3 \\ B7 \leftrightarrow B4 \\ B8 \leftrightarrow B5 \end{array} \right\} ,$$

and with the following abbreviated notation

$$B1 = \gamma - 1 + b/m \qquad 1 = \alpha$$

$$B2 = \gamma - 1 - b/m \qquad 2 = \beta$$

$$B3 = \gamma - 1 - 2\alpha - b/m \qquad 3 = \gamma$$

$$B4 = \gamma - 1 - \alpha - \beta - b/m \qquad 4 = 1 + \alpha - \gamma$$

$$B5 = \gamma - 1 - 2\beta - b/m \qquad 5 = 1 + \alpha - \beta$$

$$B6 = \gamma - 1 - 2\alpha + b/m \qquad 6 = 1 + \beta - \gamma$$

$$B7 = \gamma - 1 - \alpha - \beta + b/m \qquad 7 = 1 + \beta - \alpha$$

$$B8 = \gamma - 1 - 2\beta + b/m \qquad 8 = 2 - \gamma .$$

$$B9 = b/m$$

$$B10 = -b/m$$

$$B11 = 1 - \gamma + b/m$$

$$B12 = 1 - \gamma - b/m$$

The symbol {↔} represents an interchange in the immediately preceding braces.

APPENDIX C

SCATTERING COEFFICIENTS OF THE TURBULENT EPSTEIN LAYER

C.1 *THE TURBULENT HALFSPACE*

From the theory in Section 6.1.3, the incoherent and coherent reflection coefficients are given by

$$W_I = \zeta^2 4\Omega_p^4 (1+\Omega_c^2)^{-1} [1+2\alpha'+(\alpha'^2+\beta'^2)]^{-2}$$
$$\times [(k_0 a)^{-4} + 8(\alpha'^2-\beta'^2)/(k_0 a)^2 + 16(\alpha'^2+\beta'^2)]$$
$$\times \{[(1/2\beta' k_0 a)-1]/(k_0 a)$$
$$+ 4(\alpha'^2+\beta'^2) + 2(\alpha'^2-\beta'^2)/\beta' k_0 a\} \qquad \text{(C1)}$$

$$W_c = |(1-X)/(1+X)|^2$$
$$+ 2\mathrm{Re}\left\{\zeta^2 [(X^*-1)/(X^*+1)][\Omega_p^2(1-i\Omega_c)/(1+\Omega_c^2)]^2\right.$$
$$\times [2/(1+X)]^2 (4X)^{-1} [(2X)^2+(k_0 a)^{-2}]^{-1}$$
$$\times \{[(X-1)/(X+1)][i/2k_0 aX-1] \qquad \text{(C2)}$$
$$\left. -i2k_0 aX-1 + i(2X)^{-1}(4X^2 k_0 a + 2/k_0 a + i2X)\right\}$$

where

$$X = +[1-N]^{\frac{1}{2}}$$
$$\alpha' = \mathrm{Re}\{1-N\} \qquad \text{(C3)}$$
$$\beta' = \mathrm{Im}\{1-N\}$$

C.2 *THE TURBULENT SYMMETRIC AND TRANSITION LAYERS*

From the theory in Section 5.2, and using the notation introduced in Appendix B, the incoherent reflection coefficient is given by

$$\begin{aligned} W_I = (k_0/2m)^2 |N|^2 \zeta^2 \Big\{ & \\ & Q[1,2,3,1,2,3,(1,2,3,1,2,3)^*,B_1,B_2,B_1,B_2] \\ + 4|B_2/B_4|^2 & \\ & \times Q[1,2,3,4,6,8,(1,2,3,4,6,8)^*,B_1,B_2,B_3,B_4] \\ + |B_2/B_4|^4 & \\ & \times Q[4,6,8,4,6,8,(4,6,8,4,6,8)^*,B_3,B_4,B_3,B_4] \end{aligned} \qquad \text{(C4)}$$

+

$$
\begin{aligned}
&2\mathrm{Re}\{-2(-1)^{iS}(B_2/B_4) \\
&\qquad \times Q[1,2,3,4,6,8,(1,2,3,1,2,3)^*,B_1,B_2,B_3,B_4] \\
&+ (-1)^{i2S}(B_2/B_4)^2 \\
&\qquad \times Q[4,6,8,4,6,8,(1,2,3,1,2,3)^*,B_3,B_4,B_3,B_4] \\
&- 2(-1)^{iS}|B_2/B_4|^2(B_2/B_4) \\
&\qquad \times Q[4,6,8,4,6,8,(1,2,3,4,6,8)^*,B_3,B_4,B_3,B_4]\}\Big\}
\end{aligned}
$$

Likewise, the incoherent transmission coefficient is

$$
\begin{aligned}
V_I = (mk_0^2/2)^2 |N/k_1 B_4|^2 \zeta^2 \Big\{ & \\
&Q[1,2,3,4,6,8,(1,2,3,4,6,8)^*,B_1,B_2,B_3,B_4] \\
&+ |B_2/B_4|^2 \\
&\qquad \times Q[4,6,8,4,6,8,(4,6,8,4,6,8)^*,B_3,B_4,B_3,B_4] \\
&+ 2\mathrm{Re}\{(-1)^{iS}(B_2/B_4) \\
&\qquad \times Q[4,6,8,4,6,8,(1,2,3,4,6,8)^*,B_3,B_4,B_3,B_4]\}\Big\}
\end{aligned}
\qquad \text{(C5)}
$$

ILLUSTRATIONS

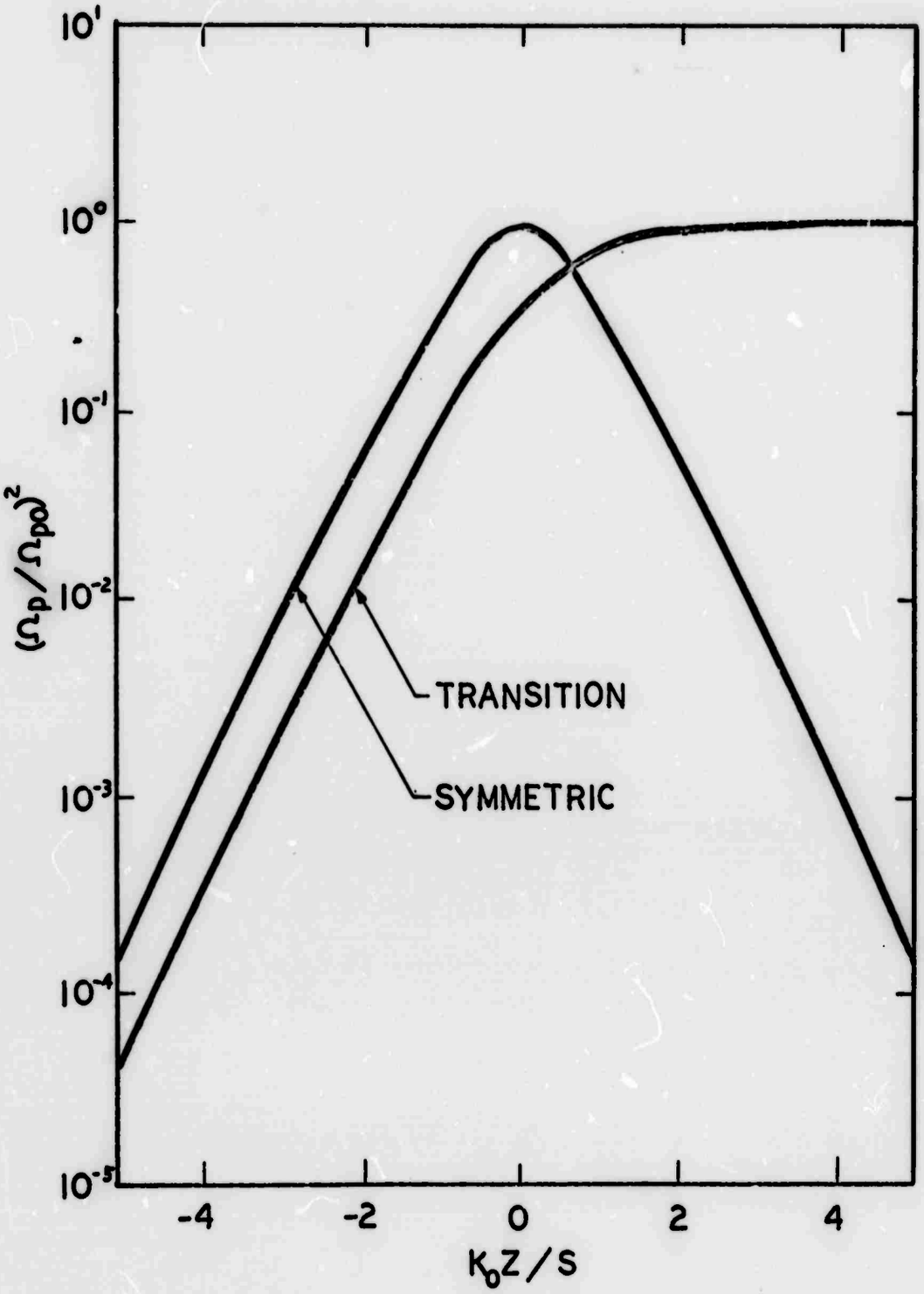


FIG. 1 THE EPSTEIN PROFILE DISTRIBUTIONS

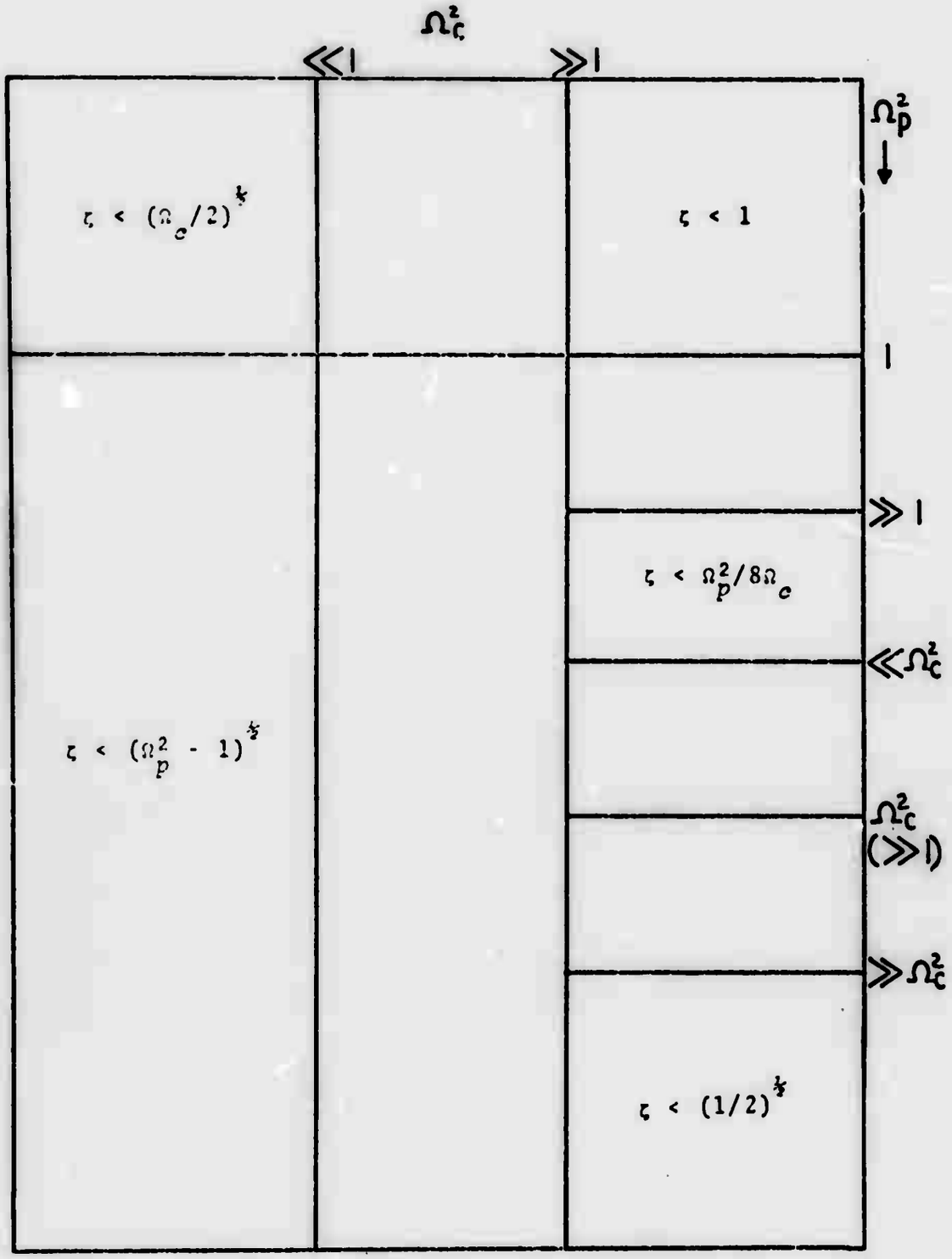


FIG. 2 LIMITING CASES OF THE BOUNDS UPON THE HALF-SPACE SOLUTION

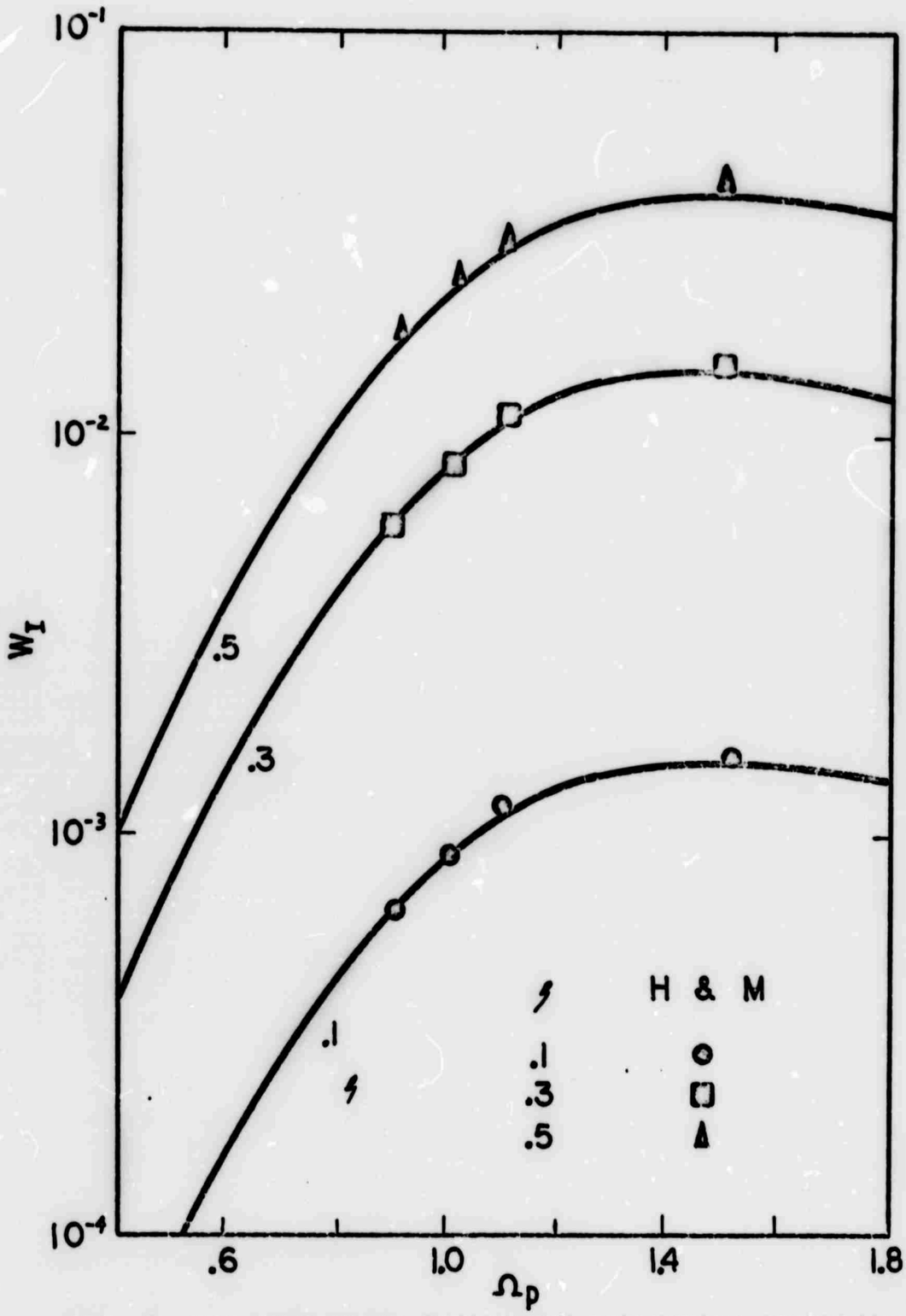


FIG. 3 THE HALF-SPACE INCOHERENT REFLECTION COEFFICIENT COMPARED TO EXPERIMENT (Ω_c = 1.0)

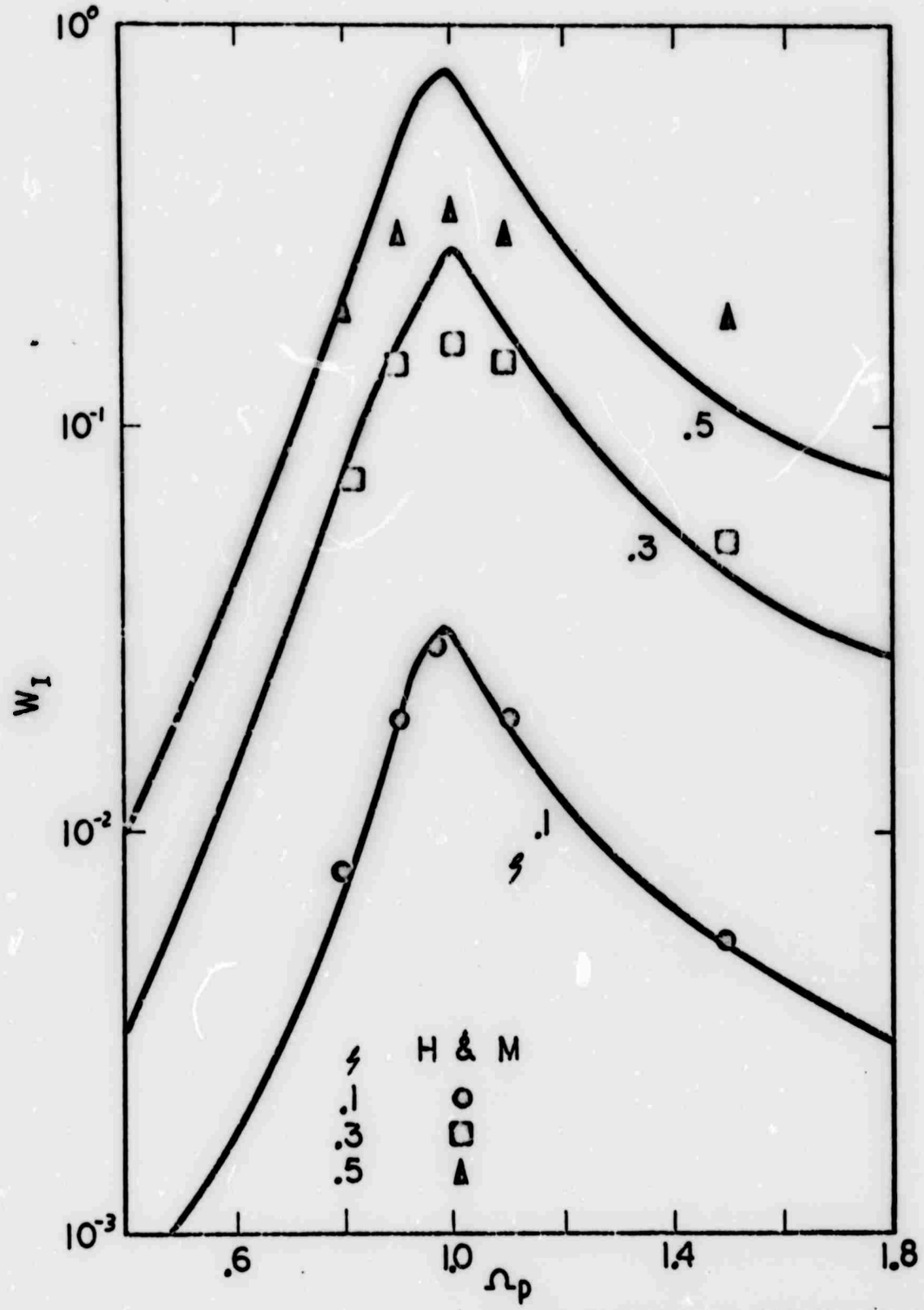


FIG. 4 THE HALF-SPACE INCOHERENT REFLECTION COEFFICIENT COMPARED TO EXPERIMENT (a_e = .1)

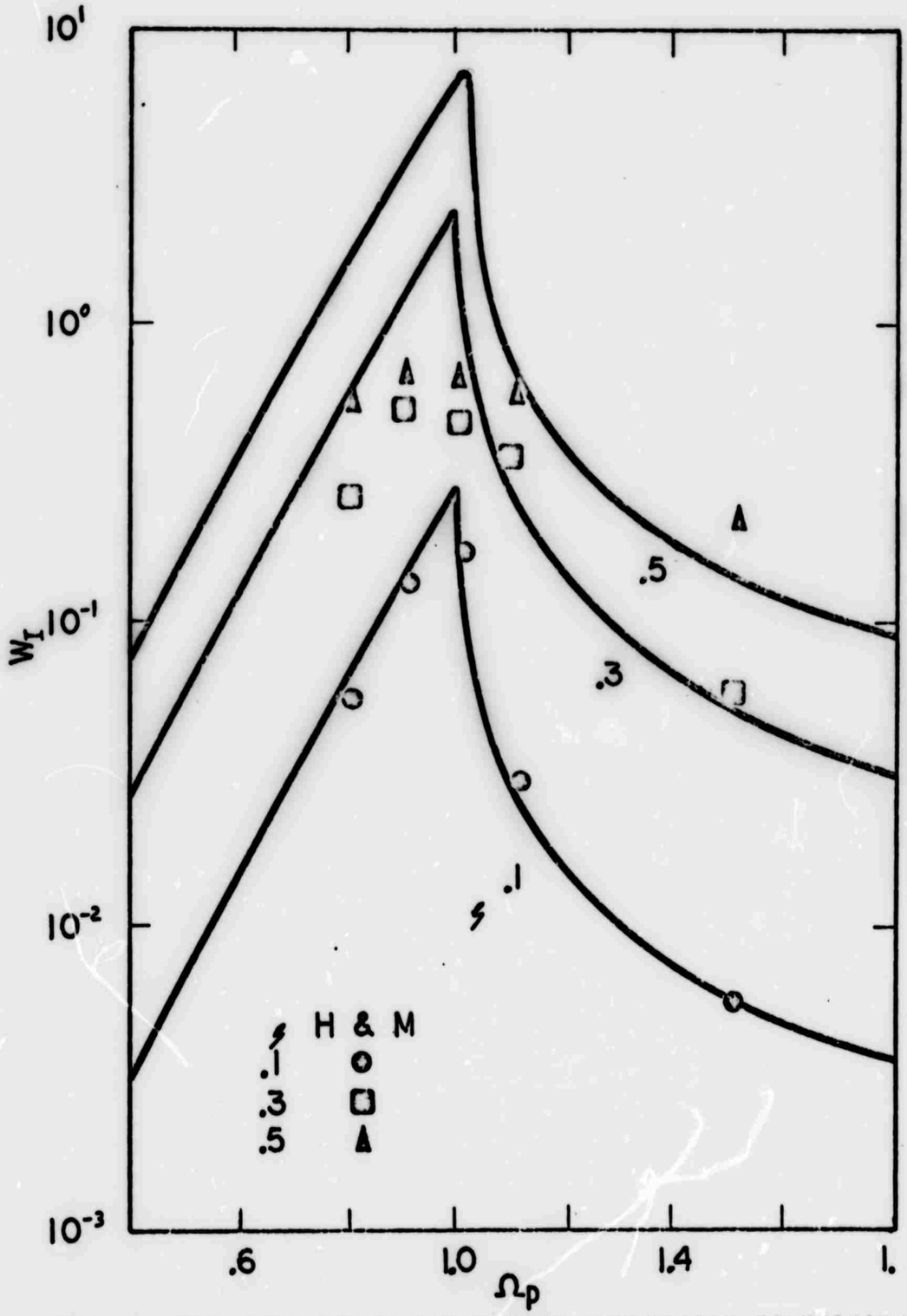


FIG. 5 THE HALF-SPACE INCOHERENT REFLECTION COEFFICIENT COMPARED TO EXPERIMENT (a_o = .01)

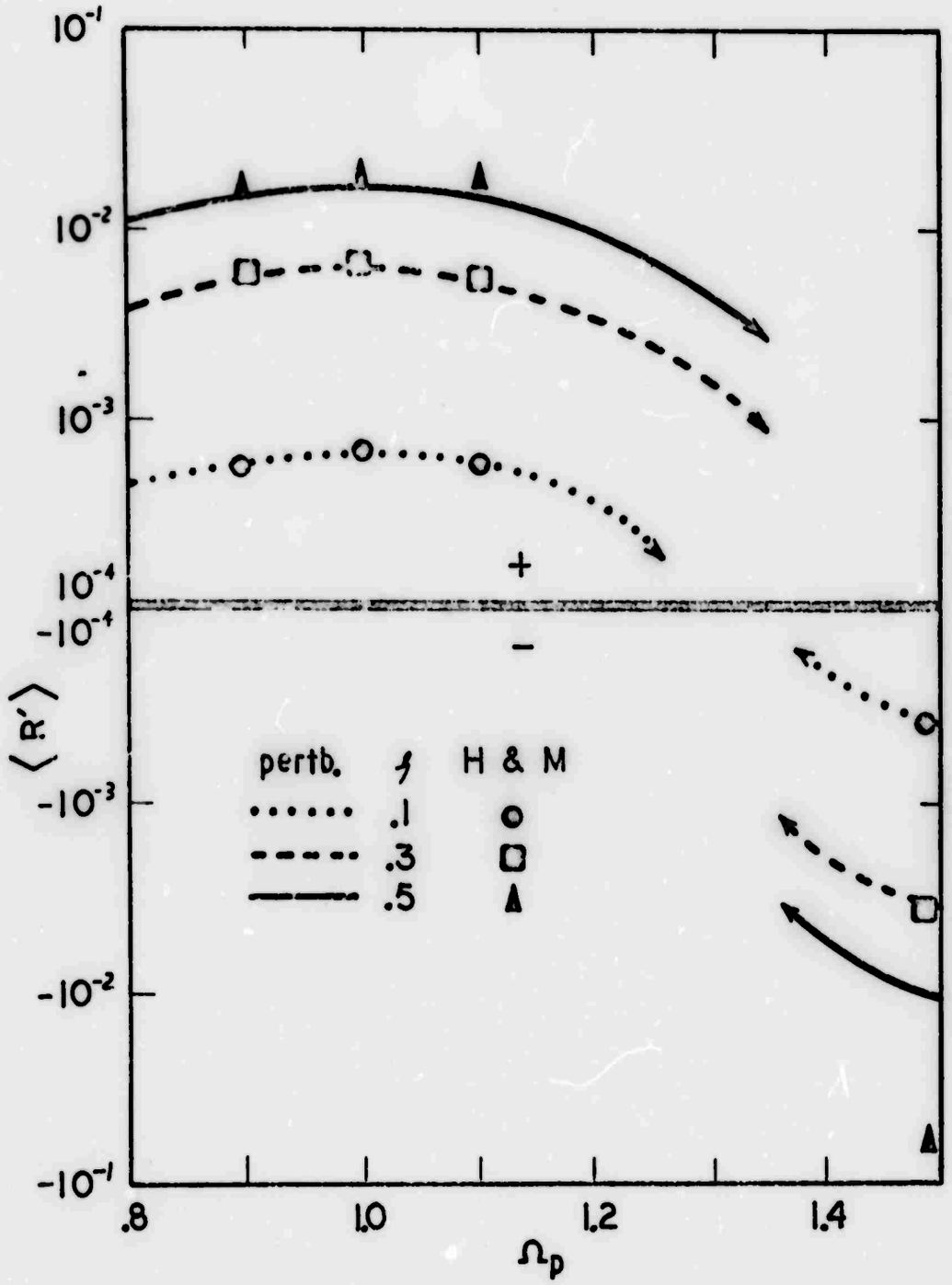


FIG. 6 THE HALF-SPACE REFLECTION COEFFICIENT <R'> COMPARED TO EXPERIMENT (Ω_c = 1.0)

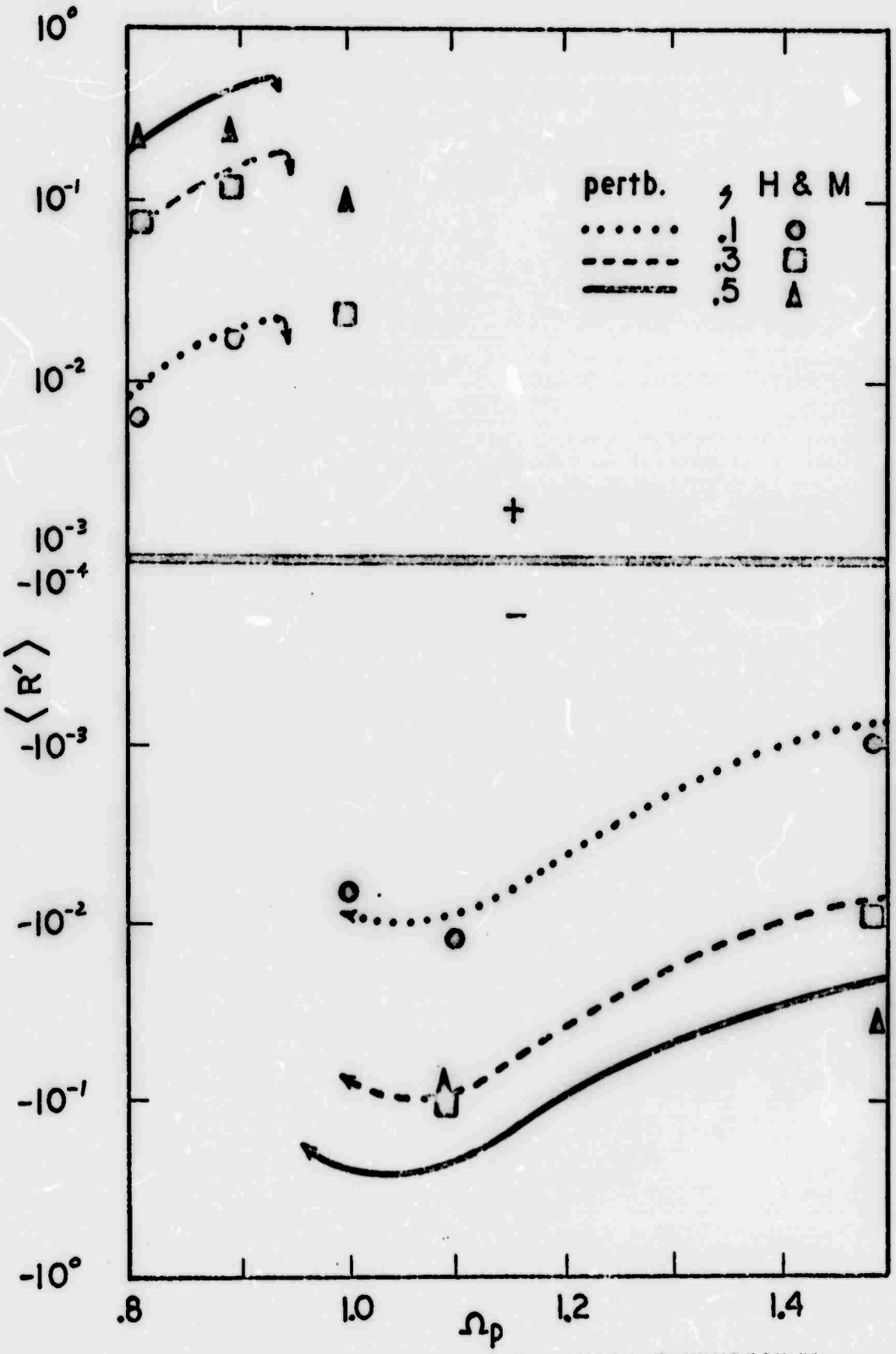


FIG. 7 THE HALF-SPACE REFLECTION COEFFICIENT <R'> COMPARED TO EXPERIMENT (Ω_c = .1)

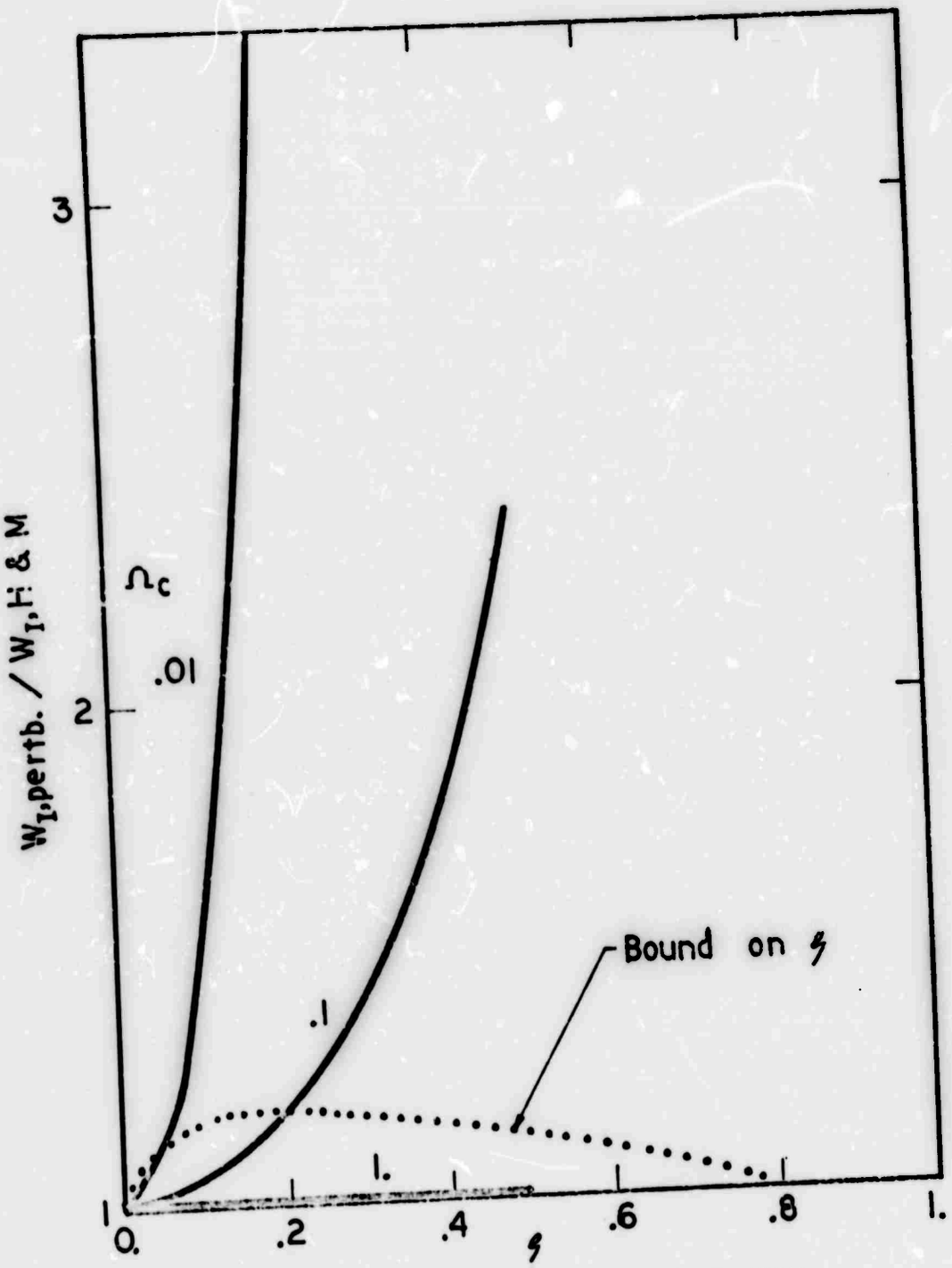


FIG. 5 EXAMPLE OF THE BOUNDS UPON THE HALF-SPACE SOLUTION (Ω_p = 1.0)

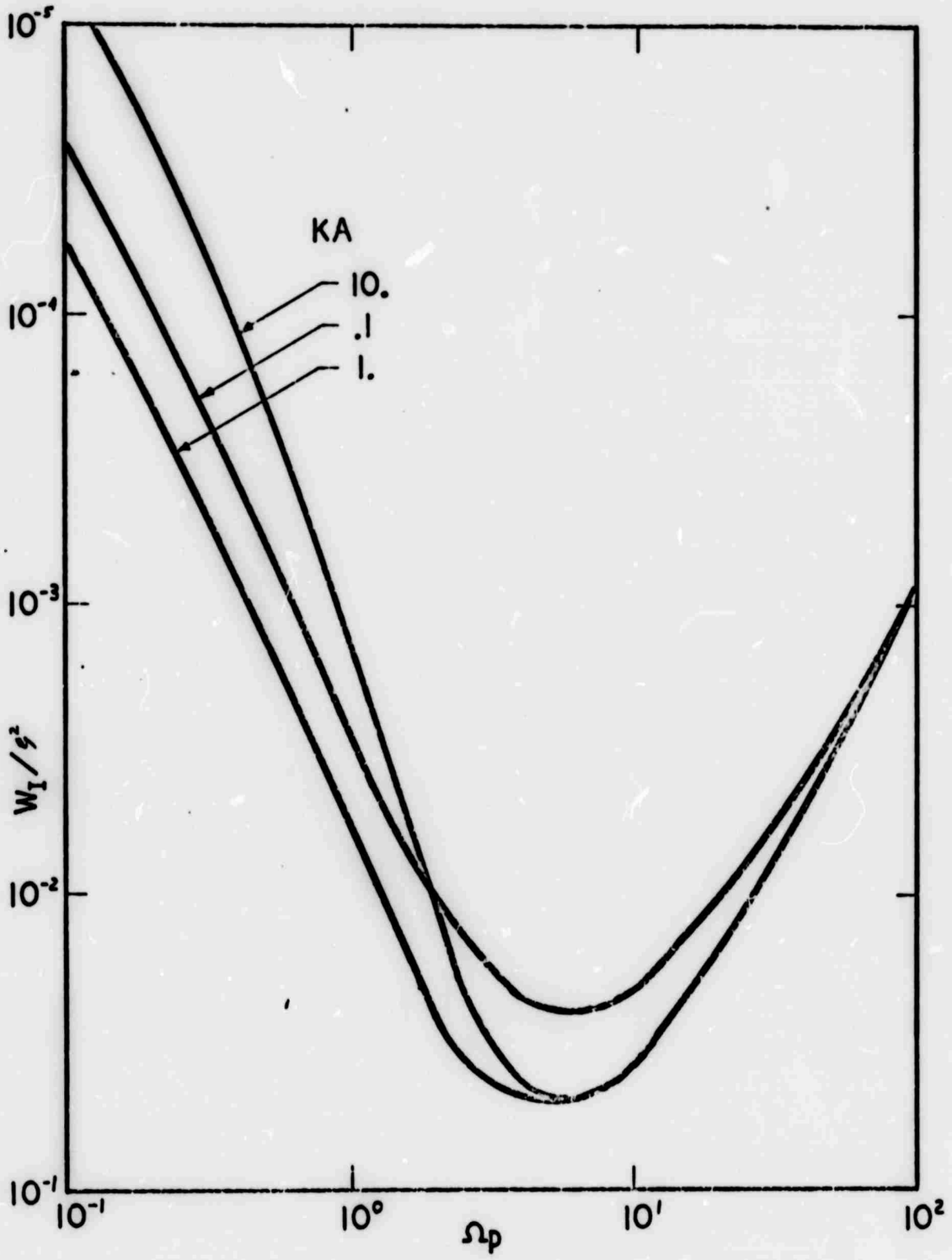
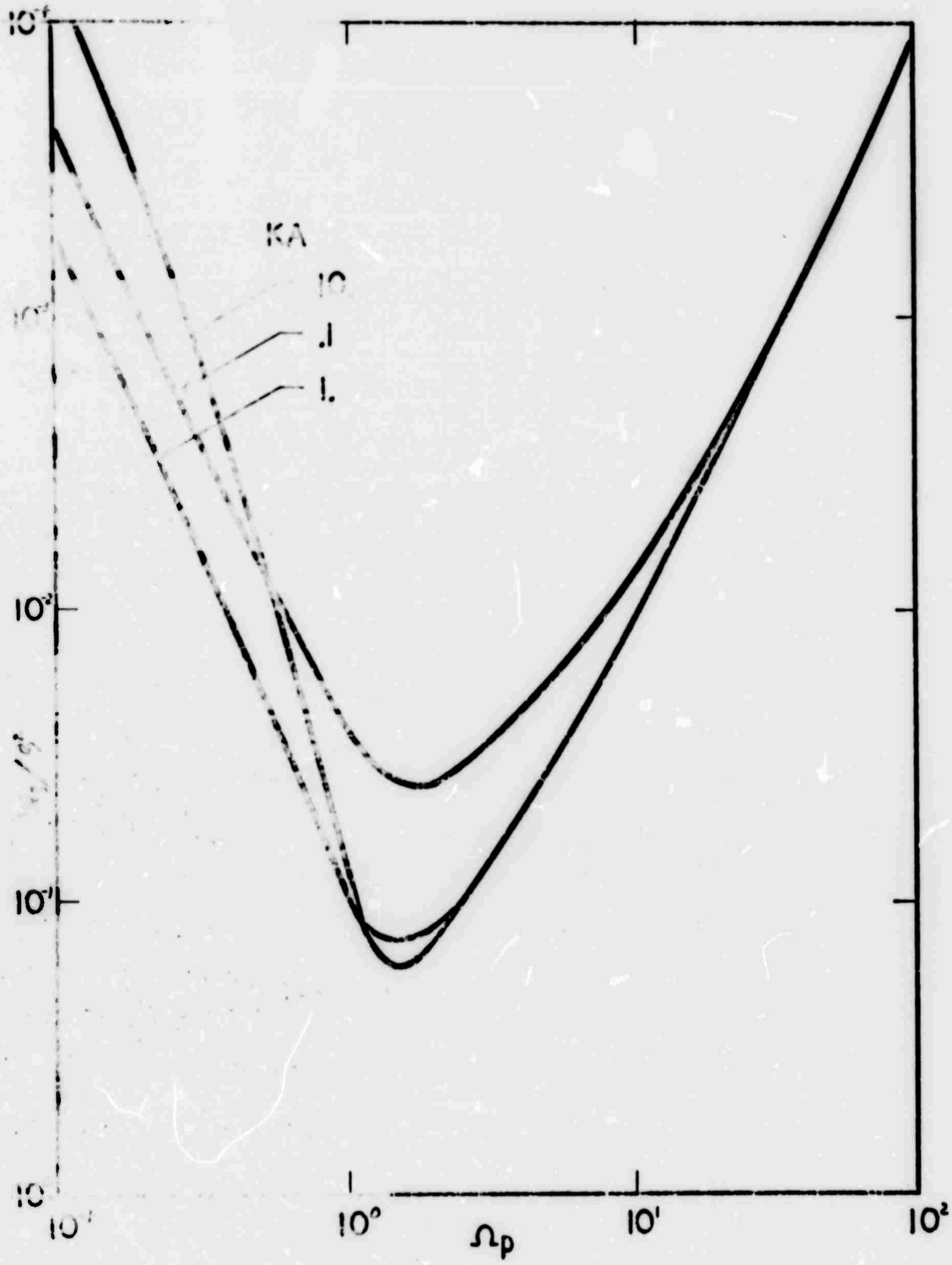


FIG. 9 THE HALF-SPACE INCOHERENT REFLECTION COEFFICIENT ($\Omega_c = 10.0$)



ALL-SPACE INCOHERENT REFLECTION COEFFICIENT

(Ω_c = 1.0)

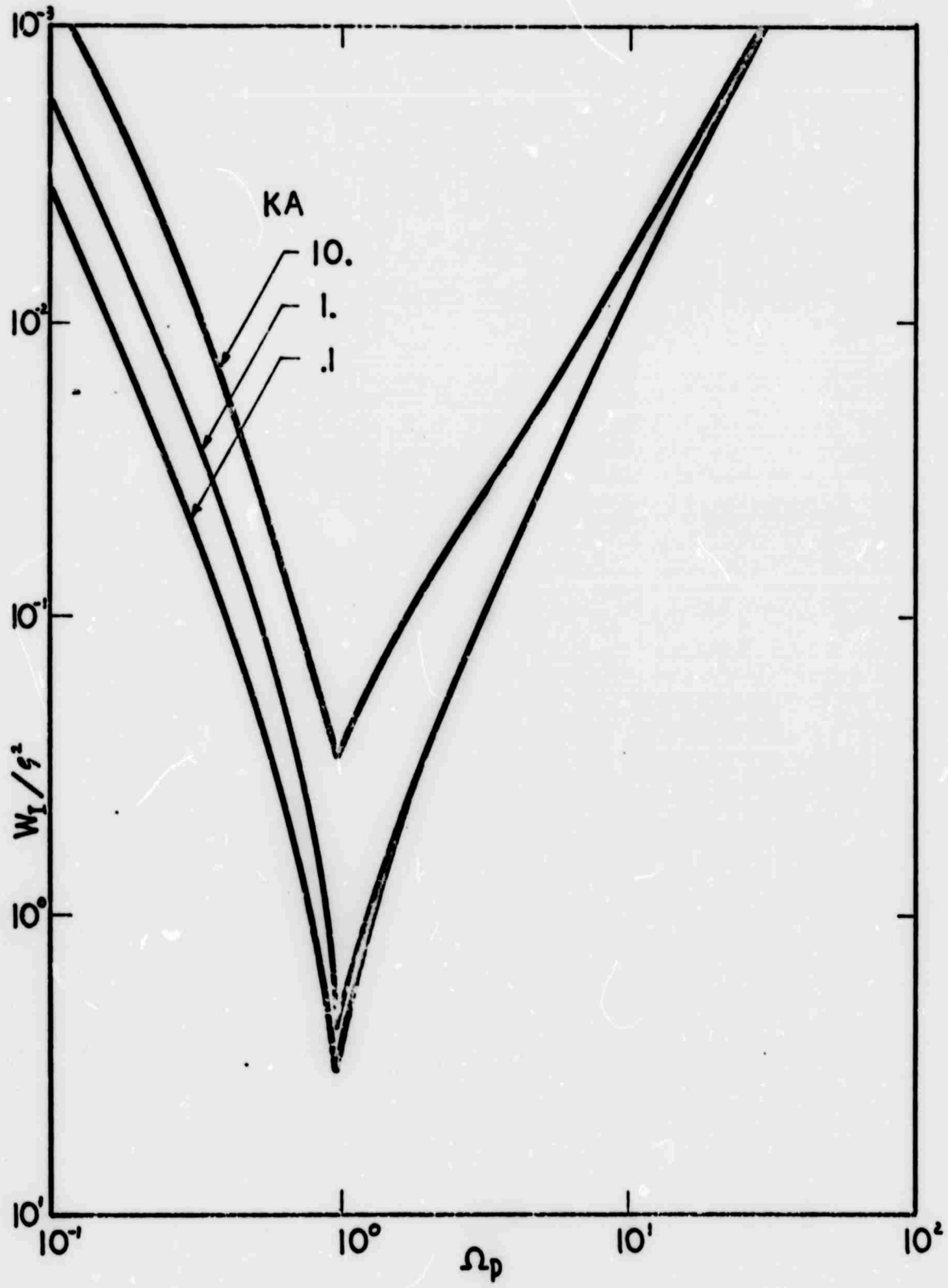


FIG. 11 THE HALF-SPACE INCOHERENT REFLECTION COEFFICIENT (Ω_σ = .1)

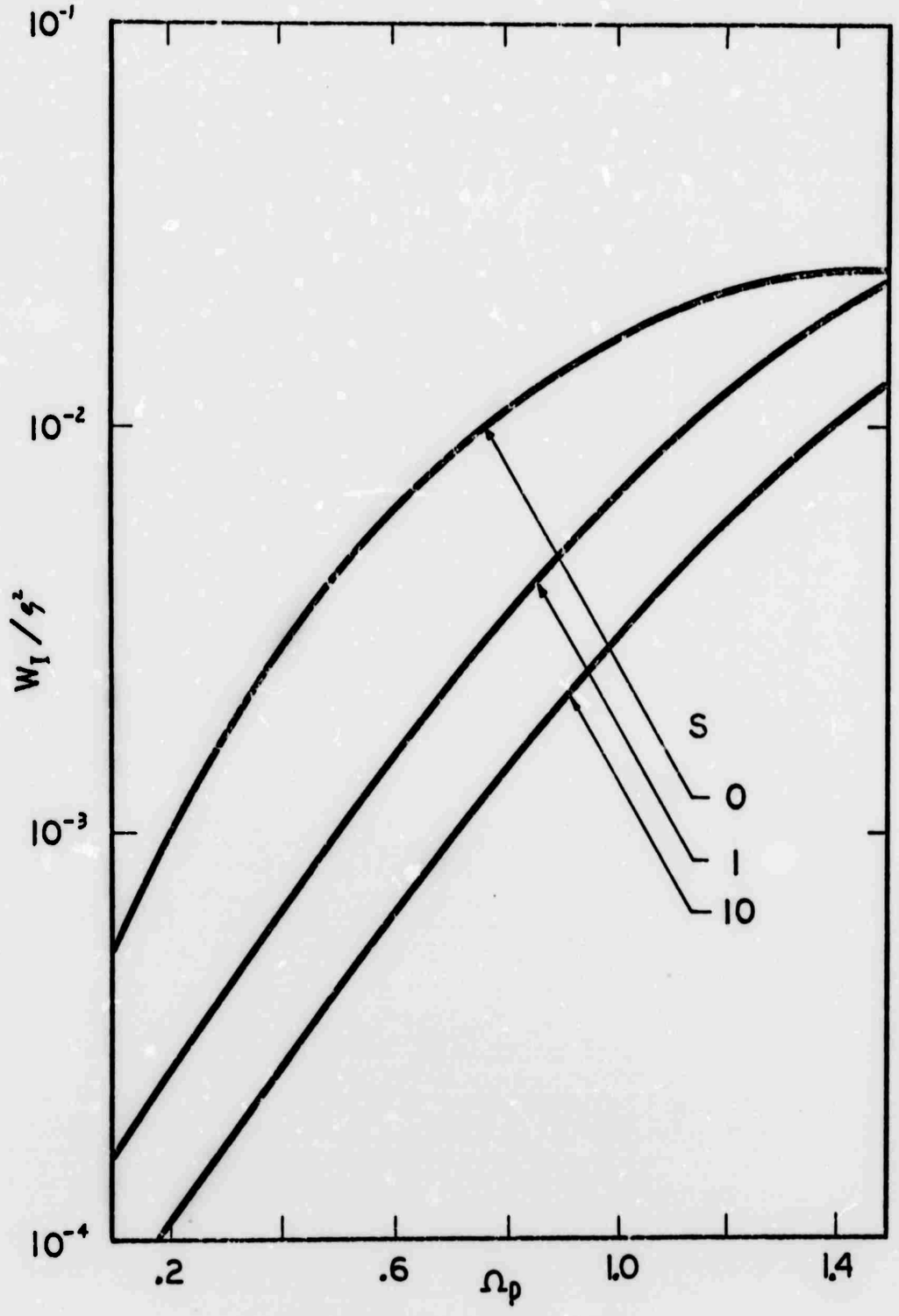


FIG. 12 THE TRANSITION LAYER
INCOHERENT REFLECTION COEFFICIENT (Ω_c = 1.0)

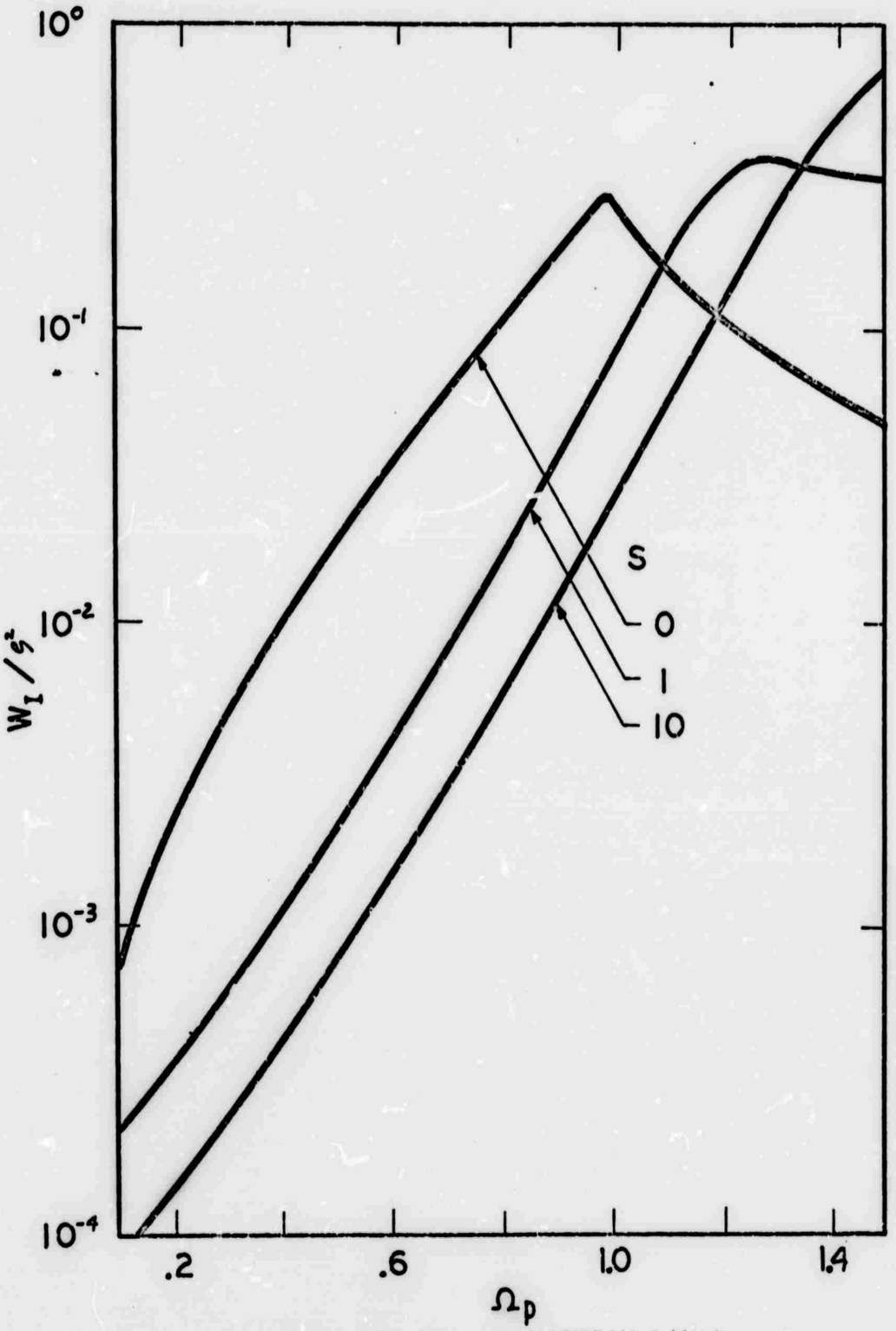


FIG. 13 THE TRANSITION LAYER
INCOHERENT REFLECTION COEFFICIENT (α_c = .1)

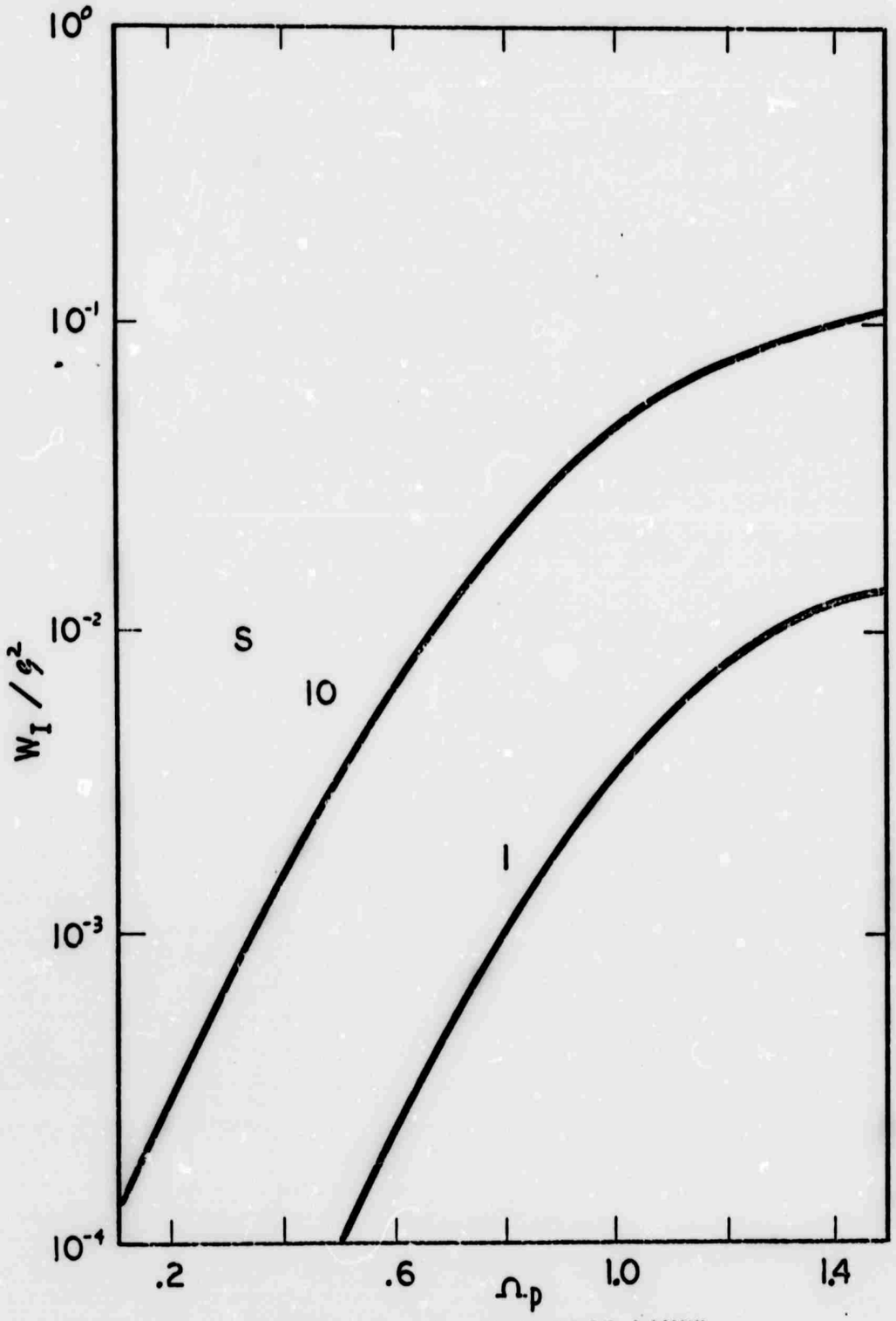


FIG. 14 THE SYMETRIC LAYER
INCOHERENT REFLECTION COEFFICIENT (Ω_s = 1.0)

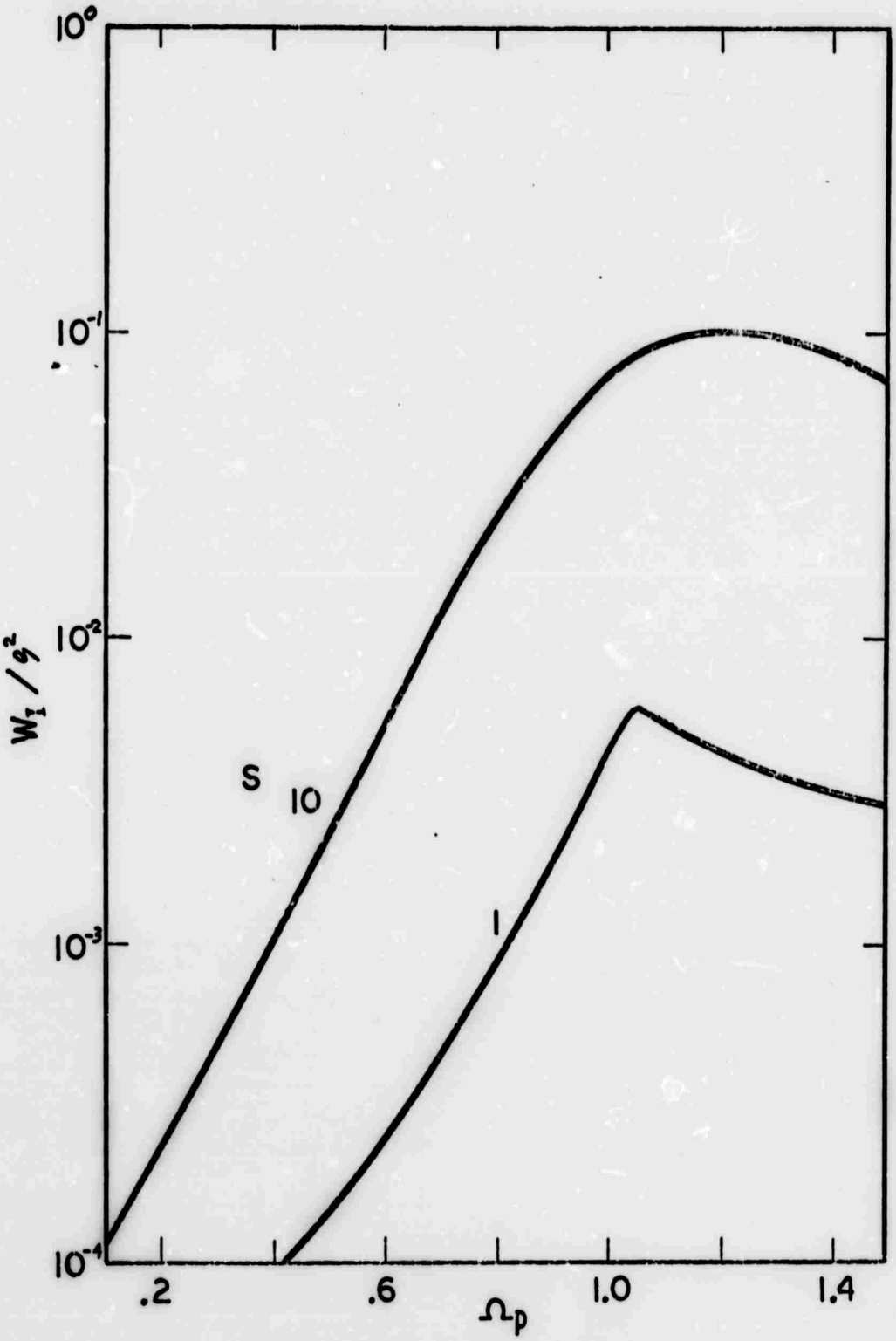


FIG. 13 THE SYMMETRIC LAYER
INCOHERENT REFLECTION COEFFICIENT (Ω_c = .1)

VITA

VITA

[illegible] Wasneski was born in Riverside, New [illegible], on March 19, 1943. He received the B.S. degree in [illegible] degree in 1968, both in electrical [illegible] from Drexel University.

[illegible] 1966, he was a part-time Technician and Research Technician with AeroChem Research Laboratories, Inc., Princeton, New Jersey. From 1966 to 1970 he was associated with that laboratory as a part-time Research Engineer.

[illegible] is a member of Phi Kappa Phi, Tau Beta Pi, Eta Kappa Nu and IEEE.