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FOREIGN TECHNOLOGY DIVISION

AIR FORCE SYSTEMS COMMAND

DYNAMICS OF THE DESIGN OF A FLIGHT VEHICLE

by

V. F. Gladkiy

FOREIGN TECHNOLOGY DIVISION

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DYNAMICS OF THE DESIGN OF A FLIGHT VEHICLE

By: V. F. Gladkiy

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U. S. BOARD ON GEOGRAPHIC NAMES TRANSLITERATION SYSTEM

Block	Italic	Transliteration	Block	Italic	Transliteration
А а	*А а*	A, a	Р р	*Р р*	R, r
Б б	*Б б*	B, b	С с	*С с*	S, s
В в	*В в*	V, v	Т т	*Т т*	T, t
Г г	*Г г*	G, g	У у	*У у*	U, u
Д д	*Д д*	D, d	Ф ф	*Ф ф*	F, f
Е е	*Е е*	Ye, ye; E, e*	Х х	*Х х*	Kh, kh
Ж ж	*Ж ж*	Zh, zh	Ц ц	*Ц ц*	Ts, ts
З з	*З з*	Z, z	Ч ч	*Ч ч*	Ch, ch
И и	*И и*	I, i	Ш ш	*Ш ш*	Sh, sh
Й й	*Й й*	Y, y	Щ щ	*Щ щ*	Shch, shch
К к	*К к*	K, k	Ъ ъ	*Ъ ъ*	"
Л л	*Л л*	L, l	Ы ы	*Ы ы*	Y, y
М м	*М м*	M, m	Ь ь	*Ь ь*	'
Н н	*Н н*	N, n	Э э	*Э э*	E, e
О о	*О о*	O, o	Ю ю	*Ю ю*	Yu, yu
П п	*П п*	P, p	Я я	*Я я*	Ya, ya

* ye initially, after vowels, and after ъ, ь; e elsewhere. When written as ё in Russian, transliterate as yё or ё. The use of diacritical marks is preferred, but such marks may be omitted when expediency dictates.

FOLLOWING ARE THE CORRESPONDING RUSSIAN AND ENGLISH DESIGNATIONS OF THE TRIGONOMETRIC FUNCTIONS

Russian	English
sin	sin
cos	cos
tg	tan
ctg	cot
sec	sec
cosec	csc
sh	sinh
ch	cosh
th	tanh
cth	coth
sch	sech
csch	csch
arc sin	sin^{-1}
arc cos	cos^{-1}
arc tg	tan^{-1}
arc ctg	cot^{-1}
arc sec	sec^{-1}
arc cosec	csc^{-1}
arc sh	$sinh^{-1}$
arc ch	$cosh^{-1}$
arc th	$tanh^{-1}$
arc cth	$coth^{-1}$
arc sch	$sech^{-1}$
arc csch	$csch^{-1}$
rot	curl
lg	log

DYNAMICS OF THE DESIGN OF A
FLIGHT VEHICLE

Gladkiy, V. F.

Publishing House "Science," Main
Editorial Office of Physico- and
Mathematical Literature,
M., 1969

The book is devoted to the problem of determining the necessary carrying capacity and rigidity of design of a flight vehicle. Discussed in it are theoretical bases and practical methods of the calculation of internal force factors from external forces acting on the vehicle in the process of operation, and methods of development of calculation cases of loading are given. In this case the main attention is given to problems of the dynamics of design, in particular the selection of design configurations, the formulation of equations of dynamic equilibrium and determination of the dynamic reaction of design on the effect of external perturbations. We consider those limitations which are imposed by conditions of strength of the design on values of certain parameters of the propulsion system, automatic control system, complex of ground equipment, and also on conditions of operation of flight vehicles of different types.

There are 118 figures and a bibliography of 93 names.

PREFACE

The necessary carrying capacity and rigidity of design of any flight vehicle are determined basically those internal force factors which appear in its elements in the process of operation under the action of external loads. Due to the random character of the external forces and complexity of the theoretical description of real conditions of operation of the design, the solution to the problem of its strength with low weight is very difficult. Therefore, in practice in the designing of the vehicle, it is necessary to resort to the standardization of their carrying capacity and rigidity.

The appropriate norms of strength establish (for all main centers of design of a flight vehicle of a given type) the calculation cases of loading, the largest magnitude of external loads and values of safety factors. The existing norms of strength of the aircraft were created over prolonged time by a comparatively large collective of scientists and engineers under the leadership of V. P. Vetchinkin, S. N. Shishkin, M. V. Keleysh, Ye. P. Grossman, A. I. Makarevskiy, A. A. Goriyanov, N. N. Korchemkin, T. A. Frantsuz and others. It is natural that they are constantly being refined and are perfected with the development of technology, the accumulation of experimental material and the fulfillment of the necessary theoretical research.

At present in connection with the appearance of new models of guided and unguided flight vehicles of the ballistic type, satellites, spaceships, and landing vehicles, for which the problem of weight of the design is even more critical than that for aircraft, questions of the theoretical foundation of standard methods of calculation for strength have become very urgent. To this one should add that the specific peculiarities of operation of the mentioned types of vehicles substantially limit the possibilties of a natural experimental development of the strength of their design and in many cases practically exclude the statistical processing of their results.

With an increase in speeds of flight, powers of the propulsion systems and especially dimensions of flight vehicles of the ballistic type (the launching weights of which are measured in hundreds and even thousands of tons), an even greater importance in the solution of the considered problem is inherent in the problems of the dynamics of design. Elastic oscillations of the design of a flight vehicle on the whole and its separate elements and also oscillations of liquid fuel tanks have an effect not only on the strength, but also on operating conditions of the control system and sometimes the propulsion system. Therefore, for similar vehicles the problem of the determination of the necessary carrier ability and rigidity of design proves to be closely connected with the selection of optimum values of parameters of automaton of stabilization, ground equipment, trajectory of flight, transitional regimes of operation of engines and with establishment of general conditions of operation of the vehicle.

This book covers the theoretical bases and practical methods of determining the necessary carrier ability and rigidity of design of the aforementioned types of vehicles and solutions of certain problems of their dynamics. Here there is no discussion of problems of dynamic strength of elements of design of these vehicles, caused by the effect of turbulence of the boundary layer, turbulence of the atmosphere, acoustic field of pressure of the stream of a jet engine, local separation of air flow, and also many problems of aeroelasticity, which are the subject of independent investigation.

The book consists of five parts. In the first part an analysis of external forces acting on the vehicle in the process of operation is conducted. The effect of the mobility of the liquid in fuel tanks on inertial characteristics of vehicles of the ballistic type is estimated. Approximation methods of determining the general internal forces in elements of their housings are discussed.

The second part gives basic informations about methods of the solution of the problem on the reaction of the elastic vehicle on external disturbance.

In the third part problems of the dynamics of design for ground cases of the loading of such as the transporting, installation in the launch device, prelaunch servicing, and ground and silo launch are examined. Due to the effect on the character of loading of the vehicle and elasticity of elements of corresponding ground equipment is investigated.

The fourth part is devoted to problems of loading of the design of vehicles in flight. In it approximate perturbation equations of the flight vehicle, which consider the effect of elasticity of its design and mobility of the liquid in fuel tanks are derived. The reaction is analyzed of the design on the effect of perturbing forces and moments appearing in the motion of the vehicle in restless air, in process of uncoupling of the stages, on the phase of orbit and descent, during landing on the hard surface of the planet and on water, and also with dynamic instability (in small) systems: elastic vehicle-automaton of stabilization, elastic body of the vehicle – system of fuel feed – propulsion system.

In the last part the problems connected with the establishment of the necessary carrier ability of design are discussed. In particular, methods of the finding of calculation cases of loading and determination of limitations on operating conditions of different systems of the vehicle are given. General information concerning the selection of safety factors and experimental methods of the solution of dyanmic problems is given. In conclusion an approximate list of calculation cases of loading for vehicles of the ballistic type is applied.

In the book only those problems are considered which at present are widely discussed in Soviet and foreign, especially periodic literature. As illustrations in it component diagrams of only hypothetical and foreign vehicles are used. For foreign literature all concrete data about conditions of the operation of vehicles and cases of their loading are borrowed. Digital material is tentative.

The book uses double numeration of paragraphs, figures and formulas. In this case the first number indicates the number of the chapter and with reference to the formula the chapter considered is omitted.

The book is intended for engineers interested in problems of strength, dynamics and optimum design structures of flight vehicles. The author trusts that it will be useful also to students of higher educational institutions specializing in the indicated field.

The author expresses his sincere gratefulness to Professor I. I. Gol'denblat and L. A. Chul'skiy for their valuable remarks made in the examination of the manuscript.

The author will be thankful to all readers who send any remarks and requests.

SECTION I

LOADS OF FLIGHT VEHICLES

C H A P T E R I

EXTERNAL FORCES

§ 1.1. Systems of Coordinates. Classification of Forces

Movement of the center of gravity of a vehicle in space is described in the terrestrial system of coordinates $x_0y_0z_0$. Usually the origin of this rectangular system of coordinates on phase of powered flight is placed at the point of launch and on the phase of descent – at the point of landing. The axis x_0 is directed along the tangent to the arc conducted from the center of earth and connecting the launch point with some characteristic point, for instance, with the touchdown point, axis y_0 – vertically upwards (Fig. 1.1) and z_0 – perpendicular to the plane firing x_0y_0 (according to rule of right-handed system).

Movement of the vehicle with respect to the center of gravity is considered in the so-called continuous or natural system of coordinates xyz, the origin of which coincides with the instantaneous true position of the center of gravity of the vehicle. The x axis of this system of coordinates is directed along the tangent to the trajectory (along the velocity vector) in the plane of flight, y axis – along the normal upwards and z axis – along the binormal (perpendicular to the plane xy). The plane xy is called the pitch plane, plane xz – yawing plane, and plane yz – rolling plane.

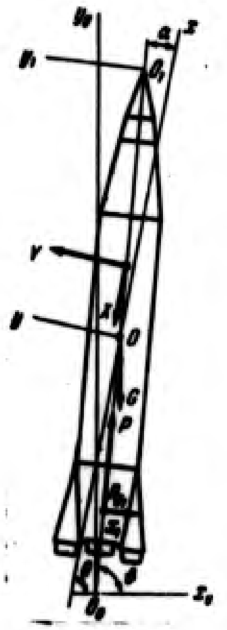

Fig. 1.1. Diagram of forces acting on the flight vehicle in flight (case R). Systems of coordinates.

In calculations of internal forces the connected system of coordinates $x_1y_1z_1$ is usually used. The origin of this auxiliary system of coordinates is placed at the vertex of the body of the vehicle (Fig. 1.1). The x_1 axis is directed along its longitudinal axis (to the tail), axis y_1 – in plane xy (upwards) and axis z_1 – perpendicular to x_1y_1. In this case by longitudinal axis we understand as the building axis of the body, or axis parallel to the chord of the wing and passing through the center of gravity. Axis y_1 is usually called lateral axis and axis z_1 – side-force axis.

The mutual direction of axes of the terrestrial and connected systems of coordinates is determined by the pitch angle ϑ (angle between the projection of axis x_1 onto plane x_0y_0 and axis x_0) and yawing angles ψ (angle between axis x_1 and plane x_0y_0). Angles between axes of the connected and continuous systems of coordinates ϕ, β, α can be called respectively, angle of role (angle between axis y_1 and plane xy), angle of slip (angle between axis x_1 and plane xy) and angle of attack (angle between the projection of axis x_1 onto plane xy and axis x). As can be seen from the figure, the angle of attach is determined by formula $\alpha = \nu - \theta$, where θ – angle between the projection of the velocity vector onto plane x_0y_0 and axis x_0. In examining the disturbed motion at the angle of attack we understand the angle between projections of axis x_1 and velocity vector of the incident flow of air onto plane xy.

Pitch, yawing and band angles are considered positive with rotation counterclockwise, if one were to examine them from positive ends of corresponding axes. Angles β and α are counted off from the velocity vector according to the rule of the right-handed system.

In the investigation of fluctuations of liquid in fuel tanks, we use the auxiliary connected systems of coordinates $x_jy_jz_j$, the origin of which we place in centers of the free surface of the liquid, directing axis x_j along the longitudinal axis of j-th tank upwards to the vertex of the body and axes y_j and z_j as is shown on Fig. 1.2.

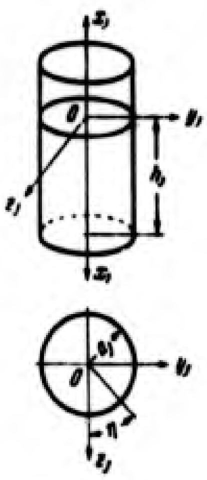

Fig. 1.2. Auxiliary system of coordinates.

In accordance with the designation of axes of continuous and connected system of coordinates, the designation of projections of resultant external and internal forces is conducted. In this case the positive forces are considered to be those forces whose direction of action coincides with the direction of corresponding axes of the coordinates.

A diagram of external forces acting on the flight vehicle is given in Fig. 1.1, where G – gravity, P – thrust force, R_{v_1} – control force and X and Y – components of resultant aerodynamic loads.

All these forces are external and characterize the effect on flight vehicle of the environment.

Internal efforts appearing in carrier elements of the structure depend on the magnitude and distribution of these external forces with respect to body and on the nature of their change with time.

According to the nature of the distribution all external forces can be divided into three categories.

1. Volume, or mass forces, distributed over the whole volume of the vehicle and proportional to the density of its material. Gravity and inertial forces pertain to them.

2. Surface forces – forces distributed over the surface of the structure. This category of forces pertains to aerodynamic forces, hydrodynamic forces (pressure or liquid) and others.

3. Concentrated forces – force of comparatively great magnitude distributed over a relatively small surface. Theoretically these are forces applied at a point. An example of concentrated forces can be contact forces, which appearing at the place of the transmission to the body (through rods of the frame of the propulsion system) of thrust force.

According to the nature of the change with time, all external forces can be conditionally divided into two classes:

a) statically acting forces,

b) dynamically acting force.

First class pertains to slowly variable forces, i.e., forces whose time of application is great as compared to a certain characteristic time for a considered design. The second class pertains to rapidly variable forces, the time of application of which (or substantial change with time of their magnitude) is comparable with this characteristic time. For vehicles such characteristic time is the period of natural elastic oscillations of the design of any tone. An example of dynamically acting forces exciting elastic oscillations of the design is the change in thrust force due to the change in the altitude of flight and others. The same force can belong in one case to the class of static forces and in the other case – to the class of dynamic forces.

Gravity, certain dynamic forces, pressure in tanks and thrust force are constantly acting forces. Nominal values of these forces, which vary little from one copy of the given design to the other, are always considered in the determination of parameters of the so-called undisturbed (program) motion of a flight vehicle. Therefore, they are frequently called "program" forces. Any deviation of real values of external forces from the program values refers to the category of external disturbing forces. With respect to their nature disturbing forces are random time functions.

Thus, for instance, the effect on the vehicle at a defined moment of flight of a gust of wind of a determinate structure and intensity or defined deviation of the line of force of traction from the longitudinal axis and others are random.

§ 1.2. Dynamic Parameters of the Atmosphere

Conditions of operation of design in flight in many respects depend on the state of the surrounding atmosphere extending usually to very great altitudes. Values of aerodynamic forces acting on the vehicle in the atmosphere are determined by the speed of its motion and altitude of the flight, in particular, air density and its temperature. The joint effect of air density ρ and true air speed of flight v on these forces is characterized by the magnitude of impact pressure $q = \frac{1}{2}\rho v^2$. Since ρ decreases greatly with altitude, q can attain comparatively large values only in the surface layer of atmosphere with a thickness of about 20-30 km, which is called the dense layer of the atmosphere, since in it the basic mass of air is contained. The temperature affects the magnitude of heating of the wheel of the body of the vehicle, the speed of sound propagation, i.e., the flight Mach number $(M_\infty = \frac{v}{a})$, and air density.[1]

In view of continuous oscillations of values of parameters of the atmosphere, in the calculation of the nominal trajectory there are used certain average dependencies of air density and speed of sound propagation on heights, which are usually assigned in the form of tables of the so-called standard motionless atmosphere (see All-Union Government Standard 4401-64). For example, on Fig. 1.3 gives curves of the change in relative density $\Delta_0 = \frac{\rho}{\rho_0}$ of the air and relative speed of sound propagation $\bar{a} = \frac{a}{a_0}$ depending upon altitude (when $\rho_0 = 0.125$ kgs^2/m^4 and $t_0 = 20°C$) for the atmosphere of the earth. Sometimes these dependences are established separately for "hot" ($t_{0н} = 50°C$) and "cold" ($t_{0x} = 50°C$) days.

Deviation of the real values of elements of the atmosphere from the standard is considered by means of the introduction of additional aerodynamic loads (disturbing forces). In many cases these additional loads, in particular, from wind, prove to be the determining ones for the strength of the structure of a heavy flight vehicle.

The action of wind on the flight vehicle in flight is reduced to a change in magnitude and direction of the vector of its speed with respect to the air. Approximately its effect can be estimated by a change in angles of attack and slip on magnitudes $\Delta\alpha = \frac{u_y}{v}$ and $\Delta\beta = \frac{u_z}{v}$ and by corresponding change in impact pressure $\frac{1}{2}\rho\,[(v + u_x)^2 + u_y^2 + u_z^2]$, where u_y and u_z — components of the speed of wind perpendicular to v, and u_x — parallel to v.

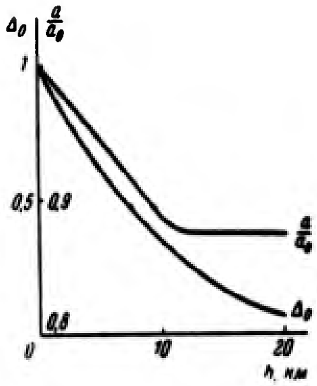

Fig. 1.3. Change in relative air density Δ_0 and relative speed of sound $\bar{a}$ with respect to altitude.

The velocity of the wind depends on the geographic latitude of place of launch, time of the year and even day, and basically due to altitude h of flight. With an increase in the latter it at first increases, attaining its greatest value in the region of the tropopause h_T, and then decreases. An approximate graph of the change in average speed of the wind with respect to relative altitude is shown in Fig. 1.4. It is noted that in many regions of the earth in the region of the tropopause (in certain seasons) large in thickness, of the order of several kilometers) established air flows of great extent – so-called jet streams, are observed. The width of these flows sometimes reaches 600 km. Maximums of average airspeeds in these jet streams reach 70-100 m/s. The most intense jet streams are observed in the winter in the region of the Pacific Ocean (Japan and others) where maximum wind speeds of 120 m/s and even 180 m/s are recorded. Maximum wind speeds above the eastern part of North America reach 130 m/s and above the central part 160 m/s. In large limits values of shifts $\frac{du}{dh}$ of wind speed with respect to altitude are changed. In practice magnitudes of these shifts are defined as the difference of wind speeds at two comparatively closely located altitudes (300-500 m) divided by the thickness of the layer.

The most frequently encountered (with a probability of 40-50%) shifts are of the order of 0.015-0.025 1/s. In certain windy regions (with a probability of 2-3%) values $\frac{du}{dh}$ and of the order of 0.07-0.08 1/s. There are assumptions that high wind speeds correspond to greater shifts. Data on shifts are an important characteristic of wind flow, since they determine its profile and, consequently, the nature of its effects on the flight vehicle.

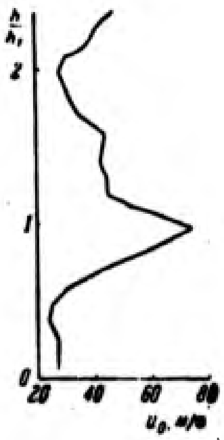

Fig. 1.4. Approximate change in average wind speed with respect to altitude (h_T – altitude of the tropopause).

In the selection of the calculation wind regime, it should be kept in mind that the quantity of measurements of wind speeds at various meteorological stations (in various seasons are very nonuniformly distributed over the territory of the earth. Therefore, it is at present very difficult to obtain reliable statistical characteristic parameters of even steady wind. Various methods of processing these measurements give various values of average speeds. There are mean annual wind speeds for a defined region. Mean values of speeds, found over many years for whole countries and even continents, are encountered. Similar data, of course, do not have special practical importance. Of interest are only data of maximum speeds obtained for a standard windy region and in a season most unfavorable from this point of view.

Along with the steady shifts in large masses of air, in the atmosphere there exist local vortex flows of small extent but with comparatively high speeds, which are called wind gusts. The presence of great turbulences is noted in all layers of the atmosphere in the zone of cumulonimbi, above rugged terrain and especially on flangs of jet streams. Parameters of these vortexes are studied very little. This is explained by the fact that the basic method of measuring wind speed at great altitudes is by radar. It permits obtaining values of wind over comparatively large (for vehicles of the ballistic type) time intervals calculated in the tens of seconds. In this case the gustiness of the wind, i.e., maximum instantaneous values of wird speeds, are usually not detected. This must always be remembered when it is necessary to use actual data of measurements of wind speed.

Investigation of the structure of atmospheric turbulence is conducted by the following:

1. Indirect method, which received widespread use in aviation technology. The essence of it consists in the determination of certain conditional calculation wind gusts or spectral density of energy of external forces on the basis of analytic analysis of systematic measurements of the reaction of different types of aircraft on the effect of atmospheric turbulence.

2. Direct method of measuring the pulsation of speed or impact pressure of wind flow and obtaining estimators of atmospheric turbulence. Here in the majority of the cases, special attention is given to the investigation of discrete wind gusts. Usually parameters of these gusts is determined approximately by means of the elementary mathematical processing of data of measurements (lateral accelerations of the aircraft).

By means of the analysis of a comparatively large number of such measurements, obtained for different types of aircraft at different altitudes, the conclusion was drawn [86] that wind gusts of identical intensity are equiprobable in any direction (horizontal and vertical). In this case dimensions of gusts to which an aircraft reacts, expressed as a function of the mean geometric chord of a wing, depend little on parameters of the aircraft and altitude of its flight. An estimate of the intensity of turbulence with these investigations was conducted by means of the introduction of effective speed u_c of a certain conditional gust of wind instantly enveloping the whole aircraft, namely:

$$u_e = \frac{2m\ddot{y}}{\rho_0 S v_e c_y^{\alpha}},$$

where m – mass of the aircraft, $\ddot{y}$ – lateral acceleration of its center of gravity),[2] v_e – calibrated speed of the aircraft, c_y^{α} – derivative with respect to the angle of attack from the coefficient of lift of the aircraft and S – area of its wings to which the nondimensional coefficient c_y^{α}.

According to the nature of the change in acceleration of the aircraft with time, it is established that in the first approximation during the calculation of loads it is possible to use conditionally either the triangular or sinusoidal law of the change in true speed of the wind gust in the direction of movement of the aircraft (Fig. 1.5). However, the most widespread is the form of the gust determined by expression

$$u(x) = \frac{u_m}{2}\left(1 - \cos\pi\frac{x}{H}\right). \quad (1.1)$$

Here it was found that distance H, during the period of which the speed of the wind gust is changed from zero to a maximum magnitude, lies within limits of 30-60 m. The value of it is increased with an increase in u_e. It was determined that the effect of the altitude of flight (to 12.5 km) and speed of the steady wind on magnitude u_e in practice can be disregarded.

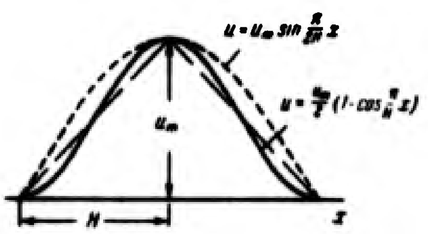

Fig. 1.5. Conditional profiles of wind gust.

The true value of the calibrated speed of wind gust u_m with such a method of processing of experimental data will be approximately 1.3-1.8 times more effective than that of u_e. Since the largest measured values u_e in the indicated range of altitudes were about 15 m/s [86], it is possible to assume that maximums u_m will be of the order of 15-30 m/s. There are bases to consider that the indicated data on the speed of wind gusts are nevertheless not the highest possible in view of the interference of pilots into the regime of motion of the aircraft entering into the zone of atmospheric turbulence.

The given brief information about the structure of wind gusts, (basically vertical) pertain, as was already stressed, to aircraft and therefore cannot be directly transferred to other flight vehicles which essentially differ from aircraft not only in design but also in conditions of motion and type of trajectory. Data on results of measurements of the structure of atmospheric turbulence with the help of rockets as yet found is not in literature.

Proceeding from the established isotropism of the atmosphere with respect to wind gusts, it is possible to consider the direction of the speed of the wind gust always perpendicular to the velocity vector of the flight vehicle, i.e., to estimate the effect of its action by the equivalent change in the angle of attack $\Delta\alpha = \frac{u}{v}$ (or angle of slip $\Delta\beta$).

The absence of sufficient information on the real characteristics of atmospheric turbulence led to the necessity of the application of conditional methods of calculation of its effect on flight vehicles and to the standardization of its parameters. As a result frequently during calculations of strength the reaction of flight vehicles on single wind gusts and so-called cyclical gusts is considered separately, simulating by this in some measure the action of the continuous atmospheric turbulence. In this case the cyclical gust is represented in the form of several wind gusts following one after the other (at a distance of 70-300 m) of comparatively small intensity, having opposite directions. Thus, for instance, at altitudes up to 4 km we take u_e = 8-9 m/s and above (up to 9-10 km) of the order of 2-3 m/s.

In general, the selection of the rated wind regime depends on specific requirements placed on the flight vehicle. Peculiarities of the wind regime of the atmosphere in the surface layer is discussed in detail in Chapter VI.

§ 1.3. Gravity

Before proceeding to the determination of loads and calculation of dynamic characteristics of the design, it is necessary to compile diagrams of the distribution of mass along the length of the body for all cases of loading. In the stage of sketch designing, according to approximate formulas of gravimetric analysis the total weight of the flight vehicle, the weight of components of fuel and limits of weights for main units of its design are established: propulsion system, control system, payload and all sections of the body. In the process of development of the project and fulfillment of the design calculations for strength, these weights are refined, and the nature of their distribution becomes clearer. At the stage of preparation of engineering drawings of design, an exact calculation is conducted of the weight of all elements of the body, position of the center of gravity of the vehicle and moments of inertia relative to the longitudinal (J_x) and lateral (J_y and J_z) axes passing through the center of gravity.

Diagrams of weight for each section are constructed taking into account the nature of transfer of weight of loads to the power section of the body. Since usually the bracing of many loads to the body in longitudinal and lateral directions is carried out by different design elements, the distribution of the component force of weight in these directions will also be different. For instance, the weight of the liquid for vehicles of the ballistic type with carrier tanks in a longitudinal direction is transferred to the body (at the place of the connection of the lower bottom of the tank to the shell) in the form of a load distributed along the contour of the cross section. In a lateral direction it is transferred in the form of a surface load (hydrostatic pressure), distributed along the section of the length of the tank below the mirror of the liquid. For vehicles with suspension tanks both in a lateral and longitudinal direction, the weight of the liquid and weight of the structure of the tanks themselves are transferred to the body in the form of a system of concentrated forces (at places of bracing the tanks to the body). The method of construction of weight diagrams depends on the necessary accuracy of the initial data. One of the most widespread is the following: the length of the body of the vehicle is divided into a comparatively large number of zones (about 50). The weight of all elements of the design entering into the zone is calculated and is evenly distributed over its length. The obtained step diagram is sometimes smoothed. The number of concentrated forces with such method of determination of the weight per unit length $q_G(x_1)$ refers to only those weights the magnitude of which is great as compared to $q_G(x_1)$, for instance, propellant weight of suspension tanks, weight of the propulsion system and others.

By means of numerical or graphic integration of the indicated diagrams, the lateral and longitudinal forces are determined in any section x_1 of the body from the force of weight. They will be equal to the weight of all elements of the structure lying on one side of this section:

$$\left.\begin{aligned} Q_G(x_1) &= \int_0^{x_1} q_{Gy}(x_1)\,dx_1 + \sum_{i=1}^{k} G_{yi}, \\ N_G(x_1) &= \int_0^{x_1} q_{Gx}(x_1)\,dx_1 + \sum_{i=1}^{k} G_{xi}, \end{aligned}\right\} \qquad (1.2)$$

whereby G_{xi} and G_{yi} loads applied to the body on section Ox_1 are designated. With the bracing of these loads to the body at several sections as G_{xi} and G_{yi}, the corresponding reference loads are taken.

Repeated integration of diagram $Q_G(x_1)$ gives values of the bending moment $M_G(x_1)$ in the same section:

$$M_G(x_1) = \int_0^{x_1}\int_0^{x_1} q_{Gy}(x_1)\,dx_1\,dx_1 + \sum_{i=1}^{k} G_{yi}(x_1 - x_{1i}) + \sum_{j=1}^{k} \Delta M_{Gj}. \tag{1.3}$$

Here x_{1i} – abscissa of the place of bracing (in a lateral direction) of the concentrated load to the body, ΔM_{Gj} – concentrated moments from weight of the loads in sections x_{1j} on sections $(0,\ x_1)$. In the end section of the body (when $x_1 = l$) the lateral force $Q_G(l)$ will be equal to the combining weight of the apparatus G, and the bending moment $M_G(l) = G(l - x_{1T})$, where x_{1T} – abscissa of the center of gravity of the flight vehicle and l – its length.

The total weight of the aircraft in flight is defined as the sum of the weight of the structure G_K (with payload G_Γ) and current propellant weight G_T:

$$G(t) = G_K + G_T(t) = G_K + G_{T0} - g_h \int_0^t \frac{dm}{dt}\,dt, \tag{1.4}$$

where G_{T0} – initial (launching) propellant weight, which in general depends on the method of servicing and its temperature, $\frac{dm}{dt}$ – mass flow rate per second, g_h – acceleration of gravity at the considered altitude of flight h:

$$g_h = g_0 \frac{R^2}{(R+h)^2},$$

g_0 – acceleration of gravity at the surface of the earth, R – its radius.

In most cases during calculations of the structure for strength, it is possible with sufficient accuracy to consider $g_h = g_0$. Instead of time t it is convenient to use the relative time $\bar{t} = \frac{t}{T}$, taking as T a certain fictitious time equal to the time during which would burn the whole launching weight of the flight vehicle ($G_0 = G_K + G_{T0}$) with the indicated flow rate per second $\frac{dG}{dt} = g_0\,\frac{dm}{dt}$:

$$T=\frac{G_0}{\frac{dG}{dt}}. \qquad (1.5)$$

In this case with constant per second fuel consumption formula (4) can be recorded in a simpler form:

$$G(t)=G_{\text{к}}+G_{\text{т}0}\left(1-\frac{t}{G_{\text{т}0}}\frac{dG}{dt}\right)=G_0(1-\bar{t}). \qquad (1.6)$$

It is obvious that when $\bar{t}=0$, $G(0)=G_0$, and when $\bar{t}=\bar{t}_{\text{к}}$ (at the end of the powered-flight trajectory phase)

$$G(\bar{t}_{\text{к}})=G_0(1-\bar{t}_{\text{к}}). \qquad (1.6')$$

Usually the magnitude of the final weight of the flight vehicle depends not only on the weight of the actual structure $G_{\text{к}}$, but on the weight of remainders of fuel in the tanks, and only in the limiting case

$$G(\bar{t}_{\text{к}})=G_{\text{к}}.$$

Parameter $\bar{t}_{\text{к}}$ is an important characteristic for design of flight vehicles, especially, of the ballistic type. It is connected by a simple formula with a relative final weight $\mu_{\text{к}}=1-\bar{t}_{\text{к}}$, which in accordance with the known Tsiolkovsky formula determines terminal velocity of the rocket. In other words, parameter $\bar{t}_{\text{к}}$ is its own kind of the criterion of the quality of design, its weight "culture." The larger $\bar{t}_{\text{к}}$, the more perfect (at the assigned value G_{Γ}) is the design of the flight vehicle.

Mass moments of inertia of the flight vehicle with respect to axes of the connected system of coordinates, for instance z_1, are calculated by the formulas of the form

$$J_z=\frac{1}{K_0}\left[\int_0^l q_{0y}(x_1)(x_1-x_{1\text{т}})^2\,dx_1+\sum_{i=1}^{k}G_{yi}(x_{1\text{т}i}-x_{1\text{т}})^2\right]. \qquad (1.6'')$$

where $x_{\text{цт}i}$ – abscissa of centers of gravity of the concentrated loads.

Moments of inertia, mass and the position of the center of gravity characterize dynamic properties of the flight vehicle, considered as solid body.

§ 1.4. Thrust Force

From the equation of motion of the point of variable mass (Meshcherskiy equation) it follows that with the separation from the body of a particle of mass $\frac{dm}{dt}$ with relative speed w_e, acting on this will be reactive force equal to

$$P_{д} = w_e \frac{dm}{dt}. \qquad (1.7)$$

With flight in dense layers of the atmosphere, acting on supports of the combustion chamber, besides force $P_{д}$, will act additional force, proportional to the difference in pressure of the external medium on the external surface of the body of the nozzle p_h and pressure of gases on the nozzle section p_c:

$$P_{ст} = (p_c - p_h) F_0, \qquad (1.8)$$

where F_0 – area of output section of nozzle of motor.

Usually we operate with the total value of these forces, which we call thrust force P:

$$P = P_{д} + P_{ст}. \qquad (1.9)$$

The magnitude of this total force for any height of flight can be expressed in terms of the test stand value of thrust force of the engine P_0, which is found by means of the measurement of forces in rods of the frame of the propulsion system during ground tests.

From formulas (8) and (9) it follows that when $p_h = p_0$ (where p_0 – atmospheric pressure at earth) the test stand value of thrust is equal to

$$P_0 = P_д + F_0(p_c - p_0).$$

Determining from this expression the component $P_д$ (not depending on h) substituting it into (9), we obtain

$$P = P_0 + F_0(p_0 - p_h). \tag{1.10}$$

Thrust force by its nature is a surface force. However, it is transmitted to a body either in the form of concentrated forces (at places of the connection of rods of the frame of the propulsion system) or in the form of a load distributed over the contour of the cross section of the body (in the presence of a comparatively large number of support rods of the frame or with the use of a reinforced shell instead of a rod system).

Since the relative speed w_e depends basically on the calorific value of fuel and is constant for every engine, the character of the change in thrust force with time will be determined by a change in magnitude of flow rate per second, in other words, the design of the engine, the scheme of its starting and turning off and the regime of control of the consumption of fuel components in the process of flight. Reactive force of solid-propellant engines depends on the temperature of the charge.

Laws of the buildup and drop in thrust force in the process of starting and turning off of engines are established usually experimentally. In the first approximation they can be presented in the form of a linear dependence P on t, namely:

$$\left.\begin{aligned} P(t) &= P_0 \frac{t}{T_н} \text{ when } 0 < t < T_н, \\ P(t) &= P_0 \text{ when } t > T_н \end{aligned}\right\} \tag{1.11}$$

with putting the engine into operation (with start) and

$$\left.\begin{array}{ll} P(t)=P_{\kappa} & \text{when } 0<t<t_{\kappa}, \\ P(t)=P_{\kappa}\left(1-\frac{t-t_{\kappa}}{T_{c}}\right) & \text{when } t_{\kappa}<t<(t_{\kappa}+T_{c}), \\ P(t)=0 & \text{when } t>(t_{\kappa}+T_{c}) \end{array}\right\} \tag{1.12}$$

with the turning off of the engine (at the end of the operation).

In certain cases these laws can be described by equations of the form

$$P(t)=P_0(1-e^{-\gamma t}). \tag{1.13}$$

For certain (in particular, for solid-propellent) engines starting peaks of thrust force exceeding P_0 can sometimes be observed.

If one were not to consider gas-dynamic deviations of the vector of thrust force, it is possible to assume that the line of its action coincides with the longitudinal axis of the nozzle and combustion chamber. Then the accuracy of the coincidence of the direction of the action of the thrust force with the longitudinal axis of the flight vehicle passing through the center of gravity will be determined basically by errors in assembling of the very propulsion system and sections of the body of the flight vehicle. Thus, in general it is necessary to take into account the presence of both the lateral component force of thrust $P_{y1}=\Delta\beta P$, and static disturbing moment (relative to the lateral axis passing through the center of gravity of the flight vehicle equal to

$$M_{p\text{в}}=P\left[\Delta y_1-\Delta\beta(x_{1\text{п}}-x_{1\text{т}})\right], \tag{1.14}$$

where $\Delta y_1, \Delta\beta$ – eccentricity and angle of inclination of the vector of the thrust force, and $x_{1\text{п}}$ – abscissa of the place of bracing of the frame of the propulsion system to the body (Fig. 1.6). In the case of the use of several propulsion systems, the appearance is possible of both static (in operating conditions) and dynamic (in transitional regimes of operation) disturbing moments, caused by the scattering of values of thrust force of separate engines.

Fig. 1.6. Component forces of thrust in a connected system of coordinates.

§ 1.5. Aerodynamic Loads

With movement in dense layers of the atmosphere, external surface aerodynamic forces will act on the flight vehicle. Values of these forces at every point of the surface of the body are characterized by the magnitude of normal pressure p^{H} and stress of frictional force p^{τ} (Fig. 1.7).

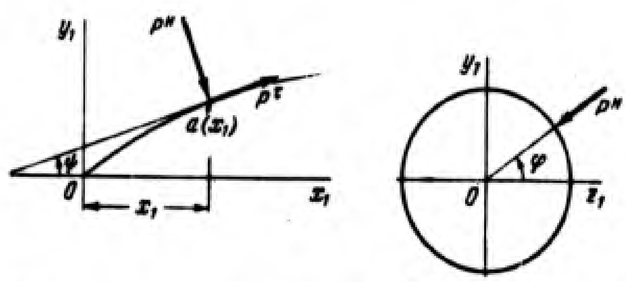

Fig. 1.7. Components of an elementary aerodynamic force.

Normal pressure can be represented in the form of the sum of static atmosphere air pressure p_h and a certain excess in aerodynamic pressure Δp^{H} proportional to the impact pressure q:

$$p^{H} = p_h + \Delta p^{H}.$$

The value of this excess pressure will depend on the angle of attack α, number M_∞ and form of the body. The nature of the distribution Δp^H over the contour of the surface of the body and law of the change in its magnitude with respect to M_∞ and α are usually established experimentally by means of test blowings in wind tunnels of special models. Only for certain forms of bodies can the steady-state values Δp^H be determined theoretically with sufficient accuracy, for instance, for a conical surface at small angles of attack. In particular, with $\alpha = 0$ this pressure, identical for all points of the cone, will depend only on the number M_∞ of incident flow and half-angle of the cone θ_s. The indicated dependence of p_h/p^H on M_∞ and θ_s, determined by tables of Kopal [91], is represented on Fig. 1.8. It is possible to use this figure for the approximation calculus p^H for the front part of the body made from truncated cones, taking local values θ_s. In the presence of blunting in the form of a sphere, the pressure at the front point of this sphere p_A^H can be approximately determined by the Rayleigh formula

$$\frac{p_A^H}{p_h} = 1{,}2M_\infty^2\left(\frac{7{,}2M_\infty^2}{7M_\infty^2 - 1}\right)^{2{,}5}.$$

With angles of attack different from zero, pressure p^H with sufficient accuracy (with $\alpha < 5°$ and $M_\infty < 5°$) can be found by formula

$$p^H(\alpha) = p^H(\alpha = 0) - k\alpha\cos\theta_s.$$

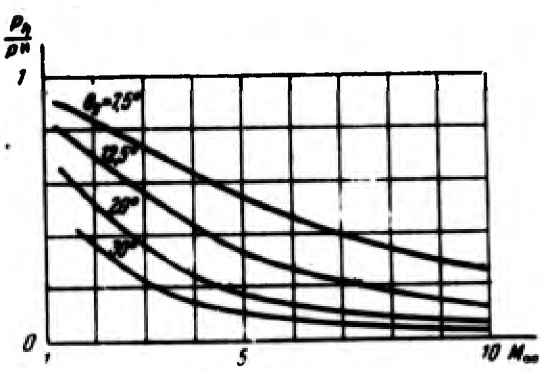

Fig. 1.8. Dependence of relative pressure on M_∞ and θ_s for a cone.

Values of coefficient k are given on Fig. 1.9 in the form of $\frac{p_h}{k}$ as a function of θ_s and M_∞. For $M_\infty > 5$ with success it is possible to use the Newtonian theory, sometimes not making limitations on the angle of attack.

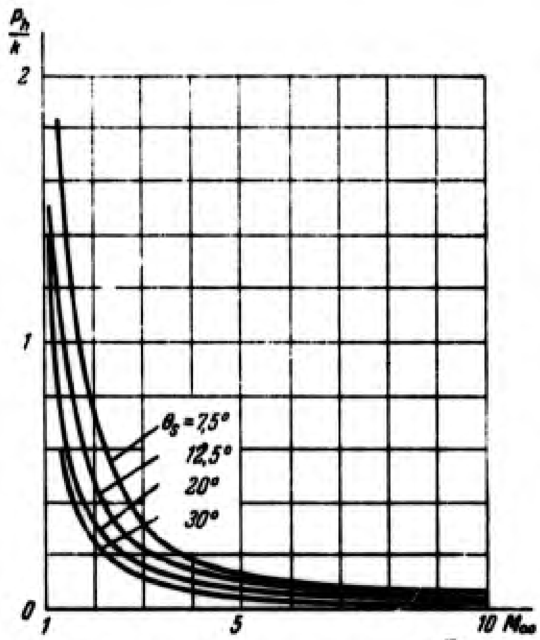

Fig. 1.9. Dependence of $\frac{p_h}{k}$ on M_∞ and θ_s for a cone.

Usually on the front (conical) part of the body, p^H considerably exceeds p_h. On the cylindrical part of the body the normal pressure at supersonic speeds differs little from the atmopsheric pressure. Large negative values of Δp^H (with $M_\infty > 1$) are observed only on the bottom of the body and especially during free flight of the vehicle. The presence of a stream of gases coming out of the nozzle substantially changes the magnitude of the base pressure $p^H_Д$. In certain cases it can even exceed p_h. The dependence $p^H_Д$ on the number M_∞, taking into account the interaction of the external flow with the stream of the engine, is determined experimentally in every concrete case. At high subsonic speeds of flight, large negative values of Δp^H appear beyond corner points of the contour of the body (at places of the joint of conical parts with the cylindrical). In the zone $0.7 < M_\infty < 1.1$ at places of the joint of the cylindrical surface with the conical surface, a sharp change in the local static pressure is observed. The magnitude of this pressure is characterized basically a half-angle of the conical part of the body and angle of attack of the vehicle. For comparatively small values of indicated parameters, the maximum magnitude of it is determined approximately by formula

$$p_{max} = p_h\left[0{,}25 + \sqrt{2}(\sin\theta_s + \alpha\cos\theta_s)^2\right].$$

In most cases to decrease the volume of expensive drain wind tunnel tests, we are limited by the determination of the relative excess pressure $\overline{\Delta p}^H = \frac{\Delta p^H}{q}$ for one or several values of M_∞, assuming that max Δp^H coincides with q_{max}. This assumption is valid for comparatively large numbers M_∞ ($M_\infty > 1.5$). Statistics shows that the nature of the change in $\bar{q} = \frac{q}{q_{max}}$ with respect to $\bar{t}$ for rockets of the ballistic type on the phase of powered flight is approximately identical. In this case usually the maximum of the impact pressure is observed in the region $2.5 > M_\infty > 1.5$ when $\bar{t} = 0.3$-0.5 (Fig. 1.10).

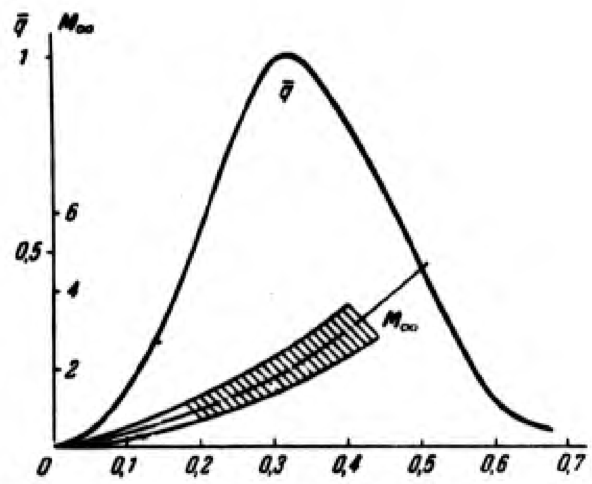

Fig. 1.10. Approximate change with time of the relative impact pressure on the phase of powered flight.

The tangential component p^{τ} of the elementary aerodynamic force is practically identical for all points of the surface of the body. The magnitude of it depends only on the speed and altitude of flight. During calculations for the strength of elements of the structure absorbing aerodynamic forces, the effect of the component can be disregarded almost in all cases.[3]

In the investigation of general strength of the structure, it is necessary most frequently to operate with total forces from aerodynamic forces. Having graphs of the distribution of pressure $\Delta p^{\text{н}}(x_1, \varphi)$ for different contours, it is easy to calculate linear values of components of aerodynamic loads (longitudinal and lateral forces), using the approximate expressions

$$\frac{\partial X(x_1)}{\partial x_1} = 2\pi a(x_1) p^{\tau} + a(x_1) \int_0^{2\pi} \Delta p^{\text{н}}(x_1, \varphi) \operatorname{tg} \psi(x_1)\, d\varphi, \tag{1.15}$$

$$\frac{\partial Y(x_1)}{\partial x_1} = a(x_1) \int_0^{2\pi} \Delta p^{\text{н}}(x_1, \varphi) \sin\varphi\, d\varphi, \tag{1.16}$$

where $a(x_1)$ – radius of the body in section x_1, $\psi(x_1)$ – angle between the x_1 axis and tangent to the generator of the shell of the body in the same section (Fig. 1.7).

Considering the dependence of Δp^{H} on α and q, these formulas are usually written in the form

$$\frac{\partial X_1}{\partial x_1} = qS \frac{\partial c_{x_1}}{\partial x_1},$$

$$\frac{\partial Y_1}{\partial x_1} = qS\alpha \frac{\partial c_n^{\alpha}}{\partial x_1}.$$

Here S – a certain characteristic area, for instance, a wing area of midsection of the body, c_{x_1} – drag coefficient, and c_n^{α} (or $c_{y_1}^{\alpha}$) – derivative with respect to the angle of attack from the coefficient of lateral aerodynamic force so that

$$\frac{\partial c_n^{\alpha}}{\partial x_1} = \frac{a(x_1)}{\alpha S}\int_0^{2\pi} \overline{\Delta p}^{\text{н}}(x_1, \varphi)\sin\varphi \, d\varphi. \qquad (1.17)$$

The main part of the aerodynamic load acts on the head (conical) part of the body. At a distance of two to three diameters from the place of the joint of the conical part with the cylindrical, lift at small angles of attack practically no longer acts. The role of the cylindrical part during calculation $Y_1(x_1)$ substantially increase only at large angles of attack.

Integrating (17) with respect to x_1, let us find values of components of the longitudinal and lateral force and bending moment in any section x_1 of the body from the aerodynamic load:

$$N_a(x_1) = qS\int_0^{x_1} \frac{\partial c_{x_1}(x_1)}{\partial x_1}\, dx_1, \qquad (1.18)$$

$$Q_a(x_1) = qS\alpha\int_0^{x_1} \frac{\partial c_n^{\alpha}(x_1)}{\partial x_1}\, dx_1, \qquad (1.19)$$

$$M_a(x_1) = qS\alpha\int_0^{x_1} dx_1 \int_0^{x_1} \frac{\partial c_n^{\alpha}(x_1)}{\partial x_1}\, dx_1. \qquad (1.20)$$

The force of the base drag

$$X_{\text{п}} = (p_{\text{д}}^{\text{н}} - p_h)\Delta F_{\text{д}},$$

where $\Delta F_{\text{д}}$ – area of the bottom of the body free from the effect of the stream of the engine, with hermetic body of the engine section is applied in the end section (when $x_1 = l$). With a nonhermetic body of the engine section, i.e., with a small distinction in pressure inside the section from the external base pressure, the place of the application of force $X_{\text{п}}$ can be considered the place of connection to the body of the bottom of the hermetic section nearest to the tail.

The sum of $N_a(l)$ and $X_{\text{п}}$ is equal to the total drag of the flight vehicle X. Usually this drag reaches its maximum value in the region of high impact pressures. In most cases the change in relative magnitude $\bar{X} = \frac{X}{X_{max}}$ with respect to $\bar{t}$ for flight vehicles of the ballistic type similar to the change in $\bar{q}(\bar{t})$. In this case on the phase of powered flight it comparatively little depends on flying range of the vehicle and even design of the vehicle. On the phase of free flight the magnitude of this drag depends greatly on speed, and therefore it can reach very high values.

The ratio of $M_a(l)$ to $Q_a(l)$ determines the position of the so-called <u>center of pressure</u> (point of intersection of resultant aerodynamic forces with the longitudinal axis of the flight vehicle). The mutual location of this center of pressure $x_{1\text{д}}$ and center of gravity $x_{1\text{т}}$ characterizes the degree of aerodynamic stability of the flight vehicle, i.e., its ability (under the action of only aerodynamic forces) to return to the initial position after the removal of the external disturbing force, which deflected it from the state of equilibrium (undisturbed motion). If the center of pressure is behind the center of gravity $(x_{1\text{д}} > x_{1\text{т}})$, the vehicle will be aerodynamically (frequently it is said statically) stable, if $x_{1\text{д}} <$ $< x_{1\text{т}}$, then – aerodynamically unstable.

For displacement of the center of pressure back, i.e., for an increase in $x_{1\text{д}}$, sometimes on the tail section of nonmaneuverable flight vehicles stabilizers are specially installed.

For maneuvering flight vehicles lifting surface (wings) of different shape and area are joined to the body. The calculation for strength of these surfaces is usually produced on loads corresponding to the case of flight in the zone of high impact pressures with the highest possible angles of attack (taking into account the effect of wind speed and wing downwash) equal to

$$\left.\begin{aligned} \max Y_{1i} &= q_{max} S_i c_{ni}^{\alpha} \alpha_{max}, \\ \max X_{1i} &= q_{max} S_i c_{x_i i}. \end{aligned}\right\} \quad (1.21)$$

where S_1 – surface area, $c_{x,1}$ and c_{n1}^{α} – drag coefficient and derivative with respect to the angle of attack from the coefficient of lateral force of the lifting surface.

Forces X_{1i}, Y_{1i} are applied to the body at places of the connection of airfoil surfaces to the frames. In this case additional aerodynamic loads, caused by the effect of the body, are distributed along the span of the lifting surface inversely proportional to the square of the distance from the longitudinal axis of the vehicle.

In the investigation of different problems of dynamics of flight, total values of components of aerodynamic forces in a continuous system of coordinates are usually used:

$$\left.\begin{aligned} X &= qSc_x, \\ Y &= qSac_y^{\alpha}, \\ M_z &= qSac_y^{\alpha}(\bar{x}_{1T} - \bar{x}_{1Д})\,l, \end{aligned}\right\} \qquad (1.22)$$

where c_x – drag coefficient, c_y^{α} – derivative with respect to the angle of attack from the coefficient of lift, $\bar{x}_{1Д} = \frac{x_{1Д}}{l}$, and $\bar{x}_{1T} = \frac{x_{1T}}{l}$. Here values of coefficients c_x, c_y^{α} and $\bar{x}_{1Д}$ are determined experimentally by means of weight tests in wind tunnels of special models and not by means of integration of Δp^{H} (17), since weight tests are considerably simpler and more accurate than drain.

In the calculation of loads it is necessary to use both general and distributed aerodynamic properties. Therefore one should always pay serious attention to the coordination of calculations of components $Y_1(x_1)$ and $M_z(x_1)$ according to data of drain and weight tests. In practice this is carried out by means of a certain correction of the diagram $\frac{\partial c_n^{\alpha}(x_1)}{\partial x_1}$ (basically on the cylindrical section of the body).

For flight vehicles of packet configuration equipped with lateral accelerators, the calculation of longitudinal and lateral forces and bending moments from aerodynamic loads is conducted separately for each accelerator (taking into account corresponding reactions at places of their connection).

The nature of the application of lateral stationary aerodynamic forces to the body is basically determined by the law of the change with time of angles of attack.

The effect of angular velocity of the flight vehicle (for instance, $\dot{\vartheta}$) on the magnitude of aerodynamic load can be approximately considered by means of the introduction of additional local angles of attack, since the motion of any section of the body x_1 with linear speed $\dot{\vartheta}(x_{1T}-x_1)$ is equivalent to the motion of air with respect to the vehicle with the same speed but in opposite direction. In other words, it is equivalent to the change in direction of resultant flow rate on angle

$$\frac{\dot{\vartheta}}{v}(x_1 - x_{1T}).$$

Thus, with rotation of the flight vehicle, damping aerodynamic forces will act on it. The moment of these forces with respect to the center of gravity (damping moment) is determined approximately by formula

$$M_{z\text{д}} = -qSl^2 m_z^{\omega} \frac{\dot{\vartheta}}{v}, \qquad (1.23)$$

where

$$m_z^{\omega} = \frac{1}{l^3} \int_0^l \frac{\partial c_n^{\alpha}(x_1)}{\partial x_1} (x_1 - x_{1T})^2 \, dx_1.$$

The forward lateral motion of the vehicle, for instance, with speed $\dot{z}$, leads to a change in the total angle of attack on magnitude

$\Delta\alpha_T = -\frac{\dot{z}}{V}$, and, consequently, to the appearance of a corresponding damping lateral force

$$Z_{\text{да}} = -qSc_z^{\beta}\frac{\dot{z}}{v}.$$

On the section of powered flight angular velocities of the guided flight vehicle are usually small (less than 10°/s). Therefore, the influence of $\dot{\vartheta}$, $\dot{\psi}$ on the magnitude and nature of distribution of the lateral component of aerodynamic force in most cases can be disregarded. On descent phase the angular velocities can be considerable, and, consequently, the damping aerodynamic forces can have a substantial effect on parameters of disturbed motion of the flight vehicle. The question of the necessity of calculation of their effect on $Q_a(x_1)$ and $M_a(x_1)$ is solved in each concrete case.

In the process of ground operation, acting on the flight vehicle will be an aerodynamic force equal to

$$Z = qS_M c_z, \tag{1.24}$$

where q – impact pressure of air flow, S_M – projection of the surface area blowed by wind, and c_z – drag coefficient referred to S_M. Values of this coefficient depend on the state of the surface of the vehicle and its form behind the flow. Besides the force (24) proportional to the impact pressure, in the given case of loading on the vehicle there will act forces proportional to the acceleration of the flow and dynamic forces caused by the formation vortexes outgoing from boundaries of flow separation. This question is investigated in detail in Chapter VI.

Nonstationary aerodynamic loads appearing in flight, conditioned by the turbulence of the boundary layer, instability of the interaction of the shock wave with the boundary layer and flow separations, affect basically the vibration and local strength of construction of the body of the flight vehicle and in this book are not considered.

§ 1.6. Acoustic Loads

On the boundary of the gas stream passing from the nozzle of the jet or rocket engine with supersonic speed, a zone of turbulent flow, which generates in the surrounding air space sound waves of a different frequency will usually be formed. The thickness of this turbulent zone (and, consequently, limiting scales of the vortexes spreading downwards over the flow) is increased continuously with distance from the section of the nozzle. The spectrum of frequencies of pulsations of the stream of acoustic pressure radiated by points is correspondingly changed. In the region of little developed (small-scale) turbulence, located near the section of the nozzle, sources of high-frequency sound waves are found, and in the region of a stream with completely developed turbulence – sources of basically low-frequency sound waves. In the region of mixing in which the flow remains supersonic, the generating of additional pulsations of pressure, caused by the interaction of shock waves with turbulence is possible.

The intensity of the noise created by such a stream is proportional to its average speed in the degree from 6 to 8. Here its acoustic power is 0.4-0.8% of the mechanical power of motor. It follows from this that with an increase in thrust force of the engines of the flight vehicles, a proportional increase of the total level of acoustic load on the surface of the body of their structure is possible. In those cases when these levels will have an order of 150-160 and more decibels,[4] the effect of acoustic loads will have a substantial effect on conditions of vibrations of elements of construction of the flight vehicle, on the functioning of different instruments and on the fatigue longevity of separate parts of the body of the vehicle located near the stream of the engine. In certain cases, for instance, with the annular location of nozzles of a multiple-chamber engine (when there are comparatively large free areas of the bottom of the body of the flight vehicle between the nozzles), the pulsation of acoustic pressure inside this region can lead to forced longitudinal elastic oscillations of the structure as a whole. In other words, even those units of construction which are inside the flight vehicle can have an effect on the strength.

It is not possible to establish by means of calculation the field of sound pressure near the stream of the engine with sufficient accuracy, mainly because of difficulties in the determination of parameters of the field of turbulence of stream. Therefore, usually it is necessary to use experimental data. The latter can be obtained not only with natural tests (test stand or flying) the vehicles or their engines, but also by means of appropriate recalculation of results of measurements of parameters of random pulsation of the acoustic pressure generated by streams of the engine of a similar flight vehicle. In certain cases these data can be obtained by means of modeling of the process by jet or cold or hot air (in scales from 1/5 to 1/10) with the use as the scale of frequency of the Strouhal number

$$Sh_a = \lambda \frac{d_c}{w_e},$$

where d_c – diameter of the output section of the nozzle, λ – frequency in cycles per second and w_e – average speed of the jet. There are experimental data[5] which show that for $Sh_a < 0.4$ corresponding spectra of the pulsation of acoustic pressure radiated by the stream of air and stream of the actual engine practically coincide. In this case the essential dependence of acoustic power (at equal average speeds) on the temperature of the jet is not revealed.

The method of the mentioned recalculation ensues from the aforementioned simplified physical model of the process and theory of Lighthill, according to which a moving turbulent field of the jet can be schematically represented in the form of a fixed field sources (quadrupoles) located along the axis of the stream, each of which generates sound waves of only a definite frequency.

In general parameters of a sound field of pressure are functions of not only the power of the propulsion system, but also distances of the considered point of space from points of the axis of the jet. If the mentioned distances are quite great, the sound field of pressure (far field) radiated by the jet will practically be equivalent to the field of one point source. If the distance of the point from the sound sources is comparable to the distance between the sources, the field of pressure at this point (the near field) will be determined by the entire totality of the sources. In this case its intensity will essentially depend on the presence near the given point of different barriers capable of reflecting sound waves in the direction of this point.

The near field has a directional nature. The largest value of the total level of pressure in it is noted in a direction forming with the axis of the jet an angle of the order of 50-70°. The magnitude of this total level drops with an increase in distance of the considered point of the surface of the flight vehicle from the section of the nozzle basically due to damping of high-frequency pulsations of pressure. Usually for strength of structure the greatest interest is in the low-frequency part of the spectrum of the acoustic load. Therefore, in the first approximation the spectral density of the pulsation of sound pressure at comparatively low speeds of the flight vehicle can be presented in the form

$$\Phi_a(\mathrm{Sh}_a) = \varkappa_a \frac{B_a(\mathrm{Sh}_a)}{\bar{l} - \bar{x}_1 - \bar{x}_\lambda(\mathrm{Sh}_a)}, \qquad (1.25)$$

where

$$\bar{l} = \frac{l}{d_c}, \quad \bar{x}_1 = \frac{x_1}{d_c}, \quad \bar{x}_\lambda = \frac{x_\lambda}{d_c},$$

and $B_a(\mathrm{Sh}_a)$ is a certain function of the Strouhal number, which characterizes the energy generated by the source located at distance x_λ from the nozzle section. Mean values of these functions B_a and $\bar{x}_\lambda$, obtained[6] by means of averaging of data of measurements of spectral density of pressure recounted by the formula (25) at different points of the surface for different flight vehicles of the ballistic type shown on Fig. 1.11. In this case the effect of such factors as wave drag of the environment ρa, exhaust velocity of the gas w_e and diameter of the outlet section of the nozzle d_c is considered by the coefficient

$$\varkappa_a = \frac{\pi}{8} \rho a \rho_c d_c w_e^2 = \frac{\rho a}{2d_c} P_{\text{Д}}, \qquad (1.26)$$

where ρ_c – density of gas of the jet, a – speed of sound in the environment, $P_{\text{Д}}$ – reactive force.

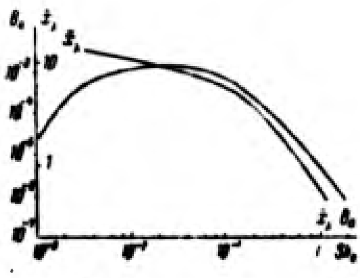

Fig. 1.11. Mean values of functions $B_a(Sh_a)$ and $\bar{x}_1(Sh_a)$.

As can be seen from this formula, the value of spectral density (with constant exhaust velocity) is changed in proportion to the diameter of the nozzle d_c, where with an increase in d_c the peak of the spectral density is displaced into the region of lower frequencies.

The standardized value of coefficients of correlation (see Chapter IV) of the pulsation of acoustic pressure along the axis of the flight vehicle can be approximately described by the function, and the coefficient

$$\sigma(x_1, x_{12}) = \cos\frac{2\pi\lambda}{a}(x_1 - x_{12}),$$

and the coefficient of correlation in a circumferential direction when

$$\frac{\pi a(x_1)}{a}(\varphi - \varphi_1) \leqslant 1$$

in the first approximation can be taken equal to one. Usually the most intensive acoustic loads are observed in the process of launch of the flight vehicle. Peak values of sound pressure in the period of launch of the flight vehicle. Peak values of sound pressure in the period of starting of the engine can in certain cases exceed several times pressures corresponding to its established operating conditions. With an increase in flight speed the total level of sound pressure drops, and when $M_\infty > 1$ sound waves generated by the jet of the reaction engine do not reach parts of the body of flight vehicle located ahead of this engine.

§ 1.7. Control Forces

For the automatic guiding of the flight vehicle over a defined trajectory in accordance with the assigned flight program or to accomplish required maneuvers and also for providing stability to its motion under action of external disturbances of a static nature, a special system of control is used. It usually consists of sensing devices recording deviations of flight vehicle from the assigned position in space, amplifying-converting devices and effectors, which create the necessary control forces in planes of pitch R_{y_1}, yawing R_{z_1} and bank ΔR_{y_1}. Values of necessary controlling moments are determined basically by the assignment of the flight vehicle and its aerodynamic and dynamic characteristics. The correct selection of maximum magnitudes of control forces, the rate of their change and place of the application to the body are important for the strength of the structure as a whole, especially in cases of loading connected with the low-quality of emergency operation of the control system (see Chapter XIII).

As a rule, the effectors are located in the tail section of the flight vehicle, further away from the center of gravity, which makes it possible to obtain the maximum controlling moment with minimum values of control forces. It is seldom found in the nose section. Control forces creating the moment in the plane of bank are usually applied at places most remote from the longitudinal axis of the vehicle (on wings, stabilizers).

Used as effectors can be aerodynamic ("air") vanes, so-called gas-jet vanes, working in the jet of the engines, special oscillatory jet engines of low thrust force, and finally, basic engines (turning, adjustable and others).

Aerodynamic vanes are used only for control in atmospheric phase of motion of the flight vehicle. They are practically ineffective in the beginning of flight at low speeds and at comparatively high altitudes at low air density. Values of the control force created by these vanes depend basically on their area, magnitude of impact pressure and angle of deviation δ:

$$R_{y_1} = qS_p c_{yp}(\delta),$$
$$R_{x_1} = qS_p c_{xp}(\delta),$$

where S_p – area of the vane, c_{yp} – coefficient of lift of the vane, c_{xp} – drag coefficient of the vane. Here coefficients c_{xp}, c_{yp} can be nonlinear functions of the angle δ. The form of the vanes in the plan is selected in order to obtain the maximum value $\frac{\partial R_{y_1}}{\partial \delta}$ with minimum area S_p and to ensure a small change with respect to M_∞ and δ of the hinge moment on the shaft of vane: $M_{\text{шр}} = Y_p \Delta x_{\text{др}}$. Here $\Delta x_{\text{др}}$ denotes the distance from the center of pressure on the vane to the axis of rotation, and Y_p – lateral load on the vane equal to

$$Y_p = R_{y_1}\cos\delta + R_{x_1}\sin\delta. \tag{1.27}$$

In the preliminary calculation for the strength of the structure of the actual vanes and elements of their bracing to the body, it is possible to be oriented on maximum values of R_{y_1}, and in calculation of aerodynamic loads on the body of the vehicle – on balancing values. The latter are understood to be those values of control forces at which the equality of controlling and disturbing moments in the considered plane of stabilization is satisifed.

The direction of the action of the balancing lateral load is taken depending upon the mutual location of sections x_{1p}, $x_{1д}$ and $x_{1т}$. When $x_{1p} > x_{1д} > x_{1т}$, i.e., for an aerodynamically stable vehicle, direction R_{y_1} is opposite to the direction of action of the total lateral force Y_1. When $x_{1p} > x_{1т} > x_{1д}$, i.e., in case of an aerodynamically unstable vehicle, the direction of the action of forces R_{y_1} and Y coincide. In this case loads on the aerodynamic vanes prove to be larger than those in the first case:

$$Y_{p\text{б}} = qS_{p}c_{yp}(\delta_{\text{б}}+\alpha)\cos\delta_{\text{б}} + qS_{p}c_{xp}(\delta_{\text{б}}+\alpha)\sin\delta_{\text{б}}, \tag{1.28}$$

$$M_{\text{шp}} = Y_{p\text{б}}\Delta x_{\text{д}p}, \tag{1.29}$$

where $\delta_{\text{б}}$ – balancing value of the angle of deviation of vanes.

The nature of distribution of the lateral load Y_{1p} over the chord and span of the vane is determined experimentally. The normal pressure at every point of its surface considerably depends not only on the number M_∞ of incident flow, but also on the angle of attack of the actual vane $(\delta + \alpha)$. Usually the maximum of lateral load for nonmaneuvering flight vehicles is observed in the zone of high impact pressures, and the maximum of the hinge moment, due to the dependence of $\Delta x_{\text{д}p}$ on M_∞ (especially in the region $M_\infty = 0.6\text{-}1.2$), cannot coincide with max Y_{1p}.

Lift R_{y_1} and drag R_{x_1} of the gas-jet vanes, which operate in the flow of gases flowing from the nozzle of the engine, depend only on the angle of their deviation δ and on the time of operation of the engine and do not depend on the al itude and speed of flight.

The effect of the time of operation of the engine is expressed on the nature of the burnout of vanes, i.e., on the magnitude of the area S_p of the vane and its form in the plan. The impact pressure of the bas jet $q_{\text{г}}$ with the established regime of operation of the engine practically does not change. Therefore, the calculation for strength of gas-jet vanes and elements of their bracing to the body is produced on loads corresponding to the maximum possible value of δ on the initial phase of the flight.

Substantial deficiencies of gas-jet vanes are the loss of part of the thrust force on drag, limitedness of the time of their operation and limitedness of the angle of their turn (conditions of continuous flow). These deficiencies are not inherent in actuating control devices made in the form of turning motors, which create a lateral control force R_{y_1} by means of turning the axis of thrust:

$$\left.\begin{aligned} R_{y_1} &= P_y \sin\delta, \\ R_{x_1} &= -P_y \cos\delta, \end{aligned}\right\} \qquad (1.30)$$

where P_y – thrust of the controlling motor. Here the minus sign indicates the direction of force R_{x_1} opposite the axis x_1, i.e., in the direction of force P. Here the value of δ is entirely determined by the magnitude of the thrust force P_y.

If special controlling motors of low thrust are used, then these angles can reach $\pi/2$. At large values of P_y, for instance, with a turn of the combustion chamber or nozzle of the main engine, necessary angles of deviation δ prove to be small, and it is possible with sufficient accuracy to consider that

$$\left.\begin{aligned} R_{y_1} &= P\delta, \\ R_{x_1} &= 0. \end{aligned}\right\} \qquad (1.31)$$

The nature of the application of control forces to the body in general depends on the nature of the change with time of external disturbing forces but basically on parameters of drive of the vanes, in particular, the rate of change of angles δ. The time of the buildup of R_{y_1} from zero to a value corresponding to any δ cannot be less than $t_{min} = \frac{\delta}{\max \dot{\delta}}$, and the amplitude of the steady-state oscillations with frequency ω greater than $\Delta\delta_m = \frac{\max \dot{\delta}}{\omega}$. From the point of view of strength of the structure, it is desirable to have minimum values of $\dot{\delta}$ and δ.

CHAPTER II

INERTIAL LOADS

§ 2.1. Dynamic Factors

In the proceeding chapter a brief characteristic of external forces acting on the flight vehicle in flight was given, and methods of the determination of forces in cross sections of the body separately from aerodynamic forces and gravity are shown. To find the total internal forces appearing as a result of the interaction of parts of the body with accelerated motion of the vehicle, it is necessary to use the d'Alembert principle, i.e., conditionally introduce appropriate forces from forces of inertia. The value of these forces of inertia in a number of cases can be obtained directly from conditions of dynamic equilibrium.

It is obvious that components of inertial forces in the continuous system of coordinates will be equal to the sum of projections on axes x, y, and z all external forces acting on the flight vehicle. According to Fig. 1.1 for the phase of powered flight at small angles of attack and slip, i.e., when $\sin\alpha \approx \alpha$, $\sin\beta \approx \beta$, $\cos\alpha \approx \cos\beta \approx 1$ these equalties will take the following form:

$$-m\dot{v} + P - R_{x_1} - X - G\sin\theta = 0, \tag{2.1}$$

$$-m\ddot{y} + (P - R_{x_1} + qSc_y^{\alpha})\alpha + R_{y_1} - G\cos\theta = 0, \tag{2.2}$$

$$-m\ddot{z} - (P - R_{x_1} + qSc_z^{\beta})\beta - R_{z_1} = 0. \tag{2.3}$$

Equating the sums of moments of all external forces with respect to axes y and z, which pass through the center of gravity of the flight vehicle, by corresponding inertial moments, we obtain equations

$$J_z\ddot{\vartheta} + qSl^2 m_z^{\bar{\omega}}\frac{\dot{\vartheta}}{v} + qSc_{y_1}^{\alpha}\alpha(x_{1\text{д}} - x_{1\text{т}}) + R_{y_1}(x_{1\text{р}} - x_{1\text{т}}) = 0, \tag{2.4}$$

$$J_y\ddot{\psi} + qSl^2 m_y^{\bar{\omega}}\frac{\dot{\psi}}{v} - qSc_{z_1}^{\beta}\beta(x_{1\text{д}} - x_{1\text{т}}) - R_{z_1}(x_{1\text{р}} - x_{1\text{т}}) = 0. \tag{2.5}$$

For the atmospheric phase of free flight (with a nonoperating propulsion system) at a small angle of roll and, in general, large angles of attack and slip, in accordance with Fig. 2.1.

$$\left.\begin{aligned}
-m_{\text{г}}\dot{v} + G_{\text{г}}\sin\theta - X = 0,\\
m_{\text{г}}\ddot{y} + G_{\text{г}}\cos\theta - Y = 0,\\
m_{\text{г}}\ddot{z} - Z = 0,\\
J_{z\text{г}}\ddot{\vartheta} + (Y\cos\alpha + X\sin\alpha)(x_{1\text{д}} - x_{1\text{т}}) = 0,\\
J_{y\text{г}}\ddot{\psi} + (Z\cos\beta + X\sin\beta)(x_{1\text{д}} - x_{1\text{т}}) = 0.
\end{aligned}\right\} \tag{2.6}$$

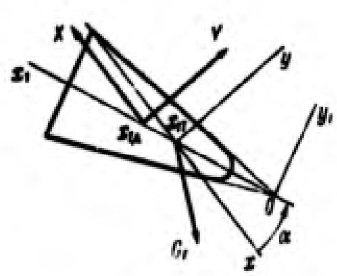

Fig. 2.1. Diagram of forces acting on a descending vehicle during free flight.

In practice it is convenient to consider the inertial force jointly with gravity, characterizing their total values of a certain vector the n, called dynamic factor. In magnitude the dynamic factor will be equal to the ratio of full acceleration, which would be obtained by the flight vehicle outside the field of terrestrial gravitation under the action of external surface forces, to the acceleration of gravity. The direction of the action of this overload is opposite to the direction of full acceleration.

Let us designate the components of the dynamic factor of the center of gravity of the flight vehicle in the continuous system of coordinates by n_x^0, n_y^0, n_z^0 and call them tangential, normal, and binormal accelerations respectively. Thus, by definition, the **tangential overload in the center of gravity of the vehicle will be equal to**

$$n_x^0 = -\frac{1}{g_0}(\dot{v} + g_0 \sin\theta). \tag{2.7}$$

Here $\dot{v}$ is the real acceleration of the center of gravity of the flight vehicle in the direction of the x axis, determined from equation (1), and the second term is the tangential component of acceleration of gravity. The normal acceleration (in the pitch plane) will be proportional to the sum of the normal (centripetal) acceleration and normal component of the acceleration of gravity

$$n_y^0 = -\frac{1}{g_0}(\ddot{y} + g_0 \cos\theta). \tag{2.8}$$

The binormal overload in the center of gravity of the vehicle (in the yawing plane) is determined by formula

$$n_z^0 = -\frac{1}{g_0}\ddot{z}. \tag{2.9}$$

If the aforementioned conditions of dynamic equilibrium are used, then the expression for these components of the overload and in terms external surface forces can be obtained. Actually, from formula (7) and equation (1), it follows that on the phase of powered flight n_x^0 is equal to

$$n_x^0 = -\frac{1}{G}(P - R_{x_1} - X). \tag{2.10}$$

On phase of free flight

$$n_x^0 = \frac{X}{G}. \tag{2.10'}$$

In this case, as one can see, the direction of the tangential overload (10) and (10') on the indicated phases of flight are opposite. For the undisturbed motion of a guided flight vehicle when $= \ddot{\psi} = 0$ from equations (2), (4) and (8) it follows that

$$n_y^0 = -\frac{1}{G}\left(P - R_{x_1} + qSc_y^\alpha \frac{x_{1p} - x_{1\text{д}}}{x_{1p} - x_{1\text{т}}}\right)\alpha. \tag{2.11}$$

During free unguided flight the normal and binormal components of overload in the center of gravity of the flight vehicle correspondingly will be equal to

$$\left.\begin{aligned} n_y^0 &= -\frac{Y}{G}, \\ n_z^0 &= -\frac{Z}{G}. \end{aligned}\right\} \tag{2.12}$$

The minus sign in formulas (11) and (12) indicates the difference in directions of action of the total mass force and resultant of external surface forces. From these formulas it follows that with the equality to zero of any projection of the resultant of external surface forces, a corresponding component of the dynamic factor will be equal to zero.

In calculation of loads and experimental determination of mass forces in flight, it is more convenient to use components of the dynamic factor in the connected system of coordinates $x_1y_1z_1$. By analogy let us designate them by n_{x_1}, n_{y_1}, n_{z_1} and call them

longitudinal, lateral, and side accelerations, respectively. The expression of these accelerations n_x, n_y, and n_z can easily be found if one were to use known formulas of the transformation of coordinates. Considering designations given in § 1 of Chapter I of angles between axes of the continuous and connected systems of coordinates, we will have when $\phi = 0$

$$n^0_{x_1} = -n^0_x \cos\alpha\cos\beta - n^0_y \sin\alpha\cos\beta + n^0_z \sin\beta,$$
$$n^0_{y_1} = -n^0_x \sin\alpha + n^0_y \cos\alpha,$$
$$n^0_{z_1} = -n^0_x \cos\alpha\sin\beta - n^0_y \sin\alpha\sin\beta - n^0_z \cos\beta.$$

Dynamic factors $n^0_{x_1}$ and $n^0_{y_1}$ are the most important dimensionless parameters of flight vehicles, which in many respects determine their assembly scheme and necessary strength of the structure.

The magnitudes of these overloads are different for various classes of flight vehicles. Thus, for instance, for maneuvering flight vehicles the presence of great lateral overloads is characteristic: of the order of 6-9 for piloted vehicles (maneuvering aircraft, launched spacecraft) and of the order of 20-30 for unpiloted (surface-to-air guided rockets and others).[1] For nonmaneuvering pilotless flight vehicles there are great longitudinal accelerations, which (depending upon the assignment of the craft) can be measured in the tens and even hundreds of units.

At small angles α and β formulas for the above accelerations can approximately be represented in the form (Fig. 2.2)

$$n^0_{x_1} = -(n^0_x + \alpha n^0_y) + \beta n^0_z, \qquad (2.13)$$
$$n^0_{y_1} = n^0_y - \alpha n^0_x, \qquad (2.14)$$
$$n^0_{z_1} = -n^0_z - \beta n^0_x. \qquad (2.15)$$

The minus sign in (13) considers the different direction of axis x_1 and x. From (11) it follows that at small α (or β) magnitudes of normal (or binormal) accelerations will also be small. In this case terms αn^0_y and βn^0_z can be disregarded as compared to n^0_x, i.e., one can take $n^0_{x_1} = -n^0_x$.

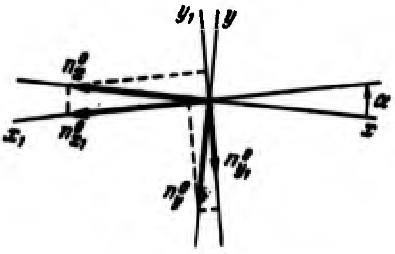

Fig. 2.2. Components of the dynamic factor in high-speed and connected systems of coordinates.

If n_x^0 is great as compared to n_y^0, then the effect of the term αn_x^0 on $n_{y_1}^0$ $(n_{x_1}^0)$ cannot be disregarded.

The formula for the longitudinal acceleration can be simplified if one were to consider that the influence of drag is expressed by $n_{x_1}^0$ (on the phase of powered flight) only in the zone of maximum impact pressures. Since on this phase of the flight in most cases values of q_{max} are comparatively small and limited, then this effect of drag will noticeably decrease with an increase in lateral force and initial weight of the flight vehicle. Therefore, for heavy vehicles the calculation of $n_{x_1}^0$ with sufficient accuracy can be conducted according to the approximate formula

$$n_{x_1}^0 = \frac{P - R_{x_1}}{G}, \tag{2.16}$$

which gives somewhat oversized values of the longitudinal acceleration.

Using the concept of specific thrust $P_{уд} = \frac{P - R_{x_1}}{G}$ and formulas (16), (1.5), (1.6), the following expression for this component of the acceleration can be obtained:

$$n_{x_1}^0 = \frac{P_{уд}}{T(1-\bar{t})}. \tag{2.16'}$$

Let us designate the components of the dynamic factor of the center of gravity of the flight vehicle in the continuous system of coordinates by n_x^0, n_y^0, n_z^0 and call them tangential, normal, and binormal accelerations respectively. Thus, by definition, the tangential overload in the center of gravity of the vehicle will be equal to

$$n_x^0 = -\frac{1}{g_0}(\dot{v} + g_0 \sin\theta). \qquad (2.7)$$

Here $\dot{v}$ is the real acceleration of the center of gravity of the flight vehicle in the direction of the x axis, determined from equation (1), and the second term is the tangential component of acceleration of gravity. The normal acceleration (in the pitch plane) will be proportional to the sum of the normal (centripetal) acceleration and normal component of the acceleration of gravity

$$n_y^0 = -\frac{1}{g_0}(\ddot{y} + g_0 \cos\theta). \qquad (2.8)$$

The binormal overload in the center of gravity of the vehicle (in the yawing plane) is determined by formula

$$n_z^0 = -\frac{1}{g_0}\ddot{z}. \qquad (2.9)$$

If the aforementioned conditions of dynamic equilibrium are used, then the expression for these components of the overload and in terms external surface forces can be obtained. Actually, from formula (7) and equation (1), it follows that on the phase of powered flight n_x^0 is equal to

$$n_x^0 = -\frac{1}{G}(P - R_{x_1} - X). \qquad (2.10)$$

On phase of free flight

$$n_x^0 = \frac{X}{G}. \qquad (2.10')$$

In this case, as one can see, the direction of the tangential overload (10) and (10') on the indicated phases of flight are opposite. For the undisturbed motion of a guided flight vehicle when $= \ddot{\psi} = 0$ from equations (2), (4) and (8) it follows that

$$n_y^0 = -\frac{1}{G}\left(P - R_{x_1} + qSc_y^{\alpha}\frac{x_{1\text{р}} - x_{1\text{д}}}{x_{1\text{р}} - x_{1\text{т}}}\right)\alpha. \qquad (2.11)$$

During free unguided flight the normal and binormal components of overload in the center of gravity of the flight vehicle correspondingly will be equal to

$$\left.\begin{aligned} n_y^0 &= -\frac{Y}{G}, \\ n_z^0 &= -\frac{Z}{G}. \end{aligned}\right\} \qquad (2.12)$$

The minus sign in formulas (11) and (12) indicates the difference in directions of action of the total mass force and resultant of external surface forces. From these formulas it follows that with the equality to zero of any projection of the resultant of external surface forces, a corresponding component of the dynamic factor will be equal to zero.

In calculation of loads and experimental determination of mass forces in flight, it is more convenient to use components of the dynamic factor in the connected system of coordinates $x_1y_1z_1$. By analogy let us designate them by n_{x_1}, n_{y_1}, n_{z_1} and call them

longitudinal, lateral, and side accelerations, respectively. The expression of these accelerations n_x, n_y, and n_z can easily be found if one were to use known formulas of the transformation of coordinates. Considering designations given in § 1 of Chapter I of angles between axes of the continuous and connected systems of coordinates, we will have when $\phi = 0$

In beginning of flight, i.e., when $\bar{t} = 0$ and $p_{уд} = p_{уд\,0}$,

$$n^0_{x_1 0} = \frac{P_0}{G_0} = \frac{p_{уд\,0}}{T}. \tag{2.17}$$

At the end of the phase of powered flight (at great altitudes) when $\bar{t} = \bar{t}_к = 1 - \mu_к$ and $p_{уд} = p_{уд.п}$

$$n^0_{x_1 к} = \frac{p_{уд.п}}{T\mu_к} = n^0_{x_1 0}\frac{p_{уд.п}}{\mu_к\, p_{уд.0}}, \tag{2.18}$$

where $p_{уд.п}$ – specific thrust of the engine in a vacuum, and $p_{уд.0}$ – near the earth (taking into account losses by control effectors).

Since the specific thrust of liquid-propellant rocket engines in a vacuum usually consists approximately 1.15 of the specific thrust near the earth, then approximately

$$n^0_{x_1 к} = 1{,}15\,\frac{n^0_{x_1 0}}{\mu_к}.$$

In the fulfillment of a preliminary calculation of longitudinal loads for nonmaneuvering flight vehicles of the ballistic type, it is possible in first approximation to take linear law of the change in $p_{уд}$ with respect to $\bar{t}$

$$\left.\begin{aligned} p_{уд} &= p_{уд\,0}\left[1 + \frac{p_0 F_0}{P_0}\left(\frac{\bar{t} - \bar{t}_{p0}}{\bar{t}_p - \bar{t}_{p0}}\right)\right] \text{ when } \bar{t}_{p0} < \bar{t} < \bar{t}_p, \\ p_{уд} &= p_{уд.п} = p_{уд\,0}\left(1 + \frac{p_0 F_0}{P_0}\right) \text{ when } \bar{t} > \bar{t}_p, \end{aligned}\right\} \tag{2.19}$$

where $\bar{t}_{p0}$, $\bar{t}_p$ – relative time of the beginning and end of the change in thrust force with the altitude of the flight (tentatively t_{p0} = = 0.04-0.09, and t_p = 0.45-0.55).

Then (for $t_{p0} < \bar{t} < \bar{t}_p$)

$$n^0_{x_1} = \frac{n^0_{x_1 0}}{(1-\bar{t})}\left[1 + \frac{p_0 F_0}{P_0}\left(\frac{\bar{t}-\bar{t}_{p0}}{\bar{t}_p-\bar{t}_{p0}}\right)\right]. \tag{2.19'}$$

The above formulas (from (10) to (19')) are derived for the case of the undisturbed motion of the flight vehicle. In the presence of the unbalanced moment of surface forces with respect to the center of gravity the local value of components of the dynamic factor in any cross section x_1 of the body not coinciding with x_{1T} will differ from the above values $n^0_{x_1}$, $n^0_{y_1}$ and $n^0_{z_1}$ due to the effect of angular velocity and regular acceleration. Components of corresponding additional linear accelerations in the connected system of coordinates will be, obviously, equal to: $\ddot{\vartheta}(x_{1T}-x_1)$ in the direction of y_1 axis, $\ddot{\psi}(x_{1T}-x_1)$ in the direction of z_1 axis and $(x_{1T}-x_1)(\dot{\vartheta}^2+\dot{\psi}^2)$ in the direction of the x_1 axis. Consequently, formulas for additional components of dynamic factors will have the form

$$\left.\begin{aligned} \Delta n_{x_1}(x_1) &= \frac{1}{g_0}(x_1 - x_{1T})(\dot{\vartheta}^2+\dot{\psi}^2), \\ n^x_{y_1}(x_1) &= \frac{\ddot{\vartheta}}{g_0}(x_1 - x_{1T}), \\ n^x_{z_1}(x_1) &= \frac{\ddot{\psi}}{g_0}(x_1 - x_{1T}). \end{aligned}\right\} \tag{2.20}$$

Thus, in the general case of the disturbed motion the later longitudinal and side accelerations at an arbitrary point x_1 of the longitudinal axis of the body will be determined by expressions

$$\left.\begin{aligned} n_{x_1}(x_1) &= n^0_{x_1} + \Delta n_{x_1}(x_1), \\ n_{y_1}(x_1) &= n^0_{y_1} + n^x_{y_1}(x_1), \\ n_{z_1}(x_1) &= n^0_{z_1} + n^x_{z_1}(x_1). \end{aligned}\right\} \tag{2.21}$$

For nonmaneuvering flight vehicles values of $\dot{\vartheta}$ and $\dot{\psi}$ on the phase of powered flight are comparatively small, and the effect of Δn_{x_1} on n_{x_1} practically proves to be insignificant. On the section of free flight Δn_{x_1} can be the same order with $n^0_{x_1}$.

At points of the body remote by distance y_1 from plane x_1z_1, with oscillations in the pitch plane, an additional rotary component of longitudinal acceleration appears equal to

$$\Delta n^{x}_{x_1}(y_1) = -\frac{1}{g_0} y_1 \ddot{\vartheta}. \tag{2.22}$$

Similarly, with oscillations in the plane of yawing

$$\Delta n^{x}_{x_1}(y_1) = -\frac{1}{g_0} z_1 \ddot{\psi}. \tag{2.22'}$$

These components of longitudinal acceleration prove to be important only for flight vehicles having comparatively large lateral dimensions, or at very high angular accelerations, for instance, for a descending unpiloted vehicle.

The sign and magnitude of the rotary component of the lateral acceleration $n^{\varepsilon}_{y_1}$ depend on the location of section x_1 with respect to the center of gravity of the flight vehicle and magnitude and direction of the action of the unbalanced moment of surface forces. Thus, for instance, with the free unguided flight of an aerodynamically stable apparatus unbalanced the aerodynamic moment will be determined by expression $Y_1(x_{1д} - x_{1т})$. In this case the lateral acceleration in any section x_1, on the basis of formulas (20), (21), the corresponding equation (4) and expression $n^0_{y_1} = -\frac{Y_1}{G}$, will be equal to

$$n_{y_1}(x_1) = -\frac{Y_1}{G}\left[1 + \frac{G}{g_0 J_z}(x_{1д} - x_{1т})(x_1 - x_{1т})\right]. \tag{2.23}$$

As can be seen from this formula and from Fig. 2.3, it decreases in the front section of the body (when $x_1 < x_{1т}$) and increases in the tail section, and at a certain point x_{11} is equal to zero. If the flight vehicle proved to be for some reason aerodynamically unstable $(x_{1д} < x_{1т})$, then n_{y_1} would be increased in the front section of the body and decreased in the tail section.

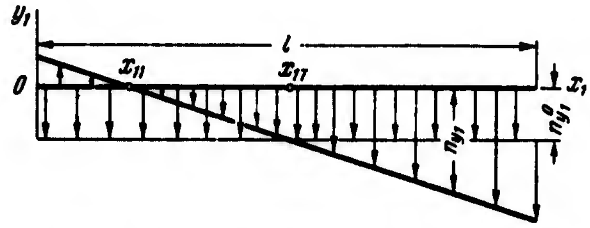

Fig. 2.3. Change in lateral acceleration along the length of a flight vehicle.

In the general case of disturbed motion of a guided flight vehicle, values of components of the acceleration can be determined only by means of solution of the corresponding system of equations of dynamic equilibrium.

§ 2.2. Perturbation Equations

The system of differential equations (1)-(5) with nominal values of coefficients and program values of the angle attack jointly with evident relations

$$\dot{y}_0 = v \sin\theta, \quad \dot{x}_0 = v \cos\theta,$$

where x_0 and y_0 are coordinates of the center of gravity in the terrestrial system of coordinates, describes the undisturbed motion of a flight vehicle on the phase of powered flight.

Integrating this system of equations with variable coefficients by some approximation method, we obtain the values of speed and altitude of flight necessary for the calculation of aerodynamic forces depending upon $\bar{t}$.

To find equations of dynamic equilibrium of a flight vehicle with disturbed motion, we proceed in the following way. Let us assume that at a certain moment of time there occurred a change in magnitude of external aerodynamic forces acting on the vehicle in the yawing plane by a certain magnitude $Z_в$. As a result the aerodynamic moment will be changed by $M_{вy}$. This will lead to a change of both parameters of motion of the flight vehicle (z, ψ) and the controlling force R_z by Δz, $\Delta\psi$ and ΔR_z, respectively. Let us substitute the obtained values $\psi + \Delta\psi$, $\beta + \Delta\beta$, $z + \Delta z$ and $R_z + \Delta R_z$ into equations (3) and (5):

$$m(\ddot{z} + \Delta\ddot{z}) + (P - R_{x_1} + qSc_z^\beta)(\beta + \Delta\beta) - R_{z_1} - \Delta R_{z_1} = Z_в,$$

$$J_y(\ddot{\psi} + \Delta\ddot{\psi}) + (R_{z_1} + \Delta R_{z_1})(x_{1р} - x_{1т}) + qSc_z^\beta(\beta + \Delta\beta)(x_{1д} - x_{1т}) + \\ + qSl^2 m_y^{\omega} \frac{1}{v}(\dot{\psi} + \Delta\dot{\psi}) = M_{вy}.$$

Then from these equations let us subtract equations (3) and (5), which correspond to the base of undisturbed motion of the flight vehicle, i.e., the case when $Z_в = 0$ and $M_{вy} = 0$. As a result we obtain the sought perturbation equations of the flight vehicle in the yawing plane in the form

$$m\Delta\ddot{z} - (P - R_{x_1} + qSc_z^\beta)\Delta\beta - \Delta R_{z_1} = Z_в,$$

$$J_y\Delta\ddot{\psi} + qSl^2 m_y^{\omega} \frac{1}{v}\Delta\dot{\psi} + \Delta R_{z_1}(x_{1р} - x_{1т}) + qSc_z^\beta\Delta\beta(x_{1д} - x_{1т}) = M_{вy}.$$

Analogously perturbation equations of the vehicle as a solid body and in the pitch plane can be found.

Thus, conditions of dynamic equilbrium of a guided fight vehicle on the phase of powered flight in the plane of yawing (pitch) will consist of the equation of the control, which connects the value of angles of deviation of control devices with parameters of lateral motion of the body of the vehicle:

$$\delta = f(\Delta z, \Delta\dot{z}, \Delta\psi, \Delta\dot{\psi}, \Delta\ddot{\psi}, \delta, \dot{\delta}, \ldots)$$

and equations describing the deviation of the actual body from the state of undisturbed motion. Disregarding changes in parameters of longitudinal motion of the flight vehicle and introducing designations $\Delta\dot{z} = v_z$, $\Delta\dot{y} = v_y$, the last equations (when $\phi = 0$) will be presented in the following form:

a) for the pitch plane

$$m\dot{v}_y + c_{\text{д}}\frac{v_y}{v} + c_\alpha \Delta\vartheta + c_\delta \delta = Y_{\text{в}}, \qquad (2.24)$$

$$J_z \Delta\ddot{\vartheta} + b_{\text{д}} \Delta\dot{\vartheta} + b_\alpha \Delta\alpha + b_\delta\, \delta = M_{\text{в}z}; \qquad (2.25)$$

b) for the yawing plane

$$\left.\begin{aligned} -m\dot{v}_z + c_{\text{д}}\frac{v_z}{v} + c_\alpha \Delta\psi + c_\delta \delta &= Z_{\text{в}}, \\ J_y \Delta\ddot{\psi} + b_{\text{д}} \Delta\dot{\psi} - b_\mu \Delta\beta - b_\delta \delta &= M_{\text{в}y}, \end{aligned}\right\} \qquad (2.26)$$

where

$$\left.\begin{aligned} & c_{\text{д}} = P + qSc_y^\alpha - R_{x_1}, \\ & c_\alpha = -(P - R_{x_1} + qSc_y^\alpha),\ c_\delta = -\frac{\partial R_y}{\partial \delta}, \\ & b_{\text{д}} = \frac{1}{v}\, qSl^2 m_z^\omega, \\ & b_\delta = (x_{1\text{р}} - x_{1\text{т}})\frac{\partial R_y}{\partial \delta}, \\ & b_\alpha = qSc_y^\alpha (x_{1\text{д}} - x_{1\text{т}}), \\ & b_\beta = qSc_z^\beta (x_{1\text{д}} - x_{1\text{т}}), \\ & \Delta\alpha = \Delta\vartheta - \frac{v_y}{v},\ \Delta\beta = \Delta\psi - \frac{v_z}{v}, \end{aligned}\right\} \qquad (2.27)$$

and $Y_{\text{в}}$ and $M_{\text{в}y}$ designate the external disturbing forces and moments. Analogously coefficients for plane xy are found.

For flight vehicles of the ballistic type, in most cases the perturbation equations in planes of pitch, yawing and bank are independent. For vehicles of the aircraft type they will be connected.

For all flight vehicles carrying large masses of liquid fuel, sometimes it is necessary to consider additionally the effect of the mobility of the liquid in the tanks both on parameters of motion of the vehicle and on the nature of the loading of the construction as a whole.

With undisturbed motion and with static action of the disturbing forces, the mirror of the liquid in the tanks remains flat and perpendicular to the direction of the total overload. With disturbed motion on the free surface of the liquid waves the parameters of which depend on the nature of the motion and geometry of the tank are formed. As a result a certain part of the work of external forces departs by the change in the energy of the liquid. Quantitatively this effect in a number of cases can be estimated by means of a change in corresponding inertial characteristics of the flight vehicle. N. Ye. Zhukovskiy showed, motion of the body containing cavities completely filled with liquid can be described by equations of the dynamics of an equivalent solid body having a certain reduced mass and reduced moment of inertia. This conclusion proves to be valid for a number of the most important particular cases of disturbed motion of the flight vehicle having cavities (tanks) partially filled with liquid. Such particular cases are cases of the appearance of motion from a state of rest and the case of steady harmonic oscillations. In the general case the reduced inertial characteristics of the flight vehicle depend on the form of its perturbed motion, due to the quantity of liquid fuel (relative time $\bar{t}$) and on the geometry of the cavity.

§ 2.3. Hydrodynamic Loads

The effect of the mobility of liquid in fuel tanks of the flight vehicle on the state of strain of its structure can be considered by means of introducing corresponding additional external surface forces. The value of these forces will be determined by the magnitude of hydrodynamic pressure of liquid at every point of the internal surface of the wall of the tank.

With undisturbed motion the total pressure at any point of the liquid near the wall of the tank in the system of coordinates $x_j y_j z_j$ (Fig. 1.2) is approximately equal to

$$p_j(x_j, y_j, z_j, t) = p_{nj}(t) + u^*(x_j, y_j, z_j, t) + \rho_j \frac{v_j^2}{2} \qquad (2.28)$$

when $0 > x_j \geq -h_j$, $y_j \leq a_j$, $z_j \leq a_j$, where h_j – level of liquid in the considered j-th tank, ρ_j – its density, $p_{Hj}(t)$ – magnitude of absolute gas pressure for the free surface of the liquid, $a_j(x_j)$ – radius of the tank in section x_j, v_j – relative rate of movement of the level of the liquid in the tank, u^* – potential of mass forces.

For nonmaneuvering the flight vehicles with undisturbed motion and at small values of n_{y_1} and n_{z_1} as compared to n_{x_1} approximately (when $x_j < 0$)

$$u^*(x_j, y_j, z_j, t) = -\rho_j g_0 x_j n_{x_1}. \tag{2.29}$$

The magnitude of gas pressure of supercharging for similar vehicle is selected basically from the condition of providing noncavitation operation of fuel feed pumps into the combustion chamber of the engine. The relative rate v_j of motion of the liquid in the tank due to fuel consumption is usually small, and its effect on p_j in most cases can be disregarded.

Thus, with undisturbed motion of the flight vehicle the pressure of the liquid on walls of the tank in practice is a function of only the acceleration and level h_j.

With disturbed motion of the vehicle, acting on walls of the tank will be an additional hydrodynamic pressure of the liquid Δp_j, proportional to the lateral acceleration $n_{y_1}(x_1, t)$. The magnitude of this pressure in the case of transient potential flow of an ideal liquid can be found by using the Lagrange integral:

$$\Delta p_j = -\rho_j \frac{\partial \Phi_j}{\partial t}, \tag{2.30}$$

where Φ_j – velocity potential of the liquid.

As is known [21], this potential should satisfy the condition of incompressibility of the liquid (Laplace equation)

$$\Delta\Phi_j = \frac{\partial^2\Phi_j}{\partial x_j^2} + \frac{\partial^2\Phi_j}{\partial y_j^2} + \frac{\partial^2\Phi_j}{\partial z_j^2} = 0 \qquad (2.31)$$

and the defined boundary conditions on the surface of the tank moistened by the liquid. These conditions ensure the equality of the normal (to the surface of the tank) component of the velocity of liquid particles of the corresponding component velocity of motion of points of the wall and bottom of the tank

$$\frac{\partial\Phi_j}{\partial n} = v_n.$$

Furthermore, the potential Φ_j should satisfy the boundary condition on the free surface of the liquid, which consists in the equality of pressure p_j to the gas pressure of supercharging p_{Hj}.

$$q_\tau(x_j, t) = -a_j(x_j)\rho_j \int_0^{2\pi} \frac{\partial\Phi_j(x_j, y_j, z_j, t)}{\partial t} \sin\eta \, d\eta, \qquad (2.32)$$

where $\eta = \operatorname{arc\,tg} \frac{y_j}{z_j}$.

The lateral force and bending moment from this load on the section of the tank filled with liquid $0 > x_j > -h_j$ will be determined by expressions

$$-Q_{\tau j}(x_j, t) = \int_0^{x_j} q_\tau(x_j, t)\, dx_j, \qquad (2.33)$$

$$-M_{\tau j}(x_j, t) = \int_0^{x_j} Q_{\tau j}(x_j, t)\, dx_j. \qquad (2.34)$$

Due to the nonuniform distribution of the hydrodynamic pressure of the liquid on the bottom of the tank at the place of bracing of it to the casing acting on the body will be the additionally concentrated moment

$$\Delta M_{xj}(t)=\int_0^{a_j}\int_0^{2\pi}[\Delta p_j(-h_j,\ r_j,\ \eta)-\Delta p_j(0,\ r_j,\ \eta)]\,r_j^2\sin\eta\, d\eta\, dr_j. \qquad (2.35)$$

Here $r_j^2 = y_j^2 + z_j^2$.

On section $x_j < -h_j$, i.e., when $x_1 > x_{1\text{д}j}$, the bending moment from the surface load $q_\text{т}(x_j\ t)$ will be equal to

$$M_{\text{т}j}(x_j,\ t)=-\Delta M_{xj}(t)+M_{\text{т}j}(-h_j,\ t)+Q_{\text{т}j}(-h_j,\ t)(x_1-x_{1\text{д}j}), \qquad (2.36)$$

where $x_{1\text{д}j}$ – abscissa of the place of connection of the bottom of the tank to the casing.

The total lateral load on the body of the vehicle from hydrodynamic forces acting on each tank will be determined by formula

$$Y_\text{в}^* = \sum_{j=1}^{n_j} Q_{\text{т}j}(-h_j), \qquad (2.37)$$

where n_j – number of the tanks.

The total moment of this force with respect to the center of gravity of the flight vehicle will be equal to

$$M_\text{в}^* = \sum_{j=1}^{n_j} M_{\text{т}j}(x_{1\text{т}}). \qquad (2.38)$$

Including these values $Y_\text{в}^*$ and $M_\text{в}^*$ in the right side of equations (24) and (25) in the form of disturbing factors, we obtain the unknown conditions of dynamic equilibrium of the flight vehicle, taking into account the mobility of the liquid in the tanks in the following form:

$$\left.\begin{aligned} m_\text{к}\dot{v}_y + c_\lambda\frac{v_y}{v} + c_\alpha\Delta\vartheta + c_\delta\delta &= Y_\text{в} + Y_\text{в}^*,\\ J_{z\text{к}}\Delta\ddot{\vartheta} + b_\lambda\Delta\dot{\vartheta} + b_\alpha\Delta\alpha + b_\delta\delta &= M_{\text{в}z} + M_\text{в}^*.\end{aligned}\right\} \qquad (2.39)$$

Since potential Φ_j is a function of parameters of lateral oscillations of the tank, expressions of external force Y^*_B and moment M^*_B in these equations will be certain functions of parameters of motion of the flight vehicle.

The problem of the determination of velocity potential Φ_j for the case of small oscillations of vessels of different form, which partially were filled by an ideal incompressible liquid, was considered by a number of authors [43, 47, 56, 66, and others].

Most frequently on nonmaneuvering flight vehicles fuel tanks of the cylindrical form are used. The approximate expression of the velocity potential of the liquid for such tanks during small oscillations of the liquid is determined comparatively simply, if replacement of the true form of bottoms of the tanks by flat bottoms is allowed. Similar schematization gives large errors only at relatively small levels of liquid in the tanks ($h_j < 0.5\ a_j$), i.e., at the end of the powered flight when the ratio of the weight of remaining fuel to the weight of the construction of the flight vehicle becomes comparatively small, and when the effect of the mobility of liquid in the tanks on loads proves to be insignificant.

§ 2.4. Equations of Forced Oscillations of the Liquid

Thus, let us consider small oscillations of a cylindrical tank with flat bottoms in a "motionless" system of coordinates xyz (for instance, in the plane of yawing). Let us assume that the absolute motion of liquid in this tank is irrotational, and forces of viscous friction in the liquid are absent.

The potential of absolute velocity of the liquid Φ_j, as is shown in § 2.3, should satisfy the continuity equation (31), which in cylindrical coordinates will be written in the form

$$\frac{\partial^2\Phi_j}{\partial r_j^2}+\frac{1}{r_j}\frac{\partial\Phi_j}{\partial r_j}+\frac{1}{r_j^2}\frac{\partial^2\Phi_j}{\partial\eta^2}+\frac{\partial^2\Phi_j}{\partial x_j^2}=0, \qquad (2.40)$$

and corresponding boundary conditions. In this case (plane motion of the tank) these conditions will be equal to

$$\frac{\partial\Phi_j}{\partial r_j}\Big|_{r_j=a_j} = v_y(x_j), \quad \frac{\partial\Phi_j}{\partial x_j}\Big|_{x_j=-h_j} = v_{0x} \qquad (2.40')$$

on walls of the tank and

$$-\frac{\Delta p_j}{\rho_j}\Big|_{x_j=0} = n_{x_1} g_0 \varepsilon_x + \frac{\partial \Phi_j}{\partial t}\Big|_{x_j=0} = 0 \qquad (2.40'')$$

on the free surface of the liquid. Here ε_x – displacement of a particle of liquid along the x_j axis from the plane $x_j = 0$, is approximately equal to

$$\varepsilon_x = \int_0^t \frac{\partial \Phi_j}{\partial x_j}\Big|_{x_j=0} dt,$$

and v_y – projection of the speed of points of the wall of the tank on the y axis, v_{0x} – projection of the speed of points of the bottom of the tank on the x axis.

For small oscillations of the tank, completely filled with liquid, the value of the potential Φ_j was found by N. Ye. Zhukovskiy [21] in the form

$$\Phi_{j1} = (F_j + x_j y_j)\Delta\dot{\Theta} + v_{0y} y_j, \qquad (2.41)$$

where

$$F_j = 4a_j^2 \sum_{k=1}^{\infty} \frac{J_1(\xi_k \bar{r}_j)\,\mathrm{sh}(\xi_k \bar{x}_j)\sin\eta}{J_1(\xi_k)\,\xi_k(\xi_k^2 - 1)\,\mathrm{ch}\,\mu_{kj}},$$

$$\bar{x}_j = \frac{x_j}{a_j},\ \bar{r}_j = \frac{r_j}{a_j},\ \mu_{kj} = \frac{h_j}{a_j}\xi_k.$$

v_{0yj} – projection of the speed of the origin of the system of coordinates x_jy_j connected with the tank (Fig. 1.2) on the y axis, J_1 – bessel function of the first kind of the first order, ξ_k – roots of equation

$$\frac{dJ_1(\xi_k)}{d\xi_k}=0.$$

for k = 1, ξ_k = 1.84; where k = 2, ξ_k = 5.33; ...

The above expression for function F_j is obtained in the following way. Substituting (41) into equation (40), we find that function F_j should satisfy equation

$$\frac{\partial^2 F_j}{\partial r_j^2}+\frac{1}{r_j}\frac{\partial F_j}{\partial r_j}+\frac{1}{r_j^2}\frac{\partial^2 F_j}{\partial \eta^2}+\frac{\partial^2 F_j}{\partial x_j^2}=0 \qquad (2.42)$$

and boundary conditions, which result from conditions (40') after substituting in them (41) of the corresponding values of v_{0yj}

$$\left.\begin{aligned}&\frac{\partial F_j}{\partial r_j}=0 \text{ when } r_j=a_j,\\&\frac{\partial F_j}{\partial x_j}=2y_j \text{ when } x_j=\pm h_j,\\&v_{0yj}=L_j\Delta\dot{\varphi}.\end{aligned}\right\} \qquad (2.43)$$

Here L_j denotes the distance origin 0 of the system of coordinates x_jy_j from the axis of oscillations of the tank as a solid body

$$L_j=x_{1T}-x_{10j}.$$

The solution of equation (42) is sought in the form of the product of functions, each of which depends only on one of the variables x_j, r_j or η

$$F_j=Z(x_j)H(\eta)R(r_j). \qquad (2.44)$$

Substituting (44) into equation (42), we have

$$\frac{d^2R}{dr_j}+\frac{1}{r_j}\frac{dR}{dr_j}=-R\left(\frac{1}{r_j^2}\frac{1}{H}\frac{d^2H}{d\eta^2}+\frac{1}{Z}\frac{d^2Z}{dx_j^2}\right).$$

Assuming that

$$\left.\begin{aligned}\frac{d^2H}{d\eta^2}&=-m^2H,\\ \frac{d^2Z}{dx_j}&=\lambda^2Z,\end{aligned}\right\} \qquad (2.45)$$

we obtain the following Bessel equation for the determination of function R:

$$\frac{d^2R}{dr_j^2}+\frac{1}{r_j}\frac{dR}{dr_j}+\left(\lambda^2-\frac{m^2}{r_j}\right)R=0. \qquad (2.46)$$

Solutions of equations (45) and (46) will be functions

$$\begin{aligned}&H(\eta)=\sin(m\eta+\varphi_0),\\ &Z(x_j)=\mathrm{sh}\,\lambda x_j,\\ &R(r_j)=J_m(\lambda r_j)+\gamma N_m(\lambda r_j),\end{aligned}$$

where J_m and N_m - Bessel functions of the first and second kind of the order m.

In virtue of the periodicity of function $H(\eta)$ m will be the integer. It is possible to take m = 1 and $^{\wedge}_{j}$ = 0. Since the function $N_m(\lambda r_j)$ takes an infinite value when $r_j = 0$, and with respect to the meaning of the problem the solution in this case should be finite, then $\gamma = 0$. Constant λ is from boundary conditions on the wall of the tank (43). From the first condition $\frac{\partial F_j}{\partial r_j} = 0$ when $r_j = a_j$ it follows that

$$\frac{dJ_m(\lambda \bar{r}_j)}{dr_j}=0.$$

This equation gives an infinite number of roots ξ_k ($k = 1, 2, \ldots$).

Function $R(r_j)$ is standardized in such a way so that it is equal to one when $r_j = a_j$, i.e., we assume

$$R(\bar{r}_j) = \frac{J_1(\xi_k \bar{r}_j)}{J_1(\xi_k)}.$$

Thus, as a result we obtain that

$$F_j = \sum_{k=1}^{\infty} C_k \operatorname{sh}(\xi_k \bar{x}_j) R_k(\bar{r}_j) \sin \eta, \qquad (2.47)$$

where C_k – certain coefficient which is selected in such a manner that the second boundary condition (43) is satisfied.

Expanding y_j in series with respect to cylindrical functions $R(r_j)$

$$y_j = 2a_j \sum_{k=1}^{\infty} \frac{1}{(\xi_k^2 - 1)} \frac{J_1(\xi_k \bar{r}_j)}{J_1(\xi_k)} \sin \eta,$$

we find that

$$C_k = \frac{4a_j^2}{\xi_k(\xi_k^2 - 1) \operatorname{ch} \mu_{kj}} \quad (k = 1, 2, \ldots).$$

Fulfillment of the boundary condition on the free surface of the liquid is ensured by the selection of function $Z(x_j)$.

For the case when the liquid fills the tank partially, we present [47] potential Φ_j in the form of a sum of two potentials Φ_{j1} and Φ_{j2}:

$$\Phi_j = \Phi_{j1} + \Phi_{j2}. \qquad (2.48)$$

The second potential determines the motion of the liquid (with a motionless tank), which is conditioned by the irregularity of the pressure appearing during oscillations with the first potential Φ_{j1}.

Potential Φ_{j2} will be defined in such a manner so that it satisfies the continuity equation (31) and zero boundary conditions on the walls and bottom of the tank and together with potential Φ_{j1} – condition (40") on the free surface of the liquid.

The expression for Φ_{j2} will be formed in a form similar to Φ_{j1} with the introduction of a certain unknown time function $\beta(t)$, i.e.,

$$\Phi_{j2}=\dot{\beta}_{kj}(t)Z(x_j)H(\eta)R(r_j). \tag{2.49}$$

In this case function $Z(x_j)$ is selected in such a manner so that zero boundary conditions on the bottom of the tank are fulfilled (when $x_j = -h_j$)

$$Z(x_j)=\operatorname{ch}[\xi_k(\bar{x}_j+\bar{h}_j)].$$

Thus, we have

$$\Phi_{j2}=2a_j\sum_{k=1}^{\infty}\dot{\beta}_{kj}\frac{\operatorname{ch}[\xi_k(\bar{x}_j+\bar{h}_j)]}{(\xi_k^2-1)\operatorname{ch}\mu_{kj}}\frac{J_1(\xi_k\bar{r}_j)}{J_1(\xi_k)}\sin\eta, \tag{2.50}$$

where $\bar{h}_j = \dfrac{h_j}{a_j}$.

Satisfaction of the required condition on the free surface of the liquid (40") is ensured by an appropriate selection of the time function $\beta_{kj}(t)$. As a result the following linear differential equation is obtained

$$n_{x_1}g_0\left(\int_0^t\frac{\partial\Phi_{j1}}{\partial x_j}\Big|_{x_j=0}dt+\int_0^t\frac{\partial\Phi_{j2}}{\partial x_j}\Big|_{x_j=0}dt\right)+\frac{\partial(\Phi_{j1}+\Phi_{j2})}{\partial t}\Big|_{x_j=0}=0,$$

which in the case considered by us (after fulfillment of the appropriate transformations) we will have the form

$$\ddot{\beta}_{kj} + \omega^2_{kj}\beta_{kj} + \ddot{\vartheta}_y + \Delta\ddot{\vartheta}(x_{1T} - x_{10j}) + \sigma^2_{kj}\Delta\vartheta = 0.$$

This equation actually describes the forced small oscillations of liquid in the tank. In it ω_{kj} denotes the angular frequency of natural oscillations of liquid in a motionless tank equal to

$$\omega_{kj} = \left(\xi_k \frac{n_{x_1} g_0}{a_j} \operatorname{th} \mu_{kj}\right)^{\frac{1}{2}}. \tag{2.51}$$

and σ^2_{kj} – denotes the coefficient

$$\sigma^2_{kj} = n_{x_1} g_0 \left(1 + \frac{2}{\operatorname{ch} \mu_{kj}}\right) \quad (k = 1, 2, \ldots). \tag{2.52}$$

As can be seen from formula (51), ω_{kj} essentially depends on the radius of the tank. Any increase in this radius will be accompanied by a lowering of the frequency of natural oscillations of liquid in the tank. The nature of the change in ω_{kj} with respect to the time of flight is basically determined by the nature of the change in longitudinal acceleration n_{x_1}. The effect of the change in relative level of the liquid $\bar{h}_j(t)$ appears only when $\bar{h}_j < 1$, since when $\bar{h}_j \geq 1$ it is possible to consider

$$\omega_{kj} = \left(\xi_k g_0 \frac{n_{x_1}}{a_j}\right)^{\frac{1}{2}}.$$

Each frequency ω_{kj} corresponds to the fully defined form of oscillations of free surface of liquid in the tank. Thus, for instance, the first tone of natural oscillations of this surface (k = 1) in the cylindrical tank is characterized by the presence (in the cross section of the wave) of one nodal point, the second (k = 2) – two nodal points, etc. In this case with an increase in amplitude of oscillations of particles of liquid, these forms are considerably deformed, and they become asymmetric (the crest is larger than the trough). In other words, the problem becomes clearly nonlinear. Calculation of this distortion of the form of free surface of the liquid is of practical importance basically only in the investigation of certain problems of the stability of motion of flight vehicles and designing of intake devices for fuel feed into pumps of the engine. Therefore, specially do not discuss this problem, referring those who are interested to the available works ([44] and others).

Substituting the value of potentials (41) and (49) into (48) and then into (32), will find the formula for the calculation of the total linear lateral load on the body of the cylindrical tank from the liquid

$$q_{\tau j}(x_j, t) = -m_\tau(x_j)\dot{v}_y(x_j, t) + \Delta q_{\tau j}(x_j, t), \tag{2.53}$$

where $m_\tau(x_j) = \pi\rho_j a_j^2(x_j)$.

The first term of this formula determines the lateral inertial load from the mass of hardening fuel, which is proportional to the local value of lateral linear acceleration of the tank

$$\dot{v}_y(x_j, t) = \dot{v}_{0yj}(t) + \Delta\ddot{\vartheta}(t)x_j,$$

and the second term – the additional load caused by the motion of liquid in the tank equal to

$$\Delta q_{\tau j}(x_j, t) = -2m_\tau(x_j)\sum_{k=1}^{\infty}\left\{2\Delta\ddot{\vartheta}(t)\frac{\operatorname{sh}(\xi_k\bar{x}_j)\,a_j(x_j)}{\xi_k(\xi_k^2-1)\operatorname{ch}\mu_{kj}} + \right.$$
$$\left. + \beta_{kj}\frac{\operatorname{ch}[\xi_k(\bar{x}_j+\bar{h}_j)]}{(\xi_k^2-1)\operatorname{ch}\mu_{kj}}\right\}. \tag{2.54}$$

Using expressions (33), (34) and (54), it is easy to obtain values corresponding to this additional load of lateral force $\Delta Q_{\tau j}(x_j)$ and bending moment $\Delta M_{\tau j}(x_j)$. At the end of the tank when $x_j = -h_j$ they will be equal to

$$-\Delta Q_{\tau j}(-h_j)=2\pi\rho_j a_j^3\sum_{k=1}^{\infty}\left[\beta_{kj}\frac{\operatorname{th}\mu_{kj}}{\xi_k(\xi_k^2-1)}-2a_j\Delta\vartheta\frac{\left(1-\frac{1}{\operatorname{ch}\mu_{kj}}\right)}{\xi_k^2(\xi_k^2-1)}\right], \tag{2.55}$$

$$-\Delta M_{\tau j}(-h_j)=2\pi\rho_j a_j^4\sum_{k=1}^{\infty}\left[\beta_{kj}\frac{1-\operatorname{ch}\mu_{kj}+\mu_{kj}\operatorname{sh}\mu_{kj}}{\xi_k^2(\xi_k^2-1)\operatorname{ch}\mu_{kj}}-\right.$$
$$\left.-2a_j\Delta\vartheta\frac{\operatorname{sh}\mu_{kj}-\mu_{kj}}{\xi_k^3(\xi_k^2-1)\operatorname{ch}\mu_{kj}}\right]. \tag{2.56}$$

Taking into account the effect of additional pressure equal to $n_{x_1}g_0\rho_j\varepsilon_{xj}$, and relation

$$\int_0^{a_j} r^2 J_1(\xi_k r)\,dr=\frac{a_j^2}{\xi_k^2}J_1(\xi_k),$$

from formula (35) we find the following expanded expression for moment $\Delta M_{\text{д}j}$ concentrated in the section $x_j = -h_j$:

$$\Delta M_{\text{д}j}=\Delta\vartheta n_{x_1}g_0\pi\rho_j a_j^4\sum_{k=1}^{\infty}\frac{\left(\frac{2}{\operatorname{ch}\mu_{kj}}+1\right)}{\xi_k^2(\xi_k^2-1)}+\sum_{k=1}^{\infty}\beta_{kj}\frac{2\pi\rho_j g_0 a_j^3 n_{x_1}}{\xi_k(\xi_k^2-1)}\operatorname{th}\mu_{kj}+$$
$$+2\pi\rho_j a_j^4\sum_{k=1}^{\infty}\left[\Delta\vartheta\frac{2a_j\operatorname{th}\mu_{kj}+h_j\xi_k}{\xi_k^2(\xi_k^2-1)}+\beta_{kj}\frac{1}{\xi_k^2(\xi_k^2-1)}\left(1-\frac{1}{\operatorname{ch}\mu_{kj}}\right)\right]. \tag{2.57}$$

Using formulas (33), (54), and (57), the unknown expressions for external force Y_B^* (37) and moment M_B^* (38) as a function of independent variables $\Delta\vartheta$, β_{kj} and v_y can be obtained. After substitution of them into equations (39) and carrying out certain elementary transformations, which here we omit, the system of perturbation equations of the flight vehicle with cylindrical tanks (or with conical tanks with a small angle of conicity) will take the following form:

$$m\dot v_y+c_1\frac{v_y}{v}+c_\alpha\Delta\vartheta+c_{\dot\vartheta}\Delta\dot\vartheta=-c_\delta\delta+Y_B+\sum_{k=1}^{\infty}\sum_{j=1}^{n_j}c_{kj}\beta_{kj}. \tag{2.58}$$

$$I_z^* \Delta\ddot{\vartheta} + b_{\text{д}} \Delta\dot{\vartheta} + b_\alpha^* \Delta\vartheta - \frac{v_y}{v} b_\alpha = -b_\delta \delta + M_{\text{в}} + + \sum_{k=1}^{\infty} \sum_{j=1}^{n_j} b_{kj} \beta_{kj} + \sum_{k=1}^{\infty} \sum_{j=1}^{n_j} b_{k0j} \ddot{\beta}_{kj}, \tag{2.59}$$

$$\ddot{\beta}_{kj} + \omega_{kj}^2 \beta_{kj} + \Delta\ddot{\vartheta} (x_{1\text{т}} - x_{10j}) + \dot{v}_{0y} + \sigma_{kj}^2 \Delta\vartheta = 0 \tag{2.60}$$

$$(k = 1, 2, \ldots;\ j = 1, 2, \ldots, n_j),$$

where

$$\left.\begin{aligned} & c_{kj} = -2\pi\rho_j a_j^3 \frac{\operatorname{th}\mu_{kj}}{\xi_k(\xi_k^2 - 1)}, \quad b_{k0j} = -n_{x_1} \frac{2\pi\rho_j g_0 a_j^3}{\xi_k(\xi_k^2 - 1)} \operatorname{th}\mu_{kj}, \\ & c_0 = -4\pi \sum_{j=1}^{n_j} \rho_j a_j^4 \sum_{k=1}^{\infty} \frac{1}{\xi_k^2(\xi_k^2 - 1)} \left(1 - \frac{1}{\operatorname{ch}\mu_{kj}}\right), \\ & b_{kj} = -\frac{2\pi\rho_j a_j^3}{\xi_k(\xi_k^2 - 1)} (x_{1\text{т}} - x_{10j}) \operatorname{th}\mu_{kj}, \\ & I_z^* = I_z - 4\pi \sum_{j=1}^{n_j} \rho_j a_j^4 \sum_{k=1}^{\infty} \frac{1}{\xi_k^2(\xi_k^2 - 1)} \left[(\bar{x}_{1\text{т}} - \bar{x}_{10j})\left(1 - \frac{1}{\operatorname{ch}\mu_{kj}}\right) - h\right], \\ & b_\alpha^* = b_\alpha + 2\pi \sum_{j=1}^{n_j} \sum_{k=1}^{\infty} \rho_j a_j^4 \frac{\sigma_{kj}^2}{\xi_k^2(\xi_k^2 - 1)}. \end{aligned}\right\} \tag{2.61}$$

The calculations show that the specific gravity of additional bending moments (56) and (57) and lateral forces (55), caused by the direct effect on the construction of hydrodynamic loads, in many cases is small, and it can be disregarded. Oscillation of the liquid in fuel tanks usually has a basic effect on parameters of the disturbed motion of a flight vehicle, in particular, on magnitudes of lateral accelerations and angles of attack.

§ 2.5. Estimate of the Effect of Mobility of the Liquid on Parameters of Disturbed Motion

To get rid of an additional equation (60), i.e., to replace the complicated system flight vehicle-liquid by some equivalent solid body, is possible when condition

$$\beta_{kj} = A_{kj}(x_{1\text{т}} - x_{10j}) \Delta\ddot{\vartheta} + B_{kj}\dot{v}_y,$$

is executed, in other words, when the reaction of the liquid on walls of the tank is proportional to the acceleration of the vehicle.

It is easy to check that the indicated condition is realized with established harmonic lateral oscillations of the tank with the frequency ω different from ω_{kj}. In this particular case of disturbed motion of the flight vehicle, amplification factors A_{kj} and B_{kj} will be determined by formula

$$\left.\begin{aligned} -A_{kj} &= \frac{\omega_{kj}^2/L_j-\omega^2}{\omega_{kj}^2-\omega^2}, \\ B_{kj} &= \frac{\omega^2}{\omega_{kj}^2-\omega^2}. \end{aligned}\right\} \tag{2.62}$$

As can be seen, their values do not absolutely depend on time but are only functions of the relation of frequencies of forced and natural oscillations of the liquid in the tank.

The second particular case of disturbed motion at which A_{kj} and B_{kj} are not time functions will be the case of sudden lateral deviation of the tank from the state of rest (undisturbed motion) for which $A_{kj} = B_{kj} = 1$.

The noted particular cases of disturbed motion of the tank in reality are most frequently encountered with the operation of the flight vehicles. Therefore, a detailed consideration of them is of great practical interest, since it permits noticeably simplfying the solution of a number of problems of dynamic calculation of the design of flight vehicles with engines operating on liquid fuel. This simplification[2] leads to the use in equations (24) and (25) of reduced inertial characteristics: reduced mass m^* and reduced moment of inertia J_z^*. The last are understood as corresponding coefficients with accelerations $\dot{v}_y$ and $\Delta\ddot{\theta}$ in equations (58) and (59).

Let us consider at first the case $A_{kj} = B_{kj} = 1$. In this case the hydrodynamic pressure Δp_j on walls of the tank is entirely determined by the local value of coefficient of lateral acceleration

$$\Delta p_j = \rho_j g_0 a_j (n^0_{y_1} \eta^0_j + n^x_{y_1} \eta^x_j), \tag{2.63}$$

where

$$\eta^0_j = 1 - 2\sum_{k=1}^{\infty} \frac{B_{kj} \operatorname{ch}[\xi_k (\bar{x}_j + \bar{h}_j)]}{(\xi_k^2 - 1) \operatorname{ch}\mu_{kj}},$$

$$\eta^x_j = 1 - 2\sum_{k=1}^{\infty} \frac{1}{\xi_k (\xi_k^2 - 1)(\bar{L}_j + \bar{x}_j)} \left\{ A_{kj}\xi_k \frac{\operatorname{ch}[\xi_k (\bar{x}_j + \bar{h}_j)]}{\operatorname{ch}\nu_{kj}} \bar{L}_j + 2\frac{\operatorname{sh}(\xi_k \bar{x}_j)}{\operatorname{ch}\mu_{kj}} \right\}$$

$$(k = 1, 2, \ldots),$$

$$n^0_{y_1} = -\frac{\dot{v}_y}{g_0}, \quad n^x_{y_1} = -a_j \frac{\Delta\dot{\Phi}}{g_0} (\bar{L}_j + \bar{x}_j).$$

It is easy to establish that the magnitude of this pressure will be comparable to hydrostatic pressure u^*_j (29) only in the presence of lateral acceleration close to $\frac{n_{x_1} h_j}{2}$. For flight vehicles of the ballistic type a similar phenomenon can take place only before the end of the operation of engines with small $\bar{h}_j$. In beginning of flight, when $\bar{h}_j$ usually differs little from unity, the value of $n^0_{y_1}$, and, consequently, Δp_j is small, and it can be disregarded. Even at the end of the flight the effect of the indicated hydrodynamic pressure on the strength of structure of the tank can be noticeable only when u^*_j is comparable to p_{Hj}.

The basic effect on the magnitude of hydrodynamic pressure render only members, corresponding $k = 1$. Therefore, in the estimate of the general reaction of the liquid on the flight vehicle, with sufficient accuracy it is possible to be limited to the calculation of only first terms of the series of expansion of the actual form of oscillations of the free surface of the liquid with respect to functions $J_m(\xi_k \bar{r}_j)$. Calculation of the second tone of natural oscillations of the liquid ($k = 2$) introduces a correction of the order of 5%. It is difficult to be convinced of this, if one were to compare values m_j^* and J_j^* at various k with each other. Similar conclusions are confirmed by experimental data. Subsequently, for simplification of the letter we will omit the symbol of the sum with respect to k and consider k = 1, perserving, however, subscript k with the designation of all parameters characterizing oscillations of the liquid.

The expression for the reduced weight of the liquid is obtained directly from the formula for the lateral hydrodynamic force $Q_{Tj}(x_j)$ in the section located at the bottom of the tank ($x_j = -h_j$),

$$Q_{\tau j}(-h_j) = G_{\tau j}\left(1 - 0{,}456\frac{B_{kj}}{\bar{h}_j}\operatorname{th}\mu_{kj}\right)n_{y_1}^0 - \pi\rho_j\Delta\vartheta a_j^4\left[\bar{h}_j(\bar{h}_j - 2\bar{L}_j) - 0{,}5\left(1 - \frac{1}{\operatorname{ch}\mu_{kj}}\right) + 0{,}456\bar{L}_j A_{kj}\operatorname{th}\mu_{kj}\right]. \qquad (2.6\ )$$

Here

$$G_{\tau j} = \pi\rho_j g_0 \bar{h}_j a_j^3.$$

The first term of the formula, which characterizes the effect of the liquid on the body with forward motion of the vehicle can be represented in the form

$$n_{y_1}^0 G_{\tau j}^* = k_j^0 G_{\tau j} n_{y_1}^0, \qquad (2.65)$$

where G_{Tj}^* is the reduced weight of the liquid, and k_j^0 is the coefficient considering the effect of mobility of the liquid in the tank on magnitude G_{Tj}^*,

$$k_j^0 = 1 - \frac{2B_{kj}}{\xi_k(\xi_k^2 - 1)\bar{h}_j}\operatorname{th}\mu_{kj}. \qquad (2.66)$$

At small $\bar{h}_j$ the coefficient k_j^0 of the effect (and consequently, and reduced weight of the liquid) decreases with a growth in amplification factor B_{kj}. This indicates that the liquid is not completely attracted by walls of the tank during oscillations of the flight vehicle. With a sudden lateral deviation of the vehicle from the state of undisturbed motion (when $B_{kj} = 1$) the liquid for the free surface will lag behind the motion of walls of the tank, and this lag will be greater, the less $\bar{h}_j$. At large $\bar{h}_j$ almost all the liquid is entrained by the tank, and coefficient k_j^0 differs little from unity.

With steady-state oscillations of the flight vehicle, on the free surface of the liquid waves are formed whose amplitude of oscillations depends mainly on the relation of frequencies of forced and natural oscillations ω/ω_{kj}. Values of the coefficient of effect k_j^0 in this case of motion can be changed in wide limits. With transition through resonance they become even negative. Physically this means that the direction of the external force and direction of the acceleration caused by it become opposite. With ω/ω_{kj} close to unity, oscillations of the liquid become large, and the above linear theory, which assumes the smallness of all parameters of motion of the flight vehicle, gives incorrect results. According to this theory, the magnitude of the coefficient k_j^0 of effect in the indicated case (with small oscillations of vehicle) will approach infinity. In reality it always remains limited. This circumstance can be considered either by means of the solution of the hydrodynamic problem in a nonlinear setting, i.e., accurate satisfaction of boundary conditions on the free surface of the liquid without the drift of them onto plane $x_j = 0$ or by means of the conditional introduction (on the basis of experiments) of damping forces, which limit the amplitude of oscillations of the free surface of the liquid. With steady-state great oscillations of the liquid a limiting wave will be formed. The crest of such a wave is always greater than the trough. For illustration Fig. 2.4 gives a computed value of the form of free surface of the liquid in a tank when $k = 1$ and $\frac{\omega}{\omega_{kj}} = 1$. With small oscillations outside the zone of the resonance, the crest and trough of the wave are identical. Calculations show that values of the greatest deviations of particles of the liquid of the free surface in the direction of the longitudinal axis of the tank in this case depend mainly on the radius of the tank. The larger the radius of the tank, the less these movements. The effect of longitudinal acceleration is little.

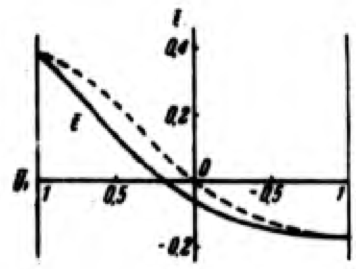

Fig. 2.4. Form of large oscillations of the free surface of the liquid in a cylindrical tank.

The reduced moment of inertia of the liquid on the tank relative to the lateral axis, which is at distance L_j from the free surface of the liquid, is equal to

$$J^{*}_{\tau j} = J_{\tau j}(d_j - b_j A_{kj}), \tag{2.67}$$

where

$$J_{\tau j} = \pi\rho_j a_j^5 \bar{h}_j\left(\bar{L}_j^2 - \bar{h}_j\bar{L}_j + \frac{\bar{h}_j^2}{3} + \frac{1}{4}\right),$$

$$d_j = 1 - \frac{4\left[\bar{L}_j\left(1 - \frac{1}{\operatorname{ch}\mu_{kj}}\right) - \bar{h}_j\right]}{\bar{h}_j\xi_k^2(\xi_k^2 - 1)\left(\bar{L}_j^2 - \bar{L}_j\bar{h}_j + \frac{\bar{h}_j^2}{3} + \frac{1}{4}\right)},$$

$$b_j = \frac{2\bar{L}_j^2 \operatorname{th}\mu_{kj}}{(\xi_k^2 - 1)\xi_k\bar{h}_j\left(\bar{L}_j^2 - \bar{L}_j\bar{h}_j + \frac{\bar{h}_j^2}{3} + \frac{1}{4}\right)}.$$

As can be seen, d_j is a positive function of $\bar{h}_j$ and $\bar{L}_j$. The magnitude of it with an increase in modulus $\bar{L}_j$ tends to unity. In the region of small $|\bar{L}_j|$ value d_j essentially depends on $\bar{h}_j$. Function $b_j(\bar{k}_j, \bar{L}_j)$, remaining also all the time positive, grows with an increase in h_j when $\bar{L}_j > 0$ and decreases when $\bar{L}_j < 0$, so that the reduced moment of inertia of the liquid will be positive when $A_{kj} < d_j/b_j$ and negative when $A_{kj} > d_j/b_j$. With a tendency of A_{kj} to $\frac{d_j}{b_j}$ it will tend to zero.

Let us consider now how the mobility of the liquid in the tanks affects the inertial characteristics of the flight vehicle as a whole. Let us assume that the vehicle has n_j cylindrical fuel tanks. With undisturbed motion the weight of it is equal to

$$G = G_\kappa + \sum_{j=1}^{n_j} G_{\tau j}.$$

where G_κ – weight of the vehicle without fuel in the tanks. With disturbed motion the reduced weight of the vehicle, according to (65), will be determined by expression

$$G^{*} = G_\kappa + \sum_{j=1}^{n_j} G_{\tau j} k_j^0. \qquad (2.68)$$

Using formula (66), we represent it in the form

$$G^{*} = G - 1{,}44 \sum_{j=1}^{n_j} \rho_j a_j^3 g_0 B_{kj} \operatorname{th} \mu_{kj}.$$

Then the coefficient of effect (for the flight vehicle as a whole) will be equal to

$$k^0 = 1 - \frac{0{,}456 g_0}{l\gamma_c} \sum_{j=1}^{n_j} \rho_j B_{kj} \left(\frac{a_j}{a}\right)^3 \operatorname{th} \mu_{kj}, \qquad (2.69)$$

where $\gamma_c = G/lS$ – the specific weight.

If all the tanks (in part filled with liquid) have identical radii ($a_j = a$), then

$$k^0 = 1 - \frac{0{,}456 g_0}{\gamma_c l} B_k \sum_{j=1}^{n_j} \rho_j \operatorname{th} \mu_{kj}. \qquad (2.70)$$

From this formula it follows that when $B_k = \lambda_0(1 - \bar{t})$, where

$$\lambda_0 = \frac{G_0}{0{,}456 a^3 g_0 \pi \sum\limits_{j=1}^{n_j} \rho_j \operatorname{th} \mu_{kj}}, \tag{2.71}$$

k^0 and, consequently, G become equal to zero. According to the linear theory, this can take place during steady-state oscillations of the vehicle with the period equal to

$$T_0 = 1{,}48 \left\{ \frac{a}{n_{x_1}} \frac{1 + \lambda_0 (1 - \bar{t})}{\lambda_0 (1 - \bar{t})} \right\}^{\frac{1}{2}}. \tag{2.72}$$

When $B_k = 1$

$$k^0 = k_n^0 = 1 - \frac{1}{\lambda_0 (1 - \bar{t})}$$

is a function of only $\bar{t}$ and parameter λ_0. Although a change in magnitude of k_n^0 with respect to t occurs in this case in comparatively small limits $0.8 < k_n^0 < 1$, nevertheless, one should not disregard its effect in the calculation of lateral loads.

Reference of the surface hydrodynamic load to the category of inertial forces, i.e., introduction of corresponding reduced values of the force of weight and moment of inertia, requires the introduction of the concept of the reduced center of gravity of the system. The center of pressure of the liquid in the cylindrical tank with undisturbed motion of the flight vehicle is located at the distance $x_{\tau j} = -\frac{h_j}{2}$ from the free surface. With disturbed motion of the vehicle, the abscissa of the center of pressure $x_{\text{Д}j}$ of hydrodynamic forces will not coincide with $x_{\tau j}$. Its position is easy to find if expressions for $\Delta Q_{\tau j}(-h_j)$ and $\Delta M_{\tau j}(-h_j)$, when $\ddot{\vartheta} = 0$

$$\bar{x}_{\text{Д}j} = -\left\{ \frac{h_j}{2} - \frac{B_{kj}}{\xi_k} \left[\operatorname{th} \mu_{kj} - \frac{1}{\xi_k h_j} \left(\frac{1}{\operatorname{ch} \mu_{kj}} - 1 \right) \right] \right\} \frac{1}{k_j^0}. \tag{2.73}$$

Due to the effect of mobility of the liquid on the position of the center of pressure and especially on the magnitude of the reduced propellant weight in the tanks, the position of the reduced center of gravity of the flight vehicle in the whole $x^{\bullet}_{1т}$ can differ from the position of the center of gravity $x_{1т}$ corresponding to the undisturbed motion. On the basis of formula (73)

$$x^{\bullet}_{1т} = x_{1тк}\frac{G_k}{k^0 G} + \sum_{j=1}^{n_j}\frac{G_{тj}}{G}\frac{k^0_j}{k^0}(x_{10j} - x_{kj}). \tag{2.74}$$

Here $x_{1тк}$ – coordinate of the center of gravity of "dry" flight vehicle (without fuel in the tanks).

For an illustration Fig. 2.5 gives values $\bar{x}^{\bullet}_{1т} = \frac{x^{\bullet}_{1т}}{l}$ and $\bar{x}_{1т} = \frac{x_{1т}}{l}$ as a function of $\bar{t}$ for the flight vehicle "V-2." As can be seen, $x^{\bullet}_{1т}$ is constant almost on the entire phase of powered flight and equal approximately to 0.6 l with the average magnitude of $x_{1т} = 0{,}56\,l$.

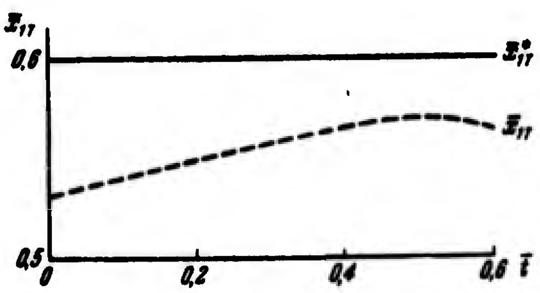

Fig. 2.5. Effect of mobility of liquid in the fuel tanks on the position of the effective center of gravity of the flight vehicle when A = B = 1.

The indicated change of the position of the reduced center of gravity with disturbed motion of the flight vehicle should (in the case of the use of coefficients of effect) be considered both in the determination of moments from external forces acting on the flight vehicle in flight and during calculation of the reduced moment of inertia J^{*}_{z}.

The effect of influence of the mobility of the liquid on magnitude J_z^* or on the value of the corresponding coefficient k^1 of effect to a considerable degree depends on the ratio of moments of inertia of the fuel and structure $J_т/J_к$. With a decrease in the latter (for instance, due to fuel consumption), the distinction J_z^* from J_z will decrease. Let us designate the moment of inertia of a "dry" flight vehicle (taking into account the liquid found only in the fuel lines and jacket of the combustion chamber of the engine) relative to the reduced center of gravity by $J_к^*$. Then the reduced moment of inertia of the flight vehicle on the whole can be presented in the form

$$J_z^* = J_к^* + \sum_{j=1}^{n_j} J_{тj}^* = J_к^*\left[1 + \sum_{j=1}^{n_j} \frac{J_{тj}}{J_к^*}(d_j - b_j A_{kj})\right], \tag{2.75}$$

where

$$J_к^* = J_к + m_к (x_{1т}^* - x_{1тк})^2.$$

When $A_{kj} = B_{kj} = 1$

$$J_z^* = J_к^*\left[1 + \sum_{j=1}^{n_j} (d_j - b_j)\cdot\frac{J_{тj}}{J_к^*}\right]. \tag{2.76}$$

In this particular case always $1 > d_j - b_j > 0$, and, consequently, $k_п^1 = \frac{J_z^*}{J_z} > 0$. Usually coefficient $k_п^1$ differs comparatively little from unity (not more than 15%), and in the first approximation the effect of mobility of the liquid on the moment of inertia with an impulse load of the flight vehicle can be disregarded.

To illustrate the degree of the effect of mobility of the liquid on inertial characteristics of the considered flight vehicle with such form of motion, Fig. 2.6 gives approximate graphs $k_п^0(\bar{t})$ and $k_п^1(\bar{t})$.

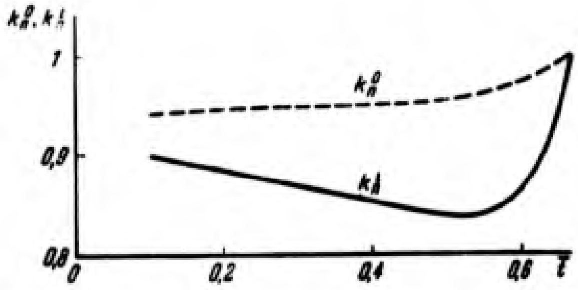

Fig. 2.6. Change in coefficients k_n^0 and κ_n^1 with respect to $\bar{t}$.

During steady-state oscillations of the flight vehicle the effect of mobility of the liquid on its reduced moment of inertia can be substantial, and k^1 can considerably differ from unity and even take negative values. Usually the greatest effect on magnitude k^1 is by that tank whose free surface of the liquid is further from the center of gravity of the flight vehicle.

The dependence of the given inertial characteristics of the flight vehicle from the nature of the disturbed motion is reflected on the magnitude of the angle of attack and on values of lateral forces and bending moments in sections of the body. An appraisal of this effect can be conducted only by knowing the dependence of parameters of motion of the flight vehicle due to its inertial characteristics, i.e., the solution of the system of equations (58), (59), and (60).

For the case of impulse loading of an unguided flight vehicle, it is possible to estimate directly the effect of mobility of the liquid on the magnitude of the lateral acceleration, using coefficients k_n^0 and k_n^1 and values of $\dot{x}_{1T}$. Actually, from formulas (12) and (65) it is clear that in this case loading of the value of the lateral acceleration of the center of gravity of the flight vehicle depend only on the magnitude k_n^0:

$$(n^0_{y_1})^* = \frac{n^0_{y_1}}{k^0_n}.$$

Since $k_n^0 \geq 0.8$, the actual value of the acceleration can differ from $n_{y_1}^0$ by not more than 20%. The change in the rotary component of the acceleration in the given case of loading is characterized by the ratio

$$c^x = \frac{(\overset{\bullet}{x}_{1T} - x_{1Д})(\overset{\bullet}{x}_{1T} - x_1)}{k_n^l (x_{1T} - x_{1Д})(x_{1T} - x_1)},$$

the magnitude of which, depending upon the position of the center of pressure, $x_{1Д}$ can be changed in comparatively wide limits.

The analysis conducted above shows that the effect of mobility of the liquid on parameters of disturbed motion of the flight vehicle depends on the place of location of the tanks with respect to the center of gravity and their geometry. The form and dimensions of the tanks determine both the frequency of the natural oscillations and the mass of liquid participating in these oscillations. With an increase in the indicated mass and decrease in the frequency ω_{kj}, the effect of mobility of the liquid increases. With the approach of frequency ω_{kj} to frequencies of regulating the control system of characteristics of the stability of motion of the flight vehicle substantially worsen. Therefore, we usually take appropriate measures to increase values ω_{kj} and limiting the mass of freely fluctuating liquid, for instance, by means of installation in the tanks (remote from the center of gravity) of different partitions.[3] As a result for the majority of the flight vehicles (including vehicles having the form of their tanks different from cylindrical) the effect of mobility of the liquid on parameters of their motion practically proves to be very limited, and it frequently can be disregarded.

<u>Footnotes</u>

[1]The indicated values are the maximum and determining strength of the flight vehicle. They are realized only during a relatively small time of operation and for many types of flight vehicles (for instance, aircraft) by no means not in every flight and not for each type of the vehicle. This permits in a number of cases, due to corresponding lowering (usually small) of the probability of fulfillment of the problem, decreasing the maximums of overloads and, consequently, improve weight and flying characteristics of the vehicle.

[2]It is necessary to note that with selected system of coordinates these equations remain, nevertheless, connected with each other. However, this connection is weak, and in the first approximation it can be disregarded.

[3]Abramson H. N., Garza L. R., Some measurements of the effects of Ring Baftles in Cylindrical Tanks. Journal of Spacecraft and Rockets 1, 1964.

CHAPTER III

INTERNAL FORCE FACTORS. CASES OF LOADING

§ 3.1. Methods of the Determination of Internal Force Factors

After longitudinal and lateral forces and bending moments in sections of the body of the flight vehicle are determined separately from aerodynamic, hydrodynamic and inertial forces, it is possible to turn to the question of finding the total internal force factors in these sections from the totality all external forces and forces acting on the vehicle, which are caused by elastic oscillations of structure. With the dynamic action of disturbing forces and moments, the actual deformations of the structure will not correspond to those deformations which are obtained in the static application of these loads. Real values of forces in cross sections of the body of the flight vehicle will differ from forces obtained as a result of the static calculation. This distinction increases with an increase in dimensions of the flight vehicles and decrease in relative rigidity of their structure. Appearing with rapid application of external forces, elastic oscillations of the structure have both a direct and indirect effect on values of the internal force factors. In this case the indirect effect (in terms parameters of disturbed motion of the flight vehicle) becomes especially noticeable in the case when the frequency of the natural bending oscillations of the body of the flight vehicle prove to be close to frequencies of regulation of the control system. In similar cases it is impossible to use the ~~approximate~~ system of perturbation equations of the form (2.58) and (2.59), which describes the motion of the flight vehicle as a solid body.

For a linear deformed system, i.e., for a system, to which there will be applied the superposition principle, it is possible to indicate two methods of the calculation of these forces: the method of <u>displacements</u> and the method of <u>overloads</u> (accelerations). According to the method of displacements the total force factors in cross sections of the body of the flight vehicle (longitudinal and lateral forces, bending and torsional moments) are determined in terms of deformation according to formulas known from the elementary theory of the strength of materials, namely,

$$\left.\begin{aligned} N(x_1, t) &= -E(x_1)F_c(x_1)\frac{\partial u(x_1, t)}{\partial x_1},\\ Q(x_1, t) &= \frac{\partial M(x_1, t)}{\partial x_1},\\ M(x_1, t) &= B(x_1)\frac{\partial^2 y_1(x_1, t)}{\partial x_1^2},\\ M_{\text{кр}}(x_1, t) &= -G(x_1)I_p(x_1)\frac{\partial \theta(x_1, t)}{\partial x_1}, \end{aligned}\right\} \tag{3.1}$$

where E — modulus of longitudinal elasticity, G — shear modulus, F_c — area of cross section of the carrier part of the structure $B(x_1)=E(x_1)I(x_1)$ — rigidity of construction of the body of the flight vehicle with bending, $I(x_1)$ — moment of inertia of the area of the cross section of the carrier part of the body of the relative lateral axis z_1, $I_p(x_1)$ — polar moment of inertia of this area of the cross section, $u(x_1, t)$ — longitudinal elastic displacement of the cross section, $y_1(x_1, t)$ — lateral displacement of points of the longitudinal axis of the body due to bending of the structure, and $\theta(x_1, t)$ — angle of twist of the considered section x_1.

The essence of the method of overloads consists of a separate calculation of the so-called <u>static</u> and <u>dynamic</u> forces. In accordance with this method static values of force factors are determined in terms of external forces proceeding from conditions of dynamic equilibrium of the flight vehicle as a solid body and dynamic values by overloads caused by elastic oscillations of the structure. The total forces here are equal to the sum of the static and dynamic components

$$
\left.\begin{aligned}
N(x_1, t) &= N_{\text{с}}(x_1, t) + N_{\text{д}}(x_1, t),\\
Q(x_1, t) &= Q_{\text{с}}(x_1, t) + Q_{\text{д}}(x_1, t),\\
M(x_1, t) &= M_{\text{с}}(x_1, t) + M_{\text{д}}(x_1, t),\\
M_{\text{кр}}(x_1, t) &= M_{\text{кр с}}(x_1, t) + M_{\text{кр д}}(x_1, t).
\end{aligned}\right\} \tag{3.2}
$$

In this case dynamic components of these forces (designated by subscript "д") are determined in terms of forces of inertia by formulas

$$
\left.\begin{aligned}
N_{\text{д}}(x_1, t) &= -\int_0^{x_1} m(x_1)\frac{\partial^2 u(x_1, t)}{\partial t^2}\,dx_1,\\
Q_{\text{д}}(x_1, t) &= -\int_0^{x_1} m(x_1)\frac{\partial^2 y_1(x_1, t)}{\partial t^2}\,dx_1,\\
M_{\text{д}}(x_1, t) &= \int_0^{x_1} Q_{\text{д}}(x_1, t)\,dx_1,\\
M_{\text{кр д}}(x_1, t) &= -\int_0^{x_1} I_m(x_1)\frac{\partial^2 \theta(x_1, t)}{\partial t^2}\,dx_1,
\end{aligned}\right\} \tag{3.3}
$$

where $I_m(x_1)$ – polar mass moment of inertia of the cross section of the body of the flight vehicle.

§ 3.2. Longitudinal Forces

The total longitudinal force $N(x_1, \bar{t})$ in any cross section of the body of the flight vehicle x_1 is equal to the algebraic sum of longitudinal forces from external surface forces acting on the cutoff part of the body and from corresponding mass forces. The latter in the general case of loading are determined by expression

$$
N_{\text{и}}(x_1, \bar{t}) = \int_0^{x_1} q_{Gx}(x_1, \bar{t})\, n_{x_1}(x_1, \bar{t})\,dx_1, \tag{3.4}
$$

where $n_{x_1}(x_1, \bar{t})$ is the local value of the longitudinal acceleration, and $q_{Gx}(x_1, \bar{t})$ – weight per unit length of the vehicle (taking into account the concentrated loads).

Static values of longitudinal forces in most cases with sufficient accuracy are determined by the product of the weight of the cutoff part of the structure $G(x_1, \bar t)$ by the coefficient of longitudinal acceleration of the center of gravity $n^0_{x_1}(\bar t)$

$$N_{\text{мс}}(x_1, \bar t) = n^0_{x_1}(\bar t)\, G(x_1, \bar t).$$

In sections located above the place of the application to the body of effective force (thrust force, forces from accelerators) the longitudinal force is determined by formula

$$N(x_1, \bar t) = N_{\text{а}}(x_1, \bar t) + N_{\text{м}}(x_1, \bar t) - \Delta p^{\text{в}}(x_1, t)\, S(x_1) \quad \text{when } x_1 < x_{1\text{п}}, \tag{3.5}$$

where $S(x_1)$ – area of midsection of the body in section x_1, and $\Delta p^{\text{в}}(x_1, \bar t)$ – excess (with respect to atmospheric) gas pressure inside the body in the same section (for instance, boost pressure in carrier fuel tanks and others).

In sections of the body located below the place of application of effective force P, this force is equal to

$$N(x_1, t) = N_{\text{м}}(x_1, \bar t) + N_{\text{а}}(x_1, \bar t) - \Delta p^{\text{в}}(x_1, \bar t)\, S(x_1) - P(\bar t) \quad \text{when } x_1 > x_{1\text{п}}. \tag{3.6}$$

And, finally, $N(l) = 0$ when $x_1 = l$. With the selected direction of x_1 axis of the connected system of coordinates the positive value N will correspond to compression and negative to tension.

Usually in the calculation of longitudinal static forces, certain difficulties are caused by the calculation of the aerodynamic component $N_{\text{а}}(x_1, t)$. For its calculation, besides parameters of the trajectory of the flight vehicle, it is necessary to have diagrams of the distribution of aerodynamic pressure over the surface of the body, at least for several M_∞ numbers.

As follows from formulas (5) and (6), a disregard of this component $N_a(x_1, t)$ will lead to an understating of the magnitude of longitudinal force $N(x_1, t)$. However, on the other hand, a disregard of drag of the vehicle in the determination of longitudinal acceleration (2.16) leads to a certain increase in the component of longitudinal force from mass forces. Therefore, naturally, there appears the question concerning in what cases the effect of $N_a(x_1, t)$ on $N(x_1, t)$ can be disregarded not to the detriment of the strength of the structure. It is obvious that this is permissible in those cases when the magnitude $N_a(x_1, t)$ is small as compared to $N_M(x_1, t)$ or when its effect is compensated by a corresponding increase in component $N_M(x_1, t)$ due to the growth in $n_{x_1}(t)$. The last condition on the basis of formulas (1.18), (1.6) and (2.16) has the form

$$\frac{1}{c_x}\int_0^{x_1}\frac{\partial c_x(x_1)}{\partial x_1}dx_1 \leqslant \frac{G(x_1)}{G} - \frac{1}{1-t}\left[\mu(x_1) - \frac{t}{T_1(x_1)}\right], \tag{3.7}$$

where

$$\mu(x_1) = \frac{G_0(x_1)}{G_0}, \quad T_1(x_1) = \frac{\dot{G}}{\dot{G}(x_1)},$$

$G_0(x_1)$ and $\dot{G}(x_1)$ are the initial weight and flow rate per second of weight of the cutoff (section x_1) section of the vehicle. At constant weight, i.e., when $\dot{G}(x_1) = 0$

$$\frac{1}{c_x}\int_0^{x_1}\frac{\partial c_x(x_1)}{\partial x_1}dx_1 \leqslant \frac{\mu(x_1)}{1-t}.$$

Usually the indicated inequality is not fulfilled only for those parts of the flight vehicle which possess relatively low weight of the structure and absorb large aerodynamic loads. These include different cowls, protective caps, stabilizers, aerodynamic and gas current vanes, and in a number of cases the body covering the engine section, and others. For carrier elements of the structure of the body of a heavy flight vehicle of the ballistic type, this inequality is almost always fulfilled. In any case, nonfulfillment of it does not lead to great errors. Therefore, on the section of powered flight longitudinal forces in main sections of the body of such a flight vehicle with sufficient accuracy can be calculated according to the following formula:

$$N(x_1, \bar{t}) = \frac{P(\bar{t})}{1-\bar{t}}\left[\mu(x_1) - \frac{\bar{t}}{T_1(x_1)}\right] - \Delta p^*(x_1, \bar{t})\, S(x_1). \tag{3.8}$$

For light flight vehicles for which the ratio X/G is great, one should use an expression of the form (5).

In carrying out calculations for strength, frequently knowledge of only the largest values of longitudinal forces is required. In this case it is sufficient that inequality (7) be fulfilled only in the region of the largest values of the fictitious mass force. It is not difficult to note that when $G(x_1) = \mathrm{const}$ and $\Delta p^*(x_1) = \mathrm{const}$ function $N(x_1, \bar{t})$ will have extrema at min n_{x_1} and max n_{x_1}, i.e., when $\bar{t} = 0$ and $\bar{t} = \bar{t}_к$.

For finding the largest magnitude of this function when $\dot{G}(x_1) \neq 0$ and $\Delta p^*(x_1) = \mathrm{const}$, one should equate to zero the derivative of $N(x_1, \bar{t})$ with respect to $\bar{t}$:

$$(1-\bar{t})\left[\mu(x_1) - \bar{t}\,\frac{\dot{G}(x_1)}{\dot{G}}\right]\dot{P} + P\left[\mu(x_1) - \frac{\bar{t}\dot{G}(x_1)}{\dot{G}}\right] = \frac{P(1-\bar{t})}{T_1(x_1)}.$$

Substituting into this equation the expression for function $P(\bar{t})$, for instance, in the form (2.19), we obtain the approximate value $\bar{t} = \bar{t}_1$, at which function $N(x_1, \bar{t})$ with respect to (when $\Delta p^* = 0$) has the maximum

$$\bar{t}_1 = 1 - \{(1 - a_p)[1 - \mu(x_1)\,T_1(x_1)]\}^{\frac{1}{2}}, \tag{3.9}$$

where

$$a_p = \bar{t}_{p0} - \frac{P_0}{p_0 F_0}(\bar{t}_p - \bar{t}_{p0}),$$

in which one can be certain, having taken the second derivative of $N(x_1, \bar{t})$ according to $\bar{t}$. It is necessary to note that with a small change in $G(x_1)$ according to $\bar{t}$, it is possible to take $\bar{t}_i = \bar{t}_k$.

The considered case $\dot{G}(x_1) \neq 0$ in practice pertains only to sections located between the fuel tanks. Longitudinal forces in sections found after the tanks, in region of the engine section, are basically determined by the magnitude of force $N_p(\bar{t})$ arriving at the body of the flight vehicle from the propulsion system:

$$N_p(\bar{t}) = P(\bar{t}) - n_{x_1}(\bar{t}) G_{\text{дв}} - \Delta p^{*}(\bar{t}) F_0 = \\ = P_0 + F_0 \left[p_0 - p_x^{*}(\bar{t})\right] - n_{x_1}(\bar{t}) G_{\text{дв}}, \qquad (3.10)$$

where $G_{\text{дв}}$ – flying weight of the propulsion system (taking into account filling with fuel), p_x^{*} – absolute pressure inside the tail section. This force is changed comparatively little with time and practically does not depend on $\dot{G}(x_1, \bar{t})$.

Longitudinal forces in cross sections of the body of the side-mounting booster of the flight vehicle of packet configuration in general will be determined by expression

$$N_i(x_1, \bar{t}) = P_{\text{б}i}(\bar{t}) - n_{x_1}(\bar{t}) \left[G_{\text{б}i}(\bar{t}) - G_i(x_1, \bar{t})\right] - \\ - \left[X_{\text{б}i}(\bar{t}) - N_{\text{а}i}(x_1, \bar{t})\right] - \Delta p^{*}(x_1, \bar{t}) S_i(x_1) - R_{x_1 i}(\bar{t}), \qquad (3.11)$$

where $P_{\text{б}i}$ – tractive force of i-th booster, $G_{\text{б}i}$ – its flying weight, $X_{\text{б}i}$ – drag of the booster, n_{x_1} – longitudinal accelerator of the center of gravity of the flight vehicle equal to

$$n_{x_1} = \frac{P_A + \sum_{i=1}^{m} P_{бi} - \left(X_A + \sum_{i=1}^{m} X_{бi}\right) - R_{x_1 A} - \sum_{i=1}^{m} R_{x_1 i}}{G_A + \sum_{i=1}^{m} G_{бi}}, \tag{3.12}$$

m – number of side-mounting boosters.

Calculated according to an analogous formula are longitudinal forces in cross sections of the body of the flight vehicle located below the place of application of longitudinal loads from side-mounting boosters, i.e., when $x_{1p}>x_1>x_{1n}$:

$$N_A(x_1, \bar{t}) = P_A(\bar{t}) - R_{x_1 A}(\bar{t}) - n_{x_1}\left[G_A(\bar{t}) - G_A(x_1, \bar{t})\right] - \Delta p^*(x_1, \bar{t}) S_A(x_1) - X_{1A}(\bar{t}) + X_{1A}(x_1, \bar{t}). \tag{3.13}$$

Here $[G_A(\bar{t}) - G_A(x_1, \bar{t})]$ and $[X_{1A}(\bar{t}) - X_{1A}(x_1, \bar{t})]$ is the weight and drag of the section of the vehicle located between sections x_1 and $x_1 = l$, respectively. Signs in formulas (11) and (13) correspond to signs accepted above for compressive and tensile forces.

In sections $x_1>x_{1n}$ of the body of the engine (tail) section (when $x_{1p} = \bar{l}$), unloaded by the force of thrust,

$$N_x(x_1, \bar{t}) = -\left[n_{x_1}(\bar{t}) \Delta G_x(x_1) + R_{x_1}(\bar{t}) + \Delta X_x(x_1, \bar{t})\right]. \tag{3.14}$$

where $\Delta G_x(x_1)$ and $\Delta X_x(x_1, \bar{t})$ denote the weight and drag of the considered section of the body of the tail section (between sections x_1 and $x_1 - l$). If $\Delta G_x(x_1)$ is small, the value of this force will basically depend on magnitude $R_{x1}(\bar{t})$, in particular, on angles of deviation of control devices. Since the latter are usually a random time function, then in this case one should be oriented on possible limiting values of the longitudinal force $N_x(x_1, \bar{t})$, corresponding to $\delta = 0$ and $\delta = \delta_{max}$ (simultaneously for all control devices – vanes, specially controlling engines joined to the body of the tail section).

On the phase of free flight (outside the dense layers of the atmosphere) longitudinal forces will be determined only by the magnitude of excess pressure inside the body of the flight vehicle. When motion in dense layers of the atmosphere they are determined by the difference in longitudinal components of aerodynamic and mass forces. In the latter case their magnitude will depend basically on the weight and aerodynamic arrangement of the vehicle.

Longitudinal loads on elements of bracing of all loads placed inside the body of the flight vehicle on all sections of the flight are completely determined by the local value of the longitudinal accelerator.

<u>Example</u>. If the body of the flight vehicle consists of separate comparatively small technological sections, then sometimes (for instance, for static tests) there is assigned not a diagram of longitudinal forces but a diagram of external longitudinal forces transmitted to the considered section on the side of adjacent sections. Let us conduct as an example calculations of such loads for the body of the fuel section of a flight vehicle of the ballistic type "V-2" (Fig. 3.1). The indicated section constitutes a shell strengthened by a longitudinal and lateral assembly, inside which suspension fuel tanks are located. The upper tank Г (for fuel) is suspended to the upper joint frame 4 of the section. The lower tank with the help of rods 2 is connected to the lower joint frame 1. In the section of the powered powered flight, acting on the upper part of this phase are forces N_{Γ} concentrated (at places of the suspension of rods of the fuel tank, the total quantity of which is equal to $n_{x_1} G_{\Gamma\text{р}}$, and compressing force on the side of the instrument section

$$N_{\text{пр}} = n_{x_1}(G_{\Gamma} + G_{\text{пр}}) + X_{1\text{п}},$$

where G_{Γ} – weight of the nose cone, $G_{\text{пр}}$ – weight of the instrument section, $G_{\Gamma\text{р}}$ – weight of tank of the fuel with liquid, and $X_{1\text{п}}$ – wave drag of the nose cone and instrument section. Since the construction of the instrument section of this vehicle consists of four longerons absorbing all longitudinal loads and four detachable covers absorbing only external aerodynamic pressure, then load N in practice is transmitted to the body of the fuel section in the form of four concentrated forces distributed on small length of the joint frame.

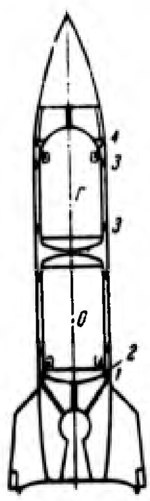

Fig. 3.1. Diagram of fuel section of the flight vehicle with suspension tanks ("V-2").

Acting on the lower part of the fuel section are compressing concentrated forces (at the place of the connection of four rods of the frame of the propulsion system and four rods of the tank of oxidizer), each of which is equal to

$$N_{н} = \frac{1}{4}\left[P_0 - n_{x_1}G_{дв} - n_{x_1}G_{ок} + F_0(p_0 - p_x^{*})\right],$$

and tensile load of an intensity evenly distributed over the contour of the section

$$q = \frac{1}{2\pi a}\left(n_{x_1}G_{хв} + X_{хв} + 4R_{xрв} + 4R_{xг}\right),$$

where a – radius of the middle surface of the shell of the body of the fuel section, $G_{хв}$, $G_{дв}$ – weight of the body of the tail section and propulsion system, $G_{ок}$ – weight of tank for the oxidizer with liquid, $R_{xpв}$ and $R_{xг}$ – drag of each of the four aerodynamic and gas-jet vanes (when $\delta = 0$), $X_{xв}$ – force of aerodynamic drag of the body of the tail section with stabilizers (taking into account the resisting force of suction according to ogival section of the body and corresponding part of the end section of the body).

On the phase of free flight in the case of undisturbed motion of this flight vehicle, acting on the upper section of the fuel section will be concentrated forces the total quantity of which is equal to

$$N_{\text{п}} = -\frac{X_1}{G}(G_{\text{г}} + G_{\text{пр}} + G_{\text{гр}}) + X_{1\text{в}}.$$

On the lower section the compressing concentrated forces

$$N_{\text{н}} = \frac{1}{4}\left[\frac{X_1}{G}(G_{\text{дв}} + G_{\text{ок}}) - F_0(p_x^{\text{в}} - p_x^{\text{н}})\right]$$

and distributed load of intensity

$$q = \frac{1}{2\pi a}\left(\frac{X}{G}G_{\text{хв}} - X_{\text{хв}} - 4R_{x\text{рв}}\right),$$

where $p_x^{\text{н}}$ – air pressure on the nozzle section (base pressure), $p_x^{\text{в}}$ – pressure on walls of the nozzle inside the tail section. In general the sign of force $N_{\text{п}}$ will depend on magnitude $G_{\text{гр}}$, i.e., due to the quantity of liquid remaining in the tank.

§ 3.3. Lateral Forces and Bending Moments

Static values of the lateral force and bending moment in cross sections of the body in plane x_1y_1 on the basis of the superposition principle will be equal to the algebraic sum of components Q and M from lateral aerodynamic (1.19) and (1.20), hydrodynamic (2.55), (2.56) and (2.57) and mass forces, and also external disturbing forces and moments applied to the apparatus on one side of the considered section:

$$\left.\begin{aligned} Q_{\text{c}}(x_1) &= Q_{\text{а}}(x_1) + Q_{\text{м}}(x_1) + \Delta Q_{\text{т}j}(x_1) + Q_{\text{в}}(x_1), \\ M_{\text{c}}(x_1) &= M_{\text{а}}(x_1) + M_{\text{м}}(x_1) + \Delta M_{\text{т}j}(x_1) + M_{\text{в}}(x_1), \end{aligned}\right\} \tag{3.15}$$

where

$$\left.\begin{aligned} Q_{\text{м}}(x_1) &= \int_0^{x_1} q_{Oy}(x_1)\, n_{y_1}(x_1)\, dx_1, \\ M_{\text{м}}(x_1) &= \int_0^{x_1} Q_{\text{м}}(x_1)\, dx_1. \end{aligned}\right\} \tag{3.16}$$

Here $q_{Gy}(x_1)$ denotes the lateral linear loading density of the vehicle, taking into account corresponding loads (real and conditional) applied to the body on section Ox_1. Conditional loads can be introduced when only the total weight of some load or even the whole section and position of its center of gravity are known, and the nature of its distribution is unknown. This is exactly how the lateral force and bending moment in plane x_1z_1 perpendicular to x_1y_1 are calculated.

The total value of the static bending moment in any section x_1 of the body of the flight vehicle will be equal to

$$M_{\text{c}}(x_1) = \left[M_{y\text{c}}^2(x_1) + M_{z\text{c}}^2(x_1)\right]^{\frac{1}{2}}. \tag{3.17}$$

It is expedient to calculate lateral forces $Q_{\text{м}}(x_1)$ and bending moments $M_{\text{м}}(x_1)$ by means of graphic integration of diagram $n_{y_1}(x_1)q_{Oy}(x_1)$ or by some method of approximation of numerical integration. The latter is selected in every concrete case depending upon the form of the diagram $q_{Oy}(x_1)$, the necessary accuracy of the calculations and the accuracy of the initial data.[1]

With the fulfillment of designed calculations it is frequently necessary to modify certain parameters of the flight vehicle. Therefore it is more convenient to use unit diagrams of lateral forces and bending

[1]In most cases in practice the formula of trapezoids with the number of steps of 10-20 proves to be quite sufficient for the distributed component of the load.

moments plotted for each external force separately. Values of such unit lateral forces and bending moments from aerodynamic forces are determined by formulas (1.19) and (1.20) when $\alpha = 1°$. It is somewhat more difficult to calculate unit values of the lateral force and bending moment from mass forces (16), since the rotary component of the lateral acceleration, weight and position of the center of gravity of the vehicle can be substantially changed in flight. In similar cases it is expedient to consider separately the constant and variable components of mass forces. For this one should separate from $n^x_{y_1}(x_1)$ the rotary component of the acceleration $n_{y_1}(x_1)$ in such a way so that the magnitude of it does not depend on the position of center of gravity of the flight vehicle, and, namely,

$$n_{y_1}(x_1) = \bar{n}^0_{y_1} + \bar{n}^x_{y_1}(x_1), \tag{3.18}$$

where

$$\bar{n}^0_{y_1} = n^0_{y_1} - \Delta\ddot{\vartheta}\frac{x_{1\text{т}}}{g_0},$$
$$\bar{n}^x_{y_1} = \Delta\ddot{\vartheta}\frac{x_1}{g_0}.$$

Having diagrams of lateral force and bending moment from the constant component of the force of weight $Q_{\text{мк}}(x_1)$ and $M_{\text{мк}}(x_1)$ and from the variable propellant weight $Q_{\text{мт}}(x_1, \bar{t})$ and $M_{\text{мт}}(x_1, \bar{t})$ with unit values of the above components of lateral acceleration (for instance, $\bar{n}^0_{y_1} = 1$, $\bar{n}^x_{y_1} = \frac{x_1}{g_0}$), the required values $Q_{\text{м}}(x_1, \bar{t})$ and $M_{\text{м}}(x_1, \bar{t})$ for any moment of time $\bar{t}$ can be obtained by using relation

$$\left.\begin{aligned} Q_{\text{м}}(x_1, \bar{t}) = {} & [Q^0_{\text{мк}}(x_1) + Q^0_{\text{мт}}(x_1, \bar{t})]\,\bar{n}^0_{y_1}(\bar{t}) + \\ & + [Q^x_{\text{мк}}(x_1) + Q^x_{\text{мт}}(x_1, \bar{t})]\,\Delta\ddot{\vartheta}(\bar{t})\frac{1}{g_0}, \\ M_{\text{м}}(x_1, \bar{t}) = {} & [M^0_{\text{мк}}(x_1) + M^0_{\text{мт}}(x_1, \bar{t})]\,\bar{n}_{y_1}(\bar{t}) + \\ & + [M^x_{\text{мк}}(x_1) + M^x_{\text{мт}}(x_1, \bar{t})]\,\Delta\ddot{\vartheta}(\bar{t})\frac{1}{g_0}, \end{aligned}\right\} \tag{3.19}$$

where

$$
\left.\begin{aligned}
&Q^{0}_{\text{м}i}(x_1) = \int\limits_0^{x_1} q_{Gy}(x_1)\,dx_1, \quad Q^{x}_{\text{м}i}(x_1, t) = \int\limits_0^{x_1} q_{Gy}(x_1, t)\,x_1\,dx_1, \\
&M^{j}_{\text{м}i} = \int\limits_0^{x_1} Q^{j}_{\text{м}i}\,dx_1 \qquad (i = \text{к},\ \text{т};\ j = 0,\ x).
\end{aligned}\right\} \tag{3.19'}
$$

In a number of cases it is convenient to present expressions for lateral forces and bending moments in canonical form, i.e., in the form

$$
\left.\begin{aligned}
Q(x_1) &= Q^{\alpha}(x_1)\alpha + Q^{\delta}(x_1)\delta, \\
M(x_1) &= M^{\alpha}(x_1)\alpha + M^{\delta}(x_1)\delta,
\end{aligned}\right\} \tag{3.20}
$$

where $Q^{\alpha}(x_1)$ and $M^{\alpha}(x_1)$ – total values of lateral force and bending moment from the action of the unit lateral aerodynamic force, and $M^{\delta}(x_1)$ and $Q^{\delta}(x_1)$ – from the action of the unit lateral control force.

Let us note that with correct fulfillment of calculations of lateral forces and bending moments in the end section of the body of any flight vehicle, there must be fulfilled the equalities

$$
\begin{aligned}
Q(l, t) &= 0, \\
M(l, t) &= 0.
\end{aligned}
$$

For this it is necessary always to make sure that the inertial characteristics of the vehicle G, J_z and $x_{1т}$ correspond to the assigned distribution of weight and the total aerodynamic properties c_n^{α}, $x_{1д}$ – to the assigned diagram of aerodynamic forces.

For composite flight vehicles equipped with side-force booster, which, as a rule, are less in dimensions and weight than the vehicle itself, are considered as beams located on two or three supports depending on the scheme of the bracing of them to the flight vehicle in a lateral direction. The plotting of diagrams of lateral forces and bending moments for the actual (carrier) vehicle is conducted taking into account corresponding loads R_{yi} from these side-mounting boosters. Thus, for instance, when $x_{1н}>x_1>x_{1в}$

$$\left.\begin{aligned} Q_{Ac}(x_1)&=Q_c(x_1)+\sum_{i=1}^{m}R_{yiв},\\ M_{Ac}(x_1)&=M_c(x_1)+\sum_{i=1}^{m}R_{yiв}(x_1-x_{1в}), \end{aligned}\right\} \qquad (3.21)$$

where $x_{1в}$ and $x_{1н}$ – abscissas of those sections of the body of the flight vehicle in which side-mounting boosters are mounted, m – number of side-mounting boosters. The indicated support loads determine the required strength of points of attachment of boosters to the vehicle. In the presence of two supports they ($R_{yiв}$ and $R_{yiн}$) are found from the equation of static equilibrium of the system of all external forces acting on the side-mounting booster: lateral aerodynamic force Y_{1i}, control force R_{y_1i} and corresponding mass forces:

$$\left.\begin{aligned} R_{yiв}&=Y_{1i}\left(1-\frac{x_{1дi}}{a}\right)+R^{\delta}_{y_1i}\left(1-\frac{l_i}{a}\right)\delta_i+\\ &\qquad+G_{бi}\left[n_{y_1}(x_{1тi})-\frac{x_{1тi}}{a}n_{y_1}(x_{1в})\right]-\Delta\ddot{\vartheta}\,\frac{J_{бi}}{a},\\ R_{yiн}&=\frac{1}{a}\left[Y_{1i}x_{1дi}+R^{\delta}_{y_1i}l_i\delta_i+G_{бi}n_{y_1}(x_{1в})\,x_{1тi}\right]+\Delta\ddot{\vartheta}J_{бi}, \end{aligned}\right\} \qquad (3.22)$$

where $G_{бi}$ – weight of the i-th side-mounting booster, l_i – its length, Y_{1i} – lateral aerodynamic load on this booster, δ_i – angle of deviation of the control device of the booster, $x_{1тi}$ and $x_{1дi}$ – coordinates of the center of gravity and center of pressure of the booster in the auxiliary system of coordinates, the origin of which is located in the vertex of the booster, and the x_{1i} axis is directed along its longitudinal axis to the tail, $J_{бi}$ – mass moment of inertia of the side-mounting booster relative to the lateral axis passing through the origin of the indicated auxiliary system of coordinates,

$$n_{y_1}(x_{в})=n^0_{y_1}-\frac{\Delta\ddot{\vartheta}}{g_0}(x_{1т}-x_{1в}),$$

$$n_{y_1}(x_{1тi})=n^0_{y_1}-\frac{\Delta\ddot{\vartheta}}{g_0}(x_{1т}-x_{1в}-x_{1тi}),$$

$$a=x_{1н}-x_{1в}.$$

$x_{1т}$ – coordinate of the center of gravity of the vehicle as a whole,
$R^{\delta}_{y,i} = \frac{\partial R_{y,i}}{\partial \delta_i}$.

As can be seen from these formulas, in general values of reactions ($-R_{y1в}$, $-R_{y1н}$) and, consequently, forces in cross sections of the body of side-mounting boosters depend not only on parameters of motion of the flight vehicle, but also due to the magnitude and direction of action of the control force, i.e., value δ_1. In the case when control devices (turning chambers) of side-mounting boosters are used for the obtaining of a controlling moment in the rolling plane, these angles δ_1 can noticeably differ from values resulting from equations of dynamic equilibrium of the flight vehicle in pitch or yawing planes. Therefore, in the calculation of total forces in bodies of boosters one should take into consideration the cause of the additional deviation of control devices and when necessary consider both the presence of additional disturbing forces and the effect of angular acceleration (relative to the longitudinal axis x_1).

If the body of the flight vehicle possesses large lateral dimensions, then in the calculation of bending moments from mass forces $M_м(x_1)$ it is necessary to consider additional bending moments caused by angular accelerations in pitch and yawing planes of the form

$$\frac{d}{dx_1} \Delta M_м(x_1) = -\Delta\ddot{\vartheta} G_к(x_1) \frac{R^2(x_1)}{2g_0}, \qquad (3.23)$$

where $G_к(x_1)$ – all parts of the body located on sections dx_1, and $R(x_1)$ – radius of the body in section x_1. Thus, for instance, with the parallel location (around the flight vehicle) in plane x_1y_1 of two side-mounting boosters this moment will be equal to

$$\Delta M(x_в) = -2r(x_в) G_{бi} \frac{\Delta\ddot{\vartheta}_i}{g_0}, \qquad (3.24)$$

where $r(x_в)$ – distance between the longitudinal axes of the body of the carrier vehicle and side-mounting booster at place of attachment of this booster (in a longitudinal direction) to the vehicle.

In a number of particular cases of loading, static bending moments and lateral forces are determined very simply. Let us consider as a first example the case of powered flight of a single-stage flight vehicle of the ballistic type with a solid-propellant engine outside the atmosphere. In this case the source of disturbing forces and moments can be only the propulsion system and control system. With steady-state operating conditions of engine the lateral component of the force of thrust P_{y_1} and disturbing moment $M_{pв}$ (1.14) will be slowly changed with time of the flight. Similarly, the corresponding control forces will be changed.

In this case, obviously,

$$\left.\begin{array}{l}\Delta n^x_{y_1}=0,\ -n^0_{y_1}=\frac{1}{G}(R^\delta_{y_1}\delta\pm P_{y_1}),\\ \left.\begin{array}{l}Q(x_1)=n^0_{y_1}G(x_1),\\ M(x_1)=n^0_{y_1}M^0_м(x_1)\end{array}\right\}\quad \text{when}\quad x_1<x_{1в},\\ \left.\begin{array}{l}Q(x_1)=n^0_{y_1}G(x_1)\pm P_{y_1},\\ M(x_1)=n^0_{y_1}M^0_м(x_1)\pm M_{pв}\end{array}\right\}\quad \text{when}\quad x_1>x_{1в}.\end{array}\right| \tag{3.25}$$

Since we usually try to ensure small values of P_{y_1} and $M_{pв}$, the indicated static lateral forces and bending moments will also be small.

With motion of the flight vehicle outside the atmosphere any deviation of the control devices from the program position, caused by errors in the operation of the actual control system, leads to the appearance of static lateral forces and bending moments equal to

$$Q_c(x_1)=-\frac{R^\delta_{y_1}\delta}{G}\left\{Q^0_м(x_1)\left[1-\frac{x_{1т}(x_{1p}-x_{1т})}{r^2}\right]+Q^x_м(x_1)\frac{x_{1p}-x_{1т}}{r^2G}\right\}, \tag{3.26}$$

$$M_c(x_1)=-\frac{R^\delta_{y_1}\delta}{r^2G}(x_{1p}-x_{1т})\left\{M^0_м(x_1)\left[\frac{r^2}{x_{1p}-x_{1т}}-x_{1т}\right]+M^x_м(x_1)\right\}. \tag{3.27}$$

where

$$r^2 = g_0 \frac{J_z}{G}.$$

Here

$$Q_c(x_1) = Q_c^{\delta}(x_1)\delta, \quad M_c(x_1) = M_c^{\delta}(x_1)\delta.$$

A second example can be the case of the free flight of an aerodynamically stable flight vehicle in dense layers of the atmosphere. Having set in equation (2.4) $R_{y_1} = 0$ ($\delta = 0$) and using formulas (2.12), (2.14) and (18), we obtain that

$$Q_c(x_1) = qSc_n^{\alpha}\alpha \left\{ \frac{1}{c_n^{\alpha}} \int_0^{x_1} \frac{\partial c_n^{\alpha}(x_1)}{\partial x_1} dx_1 - \right.$$
$$\left. - \left[Q_{\text{м}}^0(x_1) \left(\frac{r^2}{x_{1\text{д}} - x_{1\text{т}}} - x_{1\text{т}} \right) + Q_{\text{м}}^x(x_1) \right] \frac{x_{1\text{д}} - x_{1\text{т}}}{r^2 G} \right\}, \tag{3.28}$$

$$M_c(x_1) = \int_0^{x_1} Q_c(x_1)\, dx_1 = M_c^{\alpha}(x_1)\alpha. \tag{3.29}$$

Expressions for the lateral force and bending moment are similarly found for the case of guided flight of the flight vehicle in dense layers of the atmosphere with a slow change in the disturbing aerodynamic force and balancing values of the control forces. Angular accelerations of the flight vehicle with such a method of loading prove to be small, and in practice it is possible to consider $n_{y_1}^x = 0$. Using the first formula of (2.12) and expressions (2.14) and (2.11), we find

$$Q_c(x_1) = qSc_n^{\alpha}\alpha \left[\frac{1}{c_n^{\alpha}} \int_0^{x_1} \frac{\partial c_n^{\alpha}(x_1)}{\partial x_1} dx_1 - \frac{Q_{\text{м}}^0(x_1)}{G} \frac{(x_{1\text{р}} - x_{1\text{д}})}{(x_{1\text{р}} - x_{1\text{т}})} \right], \tag{3.30}$$

$$M_c(x_1) = qSc_n^{\alpha}\alpha \left[\frac{1}{c_n^{\alpha}} \int_0^{x_1} \int_0^{x_1} \frac{\partial c_n^{\alpha}(x_1)}{\partial x_1} dx_1\, dx_1 - M_{\text{м}}^0(x_1) \frac{1}{G} \frac{x_{1\text{р}} - x_{1\text{д}}}{x_{1\text{р}} - x_{1\text{т}}} \right] \tag{3.31}$$

Basic difficulties of the calculation of static lateral forces and bending moments for flight cases of loading consist in the determination of local values of the lateral acceleration, angles of attack of the flight vehicle and values of control forces.

§ 3.4. Load Cases of Flight Vehicles

On the basis of the analysis of conditions of operation of the structure of the flight vehicle in the process of operation it is possible to distinguish a number of characteristic from the point of view of strength of moments of its load, so-called load cases. In each such case the structure is under the effect of a fully defined (peculiar to only a given case) combination of external forces and surrounding conditions. It is natural that during the time of operation the flight vehicle will be subjected to a large number of different load cases. However, if one were not to take into account questions of fatigue strength (having importance mainly for vehicles of reusable application), for every element of the structure only in one of numerous load cases there will appear such internal forces of deformations which will determine its necessary carrier ability or rigidity. This load case characteristic for the considered element is called computed case load or simply computed case.

With the existing variety of designs of flight vehicles and conditions of their operation, it is difficult beforehand to establish all computed load cases not being attached to the specific design or at least to the very narrow class of flight vehicles. For a more or less broad class of these vehicles, it is possible to indicate only a number of generalized load cases, which determine not the separate characteristic moments of the load, but describes the characteristic processes of load of the design.

Let us divide all load cases into two categories: ground and flight cases. In this case we separate the basic load cases describing normal conditions of operation of the flight vehicle and additional cases corresponding to mainly emergency conditions of operation of its systems. For abbreviation all generalized load cases will be given designation by letters (letters of the Latin alphabet) with corresponding subscripts (basically letters of the Russian alphabet).

a) Ground load cases.

The category of ground load cases will refer to the following generalized cases.

Transport (T) – load cases of the structure in the process of the transport of flight vehicles in assembled form or separate parts (units) by land, water and air transport, namely: case $T_{\text{а}}$ – self-propelled ground transport, case $T_{\text{ж}}$ – railroad transport, case $T_{\text{к}}$ – water transport (above water or underwater), case $T_{\text{у}}$ – special transporter-erector over standard or specially built roads (iron, concrete), case $T_{\text{п}}$ – load of the structure in the process of raising and lowering the vehicle by a crane.

Case U – load of the structure in the process of installation of the flight vehicle from the transport position to the position accepted as the initial one for launch and in the process of the removal of it from the firing position.

Case V – load of the structure in the period of the servicing procedure with open ground launch (case $V_{\text{з}}$ – for a fueled flight vehicle and case $V_{\text{с}}$ – for unfueled). With open launch in a storm from the deck of a ship – cases $V_{\text{кз}}$ and $V_{\text{кс}}$, respectively.

Case S – load of the flight vehicle in the process of launch (up to the moment of breakaway from the launcher), including the process of the emergency shutdown of engines on the launcher.

Case E – load of flight vehicle or its descending part with landing on the surface of the planet. Case E' – with landing on water.

<u>Test (0)</u> – load cases of separate parts of the structure of the flight vehicles in the process of factory (technological, finishing or control) tests (for airtightness (0_{Γ}), external atmospheric pressure) with storage (0_{x}), vibration tests ($0_{в}$) and hot bench tests of the engines (0_{c}).

Additional load cases of this type can be cases of the effect on flight vehicles installed in closed structures (silo, hold of a ship and others), impulse or seismic forces caused by some explosion.

b) <u>Flight load cases</u>.

The category of basic generalized flight load cases can be the following cases:

<u>Case A</u> – load of the structure on the phase of powered flight in dense layers of the atmosphere. For maneuvering flight vehicles this case corresponds to a flight with max [illegible], for nonmaneuvering flight vehicles of the ballistic type – the case of flight through a jet stream, for other types of flight vehicles – the case max [illegible].

<u>Case A'</u> corresponds to a flight with max n_{y_1}, and <u>case A"</u> – flight of the vehicle with high subsonic speeds.

<u>Case B</u> – load of the structure on the final phase of powered flight in the region max n_{x}. For multistage flight vehicles this case (case B_{i}) corresponds to the end of operation of engines of the i-th stage.

<u>Case W</u> – load of structure during flight in bumpy air (turbulent atmosphere). Essentially this case is a derivative of case A and is distinguished specially in view of its great significance for flight vehicles of the aircraft type.

<u>Case K</u> – load of the structure in the process of separation of the boosters. For composite flight vehicles (case K_{i}) – in the process

of uncoupling of stages or separation of boosters (from the moment of the beginning shutdown of the engines of the i-th stage prior to the moment of breaking of the power connection between this stage and the vehicle).

Case K_r corresponds to the separation of the last stage (nose cone, spaceship and others), and case K_a – process of separation of the recoverable section during operation of the system of emergency rescue of the crew.

Case P – load of the structure in the process of attachment or mooring of the vehicle.

Case L – load of the structure on the initial section of powered flight with min n_{x_1}. For multistage vehicles this case (case L_i) will correspond to the load of the structure directly after breakaway of the vehicle from the launcher (i = I) or after separation of the (i - I)-th stage (for i = II, III, ...). It covers completely (with "cold" launch or separation of the stages) or partially (with "hot" launch or separation of the stages) the process of conditions of engines of the i-th stage.

Case L' – load of the recoverable section of the structure directly after its separation from the flight vehicle (with emergency rescue of the crew).

Case H – load of the structure of the flight vehicle in the process of realization of the limiting (available) controlling moment.

Case C – load of the structure during flight with zero angles of attack in the region of maximum impact pressures on the descent phase. For maneuvering flight vehicles this case corresponds to the case of diving with $c_{y_1}=0$ and for nonmaneuvering – to the moment of achievement of the limiting longitudinal acceleration with undisturbed motion. Case C' corresponds to the moment of achievement of the maximum value of longitudinal acceleration with disturbed motion of the flight vehicle.

Case D – load of the structure of the flight vehicle on the descent phase during flight with limiting values of lateral acceleration.

Case F_j – load of the structure of the vehicle descending to earth in the process of the inclusion of a parachute system: main (F_o), brake (F_T), reserve (F_3) – and in the process of the inclusion of retrorocket installation (F').

The additional flight generalized load cases will include cases corresponding to the following: 1) low-quality work of certain systems of the flight vehicle and 2) complete failure of these systems.

The first group of these cases determines the additional dynamic loads on the structure caused by the presence of the dynamic instability of the system – body of the flight vehicle and propulsion system (case J), the system – elastic body of the flight vehicle and the control system (case G) and the system – the vehicle with liquid filling and control system (case G').

Conditionally referred to this group can be cases G", which corresponding to the load of the structure as a whole or its parts with dynamic instability of the flutter type. All cases of the indicated group do not have independent importance for the strength of the structure of the flight vehicle and are considered only jointly with basic flight load cases of the phase of powered flight.

Cases of second group describe the load of the structure in emergency stages and are of practical importance only for flight vehicles equipped with a crew escape system. These include all forms of crash landing of piloted and unpiloted flight vehicles on the surface of the planet (case E_a), load cases of the structure in the process of partial or complete failure of the propulsion system (case Q), and in the process of complete or partial failure of the control system (case R).

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S E C T I O N II

DYNAMIC PROPERTIES OF FLIGHT VEHICLES CONSTRUCTION

CHAPTER IV

DYNAMIC REACTION OF THE CONSTRUCTION ON EXTERNAL INFLUENCE

This chapter contains general information about the dynamic characteristics and dynamic properties of elastic systems, which it is impossible to do without when reading subsequent material. In particular, methods of formulation of equations of dynamic equilibrium of elastic systems and calculations of their reaction to an external influence are indicated. Methods of approximation of détermination of frequencies and forms of natural elastic oscillations of simple and complicated constructions are discussed, and questions connected with account of the effect of damping are also touched upon.

§ 4.1. Selection of Dynamic Scheme

Selection of the dynamic scheme, i.e., formulation of a system of equations, describing conditions of dynamic equilibrium of the construction under the effect of external dynamic forces, is the most complicated and important problem of dynamic calculations of the construction of any flight vehicle. The dynamic model should, first of all, correctly reflect the basic properties of the initial real system and, secondly, be sufficiently simple, since dynamic calculation of the construction is an applied discipline and is occupied with solution of practical problems. It is possible to find the solution of the shown problem only by means of introduction of large simplifications. One of the basic of such simplifications is reduction of a system with an infinite number of degrees of freedom, described by differential equation in partial derivatives, to a system with a limited number of degrees of freedom, motion of which can be represented by a finite number of ordinary differential equations.[1]

There are two methods of such reduction. The first method is called the method of lumped parameters. According to this method the real construction with distributed mass and rigidity is replaced by a system of lumped masses, connected together with weightless springs, simulating the rigidity of elements of the system. The second method consists of decomposition of deformation of the system with respect to types of natural oscillations and calculation of only a finite number of the lowest forms of these oscillations. This finite number of degrees of freedom is selected in each case depending upon loading conditions, dynamic characteristics of the system and the required accuracy of calculations.

In this book we will use two methods of formulation of conditions of dynamic equilibrium of elastic systems.

For simple systems the equation of dynamic equilibrium is easily obtained from conditions of static equilibrium by means of inclusion (according to d'Alembert principle) in the number of external loads of corresponding forces of inertia. This method was already used by us during formulation of equations of motion of a flight vehicle as a solid body and equations of oscillations of an elastic beam. For complicated systems, for which the first method loses its apparent simplicity and clarity, the second method is very effective, with which conditions of dynamic equilibrium are written in the form of Lagrange equations of the second type, using the expression for work of external forces, kinetic and potential energy of the system:

$$-\frac{\partial T_0}{\partial q_j}+\frac{\partial}{\partial t}\left(\frac{\partial T_0}{\partial \dot{q}_j}\right)+\frac{\partial U_0}{\partial q_j}+\frac{\partial \Phi_0}{\partial \dot{q}_j}=Q_j, \tag{4.1}$$

where T_0 – is kinetic energy of the entire system, U_0 – potential energy, Φ_0 – function of energy dissipation, q_j – generalized coordinates and Q_j – generalized forces, corresponding to these generalized coordinates.

The application of these equations to elastic systems with distributed parameters assumes the possibility of presentation of any shift of elements of these systems, satisfying the condition of continuity of material, in the form of the sum of products of functions of only time and functions of only coordinate (sum of standing oscillations). In other words, it assumes the possibility of use of <u>method of expansion</u> of elastic deformations with respect to forms of natural oscillations of the construction. Moreover, as generalized coordinates there are taken independent functions of only time, the number of which will obviously correspond to the number of degrees of freedom of the system.

Similar transfer to a system with a limited number of degrees of freedom is justified by the fact that the time of application and removal of all the external forces affecting the flight vehicle is usually comparable with periods of only the first tones of natural oscillations of the construction. Therefore, the effect of the highest tones of these oscillations on internal forces turns out to be very small, and it can be disregarded. Only in special cases, when the time of action of external loads will be small as compared to periods of the first tones of natural oscillations of the system, can this method become little effective due to the necessity of calculation of a large number of harmonics.

With dynamic calculation of construction it is usually assumed that the system accomplishes small oscillations near the position of stable equilibrium, in which all generalized coordinates q_j are equal to zero.

Total kinetic energy of the system in general will consist of the sum of kinetic energies of its separate parts. At small oscillations the kinetic energy of each component part of the system will be equal

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to the sum of kinetic energies of its points. In case of stationary connections the kinetic energy of the system can be represented in the form of functions of the second degree from generalized velocities $\dot{q}_j$

$$T_0 = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} A_{ij} \dot{q}_i \dot{q}_j. \qquad (4.1')$$

In this case, if T_0 is reduced to the sum of squares, then equations are obtained in direct form

$$a_{ii}\ddot{q}_i + \sum_{j=1}^{\infty} b_{ij} q_j = Q_i \quad (i = 1, 2, \ldots), \qquad (4.2)$$

and if to the sum of squares there is reduced only potential energy U_0 of the system, then these equations take the reverse form

$$b_{ii} q_i + \sum_{j=1}^{\infty} a_{ij} \ddot{q}_j = Q_i \quad (i = 1, 2, \ldots), \qquad (4.3)$$

where a_{ij} – coefficients determining the inertial interaction of elements of the system, and b_{ij} – coefficients of elastic interaction of these elements.

If to the sum of squares there are simultaneously reduced potential and kinetic energies of the system, then equations (1) take the simplest form, and namely represent a system of independent second order equations

$$M_m(\ddot{q}_m + \omega_m^2 q_m) = Q_m \quad (m = 1, 2, \ldots), \qquad (4.4)$$

where ω_m – angular frequency of natural oscillations of system of m-th tone, M_m – generalized mass of the system. In this case generalized coordinates q_m are usually called normal coordinates.

The use of normal coordinates considerably simplifies the solution of all problems connected with investigation of forced oscillations of a complicated system.

Reduction of kinetic and potential energies of the system to the sum of squares, i.e., reduction of the system of equations in generalized coordinates q_j to a system of equations in normal coordinates q_m, can be carried out by means of simple linear transformation of coordinates

$$q_j = \sum_{m=1}^{\infty} q_m A_{jm}. \tag{4.5}$$

Potential energy of any elastic system basically consists of strain energy, accumulated in its elements. Therefore, it is determined by the work of external forces expended for deformation of these elements. It is necessary to emphasize that the difficulty of formulation of conditions of dynamic equilibrium by the second method consists not only of the selection of generalized coordinates, but obtaining an expression for potential energy of the entire system. Therefore, we will specially elaborate on this question in the examining concrete problems. Here we are limited only by the fact that we will give expressions of strain energy for a thin-walled rod of variable cross sections. Potential energy of such a rod during bending is determined by expression

$$U_0(t) = \frac{1}{2}\int_0^l B(x_1)\left[\frac{\partial^2 y_n(x_1, t)}{\partial x_1^2}\right]^2 dx_1 + \frac{1}{2}\int_0^l G(x_1)F_c(x_1)\left[\frac{\partial y_c(x_1, t)}{\partial x_1}\right]^2 dx_1, \tag{4.6}$$

where G – shear modulus, $y_n(x_1, t)$ and $y_c(x_1, t)$ – deflections of section x_1 from bending and shear.

With longitudinal deformation – formula

$$U_0(t) = \frac{1}{2}\int_0^l E(x_1)F_c(x_1)\left[\frac{\partial u(x_1, t)}{\partial x_1}\right]^2 dx_1. \tag{4.7}$$

Strain energy of torsion has a similar form

$$U_0(t) = \frac{1}{2}\int_0^l G(x_1)I_p(x_1)\left[\frac{\partial \varphi(x_1, t)}{\partial x_1}\right]^2 dx_1. \tag{4.8}$$

Here $I_p(x_1)$ – polar moment of inertia of the area of the supporting part of the cross section of construction, $u(x_1, t)$ – longitudinal shift of section, $\varphi(x_1, t)$ – angle of rotation of the cross section relative to the longitudinal axis.

Work of external surface forces is expended not only on increase of potential energy, but also on change of kinetic energy of the system. If the load is changed slowly (statically), then the rate of shifts of points of the system will be small and the role of kinetic energy in the overall work will be negligible. With rapid change of the magnitude of external surface load (dynamic action of force) the role of kinetic energy of elastic oscillations of the construction can already be great. In the process of free harmonic oscillations of the system, i.e., in the absence of external forces and in the absence of energy losses for damping, kinetic energy changes into potential and back. In this case the sum of these energies remains constant. Bearing in mind that with harmonic oscillations maximum value T_0 is attained when $q_j = 0$, i.e., at the moment of transition of the elastic system through the position of stable equilibrium, in which $U_0 = 0$, we obtain that

$$\max T_0 = \max U_0. \qquad (4.9)$$

This equality plays an important role in the dynamics of elastic bodies. It is the basis of almost all methods of approximation of determination of the frequencies of natural elastic oscillations of elements of different constructions, including flight vehicles.

In the conclusion of this paragraph let us note that for finding generalized forces Q_j, Q_m it is sufficient to formulate an expression of work of all external surface forces on virtual displacements of the system and to take coefficients at variations of corresponding generalized coordinates. In this case the work is considered positive if the shift and projection of external force to this shift have identical signs. Work of gravity is usually considered during formulation of the expression for potential energy of the system.

§ 4.2. Reaction to External Influence

Basic properties of the linear dynamic system, behavior of which is described by an ordinary differential equation with constant coefficients, can be obtained by considering its reaction to elementary external influences, namely: to unit pulse, harmonic function and random function of time. By using these simplest forms of reaction, it is possible on the basis of the superposition principle to find the reaction of linear system to any external influence, which is an arbitrary function of time.

1. Reaction to unit pulse. By pulse we usually mean such a disturbance, the time of application of which is infinitesimal. Unit pulse is determined by the following expression:

$$\int_0^t \delta(t-\tau)\,dt = 1,$$

and

$$\int_0^t u(t)\,\delta(t-\tau)\,dt = u(\tau) \qquad (4.10)$$

when τ is inside the interval of integration, and

$$\int_0^t \delta(t-\tau)\,dt = 0$$

when τ is outside this interval. In these expressions $\delta(t-\tau)$ designates delta function, equal to zero everywhere except $t = \tau$, where it becomes infinite, and τ – moment of time corresponding to the application of unit external influence to the system.

As a result of such impulsive influence on a system with one degree of freedom, a change of its quantity of motion will occur by a magnitude equal to $M_m \dot{q}_m$, where M_m – generalized mass, and $\dot{q}_m$ – generalized velocity. Consequently, if a similar system was at rest, then its reaction $K_m(t)$ to unit pulse applied at moment $t = 0$ can be obtained as the solution of the equation describing free oscillations of the system at initial conditions, corresponding to $q_m(0)=0$ and $\dot{q}_m(0)=\frac{1}{M_m}$. This reaction carried the name <u>impulsive transient function</u> and is determined by the following formulas;

$$K_m(t)=\frac{1}{\omega_m M_m}\sin\omega_m t \tag{4.11}$$

for conservative system and

$$K_m(t)=\frac{1}{\omega_{m1} M_m}e^{-h_m t}\sin\omega_{m1} t, \tag{4.11'}$$

where

$$\omega_{m1}=\sqrt{\omega_m^2-h_m^2}$$

for a system possessing drag, proportional to the first degree of velocity, i.e., for a system whose motion is described by equation

$$\ddot{q}_m+2h_m\dot{q}_m+\omega_m^2 q_m=Q_m\frac{1}{M_m}. \tag{4.11''}$$

A graph of any function, representing an external disturbance on the system (in a certain time interval $0 < \tau < t$), can be presented in the form of sequence of pulses $Q_m(\tau)\Delta\tau$, applied at moments τ. Total reaction of the conservative system to this sequence of pulses will be determined by expression

$$q_m(t)=\sum_{i=1}^{k}Q_m(\tau_i)K_m(t-\tau_i)\Delta\tau_i.$$

By transferring to the limit when $\Delta\tau \to 0$, we obtain

$$q_m(t)=\int_0^t Q_m(\tau)K_m(t-\tau)\,d\tau. \tag{4.12}$$

Further, having taken into account expression (11), we will have

$$q_m(t)=\frac{1}{\omega_m M_m}\int_0^t Q_m(\tau)\sin\omega_m(t-\tau)\,d\tau. \tag{4.12'}$$

By means of replacement of the variable of integration, expression (12) can lead to the form

$$q_m(t) = \int_0^t Q_m(t-\tau) K_m(\tau)\, d\tau.$$

Reaction of unconservative system in general with arbitrary initial conditions $q_m(0) = q_{m0},\ \dot{q}_m(0) = \dot{q}_{m0}$, will be equal to

$$q_m(t) = \frac{1}{M_m \omega_{m1}} \int_0^t e^{-h_m (t-\tau)} Q_m(\tau) \sin \omega_{m1} (t-\tau)\, d\tau + e^{-h_m t}\left[q_{m0} \cos \omega_{m1} t + \frac{1}{\omega_{m1}} (\dot{q}_{m0} + h_m q_{m0}) \sin \omega_{m1} t\right]. \qquad (4.13)$$

2. Reaction to harmonic function. Basic dynamic characteristic of any linear oscillatory system is its reaction to unit harmonic function $e^{i\omega t}$, since it is well known that a very large group of external influences can be represented in the form of imposition of harmonic oscillations. In general the shown reaction of the system can be written in the form

$$q_m(t) = G_m(i\omega) e^{i\omega t},$$

where $G_m(i\omega)$ is a function only of ω, called complex transfer function.

Transfer function is a very important frequency-response curve of the construction of any flight vehicle. It is connected with pulse transient function by known Fourier transformation

$$G_m(i\omega) = \int_{-\infty}^{\infty} K_m(t) e^{-i\omega t}\, dt. \qquad (4.14)$$

In the particular case of a system with one degree of freedom, forced oscillations of which are described by equation

$$\ddot{q}_m + \omega_m^2 q_m = \frac{1}{M_m} e^{i\omega t},$$

the search reaction on the basis of formula (12) is determined by expression

$$q_m(t) = \frac{e^{i\omega t}}{M_m \omega_m^2} \left[\frac{1}{1 + \left(\frac{i\omega}{\omega_m} \right)^2} \right].$$

The first factor of this expression determines the reaction of the system to the static action of the considered disturbance, and the second factor, called amplification factor or dynamic coefficient, shows how many times the dynamic system response in this case of external influence is greater than static. As can be seen, the magnitude of this amplification factor depends on the relationship of frequencies ω and ω_m. For a conservative system with ω approaching ω_m, the amount of amplification factor will be increased infinitely. With the presence of energy dissipation in the system, for instance, proportional to generalized velocity, this amplification factor η will take limited values in the zone of resonance when $\omega = \omega_m$.

Indeed, as it is easy to establish from (11"), in this case

$$\eta_m = \frac{1}{\left[\left(1 - \frac{\omega^2}{\omega_m^2} \right)^2 + \gamma_m^2 \left(\frac{\omega}{\omega_m} \right)^2 \right]^{\frac{1}{2}}}, \quad \text{where} \quad \gamma_m = \frac{2h_m}{\omega_m}.$$

The magnitude of h_m determines phase shift ϕ_m between disturbance and generalized shift

$$\varphi_m = \operatorname{arctg} \frac{\gamma_m \frac{\omega}{\omega_m}}{1 - \left(\frac{\omega}{\omega_m} \right)^2}.$$

The relationship of η_m to ω determines the amplitude frequency characteristic of the system, and the relationship of ϕ_m to ω – phase frequency characteristic of the system.

3. <u>Reaction to random function</u>. If the external influence cannot be represented in the form of some known function of time, or its values are continuously changed and turn out to be different with each new observation, then such an influence can be related to the category of random.

System response to the action of discrete random disturbances (for instance, gusts of wind), appearing through intervals of time that are sufficient so that the system would return to initial undisturbed state of motion, is expediently considered as a usual individual transient process. System response to continuous random influence, for instance, to continuous atmospheric turbulence, will constitute sustained oscillations with continuous spectrum. During calculation of such reaction one should use already statistical methods.

The basic characteristic of a continuous random process is the probability distribution function for magnitudes of perturbations at different moments of time. If the form of this probability distribution function does not depend on the selection of the beginning of reading of time t, then such a random process is called <u>stationary</u>. Subsequently in examining the effect of continuous atmospheric turbulence on the flight vehicle we will assume that we are dealing only with stationary random processes, for which the so-called <u>ergodic hypothesis</u> is valid. In accordance with this hypothesis one may assume that statistical properties of atmospheric turbulence, determined during observation of one realization of the process for a comparatively large interval of time, coincide with the statistical properties obtained during observation of many similar realizations at the same moment of time.

The structure of random stationary influence is characterized by the value of perturbation function at every moment of time u(t) and the degree of interconnection between these values at moments t and t + τ. The shown degree of interconnection of values of u(t) and u(t + τ) is established by <u>correlation function</u> R(τ), which is defined as the time average from products of u(t) by u(t + τ):

$$R(\tau)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}u(t)u(t+\tau)\,dt. \qquad (4.15)$$

Correlation function of continuous stationary random process is an even function of τ with maximum when τ = 0. This maximum is equal to the mean square value of random process

$$R(0)=\overline{u}^2=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}u^2(t)\,dt.$$

With increase of the magnitude of τ the degree of interconnection of u(t) and u(t + τ) is naturally decreased, and consequently, the magnitude of R(τ) is decreased. Therefore, the correlation function of random process always has the form of an attenuated curve. For purely random process R(τ) = 0.

From formula (15) it is clear that with the presence in function u(t) of a constant and periodic components

$$u(t)=u_1(t)+C_0+C_1\sin\omega t,$$

analogous components will be present in function R(τ):

$$R(\tau)=R_1(\tau)+C_0^2+\frac{C_1^2}{2}\cos\omega\tau.$$

For finding the correlation function of a real random process the graph of concrete realization of random function u(t) at a sufficiently large time interval T_1 should be divided into a comparatively large number n of equal intervals Δt. In this case the length of the interval must be selected so that function u(t) changed little within limits of t and t + Δt. The less Δt is, the more exact the values of R(τ) will be. Thus, t = νΔt and τ = μΔt, where ν and μ – integers. Having assumed that u(t) = 0 when t > T_1, i.e., $\nu<n-\mu$, and being limited by consideration of only positive values of μ, instead of (15) we obtain the following formula for statistical appraisal of the sought correlation function:

$$R(\mu)=\frac{1}{n-\mu}\sum_{\nu=1}^{n-\mu}u_\nu u_{\nu+\mu}. \qquad (4.16)$$

This function is usually normalized in such a manner that it is dimensionless, for instance,

$$\rho(\tau)=\frac{R(\tau)}{R(0)}.$$

For providing the required accuracy of calculations $R(\mu)$ one should pay special attention to selection of interval T_1. In this case it is recommended to take μ less than $n/5$.

During determination of the system response to random influence it is convenient to use not the correlation function, but spectral density of random process $\Phi(\omega)$. The latter constitutes Fourier transformation from correlation function

$$\Phi(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}R(\tau)e^{-i\omega\tau}d\tau=\frac{2}{\pi}\int_0^{\infty}R(\tau)\cos\omega\tau\, d\tau. \qquad (4.17)$$

Spectral density is its own statistical characteristic of energy distribution of the process with respect to frequencies of continuous spectrum. It shows what portion of energy the component with frequency ω introduces into total energy of the system. Calculation of $\Phi(\omega)$ is carried out by graphic integration, and if $R(\tau)$ can be approximated by some function, then analytically. It is best of all, of course, for finding $R(\tau)$ and $\Phi(\omega)$ to apply electronic digital computers. Then

$$\Phi(\omega)=\frac{2}{\pi}\sum_{\mu=1}^{n/5}R(\mu)\cos(\omega\mu\Delta t)\Delta t. \qquad (4.18)$$

In certain cases it is more convenient to find the spectral function by applying Fourier transformation directly to function $u(t)$:

$$\Phi(\omega)=\lim_{T_1\to\infty}\frac{2}{T_1}\left|\int_0^{T_1}u(t)e^{-i\omega t}dt\right|^2\frac{1}{\pi}.$$

For a purely random process (white spectrum) $\Phi(\omega)$ = const; for a process containing periodic component of frequency ω_1, the spectral density will theoretically have discontinuity at this frequency. It is practically always continuous because of the finiteness of the section of recording and processing.

In order to obtain the spectral density of reaction $\Phi(\omega)$ of the system to random external influence, the spectral density of which $\Phi_B(\omega)$, one should calculate the value of correlation function of the system response by formula (15), having substituted expressions for q(t) and q(t + τ) instead of u(t) and u(t + τ), determined by formula (13) with corresponding replacement of variable of integration τ by other variables. Then, by using formulas (17) and (14), find the expressions for the energy spectrum of reaction

$$\Phi(\omega)=\Phi_{\text{в}}(\omega)|G(i\omega)|^2. \qquad (4.19)$$

By knowing $\Phi(\omega)$, it is easy to calculate the mean value of the square of generalized shift of system

$$\overline{q^2}=\int_0^\infty \Phi(\omega)\,d\omega,$$

which is a measure of dispersion of random variable from mean value

$$\bar{q}=\lim_{T_1\to\infty}\frac{1}{2T_1}\int_{-T_1}^{T_1} q(t)\,dt.$$

By using known Rays formula [58], we can find the average of cases of exceeding the definite level of reaction, for instance, longitudinal force N_1 for unit of time Δt:

$$n(\Delta t)=\frac{\Delta t}{2\pi}\left[\frac{1}{\sigma_N^2}\int_{-\infty}^{\infty}\omega^2\Phi(\omega)\,d\omega\right]^{\frac{1}{2}} e^{-\frac{N_1^2}{2\sigma_N^2}}, \qquad (4.20)$$

where σ_N – standard deviation of random value. The probability of exceeding the given value of force N_1 with normal law of distribution is determined by formula

$$P(N>N_1)=\frac{1}{\sigma_N\sqrt{2\pi}}\int\limits_{N_1}^{\infty} e^{-\frac{(N-\bar{N})^2}{2\sigma_N^2}}\,dN.$$

By means of substitution of

$$u=\frac{N-\bar{N}}{\sigma_N} \quad \text{and} \quad du=\frac{dN}{\sigma_N}$$

it is reduced to the following dimensionless form:

$$P(N>N_1)=\frac{1}{\sqrt{2\pi}}\int\limits_{u_1}^{\infty} e^{-\frac{u^2}{2}}\,du. \qquad (4.20')$$

From this formula it follows that events $N>N_1$ and $P>P_1$ will have identical probability if the lower limits of corresponding integrals of probability (20') will be equal, i.e., if

$$\frac{N_1-\bar{N}}{\sigma_N}=\frac{P_1-\bar{P}}{\sigma_P}, \qquad (4.21)$$

where $\bar{N}$, $\bar{P}$ — mean values of considered functions N and P. This condition will be subsequently used for appraisal of dynamic coefficients (Chapter VI).

§ 4.3. Equations of Elastic Oscillations

As it is known, an elastic body with distributed mass has an infinite number of degrees of freedom and for a description of its oscillations in general it is necessary to indicate the position of all its points. For certain types of elastic bodies, in particular for a beam of large elongation, it is possible to be limited by indication of the position of only the points of one line (axis of rigidity of the beam).

Numerous experimental data, accumulated in shipbuilding, aviation technology, rocket construction, and also certain theoretical research ([30] and others) show that with sufficient accuracy the bodies (fuselages) of flight vehicles, wings and stabilizers, having aspect ratio over four, can be schematically represented in the form of thin-walled beams of variable section with mass and rigidity arbitrarily distributed along the length. Simplified equations of elastic oscillations of such constructions are simple to obtain by using expressions of forces through deformations (3.1) shown in § 3.1. For this it is sufficient by means of differentiation of lateral force, longitudinal force and torsional moment with respect to x_1 to find expressions connecting the components of deformation of beam with external distributed loads, and then to these loads to introduce the corresponding distributed forces of inertia. For instance, by differentiating the expression for $Q(x_1, t)$ with respect to x_1, we will have

$$q(x_1, t) = \frac{\partial^2}{\partial x_1^2}\left[B(x_1)\frac{\partial^2 y_1(x_1, t)}{\partial x_1^2}\right]. \tag{4.22}$$

Having representing the external load in the form

$$q(x_1, t) = q_a(x_1, t) - m(x_1)\frac{\partial^2 y_1(x_1, t)}{\partial t^2}, \tag{4.22'}$$

where $q_a(x_1, t)$ — external linear surface load, we obtain the sought equation of bending oscillations of the body of a ballistic type flight vehicle in plane x_1y_1

$$\frac{\partial^2}{\partial x_1^2}\left[B(x_1)\frac{\partial^2 y_1(x_1, t)}{\partial x_1^2}\right] + m(x_1)\frac{\partial^2 y_1(x_1, t)}{\partial t^2} = q_a(x_1, t). \tag{4.23}$$

The values of function $y_1(x_1, t)$ will depend on the initial values of coordinates and velocity of points of longitudinal axis of the beam $y_1(x_1, t_0)$ and $\dot{y}_1(x_1, t_0)$ and on boundary conditions on the ends of the beam.

For the body of a flight vehicle, both ends of which are free, these boundary conditions will consist of equality of lateral forces and bending moments to zero

$$\left.\begin{aligned} &\frac{\partial}{\partial x_1}\left[B(x_1)\frac{\partial^2 y_1(x_1, t)}{\partial x_1^2}\right] = 0, \\ &\left[B(x_1)\frac{\partial^2 y_1(x_1, t)}{\partial x_1^2}\right] = 0 \end{aligned}\right\} \text{ when } x_1 = 0, \quad x_1 = l. \tag{4.24}$$

Function $y_1(x_1, t)$ can be represented in the form of the sum of

$$y_1(x_1, t) = y_{11}(x_1, t) + y_{12}(x_1, t), \qquad (4.25)$$

the first term of which describes free elastic transverse oscillations of the beam, and the second-forced oscillations. Function $y_{11}(x_1, t)$ should satisfy homogeneous differential equation

$$\frac{\partial^2}{\partial x_1^2}\left[B(x_1)\frac{\partial^2 y_{11}(x_1, t)}{\partial x_1^2}\right] + m(x_1)\frac{\partial^2 y_{11}(x_1, t)}{\partial t^2} = 0 \qquad (4.26)$$

and the above-mentioned boundary and initial conditions, function $y_{12}(x_1, t)$ – nonhomogeneous equation (23), boundary conditions (24) and homogeneous initial conditions

$$y_{12}(x_1, t) = 0, \quad \dot{y}_{12}(x_1, t) = 0 \text{ when } t = t_0.$$

The solution of linear partial differential equations of such type can be obtained by the method of separation of variables. According to this method the value of $y_1(x_1, t)$ is represented in the form of an infinite sum of products of functions only of coordinate x_1 and only of time t, i.e.,

$$y_{11}(x_1, t) = \sum_{n=1}^{\infty} S_{n0}(t) f_n(x_1), \qquad (4.27)$$

$$y_{12}(x_1, t) = \sum_{n=1}^{\infty} S_n(t) f_n(x_1), \qquad (4.28)$$

where S_{n0} and S_n – functions of only time, and f_n – function of only abscissa x_1. By putting expression (27) in equation (26) and dividing the variables, we obtain the following ordinary linear differential equation for determination of function $f_n(x_1)$

$$\frac{d^2}{dx_1^2}\left[B(x_1)\frac{d^2 f_n(x_1)}{dx_1^2}\right] - m(x_1)\omega_n^2 f_n(x_1) = 0 \quad (n = 1, 2, \ldots), \qquad (4.29)$$

where by ω_n there is designated the frequency of natural oscillations of n-th tone.

On the basis of formulas (25), (27) and (28) the additional vibration component of transverse overload will be determined by formula

$$n_{y_1 \text{в}}(x_1, t) = -\frac{1}{g_0} \sum_{n=1}^{\infty} f_n(x_1) [\ddot{S}_{n0}(t) + \ddot{S}_n(t)], \qquad (4.30)$$

and the expressions for dynamic components of lateral forces $Q_{\text{д}}(x_1, t)$ and bending moments $M_{\text{д}}(x_1, t)$ (3.3) will have the form

$$Q_{\text{д}}(x_1, t) = -\sum_{n=1}^{\infty} [\ddot{S}_{n0}(t) + \ddot{S}_n(t)] \, Q_{nx}(x_1), \qquad (4.31)$$

$$M_{\text{д}}(x_1, t) = -\sum_{n=1}^{\infty} [\ddot{S}_{n0}(t) + \ddot{S}_n(t)] \, M_{nx}(x_1), \qquad (4.32)$$

where

$$Q_{nx}(x_1) = \int_0^{x_1} m(x_1) f_n(x_1) \, dx_1, \qquad (4.33)$$

$$M_{nx}(x_1) = \int_0^{x_1} \int_0^{x_1} m(x_1) f_n(x_1) \, dx_1 \, dx_1 \qquad (4.34)$$

are corresponding unit values of the dynamic component of lateral force and bending moment.

Usually during fulfillment of practical calculations we are limited by a finite number of terms of these series. In this case the quantity of retained terms is determined basically by the required accuracy of calculations. In certain cases it depends on the method of calculation of dynamic forces. The method of deformations, as a rule, requires the calculation of a greater quantity of types of natural oscillations than the method of overloads. In other words, the method overloads, giving the best convergence, is more precise with the same quantity of terms of the series (28). This is explained by the fact that the dynamic component of bending moment is described rather well by the lowest tones of oscillations. For description of the static component of bending moment in many cases it is required to consider the highest forms of oscillations.

All the above-stated, concerning methods of formulation and solution of equations of elastic bending oscillations, pertains also to longitudinal and torsional oscillations of the construction. In accordance with formulas (3.1), (3.4) and (22') the differential equations describing these oscillations will have the form

$$\frac{\partial}{\partial x_1}\left[E(x_1)F_c(x_1)\frac{\partial u(x_1, t)}{\partial x_1}\right] - m(x_1)\frac{\partial^2 u(x_1, t)}{\partial t^2} = q_x(x_1, t), \qquad (4.35)$$

$$\frac{\partial}{\partial x_1}\left[G(x_1)I_p(x_1)\frac{\partial \theta(x_1, t)}{\partial x_1}\right] - I_m(x_1)\frac{\partial^2 \theta(x_1, t)}{\partial t^2} = q_{\text{кр}}(x_1, t). \qquad (4.36)$$

Here $q_x(x_1, t)$ and $q_{\text{кр}}(x_1, t)$ – functions of external linear load.

According to the method of separation of variables, their solutions can be represented in the form

$$u(x_1, t) = \sum_{m=1}^{\infty} T_m(t)X_m(x_1), \qquad (4.37)$$

$$\theta(x_1, t) = \sum_{p=1}^{\infty} \mu_p(t)\varphi_p(x_1), \qquad (4.38)$$

where $X_m(x_1)$ and $\varphi_p(x_1)$ – functions of the form of oscillations, and T_m and μ_p – functions of only time, satisfying the required boundary and initial conditions respectively.

If the beam is not axially symmetric and its axis of rigidity does not coincide with the axis of centers of gravity of cross sections (for instance, the wing of an aircraft), then bending oscillations of such a beam will be accompanied by torsional oscillations relative to the axis of rigidity and conversely. The compatibility of shown oscillations is explained by the presence of an inertial connection between them, proportional to the distance between axes of rigidity and centers of gravity (in each cross section). Bending-torsional oscillations of a similar construction will be described by system of equations

$$\frac{\partial^2}{\partial x_1^2}\left[B(x_1)\frac{\partial^2 y_1(x_1, t)}{\partial x_1^2}\right] + m(x_1)\frac{\partial^2 y_1(x_1, t)}{\partial t^2} -$$

$$- m(x_1)\sigma(x_1)\frac{\partial^2 \theta(x_1, t)}{\partial t^2} = q_s(x_1, t), \qquad (4.39)$$

$$\frac{\partial}{\partial x_1}\left[GI_p(x_1)\frac{\partial \theta(x_1, t)}{\partial x_1}\right] - I_m(x_1)\frac{\partial^2 \theta(x_1, t)}{\partial t^2} +$$

$$+ m(x_1)\sigma(x_1)\frac{\partial^2 y_1(x_1, t)}{\partial t^2} = q_{\kappa p}(x_1, t), \qquad (4.40)$$

the solution of which can also be represented in the form of series (28) and (38). Dynamic components of lateral force, bending moment and torsional moment in this case will be expressed by formulas

$$Q_s(x_1, t) = -\int_0^{x_1} m(x_1)\left[\frac{\partial^2 y_1(x_1, t)}{\partial t^2} - \sigma(x_1)\frac{\partial^2 \theta(x_1, t)}{\partial t^2}\right]dx_1,$$

$$M_s(x_1, t) = \int_0^{x_1} Q_s(x_1, t)\,dx_1, \qquad (4.41)$$

$$M_{\kappa p}(x_1, t) = -\int_0^{x_1}\left[I_m(x_1)\frac{\partial^2 \theta(x_1, t)}{\partial t^2} - m(x_1)\sigma(x_1)\frac{\partial^2 y_1(x_1, t)}{\partial t^2}\right]dx_1. \qquad (4.42)$$

Equations of bending (23), longitudinal (35), torsional (36) and bending-torsional (39) and (40) oscillations of beams of arbitrary cross section can be directly used for description of low-frequency elastic oscillations of constructions of flight vehicles, the dynamic scheme of which can be approximated by one beam or a system of beams. In the latter case oscillations of the vehicle will be described by the system of shown equations, connected together by corresponding geometric and power conditions of coupling.

These coupling conditions are established in each concrete case depending upon peculiarities of the structure diagram of flight vehicle and the character of its loading. For an example let us consider the transverse elastic oscillations of the construction of an aircraft with sweptback wing (Fig. 4.1). Let us place, as usual, the origin of connected system of coordinates at the apex of the body of the fuselage, let us direct axis x_1 along its axis of rigidity (toward the tail), and axis y_1 upwards. Let us still introduce auxiliary connected systems of coordinates $O_\kappa\xi_\kappa\eta_\kappa$ and $O_0\xi_0\eta_0$. Let us direct axes ξ_κ and ξ_0 of these systems along the axes of rigidity of wing and horizontal stabilizer brackets, and arrange the origin of coordinates at places of intersection of the shown axes with axis x_1 (at points x_κ and x_0). Lateral axes η_κ and η_0 will be directed parallel to axis y_1. Let us assume that cantilevers of the wing and stabilizer are rigidly fixed in the fuselage body (at places of intersection of axes of rigidity). Let us assume further that these cantilevers can accomplish only bending-torsional oscillations, and fuselage – only bending oscillations in plane x_1y_1. (This problem was solved for the first time in a similar formulation by N. N. Korchemkin.) Transverse elastic oscillations of the construction of an aircraft on the whole in this case will be described by a system of differential equations of the form (33), (39) and (40):

$$\frac{\partial^2}{\partial x_1^2}\left[B(x_1)\frac{\partial^2 y_1(x_1, t)}{\partial x_1^2}\right]+m(x_1)\frac{\partial^2 y_1(x_1, t)}{\partial t^2}=q_\text{в}(x_1, t), \quad (4.43)$$

$$\frac{\partial^2}{\partial \xi_\kappa^2}\left[B_\kappa(\xi_\kappa)\frac{\partial^2 \eta_\kappa(\xi_\kappa, t)}{\partial \xi_\kappa^2}\right]+m_\kappa(\xi_\kappa)\frac{\partial^2 \eta_\kappa(\xi_\kappa, t)}{\partial t^2}-$$
$$-m_\kappa(\xi_\kappa)\sigma_\kappa(\xi_\kappa)\frac{\partial^2 \theta_\kappa(\xi_\kappa, t)}{\partial t^2}=q_{\text{в}\kappa}(\xi_\kappa, t), \quad (4.44)$$

$$\frac{\partial}{\partial \xi_\kappa}\left[GI_{p\kappa}(\xi_\kappa)\frac{\partial \theta_\kappa(\xi_\kappa, t)}{\partial \xi_\kappa}\right]-I_m(\xi_\kappa)\frac{\partial^2 \theta_\kappa(\xi_\kappa, t)}{\partial t^2}+$$
$$+m_\kappa(\xi_\kappa)\sigma_\kappa(\xi_\kappa)\frac{\partial^2 \eta_\kappa(\xi_\kappa, t)}{\partial t^2}=q_{\text{кр.}\kappa}(\xi_\kappa, t), \quad (4.45)$$

$$\frac{\partial}{\partial \xi_0}\left[GI_{p0}(\xi_0)\frac{\partial \theta_0(\xi_0, t)}{\partial \xi_0}\right]-I_m(\xi_0)\frac{\partial^2 \theta_0(\xi_0, t)}{\partial t^2}+$$
$$+m_0(\xi_0)\sigma_0(\xi_0)\frac{\partial^2 \eta_0(\xi_0, t)}{\partial t^2}=q_{\text{кр}0}(\xi_0, t), \quad (4.46)$$

$$\frac{\partial^2}{\partial \xi_0^2}\left[B_0(\xi_0)\frac{\partial^2 \eta_0(\xi_0, t)}{\partial \xi_0^2}\right]+m_0(\xi_0)\frac{\partial^2 \eta_0(\xi_0, t)}{\partial \xi_0^2}-$$
$$-m_0(\xi_0)\sigma_0(\xi_0)\frac{\partial^2 \theta_0(\xi_0, t)}{\partial t^2}=q_{\text{в}0}(\xi_0, t). \quad (4.47)$$

The solution of this system of equations can be sought in the form of series similar to (28) and (38).

$$
\left.\begin{aligned}
y_1(x_1, t) &= \sum_{n=1}^{\infty} q_n(t) f_n(x_1),\\
\eta_\kappa(\xi_\kappa, t) &= \sum_{n=1}^{\infty} q_n(t) \Phi_n(\xi_\kappa),\\
\eta_0(\xi_0, t) &= \sum_{n=1}^{\infty} q_n(t) \Psi_n(\xi_0),\\
\theta_\kappa(\xi_\kappa, t) &= \sum_{n=1}^{\infty} q_n(t) \varphi_n(\xi_\kappa),\\
\theta_0(\xi_0, t) &= \sum_{n=1}^{\infty} q_n(t) \varkappa_n(\xi_0),
\end{aligned}\right\} \qquad (4.48)
$$

where q_n – function of only time, f_n – function of the form of bending oscillations of the fuselage, Φ_n and Ψ_n – functions of the form of bending oscillations of the cantilever of wing and cantilever of stabilizer, ϕ_n and κ_n – functions of the form of torsional oscillations of wing and stabilizer cantilever respectively.

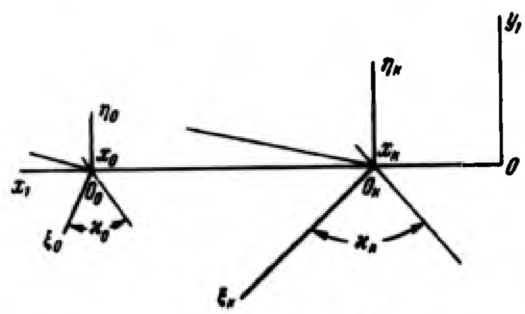

Fig. 4.1. Systems of coordinates for an aircraft with sweptback wings.

These solutions should satisfy boundary conditions on free ends of:

a) body of fuselage

$$\frac{\partial}{\partial x_1}\left[B(x_1)\frac{\partial^2 y_1(x_1, t)}{\partial x_1^2}\right] = 0, \quad \frac{\partial^2 y_1(x_1, t)}{\partial x_1^2} = 0 \quad \text{when} \quad x_1 = 0, \; x_1 = l, \qquad (4.49)$$

b) cantilever of wing

$$\frac{\partial}{\partial\xi_{\text{к}}}\left[B_{\text{к}}(\xi_{\text{к}})\frac{\partial^2\eta_{\text{к}}(\xi_{\text{к}},t)}{\partial\xi_{\text{к}}^2}\right]-\frac{\partial^2\eta_{\text{к}}(\xi_{\text{к}},t)}{\partial\xi_{\text{к}}^2}-\frac{\partial\theta_{\text{к}}(\xi_{\text{к}},t)}{\partial\xi_{\text{к}}}=0 \quad \text{when} \quad \xi_{\text{к}}=l_{\text{к}}, \tag{4.50}$$

c) cantilever of stabilizer

$$\frac{\partial}{\partial\xi_0}\left[B_0(\xi_0)\frac{\partial^2\eta_0(\xi_0,t)}{\partial\xi_0^2}\right]-\frac{\partial^2\eta_0(\xi_0,t)}{\partial\xi_0^2}-\frac{\partial\theta_0(\xi_0)}{\partial\xi_0}=0 \quad \text{when} \quad \xi_0=l_0 \tag{4.51}$$

and coupling conditions. The latter in this case will consist of:

a) equality of lateral shifts and angles of rotation of cross sections at places of connection of fuselage and cantilever of wing, fuselage and cantilever of stabilizer:

$$\left.\begin{aligned}&y_1(x_1,t)=\eta_{\text{к}}(\xi_{\text{к}},t),\\&-\frac{\partial y_1(x_1,t)}{\partial x_1}=\theta_{\text{к}}(\xi_{\text{к}},t)\cos\varkappa_{\text{к}}-\frac{\partial\eta_{\text{к}}(\xi_{\text{к}},t)}{\partial\xi_{\text{к}}}\sin\varkappa_{\text{к}}\end{aligned}\right\} \quad \text{when} \quad x_1=x_{\text{к}},\ \xi_{\text{к}}=0, \tag{4.52}$$

$$\left.\begin{aligned}&y_1(x_1,t)=\eta_0(\xi_0,t),\\&-\frac{\partial y_1(x_1,t)}{\partial x_1}=\theta_0(\xi_0,t)\cos\varkappa_0-\frac{\partial\eta_0(\xi_0,t)}{\partial\xi_0}\sin\varkappa_0\end{aligned}\right\} \quad \text{when} \quad x_1=x_0,\ \xi_0=0, \tag{4.53}$$

where $\varkappa_{\text{к}}$ and $\varkappa_0$ – angles of sweep of wing and stabilizer respectively;

b) equality of the sum of lateral forces and the sum of bending moments, taken on the left (sign "л") and on the right (sign "п") from sections $x_{\text{к}}$ and x_0 to lateral forces and corresponding components of bending moments, arriving from both cantilevers of the wing and stabilizer:

$$\begin{aligned}&\frac{\partial}{\partial x_1}\left[B(x_1)\frac{\partial^2 y_1(x_1,t)}{\partial x_1^2}\right]_{\text{л}}-\frac{\partial}{\partial x_1}\left[B(x_1)\frac{\partial^2 y_1(x_1,t)}{\partial x_1^2}\right]_{\text{п}}=\\&\qquad=2\cdot\frac{\partial}{\partial\xi_{\text{к}}}\left[B_{\text{к}}(\xi_{\text{к}})\frac{\partial^2\eta_{\text{к}}(\xi_{\text{к}},t)}{\partial\xi_{\text{к}}^2}\right] \quad \text{when} \quad x_1=x_{\text{к}},\ \xi_{\text{к}}=0,\end{aligned} \tag{4.54}$$

$$\begin{aligned}&\left[B(x_1)\frac{\partial^2 y_1(x_1,t)}{\partial x_1^2}\right]_{\text{л}}-\left[B(x_1)\frac{\partial^2 y_1(x_1,t)}{\partial x_1^2}\right]_{\text{п}}=2B_{\text{к}}(\xi_{\text{к}})\frac{\partial^2\eta_{\text{к}}(\xi_{\text{к}},t)}{\partial\xi_{\text{к}}^2}\sin\varkappa_{\text{к}}-\\&\qquad-2GI_{p\text{к}}(\xi_{\text{к}})\frac{\partial\theta_{\text{к}}(\xi_{\text{к}},t)}{\partial\xi_{\text{к}}}\cos\varkappa_{\text{к}} \quad \text{when} \quad x_1=x_{\text{к}},\ \xi_{\text{к}}=0,\end{aligned} \tag{4.55}$$

$$\begin{aligned}&\frac{\partial}{\partial x_1}\left[B(x_1)\frac{\partial^2 y_1(x_1,t)}{\partial x_1^2}\right]_{\text{л}}-\frac{\partial}{\partial x_1}\left[B(x_1)\frac{\partial^2 y_1(x_1,t)}{\partial x_1^2}\right]_{\text{п}}=\\&\qquad=2\cdot\frac{\partial}{\partial\xi_0}\left[B_0(\xi_0)\frac{\partial^2\eta_0(\xi_0,t)}{\partial\xi_0^2}\right] \quad \text{when} \quad x_1=x_0,\ \xi_0=0,\end{aligned} \tag{4.56}$$

$$\left[B(x_1)\frac{\partial^2 y_1(x_1, t)}{\partial x_1^2}\right]_{\text{л}} - \left[B(x_1)\frac{\partial^2 y_1(x_1, t)}{\partial x_1^2}\right]_{\text{п}} = 2B_0(\xi_0)\frac{\partial^2 \eta_0(\xi_0, t)}{\partial \xi_0^2}\sin\varkappa_0 - -2GI_{p0}(\xi_0)\frac{\partial \theta_0(\xi_0, t)}{\partial \xi_0}\cos\varkappa_0 \quad \text{when} \quad x_1 = x_0, \quad \xi_0 = 0. \qquad (4.57)$$

In conclusion let us note that the dynamic scheme of constructions of those flight vehicles, aspect ratio of which is less than four, can be represented in the form of a system of rigid bodies elastically connected together. The motion of each link of such a system (Fig. 4.2) will be described by an equation of the form

$$m_i\ddot{x}_i - c_{i+1}(x_{i+1} - x_i) + c_i(x_i - x_{i-1}) = 0, \qquad (4.58)$$

where m_i – mass of link, c_i – generalized rigidity of elastic connection, x_i – absolute shift of mass of i-th link. We will elaborate in more detail on the question of the formulation of equations of oscillations of such systems in Chapter XI. Let us note only that equations of elastic oscillations in a similar form can be obtained (in particularly, for separate nodes of the construction, connected to elastic elements of the body of the flight vehicle), proceeding from expressions for shifts in canonical form

$$x_i = \sum_{j=1}^{k}\delta_{ij}X_j, \qquad i = 1, 2, \ldots, k,$$

where k – the number of concentrated masses in the considered system, x_i – sought shift of i-th mass in the assigned direction, δ_{ij} – shift of this mass in the mentioned direction from the unit value of power factor, applied to j-th mass, and X_j – total value of the shown generalized power factor. For the case of forward motion of j-th mass this power factor is described by expression

$$X_j = P_j - m_j\ddot{x}_j - b_j\dot{x}_j.$$

Here P_j – external force applied to the given mass, and b_j – corresponding coefficient of resisting force. Methods of determination of the values of δ_{ij} for different typical constructions are usually discussed in courses on structural mechanics and the strength of materials.

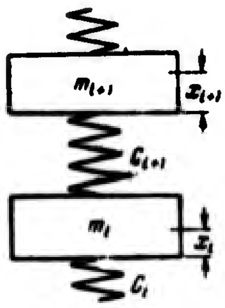

Fig. 4.2. Model of elastically suspended mass.

The equations of elastic oscillations of constructions of flight vehicles given in this paragraph give the possibility of determining the dynamic components of internal forces, caused by the effect of external disturbing forces and moments. For this it is only necessary to know the dynamic characteristics of the construction of the flight vehicle itself and laws of change of external influences in time.

§ 4.4. Dynamic Characteristics of the Construction

Many approximate and exact methods of calculation of the forms and frequencies of natural elastic oscillations of different constructions are known. Each of them has its advantages and disadvantages, i.e., its own field of application. Since in most cases the dynamic calculation of any construction is reduced to removal of the various types of resonance phenomena, then the methods of approximation, permitting estimation of the value of only the lowest frequencies of natural oscillations with some accuracy, received predominant development. During dynamic calculation of the construction of a flight vehicle there are also considered and resonance conditions of oscillations (with frequencies very close to the natural frequencies of elastic oscillations). Therefore, it is often necessary to pay considerable attention to the accuracy of calculation of frequencies and especially the forms of natural oscillations of these constructions. From this point of view the best method of calculation of dynamic characteristics is undoubtedly the method of <u>successive approximations</u> (method of <u>iterations</u>), which permits obtaining the forms and frequencies of oscillations with any accuracy. The basic disadvantage of the method of iterations – time consumption. At present with the presence of electronic digital computers this disadvantage is easily eliminated. Therefore, in this work we will not pause on analysis of the existing methods of calculation of dynamic characteristics of elastic systems, but will give only those methods which received the widest application during investigation of questions of the dynamics of construction of flight vehicles. For other methods of interest it is possible to refer to corresponding literature [3, 31, 82 and others].

By form of natural oscillations we usually mean the function of only coordinate x_1, which represents the form of deformation of the body from forces of inertia, appearing with these oscillations. Since this deformation is calculated with accuracy to arbitrary constant factor ($\ddot{S}_n$ or $\ddot{q}_m$), usually it is normalized in some way, for instance, we divide the shift of all points by maximum shift or by the value of this function at a fixed point (for instance, $x_1=0$ or $x_1=l$), which we call the <u>point of reduction</u>. Therefore, in an expression determining the decomposition of any shift with respect to forms of natural oscillations, the function of only time will represent corresponding shifts of the points of reduction and have the dimension of length.

Basic difference of the forms of natural oscillations from each other consists of the quantity of nodes, i.e., points, at which these functions are equal to zero. Thus, for instance, the form of natural bending oscillations of a beam with free ends of the first tone ($n=1$) will have two nodes (Fig. 4.3), the second tone ($n = 2$) – three nodes, etc. Since the function of form represents an elastic line of the deformed construction at oscillations with natural frequency (principal oscillations), then it obviously should satisfy the corresponding boundary conditions (kinematic and dynamic). These boundary conditions constitute limitations placed on shifts of cross sections of the body ($x_1=0,\ x_1=l$) and on the values of power factors (longitudinal and lateral forces, bending and twisting moments) under these sections. For the free end (shift of which can be any) the boundary conditions will consist of the absence of any forces. On the basis of formulas (3.1), (28), (37) and (38) this is possible only when

$$\frac{dX_m(x_1)}{dx_1}=0,\quad \frac{d^2f_n(x_1)}{dx_1^2}=0,\quad \frac{d\varphi_p(x_1)}{dx_1}=0,$$
$$\frac{d}{dx_1}\left[B(x_1)\frac{d^2f_n(x_1)}{dx_1^2}\right]=0 \qquad (4.59)$$
$$(n=1,\ 2,\ \ldots;\ p=1,\ 2,\ \ldots;\ m=1,\ 2,\ \ldots),$$

where $X_m(x_1)$ – function of the form of longitudinal natural oscillations, $\varphi_p(x_1)$ – function of the form of torsional oscillations, and $f_n(x_1)$ – function of the form of bending transverse natural oscillations.

Fig. 4.3. Forms of the first three tones of natural transverse oscillations of a construction with free ends.

For an end section hinged on a rigid support the boundary conditions are mixed and reflect the absence of the possibility of lateral shift of this section and equality of bending moment and longitudinal force to zero:

$$f_n(x_1)=0,\quad \frac{d^2f_n(x_1)}{dx_1^2}=0,\quad \frac{dX_m(x_1)}{dx_1}=0$$
$$(n=1,\ 2,\ \ldots;\ m=1,\ 2,\ \ldots). \qquad (4.60)$$

s)

ns,

In this case the angle of rotation of the section and the magnitude of lateral force can be any. For a rigidly attached end cross section the boundary conditions will consist of the equality of lateral and longitudinal shifts to zero and the angles of rotation of this section

$$f_n(x_1)=0, \quad X_m(x_1)=0, \quad \varphi_p(x_l)=0, \quad \frac{df_n(x_1)}{dx_1}=0. \tag{4.61}$$

Due to the mutual independence of the forms of deformation of the body with oscillations with different natural frequencies the work of forces of inertia appearing at one of the natural oscillations on shifts, corresponding to another natural oscillation, is always equal to zero. For transverse oscillations this condition is written in the form

$$\int_0^l m(x_1) f_n(x_1)\ddot{q}_n f_m(x_1) q_m\, dx_1 = 0 \quad \text{when} \quad n \neq m.$$

From this expression (considering that $\ddot{q}_n$ and q_m are not equal to zero) ensues the following very important condition (condition of orthogonality), which all functions of forms of natural bending oscillations must satisfy in this case:

$$\int_0^l m(x_1) f_n(x_1) f_m(x_1)\, dx_1 = 0 \quad \text{when} \quad n \neq m \tag{4.62}$$

$$(n=1,\ 2,\ \ldots;\quad m=1,\ 2\ \ldots).$$

Taking into account rotary motions of masses this condition takes the form

$$\int_0^l m(x_1) f_n(x_1) f_m(x_1)\, dx_1 + \int_0^l i(x_1) \frac{df_n(x_1)}{dx_1}\frac{df_m(x_1)}{dx_1}\, dx_1 = 0 \quad \text{when} \quad n \neq m$$

$$(n=1,\ 2\ \ldots;\quad m=1,\ 2,\ \ldots),$$

where $i(x_1)$ — moment of inertia of the mass relative to lateral axis of the section.

If natural oscillations of the construction in longitudinal direction, in lateral plane, and also relative to the longitudinal axis are mutually independent, analogous conditions of orthogonality are obtained for functions $X_m(x_1)$ and $\varphi_p(x_1)$, in particular,

$$\int_0^l m(x_1)X_n(x_1)X_m(x_1)dx_1 = 0 \quad \text{when} \quad n \neq m \qquad (4.63)$$

$$(n = 1, 2, \ldots; \quad m = 1, 2, \ldots).$$

Such are the basic requirements, which functions of the forms of natural oscillations should satisfy. During principal oscillation with frequency ω_m, form $f_m(x_1)$ and amplitude of the point of reduction q_{m0}, the kinetic and potential energy of the system will be equal to

$$T_m = \frac{1}{2} M_m \dot{q}_m^2, \quad U_m = \frac{1}{2} E_m q_m^2,$$

where

$$\left.\begin{aligned} q_m &= q_{m0} \sin \omega_m t, \\ \dot{q}_m &= q_{m0}\omega_m \cos \omega_m t, \\ M_m &= \int_0^l m(x_1) f_m^2(x_1)dx_1, \\ E_m &= \int_0^l B(x_1)\left[\frac{d^2 f_m(x_1)}{dx_1^2}\right]^2 dx_1. \end{aligned}\right| \qquad (4.64)$$

Having equated the maximums of these energies to each other, we obtain the expression for the square of angular frequency of natural oscillations in the form of ratio of the reduced rigidity of construction E_m to reduced mass M_m:

$$\omega_m^2 = \frac{E_m}{M_m}. \qquad (4.65)$$

This formula is frequently used both for more precise definition of the magnitude of frequency ω_m, and for obtaining its approximate value. Actually, if in (65) instead of function $f_m(x_1)$, determining the real form of natural oscillations of m-th tone, we place some other function, only approximately describing the form of oscillations, but satisfying the boundary conditions and having the same quantity of nodes, then we will obtain the approximate value of frequency ω_m. The closer this function is to the true form of oscillations, the more exact the value of ω_m will be.

§ 4.5. Determination of Forms and Frequencies by the Method of Iterations

It is impossible to exactly solve the equation determining the form of natural elastic oscillations of the construction when the rigidity and mass are arbitrary functions of coordinate x_1. Of the existing methods of approximation of the solution of this equation the most suitable is the method of successive approximations by forms of oscillations (method of iterations). It permits determining both the values of function $f_n(x_1)$ and its derivatives, and the value of frequency ω_n with the accuracy necessary for engineering calculations.

The essence of this method consists of the following [82]. Let us present an equation, for instance, of bending oscillations of a rod in a void (29) with boundary conditions

$$\frac{d}{dx_1}\left[B(x_1)\frac{d^2 f_n(x_1)}{dx_1^2}\right]=0, \quad \frac{d^2 f_n(x_1)}{dx_1^2}=0 \text{ when } x_1=0, \quad x_1=l \tag{4.66}$$

in the form

$$\frac{d^2}{dx_1^2}\left[B(x_1)\frac{d^2 f_n(x_1)}{dx_1^2}\right]=m(x_1)\omega_n^2 f_n(x_1) \quad (n=1,\ 2,\ \ldots). \tag{4.67}$$

Having twice integrated it with respect to x_1, we obtain the expression for dynamic bending moment from inertial load, equal to $\omega_n^2 m(x_1) f_n(x_1)$.

$$M_{\text{и}n}(x_1)=\omega_n^2 M_{nx}(x_1). \tag{4.67'}$$

where by M_{nx} there is designated unit dynamic bending moment (34):

$$M_{nx}(x_1)=\int_0^{x_1}\int_0^{x_1} m(x_1)f_n(x_1)\,dx_1\,dx_1+A_nx_1+B_n. \qquad (4.68)$$

Since $B(x_1)\neq 0$, then, by integrating expression (67) further with respect to x_1, we find the sought value of $f_n(x_1)$ in the form

$$f_n(x_1)=\omega_n^2 J_n(x_1), \qquad (4.69)$$

where

$$J_n(x_1)=\int_0^{x_1}\int_0^{x_1}\frac{M_{nx}(x_1)}{B(x_1)}\,dx_1\,dx_1+D_n+C_nx_1+B_n\frac{x_1^2}{2}+A_n\frac{x_1^3}{6} \qquad (4.69')$$

is the elastic bending curve of the body from load $m(x_1)f_n(x_1)$.

Arbitrary constants, entering (68) and (69'), are determined by boundary conditions (66). From these conditions it follows that $A_n=B_n=0$,

$$\left.\begin{aligned}
D_n&=\frac{1}{J_z-mx_{1\text{т}}^2}\Bigg[x_{1\text{т}}\int_0^l m(x_1)\int_0^{x_1}\int_0^{x_1}\int_0^{x_1}\frac{M_{nx}(x_1)}{B(x_1)}\,dx_1\,dx_1\,dx_1\,dx_1-\\
&\qquad-\frac{J_z}{m}\int_0^l m(x_1)\int_0^{x_1}\int_0^{x_1}\frac{M_{nx}(x_1)}{B(x_1)}\,dx_1\,dx_1\,dx_1\Bigg],\\
C_n&=\frac{1}{J_z-mx_{1\text{т}}^2}\Bigg[x_{1\text{т}}\int_0^l m(x_1)\int_0^{x_1}\int_0^{x_1}\frac{M_{nx}(x_1)}{B(x_1)}\,dx_1\,dx_1\,dx_1-\\
&\qquad-\int_0^l m(x_1)\int_0^{x_1}\int_0^{x_1}\int_0^{x_1}\frac{M_{nx}(x_1)}{B(x_1)}\,dx_1\,dx_1\,dx_1\,dx_1\Bigg],
\end{aligned}\right\} \qquad (4.70)$$

where m and J_z – mass and moment of inertia of the flight vehicle (1.6").

Function $f_n(x_1)$ is normalized in such a manner that at the point of reduction ($x_1=0$ or $x_1=l$) $f_n(x_1)=1$.

Thus, having taken arbitrary (normalized) function $f_n^{(0)}(x_1)$, as initial, by using formulas (69), (69'), (70), and (68), let us calculate the values of function $f_n^{(0)}(x_1)$. Then, having taken the obtained value of $f_n^{(0)}(x_1)$ for the new original function $f_n^{(1)}(x_1)$, let us repeat the calculation until relationship $\frac{f_n^{(k+1)}(x_1)}{f_n^{(k)}(x_1)}$ is identical for all x_1 or the relative error $\frac{f_n^{(k)}(x_1)-f_n^{(k+1)}(x_1)}{f_n^{(k+1)}(x_1)}$ becomes less than specified. The shown process will converge faster, than closer the original function is to the sought.

By the described method we find the form and frequency of principal oscillation of only the first tone. For determination of the form and frequency of principal oscillation of the second tone it is necessary to additionally satisfy the condition of orthogonality (62) of normal functions $f_1(x_1)$ and $f_2(x_1)$:

$$\int_0^l m(x_1)f_1(x_1)f_2(x_1)dx_1=0. \tag{4.71}$$

For this both into initial $f_2^{(0)}(x_1)$, and into all subsequent functions $f_2^{(k)}(x_1)$ one should introduce correction

$$f_2^{(k)}(x_1)=f_{21}^{(k)}(x_1)+\Delta_{2k}f_1(x_1), \tag{4.72}$$

where $f_{21}^{(k)}(x_1)$ — orthogonalized value of function $f_2(x_1)$, Δ_{2k} — correction factor, determined from condition (71):

$$\Delta_{2k}=-\frac{\int_0^l m(x_1)f_1(x_1)f_{21}^{(k)}(x_1)dx_1}{\int_0^l m(x_1)f_1^2(x_1)dx_1}, \tag{4.72'}$$

and k — number of approximation.

Analogously we find the forms and frequencies of natural oscillations of the third, fourth and higher tones. It is necessary only to strictly see that the sought function $f_n(x_1)$ be orthogonal with all functions of the preceding tones with weight $m(x_1)$. It is necessary to note that function $f(x_1)=ax_1+b$ also satisfies equation (67) and boundary conditions (66). It determines the form of oscillations of the construction as a solid body, i.e., the form of principal oscillations of zero tone with frequency $\omega = 0$.

Calculation of functions $f_n(x_1)$ by the above-mentioned method is the best of all by the tabular method, by selecting a comparatively small step of integration, approximately $h=\frac{l}{50}$. In this case, to avoid obtaining small differences of large numbers, it is sometimes expedient to place the origin of the system of coordinates in the middle of the body. The form of these functions (for $n = 1$, $n = 2$, $n = 3$) is shown on Fig. 4.3, and the values of unit dynamic bending moments $\bar{M}_{nx}(\bar{x}_1)$ – on Fig. 4.4.

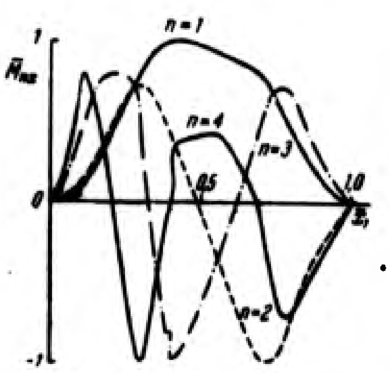

Fig. 4.4. Diagrams of unit dynamic bending moments for a beam with free ends.

In the process of finding the forms of principal oscillations by the method of iterations simultaneously by formula (69) we determine the frequencies ω_n of natural oscillations corresponding to them. However, sometimes the calculation of these frequencies is more convenient by the power method, based on equality of maximum values of kinetic and potential energies for each principal oscillation (65). Taking into account (67), we obtain

$$\omega_n^2 = \frac{\int\limits_0^l m(x_1) f_n^2(x_1)\,dx_1}{\int\limits_0^l \frac{M_{nx}^2(x_1)}{B(x_1)}\,dx_1}. \tag{4.73}$$

Taking into account the effect of lateral forces $Q_{nx}(x_1)$,[2] it will have the form

$$\omega_n^2 = \frac{\int_0^l m(x_1) f_n^2(x_1)\,dx_1}{\int_0^l \frac{M_{nx}^2(x_1)}{B(x_1)}\,dx_1 + \int_0^l \frac{Q_{nx}^2(x_1)}{G(x_1)F_c(x_1)}\,dx_1}, \tag{4.73'}$$

where

$$Q_{nx}(x_1) = \int_0^{x_1} m(x_1) f_n(x_1)\,dx_1 \qquad (n = 1, 2, \ldots).$$

Calculations show that an account of lateral forces leads to a decrease of the frequencies of free bending oscillations of the first tone by all of 3%, and second tone by approximately 15-20%.

It is possible to similarly evaluate the effect of longitudinal forces $N(x_1)$ on the magnitude of ω_n:

$$\omega_n^{*2} = \omega_n^2 - \frac{\int_0^l N(x_1)\left[\frac{d^2 f_n(x_1)}{dx_1^2}\right]^2 dx_1}{\int_0^l m(x_1) f_n^2(x_1)\,dx_1}. \tag{4.74}$$

As can be seen, longitudinal tensile forces $N(x_1)<0$, will increase the frequencies of natural bending oscillations, and compressive $N(x_1)>0$ – lower them. In most cases the effect of longitudinal forces on ω_n and especially on the form of transverse oscillations of the construction turns out to be small, and in the first approximation it can be disregarded. An exception is only case V, examined in Chapter VI.

Taking into account the inertia of rotation of masses of cross sections of the body $I(x_1)$ and concentrated loads equation (67) is written in the form

$$\frac{d^2}{dx_1^2}\left[B(x_1)\frac{d^2 f_n(x_1)}{dx_1^2}\right] = \omega_n^2\left\{m(x_1)f_n(x_1) + \sum_{j=1}^{k_j} m_j f_n(x_j)\delta(x_1 - x_j) - \right.$$

$$\left. - \frac{d}{dx_1}\left[i(x_1)\frac{df_n(x_1)}{dx_1}\right] - \sum_{j=1}^{k_j} i_j \frac{d^2 f_n(x_j)}{dx_1^2}\delta''(x_1 - x_j)\right\}, \qquad (4.75)$$

where by m_j is designated the mass concentrated in section x_j, by i_j – its moment of inertia relative to the lateral axis, passing through the center of gravity of cross section x_j, and by $i(x_1)$ – mass moment of inertia of the cross section relative to the same axis.

The calculation of the forms of natural bending oscillations is noticeably complicated if it is necessary to calculate the effect of lateral forces. In this case the kinetic energy of the system will be determined by formula

$$T_0(t) = \frac{1}{2}\int_0^l m(x_1)\left[\frac{\partial y(x_1, t)}{\partial t}\right]^2 dx_1 + \frac{1}{2}\int_0^l i(x_1)\left[\frac{\partial^2 y_{\text{и}}(x_1, t)}{\partial x_1 \partial t}\right]^2 dx_1, \qquad (4.76)$$

and potential energy by expression (6)

$$U_0(t) = \frac{1}{2}\int_0^l B(x_1)\left[\frac{\partial^2 y_{\text{и}}(x_1, t)}{\partial x_1^2}\right]^2 dx_1 +$$

$$+ \frac{1}{2}\int_0^l k_0 G(x_1) F_c(x_1)\left[\frac{\partial y_c(x_1, t)}{\partial x_1}\right]^2 dx_1,$$

where $y_{\text{и}}$ – deflection from bending, and y_c – deflection from shear. From these formulas it is clear that it is necessary to determine the deflections from bending and from shear separately. Furthermore, it is necessary to consider the lag on angle of shift of the angles of rotation of cross section of the body behind angles of slope of the tangent to elastic line, i.e., perpendicularity of cross sections to longitudinal axis.

Angle of rotation of the cross section in the considered case will be equal to

$$\frac{\partial y(x_1, t)}{\partial x_1} - \frac{\partial y_c(x_1, t)}{\partial x_1} = \frac{\partial y_{\text{и}}(x_1, t)}{\partial x_1},$$

and total deflection of the body in section x_1

$$y(x_1, t) = y_u(x_1, t) + y_c(x_1, t).$$

In this case the bending moment and lateral force will be determined by formulas

$$M(x_1, t) = B(x_1)\frac{\partial^2 y_u(x_1, t)}{\partial x_1^2},$$

$$Q(x_1, t) = -G(x_1)k_0F_c(x_1)\frac{\partial y_c(x_1, t)}{\partial x_1},$$

where k_0 – some correction factor for the shape of cross section.

Corresponding equations, describing free bending oscillations of the construction in a void, in this case will have the form

$$\left.\begin{aligned}\frac{\partial}{\partial x_1}\left[k_0G(x_1)F_c(x_1)\frac{\partial y_c(x_1, t)}{\partial x_1}\right] - m(x_1)\frac{\partial^2 y(x_1, t)}{\partial t^2} = 0,\\ \frac{\partial^2}{\partial x_1^2}\left[B(x_1)\frac{\partial^2 y_u(x_1, t)}{\partial x_1^2}\right] + m(x_1)\frac{\partial^2 y(x_1, t)}{\partial t^2} - \frac{\partial}{\partial x_1}\left[I(x_1)\frac{\partial^3 y_u(x_1, t)}{\partial x_1\,\partial t^2}\right] = 0.\end{aligned}\right\} \quad (4.77)$$

From them ensue the following equations for determination of the forms and frequencies of natural oscillations:

$$\left.\begin{aligned}&f_n(x_1) = f_{nu}(x_1) + f_{nc}(x_1),\\ &\frac{d^2}{dx_1^2}\left[B(x_1)\frac{d^2 f_{nu}(x_1)}{dx_1^2}\right] =\\ &\qquad = \omega_n^2 m(x_1)f_n(x_1) - \omega_n^2\frac{d}{dx_1}\left[I(x_1)\frac{df_{nu}(x_1)}{dx_1}\right],\\ &\frac{d}{dx_1}\left[G(x_1)k_0F_c(x_1)\frac{df_{nc}(x_1)}{dx_1}\right] = -\omega_n^2 m(x_1)f_n(x_1).\end{aligned}\right\} \quad (4.78)$$

During solution of the given system of equations by the method of iterations one should assign the values both of functions $f_n(x_1)$, and functions $f_{nu}(x_1)$. The shown connected system of equations is simply reduced to a system of independent equations of the form

$$\frac{d^2}{dx_1}\left[B(x_1)\frac{d^2f_{nn}(x_1)}{dx_1^2}\right]=$$

$$=\omega_n^2\left\{m(x_1)\left[f_{nn}(x_1)-\omega_n^2\int_0^{x_1}\frac{i(x_1)}{G(x_1)F_c(x_1)k_0}\cdot\frac{df_{nn}(x_1)}{dx_1}dx_1-\right.\right.$$

$$-\int_0^{x_1}\frac{\frac{d}{dx_1}\left[B(x_1)\frac{d^2f_{nn}(x_1)}{dx_1^2}\right]}{G(x_1)k_0F_c(x_1)}dx_1+$$

$$\left.\left.+E_n\int_0^{x_1}\frac{dx_1}{G(x_1)k_0F_c(x_1)}+E_{0n}\right]-\frac{d}{dx_1}\left[i(x_1)\frac{df_{nn}(x_1)}{dx_1}\right]\right\},$$

$$\frac{d}{dx_1}\left[G(x_1)k_0F_c(x_1)\frac{df_{nn}(x_1)}{dx_1}\right]=-\omega_n^2\frac{d}{dx_1}\left[i(x_1)\frac{df_{nn}(x_1)}{dx_1}\right]-$$

$$-\frac{d^2}{dx_1^2}\left[B(x_1)\frac{d^2f_{nn}(x_1)}{dx_1^2}\right],$$

where E_n and E_{0n} constants of integration.

It is recommended to calculate deflections, caused by shear, during calculation of the forms of natural bending oscillations of only the highest tones.[1] For the lowest tones ($n = 1$, and often $n = 2$) it is possible to be limited by the corresponding correction of the frequency of oscillations (73').

For determination of natural frequencies and the forms of twisting (longitudinal) oscillations by the method of iterations, i.e., for solving the equation of the form

$$-\frac{d}{dx_1}\left[G(x_1)I_p(x_1)\frac{d\varphi(x_1)}{dx_1}\right]=\omega_p^2I_m(x_1)\varphi_p(x_1)\quad(p=1,\ 2,\ \ldots),\qquad(4.79)$$

the above-indicated formulas (69), (69') should be replaced by the following:

$$\varphi_p(x_1)=\omega_p^2\bar{\varphi}_p(x_1),$$

$$\bar{\varphi}_p(x_1)=\int_0^{x_1}\frac{M_{\text{кр}\,x}(x_1)}{G(x_1)I_p(x_1)}dx_1+C_p,\qquad(4.79')$$

where

$$M_{\text{кр}\,x}(x_1)=\int_0^{x_1}I_m(x_1)\varphi_p(x_1)dx_1+A_p\quad(p=1;\ 2,\ \ldots).$$

Constants of integration A_p and C_p are found from corresponding boundary conditions, but condition of orthogonality keeps the same form:

$$\int_0^l I_m(x_1)\varphi_p(x_1)\varphi_n(x_1)\,dx_1 = 0 \quad \text{when } p \neq n$$

$$(p = 1,\ 2,\ \ldots;\quad n = 1,\ 2,\ \ldots).$$

In conclusion let us note that the widely used method of iterations, presented in this paragraph, will be directly applied to calculations of natural frequencies and forms of purely bending, twisting and longitudinal oscillations of the constructions of nonmaneuverable ballistic type flight vehicles (cantilevers of wing and stabilizer, fuselage). In this case we can consider (by means introduction of the corresponding resonance factors) the effect of different masses elastically suspended inside bodies. Thus, for instance, the initial equation for finding the forms and frequencies of longitudinal oscillations of such a system will take the form

$$\frac{d}{dx_1}\left[E(x_1)F_c(x_1)\frac{dX_n(x_1)}{dx_1}\right] + \\ + \omega_n^2\left[m(x_1)X_n(x_1) + \sum_j m_j X_n(x_j)\frac{1}{1-\left(\frac{\omega_n}{\omega_{0j}}\right)^2}\delta(x_1 - x_j)\right] = 0, \qquad (4.79'')$$

where ω_{0j} – partial frequency of natural oscillations of mass m_j, elastically suspended in section x_j.

§ 4.6. Determination of the Form of Oscillations with Respect to a Specified Frequency

Usually the frequencies and forms of natural elastic oscillations of the construction of a flight vehicle are calculated only for several flying values of weight, corresponding to cases A, B, K, L and others.

By considering the smooth change of ω_n by $\bar{t}$ for any phase of flight, it is possible to graphically determine the magnitudes of frequencies of natural elastic oscillations of a construction for any intermediate value of weight (time $\bar{t}$). It is possible to obtain the form of natural oscillations of the construction of a flight vehicle by interpolation (with acceptable accuracy) only when it is little changed by $\bar{t}$. Therefore, sometimes the necessity appears for finding the forms of natural oscillations by the known frequency of these oscillations. Most frequently such problems appear during analysis of results of flight or laboratory dynamic tests of the construction. In similar cases instead of the method of successive approximations discussed in the preceding paragraph it is expedient to use another method of approximation, making it possible to obtain the function of form of bending oscillations of the vehicle by solution of equation (67) by the method of numerical integration (Stormer method).

For understanding the idea of this method let us examine the process of calculation of the form of natural transverse oscillations of the first tone of a beam with free ends.

For application of the Stormer method to equation (67) one should preliminarily reduce this equation and boundary conditions (66) to the corresponding form [28]. Having substituted

$$B(x_1)\frac{d^2 f_n(x_1)}{dx_1^2} = \dot{w}_n(x_1), \qquad (4.80)$$

instead of (67) we obtain equation

$$\frac{d^2 w_n(x_1)}{dx_1^2} = \omega_n^2 f_n(x_1) m(x_1). \qquad (4.80')$$

From boundary conditions (66) it follows that with $x_1 = 0$

$$w_n(x_1) = 0 \text{ and } \frac{dw_n(x_1)}{dx_1} = 0. \qquad (4.80'')$$

Solution of the system of linear equations (80) and (80') can be represented in the form

$$f_n(x_1)=U_1p(x_1)+U_2\zeta(x_1)+U_3p_{11}(x_1)+U_4\zeta_{11}(x_1),$$
$$w_n(x_1)=U_1e(x_1)+U_2d(x_1)+U_3e_{11}(x_1)+U_4d_{11}(x_1).$$

In this case the linearly independent particular solutions of $p(x_1)$, $e(x_1)$, $\zeta(x_1)$, $d(x_1)$ and others can be approximately selected so that arbitraty constants U_1, U_2, U_3, U_4 were determined by equalities

$$U_1=f_n(0),\quad U_2=\left.\frac{df_n(x_1)}{dx_1}\right|_{x_1=0},\quad U_3=w_n(0),\quad U_4=\left.\frac{dw_n(x_1)}{dx_1}\right|_{x_1=0}.$$

Having used boundary conditions (80"), we find that

$$\left.\begin{aligned}f_n(x_1)=U_1p(x_1)+U_2\zeta(x_1),\\ w_n(x_1)=U_1e(x_1)+U_2d(x_1).\end{aligned}\right\}\tag{4.81}$$

Functions $p(x_1)$, $\zeta(x_1)$, $e(x_1)$, $d(x_1)$ must satisfy the following system of equations:

$$\left.\begin{aligned}&\frac{d^2p(x_1)}{dx_1^2}=\frac{e(x_1)}{B(x_1)},\\ &\frac{d^2\zeta(x_1)}{dx_1^2}=\frac{d(x_1)}{B(x_1)},\\ &\frac{d^2e(x_1)}{dx_1^2}=\omega_n^2m(x_1)p(x_1),\\ &\frac{d^2d(x_1)}{dx_1^2}=\omega_n^2m(x_1)\zeta(x_1).\end{aligned}\right|\tag{4.82}$$

Their initial values are selected so that boundary conditions when $x_1=0$, were fulfilled, i.e., so that

$$p(0)=1,\quad e(0)=0,\quad d(0)=0,\quad \zeta(0)=0,$$
$$\frac{d\zeta(x_1)}{dx_1}=0,\quad \frac{de(x_1)}{dx_1}=0,\quad \frac{dd(x_1)}{dx_1}=0,\quad \frac{dp(x_1)}{dx_1}=0.$$

The second pair of boundary conditions (when $x_1=l$) is used for finding arbitrary constants U_1 and U_2:

$$U_1 \left.\frac{d^2 p(x_1)}{dx_1^2}\right|_{x_1=l} + U_2 \left.\frac{d^2 \zeta(x_1)}{dx_1^2}\right|_{x_1=l} = 0,$$
$$U_1 \left.\frac{d^3 p(x_1)}{dx_1^3}\right|_{x_1=l} + U_2 \left.\frac{d^3 \zeta(x_1)}{dx_1^3}\right|_{x_1=l} = 0.$$

Since the determinant of this system is equal to zero, then function $f_n(x_1)$ will be determined with accuracy to the arbitrary factor. Having taken $U_1=1$, we find that

$$U_2 = -\frac{\left.\frac{d^2 p(x_1)}{dx_1^2}\right|_{x_1=l}}{\left.\frac{d^2 \zeta(x_1)}{dx_1^2}\right|_{x_1=l}}.$$

Having taken into account equations (81), we obtain the following expression for the form of natural bending oscillations, corresponding to frequency ω_n $(n=1)$:

$$f_1(x_1) = p(x_1) - \frac{\zeta(x_1)\,a(l)}{d(l)}.$$

The calculation is done in the following order. Let us divide the length of the body into k equal parts. Since when $k=0$ $p_0=1$ and $\frac{dp}{dx_1}=0$, then let us assume that $p_0=p_1=p_2=1$. According to formula

$$\sigma(x_1) = m(x_1)\,p(x_1)\,h^2\omega_n^2,$$

where $h=\frac{l}{k}$ – step of integration, we find values of $\sigma_0, \sigma_1, \sigma_2$ and corresponding differences of $\Delta\sigma_0, \Delta\sigma_1, \Delta^2\sigma_0$.

By using expressions

$$\left.\begin{aligned} \Delta a_0 &= \frac{1}{2}\sigma_0 + \frac{1}{6}\Delta\sigma_0 - \frac{1}{24}\Delta^2\sigma_0 + h\frac{da_0}{dx_1},\\ \Delta^2 a_0 &= \sigma_0 + \Delta\sigma_0 + \frac{1}{12}\Delta^2\sigma_0, \end{aligned}\right\} \qquad (4.83)$$

let us calculate $e_1, e_2 (e_0 = 0)$ and

$$e_{n+1} = e_n + \Delta e_n.$$

by them we find values of $\lambda_0, \lambda_1, \lambda_2, \Delta\lambda_0, \Delta\lambda_1, \Delta^2\lambda_0$, where

$$\lambda = h^2 \frac{e(x_1)}{B(x_1)}.$$

By formulas analogous to (83) let us calculate $\Delta p_0, \Delta^2 p_0$ and the corrected values of p_1 and p_2. Further let us find p_3 and e_3:

$$\Delta^2 e_{n-1} = \sigma_n^e + \frac{1}{12}\Delta^2\sigma_{n-2},$$

$$e_3 = e_2 + \Delta e_1 + \Delta^2 e_1,$$

etc.

In the same way we determine functions $\zeta(x_1)$ and $d(x_1)$, only in the first approximation it is accepted that $\zeta_0 = 0$, $\zeta_1 = h$ and $\zeta_2 = 2h$. This method gives good results with comparatively smooth change of mass and rigidity along the length of the body, i.e., with the absence of large concentrated loads, which naturally limits the region of its application. It can be used as a method of successive approximations, in which not the function of form, but the frequency of natural oscillations is initial. However, at least for the lowest frequencies the application of the method of approximations for frequencies instead of approximations for forms of oscillations is inexpedient.

§ 4.7. Application of the Method of Iterations to Complicated Systems

By the method of iterations we can directly determine the frequencies and forms of normal natural oscillations of complicated constructions of flight vehicles, oscillations of which are described not by one equation such as (4.23), but by a system of equations. As an illustration we examine the process of calculation by this method of the frequencies and forms of natural transverse oscillations of an aircraft on the whole, equations of oscillations of which are given in § 4.3.

Having placed the solution of (48) with replacement of $q_n(t) = e^{i\omega t}$, in the system of homogeneous equations corresponding to equations (43)-(47), we obtain the following system of ordinary differential equations for determination of eigenfunctions:

$$\frac{d^2}{dx_1^2}\left[B(x_1)\frac{d^2 f_n(x_1)}{dx_1^2}\right] = \omega_n^2 f_n(x_1)\, m(x_1), \qquad (4.84)$$

$$\frac{d^2}{d\xi_k^2}\left[B_k(\xi_k)\frac{d^2\Phi_n(\xi_k)}{d\xi_k^2}\right] = \omega_n^2 m_k(\xi_k)\left[\Phi_n(\xi_k) - \sigma_k(\xi_k)\varphi_n(\xi_k)\right], \qquad (4.85)$$

$$-\frac{d}{d\xi_k}\left[G(\xi_k) I_{pk}(\xi_k)\frac{d\varphi_n(\xi_k)}{d\xi_k}\right] = \omega_n^2\left[I_m(\xi_k)\varphi_n(\xi_k) - m_k(\xi_k)\sigma_k(\xi_k)\Phi_n(\xi_k)\right], \qquad (4.86)$$

$$\frac{d^2}{d\xi_0^2}\left[B_0(\xi_0)\frac{d^2\Psi_n(\xi_0)}{d\xi_0^2}\right] = \omega_n^2 m_0(\xi_0)\left[\Psi_n(\xi_0) - \sigma_0(\xi_0)\varkappa_n(\xi_0)\right], \qquad (4.87)$$

$$-\frac{d}{d\xi_0}\left[G(\xi_0) I_{p0}(\xi_0)\frac{d\varkappa_n(\xi_0)}{d\xi_0}\right] = \omega_n^2\left[I_m(\xi_0)\varkappa_n(\xi_0) - m_0(\xi_0)\sigma_0(\xi_0)\Psi_n(\xi_0)\right]. \qquad (4.88)$$

The corresponding boundary conditions and coupling conditions on the basis of formulas (49)-(57) will have the form

$$\frac{d^2 f_n(x_1)}{dx_1^2} = \frac{d}{dx_1}\left[B(x_1)\frac{d^2 f_n(x_1)}{dx_1^2}\right] = 0 \quad \text{when} \quad x_1 = 0,\ x_1 = l, \qquad (4.89)$$

$$\frac{d^2\Phi_n(\xi_k)}{d\xi_k^2} = \frac{d\varphi_n(\xi_k)}{d\xi_k} = \frac{d}{d\xi_k}\left[B_k(\xi_k)\frac{d^2\Phi_n(\xi_k)}{d\xi_k^2}\right] = 0 \quad \text{when} \quad \xi_k = l_k, \qquad (4.90)$$

$$\frac{d}{d\xi_0}\left[B_0(\xi_0)\frac{d^2\Psi_n(\xi_0)}{d\xi_0^2}\right] = \frac{d^2\Psi_n(\xi_0)}{d\xi_0^2} = \frac{d\varkappa_n(\xi_0)}{d\xi_0} = 0 \quad \text{when} \quad \xi_0 = l_0, \qquad (4.91)$$

and

$$\left.\begin{array}{c} f_n(x_1) = \Phi_n(\xi_k) \\ -\dfrac{d}{dx_1} f_n(x_1) = \varphi_n(\xi_k)\cos\varkappa_k - \dfrac{d\Phi_n(\xi_k)}{d\xi_k}\sin\varkappa_k \\ \text{when} \quad x_1 = x_k,\ \xi_k = 0, \end{array}\right| \qquad (4.92)$$

and

$$\left.\begin{array}{c} f_n(x_1) = \Psi_n(\xi_0) \\ -\dfrac{df_n(x_1)}{dx_1} = \varkappa_n(\xi_0)\cos\varkappa_0 - \dfrac{d\Psi_n(\xi_0)}{d\xi_0}\sin\varkappa_0 \\ \text{when} \quad x_1 = x_0,\ \xi_0 = 0, \end{array}\right| \qquad (4.93)$$

$$\frac{d}{dx_1}\left[B(x_1)\frac{d^2 f_n(x_1)}{dx_1^2}\right]_л - \frac{d}{dx_1}\left[B(x_1)\frac{d^2 f_n(x_1)}{dx_1^2}\right]_п = -2\frac{d}{d\xi_k^2}\left[B_k(\xi_k)\frac{d^2\Phi_n(\xi_k)}{d\xi_k^2}\right] \quad (4.94)$$

and

$$\left[B(x_1)\frac{d^2 f_n(x_1)}{dx_1^2}\right]_л - \left[B(x_1)\frac{d^2 f_n(x_1)}{dx_1^2}\right]_п = 2\left[B_k(\xi_k)\frac{d^2\Phi_n(\xi_k)}{d\xi_k^2}\right]\sin \varkappa_k - -2G(\xi_k) I_{pk}(\xi_k)\frac{d\varphi_n(\xi_k)}{d\xi_k}\cos \varkappa_k \quad \text{when} \quad x_1 = x_k, \quad \xi_k = 0, \quad (4.95)$$

$$\frac{d}{dx_1}\left[B(x_1)\frac{d^2 f_n(x_1)}{dx_1^2}\right]_л - \frac{d}{dx_1}\left[B(x_1)\frac{d^2 f_n(x_1)}{dx_1^2}\right]_п = 2\frac{d}{d\xi_0}\left[B_0(\xi_0)\frac{d^2\Psi_n(\xi_0)}{d\xi_0^2}\right] \quad (4.96)$$

and

$$\left[B(x_1)\frac{d^2 f_n(x_1)}{dx_1^2}\right]_л - \left[B(x_1)\frac{d^2 f_n(x_1)}{dx_1^2}\right]_п = 2\left[B_0(\xi_0)\frac{d^2\Psi_n(\xi_0)}{d\xi_0^2}\right]\sin \varkappa_0 - -2G(\xi_0) I_{p0}(\xi_0)\frac{d\varkappa_n(\xi_0)}{d\xi_0}\cos \varkappa_0 \quad \text{when} \quad x_1 = x_0, \quad \xi_0 = 0. \quad (4.97)$$

Having twice integrated equations (85) and (87) with respect to ξ_k and ξ_0 respectively and with the help of boundary conditions (90) and (91) determined arbitrary constants, let us find the following expressions for unit dynamic bending moments in the sections of cantilevers of wing and stabilizer:

$$M_{nx}^k(\xi_k) = \int_{\xi_k}^{l_k}\int_{\xi_k}^{l_k} m_k(\xi_k)\left[\Phi_n(\xi_k) - \sigma_k(\xi_k)\varphi_n(\xi_k)\right] d\xi_k\, d\xi_k, \quad (4.98)$$

$$M_{nx}^0(\xi_0) = \int_{\xi_0}^{l_0}\int_{\xi_0}^{l_0} m_0(\xi_0)\left[\Psi_n(\xi_0) - \sigma_0(\xi_0)\varkappa_n(\xi_0)\right] d\xi_0\, d\xi_0. \quad (4.99)$$

Unit dynamic torsional moments in the same sections will be determined by formulas

$$M_{kpnx}^k(\xi_k) = \int_{\xi_k}^{l_k}\left[I_m(\xi_k)\varphi_n(\xi_k) - m(\xi_k)\sigma_k(\xi_k)\Phi_n(\xi_k)\right] d\xi_k, \quad (4.100)$$

$$M_{kpnx}^0(\xi_0) = \int_{\xi_0}^{l_0}\left[I_m(\xi_0)\varkappa_n(\xi_0) - m_0(\xi_0)\sigma_0(\xi_0)\Psi_n(\xi_0)\right] d\xi_0, \quad (4.101)$$

which are obtained after integration of equations (86) and (88) and calculation of boundary conditions (90) and (91).

Unit bending moments in the sections of body of the fuselage on the basis of equation (84) and conditions (89), (95) and (97) will be equal to

$$M_{nx}(x_1) = \int_0^{x_1}\int_0^{x_1} m(x_1) f_n(x_1)\,dx_1\,dx_1 + \\ + 2\left[M_{nx}^k(0)\sin\varkappa_k - M_{kpnx}^k(0)\cos\varkappa_k\right]\sigma(x_1 - x_k) + \\ + 2\left[M_{nx}^0(0)\sin\varkappa_0 - M_{kpnx}^0(0)\cos\varkappa_0\right]\sigma(x_1 - x_0). \qquad (4.102)$$

where $\sigma(x_1 - x_i)$ – Heaviside unit function, equal to zero when $x_1 < x_i$ and $x_1 > x_i$ $(i = k, 0)$.

By further integrating system of equations (84)-(88), we find the expression for functions characterizing the forms of natural elastic oscillations of parts of the construction of aircraft, in the form (69)

$$\bar{f}_{nj}(x_j) = \omega_n^2 f_{nj}(x_j). \qquad (4.103)$$

Here $x_j = x_1, \xi_k, \xi_0$ and by $\bar{f}_{nj}$ are designated functions $\bar{f}_n$, $\bar{\Phi}_n$, $\bar{\Psi}_n$, $\bar{\varphi}_n$ and $\bar{\varkappa}_n$, determined by formulas

$$\left.\begin{aligned} \bar{f}_n(x_1) &= \int_0^{x_1}\int_0^{x_1} \frac{M_{nx}(x_1)}{B(x_1)}\,dx_1\,dx_1 + C_n x_1 + D_n, \\ \bar{\Phi}_n(\xi_k) &= \int_0^{\xi_k}\int_0^{\xi_k} \frac{M_{nx}^k(\xi_k)}{B_k(\xi_k)}\,d\xi_k\,d\xi_k + B_n^k \xi_k + E_n^k, \\ \bar{\Psi}_n(\xi_0) &= \int_0^{\xi_0}\int_0^{\xi_0} \frac{M_{nx}^0(\xi_0)}{B_0(\xi_0)}\,d\xi_0\,d\xi_0 + B_n^0 \xi_0 + E_n^0, \\ \bar{\varphi}_n(\xi_k) &= \int_0^{\xi_k} \frac{M_{kpnx}^k(\xi_k)}{G(\xi_k)\,I_{pk}(\xi_k)}\,d\xi_k + A_n^k, \\ \bar{\varkappa}_n(\xi_0) &= \int_0^{\xi_0} \frac{M_{kpnx}^0(\xi_0)}{G(\xi_0)\,I_{p0}(\xi_0)}\,d\xi_0 + A_n^0. \end{aligned}\right\} \qquad (4.104)$$

From conditions (92)-(97) it follows that

$$E_n^k = f_n(x_k), \quad E_n^0 = f_n(x_0),$$

$$A_n^k = -\int_0^{x_k} \frac{M_{nx}(x_1)}{B(x_1)}\, dx_1 \cos \varkappa_k,$$

$$B_n^k = \int_0^{x_k} \frac{M_{nx}(x_1)}{B(x_1)}\, dx_1 \sin \varkappa_k,$$

$$A_n^0 = -\int_0^{x_0} \frac{M_{nx}(x_1)}{B(x_1)}\, dx_1 \cos \varkappa_0,$$

$$B_n^0 = \int_0^{x_0} \frac{M_{nx}(x_1)}{B(x_1)}\, dx_1 \sin \varkappa_0.$$

As in the case of a beam with free ends, constants of integration C_n, D_n, determining the position of the axis of normal oscillations of system, are found from conditions of equilibrium

$$\sum_j \int_0^{l_j} m_j(x_j) f_{nj}(x_j)\, dx_j = 0,$$

$$\sum_j \int_0^{l_j} m_j(x_j) x_j f_{nj}(x_j)\, dx_j = 0,$$

i.e., will have a form analogous to (70). For finding the forms and frequencies of natural oscillations of second and subsequent tones in this case instead of (103) we use equation

$$f_{nj}(x_j) = \omega_n^2 \left[f_{nj}(x_j) - \sum_{i=1}^{n-1} \Delta_{ni} f_{ij} \right], \tag{4.105}$$

where Δ_{ni} — corresponding correction factor, analogous to (72') and equal to

$$\Delta_{ni} = \frac{M_{ni}}{M_{ii}}, \tag{4.106}$$

where

$$M_{ii} = \sum_{j} \int_{0}^{l_j} m_j(x_j) f_{ij}^2(x_j)\, dx_j,$$

$$\begin{aligned} M_{ni} = \sum_{j} \int_{0}^{l_j} m_j(x_j) f_{nj}(x_j) f_{ij}(x_j)\, dx_j = \int_{0}^{l} m(x_1) f_n(x_1) f_i(x_1)\, dx_1 + \\ + 2\left\{ \int_{0}^{l_k} m_k(\xi_k) [\bar{\Phi}_n(\xi_k) - \sigma_k(\xi_k) \bar{\varphi}_n(\xi_k)] \Phi_i(\xi_k)\, d\xi_k + \right. \\ + \int_{0}^{l_k} [I_m(\xi_k) \bar{\varphi}_n(\xi_k) - m_k(\xi_k) \sigma_k(\xi_k) \bar{\Phi}_n(\xi_k)] \varphi_i(\xi_k)\, d\xi_k + \\ + \int_{0}^{l_0} m_0(\xi_0) [\bar{\Psi}_n(\xi_0) - \sigma_0(\xi_0) \bar{\varkappa}_n(\xi_0)] \Psi_i(\xi_0)\, d\xi_0 + \\ \left. + \int_{0}^{l_0} [I_m(\xi_0) \bar{\varkappa}_n(\xi_0) - m_0(\xi_0) \sigma_0(\xi_0) \bar{\Psi}_n(\xi_0)] \varkappa_i(\xi_0)\, d\xi_0 \right\} \quad \text{when} \quad i \neq n. \end{aligned}$$

The process of calculations of functions f_{nj} (105) by the above-indicated formulas is performed by the scheme discussed in § 4.5.

With the presence of concentrated masses and sharp changes of rigidity along the length of the body of the flight vehicle the representation of its dynamic scheme (for calculation of frequencies and forms of natural oscillations) can become expedient in the form of a system of series connected beams of constant section and elastically suspended (at places of their joining) loads and for simpler constructions than those considered.

§ 4.8. Damping of Elastic Oscillations

A very important dynamic characteristic of an elastic system is its ability to dissipate energy. For each cycle of oscillations of a real elastic system a certain portion of energy is expended on surmounting drag. Although this loss of energy often turns out to be very small and does not practically affect the forms of natural oscillations of the system of lowest terms, in the final analysis it leads to damping of free oscillations of the system with a comparatively large quantity of cycles and in the case of resonance – to limitation of the amplitude of oscillations. With steady-state oscillations of the system damping is the cause of phase shift between external force and the displacement caused by it.

Resistance can be external and internal. External resistance is generated by the medium in which oscillations occur. Internal resistance of an elastic system is composed of friction at places of joining of its elements (constructional damping) and nonelastic resistance of the material of these elements. Even with small forces, due to the heterogeneity of material, leading to nonuniform distribution of stresses along microareas of the section, microplastic deformations appear, on which energy is expended. This energy is irreversibly absorbed by the material and is dispersed in the form of heat.

The magnitude of internal resisting forces is determined usually by natural dynamic tests of constructions, since the mechanism of their formation is unknown exactly. There are comparatively many applied theories of the internal friction in constructions. However, the majority of them does not satisfy the basic requirements which advances the practice of calculations, namely: requirement of simplicity simplicity of analytic description and satisfactory coordination with experimental data. In most cases during investigation of oscillations of a construction we use the hypothesis of viscous friction (Vogt hypothesis), ensuring maximum simplicity of calculations. According to this hypothesis the resisting force is a linear function of elastic deformation. Since experiments show that with cyclic deformation the forces of internal nonelastic resistance do not depend on the frequency of oscillations, the coefficient before rate in the equation of elastic oscillations is mathematically presented inversely proportional to the frequency of natural oscillations:

$$\frac{\partial^2}{\partial x_1^2}\left[B(x_1)\frac{\partial^2 y_1(x_1,t)}{\partial x_1^2}\right]+\frac{\gamma}{\omega_n}\frac{\partial^2}{\partial x_1^2}\left[B(x_1)\frac{\partial^3 y_1(x_1,t)}{\partial x_1^2\,\partial t}\right]+ + m(x_1)\frac{\partial^2 y_1(x_1,t)}{\partial t^2}=0. \qquad (4.107)$$

The application of other theories of internal friction in materials (N. N. Davidenkov [19], A. Yu. Ishlinskiy [23], I. L. Korchinskiy and others) introduces great mathematical difficulties into the solution of problems of forced oscillations of complicated elastic systems. Therefore, here we will not pause on their advantages and disadvantages. Desiring to become acquainted with the state of this question in detail, it is possible to recommend [49].

Internal friction is characterized quantitatively by the value of so-called absorption coefficient $\psi=2\pi\gamma$, equal to the ratio of energy absorbed for one cycle ΔU to total potential energy, corresponding to the considered amplitude of oscillations of deformation. Appraisal of internal friction is most frequently performed experimentally by means of investigation of damped oscillations of the construction. In this case we use the concept of logarithmic damping decrement δ, equal to the logarithm of the ratio of amplitude of oscillations of the considered cycle to the amplitude of oscillations of the preceding cycle. Connection of ψ with δ is determined by expression $\psi = 2\delta$. Sometimes the value of δ is determined from amplitude-frequency characteristic of the construction with respect to width $\Delta\omega$ of the resonance curve at a level corresponding to $1/\sqrt{2}$ of the greatest amplitude $\gamma=\frac{\Delta\omega}{\omega_0}=\frac{\delta}{\pi}$. It is interesting to note that the mean value of δ for metal constructions of the most various assignment (bridges, ships, radio masts, aircraft and others) turn out to be of approximately one order (0.05-0.15). The value of δ depends on the amplitude of oscillations of stresses. This relationship appears especially strongly in the zone of small deformations. Only with comparatively large deformations can the logarithmic decrement practically be considered constant. Figure 4.5 gives a graph of function $\delta(\sigma)$, determined by laboratory means [80] for a sample made from aluminum alloy AMG.

Since with elastic oscillation of the construction its different elements will have a different level of stresses, then it is obvious that the value of δ for a flight vehicle on the whole will depend on the type of state of stress, i.e., case of loading. In this case the relationship of the logarithmic decrement to the amplitude of oscillations of the construction can be essential in the presence of comparatively large deformations at separate points of the body. In the first approximation δ can be represented in the form of the sum of constant and variable components, considering the latter a function of the amplitude of oscillations T_{n0} (or S_{n0})

$$\delta = \delta_0 + \delta_n(T_{n0}). \qquad (4.108)$$

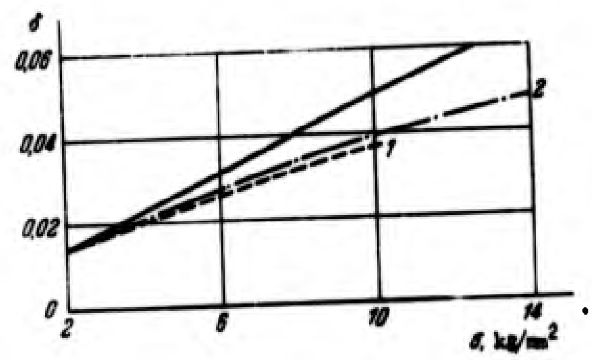

Fig. 4.5. Relationship of logarithmic damping decrement of oscillations for an aluminum alloy to stress (1 – after tensile strain 4.5%; 2 – after compressive strain 4.5%).

It is practically possible to experimentally establish the value of generalized logarithmic damping decrement of oscillations for the construction of a flight vehicle on the whole only for the first tones of natural elastic oscillations, and often only for a dry construction. Therefore, in a number of cases the question of how δ is changed with change of the form of oscillations in the process of flight or with transition to other higher tones of oscillations becomes urgent. Some idea of this can be obtained if we are limited by consideration of the particular case when damping of oscillations of the construction is caused only by nonelastic properties of its material.

As it is known, with small oscillations the absorption of energy in a unit of volume of material for one cycle of oscillations ΔU will be determined by the area of hysteresis loop. For harmonic oscillations [65] this loop has elliptical shape, for which (Fig. 4.6).

$$\Delta U = \pi R_0 \frac{\sigma_0}{E}, \qquad (4.109)$$

where R_0 – amplitude of oscillations of the force of nonelastic resistance, σ_0 – amplitude of oscillations of elastic force (normal stress), and E – modulus of normal elasticity, characterizing the relationship of elastic force to elastic deformation. Having taken into account that total energy, expended on deformation of this unit volume, is equal to

$$U = \frac{1}{2E}\sigma_0^2, \qquad (4.110)$$

we obtain the absorption coefficient for the construction of the vehicle on the whole in the form

$$\psi = \int_{\Phi} \Delta U \, d\Phi \Big/ \int_{\Phi} U \, d\Phi, \qquad (4.111)$$

where Φ – volume of supporting part of construction of the vehicle.

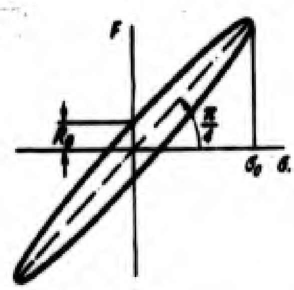

Fig. 4.6. Shape of hysteresis loop.

According to (110) and (111) the peak value of force (stress) of nonelastic resistance for an elementary volume is expressed through the value of logarithmic damping decrement of oscillations δ_0 by formula

$$R_0 = \delta_0 \frac{\sigma_0}{\pi}.$$

If we proceed from the assumption that at small elastic oscillations δ_0 linearly depends on the amplitude of stresses

$$\delta_0 = k_0 \sigma_0,$$

we obtain that

$$R_0 = \frac{k_0}{\pi} \sigma_0^2, \quad k_0 = \frac{\beta}{E}. \tag{4.112}$$

As an example let us consider how in this case the value of variable component of logarithmic damping decrement will be changed with change of the form of longitudinal oscillations of the construction of ballistic type flight vehicle.

It is obvious that stresses in the sections of the body of such a vehicle at different forms of natural oscillations (even with identical amplitudes of oscillations of the point of reduction T_{n0}) will be different. Taking into account that

$$\sigma_{0n}(x_1) = T_{n0} E \frac{dX_n(x_1)}{dx_1}, \tag{4.113}$$

in accordance with formulas (109), (110) and (112), we will have

$$\left.\begin{aligned} \Delta U(x_1) &= k_0 E^2 T_{n0}^3 \left| \frac{dX_n(x_1)}{dx_1} \right|^3, \\ U(x_1) &= \frac{1}{2} E T_{n0}^2 \left[\frac{dX_n(x_1)}{dx_1} \right]^2. \end{aligned}\right\} \tag{4.114}$$

Thus, the sought value of coefficient δ_n (108) for the construction on the whole in this case will be determined by expression

$$\delta_n = \beta \frac{\int\limits_0^l F_c(x_1) \left| \frac{dX_n(x_1)}{dx_1} \right|^3 dx_1}{\int\limits_0^l F_c(x_1) \left[\frac{dX_n(x_1)}{dx_1} \right]^2 dx_1} T_{n0} \tag{4.115}$$

or taking into account equation (79")

$$\delta_n = k_0\omega_n^2 \frac{\int\limits_0^l \frac{1}{F_c^2(x_1)} \left| N_{nx}^3(x_1) \right| dx_1}{\int\limits_0^l \frac{1}{F_c(x_1)} N_{nx}^2(x_1)\, dx_1} T_{n0}, \qquad (4.115')$$

where

$$N_{nx}(x_1) = \int\limits_0^{x_1} m(x_1) X_n(x_1)\, dx_1.$$

With equal amplitudes of oscillations of the point of reduction the change of δ_n during transition from one n-th form of oscillations to another m-th will be characterized by relation

$$\frac{\delta_n}{\delta_m} = \frac{\omega_n^2}{\omega_m^2} \frac{\int\limits_0^l \frac{1}{F_c(x_1)} N_{mx}^2(x_1)\, dx_1 \int\limits_0^l \frac{1}{F_c^2(x_1)} \left| N_{nx}^3(x_1) \right| dx_1}{\int\limits_0^l \frac{1}{F_c(x_1)} N_{nx}^2(x_1)\, dx_1 \int\limits_0^l \frac{1}{F_c^2(x_1)} \left| N_{mx}^3(x_1) \right| dx_1}. \qquad (4.116)$$

For the case of bending oscillations of the considered construction approximately

$$\delta_n = \frac{16}{3\pi} k_0\omega_n^2 \frac{\int\limits_0^l \frac{a(x_1)}{I^3(x_1)} \left| M_{nx}^3(x_1) \right| dx_1}{\int\limits_0^l \frac{1}{I(x_1)} M_{nx}^2(x_1)\, dx_1} S_{n0}. \qquad (4.117)$$

Calculation of δ_n by the shown formulas requires knowledge of the value of proportionality factor k_0. Since in reality the values of δ_0 will be affected by local concentration of stresses (especially in riveted and welded seams), and also by energy dissipation in elements of attachment of different loads and so forth, then it is expedient to determine k_0 not on samples, but from formulas (115) or (117) by means of measurement of δ_0 for the construction on the whole.

Usually the relationship of logarithmic decrement to the amplitude of oscillations will be disregarded when performing practical calculations and equation of elastic oscillations of the construction of flight vehicle is written in the form

$$\ddot{S}_n + 2h_n\dot{S}_n + \omega_n^2 S_n = \frac{F}{M_n}, \tag{4.118}$$

where

$$2h_n = \gamma\omega_n = \frac{\delta}{\pi}\omega_n.$$

Correspondingly the dynamic coefficient is determined by expression

$$\eta_n = \frac{1}{\left[\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(\frac{\delta}{\pi}\right)^2 \frac{\omega^2}{\omega_n^2}\right]^{1/2}}.$$

With resonance, when $\omega = \omega_n$, it will be equal to

$$\max \eta_n = \frac{\pi}{\delta}.$$

Subsequently when writing equations of elastic oscillations of the construction we will introduce damping factors formally (without any reservations) into the final expressions of these equations.

§ 4.9. Selection of Initial Functions of the Forms of Oscillations

The accuracy of calculation of forms and frequencies of natural elastic oscillations of the construction by the method of iterations depends on the correctness of establishment of boundary conditions and on the accuracy of initial data, namely: on the magnitude and character of distribution of weight of the flight vehicle and rigidity of the supporting part of its construction. During calculation of the frequencies and forms of bending oscillations the distribution of weight along the length of the body is taken exactly the same as during calculation of static lateral forces and bending moments, i.e., in the form of the sum of intensity of the mass of construction and mass of fuel taking into account the character of connection (in lateral direction) of each load to the supporting part of the body. Calculation in the expression for linear mass of different concentrated loads is produced by the introduction of delta functions. When finding the forms and frequencies of longitudinal natural oscillations there is used a weight diagram, intended for calculation of static longitudinal forces. The effect of mobility of liquid in the fuel tanks on the forms and frequencies of oscillations of the construction can be considered by several methods. In cases when the accuracy of calculation of forms is not very important or the effect of liquid is small, it is possible to be limited by the introduction of reduced mass m* (§2.5) into functions of frequency ω_n. Otherwise we should correct the frequencies and forms of oscillations by means of transformation of the corresponding system of equations, describing natural oscillation of the construction and liquid in tanks, to normal coordinates.

Flexural $B(x_1)$, compressive and tensile $E(x_1)F_c(x_1)$ rigidities are determined taking into account the thermal balance of work of the construction (values of temperature and time of heating).

The construction of rigidity diagram is one of the most complicated operations of calculation of the construction for strength in the process of designing the flight vehicle. For correct calculation of the effect of local stiffening of loads, local turns of sections in joints of compartments, rigidity of different frames (during bending and shear), the effect of cutaways, hatches and so forth, it is often required to carry out special investigations. Sometimes the calculation of moments of inertia and areas of cross sections of reinforced shells, the covering of which can lose stability in the process of loading, causes some difficulty. In similar cases depending upon the expected overall level of the state of stress of construction (in each particular case of loading) it is necessary to take the values of these rigidities either taking into account, or without taking into account the losses of stability of the shell.

During calculation of dynamic characteristics it is recommended to use mean values of geometric dimensions of supporting members. In this case, if the shell has a heat-shielding covering, then as average thickness δ_{cp} there is taken

$$\delta_{cp} = \delta_c + \Delta\delta \frac{E_0}{E}, \qquad (4.119)$$

where E_0 — elastic modulus of covering, δ_c — average thickness of metal part of the shell (equal to approximately 95% of nominal), and $\Delta\delta$ — thickness of covering. During calculation of definite integrals (68) and (69') by some method of approximation the step of integration is selected in each particular case depending upon the specified accuracy of calculation of the highest tone of natural oscillations and the type of diagrams of masses and rigidities. It can be different on various stages of integration, but usually not more than $\frac{1}{50} l$.

The quantity of necessary approximations is determined by the necessary accuracy of calculation of forms of natural oscillations of the construction and in many respects depends on the selection of initial functions. For a rod with variable linear mass and rigidity it is natural as the initial function to take the corresponding function of form of natural oscillations of a prismatic rod of constant section $m(x_1) = m_1$ and $B(x_1) = B_1$. Solution of equation (67) for such a rod

$$\frac{d^4 f_n(x_1)}{dx_1^4} = \frac{\omega_n^2}{p^2} f_n(x_1)$$

will have the form

$$f_n(x_1) = C_1 \operatorname{sh}\sqrt{\frac{\omega_n}{p}}\, x_1 + C_2 \operatorname{ch}\sqrt{\frac{\omega_n}{p}}\, x_1 + \\ + C_3 \sin\sqrt{\frac{\omega_n}{p}}\, x_1 + C_4 \cos\sqrt{\frac{\omega_n}{p}}\, x_1. \qquad (4.120)$$

where $p=\sqrt{\frac{B_1}{m_1}}$, C_1, C_2, C_3, C_4 – arbitrary constants.

The values of frequency ω_n and coefficients C_i $(i=1, 2, 3, 4)$ are found in such a way that function (120) satisfied the required boundary conditions and corresponding condition of normalization. Thus, for instance, for a rod with free ends from conditions

$$\frac{d^2 f_n(x_1)}{dx_1^2}=0 \text{ and } \frac{d^3 f_n(x_1)}{dx_1^3}=0 \text{ when } x_1=0$$

it follows that $C_1=C_3=C$, and $C_2=C_4=D$. Corresponding equations for determination of C and D give boundary conditions when $x_1=l$

$$\frac{d^2 f_n(x_1)}{dx_1^2}=0 \quad \text{and} \quad \frac{d^3 f_n(x_1)}{dx_1^3}=0.$$

By equating the determinant, composed of coefficients of these equations, to zero we obtain the following transcendental equation for calculation of the frequencies of natural oscillations of the construction:

$$\operatorname{ch}\sqrt{\frac{\omega_n}{p}}\,l\cos\sqrt{\frac{\omega_n}{p}}\,l=1.$$

For n = 1 we will have that $\omega_1=\left(1{,}51\frac{\pi}{l}\right)^2\sqrt{\frac{B_1}{m_1}}$, for n = 2

$$\omega_2=\left(\frac{5}{2}\frac{\pi}{l}\right)^2\sqrt{\frac{B_1}{m_1}}, \quad \text{for} \quad n=3 \quad \omega_3=\left(\frac{7}{2}\frac{\pi}{l}\right)^2\sqrt{\frac{B_1}{m_1}}.$$

Thus,

$$\begin{aligned} f_n(x_1)=&\left(\operatorname{ch}\sqrt{\frac{\omega_n}{p}}\,l-\cos\sqrt{\frac{\omega_n}{p}}\,l\right)\left(\operatorname{sh}\sqrt{\frac{\omega_i}{p}}\,x_1+\sin\sqrt{\frac{\omega_n}{p}}\,x_1\right)+\\ &+\left(\sin\sqrt{\frac{\omega_n}{p}}\,l-\operatorname{sh}\sqrt{\frac{\omega_n}{p}}\,l\right)\left(\operatorname{ch}\sqrt{\frac{\omega_n}{p}}\,x_1+\cos\sqrt{\frac{\omega_n}{p}}\,x_1\right). \end{aligned} \quad (4.121)$$

The last operation of preparation of the initial function for rods with free ends consists of its reduction to the axis of oscillations of the flight vehicle, possessing variable linear mass. For this we should use formula

$$f_n(x_1)=f_n^*(x_1)+D_{n0}x_1+C_{n0}, \qquad (4.122)$$

where

$$C_{n0}=\frac{mx_{1T}M_{n0}-Q_{n0}J_z}{m(J_z-mx_{1T}^2)}, \quad D_{n0}=\frac{x_{1T}Q_{n0}-M_{n0}}{J_z-mx_{1T}^2},$$

$$Q_{n0}=\int_0^l m(x_1)f_n^*(x_1)dx_1, \quad M_{n0}=\int_0^l m(x_1)x_1f_n^*(x_1)dx_1.$$

Coefficients D_{n0} and C_{n0} are analogous in structure to coefficients D_n and C_n (70). They are determined from condition of mutual balancing of forces of inertia, appearing with principal oscillations of a rod with frequency ω_n

$$\left.\begin{aligned}&\int_0^l m(x_1)f_n(x_1)dx_1=0,\\&\int_0^l m(x_1)x_1f_n(x_1)dx_1=0.\end{aligned}\right\} \qquad (4.122')$$

As initial function of form it is possible to take the form of static deformation of the body of a flight vehicle from external load (for instance, gravity) or corresponding forms of oscillations of similar flight vehicles, represented in the function of $\frac{x_1}{l}=\bar{x}_1$.

In conclusion let us note that in a number of cases of loading the correct selection of boundary conditions, in particular, in case V (especially in the absence of special bracing of flight vehicle to the launching pad), can present some difficulty.

Footnotes

[1]With the presence of high-speed electronic computers this simplification is not too rough, since it is practically always possible to consider the number of degrees of freedom required by conditions of accuracy.

[2]Taking into account the small influence of the error of description of function of form on the frequency of oscillations, by using this method it is possible to more precisely determine the frequencies by means of more precise determination of only the expressions for potential energy (i.e., without preliminary correction of the form of oscillations).

[1]Subsequently we will use only simplified expressions for $T_0(t)$ and $U_0(t)$, assuming that the account of the effect of lateral forces and inertia of rotation is produced with calculation of functions $f_n(x_1)$ and ω_n.

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S E C T I O N   III

GROUND CASES OF LOAD

CHAPTER V

ELASTIC VIBRATIONS OF A STRUCTURE DURING TRANSPORTATION

§ 5.1. Preliminary Remarks

During transportation the structure of a flight vehicle is acted on by two types of external loads. These are, first of all, loads which practically do not yield to adjustment (for instance, loads caused by rocking of a ship, the influence of wind, and so forth) and, secondly, adjustable loads, the magnitudes of which depend on the construction of special transport means (carriages), method of securing equipment to them, and conditions of transportation. In particular, loads are adjustable which are caused by elastic vibrations of the system carriage-flight vehicle loads appearing in the process of fulfillment of various kinds of technological operations, and so forth. Such a type of load can always be kept in the required limits by means of establishment of definite requirements for conditions of ground exploitation, for means of shock absorption for the carriage. Such a classification of forces naturally is very conditional, since in principle all ground loads yield to adjustment. However, in practice this is not always realizable, if at least from purely economic considerations.

Sometimes for flight vehicles of special assignment by means of selection of conditions of exploitation and certain parameters for the carriages it is possible to make transport cases of load noncalculable for the structure of the vehicle itself.

Thus the problem of dynamic calculation of the structure of a flight vehicle for transport cases of load usually is reduced to manifestation of requirements for the carrying capacity of vehicle structure, to conditions of exploitation of the vehicle-carriage system, and to dynamic characteristics of the transport carriages in particular to determination of places of location of connections (supports) between the vehicle and the carriage, and to determination of forces in construction of the flight vehicle.

External disturbances, acting on the transport and adjusting carriages in the process of transporting flight vehicles and in the process of pre-launch preparation of the vehicle, have a clearly expressed random nature. Therefore they can be described only by methods of the probability theory and mathematical statistics. However, the difficulties of direct measurement of the characteristics of these disturbances are so great that this description is carried out only by indirect means, and namely: by means of measurement and corresponding processing of response to external influences. The solution of this reverse problem, consisting of the determination of spectral density of disturbances $\Phi_{B}(\omega)$ through spectral density of flight vehicle-carriage response, just as the solution of the direct problem (4.19), requires a knowledge of the transfer function of the system in a comparatively wide range of frequencies. Determination of such transfer functions for complex elastic systems is also a problematic question, especially in the range of comparatively high frequencies. Therefore in most cases the dynamic calculation of the structure of flight vehicles for ground cases of load can be carried out only with the introduction of a number of simplifying assumptions. In the absence of necessary information about external disturbances these simplifications lead to rough schematization of external influences, and sometimes even to simplification of formulation of the actual problem.

Schematization of external disturbances assumes establishment of some conditional profiles of the road, "typical" (calculation) obstacles, replacement of the random process by a finite sum of sinusoidal oscillations, and so forth. The degree of simplification of the dynamic circuit depends on the parameters of the specific system and the importance of the case of load being considered. Most frequently this simplification is reduced to isolated consideration of elastic vibrations of the flight vehicle and elastic vibrations of the carriage with an absolutely rigid vehicle. Such a formulation of the problem is based mainly on the fact that the reaction of the carriage construction, which represents an elastic system with minor damping and a discrete spectrum of natural frequencies, to an external influence with a continuous spectrum of frequencies will have the form of a narrow-band spectrum. In other words, the main share of power of carriage reaction will be concentrated in narrow bands around the natural frequencies of the system. Under such conditions the vibrations of vehicle supports can be represented approximately in the form of the sum of sinusoidal oscillations with the shown frequencies and random amplitudes. The less the weight of the vehicle as compared to weight of the carriage, i.e., the less the influence of vibrations of flight-vehicle structure on the dynamic characteristics of the system on the whole, all the more founded will be such an approach to determination of possible values of forces in elements of vehicle construction. For flight vehicles, the weight of which will compare with the weight of the carriage, it follows either to be guided by experimental data or to investigate the response to "typical" disturbances. Using such a schematization, usually it is possible to establish the requirements for conditions of ground exploitation of flight vehicles. Such an approach naturally has an area of application, and therefore in practice it is frequently necessary to conduct expensive full-scale tests of the construction of flight vehicles almost for all ground load cases.

§ 5.2. Vibrations of Transport Carriages

For transportation of flight vehicles wholly and by parts from the place of manufacture or storage to a technical or launching site (airport) it is possible to use all types of transport: railroad, automotive, air, and water. From a technical position to launching site they can be transported both on ordinary railroads and on highways and dirt roads, and large flight vehicles, (for instance the carriers of spaceships) also on specially built roads. Transfer of completely assembled flight vehicles on this concluding stage of transportation is executed, as a rule, on special transporter-erectors, which sometimes are also used for erection of vehicles (in particular, vehicles of the ballistic type) in the initial launch position (for instance, vertical).

Since each form of transport possesses its own specific peculiarities, then load conditions on the structure in all cases of transportation turn out to be different. Existing transport means of general assignment permit the transporting of small flight vehicles in assembled form, and large ones - only by parts (separate units).

In the process of transportation (during loading, transporting, and installation in a launcher) the structure of a flight vehicle is loaded by gravity, aerodynamic (wind) loads, and also by inertial forces caused by vibrations of the supports.

Wind loads act directly on the structure only in those cases when transportation of the flight vehicle is carried out by the open method (on the deck of a ship, on an open platform, and so forth). In accordance with formula (1.24) these loads will be determined basically by magnitude of impact pressure, equal to $q = \frac{1}{2}\rho_0(u^2 + v^2)$ during a cross wind and $q = \frac{1}{2}\rho_0(u + v)^2$ during a headwind, where v - speed of transportation, and u - instantaneous speed of wind. Here the coefficient of lateral force c_z will depend not only on the form of the transported part of the vehicle which is circumvented by the flow, but also on the angle of incidence $\alpha = \text{arc tg } u/v$.

The values of inertial forces are determined by the type of transport utilized, construction of the carriage, and conditions of transportation. In the case of transportation by railroad (case $T_{ж}$) the source of inertial forces is vibration of a railroad car. Due to the undulating nature of the track, the oval shape of the wheels, lateral blows of wheel rims against the rails, vibrations of the tracks themselves on an elastic support, jolts on switches, and jolts of the wheels on joints on rail sections, this vibration occurs continuously. Its conditions depends on speed of the train, state of various sections of track and especially on the type of shock absorption and degree of loading of the railroad cars. Usually the most intense vibrations (with amplitudes of overload of an order of 0.5-1) are observed in vertical and longitudinal directions [11]. Comparatively large longitudinal overloads can be observed only during the sharp contact or deceleration of the train or during descent of railroad cars from hills in the period of forming a train. In most cases similar maneuvers are prohibited for railroad cars with flight vehicles. However one should be guided by them at least during calculation of elements for securing the vehicle to the railroad car. During normal deceleration of a train longitudinal overload is small. Also comparatively small are the lateral accelerations, which appear mainly on curvilinear sections of track. Vibration components of lateral accelerations, caused by wobbling oscillations of the car, have an amplitude of an order of $0.5n_n$. The static component of this overload will be proportional to the square of speed of the train v_n [km/h] and is inversely proportional to radius of turn R[m]:

$$n_{z_n} = \frac{1}{g_0 R}\left(\frac{v_n}{3.6}\right)^2.$$

Although external influence on railroad cars is a random function of time, their reaction to this influence, as was already noted, can in many cases be represented approximately in the form of the sum of steady-state oscillations with frequencies equal to the natural frequencies of oscillations of spring-suspended sections of a railroad car with the vehicles (Fig. 5.1). The greatest amplitude of such oscillations is observed during the so-called critical speeds of the train, determined from the equality of frequency of the inherent vertical oscillations of the railroad car to frequency of encounter of rail joints v_n/L_p, where L_p - length of rail. Usually the lowest frequencies of vertical and horizontal natural oscillations of railroad cars (forward and rotary oscillations of the platform of a railroad car on springs) lie within the limits of 2-8 Hz[1]. Approximate values of these frequencies can be obtained from equations of free oscillations of the spring-suspended part of the railroad car with the vehicle.

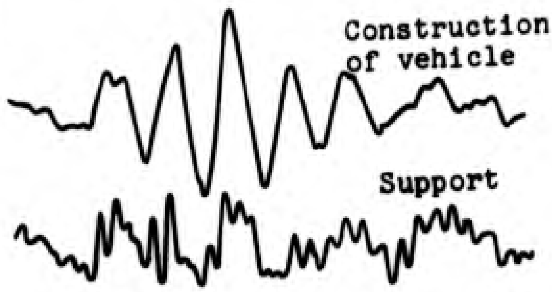

Fig. 5.1. Model picture of change of transverse overloads during transportation of vehicles by railroad.

In a simplified form vibration of this part of a railroad car can be described by a vertical η (or lateral ζ) deviation of its center of gravity (from a position of equilibrium) and angle of rotation of platform ϑ relative to this center of gravity (Fig. 5.2). Here the corresponding equations of vibrations will have the form

$$\left.\begin{aligned}(M+M_B)\ddot{\eta} &= -c(\eta+\vartheta l_1)-c(\eta-l_2\vartheta),\\ -(J_z+J_B)\ddot{\vartheta} &= c\left[l_1(\eta+l_1\vartheta)-l_2(\eta-l_2\vartheta)\right],\end{aligned}\right\} \quad (5.1)$$

where M_B - mass of spring-suspended part of the platform, c - rigidity of spring, M and J_z - mass and moment of inertia of the vehicle, and J_B - moment of inertia of platform relative to lateral axis z, passing through the center of gravity of the system (approximately $J_B = 0.08 M_B l_B^2$). The remaining designations are shown in Fig. 5.2.

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[1]Random oscillations. Collection, Editor S. Krendell, Publishing House "Mir," 1967.

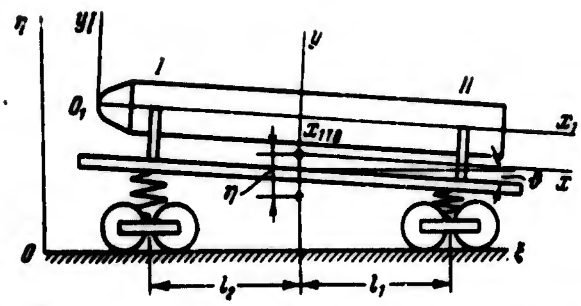

Fig. 5.2. Simplified dynamic model of the spring-suspended part of the railroad car.

Substituting $\eta=\eta_0 e^{i\omega t}$ and $\vartheta=\vartheta_0 e^{i\omega t}$, we obtain

$$[2c-(M+M_B)\omega^2]\eta_0-c\vartheta_0(l_2-l_1)=0,$$
$$-c(l_2-l_1)\eta_0+\vartheta_0[c(l_1^2+l_2^2)-\omega^2(J_z+J_B)]=0.$$

Equating to zero the determinant of this system of equations, which is made up of coefficients for η_0 and ϑ_0, we find the algebraic equation (relative to ω^2) for calculation of two frequencies of natural oscillations of the car (with a rigid vehicle)

$$a_2\omega^4+a_1\omega^2+a_0=0. \tag{5.2}$$

Here

$$\left.\begin{aligned} a_2&=(J_B+J_z)(M_B+M),\\ a_1&=-(J_B+J_z)2c-c(M_B+M)(l_1^2+l_2^2),\\ a_0&=c^2(l_1+l_2)^2. \end{aligned}\right\} \tag{5.3}$$

One of these frequencies corresponds to the forward form of motion of the railroad car (jumping), and the other - to rotary (galloping of the railroad car).

As it follows from formulas (3), the values of the mentioned frequencies will depend on rigidity of springs and inertial characteristics of the spring-suspended part of railroad car and the transported part of the vehicle. Only in that case, when the mass of the vehicle M is small as compared to M_B, its influence on

ω can be disregarded and oscillations of the vehicle considered as forced. Then approximately local values of total later overload in any section of the body of the vehicle (as a solid body) can be determined by a formula of the type

$$-n_{y_1}(x_1) = 1 \pm \frac{\ddot{\eta}}{g_0} \mp \ddot{\vartheta}(x_{1\text{тв}} - x_1)\frac{1}{g_0}. \qquad (5.4)$$

The values of overload components for a case of transporting a vehicle or its parts by air transport (case $T_в$) are set by the corresponding norms of strength, since it is natural that the strength of construction of the transported vehicle, and especially the points where it is connected to the carriage, should in some measure correspond to the strength of the transport vehicle itself (aircraft, helicopters, dirigibles).

During loading and unloading operations with the help of cranes (case $T_в$) the magnitude of the coefficient of lateral overload can be changed in wide limits (from 1.2 to 2) depending on the limitations placed on conditions of fulfillment of these operations.

During transportation on water (case $T_к$) a flight vehicle is loaded basically by inertial forces, caused by rocking of a ship due to swells of the sea. Vibrations of a ship with comparatively high frequencies of an order of 2-10 Hz (so-called running vibration of a ship), caused by operation of the screws and motors of a ship, occur with very small amplitudes of overloads and do not have any practical value on the strength of the transported vehicle.

The motion of a ship on waves consists of free and forced vibrations. Here the center of gravity of a ship shifts in orbits close to circular, with period equal to the period of waves τ and radius r equal to approximately half the height of a wave. Lateral overload, appearing in process of such vibration, can be calculated approximately by the formula

$$-n^0_{y_1} = 1 + \frac{4\pi^2 r}{g_0\tau^2}. \qquad (5.5)$$

Vibrations of a ship relative longitudinal and transverse axes, passing through its center of gravity, are usually accomplished with various periods and in practice they can be considered independently of each other. Vibrations relative to the longitudinal axis (rolling) occur with a frequency of an order of 0.15 Hz and with amplitudes from 10 to 40° (in stormy weather). Vibrations relative to the lateral axis (heaving) are characterized by frequencies of an order of 0.2 Hz and amplitudes of an order of 5°. The parameters of these vibrations are influenced greatly by the size of the ship and the condition of the sea. Values of additional overloads, caused by both rolling and heaving, depend on the location of the flight vehicle with respect to the center of gravity of the ship. Since rolling of a ship takes place with low frequencies, considerably lower than the frequencies of inherent bending oscillations in the construction of the vehicle, then these inertial forces have (according to the classification accepted by us) a static nature.

External influences in the case T_a bear a clearly expressed random nature. During movement over an uneven road the vibrations of a spring-suspended carriage with natural frequencies almost do not attenuate. Certain data [11] indicate that these oscillations have approximately identical intensity in all three directions. Besides it turns out that vibration components of transverse and lateral accelerations in such "established" conditions of oscillations do not exceed a unit in magnitude regardless of type of road. Besides these "usual" oscillations of carriages brief oscillations with large transverse overloads of an order of 3-4 and more are observed. These are caused by various types of random shocks. The magnitudes of these overloads depend mainly on the condition of the road and speed, and also on the type of shock absorption of the carriage. Since transportation of a flight vehicle by this form of transport is brief, then such random shocks will also present a basic danger to the strength of their construction.

Radical means for lowering the magnitudes of overloads in this case of loading T_a is the establishment of corresponding limitations on conditions of transportation. This selection of permissible speeds of transportation has to be conducted in each concrete case, proceeding from the strength of construction of the flight vehicle and considering that minimum permissible values of overloads should not be less than overloads corresponding to "usual" steady-state oscillations of the carriage. In first approximation the calculation of strength of construction of a flight vehicle for a given case of load can be conducted if we are given some typical profile of overload jump obtained from experiment or by means of using conditional profiles for unevenness of the road. For instance, it is possible to consider passage of several consecutive unevennessess of a sinusoidal form with a speed corresponding to lowest frequency of natural oscillations of the carriage (with the vehicle), an asymmetric incursion on an unevenness with a lateral slope of carriage, and so forth.

§ 5.3. Vibration of Light Vehicles

The dynamic calculation of construction of any vehicle for transport cases of load should begin with a calculation of static values of transverse forces $Q_c(x_1)$, bending moments $M_c(x_1)$, and support reactions R_1. The values of these forces (for instance, in a vertical plane) in the case of the presence of two supports I and II (Fig. 5.2) will be equal to (3.18), (3.19)

$$\left.\begin{aligned} Q_{yc}(x_1) &= Q^0_{\text{мк}}(x_1)\bar{n}^0_{y_1} + Q^\tau_{\text{мк}}(x_1)\bar{n}^\tau_{y_1}, \\ M_{yc}(x_1) &= M^0_{\text{мк}}(x_1)\bar{n}^0_{y_1} + M^\tau_{\text{мк}}(x_1)\bar{n}^\tau_{y_1} \end{aligned}\right\} \text{ when } x_1 < x_I, \tag{5.6}$$

$$\left.\begin{aligned} Q_{yc}(x_1) &= Q^0_{\text{мк}}(x_1)\bar{n}^0_{y_1} + Q^\tau_{\text{мк}}(x_1)\bar{n}^\tau_{y_1} + R_{Ic}, \\ M_{yc}(x_1) &= M^0_{\text{мк}}(x_1)\bar{n}^0_{y_1} + M^\tau_{\text{мк}}(x_1)\bar{n}^\tau_{y_1} + R_{Ic}(x_1 - x_I) \\ &\text{when } x_I < x_1 < x_{II}, \end{aligned}\right\} \tag{5.7}$$

$$\left.\begin{aligned} Q_{yc}(x_1) &= Q^0_{\text{мк}}(x_1)\bar{n}^0_{y_1} + Q^\tau_{\text{мк}}(x_1)\bar{n}^\tau_{y_1} + R_{Ic} + R_{IIc}, \\ M_{yc}(x_1) &= M^0_{\text{мк}}(x_1)\bar{n}^0_{y_1} + M^\tau_{\text{мк}}(x_1)\bar{n}^\tau_{y_1} + R_{Ic}(x_1 - x_I) + R_{IIc}(x_1 - x_{II}) \\ &\text{when } x_1 > x_{II}, \end{aligned}\right\} \tag{5.8}$$

where

$$
\left.\begin{aligned}
-R_{Ic} &= \frac{G_{\text{к}}}{x_{II}-x_{I}}\left[\bar{n}^{0}_{y_1}(x_{1\text{т}}-x_{II})+\bar{n}^{x}_{y_1}\left(\frac{J_{\text{к}}g_0}{G_{\text{к}}}-x_{1\text{т}}x_{II}\right)\right],\\
-R_{IIc} &= \frac{G_{\text{к}}}{x_{II}-x_{I}}\left[\bar{n}^{0}_{y_1}(x_{1\text{т}}-x_{I})+\bar{n}^{x}_{y_1}\left(\frac{J_{\text{к}}g_0}{G_{\text{к}}}-x_{1\text{т}}x_{I}\right)\right],\\
\bar{n}^{0}_{y_1} &= n^{0}_{y_1}+\frac{\ddot{\vartheta}}{g_0}x_{1\text{тв}}, \quad -\bar{n}^{x}_{y_1}=\frac{\ddot{\vartheta}}{g_0},\\
n^{0}_{y_1} &= -\left(1+\frac{\ddot{\eta}}{g_0}\right),
\end{aligned}\right\} \tag{5.9}
$$

and $x_{1\text{т}}$ - coordinate of center of gravity of transported part of vehicle, $G_{\text{к}}$ - its weight, and $J_{\text{к}}$ - mass moment of inertia of the vehicle relative to the origin of the coordinates, x_I and x_{II} - coordinates of sites of installation of vehicle supports. Unit values $Q^{0}_{\text{пк}}(x_1)$, $Q^{x}_{\text{пк}}(x_1)$, $M^{0}_{\text{пк}}(x_1)$ and $M^{x}_{\text{пк}}(x_1)$ are determined by formulas (3.19') with i = k. Analogously components of forces in lateral plane x_1z_1 are calculated, but with $\ddot{\vartheta}=0$.

Total values of static transverse forces $Q_c(x_1)$ and bending moments $M_c(x_1)$ also serve as the basis for selection of sites of distribution of supports along the length of the body of the flight vehicle. Usually these supports are mounted at those points of the construction where there are powerful frames which are able to take heavy radial forces - support reactions. Such points can be joints of sections, junctions of bottoms of carrier tanks with the conical casings, and sites of attachment to the body of power (motor) units. If there are no similar elements of construction near a support or they are insufficiently reliable, then the support reactions are distributed on two sections, applying support of the "rocking lever" type, or intermediate adjustable supports are installed. One of the transverse supports of the vehicle (usually rear) can simultaneously execute the function of longitudinal support (in the direction of axis x_1). In selecting the point for such a base support it is not difficult to find (from condition of preservation of values of power factors in the construction of the vehicle in permissible limits) the required position for the front support from formula (7)

$$x_l = \{\bar{n}^0_{y_1}[x_1 G_к (x_{1т} - x_{ll}) - x_{ll} M^0_{мк}(x_1)] + \\ + \bar{n}^x_{y_1}[x_1 (g_0 J_к - x_{1т} G_к x_{ll}) - x_{ll} M^x_{мк}(x_1)] + \\ + x_{ll} M_{доп}(x_1)\}/\{\bar{n}^0_{y_1}[G_к (x_{1т} - x_{ll}) + M^0_{мк}(x_1)] + \\ + \bar{n}^x_{y_1}[g_0 J_к - x_{1т} x_{ll} G_к + M^x_{мк}(x_1)] - M_{доп}(x_1)\}. \tag{5.10}$$

If in the case of position for supports selected in such a way the inherent frequencies of bend oscillations of the flight vehicle structure are low, one should additionally consider the influence of dynamic components of internal forces. The values of the latter will obviously be greatest when forcing frequencies are close to one of the frequencies of inherent bend oscillation of the transported part of the vehicle.

The equation of forced transverse elastic oscillations for the construction of the vehicle in this case of load will have the following form:

$$\frac{\partial^2}{\partial x_1^2}\left[B(x_1)\frac{\partial^2 y_1(x_1, t)}{\partial x_1^2}\right] + m(x_1)\frac{\partial^2 y_1(x_1, t)}{\partial t^2} + \\ + \frac{\gamma}{\omega}\frac{\partial^2}{\partial x_1^2}\left[B(x_1)\frac{\partial^3 y_1(x_1, t)}{\partial x_1^2 \partial t}\right] = -m(x_1)\frac{\partial^2 \xi(x_1, t)}{\partial t^2}, \tag{5.11}$$

where

$$\frac{\partial^2 \xi(x_1, t)}{\partial t^2} = \frac{d^2 \eta(t)}{\partial t^2} - \frac{d^2 \vartheta(t)}{dt^2}(x_{1тв} - x_1), \tag{5.12}$$

and $x_{1тв}$ - coordinate of center of gravity of the system (carriage) in the connected system of coordinates $O_1 x_1 y_1$. In accordance with the method of separation of variables we present sags of the vehicle structure $y_1(x_1, t)$ in the form (4.28).

$$y_1(x_1, t) = \sum_{n=1}^{\infty} S_n(t) f_n(x_1).$$

Multiplying the left and right sides of equation (11) by $f_n(x_1)$ and integrating it term by term by x_1 from 0 to 1, we obtain an ordinary differential equation, describing oscillations at the point of reduction:

$$\ddot{S}_n + 2h_n \dot{S}_n + \omega_n^2 S_n = -\frac{1}{M_n}\int_0^l m(x_1) f_n(x_1) \xi(x_1) dx_1 \tag{5.13}$$

$$(n = 1, 2, \ldots).$$

Here through M_n and ω_n we designate the reduced mass and frequency of inherent bending oscillations of a flight vehicle which is lying freely on articulated supports and through $f_n(x_1)$ - function of form of these oscillations. Boundary conditions in this case depend on the site of installation of the supports. If the latter (or one of them) are disposed on the ends of the body, then these conditions will consist of an equality to zero of transverse shifts and bending moments

$$f_n(x_1)=0 \text{ and } \frac{d^2 f_n(x_1)}{dx_1^2}=0$$

with $x_1=l$ and $x_1=0$ (or only with $x_1=l$).

If the supports are disposed at a certain distance from the ends of the vehicle, then the boundary conditions remain the same as for a beam with free ends (4.59), but in addition consideration is given to the conditions of joining which are imposed by the supports on deformation of the body of the vehicle. In the case of articulated supports these additional conditions will express the absence of sag in the structure of the flight vehicle at points of installation of the supports

$$f_n(x_1)=0 \text{ with } x_1=x_I, \quad x_1=x_{II}. \tag{5.14}$$

With harmonic forward oscillations of the carriage $\xi(t)=\eta(t)=\eta_m\sin(\omega_m t+\varphi_m)$ and δ = const the amplitudes of oscillations of acceleration of the reduction point will be determined by expressions (§ 4.2)

$$S_{nm}=\frac{\pi\eta_m}{\delta M_n}\int_0^l m(x) f_n(x)\,dx \quad \text{with} \quad \omega_m=\omega_n,$$

$$S_{nm}=\frac{\eta_m\int_0^l m(x) f_n(x)\,dx}{M_n\left[\left(1-\frac{\omega_m^2}{\omega_n^2}\right)^2+\frac{\delta^2\omega_m^2}{\pi^2\omega_n^2}\right]^{\frac{1}{2}}} \quad \text{with} \quad \omega_m\neq\omega_n. \tag{5.15}$$

For a case of pulse change in the acceleration of the carriage (for example, for the case T_a) function $\ddot{\eta}(t)$ can be approximated in the first approximation by formulas

$$\left.\begin{aligned}\ddot{\eta}(t)&=\eta_0\frac{t}{T_1} && \text{with } 0<t\leqslant T_1,\\ \ddot{\eta}(t)&=\eta_0\left(1-\frac{t-T_1}{T_2}\right) && \text{with } T_1\leqslant t<T_2+T_1.\end{aligned}\right\} \tag{5.16}$$

Then, using solution (4.13) of equation (13), we obtain for this case the following expressions for acceleration of reduction point:

$$\left.\begin{aligned}\ddot{S}_n(t)&=S_{nc}\frac{\omega_n}{T_1}\sin\omega_n t \quad \text{with } 0<t\leqslant T_1,\\ \ddot{S}_n(t)&=S_{nc}\left[\frac{1}{T_1}\sin\omega_n t-\frac{T_1+T_2}{T_1T_2}\sin\omega_n(t-T_1)\right]\omega_n\\ &\qquad \text{with } T_1<t<T_2+T_1,\end{aligned}\right\} \tag{5.17}$$

where

$$S_{nc}=\frac{1}{M_n}\eta_0.$$

When these accelerations are known it is not difficult to calculate the corresponding values of transverse force, bending moment, support reactions, and vibration component of transverse overload

$$\left.\begin{aligned}Q(x_1,t)&=Q_c(x_1,t)-\sum_{n=1}^{\infty}\ddot{S}_n(t)Q_{nx}(x_1),\\ M(x_1,t)&=M_c(x_1,t)-\sum_{n=1}^{\infty}\ddot{S}_n(t)M_{nx}(x_1).\end{aligned}\right\} \tag{5.18}$$

$$\left.\begin{aligned}R_I&=R_{Ic}-\frac{1}{x_{II}-x_I}\sum_{n=1}^{\infty}\ddot{S}_n\left[\int_0^l m(x_1)x_1f_n(x_1)\,dx_1-\right.\\ &\qquad\left.-x_{II}\int_0^l m(x_1)f_n(x_1)\,dx_1\right],\\ R_{II}&=R_{IIc}+\frac{1}{x_{II}-x_I}\sum_{n=1}^{\infty}\ddot{S}_n\left[\int_0^l m(x_1)x_1f_n(x_1)\,dx_1-\right.\\ &\qquad\left.-x_I\int_0^l m(x_1)f_n(x_1)\,dx_1\right],\\ \Delta n_{y_1 B}(x_1)&=-\frac{1}{g_0}\sum_{n=1}^{\infty}\ddot{S}_n f_n(x_1).\end{aligned}\right\} \tag{5.19}$$

Here (for the case T_a) the static values $Q_c(x_1, t)$, $M_c(x_1, t)$ and R_{Ic}, R_{IIc} are determined by formulas (6), (7), (8), and (9) with $\bar{n}^0_{y_1} = -\frac{1}{g_0}\ddot{\eta}$, $n^x_{y_1} = 0$, i.e., under the condition that the body of the carriage accomplishes only forward motion. If the magnitudes of the given total transverse forces or bending moments (18), or support reactions (19), exceed permissible values, then one should try to change the frequency ω_n, corresponding to the determining tone of oscillations of the vehicle, by means of changing the position of the supports (or setting up an additional support).

In conclusion let us note that selection of the point of location of supports along the length of the vehicle has important value for the strength of the actual carriage, which is especially special, since support reactions frequently are the basic load, determining the necessary carrier capacity of its structure.

§ 5.4. Dynamic Structural Load in the Case T_y

The case of load T_y embraces the process of raising flight vehicles, which are launched from a vertical or slanted position, into the initial launch position with help of special erectors (with a lifting boom, with a support-mast boom, with a lifting-directing boom, and others). The simplified dynamic arrangement of all these erectors can be presented in the form of a beam of variable section, revolving around a fixed axis under action of external moment, created by some drive (Fig. 5.3).

Fig. 5.3. Simplified dynamic model of a vehicle-erector system.

Inertial loads, appearing during a change of speed of the boom at the onset of turning (during raising) and especially at the time of stopping it (during lowering) depend strongly on the constructional shape of the system carrying out these operations (construction of the erector, type of drive) and on the dynamic characteristics of construction of the flight vehicle itself.

If the vehicle-erector system is rigid, then the greatest exploitational values of transverse overload in this case of load will be determined (at the beginning of raising and at the end of lowering) by the formula

$$-n_{y_1}(x_1) = 1 + \frac{\ddot{\vartheta}}{g_0}(l_0 - x_1).$$

Here $\ddot{\vartheta}$ designates the angular acceleration of the erector boom, and l_0 - distance from the axis of its turn to the origin of the connected system of coordinates (peak of the flight vehicle), equal to $l_0 = x_{н} + x_{1н}$.

If the rigidity of the erector boom, on the supports of which the vehicle lies, is comparatively small, then it is also necessary to consider additional accelerations caused by transverse elastic oscillations of the boom (with the vehicle). The latter can appear both during sharp application of tractive force of the driving mechanism and also when striking against the supports of the erector (during braking). Here the values of vibration accelerations will be determined not only by the speed of the boom at the time of contact with the supports, but also by the rigidity of the supports and the rigidity of construction of the boom itself.

Motion of this system we will consider in a motionless system of coordinates $\xi\eta\zeta$, the origin of which we will dispose in the center of rotation of the boom O (Fig. 5.3). Axis ξ we direct vertically upwards, axis η horizontally in the plane of rotation of the boom, and axis ζ - perpendicular to plane $\xi\eta$. Furthermore we introduce the auxiliary system of coordinates xyz with the origin in that same point O. Axis x we direct along the undeformed axis of rigidity of the boom, axis y - in plane $\xi\eta$ upwards.

We will designate transverse shifts of supports "B" and "H" of the vehicle (due to bending of the boom) in the system of coordinates xyz through $y(x_B)$ and $y(x_H)$. Assuming that the vehicle is an absolutely rigid body, it is possible to obtain the following expression for transverse overload $n_{y1}(x_1)$:

$$n_{y_1}(x_1) = -1 - \frac{\ddot{\vartheta}}{g_0}(l_0 - x_1) - \ddot{y}(x_H)\frac{x_1 - x_{1B}}{g_0 a} - \ddot{y}(x_B)\frac{x_{1H} - x_1}{g_0 a},$$

where $a = x_{1H} - x_{1B}$.

Values of ϑ are determined by the equation

$$J_0\ddot{\vartheta} + G(x_T\cos\vartheta - y_T\sin\vartheta) = M_B, \tag{5.20}$$

where G - total weight of the mobile part of the system, x_T and y_T - coordinates of its center of gravity, M_B - moment of all external forces applied to the system relative to the axis of rotation, J_0 - moment of inertia of mobile part of the system relative to this axis.

Using the method of separation of variables, we will represent sags of the erector boom y(x, t) in the form of an infinite series

$$y(x, t) = \sum_{p=1}^{\infty} K_p(t)\Phi_p(x),$$

where K_p is the unknown function of time t, and $\Phi_p(x)$ - function of coordinates, determining the form of transverse elastic oscillations of the boom which is loaded by the vehicle. This function should satisfy boundary conditions

$$\left.\begin{aligned} &\Phi_p(x) = 0 \text{ and } \frac{d^2\Phi_p(x)}{dx^2} = 0 \text{ with } x = 0,\\ &\frac{d^2\Phi_p(x)}{dx^2} = 0 \text{ and } \frac{d^3\Phi_p(x)}{dx^3} = 0 \text{ with } x = l_c \end{aligned}\right\} \tag{5.21}$$

and the ~~corresponding~~ condition of orthogonality, which will be written below. Here l_c - length of the boom.

The rates of vertical (with small values of ϑ) oscillations of any point x of the axis of rigidity of the erector boom and any point x_1 of the longitudinal axis of the vehicle in the motionless system of coordinates $\xi\eta$ (Fig. 5.3) are determined by expressions

$$\left.\begin{aligned}
&\dot{\xi}_c(x,\ t)=\dot{\vartheta}(t)\,x+\sum_{p=1}^{\infty}\dot{K}_p(t)\Phi_p(x),\\
&\dot{\xi}(x_1,\ t)=\sum_{p=1}^{\infty}\dot{K}_p(t)\Big[\Phi_p(x_н)\frac{x_1-x_{1в}}{a}+\\
&\qquad\qquad+\Phi_p(x_в)\frac{x_{1н}-x_1}{a}\Big]+\dot{\vartheta}(t)(l_0-x_1).
\end{aligned}\right\} \tag{5.22}$$

Kinetic and potential energy of the system being considered (vehicle-erector boom) in this case will be equal to

$$\left.\begin{aligned}
&T_0(t)=\frac{1}{2}\int_0^{l}m(x_1)\dot{\xi}^2(x_1,\ t)\,dx_1+\frac{1}{2}\int_0^{l_c}m_c(x)\dot{\xi}_c^2(x,\ t)\,dx,\\
&U_0(t)=\frac{1}{2}\int_0^{l_c}B_c(x)\Big[\frac{\partial^2 y(x,\ t)}{\partial x^2}\Big]^2dx+\\
&\qquad+\frac{1}{2}E_в\xi_c^2(x_в,\ t)+\frac{1}{2}E_н\xi_c^2(x_н,\ t),
\end{aligned}\right\} \tag{5.23}$$

where $E_в$ and $E_н$ - rigidity of supports "в" and "н" of the erector boom; $B_c(x)$ - rigidity of the erector boom during bending. Substituting these values T_0 and U_0 in the Lagrange equation (4.1) and taking functions $\vartheta(t)$ and $K_p(t)$ as generalized coordinates, we obtain the following system of ordinary differential equations for calculation of the unknown accelerations $\ddot{\vartheta}$ and $\ddot{K}_p$:

$$\left.\begin{aligned}
&J_0\ddot{\vartheta}+\sum_{p=1}^{\infty}a_p\ddot{K}_p+c_0\vartheta+\sum_{p=1}^{\infty}c_pK_p=0,\\
&M_p\ddot{K}_p+a_p\ddot{\vartheta}+\omega_p^2M_pK_p+c_p\vartheta+\sum_{n=1}^{\infty}c_{np}K_n=0,\\
&(p=1,\ 2,\ \ldots;\ n\neq p)
\end{aligned}\right\} \tag{5.24}$$

where

$$
\left.\begin{aligned}
M_p &= \int_0^{l_c} m_{\text{пр}}(x)\Phi_p^2(x)\,dx + 2\Phi_p(x_{\text{в}})\Phi_p(x_{\text{н}})\Delta m,\\
\Delta m &= \frac{1}{a} m(x_{\text{т}} - x_{\text{в}}) - \frac{1}{a^2} J(x_{\text{в}}),\\
m_{\text{пр}}(x) &= m_c(x) + \frac{J(x_{\text{в}})}{a^2}\delta(x - x_{\text{н}}) + \frac{J(x_{\text{н}})}{a^2}\delta(x - x_{\text{в}}),\\
c_{np} &= E_{\text{в}}\Phi_n(x_{\text{в}})\Phi_p(x_{\text{в}}) + E_{\text{н}}\Phi_n(x_{\text{н}})\Phi_p(x_{\text{н}}),\\
a_p &= \int_0^{l_c} m_{\text{пр}}(x)\Phi_p(x)\,x\,dx + \Delta m\,[\Phi_p(x_{\text{в}})x_{\text{н}} + x_{\text{в}}\Phi_p(x_{\text{н}})],\\
c_\vartheta &= E_{\text{в}}x_{\text{в}}^2 + E_{\text{н}}x_{\text{н}}^2, \quad -a = x_{\text{н}} - x_{\text{в}},\\
c &= x_{\text{в}}E_{\text{в}}\Phi_p(x_{\text{в}}) + x_{\text{н}}E_{\text{н}}\Phi_p(x_{\text{н}}),\\
\omega_p^2 &= \frac{1}{M_p}\int_0^{l_c} B_c(x)\left[\frac{d^2\Phi_p(x)}{dx^2}\right]^2 dx +\\
&\qquad + \frac{1}{M_p}\left[E_{\text{в}}\Phi_p^2(x_{\text{в}}) + E_{\text{н}}\Phi_p^2(x_{\text{н}})\right],
\end{aligned}\right| \tag{5.25}
$$

$m_c(x)$ - linear mass of erector boom, m - mass of flight vehicle, $J(x_i)$ - moment of inertia of this apparatus relative to the transverse axis passing through support x_i (i = в, н).

By the form of these formulas it is possible to establish that the calculation of forms of natural oscillations of the boom $\Phi_p(x)$ is conducted expediently taking into account apparent masses (of the vehicle) concentrated in sections $x = x_{\text{в}}$ and $x = x_{\text{н}}$ equal to $J(x_{\text{в}})/a^2$ and $J(x_{\text{н}})/a^2$ correspondingly, and with the following condition of orthogonality:

$$
\int_0^l m_{\text{пр}}(x)\Phi_p(x)\Phi_n(x)\,dx + \Delta m\,[\Phi_p(x_{\text{в}})\Phi_n(x_{\text{н}}) + \Phi_n(x_{\text{в}})\Phi_p(x_{\text{н}})] = 0
$$

$$
\text{at } n \neq p. \tag{5.26}
$$

The solution of the system of equations (24) is sought with the corresponding initial conditions. Let us substitute in (24)

$$
\vartheta(t) = \vartheta_0 e^{i\omega t}, \quad K_p(t) = K_{p0} e^{i\omega t}.
$$

Having expanded the determinant of the resulting system of equations, we find the algebraic equation (relative to ω^2) for the calculation of frequencies of natural oscillations of this system. With p = 1 it will have the form

$$\omega^4 A - \omega^2 B + C = 0,$$

where

$$A = J_0 M_p - \alpha_p^2,$$
$$B = c_\theta M_p + M_p \omega_p^2 J_0 + 2c_p \alpha_p \omega_p^2,$$
$$C = c_\theta M_p \omega_p^2 - c_p^2.$$

If the flight vehicle has comparatively low frequencies of inherent bend oscillations (on supports x_B and x_H), then during the calculation of $n_{y_1}(x_1)$ one should also consider additional vibration accelerations caused by elastic oscillations of the structure of the vehicle itself. For compiling the corresponding conditions of dynamic equilibrium of the system in this case it is necessary to record the expression for transverse speed of points of axis of the flight vehicle in the form

$$\dot{\xi}(x_1, t) = \dot{\vartheta}(t)\frac{1}{a}[x_{\text{н}}(x_1 - x_{1\text{в}}) + x_{\text{в}}(x_{1\text{н}} - x_1)] +$$
$$+ \frac{1}{a}\sum_{p=1}^{\infty}\dot{K}_p(t)[\Phi_p(x_{\text{н}})(x_1 - x_{1\text{в}}) + \Phi_p(x_{\text{в}})(x_{1\text{н}} - x_1)] + \dot{y}_1(x_1, t) \tag{5.27}$$

and to introduce additionally in the expression for potential energy of the system the potential deformation energy of the body of the flight vehicle (4.6). Let us present $y_1(x_1, t)$ in the form of series (4.28)

$$y_1(x_1, t) = \sum_{n=1}^{\infty} S_n(t) f_n(x_1),$$

where S_n - function of time, determining the amplitude of oscillations of the point of reduction of the vehicle structure, and $f_n(x_1)$ - function of inherent bending oscillations of this structure (as the beams of variable section, located on two supports $x_{1\text{в}}$ and $x_{1\text{н}}$), satisfying the corresponding boundary conditions, for instance, (4.59) or (4.60) and the condition of orthogonality (4.62).

Taking functions $S_n(t)$, $K_p(t)$, $\vartheta(t)$ as generalized coordinates and conducting transformations analogous to those which were executed during the derivation of equations (24), we obtain a system of equations of the form

$$
\left.
\begin{aligned}
&J_0\ddot{\vartheta} + \sum_{p=1}^{\infty} a_p\ddot{K}_p + c_0\vartheta + \sum_{n=1}^{\infty} c_n\ddot{S}_n + \sum_{p=1}^{\infty} c_p K_p = 0,\\
&M_p(\ddot{K}_p + 2h_p\dot{K}_p + \omega_p^2 K_p) + \sum_{n=1}^{\infty} a_{np}\ddot{S}_n + a_p\ddot{\vartheta} +\\
&\qquad + \sum_{n=1}^{\infty} c_{np}K_n + c_p\vartheta = 0,\\
&M_n(\ddot{S}_n + 2h_n\dot{S}_n + \omega_n^2 S_n) + \sum_{p=1}^{\infty} a_{np}\ddot{K}_p + c_n\ddot{\vartheta} = 0,
\end{aligned}
\right\} \qquad (5.28)
$$

where

$$
c_n = \int_0^l m(x_1)f_n(x_1)(l_0 - x_1)\,dx_1,
$$

$$
a_{np} = \frac{1}{e}\left\{[\Phi_p(x_{\text{н}}) - \Phi_p(x_{\text{в}})]\int_0^l m(x_1)f_n(x_1)x_1 + [\Phi_p(x_{\text{в}})x_{\text{н}} - \Phi_p(x_{\text{н}})x_{\text{в}}]\int_0^l m(x_1)f_n(x_1)\,dx_1\right\},
$$

$$
\omega_n^2 = \frac{1}{M_n}\int_0^l B(x_1)\left[\frac{d^2 f_n(x_1)}{dx_1^2}\right]^2 dx_1.
$$

Taking into account formula (27) and static component of transverse acceleration (acceleration due to gravity), we find the following expression for transverse overload in any section x_1 of the body of the flight vehicle:

$$
\begin{aligned}
-n_{y_1}(x_1) = 1 &+ \frac{x_1 - x_{1\text{в}}}{g_0 e}\left[x_{\text{н}}\ddot{\vartheta} + \sum_{p=1}^{\infty}\ddot{K}_p\Phi_p(x_{\text{н}})\right] +\\
&+ \frac{x_{1\text{н}} - x_1}{g_0 e}\left[x_{\text{в}}\ddot{\vartheta} + \sum_{p=1}^{\infty}\ddot{K}_p\Phi_p(x_{\text{в}})\right] + \frac{1}{g_0}\sum_{n=1}^{\infty}\ddot{S}_n f_n(x_1).
\end{aligned}
\qquad (5.29)
$$

During the calculation of transverse forces and bending moments, and also of loads on elements for securing the flight vehicle to the erector it is necessary to consider besides the mass forces proportional to n_{y_1}, the corresponding aerodynamic loads from the wind. Methods for calculation of the latter are expounded in Chapter VI.

§ 5.5. Equations for Oscillations of the Vehicle-Erector System

We will make up equations for elastic oscillations of the flight vehicle-erector system for a general case of motion. The kinetic energy of small oscillations of such a system will consist of the sum of kinetic energies of oscillations of all its sections, and potential - of the sum of potential energies of elastic deformation of the erector boom, the structure of the vehicle itself, and elastic bonds (upper "в" and lower "н"). The values of these energies are determined by speeds and shifts of points in the system.

Transverse shifts of points on the axis of rigidity of the boom due to bending in planes xy and xz can be represented in the form

$$\left.\begin{aligned} y_c(x,\ t) &= \sum_{m=1}^{\infty} K_{ym}(t)\Phi_{ym}(x), \\ z_c(x,\ t) &= \sum_{m=1}^{\infty} K_{zm}(t)\Phi_{zm}(x), \end{aligned}\right\} \qquad (5.30)$$

where Φ_{ym}, Φ_{zm} - independent systems of orthogonal functions of coordinate x, characterizing the forms of natural oscillations of the boom in planes xy and xz correspondingly, and K_{ym} and K_{zm} - indefinite functions only of time. Transverse shifts w of points of longitudinal axis of the flight vehicle in plane xy will be determined by shifts of those points $x_в$ and $x_н$ of the boom in which the upper and lower bonds are established (Fig. 5.4), deformations (extension - compression) of the bonds $w_в$ and $w_н$ themselves, and, finally, deformations during bending of the body of the flight vehicle. Expanding the sag in the construction of the flight vehicle in plane xy in forms of inherent bending oscillations

$$y_1(x_1,\ t)=\sum_{n=1}^{\infty} S_{yn}(t) f_{yn}(x_1),$$

we have approximately

$$w(x_1,\ t)=\sum_{n=1}^{\infty} S_{yn}(t) f_{yn}(x_1)+\sum_{n=1}^{\infty} K_{ym}(t)\Big[\Phi_{ym}(x_B)\frac{x_{1H}-x_1}{a}+$$
$$+\Phi_{ym}(x_H)\frac{x_1-x_{1B}}{a}\Big]+[w_B(t)+\Delta\vartheta(t)x_B]\frac{x_{1H}-x_1}{a}+$$
$$+[w_H(t)+\Delta\vartheta(t)x_H]\frac{x_1-x_{1B}}{a}, \qquad (5.31)$$

where $a=x_{1H}-x_{1B}$.

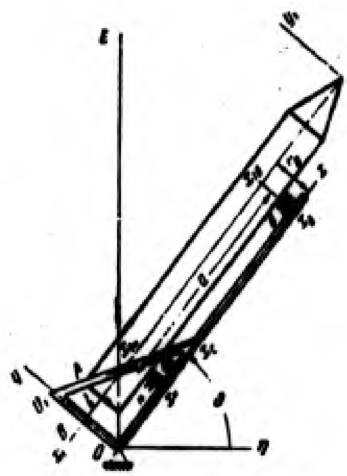

Fig. 5.4. More precise dynamic model of the vehicle-erector system.

Shifting of $v(x_1, t)$ points of longitudinal axis of the vehicle in lateral plane xz will be determined by shifts of points x_B and x_H of the boom, caused both by bending of the boom $z_C(x, t)$ and turning of its sections due to torsion $\phi(x, t)$, and also by shifts of supporting points x_{1B} and x_{1H}, caused by sags (v_B, v_H) of the corresponding bonds and bending $z_1(x_1, t)$ of the body of the flight vehicle in lateral plane. In this case, using the method of separation of variables we write

$$\left.\begin{aligned}\varphi(x,\ t)&=\sum_{p=1}^{\infty}\mu_p(t)\varkappa_p(x),\\ z_1(x_1,\ t)&=\sum_{n=1}^{\infty}S_{zn}(t)f_{zn}(x_1).\end{aligned}\right| \qquad (5.32)$$

where $\varkappa_p(x)$, $f_{zn}(x_1)$ - corresponding forms of natural oscillations of the boom and construction of the vehicle, μ_p, S_{zn} - unknown functions of time. Thus, considering various signs z and z_1

$$v(x_1, t) = \sum_{m=1}^{\infty} K_{zm}(t)\left[\Phi_{zm}(x_н)\frac{x_1 - x_{1в}}{a} + \Phi_{zm}(x_в)\frac{x_{1н} - x_1}{a}\right] + \sum_{p=1}^{\infty}\mu_p(t)\left[\varkappa_p(x_н)\frac{x_1 - x_{1в}}{a} r_н + \varkappa_p(x_в)\frac{x_{1н} - x_1}{a} r_в\right] - \sum_{n=1}^{\infty} S_{zn}(t) f_{zn}(x_1) + v_н(t)\frac{x_1 - x_{1в}}{a} + v_в(t)\frac{x_{1н} - x_1}{a}, \tag{5.33}$$

where r_H - distance between points x_H and $x_{1н}$, and r_B - distance between points x_B and $x_{1в}$.

Shifts of cross sections of the body of the flight vehicle in plane xy (along axis x) and (x_1, t) will occur due to bending strain u_H of the lower connection (with help of which the flight vehicle is held on the boom in a longitudinal direction), turning of section x_H, longitudinal deformation (extension-compression) $u_c(x, t)$ of the part of the boom located between the connection and support hinge O (Fig. 5.4), and longitudinal oscillations (compression-extension type) $u_1(x_1, t)$ of the structure of the flight vehicle. Presenting $u_1(x_1, t)$ and $u_c(x, t)$ in the form

$$\left.\begin{aligned} u_1(x_1, t) &= \sum_{p=1}^{\infty} T_p(t) X_p(x_1), \\ u_c(x, t) &= \sum_{k=1}^{\infty} T_{ck}(t) X_{ck}(x), \end{aligned}\right\} \tag{5.34}$$

where $X_p(x_1)$ - form of longitudinal oscillations in the structure of the flight vehicle, swivel-mounted in section $x_{1н}$, T_p, T_{ck} - functions of time, $X_{ck}(x)$ - form of longitudinal oscillations of the erector beam, we have

$$u(x_1, t) = \sum_{p=1}^{\infty} T_p(t) X_p(x_1) + \sum_{m=1}^{\infty} K_{ym}(t)\frac{d\Phi_{ym}(x)}{dx}\Big|_{x=x_н} r_н - \sum_{k=1}^{\infty} T_{ck}(t) X_{ck}(x_н) + u_н(t) + r_н \Delta\theta(t). \tag{5.35}$$

The kinetic energy of this system will be equal to

$$2T_0(t)=\int_0^l m(x_1)\dot{w}^2(x_1,t)\,dx_1+\int_0^l m(x_1)\dot{v}^2(x_1,t)\,dx_1+$$
$$+\int_0^l m_r(x_1)\dot{u}^2(x_1,t)\,dx_1+\int_0^L m_c(x)[\dot{y}_c(x,t)+x\Delta\dot{\theta}(t)]^2\,dx+$$
$$+\int_0^L m_c(x)\dot{u}_c^2(x,t)\,dx+\int_0^L m_c(x)\dot{z}_c^2(x,t)\,dx+\int_0^L i(x)\dot{\varphi}^2(x,t)\,dx, \qquad (5.36)$$

where l - length of the flight vehicle, L - length of boom, $m_c(x)$ - distributed mass of boom, $m(x_1)$ - linear mass of the structure of the flight vehicle, M_c - mass of the boom. If the boom is reinforced by various struts, for instance it has the form shown in Fig. 5.4, then to expression (36) one should add the kinetic energy of these additional sections ΔT_0. For sufficiently rigid struts it is possible to write approximately (with $l_A \gg l_B$)

$$2\Delta T_0(t)=M_{0c}\Delta\dot{l}_A^2+J_A(0)\Delta\dot{\theta}^2(t)+\frac{1}{l_A^2}J_{Ay}\dot{z}_c^2(x_c,t)+$$
$$+\frac{1}{l_A^2}J_{Az}[\dot{y}_c(x_c,t)\cos\alpha+\dot{u}_c(x_c,t)\sin\alpha]^2. \qquad (5.37)$$

Here l_A - length of strut A; J_{Ay}, J_{Az} - mass moments of inertia of strut A with respect to point O_1 (in planes xy and xz), α - angle of inclination of the strut to axis x, M_{0c} - reduced mass of the strut.

We will designate the rigidity of connections "B" and "H" in the direction of axis y (during extension-compression) through c_{By} and c_{Hy}, and in plane yz (during bending) - through c_{Bz} and c_{Hz} correspondingly. Let us assume that the flight vehicle is attached to the erector in a longitudinal direction (along axis x_1) by means of connection "H," the rigidity of which in this direction in plane (xy) will designate by c_{Hx}. Let us assume that the rigidity of the system of beams B and A can be described approximately by the given rigidity of one strut A for compression - extension (in plane xy), equal to c_x, and by flexural c_z (in plane xz). It is understood that these rigidities can be different for cases of raising the erector boom and for cases of mounting the flight vehicle on a launch pad when additional fastening of joint O_1 or strut A is possible.

Potential energy of the considered system will be determined by expression

$$2U_0 = c_x \Delta l_A^2 + c_{нx} u_н^2 + c_{нy} w_н^2 + c_{вy} w_в^2 + c_{нz} v_н^2 + \int_0^L E(x) F_c(x) \times \times \left[\frac{\partial u_c(x)}{\partial x}\right]^2 dx + c_z z_c^2(x_c) + c_{вz} v_в^2 + + \int_0^l B(x_1) \left[\frac{\partial^2 y_1(x_1)}{\partial x_1^2}\right]^2 dx_1 + \int_0^l E(x_1) F_c(x_1) \left[\frac{\partial u_1(x_1)}{\partial x_1}\right]^2 dx_1 + + \int_0^l B(x_1) \left[\frac{\partial^2 z_1(x_1)}{\partial x_1^2}\right]^2 dx_1 + \int_0^L B_y(x) \left[\frac{\partial^2 y_c(x)}{\partial x^2}\right]^2 dx + + \int_0^L B_z(x) \left[\frac{\partial^2 z_c(x)}{\partial x^2}\right]^2 dx + \int_0^L G(x) I_p(x) \left[\frac{\partial \varphi(x)}{\partial x}\right]^2 dx. \qquad (5.38)$$

In this formula the following designations are used: $B(x_1)$ and $B_{c1}(x)$ - flexual rigidity of body of flight vehicle and boom (B_{cz} - in plane xz, B_{cy} - in plane xy), Δl_A - change in length of strut due to deformation of the boom

$$\Delta l_A(t) = f[y_c(x_c, t), u_c(x_c, t)]. \qquad (5.39)$$

Placing the expressions for shifts (and speeds) (30), (31), (33), and (35) in the formulas of kinetic (36) and potential (38) energies and taking all the functions only of time t as generalized coordinates $q_n(t)$, we obtain from (4.1) a system of ordinary second order differential equations with constant coefficients describing oscillations of the erector jointly with the flight vehicle under the action of external disturbing forces.

Since plane $\xi\eta$ is the plane of symmetry of the particular dynamic erector - flight vehicle system, then the shown system of equations is broken up into two independent subsystems. One of these subsystems will determine longitudinal-bending oscillations

oscillations of the system in the plane of symmetry, and the second - bending-torsional oscillations of the system outside the plane of symmetry.

The first subsystem will consist of equations describing small angular oscillations of the system relative to an undisturbed state (20)

$$J\,\Delta\ddot{\vartheta} + \sum_{m=1}^{\infty} a_{m0}\ddot{K}_{ym} + \sum_{k=1}^{\infty} \ddot{T}_{ck}a_{k0} + \sum_{p=1}^{\infty} a_{p0}\ddot{T}_{p} + \sum_{n=1}^{\infty} a_{n0}\ddot{S}_{yn} + a_{в0}\ddot{w}_{в} + a_{н0}\ddot{w}_{н} + a_{u0}\ddot{u}_{н} = Q_{0}; \quad (5.40)$$

bending oscillations of the body of the flight vehicle in plane x_1y_1

$$M_{n}(\ddot{S}_{yn} + 2h_{yn}\dot{S}_{yn} + \omega_{yn}^{2}S_{yn}) + \sum_{m=1}^{\infty} a_{mn}\ddot{K}_{ym} + a_{вn}\ddot{w}_{в} + a_{нn}\ddot{w}_{н} + a_{n0}\Delta\ddot{\vartheta} = Q_{syn}; \quad (5.41)$$

longitudinal oscillations of the boom

$$M_{ck}(\ddot{T}_{ck} + 2h_{ck}\dot{T}_{ck} + \omega_{ck}^{2}T_{ck}) + \sum_{m=1}^{\infty} a_{mk}\ddot{K}_{ym} + \sum_{p=1}^{\infty} a_{pk}\ddot{T}_{p} + a_{uk}\ddot{u}_{н} + a_{k0}\,\Delta\ddot{\vartheta} = Q_{Tk}; \quad (5.42)$$

longitudinal oscillations of the body of the flight vehicle

$$M_{p}(\ddot{T}_{p} + 2h_{p}\dot{T}_{p} + \omega_{p}^{2}T_{p}) + \sum_{m=1}^{\infty} a_{mp}\ddot{K}_{ym} + \sum_{k=1}^{\infty} a_{kp}\ddot{T}_{ck} + a_{up}\ddot{u}_{н} + a_{p0}\Delta\ddot{\vartheta} = Q_{Tp}; \quad (5.43)$$

bending oscillations of the boom in plane xy

$$M_{ym}(\ddot{K}_{ym} + 2h_{ym}\dot{K}_{ym} + \omega_{ym}^{2}K_{ym}) + \sum_{n=1}^{\infty} a_{nm}\ddot{S}_{yn} + \sum_{p=1}^{\infty} a_{pm}\ddot{T}_{p} + a_{m0}\ddot{\vartheta} + a_{вm}\ddot{w}_{в} + a_{нm}\ddot{w}_{н} + a_{um}\ddot{u}_{н} = Q_{kym}; \quad (5.44)$$

extension-compression oscillations of the upper connection

$$m_{в}(\ddot{w}_{в} + \omega_{yв}^{2}w_{в}) + \sum_{m=1}^{\infty} a_{mв}\ddot{K}_{ym} + \sum_{n=1}^{\infty} a_{nв}\ddot{S}_{yn} + a_{нв}\ddot{w}_{н} + a_{в0}\Delta\ddot{\vartheta} = Q_{в}; \quad (5.45)$$

extension-compression oscillations of the lower connection

$$m_{\text{н}}(\ddot{w}_{\text{н}}+\omega_{y\text{н}}^{2}w_{\text{н}})+\sum_{m=1}^{\infty}a_{m\text{н}}\ddot{K}_{ym}+\sum_{n=1}^{\infty}a_{n\text{н}}\ddot{S}_{yn}+a_{\text{нв}}\ddot{w}_{\text{в}}+a_{\text{н}0}\Delta\ddot{\vartheta}=Q_{\text{н}}, \tag{5.46}$$

bending of the lower connection in plane xy

$$M(\ddot{u}_{\text{н}}+\omega_{\text{н}x}^{2}u_{\text{н}})+\sum_{m=1}^{\infty}a_{um}\ddot{K}_{ym}+\sum_{p=1}^{\infty}a_{\text{н}p}\ddot{T}_{p}+$$
$$+\sum_{k=1}^{\infty}a_{\text{н}k}\ddot{T}_{ck}+a_{\text{н}0}\Delta\ddot{\vartheta}=Q_{u\text{н}}. \tag{5.47}$$

The second subsystem will consist of equations describing bending oscillations of the body of the flight vehicle in plane x_1z_1

$$M_{n}(\ddot{S}_{zn}+2h_{zn}\dot{S}_{zn}+\omega_{zn}^{2}S_{zn})+\sum_{m=1}^{\infty}d_{mn}\ddot{K}_{zm}+$$
$$+d_{\text{в}n}\ddot{v}_{\text{в}}+d_{\text{н}n}\ddot{v}_{\text{н}}+\sum_{p=1}^{\infty}d_{pn}\ddot{\mu}_{p}=Q_{szn}; \tag{5.48}$$

bending oscillations of the boom in plane xz

$$M_{\text{к}m}(\ddot{K}_{zm}+2h_{zm}\dot{K}_{zm}+\omega_{zm}^{2}K_{zm})+\sum_{n=1}^{\infty}d_{nm}\ddot{S}_{zn}+$$
$$+d_{\text{в}m}\ddot{v}_{\text{в}}+d_{\text{н}m}\ddot{v}_{\text{н}}+\sum_{p=1}^{\infty}d_{pm}\ddot{\mu}_{p}=Q_{kzm}; \tag{5.49}$$

torsion of the boom with respect to axis x

$$J_{p}(\ddot{\mu}_{p}+2h_{p\mu}\dot{\mu}_{p}+\omega_{p\mu}^{2}\mu_{p})+\sum_{n=1}^{\infty}d_{pn}\ddot{S}_{zn}+$$
$$+\sum_{m=1}^{\infty}d_{pm}\ddot{K}_{zm}+d_{\text{в}p}\ddot{v}_{\text{в}}+d_{\text{н}p}\ddot{v}_{\text{н}}=Q_{\mu p}; \tag{5.50}$$

bending of the upper support connection in plane xz

$$m_{\text{в}}(\ddot{v}_{\text{в}}+\omega_{z\text{в}}^{2}v_{\text{в}})+\sum_{n=1}^{\infty}d_{\text{в}n}\ddot{S}_{zn}+\sum_{m=1}^{\infty}d_{\text{в}m}\ddot{K}_{zm}+$$
$$+\sum_{p=1}^{\infty}d_{\text{в}p}\ddot{\mu}_{p}+d_{\text{нв}}\ddot{v}_{\text{н}}=Q_{\text{в}v}. \tag{5.51}$$

bending of lower support connection in plane xz

$$m_{\text{н}}(\ddot{v}_{\text{н}}+\omega^2_{z\text{н}}v_{\text{н}})+\sum_{n=1}^{\infty} d_{\text{н}n}\ddot{\zeta} \quad +\sum_{m=1}^{\infty} d_{\text{н}m}\ddot{K}_{zm}+$$
$$+\sum_{p=1}^{\infty} d_{\text{н}p}\ddot{\mu}_p+d_{\text{нв}}\ddot{v}_{\text{в}}=Q_{\text{н}v}. \tag{5.52}$$

In these equations M and ω with subscripts designate generalized masses and angular frequencies of inherent elastic oscillations of the corresponding partial systems, the description of which is given in § 5.7. Remaining coefficients are equal to

$$\left.\begin{aligned}
&a_{ij}=a_{ji},\quad d_{ij}=d_{ji},\\
&a_{mn}=a_{\text{в}n}\Phi_{ym}(x_{\text{в}})+a_{\text{н}n}\Phi_{ym}(x_{\text{н}}),\\
&a_{\text{в}n}=\frac{1}{a}\int_0^l m(x_1)f_{yn}(x_1)(x_{1\text{н}}-x_1)\,dx_1,\\
&a_{\text{н}n}=\frac{1}{a}\int_0^l m(x_1)f_{yn}(x_1)(x_1-x_{1\text{в}})\,dx_1,\\
&a_{mk}=-a_{\text{н}m}X_{ck}(x_{\text{н}}),\\
&a_{pk}=-a_{\text{н}p}X_{ck}(x_{\text{н}}),\\
&a_{k0}=-r_{\text{н}}MX_{ck}(x_{\text{н}}),\\
&a_{m0}=(m_{\text{в}}x_{\text{в}}+a_{\text{нв}}x_{\text{н}})\Phi_{ym}(x_{\text{в}})+\int_0^L m(x)\,x\Phi_{ym}(x)\,dx+\\
&\qquad+(m_{\text{н}}x_{\text{н}}+a_{\text{нв}}x_{\text{в}})\Phi_{ym}(x_{\text{н}})+r_{\text{н}}a_{\text{н}0}\left.\frac{d\Phi_{ym}(x)}{dx}\right|_{x=x_{\text{н}}},\\
&a_{p0}=r_{\text{н}}\int_0^l m_x(x_1)X_p(x_1)\,dx_1,\\
&a_{n0}=\frac{1}{a}\int_0^l m(x_1)f_{yn}(x_1)[x_{\text{в}}(x_{1\text{н}}-x_1)+x_{\text{н}}(x_1-x_{1\text{в}})]\,dx_1,\\
&a_{\text{в}0}=\frac{1}{a}Mx_{\text{в}}(x_{1\text{н}}-x_{1\text{т}}),\\
&a_{\text{н}0}=\frac{1}{a}Mx_{\text{н}}(x_{1\text{т}}-x_{1\text{в}}),\\
&a_{\varphi 0}=r_{\text{н}}M,\\
&a_{\text{н}k}=-MX_{ck}(x_{\text{н}}),\\
&a_{mp}=r_{\text{н}}a_{\text{н}p}\left.\frac{d\Phi_{ym}(x)}{dx}\right|_{x=x_{\text{н}}},\\
&a_{\text{н}p}=\int_0^l m_x(x_1)X_p(x_1)\,dx_1,\\
&a_{\text{в}m}=m_{\text{в}}\Phi_{ym}(x_{\text{в}})+a_{\text{нв}}\Phi_{ym}(x_{\text{н}}),\\
&a_{\text{н}m}=m_{\text{н}}\Phi_{ym}(x_{\text{н}})+a_{\text{нв}}\Phi_{ym}(x_{\text{в}}),\\
&a_{\varphi m}=r_{\text{н}}M\left.\frac{d\Phi_{ym}(x)}{dx}\right|_{x=x_{\text{н}}},
\end{aligned}\right\} \tag{5.53}$$

$$
\left.\begin{aligned}
a_{\text{нв}} &= \frac{1}{a^2}\left[M\left(x_{1\text{т}}x_{1\text{н}} - x_{1\text{н}}x_{1\text{в}} + x_{1\text{т}}x_{1\text{в}}\right) - J_0\right],\\
J_0 &= \int_0^l m(x_1)\,x_1^2\,dx_1,\\
M &= \int_0^l m(x_1)\,dx_1,\\
d_{pn} &= -\frac{r_{\text{н}}}{a}\varkappa_p(x_{\text{н}})\int_0^l m(x_1)\,f_{zn}(x_1)(x_1 - x_{1\text{в}})\,dx_1 -\\
&\quad - \frac{r_{\text{в}}}{a}\varkappa_p(x_{\text{в}})\int_0^l m(x_1)\,f_{zn}(x_1)(x_{1\text{н}} - x_1)\,dx_1,\\
d_{\text{в}m} &= m_{\text{в}}\Phi_{zm}(x_{\text{в}}) + a_{\text{нв}}\Phi_{zm}(x_{\text{н}}),\\
d_{\text{н}m} &= m_{\text{н}}\Phi_{zm}(x_{\text{н}}) + a_{\text{нв}}\Phi_{zm}(x_{\text{в}}),\\
d_{nm} &= d_{\text{в}n}\Phi_{zm}(x_{\text{в}}) + d_{\text{н}n}\Phi_{zm}(x_{\text{н}}),\\
d_{\text{в}n} &= -\frac{1}{a}\int_0^l m(x_1)\,f_{zn}(x_1)(x_{1\text{н}} - x_1)\,dx_1,\\
d_{\text{н}n} &= -\frac{1}{a}\int_0^l m(x_1)\,f_{zn}(x_1)(x_1 - x_{1\text{в}})\,dx_1,\\
d_{pm} &= m_{\text{н}}r_{\text{н}}\varkappa_p(x_{\text{н}})\Phi_{zm}(x_{\text{н}}) + m_{\text{в}}r_{\text{в}}\varkappa_p(x_{\text{в}})\Phi_{zm}(x_c) +\\
&\quad + a_{\text{нв}}r_{\text{н}}\varkappa_p(x_{\text{н}})\Phi_{zm}(x_{\text{в}}) + a_{\text{нв}}r_{\text{в}}\varkappa_p(x_{\text{в}})\Phi_{zm}(x_{\text{н}}),\\
d_{p\text{в}} &= m_{\text{в}}r_{\text{в}}\varkappa_p(x_{\text{в}}) + a_{\text{нв}}r_{\text{н}}\varkappa_p(x_{\text{н}}),\\
d_{\text{н}p} &= m_{\text{н}}r_{\text{н}}\varkappa_p(x_{\text{н}}) + a_{\text{нв}}r_{\text{в}}\varkappa_p(x_{\text{в}}),\\
J &= J_{\text{c}}(0) + m_{\text{в}}x_{\text{в}}^2 + m_{\text{н}}x_{\text{н}}^2 + Mr_{\text{н}}^2,\\
m_{\text{в}} &= \frac{1}{a^2}J(x_{1\text{н}}),\\
m_{\text{н}} &= \frac{1}{a^2}J(x_{1\text{в}}),\\
J(x_{1\text{н}}) &= \int_0^l m(x_1)(x_{1\text{н}} - x_1)^2\,dx_1,\\
J(x_{1\text{в}}) &= \int_0^l m(x_1)(x_1 - x_{1\text{в}})^2\,dx_1.
\end{aligned}\right\}
\quad \text{(5.53 Cont'd)}
$$

§ 5.6. Reduction of System of Equations in Generalized Coordinates to Normal Coordinates

A particular solution of a system of uniform differential equations in generalized coordinates, given in the preceding section, will be harmonic functions of the form (with $h_{ij} = 0$)

$$\left.\begin{array}{c} S_{in}(t)=S_{inm}\sin\omega_m t, \\ K_{im}(t)=K_{imm}\sin\omega_m t, \\ \cdots\cdots\cdots\cdots \\ i=y,\ z, \end{array}\right\} \quad (5.54)$$

where ω_m - frequency of free transverse oscillations of the system, and S_{inm}, K_{imm} - coefficients, determining the amplitudes of oscillations of generalized coordinates.

Substituting the values of the cited functions in corresponding equations and reducing by $\sin \omega_m t$, we obtain a system of algebraic equation in which the unknowns will be ω_m^2, S_{inm}, K_{imm}, etc. The condition of obtaining a nontrivial solution of this system of equations relative to the enumerated unknowns consists of equality to zero of the determinant of the system composed of coefficients with S_{inm}, K_{imm}, etc. By expanding this determinant we obtain a frequency equation of the k_1-th degree.

The roots of this equation will be equal to the squares of frequencies of natural oscillations of the system. For each such root, i.e., for each frequency, there will be a corresponding specific distribution of amplitudes of oscillations of generalized coordinates. The totality of these amplitudes determines the form of natural oscillations of the system on the whole (normal form of oscillations) corresponding to the given natural frequency ω_m. The number of such forms is equal to the number of degrees of freedom, i.e., k_1.

In the case of a large number of degrees of freedom of the system the expansion of the determinant is a complex mathematical problem. At present there are a number of methods for the solution of the stated frequency equation (methods of A. N. Krylov, A. M. Danilevskiy, Sh. Ye. Mikeladze, and others) which are presented in many works on the theory of oscillations. However, all of them are very labor-consuming and complicated.

For a conservative system with a finite number of degrees of freedom, the matrix of coefficients of which is symmetric, the approximate values of frequencies and forms of natural oscillations can be found with success by the method of successive approximations. This method is very simple and with the use of high speed electronic digital computers can give sufficiently high accuracy of calculations, not only of the lowest, but also of the highest frequencies.

At first we will consider that with this method one can determine approximate values of frequencies and forms of natural oscillations of a system, described by equations of the form

$$\sum_{j=1}^{k_1} a_{ij}\ddot{q}_j + b_{ii}q_i = 0 \qquad (i = 1, 2, \ldots, k_1). \tag{5.55}$$

Any oscillation of such a system, having k_1 degrees of freedom, can be presented approximately in the form of linear superposition k_1 of the principal normal oscillations

$$q_j = \sum_{k=1}^{k_1} A_{jk}p_k \qquad (j = 1, 2, \ldots, k_1), \tag{5.56}$$

where p_k - normal coordinates of the k-th tone, and A_{jk} - coefficients, determining the contribution of each main oscillation in the overall value q_j. The first index for this coefficient designates the number of the partial system, and the second - order of form (number of tone of inherent normal oscillations).

If the system accomplishes oscillations with a frequency coinciding with one of the natural frequencies, i.e., it accomplishes one of the main oscillations, then in the absence of damping the generalized coordinates will be changed by harmonic law

$$q_j = A_{jk}\sin\omega_k t.$$

Since this value q_j satisfies equation (55), then it is possible to write that

$$\frac{1}{\omega_k^2} b_{ii}A_{ik} = \sum_{j=1}^{k_1} a_{ij}A_{jk}. \tag{5.57}$$

Substituting (56) in (55) and using the last relationship, we obtain the following equations:

$$\sum_{k=1}^{k_1} \frac{1}{\omega_k^2} b_{ii}(\ddot{p}_k + \omega_k^2 p_k) A_{ik} = 0 \quad (i = 1, 2, \ldots, k_1). \tag{5.58}$$

We will multiply each of these equations by A_{im} and will sum them for all i. As a result we will have

$$M_m(\ddot{p}_m + \omega_m^2 p_m) + \sum_{k=1}^{k_1} \sum_{i=1}^{k_1} b_{ii} A_{ik} A_{im} \frac{1}{\omega_k^2} (\ddot{p}_k + \omega_k^2 p_k) = 0 \tag{5.59}$$
$$\text{with } k \neq m \quad (m = 1, 2, \ldots, k_1),$$

where M_m designates the generalized mass, equal to

$$M_m = \sum_{i=1}^{k_1} b_{ii} A_{im}^2 \frac{1}{\omega_m^2}. \tag{5.60}$$

So that in the system of these equations (59) a connection between coordinates p_m and p_k is absent, i.e., so that free oscillations of the particular complex system are described by a system of second order differential equations with divided variables of the type

$$\ddot{p}_m + \omega_m^2 p_m = 0 \quad (m = 1, 2, \ldots, k_1), \tag{5.61}$$

it is necessary to fulfill the following condition of orthogonality of forms of free oscillations relative to potential energy:

$$\sum_{i=1}^{k_1} b_{ii} A_{ik} A_{im} = 0 \quad \text{with } k \neq m. \tag{5.62}$$

Here, obviously, the condition of orthogonality of forms of free oscillations relative to kinetic energy of the system will be executed automatically

$$\sum_{i=1}^{k_1} \sum_{j=1}^{k_1} a_{ij} A_{jk} A_{jm} = 0 \quad \text{with } k \neq m. \tag{5.63}$$

Having established the condition of orthogonality of forms of oscillations of the system, it is possible to proceed to the determination of ω_m and A_{im}. For this we will write the system k_1 of algebraic equations (57) in the form

$$\frac{A_{im}}{\omega_m^2} = y_{im} = \frac{1}{b_{ii}} \sum_{j=1}^{k_1} a_{ij} A_{jm} \qquad (i = 1, 2, \ldots, k_1). \tag{5.64}$$

We will take as the initial form of natural oscillations of the system some totality of values A_{jm}, for instance $A_{im}=1$. Placing them in the right side of formula (64), we calculate the values y_{im}. Since here the amplitudes A_{im} are determined with an accuracy of an arbitrary constant general factor, then they can be disposed in such a manner as to fulfill the condition

$$\sum_{i=1}^{k_1} A_{im}^2 = 1. \tag{5.65}$$

The system of amplitudes A_{im}, and consequently also the form of natural oscillations of the m-th tone which satisfies a similar equality is usually called standardized. From this condition we find the magnitude of the square of frequency of natural oscillations of the system

$$\omega_m^2 = \frac{1}{\left[\sum_{i=1}^{k_1} y_{im}^2\right]^{\frac{1}{2}}} \qquad (m = 1, 2, \ldots, k_1), \tag{5.66}$$

and by the formula

$$A_{im} = \omega_m^2 y_{im} \tag{5.67}$$

- values of amplitudes A_{im} of the first approximation. Then the calculations are repeated by the same scheme, taking the resulting A_{im} as new initial values for determination of the second approximation of ω_m and A_{im}, etc., as long as the approximation will not coincide with A_{im} of the $(k+1)$-th approximation with the assigned accuracy. As it is known, for equations in reverse form this process converges. Thus we obtain the frequency and standardized form of natural oscillations of a system of the first tone $(m=1)$.

If the initial system of equations has the form

$$\sum_{j=1}^{k_1} a_{ij}\ddot{q}_j + \sum_{j=1}^{k_1} b_{ij} q_j = 0 \qquad (i = 1, 2, \ldots, k_1). \tag{5.68}$$

then the corresponding system of algebraic equations (64) is presented in the form

$$y_{im} = \frac{A_{im}}{\omega_m^2} = \frac{\sum_{j=1}^{k_1} a_{ij}A_{jm}}{b_{ii}\left(1+\sum_{j=1}^{k_1} b_{ij}A_{jm}/b_{ii}A_{im}\right)}, \tag{5.68'}$$

where in the sum $\sum_{j=1}^{k_1} b_{ij}A_{jm}$ member b_{ii} is absent. In this case the condition of orthogonality will be

$$\sum_{i=1}^{k_1}\sum_{j=1}^{k_1} b_{ij}A_{jk}A_{im} = 0 \quad \text{with } k \neq m. \tag{5.69}$$

During determination of ω_m and A_{im} of the highest tones the condition of orthogonality of the form of the unknown m-th tone to all forms of oscillations of preceding tones (62) and (69) is used for obtaining the necessary correction coefficients Δ_{mk}.

We will designate by $\bar{A}_{im}$ the corrected value of A_{im}

$$\bar{A}_{im} = A_{im} + \sum_{k=1}^{m-1} A_{ik}\Delta_{mk}. \tag{5.70}$$

From condition (69) it follows that

$$\Delta_{mk} = -\frac{\sum_{i=1}^{k_1}\sum_{j=1}^{k_1}\left[b_{ij}A_{jk}A_{im} + \sum_{n=1}^{m-1} b_{ij}A_{in}A_{jk}\Delta_{mn}\right]}{\sum_{i=1}^{k_1}\sum_{j=1}^{k_1} b_{ij}A_{ik}^2} \quad \text{with } k \neq n, \tag{5.71}$$

where A_{jk} are the already known values of amplitudes. For a case when the condition of orthogonality is written in the form (62),

$$\Delta_{mk} = -\frac{\sum_{i=1}^{k_1}\left[b_{ii}A_{ik}A_{im} + \sum_{n=1}^{m-1} b_{ii}A_{in}A_{ik}\Delta_{mn}\right]}{\sum_{i=1}^{k_1} b_{ii}A_{ik}^2} \quad \text{with } k \neq n. \tag{5.72}$$

Accuracy of calculation in such a way of A_{im} and ω_m for the highest tones depends on the accuracy of fulfillment of the condition of orthogonality and will decrease with approximation of m to k_1

due to the systematic accumulation of error. Therefore it is expedient during calculation of the dynamic characteristics of a system to reject the highest frequencies. This method of consecutive approximations converges very rapidly if there is a large difference between partial frequencies of the system. Here the fundamental frequencies of the system frequently turn out to be close to partial frequencies.

The totality of frequencies of ω_m determines the spectrum of frequencies of natural oscillations of the system. Their numeration is carried out in increasing order. Partial frequencies are always found between fundamental frequencies. Whereas partial frequencies and forms of natural oscillations depend on the selection of generalized coordinates, fundamental frequencies of the system are invariants, i.e., do not depend on selection of the system of coordinates. Usually only part of this spectrum of frequencies has practical value during the investigation of dynamic loads.

§ 5.7. Dynamic Characteristics of Partial Systems

In § 5.5 the system of differential equations with a limited number of degrees of freedom will describe well the dynamic properties of the system flight vehicle - erector only in the case of special selection of all functions characterizing the form of relative oscillations of separate sections of the system. These functions from coordinates should be fundamental functions of corresponding partial systems, namely forms of natural oscillations of systems obtained from the basic system vehicle-erector by means of "freezing" of all its links besides the considered link. "Freezing" in this case is reduced to the fact that oscillations of these sections are considered only as the motion of absolutely rigid bodies. In other words, the influence of "frozen" sections on partial frequency and form of oscillations of the considered link is considered only by means of introduction of the corresponding apparent additional masses in places of joining these sections.

For the body of a flight vehicle which is located on articulated supports $x_{1\text{в}}>0$ and $x_{1\text{н}}<l$ the forms of transverse partial oscillations $f_{yn}(x_1)$, $f_{zn}(x_1)$ will coincide with the forms of natural transverse oscillations, satisfying equation (4.29), boundary conditions (4.59), and additional conditions (14). Functions $X_p(x_1)$, representing the forms of longitudinal partial oscillations of the body of a vehicle which is secured on hinges in section $x_{1\text{н}}$, are determined by equation

$$\frac{d}{dx_1}\left[E(x_1)F_c(x_1)\frac{dX_p(x_1)}{dx_1}\right]=-m_x(x_1)\omega_p^2X_p(x_1)\ (p=1,2,\ldots) \tag{5.73}$$

with boundary conditions

$$\frac{dX_p(x_1)}{dx_1}=0 \text{ with } x_1=0,\ x_1=l, \tag{5.74}$$

additional condition $X_p(x_1)=0$ with $x_1=x_{1\text{н}}$, and condition of orthogonality

$$\int_0^l m(x_1)X_p(x_1)X_k(x_1)dx_1=0 \text{ with } k\neq p.$$

Here

$$\left.\begin{aligned}M_p\omega_p^2&=\int_0^l E(x_1)F_c(x_1)\left[\frac{dX_p(x_1)}{dx_1}\right]^2dx_1,\\ M_p&=\int_0^l m_x(x_1)X_p^2(x_1)dx_1\quad (p=1,2,\ldots).\end{aligned}\right| \tag{5.75}$$

Since in the deduction of equations of oscillations of the considered system the boom of the erector was schematically represented in the form of a beam, having a certain given linear mass $m_c(x)$ and given rigidity $B_c(x)$, then the calculation of partial frequencies and forms of natural oscillations of this boom can present certain difficulties. It is natural that these difficulties will be even greater, the more that the construction of the mobile part of the boom differs from a simple beam. The schematization accepted above signifies essentially that in general partial forms of oscillations of the boom are normal forms of oscillations of a certain system of beams forming its construction. These normal forms and their corresponding frequencies for concrete construction of the boom can be determined by method of iterations presented in Chapter IV. Here it is necessary only to fulfill a series of conditions. In particular, boundary conditions have to express the absence on the free end of the boom of transverse and longitudinal forces, bending and turning moments

$$\left.\begin{array}{ll} \dfrac{d^2\Phi_{ym}(x)}{dx}=0, & \dfrac{d^2\Phi_{zm}(x)}{dx}=0, \\ \dfrac{d^3\Phi_{ym}(x)}{dx^3}=0, & \dfrac{d^3\Phi_{zm}(x)}{dx^3}=0, \\ \dfrac{dX_{ck}(x)}{dx}=0, & \dfrac{d\varkappa_p(x)}{dx}=0 \end{array}\right\} \text{ with } x=L. \tag{5.76}$$

On the other end (at the point of location of axis of rotation of the boom) - the absence in plane of rotation xy of sag and bending moment (in the case when rigidity of the drive is low)

$$\Phi_{ym}(x)=0, \quad \frac{d^2\Phi_{ym}(x)}{dx^2}=0 \quad \text{with} \quad x=0 \tag{5.77}$$

and equality to zero of shifts and angles of turn (in plane xz and with respect to axis of rigidity of the boom

$$\left.\begin{array}{ll} \Phi_{zm}(x)=0, & \varkappa_p(x)=0, \\ \dfrac{d\Phi_{zm}(x)}{dx}=0, & X_{ck}(x)=0 \end{array}\right\} \text{ with } x=0. \tag{5.78}$$

Condition of orthogonality for functions $\Phi_{im}(x)$ and $\kappa_p(x)$ should have the form

$$\begin{gathered} \int_0^l m_{inp}(x)\Phi_{im}(x)\Phi_{in}(x)\,dx + a_{нв}[\Phi_{im}(x_в)\Phi_{in}(x_н)+ \\ +\Phi_{im}(x_н)\Phi_{in}(x_в)]=0 \quad (i=y,\,z) \quad \text{with} \quad n\neq m, \\ \int_0^L i_{np}(x)\varkappa_p(x)\varkappa_n(x)\,dx + a_{нв}r_нr_в[\varkappa_p(x_н)\varkappa_n(x_в)+ \\ +\varkappa_p(x_в)\varkappa_n(x_н)]=0 \quad \text{with} \quad p\neq n, \end{gathered} \tag{5.79}$$

where $i_{\text{пр}}(x) = i(x) + r_{\text{н}}^2 m_{\text{н}} \delta(x - x_{\text{н}}) + r_{\text{в}}^2 m_{\text{в}} \delta(x - x_{\text{в}})$.

Moreover in the expression for linear reduced mass, besides the mass of construction of the boom itself one should consider additional concentrated masses (and corresponding moments) from all loads distributed on the boom, including and mass of the flight vehicle. For instance, in the case considered in § 5.5 the given linear mass of the boom will be:

in plane xy

$$m_{y\text{пр}}(x) = m_{\text{c}}(x) + m_{\text{в}}\delta(x - x_{\text{в}}) + m_{\text{н}}\delta(x - x_{\text{н}}) + \\ + m_{\text{c}}\delta(x - x_{\text{c}}) + Mr_{\text{н}}^2 \frac{d}{dx}\delta(x - x_{\text{н}}); \tag{5.80}$$

in plane xz

$$m_{z\text{пр}}(x) = m_{\text{c}}(x) + m_{\text{в}}\delta(x - x_{\text{в}}) + m_{\text{н}}\delta(x - x_{\text{н}}) + m_{\text{c}z}\delta(x - x_{\text{c}}); \tag{5.81}$$

in the direction of axis x

$$m_{x\text{пр}}(x) = m_{\text{c}}(x) + m_{\text{c}x}\delta(x - x_{\text{c}}) + M\delta(x - x_{\text{н}}), \tag{5.82}$$

where

$$m_{\text{c}} = J_{Az}\frac{1}{l_A^2}\cos^2\alpha, \quad m_{\text{c}z} = J_{Ay}\frac{1}{l_A^2}, \quad m_{\text{c}x} = J_{Az}\frac{1}{l_A^2}\sin^2\alpha,$$

and $\delta(x - x_1)$ - delta-function, equal to zero with $x \neq x_1$ and infinity with $x = x_1$, for which

$$\int_{x_i-\varepsilon}^{x_i+\varepsilon} \Phi_{ym}^2(x_i)\frac{d}{dx}\delta(x - x_i)\,dx = \left[\frac{d\Phi_{ym}(x_i)}{dx}\right]^2 \quad \text{with} \quad \varepsilon \ll 1.$$

Corresponding generalized masses will be determined by expressions

$$M_{ym} = \int_0^L m_{y\text{пр}}(x)\Phi_{ym}^2(x)\,dx, \tag{5.83}$$

$$M_{zm} = \int_0^L m_{z\text{пр}}(x)\Phi^2_{zm}(x)\,dx, \tag{5.84}$$

$$M_{ck} = \int_0^L m_{x\text{пр}}(x)X^2_{ck}(x)\,dx. \tag{5.85}$$

In the absence of struts partial frequencies of natural oscillations of boom construction will be equal to

$$\omega^2_{ym} = \frac{1}{M_{ym}}\int_0^L B_y(x)\left[\frac{d^2\Phi_{ym}(x)}{dx^2}\right]^2 dx, \tag{5.86}$$

$$\omega^2_{zm} = \frac{1}{M_{zm}}\int_0^L B_z(x)\left[\frac{d^2\Phi_{zm}(x)}{dx^2}\right]^2 dx, \tag{5.87}$$

$$\omega^2_{ck} = \frac{1}{M_{ck}}\int_0^L E(x)F_c(x)\left[\frac{dX_{ck}(x)}{dx}\right]^2 dx, \tag{5.88}$$

$$\omega^2_{\mu\rho} = \frac{1}{T_{\mu\rho}}\int_0^L G(x)I_p(x)\left[\frac{d\varkappa_p(x)}{dx}\right]^2 dx, \tag{5.89}$$

and partial frequencies of inherent elastic oscillations of connections

$$\left.\begin{aligned}&\omega^2_{z\text{в}} = \frac{c_{\text{в}z}}{m_\text{в}}, \quad \omega^2_{z\text{н}} = \frac{c_{\text{н}z}}{m_\text{н}},\\ &\omega^2_{y\text{в}} = \frac{c_{\text{в}y}}{m_\text{в}}, \quad \omega^2_{y\text{н}} = \frac{c_{\text{н}y}}{m_\text{н}}, \quad \omega^2_{x\text{н}} = \frac{c_{\text{н}x}}{M};\end{aligned}\right\} \tag{5.90}$$

§ 5.8. Determination of Internal Power Factors

Necessary magnitude of moment $M_Д$, ensuring the assigned rate of raising the boom, is determined by the value of external moments acting on the system flight vehicle - erector in the process of exploitation. Since moments of aerodynamic forces, caused by the influence of wind, can be changed in magnitude in wide limits and even change sign, then, according to equation (20), the possible required values of moment $M_Д$ (for current ϑ_0) will lie in the region

$$\min M_Д(\vartheta_0) \leqslant M_Д(\vartheta_0) < \max M_Д(\vartheta_0). \tag{5.91}$$

where

$$\left.\begin{aligned} \min M_{\text{д}}(\vartheta_0) &= G(x_{\text{т}}\cos\vartheta_0 - y_{\text{т}}\sin\vartheta_0) - J\ddot{\vartheta}_{\text{п}} - \\ &\quad - Y(\vartheta_0)x_{\text{д}} - X(\vartheta_0)y_{\text{д}}, \\ \max M_{\text{д}}(\vartheta_0) &= G(x_{\text{т}}\cos\vartheta_0 - y_{\text{т}}\sin\vartheta_0) - J\ddot{\vartheta}_{\text{д}}(\vartheta_0) + \\ &\quad + Y(\vartheta_0)x_{\text{д}} + X(\vartheta_0)y_{\text{д}}. \end{aligned}\right\} \qquad (5.92)$$

Here the necessary angular acceleration $\ddot{\vartheta}_{\text{п}}$ of the boom, which is determined by assigned time for raising the system into initial position, will depend basically on the assignment of the flight vehicle. Permissible angular acceleration $\ddot{\vartheta}_{\text{д}}$ will be limited by the carrying capacity of the vehicle construction for the given case of load, i.e., is a function of angle of rise of the boom ϑ_0, and also magnitude and direction of wind speed.

The method of realization of moment $M_{\text{д}}$, which in general is a nonlinear function of angle ϑ_0, does not have fundamental value for dynamic calculation of the system. Therefore for definitiveness we will consider that it is created by the concentrated force $P(P_x, P_y, P_z)$, applied at a certain distance from the axis of rotation of the boom. Further let us assume that the stated force is changed in magnitude and in direction in such a way that condition $\ddot{\vartheta}_0<\ddot{\vartheta}_{\text{д}}(\vartheta_0)$ is fulfilled for all ϑ_0 and for any direction of wind speed. This is ensured either by the construction of the drive, or, in an extreme case, by means of increase of the carrying capacity of the body of the flight vehicle, i.e., by an increase of acceleration $\ddot{\vartheta}_{\text{д}}$.

In certain cases, values $\ddot{\vartheta}_{\text{д}}, \ddot{\vartheta}_{\text{п}}$ themselves determine the limiting value of the static required moment $M_{\text{д}}$

$$\left(1 - \frac{J\ddot{\vartheta}_{\text{д}}}{x_{\text{т}}G}\right) < \frac{\max M_{\text{д}}}{x_{\text{т}}G} < \left(1 - \frac{J\ddot{\vartheta}_{\text{п}}}{x_{\text{т}}G}\right) \quad \text{with} \quad \vartheta_0 = 0. \qquad (5.93)$$

For light flight vehicles, especially with lifting surfaces (wings),

$$\max M_{\text{д}} > x_{\text{д}} Y_{\max}, \qquad (5.94)$$

where Y_{max} - maximum transverse aerodynamic load on the system with $\vartheta_0 = \vartheta_{max}$ and average local wind speed also being a function of angle ϑ_0. In the following chapter we will dwell in detail on the question of influence of wind on this particular elastic system flight vehicle-erector. In this section we will be limited only to the determination of expressions for the corresponding generalized forces.

Everything stated above remains true also for the case of lowering the boom of the erector with the vehicle in a horizontal position, excluding the moment of contact with supports which was considered in § 5.3. In the case of lowering the flight vehicle onto a launching pad with $\vartheta_0 = \frac{\pi}{2}$ and $\dot{\vartheta}_0 = 0$, besides the force of weight and crosswind loads the system will be influenced by additional corresponding reactions of the pad or special guide rails (R_x, R_y, R_z).

During investigation of small elastic oscillations of the system flight vehicle-erector relative to a certain "undisturbed" state, characterized by parameters ϑ_0 and $\ddot{\vartheta}_0$, the influence of the drive can be considered by means of introduction of an additional elastic connection in section x_p with rigidity c_{np}, i.e., substituting

$$-P_y = c_{np}[y(x_p) + x_p \Delta\vartheta].$$

Composing expressions for the work of all external forces on possible displacements, determined by formulas (30), (31), (33) and (35), and taking from them the partial derivatives for the corresponding generalized coordinates, we obtain the following expressions for generalized forces:

$$Q_0 = r_{\text{н}} X_1 + \frac{1}{a} Y_1 [x_{\text{в}}(x_{1\text{н}} - x_{1\text{д}}) + x_{\text{н}}(x_{1\text{д}} - x_{1\text{в}})] + x_p P_y,$$

$$Q_{\text{упп}} = S \int_0^l q_y(x_1) \frac{\partial c_{y_1}(x_1)}{\partial x_1} f_{yn}(x_1)\, dx_1,$$

$$Q_{\text{тк}} = P_x X_{\text{ск}}(x_p) - X_1 X_{\text{ск}}(x_{\text{н}}),$$

$$Q_{\text{тр}} = q_x S \int_0^l \frac{\partial c_x(x_1)}{\partial x_1} X_p(x_1)\, dx_1,$$

$$Q_{kym} = \frac{1}{a} Y_1 [\Phi_{ym}(x_в)(x_{1н} - x_{1л}) + \Phi_{ym}(x_н)(x_{1л} - x_{1в})] +$$

$$+ r_н X_1 \frac{d\Phi_{ym}(x)}{dx}\Big|_{x=x_н} + r_p P_x \frac{d\Phi_{ym}(x)}{dx}\Big|_{x=x_p} + P_y \Phi_{ym}(x_p),$$

$$Q_в = \frac{1}{a} Y_1 (x_{1н} - x_{1л}),$$

$$Q_н = \frac{1}{a} Y_1 (x_{1л} - x_{1в}),$$

$$Q_{uн} = X_1, \quad a = x_{1н} - x_{1в},$$

$$Q_{uzm} = S \int_0^l q_z(x_1) \frac{\partial c_z(x_1)}{\partial x_1} f_{zm}(x_1)\, dx_1,$$

$$Q_{kzm} = -\frac{1}{a} Z_1 [\Phi_{zm}(x_н)(x_{1л} - x_{1в}) + \Phi_{zm}(x_в)(x_{1н} - x_{1л})] + P_z \Phi_{zm}(x_p),$$

$$Q_{\varkappa p} = -\frac{1}{a} Z_1 [\varkappa_p(x_н)(x_{1л} - x_{1в}) r_н + \varkappa_p(x_в)(x_{1н} - x_{1л}) r_в] +$$

$$+ \varkappa_p(x_p)(P_y z_p + P_z y_p),$$

$$Q_{вw} = -\frac{1}{a} Z_1 (x_{1л} - x_{1в}),$$

$$Q_{нw} = -\frac{1}{a} Z_1 (x_{1н} - x_{1л}),$$

$$X_1 = c_x S q_x, \quad Y_1 = S \int_0^l \frac{\partial c_{y_1}(x_1)}{\partial x_1} q_y(x_1)\, dx_1,$$

$$q_x = \frac{1}{2} \rho_0 u_x^2, \quad q_y = \frac{1}{2} \rho_0 u_y^2(x_1),$$

$$q_z = \frac{1}{2} \rho_0 u_z^2(x_1), \quad Z_1 = S \int_0^l \frac{\partial c_z(x_1)}{\partial x_1} q_z(x_1)\, dx_1.$$

Using expression (31), it is easy to obtain the formula for transverse overload in any point of longitudinal axis of the flight vehicle

$$n_{y_1}(x_1) = n_{y_1}^0 + \frac{x_1}{a} n_{y_1}^x + n_{y_1 л}(x_1), \tag{5.95}$$

where

$$n_{y_1}^0 = -\frac{1}{g_0}\Bigg\{ \ddot{w}_в \frac{x_{1н}}{a} - \ddot{w}_н \frac{x_{1в}}{a} + \sum_{m=1}^{\infty} \frac{1}{a} \ddot{K}_{ym} [\Phi_{ym}(x_в) x_{1н} - \Phi_{ym}(x_н) x_{1в}] +$$

$$+ \frac{\Delta\ddot{\vartheta}}{a}(x_в x_{1н} - x_{1в} x_н) + g_0 \cos\vartheta_0 \Bigg\},$$

$$n_{y_1}^{x} = -\frac{1}{g_0}\left\{\ddot{w}_{\text{н}} - \ddot{w}_{\text{в}} + \Delta\ddot{\vartheta}(x_{\text{н}} - x_{\text{в}}) + \sum_{m=1}^{\infty} \ddot{K}_{ym}[\Phi_{ym}(x_{\text{н}}) - \Phi_{ym}(x_{\text{в}})]\right\},$$

$$n_{y_1\text{д}}(x_1) = -\frac{1}{g_0}\sum_{n=1}^{\infty} \ddot{S}_{yn} f_{yn}(x_1).$$

The longitudinal component of overload on the basis of formulas (34) and (35) will be equal to

$$n_{x_1}(x_1) = n_{x_1}^{0} + n_{x_1\text{д}}(x_1). \tag{5.96}$$

where

$$n_{x_1}^{0} = -\frac{1}{g_0}\left[\ddot{u}_{\text{н}} - \sum_{k=1}^{\infty} \ddot{T}_{ck} X_{ck}(x_{\text{н}}) + \sum_{m=1}^{\infty} r_{\text{н}} \ddot{K}_{ym} \left.\frac{d\Phi_{ym}(x)}{dx}\right|_{x=x_{\text{н}}} + \right.$$
$$\left. + r_{\text{н}}\Delta\ddot{\vartheta} + g_0 \sin\vartheta_0\right],$$

$$n_{x_1\text{д}}(x_1) = -\frac{1}{g_0}\sum_{p=1}^{\infty} \ddot{T}_p X_p(x_1).$$

And, finally, from formula (33) the expression is obtained for lateral component of the overload in the form

$$n_{z_1}(x_1) = n_{z_1}^{0} + \frac{x_1}{a} n_{z_1}^{x} + n_{z_1\text{д}}(x_1). \tag{5.97}$$

where

$$n_{z_1}^{0} = \frac{1}{ag_0}\left\{\sum_{m=1}^{\infty} \ddot{K}_{zm}[\Phi_{zm}(x_{\text{в}})x_{1\text{н}} - \Phi_{zm}(x_{\text{н}})x_{1\text{в}}] + \right.$$
$$\left. + \ddot{v}_{\text{в}}x_{1\text{н}} - \ddot{v}_{\text{н}}x_{1\text{в}} + \sum_{p=1}^{\infty} \ddot{\mu}_p[\varkappa_p(x_{\text{в}})x_{1\text{н}}r_{\text{в}} - \varkappa_p(x_{\text{н}})x_{1\text{в}}r_{\text{н}}]\right\},$$

$$n_{z_1}^{x} = \frac{1}{g_0}\left\{\sum_{m=1}^{\infty} \ddot{K}_{zm}[\Phi_{zm}(x_{\text{н}}) - \Phi_{zm}(x_{\text{в}})] + \ddot{v}_{\text{н}} - \ddot{v}_{\text{в}} + \right.$$
$$\left. + \sum_{p=1}^{\infty} \ddot{\mu}_p[\varkappa_p(x_{\text{н}})r_{\text{н}} - \varkappa_p(x_{\text{в}})r_{\text{в}}]\right\},$$

$$n_{z_1\text{д}}(x_1) = -\frac{1}{g_0}\sum_{n=1}^{\infty} \ddot{S}_{zn} f_{zn}(x_1).$$

Loads on the lower support will be equal to

$$\left.\begin{aligned} -R_{x\text{н}} &= c_{\text{н}x}u_{\text{н}}, \\ R_{y\text{н}} &= c_{\text{н}y}w_{\text{н}}, \\ R_{z\text{н}} &= c_{\text{н}z}v_{\text{н}}, \end{aligned}\right\} \tag{5.98}$$

and on the upper

$$\left.\begin{aligned} R_{y\text{в}} &= c_{\text{в}y}w_{\text{в}}, \\ R_{z\text{в}} &= c_{\text{в}z}v_{\text{в}}. \end{aligned}\right\} \tag{5.99}$$

Longitudinal, transverse, and lateral forces $N(x_1)$, $Q_y(x_1)$, $Q_z(x_1)$ and the corresponding bending moments $M_y(x_1)$ и $M_z(x_1)$ are determined by formulas

$$N(x_1) = n^0_{x_1}G_x(x_1) - \sum_{p=1}^{\infty}\tilde{T}_p N_{px}(x_1) - R_{x\text{н}}\int_0^{x_1}\delta(x_1 - x_{1\text{н}})\,dx_1, \tag{5.100}$$

$$\begin{aligned} Q_y(x_1) = {} & \int_0^{x_1}\frac{\partial Y_1(x_1)}{\partial x_1}\,dx_1 + n^0_{y_1}G_y(x_1) + n^x_{y_1}Q^x_{0y}(x_1) - \\ & -\sum_{n=1}^{\infty}\tilde{S}_{yn}Q_{nx}(x_1) - R_{y\text{в}}\int_0^{x_1}\delta(x_1 - x_{1\text{в}})\,dx_1 - R_{y\text{н}}\int_0^{x_1}\delta(x_1 - x_{1\text{н}})\,dx_1, \end{aligned} \tag{5.101}$$

$$\begin{aligned} Q_z(x_1) = {} & \int_0^{x_1}\frac{\partial Z_1(x_1)}{\partial x_1}\,dx_1 + n^0_{z_1}G_z(x_1) + n^x_{z_1}Q^x_{0z}(x_1) - \\ & -\sum_{n=1}^{\infty}\tilde{S}_{zn}Q_{nx}(x_1) + R_{z\text{в}}\int_0^{x_1}\delta(x_1 - x_{1\text{в}})\,dx_1 + R_{z\text{н}}\int_0^{x_1}\delta(x_1 - x_{1\text{н}})\,dx_1, \end{aligned} \tag{5.102}$$

$$\left.\begin{aligned} M_y(x_1) &= \int_0^{x_1} Q_y(x_1)\,dx_1, \\ M_z(x_1) &= \int_0^{x_1} Q_z(x_1)\,dx_1. \end{aligned}\right\} \tag{5.103}$$

CHAPTER VI

LOAD ON THE STRUCTURE OF A VEHICLE IN THE PROCESS OF PRELAUNCH PREPARATION

§ 6.1. Wind Loads

This case of load is of interest mainly in respect to flight vehicles which are launched from a vertical or slanted position from open ground sites (or from the deck of surface vessels). In the presence of wind the structure of such vehicles is loaded both in direction of flow and also in a perpendicular direction by aerodynamic forces which change chaotically in time. The developing elastic oscillations of system vehicle – erector of vehicle – launcher (Fig. 6.1) can render an essential influence on the required carrier ability of certain elements of their constructions, and also on conditions of exploitation of the flight vehicle, in particular on conditions of operation of the aiming system, the liquid fueling system, and so forth.

Parameters of these oscillations depend on aerodynamic and dynamic characteristics of construction of the actual vehicle, the erector and launching system, and also on parameters of air flow.

Instantaneous value of horizontal wind speed in any point of space in the direction of flow can be presented in the form of the sum of constant and variable components

$$u(x_1,\ t)=u_0(x_1)+\Delta u(x_1,\ t). \qquad (6.1)$$

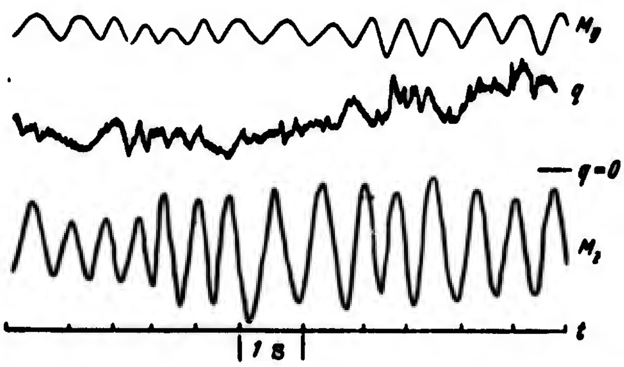

Fig. 6.1. Typical picture of pulsation of impact pressure of wind q and values of bending moment in a cross section of the body of the flight vehicle in the case V (M_y – in the plane of flow, M_z – perpendicular to the plane).

In the surface layer of the atmosphere the mean value of wind speed is changed with altitude h exponentially in certain cases, but basically by the logarithmic law

$$u_0(x_1) = u_\phi \left(\ln \frac{h}{k_0} \Big/ \ln \frac{h_\phi}{k_0}\right), \qquad (6.2)$$

where h_ϕ – altitude of weather vane (usually h_ϕ = 10 m), u_ϕ – average wind speed for altitude h_ϕ, k_0 – parameter of roughness, characterizing surface conditions. Usually this parameter is changed within limits of 0.05-0.2.

For the case of raising the flight vehicle into a vertical position altitude h is calculated by the formulas

$$h = h_0 + (l_0 - x_1)\sin\vartheta_0$$

for points of axis of rigidity of the erector boom and

$$h = h_0 + (l_0 - x_1)\sin\vartheta_0 + r_0\cos\vartheta_0$$

for points of longitudinal axis of the flight vehicle. In these formulas h_0 designates altitude of location (above the surface of the earth) of the axis of rotation of the erector boom, l_0 – distance from origin of coordinates of the system x_1y_1 to the axis of rotation of the boom, and r_0 – distance between longitudinal axes of the flight vehicle and the boom.

In scales of days the average wind speed is a random function of time. Its magnitude depends on local meteorological conditions, condition of the surface in the launch area, and temperature gradient. Selection of computed values of these average speeds is performed in each specific case taking into account the probability of accomplishing the missions assigned to the flight vehicle. Usually the permissible average wind speeds u_ϕ from the point of view of safety of exploitation (for different vehicles and military equipment) lie within the limits of 15-20 m/s. During storage they can reach up to 40 m/s (with a 4-10 minute averaging) [11]. Pulsations of wind speed Δu even within the limits of the averaging interval have a nonstationary nature. Against the background of mild continuous turbulence there are separate gusts of wind with comparatively large gradients of speeds. The speeds of such gusts (squalls) can reach 10-12 m/s. Intensity of atmospheric turbulence, representing the superposition of vortexes of different scales, essentially depends on local thermal atmospheric composition (stratification). Its statistical characteristic can differ strongly during transition from one realization to another. Therefore only within limits of a comparatively small interval of time it can be replaced by certain stationary ergodic process, the distribution of speeds of which corresponds to normal law. The dependence of average quadratic value of deviation of the horizontal component of speed of atmospheric turbulence on average flow rate of air and altitude of the point above the surface (within limits h < 50 m) is determined approximately by the formula

$$\sigma_u = (0{,}2189 - 0{,}00011h)\,u_0^{0{,}8}. \qquad (6.3)$$

Just as the average wind speed, σ_u is a random function of time in scales hours and days with a comparatively large standard deviation σ_{σ_u}.

Spectral density of energy of pulsation of speed of atmospheric turbulence, according to Davenport and Karman, is described by the following approximate formulas:

$$\Phi_{uy}(u_0, \omega) = \frac{\sigma_u^2 L^2}{u_0^2} \frac{6\omega}{\left[1 + \left(\frac{\omega L}{u_0}\right)^2\right]^{4/3}} \tag{6.4}$$

in the direction of flow of the wind and

$$\Phi_{uz}(u_0, \omega) = \frac{\sigma_u^2 L}{\pi u_0} \frac{\left[1 + \frac{8}{3}\left(1{,}34\frac{\omega L}{u_0}\right)^2\right]}{\left[1 + \left(1{,}34\frac{\omega L}{u_0}\right)^2\right]^{11/6}} \tag{6.5}$$

in vertical and lateral directions. In these formulas L designates the scale of turbulence, characterizing the average linear dimension of the vortex of the turbulent field. Quantitatively it is estimated by the integral from the standardized correlation function. Its magnitude is a function of air temperature. For pulsation of wind speed along the flow the scale of turbulence is commensurate with the altitude of the point above the surface.

Since the body of a flight vehicle of cylindrical or conical form is a bluff body in a transverse direction, then in the case of the influence of flow near it there will be a reverse gradient of pressure. As a result already at comparatively small Reynolds number Re (characterizing the relationship of forces of inertia and forces of viscous friction of flow) there will be a breaking away of the boundary layer and the formation of a vortex street in the trace after body of the flight vehicle. At small $Re = \frac{\rho_0 u_0 d}{\Delta}$ (where μ – viscosity of air, and d – diameter of body) these vortexes (Benar-Karman vortexes) move along the flow in a definite order (being detached in turn from each side of the body). The frequency of their separation in the case of the motionless body of a flight vehicle is characterized by the Strouhal number $Sh = \frac{\omega d}{2\pi u_0}$, which in this case hardly changes. In other words, the frequency of separation of vortexes at small Re numbers turns out to be proportional to incoming flow velocity.

At large Reynolds numbers (an order of $1 \cdot 10^6$), which are of primary interest to designers of flight vehicles, flow in the trace behind a motionless body becomes completely turbulent.

Thus the motionless body of a flight vehicle in a flow of wind is acted on additionally by aerodynamic forces in a lateral direction and in a direction, opposite the flow, caused by pulsations of velocity in the trace behind the body. Spectral densities of energy of these forces depend on form of the body of the flight vehicle, especially upper part, and are described by expressions analogous to (4) and (5), but with other values of scales of turbulence.

Spectral densities of external aerodynamic forces, created directly by turbulence of incident flow of the wind, are easily expressed through the corresponding spectral densities of energy of pulsation of speed (4) and (5) on the basis of formula

$$\Phi_{u^2}(u_0, \omega) = 4u_0^2\Phi_u(u_0, \omega) + 2\int_{-\infty}^{\infty} \Phi_u(u_0, \omega_1)\Phi_u(u_0, \omega - \omega_1)\,d\omega_1,$$

for which with $\sigma_u^2 \ll u_0^2$ the second member turns out to be minute. With sufficient accuracy one may assume that

$$\Phi_a(u_0, x_1, \omega) = \rho_0^2 u_0^2(x_1)\Phi_u(u_0, x_1, \omega)\,d^2(x_1)\,c_y^2(x_1). \qquad (6.6)$$

where $c_y(x_1)$ – coefficient of aerodynamic drag of the body of the vehicle in section x_1, referred to diameter $d(x_1)$. Here if the values of this coefficient are determined by means of measurement of difference of pressures in opposite points of the cross-section contour, then expression (6) will characterize the spectral density of total aerodynamic force acting on the flight vehicle in direction of flow, and the coefficient itself will be a function of the Reynolds number.

During investigation of the construction reaction of the flight vehicle to the influence of atmospheric turbulence and plane-parallel flow of wind usually it is assumed that spatial heterogeneity of pulsation of aerodynamic forces is small, i.e., it is assumed that it is possible to use one-dimensional correlation functions instead of two-dimensional.[1] This means that only the low-frequency part of the spectrum of turbulence, which is formed by large-scale eddies is considered.

With small structure oscillations of the flight vehicle the instantaneous value of distributed aerodynamic load for flow is determined by the expression

$$q_a(x_1,\ t) = \rho_0 u^2(x_1,\ t)\, c_y(\dot{x}_1)\, \frac{d(x_1)}{2}\left[1 - \frac{2\dot{y}_1(x_1,\ t)}{u(x_1,\ t)}\right], \tag{6.7}$$

in which the second member in brackets, which is proportional to the speed of transverse oscillations of section x_1 of the body, characterizes the magnitude of aerodynamic damping. In this case aerodynamic coefficients will additionally depend somehow on parameters of oscillations of the vehicle. Considering the difficulty of determination of this dependence, usually it is preferred to use spectral densities not of speeds, but of aerodynamic forces (especially in the direction perpendicular to the direction of flow of the wind, where this dependence is significant).

§ 6.2. Equations of Transverse Structure Oscillations

Approximate equations of elastic transverse oscillations, which were given in Chapter IV, can be used (with certain refinements) for investigating the reaction of a structure to the influence of wind flow also in the case of V. The solution of these equations (4.23) or (4.39), (4.40) is sought for by the method of separation of variables in the form (4.28), (4.37), and (4.38). In this case the corresponding function of form of normal inherent transverse oscillations should satisfy boundary conditions of the type (4.24) on the free end (with $x_1 = 0$) and special boundary conditions on the second end (with $x_1 = l$). The latter depending on the method of connection of the flight vehicle to the launcher can be approximated by rigid or elastic sealing or by a hinge. In particular, if the structure of the launcher (for instance a launching pad), lugs for mounting the vehicle to the launcher and corresponding support parts of the flight-vehicle structure can be considered sufficiently rigid (in comparison with the rigidity of the body of the vehicle), then these conditions will have the form

$$f_n(x_1)=0,\quad \frac{df_n(x_1)}{dx_1}=0 \text{ with } x_1=l, \tag{6.8}$$

If at least one of the enumerated support elements is elastic, then one should assume

$$f_n(x_1)=0,\quad \frac{df_n(x_1)}{dx_1}=-\frac{M_{nx}(l)}{c_0} \text{ with } x_1=l, \tag{6.9}$$

where c_0 – given rigidity, equal to the ratio of a unit static bending moment, applied in section $x_1=l$, to the angle of rotation of the support section of the vehicle caused by him. In the case when rigidity of the launching pad during bending is comparable with rigidity $B(l)$ of the body of the flight vehicle, the calculation of functions $f_n(x_1)$ should be conducted taking into account the influence of rigidity of pad construction, relating the above-indicated boundary conditions to the place of securing of the pad to the foundation.

Conditions corresponding to a hinged support are realized in practice only with free installation of the flight vehicle on the pad, i.e., in the absence of special bracing, limiting the angle of rotation of the support section.

More precise definition of the above mentioned equations is reduced to calculation of the influence longitudinal component of the force of weight, damping forces, and mobility of liquid in the fuel tanks.

In many cases additional bending moments from the force of weight, caused by transverse displacements of centers of gravity of body sections of the flight vehicles,

$$\Delta M(x_1, t)=g_0\int_0^{x_1} m_x(\xi)[y_1(\xi, t)-y_1(x_1, t)]\,d\xi$$

reach comparatively large values. Taking into account these moments the approximate equation of transverse oscillations of the structure (4.23) takes the form

$$\frac{\partial^2}{\partial x_1^2}\left[B(x_1)\frac{\partial^2 y_1(x_1, t)}{\partial x_1^2}\right] + m_y(x_1)\frac{\partial^2 y_1(x_1, t)}{\partial t^2} + m_x(x_1)\frac{\partial y_1(x_1, t)}{\partial x_1} g_0 +$$

$$+ g_0 \frac{\partial^2 y_1(x_1, t)}{\partial x_1^2}\int_0^{x_1} m_x(x_1)\,dx_1 = q_s(x_1, t). \qquad (6.10)$$

where m_x — linear (in a longitudinal direction) mass of the flight vehicle (taking into account apparent additional masses of mast or elements of the service tower, servicing communications).

Corresponding equations for determination of forms and frequencies of inherent bending oscillations of the structure will be

$$\frac{d^2}{dx_1^2}\left[B(x_1)\frac{d^2 f_n(x_1)}{dx_1^2}\right] + m_x(x_1)\frac{df_n(x_1)}{dx_1} g_0 +$$

$$+ g_0 \frac{d^2 f_n(x_1)}{dx_1^2}\int_0^{x_1} m_x(x_1)\,dx_1 = \omega_n^2 m_y(x_1) f_n(x_1) \quad (n = 1, 2, \ldots). \qquad (6.11)$$

Here the unit value of dynamic component of bending moment will be equal to

$$M_{nx}(x_1) = \int_0^{x_1}\int_0^{x_1}\Bigg\{m_y(x_1) f_n(x_1) -$$

$$- \frac{g_0}{\omega_n^2}\left[m_x(x_1)\frac{df_n(x_1)}{dx_1} + \frac{d^2 f_n(x_1)}{dx_1^2}\int_0^{x_1} m_x(x_1)\,dx_1\right]\Bigg\} dx_1\,dx_1. \qquad (6.12)$$

Additional increase of damping factor of transverse elastic oscillations of vehicle structure in direction of wind flow at the expense of aerodynamic damping force (7) in this case of load will be determined by the expression

$$\frac{\rho_0}{M_n}\int_0^{l} d(x_1)\, c_y(x_1)\, u_0(x_1)\, f_n^2(x_1)\,dx_1. \qquad (6.13)$$

Its influence turns out to be very significant at comparatively low frequencies of oscillations of the structure and especially for flight vehicles with large lifting surfaces. In the plane perpendicular to the direction of wind flow the influence of aerodynamic damping can frequently be disregarded.

Taking into account the stated corrections the equations for transverse oscillations of the point of reduction (for instance, summit of the body of the flight vehicle) in case V will have the following form:

$$\ddot{S}_n + 2h_n\dot{S}_n + \omega_n^2 S_n = \frac{1}{M_n} B_{\text{в}n} \quad (n = 1, 2, \ldots), \tag{6.14}$$

where

$$\left.\begin{aligned} &B_{\text{в}n} = \int_0^l q_{\text{в}}(x_1, t) f_n(x_1)\, dx_1, \quad \bar{u}_0 = \frac{u_0}{u_\phi} \\ &q_{\text{в}}(x_1, t) = u(x_1)\, \Delta u(x_1, t)\, \rho_0 c_y(x_1)\, d(x_1), \\ &2hn = \frac{\omega_n \delta_{\text{пр}}}{\pi}, \quad \delta_{\text{пр}} = \delta + \frac{\pi u_\phi}{\omega_n} c_{n\text{д}}, \\ &c_{n\text{д}} = \frac{\rho_0}{M_n} \int_0^l c_y(x_1)\, \bar{u}_0(x_1)\, d(x_1)\, f_n^2(x_1)\, dx_1. \end{aligned}\right\} \tag{6.15}$$

The influence of mobility of liquid in the fuel tanks on frequencies and form of transverse elastic oscillations of flight-vehicle structure in this case of load can be considered approximately by means of introduction in equation (11) of the given mass of the liquid or the corresponding influence coefficients (2.66) in the function from the unknown frequency of natural oscillations ω_n. If interest lies in oscillation of the vehicle structure in process of refueling or in parameters of oscillations of the liquid in the tanks, then instead of equations (14) it is expedient to use equations which consider the mutual influence of mobility of liquid in the tanks and elastic oscillations of the structure.

For flight vehicles with cylindrical tanks such equations can be obtained readily if one were to present the influence of the liquid on the structure of the vehicle in the form of external surface dynamic load of intensity $q_{\tau j}(x_j, t)$ (2.32) and concentrated (in places of attachment of lower ends of tanks to the body of the flight vehicle x_{1j}) moments (2.57) with $n_{x_1} = 1$

$$\Delta M_{zj} = \Delta M_{zj}(\Delta\theta, \Delta\dot{\vartheta}, \beta_{kj}, \dot{\beta}_{kj}). \qquad (6.16)$$

Then instead of equation (10) we will have

$$\frac{\partial^2}{\partial x_1^2}\left[B(x_1)\frac{\partial^2 y_1(x_1, t)}{\partial x_1^2}\right] + m_{\kappa y}(x_1)\frac{\partial^2 y_1(x_1, t)}{\partial t^2} + g_0 m_x(x_1)\frac{\partial y_1(x_1, t)}{\partial x_1} + \\ + G_x(x_1)\frac{\partial^2 y_1(x_1, t)}{\partial x_1^2} = q_a(x_1, t) + q_{\tau j}(x_j, t) - \Delta M_{zj}(t)\delta''(x_1 - x_{1\kappa}), \qquad (6.17)$$

where $m_{\kappa y}(x_1)$ – intensity of distributed mass of dry construction of the flight vehicle (without fuel) in a transverse direction. The approximate solution of this equation we present in the form of the finite series

$$y_1(x_1, t) = \sum_{n=1}^{n_m} S_n(t) f_n(x_1), \qquad (6.18)$$

where n_m – number of considered forms of inherent bending oscillations of the structure. Usually during investigation of the nature of load on the lower (tail) part of the body of the flight vehicle it is possible to be limited to a calculation of only the first member (n = 1) of this series. The influence of the second member (n = 2) becomes noticeable only for sections which are located in the front part of the body, for which given cases is rarely calculated.

Since the frequencies of natural oscillations of liquid in fuel tanks ω_k of the first tone $(k=1)$, as a rule are less than the frequencies of inherent bending oscillations of the structure ω_n, then it is possible to assume that the mutual influence of second and subsequent tones of elastic oscillations of the body and natural oscillations of the free surface of the liquid will be small.

Under these conditions one may assume with an accuracy which is sufficient for practical calculations that in the considered case of load distortion of the axis of the fuel tank on the section filled with liquid in the case of bending oscillations of the structure with a frequency of ω_n at $n=1$ will be small. In other words, one may assume that with such oscillations a section of tank which is partially filled with liquid moves like a solid body. In a similar case for determination of value of potential of absolute velocity of the liquid it is possible to use expressions (2.41), (2.48), and (2.50).

Transverse speed of center of free surface of liquid in the j-th tank in this case will be equal to

$$v_{0yjn} = \dot{S}_n(t) f_n(x_{0j}), \tag{6.19}$$

and angular velocity of oscillations of longitudinal axis of the tank is equal to

$$\Delta\dot{\vartheta}_n = \dot{S}_n(t) \frac{df_n(x_j)}{dx_j}, \tag{6.19'}$$

where approximately

$$\frac{df_n(x_j)}{dx_j} = -\frac{1}{h_j}[f_n(x_{kj}) - f_n(x_{0j})]. \tag{6.19''}$$

Here it is assumed that

$$f_n(x_{0j}) + x_j \frac{df_n(x_j)}{dx_j} = f_n(x_j), \quad 0 > x_j > -h_j.$$

Placing these expressions of speeds in formulas (2.41) and (2.50), we find that with $n_n = 1$

$$\Phi_j(x_j, y_j, z_j, t) = (F_j + x_j y_j) \dot{S}_n(t) \frac{df_n(x_j)}{dx_j} + \\ + y_i \dot{S}_n(t) f_n(x_{0j}) + 2a_j \frac{\dot{\beta}_{kj} J_1(\xi_k r_j) \operatorname{ch}[\xi_k(\bar{x}_j + \bar{h}_j)]}{(\xi_k^2 - 1) J_1(\xi) \operatorname{ch} \mu_{kj}} \sin \eta. \tag{6.20}$$

In accordance with boundary conditions (2.40") with $x_j = 0$ the equations for functions β_{kj}, characterizing the amplitude of oscillations of free surface of liquid, taking into account damping will have the form

$$\ddot{\beta}_{kj} + 2h_{kj}\dot{\beta}_{kj} + \omega_{kj}^2\beta_{kj} = -\ddot{S}_n f_n(x_{0j}) - \\ - g_0 S_n\left(1 + \frac{2}{\operatorname{ch}\mu_{kj}}\right)\frac{df_n(x_j)}{dx_j} \quad (j = 1, 2, \ldots, n_j). \tag{6.21}$$

Knowing the value of potential Φ_j, we find that

$$-q_\tau(x_j, t) = m_\tau(x_j)\left\{\left[f_n(x_1) + \frac{4a_j \operatorname{sh}(\xi_k \bar{x}_j)}{\xi_k(\xi_k^2 - 1)\operatorname{ch}\mu_{kj}}\frac{df_n(x_j)}{dx_j}\right]\ddot{S}_n(t) + \\ + 2\ddot{\beta}_{kj}(t)\frac{\operatorname{ch}[\xi_k(\bar{x}_j + \bar{h}_j)]}{(\xi_k^2 - 1)\operatorname{ch}\mu_{kj}}\right\}, \tag{6.22}$$

where $m_\tau(x_j)$ – intensity of distribution along the length of the tank of the mass of liquid, h_{kj} – damping factor of oscillations of liquid in j-th tank. Placing expressions (18), (22), and (16) in the right side of equation (17) and rejecting small magnitudes, we obtain the following approximate equation for determination of function $S_n(t)$ with $n=1$:

$$M_n^*(\ddot{S}_n + 2h_n\dot{S}_n + \omega_n^{*2}S_n) = \sum_{j=1}^{n_j} t_{kjn}\beta_{kj} + \sum_{j=1}^{n_j} t_{kjn0}\beta_{kj} + B_{nn}, \tag{6.23}$$

where

$$\left.\begin{aligned} t_{kjn} &= -\frac{2\pi\rho_j a_j^2}{(\xi_k^2 - 1)}\left[\frac{1}{\operatorname{ch}\mu_{kj}}\int_{-h_j}^{0} f_n(x_j)\operatorname{ch}[\xi_k(\bar{x}_j + \bar{h}_j)]\,dx_j + \right. \\ &\qquad \left. + \frac{a_j^2}{\xi_k^2}\frac{df_n(x_j)}{dx_j}\left(1 - \frac{1}{\operatorname{ch}\mu_{kj}}\right)\right], \\ t_{kjn0} &= -2g_0\pi a_j^3\frac{\operatorname{th}\mu_{kj}}{\xi_k(\xi_k^2 - 1)}\frac{df_n(x_j)}{dx_j}\rho_j, \end{aligned}\right\} \tag{6.23'}$$

$$
\left.
\begin{aligned}
M_n^{*} = \sum_{j=1}^{n_j} \frac{2m_{\text{т}} a_j}{\xi_k(\xi_k^2-1)} \frac{df_n(x_j)}{dx_j} \Bigg[2 \int_{-h_j}^{0} f_n(x_j) \frac{\operatorname{sh}(\xi_k \bar{x}_j)\, dx_j}{\operatorname{ch} \mu_{kj}} + \\
+ \frac{a_j}{\xi_k}\left(\frac{2a_j}{\xi_k} \operatorname{th} \mu_{kj} + h_j\right)\left(\frac{df_n(x_j)}{dx_j}\right)^2 \Bigg] + M_n, \\
\omega_n^{*2} = \frac{1}{M_n^{*}} \left[\omega_n^2 M_n + \sum_{j=1}^{n_j} \frac{2\pi\rho_j a_j^4}{\xi_k^2(\xi_k^2-1)} \sigma_{kj}^2 \left(\frac{df_n(x_j)}{dx_j}\right)^2 \right],
\end{aligned}
\right\}
\qquad (6.23'')
$$

and $f_n(x_j)$ — form of inherent bending oscillations of the flight-vehicle structure (with "hardened" liquid), determined from equation (11).

The system of approximate ordinary differential equations in generalized coordinates (21) and (23) also will describe small transverse oscillations of the flight vehicle in process of prelaunch preparation. Using the method shown in § 5.6, it is possible to reduce it to normal coordinates, i.e., to a system of independent second order differential equations, in form similar to equation (14). During fulfillment of the mentioned transformation of coordinates it is recommended that all members of equation (21) preliminarily be multiplied by coefficient c_{kj}.

§ 6.3. Oscillations of a Structure in the Direction of Flow

The dynamic reaction of the structure of a flight vehicle to a gust of wind (squall) of assigned profile is determined by the solution of equations (14) or (23). Moreover, in a number of cases the influence of additional force of an inertial nature, caused by change in the magnitude and direction of wind speed, is also considered. The value of this force is characterized by the magnitude of apparent additional mass of air. Expressions for the latter can be found by considering the transverse plane-parallel potential flow of an ideal liquid flowing around the body of the flight vehicle. Potential of speeds of such a flow can be presented in the form [64]

$$\Phi(y, z, t) = u_y(t)[\varphi_y(y, z) - y] + u_z(t)[\varphi_z(y, z) - z].$$

where u_y and u_z – components of wind speed, ϕ_y and ϕ_z – corresponding components of unit potential of disturbed speeds. It should satisfy the condition of inseparability (2.31) and the corresponding boundary conditions on surface of the body of the flight vehicle. The latter in this case will consist of equality to zero of the normal (to the surface) component flow rate

$$\left.\frac{\partial\Phi(y, z, t)}{\partial y}\right|_{y=a(x_1)}=0, \quad \left.\frac{\partial\Phi(y, z, t)}{\partial z}\right|_{z=a(x_1)}=0.$$

Furthermore, the shown potential of speeds should additionally satisfy the following condition for infinity:

$$\left.\frac{\partial\Phi(y, z, t)}{\partial y}\right|_{y=\infty}=-u_y(t), \quad \left.\frac{\partial\Phi(y, z, t)}{\partial z}\right|_{z=\infty}=-u_z(t).$$

According to [34] apparent additional masses are equal to

$$\lambda_{ii}(x)=\rho_0 a(x_1)\int_0^{2\pi}\varphi_i\left.\frac{\partial\varphi_i}{\partial n}\right|_{i=a(x_1)}d\alpha \quad (i=y,\ z).$$

Thus, for the circular cross section of a body of a flight vehicle the unknown additional aerodynamic forces will be determined by formulas

$$\Delta q_{ai}(x_i)=\rho_0\frac{\partial u_i}{\partial t}\left[\pi a^2(x_1)+\frac{\lambda_{ii}(x_1)}{\rho_0}\right]=2\rho_0\frac{\partial u_i}{\partial t}\pi a^2(x_1)\ (i=y,\ z).$$

In that case when the period of first tone of inherent transverse oscillations of the structure is small as compared to time of change in speed of the wind the effect of the influence of gusts on a flight vehicle in case V will be close to static. Therefore it is possible to expect that the basic influence on parameters of transverse oscillations of such structures in this case of load will be rendered only by continuous atmospheric turbulence. Then in accordance with the method of overloads the maximum (peak) value of bending moment in plane x_1y_1 will be determined by the expression

$$\max M_y(x_1) = M_{cy}(x_1) + \max M_{ay}(x_1), \qquad (6.24)$$

where

$$\max M_{ay}(x_1) = p\sigma_{My}(x_1),$$

and p – number of standards corresponding to the assigned level of probability of appearance of $\max M_{ay}$, M_{cy} – static bending moment from aerodynamic forces, proportional to the mean value of impact pressure of wind.

The method of determining the average quadratic value of dynamic component of bending moment $\sigma_{My}(x_1)$ depends on the method of finding the spectrum of external aerodynamic forces. If the initial spectrum is the spectrum of distributed aerodynamic load, obtained by means of measurement of pressure on the contour of cross section of the body, then for calculating the reaction of the structure it is necessary to know the correlation of this accidental load along the longitudinal axis of the vehicle. The correlation function of generalized force B_{an} of equation (14) can be determined by means of averaging the product of expressions for B_{an} (15), taken for various values of coordinates.

In the case of fulfillment of projected calculations it is possible to be limited to an approximate calculation of the correlation of these generalized forces. With this goal it follows to divide the length of the body of the flight vehicle into m such sections Δx_i, within the limits of which the correlation of aerodynamic forces is sufficiently great. Then, assuming that correlation of forces between sections is absent, it is possible to present the generalized force B_{an} (in plane x_1y_1 B_{any}) in the form

$$B_{any} = \sum_{j=1}^{m} \rho_0 \Delta u(x_j) \int_{x_1}^{x_1+\Delta x_j} u(x_1) c_y(x_1) d(x_1) f_n(x_1) dx_1. \qquad (6.25)$$

In this case spectral density of energy of the dynamic component of bending moment on the basis of formula (4.19) will be determined by the expression

$$\Phi_{uyx}(x_1, \omega) = \sum_{j=1}^{m} \Phi_{uyj}(x_1, \omega) |G_{uj}(x_1, i\omega)|^2. \qquad (6.26)$$

In the case of a low damping factor the transfer function takes large values only in the area of frequencies close to fundamental frequencies of bending oscillations in the structure, i.e., in region of frequencies $\omega = \omega_n(1 \pm \varepsilon)$, where $\varepsilon \ll 1$. Since the spectral flatness of energy of atmospheric turbulence with $\omega > 1$ changes smoothly with frequency, then if the first tones of frequencies of natural oscillations of the structure are scattered, it is possible with sufficient accuracy to disregard the mutual correlation of generalized coordinates and to take the square of the modulus of the transfer function (for instance, for bending moment) in the form

$$|G_{uj}(x_1, i\omega)|^2 = \sum_{n=1}^{n_k} \frac{\omega^4 M_{nx}^2(x_1)}{\omega_n^4 M_n^2} \frac{F_{nyj}^2(x_j)}{\left[\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \frac{4h_n^2\omega^2}{\omega_n^4}\right]}. \qquad (6.27)$$

where

$$F_{nyj}(x_j) = \rho_0 u_\Phi \int_{x_j}^{x_j + \Delta x_j} \bar{u}(x_1) c_y(x_1) f_n(x_1) d(x_1) dx_1.$$

Disregarding members of the second order of smallness, this expression can be written in the form

$$|G_{uj}(x_1, i\omega)|^2 = \sum_{n=1}^{n_k} \frac{M_{nx}^2(x_1) F_{nyj}^2(x_j)}{M_n^2 \left[4\varepsilon^2 + \left(\frac{\delta_{np}}{\pi}\right)^2\right]}.$$

Further, taking into account that spectral density of pulsation of speed of atmospheric turbulence changes little in narrow frequency bands $\frac{\omega_n \delta}{2}$, we obtain

$$\sigma^2_{uy\text{д}l}(x_1) = \sum_{n=1}^{n_n} \omega_n \Phi_{uyl}(\omega_n) F^2_{nyl}(x_l) \frac{M^2_{nx}(x_1)}{M^2_n} \int\limits_{\omega_n-\varepsilon\omega_n}^{\omega_n+\varepsilon\omega_n} \frac{d\varepsilon}{4\varepsilon^2+\left(\frac{\delta_{\text{пр}}}{\pi}\right)^2}.$$

Since

$$\int\limits_{-\varepsilon\omega_n}^{\varepsilon\omega_n} \frac{d\varepsilon}{4\varepsilon^2+\left(\frac{\delta_{\text{пр}}}{\pi}\right)^2} = \frac{\pi^2}{2\delta_{\text{пр}}},$$

then

$$\sigma^2_{uy\text{д}l}(x_1) = \sum_{n=1}^{n_n} \frac{\omega_n \pi^2 F^2_{nyl}}{2\delta_{\text{пр}} M^2_n} M^2_{nx}(x_1) \Phi_{uyl}(\omega_n) \qquad (6.28)$$

and

$$\sigma^2_{uy\text{д}}(x_1) = \sum_{l=1}^{m_1} \sigma^2_{uy\text{д}l}(x_1),$$

where m_1 – number of sections Δx_j, located on length x_1. Dispersion of the static component of bending moment from aerodynamic forces in this case will be equal to

$$\sigma^2_{uyc}(x_1) = \rho_0^2 u_\Phi^2 \sum_{l=1}^{m_1} \sigma^2_{ul}(x_l) \left[(x_1 - x_l) \int\limits_{x_j-\frac{\Delta x_j}{2}}^{x_j+\frac{\Delta x_j}{2}} \bar{u}_0(x_1) c_{y_1}(x_1) d(x_1) dx_1 \right]^2, \qquad (6.29)$$

where σ_{uj} – average quadratic value of pulsation of wind speed along the flow on altitude of location of section Δx_j of the body of the flight vehicle, and $\bar{u}_0(x_1) = \frac{u_0(x_1)}{u_\Phi}$.

Magnitude σ_{uj}, and consequently $\sigma_{\text{иус}}$, is determined basically by the low-frequency sector of the spectrum of atmospheric turbulence, whereas magnitudes $\sigma_{\text{иуд}}$ depend mainly on values of spectral density in the area of comparatively high frequencies (with ω close to ω_n). In spite of the smallness of values of function $\Phi_{uy}(\omega)$ in the area of frequencies ω_n, the magnitudes of $\sigma_{\text{иуд}}$ at small logarithmic damping decrements of transverse elastic oscillations of flight-vehicle structure exceed $\sigma_{\text{иус}}$ by an order. Therefore, considering approximately that these processes are independent with mean values equal to zero, we will represent the spectral density of energy of reaction of the structure to the influence of atmospheric turbulence in the form of the sum of spectral densities of static bending moment from aerodynamic forces and dynamic bending moment from inertial forces

$$\Phi_{\text{иу}}(x_1, \omega) = \Phi_{\text{иуд}}(x_1, \omega) + \Phi_{\text{иус}}(x_1, \omega).$$

Then the average quadratic value of deviations of bending moment in the direction of wind flow will be determined by the formula

$$\sigma_{\text{иу}}(x_1) = \left[\sigma^2_{\text{иуд}}(x_1) + \sigma^2_{\text{иус}}(x_1)\right]^{\frac{1}{2}}.$$

The interval of subdivision of Δx_j is selected in each specific case, proceeding from the required accuracy of calculation. The greatest influence on magnitudes of sections Δx_j is exerted by the form of transverse oscillations of the body of the flight vehicle. Usually in the area of location of transverse supports of the launcher or "sealing" of the tail end of the flight vehicle body on the launching pad, and namely on a length Δx_1 of an order of 25-50% from l, the values of standardized function $f_n(x_1)$ are small, and consequently the corresponding coefficients F_{nyj} and $\sigma_{\text{иуд}j}$ (at any $\Delta x_j < \Delta x_1$) will be comparatively small. In practice the parameters of transverse oscillations of the structure of similar flight vehicles will be determined by the aerodynamic forces acting on the upper part of the vehicle body. Therefore, if the length of the flight vehicle is comparatively small (order of 15-30 m), then in first approximation it is possible to take $\Delta x_j = \frac{l}{2}$.

The solution of the given problem is noticeably simplified (Δx_j becomes equal to $l, m=1$) in that case when spectral density is used not of distributed, but of total aerodynamic load acting on the vehicle from pulsation of wind speed. Such a spectral density can be obtained by means of measurement and the corresponding statistical treatment of the reaction of various flight vehicles, smoke tubes, radio masts, and so forth to the influence of atmospheric turbulence. It can be represented in the form of spectral density of generalized aerodynamic force in a function from the Strouhal number or in the form of spectral density of pulsation of flow rate (4), and namely:

$$\Phi_{uy0}(\omega)=2L_0\sigma_u^2\left(\frac{1}{\pi u_0}\right)\frac{1}{1+\left(\frac{\omega L_0}{u_0}\right)^2},$$

where σ_u – standard deviation of pulsation of wind speed on altitude h_ϕ, and L_0 – generalized scale of turbulence, characterizing the length of the section, within the limits of which the pulsations of wind speed are correlated sufficiently. For example, Fig. 6.2 shows the tentative values of such a function $\Phi_{uy0}(\omega)$, corresponding to spectral density $\Phi_{uy}(x_1, \omega)$ shown in Fig. 6.3. For a comparison in the same Fig. 6.2 the dotted line represents the values of spectral density of pulsation of wind speed, measured at a point located on standard altitude h_ϕ. As can be seen, with this realization the values Φ_{uyj} and Φ_{uy0} are similar. In the case of other realizations a difference was observed only in the area of very low frequencies. Thus, in this case

$$\sigma_{uyx}^2(x_1)=\sum_{n=1}^{n_n}\frac{\omega_n\pi^2}{2\delta_{np}}F_{ny}^2(x_1)\frac{M_{nx}^2(x_1)}{M_n^2}\Phi_{uy0}(\omega_n), \tag{6.30}$$

where

$$F_{ny}(x_1)=\rho_0u_\phi\int_0^l\bar{u}_0(x_1)c_{y_1}(x_1)d(x_1)f_n(x_1)dx_1.$$

In order to establish necessary static carrier ability in the construction of the flight vehicle for this case of load, it is necessary to find that maximum level of internal forces which can be attained at least once during the entire prelaunch period. According to (4.20) the unknown peak value of dynamic bending moment for a stationary process can be obtained from equation

$$\frac{T_0\sigma_{\dot{M}_{y\lambda}}(x_1)}{2\pi\sigma_{M_{y\lambda}}(x_1)} e^{-\frac{\max M^2_{y\lambda}(x_1)}{2\sigma^2_{M_{y\lambda}}(x_1)}} = 1,$$

where T_0 – time of influence on the flight vehicle of wind with an average speed u_0, and

$$\sigma_{\dot{M}_{y\lambda}}(x_1) = \int_0^\infty \omega^2\Phi_{M_{y\lambda}}(x_1, \omega)\,d\omega.$$

It will be equal to

$$\max M_{y\lambda}(x_1) = 1{,}4\sigma_{M_{y\lambda}}(x_1)\sqrt{\ln\frac{T_0}{T_e}}.$$

Consequently

$$p = 1{,}4\sqrt{\ln N}. \qquad (6.31)$$

where N – number of cycles, equal to $\frac{T_0}{T_e}$, and T_e – effective period of oscillations

$$T_e = 2\pi\frac{\sigma_{M_{y\lambda}}(x_1)}{\sigma_{\dot{M}_{y\lambda}}(x_1)}.$$

If oscillation of the structure occur with a frequency of the first tone, then it is possible to assume that $T_e = \frac{2\pi}{\omega_n}$ (with $n=1$), i.e., $N = \lambda_1 T_0$, where λ_1 – frequency of first tone in cycles per second.

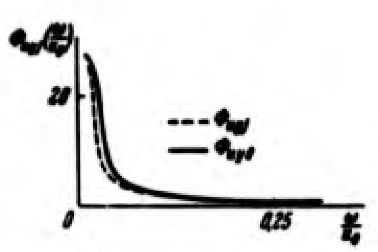

Fig. 6.2. Generalized spectral density of energy of pulsation of wind speed (along the flow) at $U_\phi = 12$ m/s.

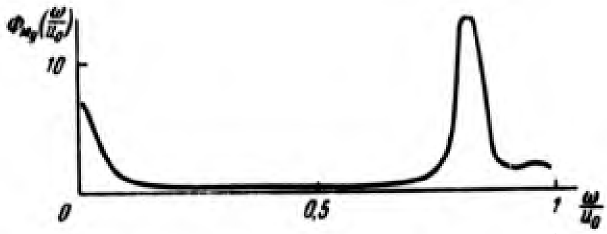

Fig. 6.3. Form of the function of spectral density of energy of bending moment in flow density of wind.

Having determined T_0, M_{yc}, $\sigma_{My\text{д}}$ in a function from u_0, it is not difficult to find possible values of limiting bending moments for the assigned range of average wind speeds and assigned duration of their action in the region of the launching site. The maximum magnitude of such a limiting bending moment also will determine the unknown required static carrier ability of the flight-vehicle structure. Here, considering that with an increase of u_0, $\delta_{\text{пр}}$ is increased and N decreases, it is possible to expect that this maximum $M_{y\text{д}}(x_1)$ in general will not coincide with max u_0.

Peak value of transverse overload in any cross section x_1 of the flight-vehicle body will be determined by the expression

$$\max n_{y_1}(x_1) = p\left[\sum_{n=1}^{n_n} \frac{\omega_n n^2}{2\delta_{\text{пр}} M_n^2} F_{ny}^2(x_1)\Phi_{uy0}(\omega_n) f_n^2(x_1)\right]^{\frac{1}{2}} \frac{1}{g_0}.$$

A rough estimate of the influence of elastic oscillations of construction on the values of bending moments in support sections of the flight-vehicle body can be obtained if one were to use the concept of coefficient of dynamic state

$$\eta(x_1) = \frac{\max M_y(x_1)}{\max M_{yc}(x_1)},$$

where $\max M_{yc}(x_1)$ – maximum static bending moment from aerodynamic forces, proportional to the peak value of impact pressure. The magnitude of this coefficient is found most simply from the condition of equiprobability of events (4.21), which in the given case takes the form

$$\max M_y(x_1) = \bar{M}_{yc}(x_1) + [\max M_{yc}(x_1) - \bar{M}_{yc}(x_1)]\,\frac{\sigma_{uy\text{д}}(x_1)}{\sigma_{uyc}(x_1)}.$$

Since approximately

$$\frac{\max M_{yc}(x_1)}{\bar{M}_{yc}(x_1)} = \frac{\max q}{\bar{q}}, \quad \text{and} \quad \bar{M}_y(x_1) = \bar{M}_{yc}(x_1),$$

then, presenting $\sigma_{uyc}(x_1)$ in a form analogous to $\sigma_{uy\text{д}}(x_1)$:

$$\sigma^2_{uyc}(x_1) = \sum_{n=1}^{n_n} F^2_{ny}(x_1)\,\sigma^2_u\,\frac{M^2_{nx}(x_1)}{M^2_n}$$

and substituting max $q = k^2\bar{q}$, where $k = 1 + p\frac{\sigma_в}{u_0}$ — coefficient of gustiness of wind, we will have

$$\eta(x_1) = \frac{1}{k^2}\left[1 - \frac{\sigma_{uy\text{д}}(x_1)}{\sigma_{uyc}(x_1)}\right] + \frac{\sigma_{uy\text{д}}(x_1)}{\sigma_{uyc}(x_1)}.$$

With such a formulation of the problem the question of establishment of limiting values of bending moment max $\dot{M}_y(x_1)$ is essentially reduced to a question of selection of magnitude of coefficient of gustiness of the wind. It is recommended [87] for case V to take k of an order of 1.4, and for case U — somewhat less.

Being limited to a calculation of only the first tone of inherent bending oscillations of the structure, we find that

$$\frac{\sigma_{uy\text{д}}(x_1)}{\sigma_{uyc}(x_1)} = \left[\frac{\omega_n\pi^2\Phi_{uy0}(\omega_n)}{2\delta_{np}\sigma^2_u}\right]^{\frac{1}{2}}.$$

Thus with the accepted assumptions the coefficient of dynamic state approximately for n = 1 will be equal to

$$\eta = \frac{1}{k^2} + \left(1 - \frac{1}{k^2}\right)\left[\frac{\pi\omega_n u_\Phi L_0}{\delta_{np}\left(u^2_\Phi + \omega^2_n L^2_0\right)}\right]^{\frac{1}{2}}. \qquad (6.32)$$

Its magnitude is a function of frequency of inherent transverse oscillations of the body of the flight vehicle, damping factor of oscillations, and parameters of wind flow. For an illustration Fig. 6.4 shows the graphs for $\eta(\lambda_1)$ with k = 1.3, u_0 = 20 m/s for l < 25 m and for a number of values of coefficient

$$c_{nx} = \frac{\omega_n}{\pi u_\Phi}(\delta_0 - 0{,}05) + \frac{\rho_0}{M_n}\int_0^l c_{y_1}(x_1)\,u_0(x_1)\,d(x_1)\,f_n(x_1)\,dx_1$$

with $n = 1$. (6.33)

As can be seen, the coefficient of dynamic state decreases with an increase of frequency $\lambda_1 = \frac{\omega_n}{2\pi}$ and the damping factor. It also drops with an increase of L_0, i.e., the length of the vehicle body. The largest values of $\eta(\lambda_1)$ are observed in the area of low frequencies of λ_1, for which the assumptions accepted above, in particular about the calculation of influence of only the first tone of inherent bending oscillations of construction, are sometimes unacceptable. For similar flight vehicles the calculation of dynamic moments should be conducted taking into account members corresponding to n = 2 and even n = 3. It is necessary to stress that in the zone of small frequencies of oscillations aerodynamic damping exerts a very large influence on the magnitude of η. The role of nonelastic resistance (constructional damping) appears only at comparatively large frequencies of ω_n.

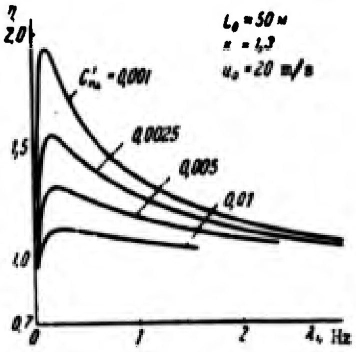

Fig. 6.4. Influence of aerodynamic damping and frequency of inherent bending oscillations of a structure on the coefficient of dynamic state in case V.

§ 6.4. Oscillations of Construction in the Plane, Perpendicular to the Direction of Flow

It has been established experimentally that at small Reynolds numbers the intensity of lateral bending oscillations of cylindrical bodies with large elongation (in a plane, perpendicular to the direction of average wind speed) increases with an increase of Re. In the region $Re_{кр}>3 \cdot 10^5$, corresponding to crisis of flow around a cylinder, i.e., in the region of a sharp drop in drag, it decreases. Then again it increases. At large Reynolds numbers (larger than critical $Re_{кр}$), which are of basic interest to designers of flight vehicles, the influence of Re on parameters of oscillations of a body becomes comparatively small, and it is frequently disregarded [83]. In the region of subcritical Reynolds numbers the amplitude of lateral oscillations of a cylindrical body in the case of a constant flow rate depends mainly on the Strouhal number Sh. It is noticed that it increases along with approximation of frequency of separation of Benar-Karman vortexes to the frequency of inherent elastic lateral oscillations of the construction. At Strouhal numbers of an order of 0.15-0.28, which are called critical, the phenomenon of "wind" resonance is observed. In this case the separation of vortexes occurs strictly in extreme positions of a fluctuating body and along the entire length simultaneously. In reality (in the case of the influence of surface wind on the body of a vertically standing flight vehicle) such a pattern of flow is not realized due to the sharp change of average wind speed in altitude (2), and consequently also the local Strouhal numbers. Under these conditions theoretically the simultaneous breaking away of vortexes along the entire length of body of a flight vehicle becomes possible only in the presence of a reverse ordered influence of motion of the body on the process of vortex formation, i.e., in the case when the system elastic body – plane-parallel flow of air is a self-oscillating system. In practice in certain cases the phenomenon of resonance can take place also in real conditions of flowing around. This is explained by the difference in degree of influence of aerodynamic forces of separate parts of the body of a flight vehicle on parameters of structural oscillations, in particular due to a strong change of function of form of lateral oscillations along the length.

In the case of supercritical Reynolds numbers the separation of vortexes occurs irregularly and flow in the trace behind a poorly streamlined body becomes completely turbulent. Results of natural tests on full-scale cylindrical bodies with large elongation testify to the absence of any discrete components in the spectrum of lateral generalized aerodynamic force (Fig. 6.5). They show that external aerodynamic influence can be described with sufficient accuracy for practical purposes by a random stationary ergodic normal process.

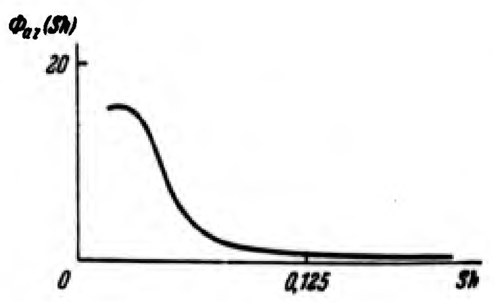

Fig. 6.5. Typical curve of spectral density of lateral generalized aerodynamic load on the body of a flight vehicle in case V with $Sh \neq Sh_{\text{кр}}$.

Spectral density of energy of lateral aerodynamic load in this case can be presented in the form of the sum or spectral densities of statistically independent surface forces, caused by peculiarities of flow around a body of a flight vehicle and the direct influence of atmospheric turbulence,

$$\Phi_z(x_1, \omega) = \Phi_{\text{в}z}(x_1, \omega) + \Phi_{\text{т}}(x_1, \omega).$$

The expressions of these spectral densities are identical and differ from each other only in scales of turbulence and dispersions. The scale of atmospheric turbulence in a lateral direction (5) is equal to half the scale of atmospheric turbulence in a longitudinal direction,[2] i.e., has an order of l, and the scale of turbulence in the trace behind the vehicle does not exceed two diameters of the body. Correspondingly the coefficients of spatial correlation differ.

Values of function $\Phi_{\alpha z}(x_1, \omega)$ are determined by formulas (6) and (5) with replacement of $c_y(x_1)$ by $c_z(x_1)$. Values $\Phi_T(x_1, \omega)$ essentially depend on the state of the body surface of the flight vehicle (its roughness) and especially on the form of nose cone. The presence of various flanges on the body of the flight vehicle distorts streamline flow and in certain cases can even lead to the appearance of a local periodic breakdown of flow. The influence of geometry of the front part of the vehicle (end effect) is noted usually in the zone $0{,}08 < \mathrm{Sh} < 0{,}15$. It is manifested in a noticeable change of lateral aerodynamic loads acting on this part of the structure. And although this influence is limited to a region of an order of one-two diameters of the body, it noticeably influences the reaction of the construction due to large values of function of form $f_n(x_1)$ in the peak of the vehicle. In connection with what was presented it is recommended that $\Phi_T(x_1, \omega)$ be determined experimentally by means of testing special drained completely aerodynamically similar models of the flight vehicle in a wind tunnel with small turbulence of flow.

Spectral densities of generalized lateral aerodynamic forces, obtained by means of measurement and corresponding statistical treatment of the reaction of dynamically similar models of flight vehicles have a limited value in this case of load. This is explained by the fact that in a wind tunnel the profile of wind flow is not modelled. Therefore these data can be used only for obtaining the upper estimate of the corresponding reaction of the structure (under the condition of equality of Strouhal numbers for the model and nature of the front part of the body of the flight vehicle). Similar spectral densities of energy of generalized lateral aerodynamic forces for cylindrical bodies of ballistic missiles (without indication of their elongation) with various forms of nose cones are given in work [83] and are reproduced partially in Fig. 6.6 in the function of Strouhal number. Here the following designation is used

$$\Phi(\mathrm{Sh}) = \frac{16\pi u_0}{d_0 S^2 q^2} \Phi_z(\omega_n).$$

Peak values of corresponding lateral forces, bending moments, transverse overloads, for angles of turn of cross sections of the body of the flight vehicle in this case will be determined by expressions

$$\left.\begin{aligned}\max Q_{zn}(x_1) &= a_n Q_{nx}(x_1),\\ \max M_{zn}(x_1) &= a_n M_{nx}(x_1),\end{aligned}\right\} \tag{6.34}$$

$$\left.\begin{aligned}\max n_{zn}(x_1) &= a_n f_n(x_1)\frac{1}{g_0},\\ \max \vartheta_n(x_1) &= a_n \frac{df_n(x_1)}{dx_1}\frac{1}{\omega_n^2},\end{aligned}\right\} \tag{6.35}$$

where

$$a_n = \rho_0 a_{n0} d_0^{\frac{3}{2}} u_0^{\frac{3}{2}} \frac{1}{4M_n}\left[\frac{\pi}{\delta_n}\omega_n \Phi(\mathrm{Sh})\right]^{\frac{1}{2}},$$

$$a_{n0} = \int_0^l f_n(x_1)\frac{d(x_1)}{d_0}\,dx_1.$$

Calculation of parameters of lateral structural oscillations of the flight vehicle, and also corresponding dynamic forces, can be carried out by the method presented in the preceding section. Thus, for instance, maximum value of lateral bending moment can be determined by formula

$$\max M_z(x_1) = p\sigma_{\text{м}z}(x_1), \tag{6.36}$$

where

$$\sigma_{\text{м}z}(x_1) = \left[\sigma^2_{\text{м}zc}(x_1) + \sigma^2_{\text{м}z\text{л}}(x_1)\right]^{\frac{1}{2}}.$$

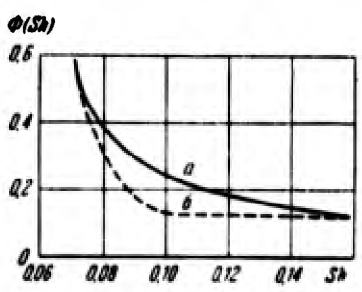

Fig. 6.6. Dependence of spectral density of lateral aerodynamic force on Strouhal number and form of nose cone of a ballistic missile (a — for conical, b — for nose cone in the form of a hemisphere.

Taking into account the mutual independence of random processes considered in that section with mean values equal to zero, it is possible to write

$$\sigma^2_{\text{мzc}}(x_1) = \sigma^2_{\text{мzcб}}(x_1) + \sigma^2_{\text{мzcв}}(z_1). \tag{6.37}$$

$$\sigma^2_{\text{мzд}}(x_1) = \sigma^2_{\text{мzдб}}(x_1) + \sigma^2_{\text{мzдв}}(z_1). \tag{6.38}$$

The values $\sigma_{\text{мzcв}}$ entering in (37) are determined by formula (29) with replacement of $c_y(x_1)$ by $c_z(x_1)$, and $\sigma_{\text{мzcб}}$ by expression

$$\sigma^2_{\text{мzcб}}(x_1) = \sum_{j=1}^{m_1} \sigma^2_{бj}(x_1 - x_j)^2 \Delta x_j^2. \tag{6.39}$$

where x_j – coordinate of center of pressure of j-th section of the body of the flight vehicle, m_1 – number of sections located higher than the considered section x_1, and

$$\sigma^2_{бj} = \int_0^\infty \Phi_{zб}(x_j, \omega)\, d\omega. \tag{6.40}$$

The average quadratic values of dynamic component of bending moment, caused by the influence of atmospheric turbulence, are found by a formula similar to (28), and namely:

$$\sigma^2_{\text{мzдв}}(x_1) = \sum_{j=1}^{m_1} \sum_{n=1}^{n_n} \frac{\pi^2 \omega_n}{2\delta} \frac{M^2_{nz}(x_1)}{M^2_n} F^2_{nzj}(x_1) \Phi_{uzj}(x_j, \omega). \tag{6.41}$$

where

$$F_{nzj}(x_j) = \rho_0 u_\Phi \int_{x_j - \frac{\Delta x_j}{2}}^{x_j + \frac{\Delta x_j}{2}} \bar{u}_0(x_1) c_z(x_1) f_n(x_1) d(x_1)\, dx_1.$$

The spatial correlation of lateral aerodynamic forces, generated by the wake, is commensurable with the scale of turbulence,[3] i.e., is comparatively weak. Therefore in the calculation of $\sigma_{\text{мгсб}}$ and $\sigma_{\text{мгдб}}$ one should take intervals Δx_j small, of an order of one-two diameters. With such a relatively small "step" it is possible in first approximation to disregard the influence of change in the function of form of inherent lateral structural oscillations of the flight vehicle on length Δx_j and take

$$\sigma^2_{\text{мгдб}}(x_1) = \sum_{j=1}^{m_1} \sum_{n=1}^{n_n} \frac{\omega_n \pi^2}{2\delta} \frac{M^2_{nx}(x_1)}{M^2_n} \Delta x_j^2 f_n^2(x_j) \Phi_{z6}(x_j, \omega_n). \qquad (6.42)$$

And in this case with small logarithmic damping decrements δ the magnitudes of $\sigma_{\text{мгдб}}$ considerably exceed $\sigma_{\text{мгсб}}$.

Peak values of lateral overloads and angles of turn of cross sections of the body of the flight vehicle are determined by expressions

$$\left.\begin{aligned} \max n_{z_1}(x_1) &= p\left[\sum_{j=1}^{m_1} \sum_{n=1}^{n_n} \frac{\pi^2 \omega_n}{2\delta} \frac{\Delta x_j^2}{M_n^2} f_n^2(x_j) f_n^2(x_1) \Phi_{z6}(x_j, \omega)\right]^{\frac{1}{2}} \frac{1}{g_0}, \\ \max \psi(x_1) &= p\left[\sum_{j=1}^{m_1} \sum_{n=1}^{n_n} \frac{\pi^2}{2\delta\omega_n^3} \frac{\Delta x_j^2}{M_n^2} f_n^2(x_j) \left[\frac{df_n(x_1)}{dx_1}\right]^2 \Phi_{z6}(x_j, \omega)\right]^{\frac{1}{2}}. \end{aligned}\right\} \qquad (6.43)$$

As experiments show [10, 83], in this case of load the values of p (even in the case of comparatively small intervals of averaging) reach 4-5.

In conclusion let us note in general neither $M_y(x_1)$, nor $M_t(x_1)$, taken separately, determine the necessary carrier ability for the construction of the flight vehicle. If the probability density functions of these components are known, then it is possible to detect those values of total bending moment $M(x_1)$, which will not be exceeded (with a certain probability) in process of the entire time of exploitation. If such functions are unknown, it follows in first approximation to be guided by limiting values of bending moments, which can be found by means of geometric addition of maximum magnitudes of the stated components

$$M_{\text{пр}}(x_1) = \left[\max M_y^2(x_1) + \max M_z^2(x_1)\right]^{\frac{1}{2}}.$$

Usually for flight vehicles which are symmetric relative to the longitudinal axis the lateral component of bending moment exceeds by a few times the dynamic component of transverse bending moment. In spite of this, its influence on magnitude of total bending moment (at comparatively large ω_n) in many cases turns out to be small. For such vehicles, and also for flight vehicles, which are asymmetric in an aerodynamic respect (with coincidence of plane of wind flow with the direction of the largest wind-action surface), in practice the case $M_y(x_1)$ will be calculated. [Translator's note: wind-action surface – that which is subjected to the action of the wind.]

In the case U both the bending moment and also the longitudinal force will additionally depend on angle of inclination of the erector boom Θ_0 (or the guide rails of the launcher). This dependence is manifested in a change by Θ_0 of the longitudinal and transverse forces from the weight of construction of the flight vehicle

$$\left.\begin{aligned} N_G(x_1) &= \sin\Theta_0 \int_0^{x_1} q_{Gx}(x_1)\,dx_1,\\ Q_G(x_1) &= \cos\Theta_0 \int_0^{x_1} q_{Gy}(x_1)\,dx_1 \end{aligned}\right\} \qquad (6.44)$$

and corresponding aerodynamic coefficients $c_y(x_1)$, $c_z(x_1)$, $c_x(x_1)$.

§ 6.5. Modeling of Oscillations of a Vehicle in a Flow of Air

In those cases when lateral structural oscillations of the flight vehicle become dangerous for the strength of its body, one should turn to the experimental method of determination of dynamic forces and amplitudes of structural oscillations. As was already noted the values of transverse forces, bending moments, and amplitudes of oscillations depend on the dynamic characteristics of construction (frequencies and forms of inherent bending oscillations and damping factors), on the parameters of flow (speed, density, and viscosity of air), on the dimensions of the flight vehicle (diameter and length) and on the aerodynamic shape of body, in particular on the form of its forward section.

The process of oscillation of a flight vehicle in an air flow or oscillation of a dynamically similar model of it in the flow of a wind tunnel are described by the same equations, which include all the above-indicated parameters. Therefore it is natural that the model should be similar to the full-scale flight vehicle in geometric form and in distribution of mass, and also possess identical forms of its inherent bending oscillations and values of logarithmic decrements of damping of oscillations.

For observance of all the conditions of similarity it is necessary that members of equations of bending oscillations in the construction of a flight vehicle differ from the corresponding members of equations of bending oscillations in the model only by a constant factor which is identical for all members. On the basis of this condition it is possible to obtain the required criterion of similarity of the model to full-scale. It turns out that for the full-scale flight vehicle and dynamically similar model it is necessary to observe the equality of dimensionless Strouhal, Reynolds, and Cauchy numbers

$$\frac{\omega d}{u_0} = \frac{\omega_{\text{м}} d_{\text{м}}}{u_{0\text{м}}}, \tag{6.45}$$

$$\frac{\rho d u_0}{\mu} = \frac{\rho_{\text{м}} d_{\text{м}} u_{0\text{м}}}{\mu_{\text{м}}}, \tag{6.46}$$

$$\frac{EI_{\text{c}}}{\rho d^4 u_0^2} = \frac{(EI_{\text{c}})_{\text{м}}}{\rho_{\text{м}} d_{\text{м}}^4 u_{0\text{м}}^2}, \tag{6.47}$$

where subscript "м" designates the parameters related to the model. For satisfying the conditions of similarity of dynamic characteristics it is necessary, in addition to observance of similarity in distribution of mass and rigidity of construction and equality of logarithmic damping decrements of oscillations, to ensure identical boundary conditions.

To reproduce exactly all these criteria of similarity and especially boundary conditions on a model is very difficult. Therefore in each specific case it is necessary to conduct a detailed investigation and proving of all deviations from conditions of similarity. It has been established, for example, that the least error is obtained in that case when similarity in Reynolds number is disturbed (under the condition that it remains larger than critical). Agreement of boundary conditions can be improved by means of modeling the launcher itself.

Geometric dimensions of the model are selected, proceeding from dimensions of the operational section of the wind tunnel. The type of tunnel determines the density and viscosity of its flow. Frequently for these experiments wind tunnels of variable density are used.

For ensuring the equality of logarithmic damping decrements of oscillations the model is made of metal. Usually its construction constitutes a tube with sections which simulate fuel tanks. The thickness of this tube is selected in accordance with the distribution of rigidity of the body. The tanks are filled with some load (fraction) or liquid. Required distribution of mass of the model is ensured by attachment of additional loads to its body. All remaining scale factors, in particular $k_\omega = \frac{\omega}{\omega_м}$, $k_u = \frac{u_0}{u_{0м}}$, are taken from conditions of similarity (45)-(47). For appraisal of the degree of accuracy in the preparation of the model an experimental determination is made of its dynamic characteristics, which are compared with calculated. The model is equipped with the necessary measuring devices — vibration pickups, strain gauges, and pressure pickups.

Tests of a dynamically similar model in a wind tunnel make it possible to determine the magnitude of lateral aerodynamic forces acting on the flight vehicle in a plane perpendicular to the flow, and also to establish the effectiveness of measures undertaken for reducing the amplitudes of these oscillations. In particular, from the formulas given in § 6.4, it follows that the lowering of these amplitudes of oscillations of the flight vehicle can be attained by means of increasing the logarithmic damping decrement of oscillations, increasing the rigidity of the body, and the corresponding selection of form of the forward section. A great influence on lateral oscillations of the flight vehicle is exerted by the roughness of its body surface, the presence of various projections, and also the presence of other bodies near it (masts, service towers, and so forth). In connection with this it is recommended to test the vehicle model taking into account the aerodynamic influence of assemblies of ground equipment which are located near the flight vehicle in the prelaunch period.

In conclusion let us note that the parameters of turbulence of flow in a tunnel essentially differ from the parameters of atmospheric turbulence (in the surface layer). Therefore the results of tunnel experiments cannot be transferred directly (without the corresponding correction) to full-scale.

§ 6.6. Dynamic Calculation of Umbilical Tower

For supplying fueling facilities and various cables to a flight vehicle, ballistic missiles in particular, frequently a special tower is used, which prior to launching is removed on a certain angle, thus avoiding collision between it and the vehicle. This removal is done under the action of its own weight or with help of external transverse force P_c. The magnitude and time of action of the latter are selected, proceeding from the values of external forces and moments acting on the tower in the process of removal and requirements, imposed for the time of removal of the tower on a safe angle $\phi_{\text{б}}$ (here clockwise).

Schematically the tower can be presented in the form of flexible rod of great elongation which is joined by a hinge to a pillar (Fig. 6.7). At the initial moment this rod can lean in several places on the body of the flight vehicle. Let us designate by Y the aerodynamic (wind) load, effective in a horizontal direction on the tower in the plane of its removal, by $R_д$ – force of reaction of the damper (brake), and by x, y, and z – axes of the mobile system of coordinates connected with the tower and with an origin at point O. Axis x we direct along the undeformed longitudinal axis of the tower.

Let us assume that the axis of rotation of the tower O_1 is found at distance y_0 from axis x. Let us assume that the law of change of arm $e_д$ of force $R_д$ with respect to point O_1 depending on the angle of rotation ϕ is known.

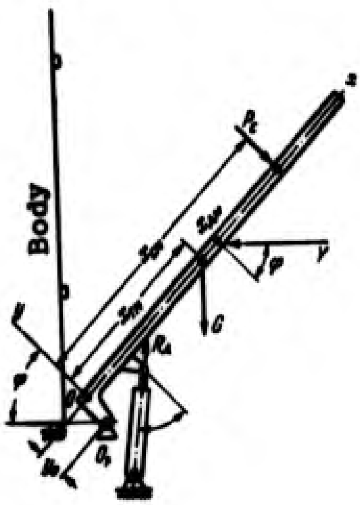

Fig. 6.7. Diagram of external forces acting on an umbilical tower in process of removal from the flight vehicle.

We obtain equation of rotation of the tower as solid body in plane xy by equating the sum of moments of all external forces with respect to point O_1 to the inertial moment:

$$\begin{aligned}&Y(x_{дм}\cos\varphi + y_0\sin\varphi) + e_д R_д -\\&- x_{см} P_c -\\&\quad - G(x_{тм}\sin\varphi - y_0\cos\varphi) +\\&\quad + (J_{zм} + M y_0^2)\ddot{\varphi} = 0,\end{aligned} \qquad (6.48)$$

where $x_{\text{см}}$ – coordinate of place of application of removing force P_c, $x_{\text{дм}}$ – coordinate of center of pressure of tower, $x_{\text{тм}}$ – coordinate of center of gravity of tower, $I_{z\text{м}}$ – mass moment of inertia of mast relative to the transverse axis passing through point O. Those forces, the direction of action of which coincides with direction of axes of coordinates are considered positive.

For compilation of equations of dynamic equilibrium of the tower taking into account elastic oscillations of its construction one should write the expression for bending moment in some section x_1 through external surface and mass forces and equate to its moment of forces of elasticity

$$M(x_1) = E(x_1) I_{\text{см}}(x_1) \frac{\partial^2 y(x, t)}{\partial x^2}\Big|_{x=x_1}, \tag{6.49}$$

where y(x, t) designates sag of the tower in section x due to bending, and $I_{\text{см}}$ – area moments of inertia of cross section of the carrier part of the tower structure relative to transverse axis z.

Components of linear mass force in the system of coordinates xy, connected with the undeformed position of the tower, are determined by formulas

$$\left.\begin{aligned} q_{\text{м}x}(x) &= m(x)(x\dot{\varphi}^2 - y_0\ddot{\varphi} - g_0\cos\varphi), \\ q_{\text{м}y}(x) &= m(x)(y_0\dot{\varphi}^2 - \ddot{y} + \ddot{\varphi}x - g_0\sin\varphi). \end{aligned}\right\} \tag{6.50}$$

The moment from these forces in the stated section $x_1 < x$ will be equal to

$$q_{\text{м}y}(x)(x - x_1) - q_{\text{м}x}(x)[y(x) - y(x_1)].$$

Bending moment from all external forces can be represented in the form

$$
\begin{aligned}
M(x_1) = &-\int_{x_1}^{l_{\text{м}}} P_{\text{c}}(x_{\text{см}} - x_1)\delta(x - x_{\text{см}})\,dx + \\
&+\int_{x_1}^{l_{\text{м}}} \frac{\partial Y(x)}{\partial x}(x - x_1)\cos\varphi\,dx + \int_{x_1}^{l_{\text{м}}} R_{\text{д}}[(x_{\text{д}} - x_1)\cos\varphi_1 + \\
&+ y_{\text{д}}\sin\varphi_1]\,\delta(x - x_{\text{д}})\,dx + \int_{x_1}^{l_{\text{м}}} (x - x_1)\,q_{\text{м}y}(x)\,dx - \\
&- \int_{x_1}^{l_{\text{м}}} q_{\text{м}x}(x)\,[y(x) - y(x_1)]\,dx,
\end{aligned} \qquad (6.51)
$$

where $\delta(x - x_{i\text{м}})$ – Dirac delta function

$$\delta(x - x_{i\text{м}}) = 0 \quad \text{with} \quad x \neq x_{i\text{м}},$$

$$\int_{x_1}^{l_{\text{м}}} F(x)\,\delta(x - x_{i\text{м}})\,dx = F(x) \quad \text{with} \quad x_{i\text{м}} > x_1 \ (i\text{м} = \text{см, д}),$$

$l_{\text{м}}$ – length of tower $x_{\text{д}}$ and $y_{\text{д}}$ – coordinates of place of application of force of reaction of the damper to the tower structure, m(x) – linear mass of tower (with filling), φ_1 – angle of inclination of line of force of damper reaction to axis y.

Integrating the expression for $q_{\text{м}x}$ (50) from x_1 to $l_{\text{м}}$, we find the following formula for calculation of longitudinal force in arbitrary section x_1:

$$N(x_1) = \dot{\varphi}^2 \int_{x_1}^{l_{\text{м}}} m(x)\,x\,dx - (\ddot{\varphi} y_0 + g_0\cos\varphi) \int_{x_1}^{l_{\text{м}}} m(x)\,dx - \sin\varphi \int_{x_1}^{l_{\text{м}}} \frac{\partial Y(x)}{\partial x}\,dx.$$

Differentiating expression (49) and (51) twice in respect to x, it is possible to obtain the unknown equation of motion of the tower in the form

$$
\begin{aligned}
&\frac{\partial^2}{\partial x^2}\left[E(x)\,I_{\text{см}}(x)\,\frac{\partial^2 y(x,t)}{\partial x^2}\right] + m(x)\left[g_0\sin\varphi + \frac{\partial^2 y(x,t)}{\partial t^2} - \right. \\
&\left. - x\frac{d^2\varphi}{dt^2} - y_0\left(\frac{d\varphi}{dt}\right)^2\right] + m(x)\left[x\left(\frac{d\varphi}{dt}\right)^2 - y_0\frac{d^2\varphi}{dt^2} - g_0\cos\varphi\right]\frac{\partial y(x,t)}{\partial x} = \\
&= \frac{\partial Y(x,t)}{\partial x}\cos\varphi - P_{\text{c}}\delta(x - x_{\text{см}}) + \\
&+ R_{\text{д}}\left[\cos\varphi_1\,\delta(x - x_{\text{д}}) + y_{\text{д}}\sin\varphi_1\,\frac{d}{dx}\,\delta(x - x_{\text{д}})\right],
\end{aligned} \qquad (6.52)
$$

where

$$\frac{d}{dx}\delta(x - x_1) = 0 \quad \text{with} \quad x \neq x_1,$$

$$\int_{x_1}^{l_M} F(x)\frac{d}{dx}\delta(x - x_1)\,dx = \frac{dF(x)}{dx}\Big|_{x=x_1} \quad \text{with} \quad x > x$$

If y_0 is small compared to l_M, then boundary conditions for a swiveling tower will consist of the absence of bending moment and transverse force on the free end and of equality to zero of bending moment and the transverse shift of the axis of the tower at the point of location of axis of rotation:

$$\left.\begin{aligned} &\frac{\partial}{\partial x}\left[E(x)\,I_{cM}(x)\frac{\partial^2 y(x, t)}{\partial x^2}\right] = 0 \quad \text{with} \quad x = l, \\ &\frac{\partial^2 y(x, t)}{\partial x^2} = 0 \quad \text{with} \quad x = l \text{ и } x = 0, \\ &y = 0 \quad \text{with} \quad x = 0. \end{aligned}\right\} \tag{6.53}$$

Under these conditions of function the forms of inherent bending oscillations of the tower $f_{nM}(x)$ will be orthogonal among themselves. Therefore for finding the solution of equation (52) it is possible to use the method of expansion of sag y(x, t) in respect to eigenfunctions $f_{nM}(x)$:

$$y(x, t) = \sum_{n=1}^{\infty} S_{nM}(t) f_{nM}(x). \tag{6.54}$$

We will place (54) in (52) and multiply its right and left sides by $f_{nM}(x)$. Then we integrate in respect to x from 0 to l_M. As a result we obtain the following approximate equation for determination of values of functions of form and frequency of inherent transverse oscillations of tower construction:

$$\frac{d^2}{dx^2}\left[E(x)\,I_{cM}(x)\frac{d^2 f_{nM}(x)}{dx^2}\right] - m(x)\,\omega_{nM}^2 f_{nM}(x) + x m(x)\left(\frac{d\varphi}{dt}\right)^2 \frac{df_{nM}(x)}{dx} - \\ - m(x)\left(y_0\frac{d^2\varphi}{dt^2} + g_0\cos\varphi\right)\frac{df_{nM}(x)}{dx} = 0 \quad (n = 1, 2, \ldots). \tag{6.55}$$

The solution of this equation in the case of boundary conditions

$$\left.\begin{array}{l}\frac{d^2 f_{n\text{м}}(x)}{dx^2}=0, \quad \frac{d}{dx}\left[E(x)I_{\text{cм}}(x)\frac{d^2 f_{n\text{м}}(x)}{dx^2}\right]=0 \quad \text{with} \quad x=l, \\ f_{n\text{м}}(x)=0, \quad \frac{d^2 f_{n\text{м}}(x)}{dx^2}=0 \quad \text{with} \quad x=0\end{array}\right\} \tag{6.56}$$

is realized by the method of successive approximations. From equation (55) it is clear that normal functions and frequencies of inherent bending oscillations of the tower depend on the rate of its rotation and also angular acceleration. However, in practice the influence of members, proportional to $\frac{d\varphi}{dt}$ and $\frac{d^2\varphi}{dt^2}$, can be disregarded.

The equation for determination of coefficients of expansion of dynamic sags of the tower in a series based on forms of natural oscillations taking into account damping will have the form

$$\begin{aligned}\ddot{S}_{n\text{м}}+2h_{n\text{м}}\dot{S}_{n\text{м}}+\omega_{n\text{м}}^2 S_{n\text{м}}= & -\frac{P_c}{M_{n\text{м}}}f_{n\text{м}}(x_{\text{cм}})+ \\ & +\frac{R_{\text{л}}}{M_{n\text{м}}}\left[f_{n\text{м}}(x_{\text{л}})\cos\varphi_1+y_{\text{лм}}\frac{df_{n\text{м}}(x_{\text{л}})}{dx}\sin\varphi_1\right]+ \\ & +\frac{\cos\varphi}{M_{n\text{м}}}\int_0^{l_{\text{м}}} f_{n\text{м}}(x)\frac{\partial Y(x)}{\partial x}dx+ \\ & +\frac{1}{M_{n\text{м}}}\left[y_0\left(\frac{d\varphi}{dt}\right)^2-g_0\sin\varphi\right]\int_0^{l_{\text{м}}} m(x)f_{n\text{м}}(x)dx+ \\ & +\frac{1}{M_{n\text{м}}}\left(\frac{d^2\varphi}{dt^2}\right)\int_0^{l_{\text{м}}} xm(x)f_{n\text{м}}(x)dx.\end{aligned} \tag{6.57}$$

The first member of the right side of this equation considers influence of removing force P_c on sag of the tower. This force is applied at the time of onset of discarding the tower and is taken when the tower reaches a certain safe angle of rotation $\phi_б$, excluding the possibility of reverse movement of it under the action of wind load Y. Time of change t_c of force $P_c^\bullet$ from zero to maximum magnitude depends on the source of energy utilized. Its point of application is selected in such a manner as to obtain as small values of bending moments (51) as possible.

The linear value of wind load is found by the formula

$$\frac{\partial Y(x)}{\partial x}=c_y(x,\varphi)\rho_0\frac{u^2}{2}F_м(x), \qquad (6.58)$$

where u – relative wind velocity $F_м(x)$ – linear area of middle section of tower, $c_y(x,\varphi)$ – linear value of coefficient of transverse aerodynamic load (at an angle of attack, equal to ϕ), referred to $F_м(x)$.

In first approach it is possible to consider only the component of wind velocity perpendicular to axis of the tower $u_y=u\cos\varphi+x\frac{d\varphi}{dt}$, and consequently, to take $c_y(x,\varphi)$ with $\varphi=\frac{\pi}{2}$. Actual value of force Y depends on the direction from which the wind is blown on the flight vehicle and can change in rather wide limits. It can be both negative and positive. Therefore during the calculation of parameters of motion of the tower and forces in its sections (especially with a constant value of P_c) it is necessary to consider particular cases of load corresponding to max Y, min Y and Y = 0, and also cases of load on the tower by aerodynamic force in plane zx perpendicular to the plane of removal.

The moment of application of damping force $t_д$ and time t_1 of change in its magnitude from zero to the greatest value also will be found with large scattering, since these times are determined by the rate of rotation $\frac{d\varphi}{dt}$ of the tower:

$$t_1=\frac{\Delta\varphi}{\frac{d\varphi}{dt}},\quad \varphi_д=\int\limits_0^{t_д}\frac{d\varphi}{dt}dt,$$

where $\Delta\phi$ – angle of rotation of the tower corresponding to change in magnitude $R_Д$ from zero to maximum value. These times will be maximum at max Y and minimum at min Y, i.e., at max $\frac{d\varphi}{dt}$. It is necessary to note that min Y depends on the diameter of the body of the flight vehicle and on how much its body covers the tower from the influence of the wind.

If braking force is created by a hydraulic device which has a limited movement of the rod, then the magnitude $R_Д$ will depend strongly on magnitude $\frac{d\varphi}{dt}$ at the time $t=t_д$. The greater $\frac{d\varphi}{dt}$, i.e., the greater the kinetic energy of motion of the tower, then the greater is the value of the braking force required. Frequently the moment of load of the tower, corresponding to max $R_Д$, turns out to be calculated for certain elements of its construction.

It is obvious that elastic oscillations of tower construction will be considerable only in those cases when t_c, t_1, are comparable with the period of inherent bending oscillations of the tower $\tau_{nм} = \frac{2\pi}{\omega_{nм}}$. Since the magnitudes of force Y and longitudinal mass forces (50) in practice are changed slowly, then in the case of approximate calculation of dynamic moments their influence on $S_{nм}$ can be disregarded.

A significant simplification of this problem is obtained in that case when it is also possible to disregard the influence of longitudinal forces on the values of forms and frequencies of inherent bending oscillations of the tower. During determination of the latter frequently the tower can be presented schematically in the form of prismatic rod with a rigidity which is constant for length and a constant linear mass. In this case the solution of equation

$$\frac{d^4 f_{nм}(x)}{dx^4} = \frac{m_0}{EI_{cм}} \omega_{nм}^2 f_{nм}(x) \qquad (6.59)$$

with boundary conditions (56) will be the function

$$f_{nм}(x) = \sin k_n x + \frac{\sin k_n l_м}{\operatorname{sh} k_n l_м} \operatorname{sh} k_n x, \qquad (6.60)$$

where k_n – roots of the transcendental frequency equation

$$\operatorname{tg} k_n l_м = \operatorname{th} k_n l_м,$$

$$k_n l = \frac{2n+1}{4}\pi, \quad k_n = \left(\frac{m_0}{EI_{cм}} \omega_{nм}^2\right)^{\frac{1}{4}} \quad (n = 1, 2, \ldots).$$

With $n = 0$ $k_n l_м = 0$ and $\omega_{nм} = 0$. With zero frequency equation (59) will be satisfied by the normal function $f_{0м}(x) = \frac{x}{l_м}$, describing the rotation of the tower as a solid body. In this case

$$S_{0м} = -\varphi l_м, \quad M_{0x} = \int_0^{l_м} m(x) f_{0м}^2(x)\,dx = I_{zм} \frac{1}{l_м^2}$$

and equation (57) is reduced to equation (48). Functions (60) can be taken as initial during calculation by the method of successive approximations of forms of natural oscillations of the tower (having rigidity and mass variable with respect to length) by equation (55).

Footnotes

[1]Similar simplification leads to a certain overstating (order of 10-15%) of the reaction of the body.

[2]Liepman H. W. On the application of statistical concepts to the buffeting problem. IAS, No. 12, 1952.

[3]Certain experiments show that its maximum is observed at critical Strouhal numbers.

CHAPTER VII

DYNAMICS OF LAUNCHING A FLIGHT VEHICLE

§ 7.1. General Information

The requirement that ground cases of load cannot be calculated for the construction of a flight vehicle naturally leads to complication of conditions of ground exploitation. Therefore in each specific case one should conduct a thorough investigation of conditions of load on the flight vehicle in the process of launching for the purpose of establishing the optimum requirements for construction of the launcher, for rigidity of its carrier elements, and also for the scheme of starting the engines.

Conditions of operation for the flight vehicle structure in this case of load depend in many respects on the layout of the launcher, on the type of launching site. Launching of flight vehicles can be carried out from open, completely closed, and semi-enclosed positions. The most widespread and the simplest is launching from an open position (ground launching, launching from the deck of a ship).

During any form of launching the launcher cannot be made absolutely rigid. Due to elastic deformation of its carrier elements there will be a shift (settling) of the vehicle in the process of fueling by a certain magnitude ξ_0. During launching with an increase of thrust force this deformation will be selected. In the case of a slow change of thrust force lift of the vehicle (at $P < G_0$) will not exceed ξ_0. With a comparatively rapid change of thrust force it turns out to be greater than static. Moreover the vehicle can in the process of this rise acquire a speed, sufficient for breakaway of its support section from the launcher with a magnitude of thrust force which is less than the initial weight. As a result the flight vehicle can be lifted a certain height ξ_n above launcher, and then (in the case of preservation of inequality $P < G_0$) fall on it, as soon as the kinetic energy obtained by it is completely expended for changing the potential energy of the position. Rate of fall will depend on magnitude ξ_n and nature of change P with respect to t.

Sometimes for increasing the reliability of launching the stepped switching on of motors[1] is used or an intermediate step[2] is introduced. As the latter they usually select a certain value of thrust force P_n which is less than the initial weight. After obtaining confidence about the normal operation of the engines in the system on lowered flow rate the main stage is switched on. If any malfunctioning is revealed automatically the command is given to shut down (emergency) all engines of the flight vehicle.

Jumping and subsequent falling of the vehicle with $P < G_0$ can be repeated periodically if the period of operation of the engines in the intermediate stage is comparatively great. Under such conditions the rapid removal of thrust force (as a result of emergency shutdown of engines) can in certain cases lead to an increase of rate of fall of the flight vehicle, and, consequently, to appearance of additional deformations of support elements of the launcher.

During launching from a closed position a system of starting and reflected waves can be formed which leads to additional load on the vehicle structure (mainly in the tail section) by external excess pressure. This increase of pressure is observed only in the initial moment of launching and for a small interval of time. Its magnitude will be even greater, the greater the power of the engine, the less the removal of gases, and the sharper the approach of the engine to operational conditions. Determination of the real nature of pulsation of pressure and temperature conditions of operation of the construction at the time of starting the engines is possible only by experimental means, by using special models.

In the process of movement of a flight vehicle of the ballistic type in a silo it will be acted on by the disturbing moment from thrust force (1.14), crosswind load (on the part of the body coming out of the silo) and gas-dynamic loads, caused by partial passage of gases emanating from the engines into the space between the body of the flight vehicle and the wall of the silo. The magnitude of this space, i.e., the internal diameter of the wall of the silo, is determined by the possible transverse shifting of the vehicle body under action of disturbing forces and moments. It is possible to decrease this diameter by means of elimination or limitation of the transverse motion of the flight vehicle with the help of a system of special elastic supports and guides. Similar guide rails can also be used during inclined launching of any flight vehicle from open positions. Therefore during the investigation of load in flight vehicles in case S considerable attention should be given namely to this type of launching.

The dynamics of vertical or horizontal takeoff of conventional aircraft (from moment of starting the engines to the moment of breakaway from surface) are not of special interest for the strength of their construction.[3]

§ 7.2. Coefficients of Dynamics for Longitudinal Support Elements of a Launcher

The most dangerous for strength of longitudinal support elements of launcher is the case of a load corresponding to emergency shutdown of engines (after approaching assigned conditions). As was already noted in the preceding section as a result of dropping of flight vehicle on the launcher considerable dynamic forces can develop in carrier elements of the launcher. The values of these forces are determined by parameters of longitudinal motion of the vehicle-launcher system.

Let us assume that P - total thrust force of flight vehicle, variable according to the law described by formulas (1.11), (1.12), (1.13), or others, E_{ci} - generalized stiffness coefficient of i-th longitudinal support element of the launcher, equal to the force causing a unitary shift of this support in the direction of axis ξ, and ξ_0 - initial shift of flight vehicle due to deformation of the launcher under the influence of force of weight G_0, i.e.,

$$\xi_0 = \frac{G_0}{\sum_{i=1}^{k_i} E_{ci}}.$$

On sector $-\xi_0 < \xi < 0$ movement of the flight vehicle occurs under the action of thrust force P and force of reaction of support elements of the launcher equal to $\sum_{i=1}^{k_i} \xi E_{ci}$, and on sector $\xi > 0$ - only under action of force of weight and thrust force.

On the basis of the d'Alambert principle the condition of dynamic equilibrium of a flight vehicle on these sectors of the trajectory in motionless launching system of coordinates ($\xi\eta\zeta$) will have the form

$$\ddot{\xi} + 2h_0\dot{\xi} + \omega_0^2\xi = g_0(n_x - 1) \quad \text{with} \quad 0 > \xi > -\xi_0, \tag{7.1}$$

$$\ddot{\xi} = g_0(n_x - 1) \quad \text{with} \quad \xi > 0, \tag{7.2}$$

where

$$\omega_0^2 = \frac{g_0}{\xi_0}, \quad n_x = \frac{P}{G_0}.$$

Solution of equation (1) in the case of initial conditions $\xi(t) = 0$ and $\xi(t) = -\xi_0$ with $t = 0$ will be function

$$\xi(t) = -\xi_0 e^{-h_0 t}\cos\omega_0 t + \frac{g_0}{\omega_0}\int_0^t (n_x - 1)e^{-h_0(t-\tau)}\sin\omega_0(t-\tau)\,d\tau. \tag{7.3}$$

Since the total thrust force remains all the time less than weight ($n_x < 1$), then motion of the flight vehicle on section $\xi > 0$ will be observed only in that case when at the time of passage of position $\xi = 0$, i.e., with $t = t_0$, its speed will be different than zero.

The value of this speed $\dot{\xi}(t) = \dot{\xi}(t_0)$ depends on the magnitude and speed of change of longitudinal overload. With comparatively large values of n_x and dn_x/dt a significant influence is exterted on $\dot{\xi}(t_0)$ by magnitude ξ_0. Other things being equal $\dot{\xi}(t_0)$ will increase with an increase of ξ_0 and increase of dn_x/dt.

Integrating equation (2) with initial conditions $\xi(\dot{t}) = 0$ and $\dot{\xi}(t) = \dot{\xi}(t_0)$ with $t = t_0$, we obtain the corresponding expression for determination of magnitude of jump of the flight vehicle above the launcher

$$\xi(t) = g_0\int_0^t\int_0^t n_x(t)\,dt\,dt + \dot{\xi}(t_0)(t - t_0) - \frac{g_0}{2}(t - t_0)^2. \tag{7.4}$$

The derivative from this function (4) based on t gives the rate of drop of the vehicle on the launcher. Designating by t_1 the moment of secondary passage of $\xi(t)$ through zero, we find the initial conditions for the second stage of motion of the flight vehicle on sector $\xi < 0$

$$\dot{\xi}(t_1) = g_0\int_{t_0}^{t_1} n_x(t)\,dt + \dot{\xi}(t_0) - g_0(t_1 - t_0), \quad \xi(t_1) = 0.$$

At the time $t = t_{01}$, when ξ again becomes equal to zero, the following jump of the vehicle above the launcher will begin with initial speed $\dot{\xi}(t_{01})$, etc. Approximately without taking into account damping the shift of the flight vehicle, and, consequently, also the deformation of support elements of the launcher will be described by formula

$$\xi(t)=\frac{1}{\omega_0}\dot{\xi}(t_1)\sin\omega_0(t-t_1)+\\+\frac{g_0}{\omega_0}\int_{t_1}^{t}n_x(\tau)\sin\omega_0(t-\tau)\,d\tau-\xi_0[1-\cos\omega_0(t-t_1)]. \qquad (7.5)$$

The relation of magnitude of this deformation $|\xi(t)|$ to static deformation determines the value of coefficient of dynamics for longitudinal support elements of the launcher.

With $n_x = n_{xn}$ = const, i.e., after approach of the engines to conditions of the assigned intermediate stage, oscillations of the flight vehicle on the elastic launcher in the absence of damping forces will have the form of steady-state oscillations. Integrating equation (2), we find that in this case with

$$\dot{\xi}(t_{1k})=-\dot{\xi}(t_{0k})$$

the time, corresponding to passage by the vehicle of position $\xi(t_{1k})$, will be determined by formula

$$t_{1k}=t_{0k}+\frac{2\dot{\xi}(t_{0k})}{g_0(1-n_{xn})}.$$

Here the greatest height of jump of the flight vehicle above the launcher will be equal to

$$\Delta h=\frac{\dot{\xi}^2(t_{01})}{2g_0(1-n_{xn})}. \qquad (7.6)$$

Since sometimes the functioning of the actual launcher (for instance, removal of support elements and so forth) is connected with movement of the flight vehicle relative to the support elements, then the magnitude of this jumping can have great practical significance for the safety of the launching.

From expression (5) it is also easy to find the magnitude of settling of the flight vehicle on the launcher

$$\xi(t) = \xi_0(n_{xn} - 1)[1 - \cos\omega_0(t - t_1)] + \frac{\dot{\xi}(t_{1k})}{\omega_0}\sin\omega_0(t - t_1). \tag{7.7}$$

Equating to zero the derivative from ξ with respect to t, we obtain value $t = t'$

$$t' = t_1 + \frac{1}{\omega_0}\,\mathrm{arctg} - \frac{\dot{\xi}(t_{1k})}{\omega_0\xi_0(n_{xn} - 1)},$$

at which this settling is maximum. Since static settling of the launcher in this case is not equal to ξ_0, but to $\xi_0(n_{xn} - 1)$, then the coefficient of dynamics η will be less than a unit. Magnitude η can be great only in the case of emergency engine cutoff.

The value of this coefficient of dynamics directly and through $\dot{\xi}(t_{0k})$ depends on three parameters: ξ_0, n_{xn} and dn_x/dt. For illustration in Fig. 7.1 there is a graph of the change of relation $\eta = \xi(t)/\xi_0$ for various moments of onset of turning off of engines. Positive values of η determine the magnitude of jump of the flight vehicle above the launcher in fractions of ξ_0, and negative - the magnitude of coefficient of dynamics for the launcher (in a vertical direction).

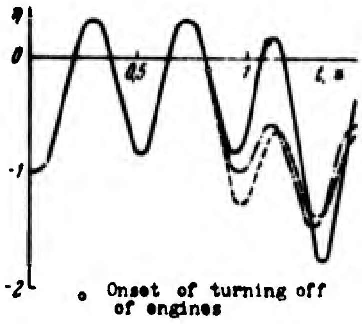

Fig. 7.1. Dependence of coefficient of dynamics for construction of a launcher on the time of onset of turning off the flight vehicle engines.

It is obvious that in general an emergency shutdown of engines can occur at any moment. Therefore during calculation of the launcher one should be guided by the possible value of coefficient of dynamics at assigned ξ_0, n_{xn} and dn_x/dt or introduce the corresponding limitations on values of these parameters. Here it should be borne in mind that the least magnitude of nx_n is determined by conditions of stable operation of the propulsion system and dn_x/dt by its construction, i.e., they are limited from below. The degree of influence on η by the rigidity of support elements of the launcher (or ξ_0) can be judged by the approximate graph shown in Fig. 7.2.

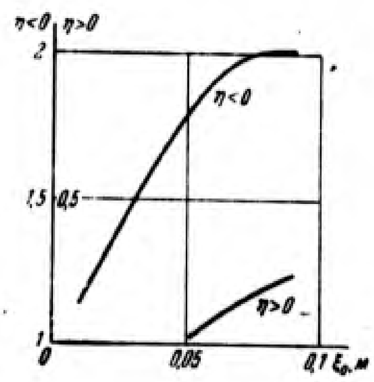

Fig. 7.2. Influence of initial static deformation of construction of the launcher on the magnitude of coefficient η.

__Example__. As an example we will consider the calculation of the coefficient of dynamics for a case of linear law of change of coefficient of static longitudinal overload (Fig. 7.3) during the start-stop operation of engines. Let us assume that

$$n_x = n_{xn}\frac{t}{T_n},$$

where T_s designates the conditional time of approach of the engine to assigned conditions. On the basis of formula (3') the speed of the vehicle at the time $t = t_0$ corresponding to $\xi(t) = 0$ will be equal to

$$\dot{\xi}(t_0) = \xi_0 \frac{n_{xn}}{T_s}(1 - \cos \omega_0 t),$$

where t_0 is the solution of equation

$$n_{xn}\frac{t_0}{T_s} - \frac{n_{xn}}{T_0\omega_0}\sin \omega_0 t_0 = 1.$$

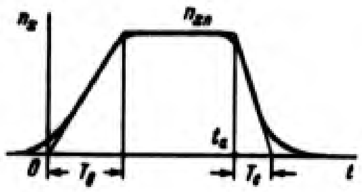

Fig. 7.3. Nature of change of static component of longitudinal overload in the process of start-stop operations of an engine.

Jumping of the flight vehicle above the launcher and speed of its motion with these initial conditions (with $t < T_s$) are determined by formulas

$$\xi(t) = \dot{\xi}(t_0)(t - t_0) - \frac{g_0}{2}(t - t_0)^2 + \frac{g_0 n_{xn}}{6T_s}(t^3 + 2t_0^3 - 3tt_0^2),$$

$$\dot{\xi}(t) = \dot{\xi}(t_0) - g_0(t - t_0) + \frac{g_0 n_{xn}}{2T_s}(t^2 - t_0^2).$$

Using the last expression, one can also determine the rate of drop of the vehicle on the launcher (at the time $t = t_1 < T_s$, corresponding to $\xi(t_1) = 0$). If it turns out that $t_1 > T_s$, then on sector $T_s < t < t_1$ the shift of $\xi(t)$ will be equal to

$$\xi(t) = g_0(n_{xn} - 1)\frac{(t - T_s)^2}{2} + \xi(T_s) + (t - T_s)\dot{\xi}(T_s). \qquad (7.8)$$

Equating this expression to zero, we obtain the equation for calculation of magnitude t_1. Differentiating (8) with respect to t, we find the following value for $\dot{\xi}(t)$ (at the time $t=t_1$)

$$\dot{\xi}(t_1)=g_0(n_{xn}-1)(t_1-T_в)+\dot{\xi}(T_в).$$

Further, using formulas (5), (7), we find the coefficient of dynamics of n in the case of an operating engine.

The change of longitudinal overload in process of emergency shutdown of motors (1.12) can be presented in the form

$$n_x=n_{xn}\left(1-\frac{t-t_a}{T_c}\right), \tag{7.9}$$

where t_a - time, corresponding to moment of onset of drop in thrust force, and T_c - conditional time of drop of thrust force to zero (Fig. 7.3). The command for turning off the engines is given in the event of any abnormalities in the starting system which were forseen by the layout (for instance, in case of failure of one of the booster engines, and so forth). In practice this shutdown can occur during any position of the flight vehicle with respect to the launcher.

If there is no jump of the flight vehicle above the launcher, for the calculation of coefficient of dynamics it is sufficient to find the solution of equation (1) with the initial conditions, determining the parameters of motion of the flight vehicle at the time of onset of emergency shutdown of the engines (e.s.e). Approximately without taking into account resisting forces

$$\xi(t)=\xi(t_a)\cos\omega_0(t-t_a)+\dot{\xi}(t_a)\frac{1}{\omega_0}\sin\omega_0(t-t_a)+ \\ +\frac{g_0}{\omega_0}\int_{t_a}^{t}\sin\omega_0(t-\tau)(n_x-1)\,d\tau.$$

Substituting expression (9) in this formula, we find that

$$\eta(t) = -\frac{\xi(t)}{\xi_0} = \left[1 - n_{xn} + \frac{\xi(t_a)}{\xi_0}\right]\cos\omega_0(t - t_a) + n_{xn}\left(1 - \frac{t - t_a}{T_c}\right) - \\ - 1 + \frac{1}{\omega_0}\left[\frac{n_{xn}}{T_c} + \frac{\dot{\xi}(t_a)}{\xi_0}\right]\sin\omega_0(t - t_a) \quad \text{with} \quad t_a < t < T_c. \tag{7.10}$$

At the time $t = t_a + T_c$

$$-\eta(t) = 1 + \left[n_{xn} - 1 - \frac{\xi(t_a)}{\xi_0}\right]\cos\omega_0 T_c - \frac{1}{\omega_0}\left[\frac{n_{xn}}{T_c} + \frac{\dot{\xi}(t_a)}{\xi_0}\right]\sin\omega_0 T_c. \tag{7.11}$$

It is convenient to present value T_c in fractions of the period of inherent longitudinal oscillations of the considered system $T_c = k_0/\omega_0$.

Formula (11), of course, does not give maximum value of $\eta(t)$. In order to find the maximum magnitude of $\eta(t)$ it is necessary to investigate the motion of the system also at $t > (t_a + T_c)$. As it is not difficult to establish, the value of coefficient of dynamics on this sector of time will depend significantly on the initial conditions of $\xi(t_a)$ and $\dot{\xi}(t_a)$. If e.s.e. is carried out after the approach of the thrust force of the main engines to some intermediate stage, i.e., with $t > T_s$, then

$$\left.\begin{aligned} \xi(t_a) &= [\xi(T_s) - \xi_0(n_{xn} - 1)]\cos\omega_0(t_a - T_s) + \\ &\qquad + \xi_0(n_{xn} - 1) + \frac{\dot{\xi}(T_s)}{\omega_0}\sin\omega_0(t_a - T_s), \\ \dot{\xi}(t_a) &= \dot{\xi}(T_s)\cos\omega_0(t_a - T_s) + \\ &\qquad + [\xi_0(n_{xn} - 1) - \xi(T_s)]\,\omega_0\sin\omega_0(t_a - T_s). \end{aligned}\right\} \tag{7.12}$$

Figure 7.4 shows the graphs of maximum values of vehicle jump above the launcher for various n_{xn} depending on ξ_0 (with $T_s = 0.1$ s). Figure 7.5 for the same n_{xn} gives the graphs of function max η (T_c/T_0), where $T_0=\frac{2\pi}{\sqrt{g_0/\xi_0}}$. If the flight vehicle has several engine installations the real law of change of n_x with respect to t can possess large scattering (since in practice not all motors will be switched on simultaneously and the nature of build-up of thrust force in them can also be different). In this case it is expedient to be guided by the limiting values of function $n_x(t)$.

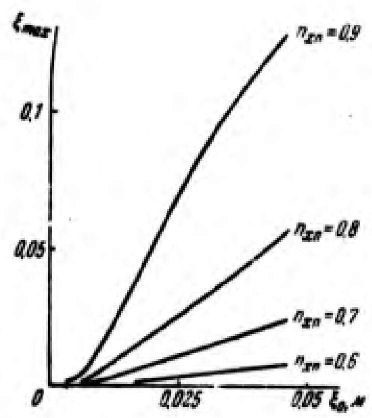

Fig. 7.4. Influence of rigidity of launcher and magnitude of thrust force on jumping of flight vehicle in the process of launching (with $T_в$ = 0.1).

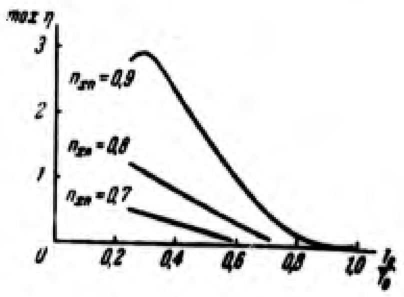

Fig. 7.5. Influence of the nature of drop in thrust force on the values of coefficient of dynamics for a launcher.

From the material presented in this section it follows that rigidity of launcher and value of n_{xn} should as far as possible be selected in such a way that jumping of the flight vehicle is absent or is insignificant. If for some reason it is not possible to carry this out, then it is necessary to calculate the possible values of coefficients of dynamics and to ensure the strength of the launcher for a load equal to nG_0.

§ 7.3. Longitudinal Dynamic Forces in Elements of the Body of the Vehicle

The stage nature of starting the engines, although it can lead to undesirable jumping of the flight vehicle in an elastic launcher, makes it possible to reduce the values of disturbing forces and moments which are acting on the construction in process of launching, in particular to decrease longitudinal dynamic forces in the body of the flight vehicle. In the case of a sharp approach of the engines to operating conditions, the so-called "cannon" starting, vehicle jump is absent, but intense longitudinal elastic oscillations of the construction develop. Even if these oscillations attenuate comparatively rapidly, they nevertheless can lead to the appearance of considerable additional dynamic longitudinal forces in the body of the flight vehicle (in the case of B_1). Application of thrust force to the body of the flight vehicle can also be sharp in the case of gradual switching on of the engine unit.

In order to determine the values of these dynamic longitudinal forces, which develop both in the process of starting and also in the process of turning off the engines, one should consider the behavior of the vehicle-launcher system in the period of launching taking into account elasticity of construction of the flight vehicle itself.

For composition of equations of longitudinal oscillations of such system we will use the method of Lagrange. Let us assume that in the case of elastic oscillations all the particles of the body accomplish shifting only in the direction of the longitudinal axis of the flight vehicle, and also that its transverse sections remain plane and perpendicular to this axis. We will designate these shifts through $u(x_1, t)$.

Proceeding from the fact that any elastic oscillation of the construction can be decomposed according to natural modes of vibration, we will represent $u(x_1, t)$ in the form of a series (4.37)

$$u(x_1, t) = \sum_{n=1}^{\infty} T_n(t) X_n(x_1), \tag{7.13}$$

where T_n - unknown function of time, determining the amplitude of oscillations of the n-th harmonic at the point of reduction, and $X_n(x_1)$ - normal function of inherent longitudinal oscillations of the body of the n-th tone, satisfying the equation

$$\frac{d}{dx_1}\left[E(x_1) F_c(x_1) \frac{dX_n(x_1)}{dx_1}\right] + m_x(x_1) \omega_n^2 X_n(x_1) = 0, \tag{7.14}$$

where $m_x(x_1)$ - linear mass of the flight vehicle taking into account the concentrated loads and zero boundary conditions (i.e., condition of equality to zero of stretching forces on the ends of the vehicle with $x_1 = 0$ and $x_1 = l$) and the condition of orthogonality (4.63).

In this case longitudinal force in a cross section of the body of the flight vehicle will be determined by expression (3.1)

$$N(x_1, T) = -E(x_1) F_c(x_1) \sum_{n=1}^{\infty} T_n(t) \frac{dX_n(x_1)}{dx_1}, \tag{7.15}$$

where $F_c(x_1)$ - area of carrier section of cross section of the body in section x_1.

The vibration component of longitudinal overload will be equal to

$$\Delta n_{xs}(x_1, t) = -\frac{1}{g_0} \sum_{n=1}^{\infty} \ddot{T}_n(t) X_n(x_1).$$

The corresponding value of dynamic longitudinal force can also be calculated by the formula (3.4).

Since at $x_1 = 0$ and $x_1 = l$ $F_c(x_1) \neq 0$ and $E(x_1) \neq 0$, then on the basis of formula (15) and equation (14) the boundary conditions for function $X_n(x_1)$ can be written in the form

$$\left.\begin{aligned} &\frac{dX_n(x_1)}{dx_1} = 0 \quad \text{with} \quad x_1 = 0, \\ &\frac{dX_n(x_1)}{dx_1} = \int_0^l m_x(x_1) X_n(x_1) dx_1 = 0 \quad \text{with} \quad x_1 = l. \end{aligned}\right\} \tag{7.16}$$

In this case, disregarding mass of fluctuating elements of the actual launcher as compared to the mass of the flight vehicle, we obtain the following expression for kinetic energy of small longitudinal oscillations of this particular system:

$$T_0(t) = \frac{1}{2} m\dot{\xi}^2(t) + \frac{1}{2} \sum_{n=1}^{\infty} \dot{T}_n^2(t) \int_0^l m_x(x_1) X_n^2(x_1) dx_1. \tag{7.17}$$

Potential energy of this system will consist of potential strain energy (extension - compression) of the body of the vehicle, potential energy of the position of center of gravity of the flight vehicle, and strain energy of elastic elements of the launcher:

$$\left.\begin{aligned} &U_0(t) = \frac{1}{2}\xi^2(t) \sum_{i=1}^{k_i} E_{ci} + G\xi(t) + \\ &+ \frac{1}{2} \sum_{n=1}^{\infty} T_n^2(t) \int_0^l E(x_1) F_c(x_1) \left[\frac{dX_n(x_1)}{dx_1}\right]^2 dx_1 \quad \text{with} \quad \xi < 0, \\ &U_0(t) = \frac{1}{2} \sum_{n=1}^{\infty} T_n^2(t) \int_0^l E(x_1) F_c(x_1) \left[\frac{dX_n(x_1)}{dx_1}\right]^2 dx_1 + G\xi(t) \\ &\qquad \text{with} \quad \xi > 0. \end{aligned}\right\} \tag{7.18}$$

Functions $T_n(t)$ and $\xi(t)$ we take as generalized coordinates. The work of thrust force on virtual displacement of the system with accepted direction of axes x_1 and ξ will be equal to

$$W(t)=P(t)\left[\xi(t)-\sum_{n=1}^{\infty} T_n(t)\,X_n(x_{1n})\right],$$

where $X(x_{1n})$ - value of function of form in the place of application of thrust force to the body the the flight vehicle. Using equation (4.1), we obtain the following system of equations for determination of unknown functions $\xi(t)$ and $T_n(t)$:

$$\ddot{\xi}+\omega_0^2\xi=g_0(n_x-1), \tag{7.19}$$

$$\ddot{T}_n+2h_n\dot{T}_n+\omega_n^2 T_n=-\frac{PX_n(x_{1n})}{M_n}. \tag{7.20}$$

As can be seen, the first of these equations coincides with equation (1), describing motion of a flight vehicle as a solid body, and the second - determines longitudinal elastic oscillations of its construction under the action of force P. For solid-fuel flight vehicle of the ballistic type the right side of equation (20) will have the form

$$-\frac{P}{M_n}\left[k_д X_n(x_{1кд})+k_c X_n(x_{1c})\right],$$

where $x_{1кд}$ and x_{1c} designate the coordinates of places of connection to the body of the flight vehicle of the upper end of the combustion chamber of the engine and the nozzle correspondingly, and $k_д$ and k_c - coefficients, showing what share of overall thrust force is applied in these sections.

The integral of equation (20) in the case of zero initial conditions, as it is known (4.12'), will be equal to

$$T_n(t)=-\frac{X_n(x_{1n})}{\omega_n M_n}\int_0^t P(\tau)\,e^{-h_n(t-\tau)}\sin(t-\tau)\,d\tau.$$

If the nature of build-up of thrust force can be described approximately by a linear law analogous to the law of change of n_x, which is shown in Fig. 7.3, then, disregarding members h_n/ω_n and $(h_n/\omega_n)^2$ as compared to a uhit, we obtain for

$$\left.\begin{aligned}
&P(t)=P_0\frac{t}{T_{\text{в}}} \quad \text{with } t\leqslant T_{\text{в}},\\
&P(t)=P_0=\text{const} \quad \text{with } t\geqslant T_{\text{в}}\\
&T_n(t)=-\frac{P_0}{T_{\text{в}}}\frac{X_n(x_{1n})}{\omega_n^2 M_n}\left[t-\frac{e^{-h_n t}}{\omega_n}\sin\omega_n t\right] \quad \text{with } t\leqslant T_{\text{в}},\\
&T_n(t)=-\frac{P_0 X_n(x_{1n})}{M_n\omega_n^2}\left\{1-\frac{e^{-h_n t}}{\omega_n T_{\text{в}}}\left[\sin\omega_n t-e^{h_n T_{\text{в}}}\sin\omega_n(t-T_{\text{в}})\right]\right\}\\
&\qquad \text{with } t\geqslant T_{\text{в}}.
\end{aligned}\right\} \qquad (7.21)$$

Differentiating these expressions twice for t and using formulas (3.3) and (13), we obtain the following expressions for calculation of dynamic component of longitudinal force:

$$\left.\begin{aligned}
&N_{\text{д}}(x_1,t)\approx\frac{P_0}{T_{\text{в}}}\sum_{n=1}^{\infty}\frac{X_n(x_{1n})}{M_n\omega_n}N_{nx}(x_1)e^{-h_n t}\sin\omega_n t \quad \text{with } t\leqslant T_{\text{в}},\\
&N_{\text{д}}(x_1,t)\approx\frac{P_0}{T_{\text{в}}}\sum_{n=1}^{\infty}\frac{X_n(x_{1n})}{M_n\omega_n}N_{nx}(x_1)e^{-h_n t}\times\\
&\qquad\times\left[\sin\omega_n t-e^{h_n T_{\text{в}}}\sin\omega_n(t-T_{\text{в}})\right] \quad \text{with } t\geqslant T_{\text{в}},
\end{aligned}\right\} \qquad (7.21')$$

where

$$N_{nx}(x_1)=\int_0^{x_1} m(x_1)X_n(x_1)\,dx_1 \qquad (n=1,\,2,\,\ldots).$$

In the case when the dependence of P on t can be presented in the form (1.13),

$$T_n(t)=-\frac{P_0 X_n(x_{1n})}{\omega_n^2 M_n}\times\left\{1-\frac{1}{1+\frac{\tau^2}{\omega_n^2}}\left[e^{-\tau t}-e^{-h_n t}\frac{\tau}{\omega_n}\left(\sin\omega_n t-\frac{\tau}{\omega_n}\cos\omega_n t\right)\right]\right\}. \qquad (7.22)$$

longitudinal dynamic force will be determined by expression

$$-N_{\text{д}}(x_1, t) \approx P_0 \sum_{n=1}^{\infty} \frac{\tau^2 X_n(x_{1n})}{M_n(\omega_n^2+\tau^2)} \times$$
$$\times \left[e^{-\tau t} + e^{-h_n t}\left(\frac{\omega_n}{\tau}\sin\omega_n t - \cos\omega_n t\right)\right] N_{nx}(x_1). \qquad (7.22')$$

In the case of turning off the engines (emergency or programmed), according to law (1.12),

$$P = P_0\left(1 - \frac{t-t_a}{T_c}\right),$$
$$T_n(t) = \left[T_n(t_a) - \frac{P_0 X_n(x_{1n})}{M_n\omega_n^2}\right] e^{-h_n(t-t_a)} \cos\omega_n(t-t_a) +$$
$$+ \frac{P_0}{M_n\omega_n^2}\left(1 - \frac{t-t_a}{T_c}\right) X_n(x_{1n}) + \qquad (7.23)$$
$$+ \left[\frac{P_0 X_n(x_{1n})}{\omega_n^3 M_n T_c} + \frac{\dot{T}_n(t_a)}{\omega_n}\right] e^{-h_n(t-t_a)} \sin\omega_n(t-t_a) \text{ with } t < t_a + T_c,$$

where $T_n(t_a)$ and $\dot{T}_n(t_a)$ - values of function T_n at the time of onset of a drop in thrust force. For this (with $T_c > 1/\omega_n$)

$$N_{\text{д}}(x_1, t) \approx \sum_{n=1}^{\infty} \left\{\left[T_n(t_a)\omega_n^2 - \frac{P_0 X_n(x_{1n})}{M_n}\right]\cos\omega_n(t-t_a) + \right.$$
$$\left. + \left[\frac{P_0 X_n(x_{1n})}{M_n\omega_n T_c} + \dot{T}_n(t_a)\omega_n\right]\sin\omega_n(t-t_a)\right\} N_{nx}(x_1) e^{-h_n(t-t_a)}. \qquad (7.24)$$

Usually during determination of $N_{\text{д}}$ they are limited only to the calculation of lowest forms of inherent longitudinal oscillations of the flight vehicle.

The coefficient of longitudinal overload of the center of gravity of the flight vehicle, determining the magnitude of static component of longitudinal force, is found by the formula

$$n_x(t) = 1 - \frac{\ddot{\xi}(t)}{g_0}. \qquad (7.25)$$

In conclusion let us note that the values of longitudinal forces in elements of construction of flight vehicle in the case S (in the process of start-stop operation of engines) depend not only on laws of change of total thrust force, but also on the assembly layout of the vehicle. The load picture turns out to be especially complex for flight vehicles equipped with lateral accelerators having a one-sided power connection with the body of the vehicle in a longitudinal direction.

§ 7.4. Load on a Vehicle During Free Launching from a Silo

Although by definition the examined generalized case of load S embraces motion of a flight vehicle only up to the moment of its final breakaway from the launcher, there is meaning in also extending to it the case of flight of a vehicle inside a silo during free launching (case S_0).

Motion of a flight vehicle in a silo up to the moment of complete approach to the surface occurs under the action of the same disturbing forces and moments as the flight of a vehicle in the case of launching from a ground launcher. However, the values of these forces and the nature of their distribution on the body in this case of load will be different.

On the bottom of the tail section of a flight vehicle during launching from a silo an additional longitudinal disturbing force of a pulsed nature $P_{н}$ can act.[4] The graph of change in total force P (thrust force P and force $P_{н}$) in this case has the form shown in Fig. 7.6 by the solid curve.

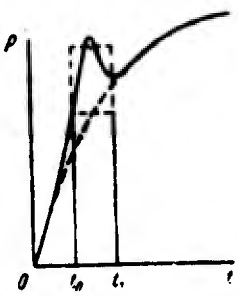

Fig. 7.6. Possible nature of change of thrust force in the process of launching (solid curve).

In a particular case, when the time of action of force $P_{\text{и}}$ is great as compared to period τ_n of inherent longitudinal oscillations of the lowest tone on the structure of the flight vehicle, one can assume that this force belongs to the static class. In all remaining cases one should estimate influence of the time of its action. If this time is very small, for example, less than $0.1\tau_n$, then the calculation of the corresponding reaction of the construction of the flight vehicle can be conducted, using the magnitude of pulsed power $P_{\text{и}}$,

$$U_p(t_1) = \int_{t_0}^{t_1} P_{\text{и}}(t)\,dt.$$

In other words, it is limited to a consideration of oscillations of a construction which appear as a result of attaching to it an instantaneous speed equal to

$$\Delta \dot{T}_{n0}(t) = -\frac{U_p(t_1)}{M_n} X_n(x_{1n}).$$

In the case of uniform initial conditions the corresponding shifts of point of reduction and value of additional longitudinal forces $\Delta N_n(x_1)$ will be determined by formulas

$$\left.\begin{aligned} \Delta T_n(t) &= e^{-h_n t}\frac{U_p(t_1)}{M_n\omega_n} X_n(x_{1n})\sin\omega_n t, \\ \Delta N_n(x_1,\ t) &\approx X_n(x_{1n})\frac{\omega_n}{M_n} e^{-h_n t} U_p(t_1) N_{nx}(x_1)\sin\omega_n t. \end{aligned}\right\} \quad (7.26)$$

It is obvious that in a similar case maximum value ΔN_n will be obtained already after disappearance of force $P_{\text{и}}$. With this the greater M_n the less ΔN_n will be. If the time of action of this force $\Delta t = t_1 - t_0$ is close to the half-period of inherent longitudinal oscillations of the construction of the flight vehicle of any tone, then this reaction will depend on the form of function $P_{\text{и}}$ changes with respect to t as is shown by the dotted line in Fig. 7.6, i.e., is applied suddenly at the time t_0, during a certain small interval Δt remains constant, and then at the time t_1 is instantly removed.

According to formula (4.13) in the case of zero initial conditions the solution of an equation of the form (20) on sector $t_0<t<t_1$ will be

$$-\Delta T_n(t)=\frac{P_n X_n(x_{1n})}{\omega_{n1}M_n}\int_{t_0}^{t}e^{-h_n(t-\tau)}\sin\omega_{n1}(t-\tau)\,d\tau.$$

Since Δt is small, then, disregarding the influence of the damping, we will have

$$\Delta\dot{T}_n(t)=\frac{P_n X_n(x_{1n})}{\omega_n M_n}\sin\omega_n(t-t_0).$$

On the subsequent sector in the case of $t>t_1$ after cessation of the action of force P_H the system will accomplish free damped oscillations. Knowing $\Delta T_n(t_1)$ and $\Delta\dot{T}_n(t_1)$, it is not difficult to obtain

$$\Delta\ddot{T}_n(t)=-\frac{2P_n}{M_n}e^{-h_n(t-t_1)}\sin\left(\omega_{n1}\frac{\Delta t}{2}\right)\left[\sin\omega_{n1}\left(t-\frac{t_0+t_1}{2}\right)-\frac{h_n}{\omega_{n1}}\cos\omega_{n1}\left(t-\frac{t_0+t_1}{2}\right)\right]X_n(x_{1n}). \quad (7.27)$$

Since in the case of nonelastic drag coefficient h_n/ω_{n1} will be of an order of δ/π, i.e., small as compared to a unit, then the second member in the brackets can be disregarded. Furthermore, it is possible with sufficient accuracy to consider that $\omega_{n1}=\omega_n$. Thus, in the period of starting the engines, i.e., in beginning of motion of the flight vehicle in the silo, the longitudinal force in cross sections of the body at $n_x > 1$ will be determined by the expression

$$N(x_1, t)=n_{x_1}(t)\,G(x_1)+N_n(x_1, t)+\Delta N_n(x_1, t). \quad (7.28)$$

where

$$\Delta N_{\text{л}}(x,\ t) = -\sum_{n=1}^{\infty} \Delta \ddot{T}_n(t)\, N_{nx}(x_1).$$

Here

$$\max \Delta \ddot{T}_n \leqslant -2\,\frac{P_{\text{н}} X_n(x_{1n})}{M_n} \sin \omega_n \frac{\Delta t}{2},$$
$$n_{x_1}^0 = \frac{1}{G_0}(P + P_{\text{н}}), \qquad (7.29)$$

and $N_{\text{л}}(x_1, t)$ is found by the formula (22'). If the graph of function $P_{\text{н}}(t)$ has a complex form, then it is possible on the basis of the superposition principle to present it approximately in the form of sum of step-functions.

Since the possibility of the flight vehicle jumping above the launching pad in the case of additional force $P_{\text{н}}$ of a pulsed nature is considerably expanded, then it may be expedient to hold it on the pad up to a specific moment. Such a delay of the flight vehicle on the pad not only limits its longitudinal shift, but promotes a decrease in the magnitude of disturbing moments caused by the nonsimultaneity of increase of pressure in different combustion chambers of the engine. It is also possible, of course, to change thrust force by stages, for example, by means of the nonsimultaneous inclusion of combustion chambers. This makes it possible sometimes to essentially lower the magnitude of force $P_{\text{н}}$ and the value of additional starting pressure acting on the body of the flight vehicle. A radical means of decreasing starting pressure on the body with the simultaneous reduction of transverse dimensions of the silo is, of course, the preliminary lifting of the flight vehicle (for starting) from the silo onto the surface of the earth.

In the selection of magnitude of thrust force or n_{xn} and time of operation of the engine in the intermediate stage one should consider not only those wishes which were mentioned in § 2, but also the requirement of damping the elastic longitudinal oscillations of the construction (up to some specific level). Calculation of limitations, imposed by the strength of the body of the flight vehicle, permits the exclusion of this case of load from number calculated.

Directly after breakaway from the launching pad the motion of the flight vehicle in the silo proceeds for a certain time only under the action of forces and moments which are conditioned by the operation of the engine installation and the control system. Then with the emergence of the body of the vehicle from the silo, it starts to be influenced by aerodynamic forces $X(x_1)$ and $Y(x_1)$ and tilting moment

$$M(x_1)=Y(x_1)[x_{1\text{т}}-x_{1\text{д}}(x_1)].$$

The magnitude of longitudinal component of aerodynamic force of the vehicle X in this case of load is usually small (in comparison with P), and its influence can be disregarded. Values of transverse component of aerodynamic force $Y(x_1)$ in the case of launching from a silo depend not only on the structure of wind flow, but also on the speed of the flight vehicle and even the construction of the silo itself (place of removal of gases).

Magnitude $Y(x_1)$ can be represented approximately as a function of coordinate x_1 in the form

$$Y(x_1)=\frac{1}{2}v_0^2\rho_0 Sc_y(x_1,\ \alpha_0), \tag{7.30}$$

where

$$x_1=\int\limits_{t_{\text{н}}}^{t} v\,dt.$$

Here v_0 - relative speed of the flight vehicle taking into account flow rate of air in the silo v_e and wind speed u, i.e., equal to

$$v_0 = [(v + v_e)^2 + u^2]^{\frac{1}{2}}, \tag{7.30'}$$

α_0 - angle of attack, c_y - coefficient of transverse aerodynamic load of the section of the flight vehicle body projecting out of the silo at an angle of attack α_0. Its values, and also the center of pressure $x_{1д}$ in function x_1 and α_0 at large angles of attack are usually found experimentally. The designation $t_н$ indicates the time of onset of exit of the summit of the flight vehicle out of the silo. The magnitude of this time can be found from equation

$$\int_0^{t_н} v\,dt = H_ш - l,$$

where $H_ш$ - depth of silo, l - length of vehicle. Subsequently for simplification of the letter we will consider that $H_ш = l$. For flight vehicles with a relative brief period of stay in the silo it is possible with sufficient accuracy to consider G_0 = const, and consequently to take

$$v = \frac{1}{m}\int_0^t P(t)\,dt - g_0 t. \tag{7.31}$$

With the law of change of thrust force P (1.13) we obtain

$$v = g_0 n_{x0}\left[t\left(1 - \frac{1}{n_{x0}}\right) + \frac{1}{\tau}(e^{-\tau t} - 1)\right].$$

We will assume (as this is usually done for obtaining the greatest probable deviations of a flight vehicle from the undisturbed state) that aerodynamic disturbing forces and moments, caused by operation of the engine installation, act in one plane. Then the approximate equations of vertical and transverse shift of the flight vehicle as a solid body will have the form

$$m\dot{v} = P - R_{x_1} - G - X(x_1), \tag{7.32}$$

$$m\ddot{\eta}_{\text{т}} = P\vartheta + R_{y_1} + Y(x_1), \tag{7.32'}$$

$$J_z\ddot{\vartheta} = Y(x_1)\left[x_{1\text{т}} - x_{1\text{д}}(x_1)\right] + M_{\text{рв}} - R_{y_1}(x_{1\text{р}} - x_{1\text{т}}). \tag{7.32"}$$

Here the additional surface forces, caused by the influence of the engine stream, i.e., change of pressure inside the silo, are included in the magnitude of thrust force P.

In deciding the system of these equations, we obtain the necessary values of components of transverse force $Q_a(x_1)$ and bending moment $M_a(x_1)$ from aerodynamic forces. Total bending moments and transverse force are found by the formulas shown in Sections 2 and 3 of Chapter III.

External excess pressure from the starting wave will obviously determine the local strength of those elements of the vehicle body which are not under internal excess pressure. Here the calculation of local and general strength of construction of the flight vehicle for this case of load should be conducted taking into account its possible heating by exhaust gases. The solution of system of equations (32), (32'), and (32") also determines the transverse shift of points of the body of the flight vehicle in the system of coordinates connected with the silo

$$\eta(x_1, t) = \eta_{\text{т}}(t) + \vartheta(t)(x_{1\text{т}} - x_1). \tag{7.33}$$

The maximum value of this shift will apparently establish the necessary transverse dimensions of the silo, more exactly, the necessary space between the exiting sections of the flight vehicle and the walls of the silo, thus ensuring that they do not collide. It is clear that the least internal diameter of silo will be only in the presence of special guide rails, eliminating, or in any case considerably limiting, the transverse motion of the body of the flight vehicle.

7.5. Peculiarities of Load in a Vehicle in the Case of Movement on Guide Rails

As support elements, limiting the transverse shifting of a flight vehicle, it is possible to use the inner surface of the jacket of the silo or special guiding rods, girders. The use of guide rails is economically expedient, since it does not require the thorough treatment of all the inner surface of the jacket of the silo. Two guiding rods are sufficient to ensure comparatively exact preparation and accurate adjustment in one diametrical plane. For ensuring the connection of the flight vehicle with these support guide rails special support devices are installed on several sections of its body, in the form of rollers, lugs, and other similar elements.

We will not dwell in detail on the construction of a silo, construction of support guide rails, and support units of the vehicle, since this question does not have fundamental value for the selection of a system for dynamic calculation of construction of the actual flight vehicle. Let us note only that in the presence of guide rails this case of load already embraces the whole process of movement of the flight vehicle in the silo (up to the moment of breakaway of its lower transverse support from the guiding device) not formally, but in strict conformity with calculation.

As was already noted, it is most expedient to install two support guiding rods, the lateral walls of which are used for limiting the motion of the flight vehicle in plane perpendicular to the plane of their location, for instance in plane $\xi\eta$, and the face surface - in plane $\xi\zeta$.

For the best understanding of the arrangement of load on the body of the flight vehicle during movement on such guide rails we will consider a case when the support elements of the flight vehicle (lugs) are absolutely rigid and slide on guide rails without a gap. We will consider the vehicle itself absolutely rigid. Let us assume that there are two such support elements and they are fixed in sections $x_{1\text{в}}$ and $x_{1\text{н}}$ (in a connected system of coordinates x_1y_1). Further let us assume that in plane $\xi\eta$ transverse shifting of the flight vehicle is limited by only one main guide rail, the cross section of which has the form shown in Fig. 7.7. Since the support elements of the flight vehicle follow strictly the profile of the guide rail, then the transverse shifting of any point of it in the case of the assumptions made will be written in the form

$$\eta(x_1)=\frac{x_1-x_{1\text{в}}}{x_{1\text{н}}-x_{1\text{в}}}\eta(x_{1\text{н}})+\eta(x_{1\text{в}})\frac{x_{1\text{н}}-x_1}{x_{1\text{н}}-x_{1\text{в}}}, \qquad (7.34)$$

where $\eta(x_{1\text{в}})$, $\eta(x_{1\text{н}})$ - transverse shifts of support stirrups. Consequently transverse overload in the considered point x_1 of longitudinal axis of the flight vehicle (in the plane of location if the guide rails) will be

$$n_{y_1}(x_1)=-\frac{1}{g_0}\left[\ddot{\eta}(x_{1\text{в}})\frac{x_{1\text{н}}-x_1}{x_{1\text{н}}-x_{1\text{в}}}+\ddot{\eta}(x_{1\text{н}})\frac{x_1-x_{1\text{в}}}{x_{1\text{н}}-x_{1\text{в}}}\right].$$

We will present it in the form

$$n_{y_1}(x_1)=n_{y_1}^0+\frac{1}{g_0}A\frac{x_1}{x_{1\text{н}}-x_{1\text{в}}}. \qquad (7.35)$$

Knowing $n^0_{y_1}$ and A, it is not difficult to calculate the corresponding support reactions from inertial forces. From equations

$$\left.\begin{aligned} R_{\text{вн}} + R_{\text{нн}} &= \int\limits_0^l q_{Gy}(x_1)\, n_{y_1}(x_1)\, dx_1, \\ R_{\text{вн}} a &= \int\limits_0^l q_{Gy}(x_1)\, n_{y_1}(x_1)(x_{1\text{н}} - x_1)\, dx_1, \end{aligned}\right\} \tag{7.36}$$

where

$$a = x_{1\text{н}} - x_{1\text{в}},$$

it ensues that

$$\left.\begin{aligned} R_{\text{вн}} &= \frac{G}{a}\left[(x_{1\text{н}} - x_{1\text{т}})\, n^0_{y_1} + \frac{A}{a g_0} x_{1\text{т}} x_{1\text{н}}\right] - \frac{A}{a^2} J_z, \\ R_{\text{нн}} &= G\left[\left(1 - \frac{x_{1\text{н}} - x_{1\text{т}}}{a}\right) n^0_{y_1} + \frac{A x_{1\text{т}}}{a g_0}\left(1 - \frac{x_{1\text{н}}}{a}\right)\right] + \frac{A}{a^2} J_z. \end{aligned}\right\} \tag{7.37}$$

Here J_z - mass moment of inertia of the flight vehicle relative to transverse axis z, passing through its center of gravity $x_{1\text{т}}$, and G - weight of vehicle.

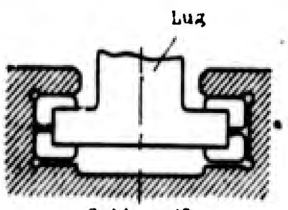

Fig. 7.7. Cross section of guide rail.

We will assume that external disturbing forces and moments act in one plane, namely in plane $\xi\zeta$. Then the values of components of reactions from aerodynamic loads acting on the flight vehicle during exit from the silo, from disturbing moment $M_{рв}$ (1.14), and from control forces will equal correspondingly

$$\left.\begin{aligned}
R_{вв} &= -\frac{1}{a}\left[x_{1н} - x_{1д}(\xi)\right] Y(\xi),\\
R_{нв} &= -\frac{1}{a}\left[x_{1д}(\xi) - x_{1в}\right] Y(\xi),\\
R_{вт} &= R_{нт} = \frac{M_{рв}}{a},
\end{aligned}\right\} \qquad (7.38)$$

$$\left.\begin{aligned}
R_{ву} &= R^{\delta}_{y_1}\frac{\delta}{a}(x_{1р} - x_{1н}),\\
R_{ну} &= R^{\delta}_{y_1}\delta\left(1 - \frac{x_{1р} - x_{1н}}{a}\right),
\end{aligned}\right\} \qquad (7.38')$$

where $x_{1д}(\xi)$ - coordinates of center of pressure of aerodynamic forces applied to the section of the flight-vehicle body coming out of the silo $(x_1 < \xi)$. Total reactions will be

$$\left.\begin{aligned}
R_в &= R_{вн} + R_{вв} + R_{вт} + R_{ву},\\
R_н &= R_{нн} + R_{нв} + R_{нт} + R_{ну}.
\end{aligned}\right\} \qquad (7.39)$$

Dependence of accelerations $\ddot{\eta}(x_{1в})$ and $\ddot{\eta}(x_{1н})$ on t will be determined basically by the nature of uveneness of the guide rails. To ensure ideal rectilinearity of each guide rail and parallelism of their working surfaces to each other is practically impossible. It is necessary always to establish a certain system of allowances for the accuracy of their manufacture and assembling. Let us assume, for instance that curvature of a guide rail can be described approximately by equation

$$\Delta = \frac{\Delta_0}{2}\left(1 - \cos\frac{4\pi}{L}\xi\right), \qquad (7.40)$$

where L - length of that basic section of the guide rail on which control of the quality of its manufacture is realized, Δ_0 - maximum sag of this section, and ξ - coordinate of the considered point of the guide rail (Fig. 7.8).

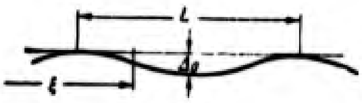

Fig. 7.8. Schematic representation of the form of the carrying surface of a guide rail (in the direction of movement of the flight vehicle).

Then transverse shift of upper transverse supporting point of the vehicle $x_{1в}$, coinciding at the particular moment with point ξ of the guide rail, will obviously equal

$$\eta(x_{1в}, \xi) = \frac{\Delta_0}{2}\left(1 - \cos\frac{4\pi}{L}\xi\right). \tag{7.41}$$

Analogously the transverse shift of point $x_{1н}$ of the body of the flight vehicle is found

$$\eta(x_{1н}, \xi) = \frac{1}{2}\Delta_0\left[1 - \cos 4\pi(\xi - a)\frac{1}{L}\right]. \tag{7.42}$$

Differentiating these expressions (41) and (42) twice in terms of t we obtain

$$\left.\begin{aligned}\ddot{\eta}(x_{1в}, \xi) &= 2\pi\frac{\Delta_0}{L}\left[4\frac{\pi}{L}\dot{\xi}^2\cos\left(4\pi\xi\frac{1}{L}\right) + \ddot{\xi}\sin\left(4\pi\xi\frac{1}{L}\right)\right],\\ \ddot{\eta}(x_{1н}, \xi) &= 2\pi\frac{\Delta_0}{L}\left\{4\frac{\pi}{L}\dot{\xi}^2\cos\left[4\pi\frac{1}{L}(\xi - a)\right] + \right.\\ &\left. + \ddot{\xi}\sin\left[4\pi\frac{1}{L}(\xi - a)\right]\right\}.\end{aligned}\right\} \tag{7.43}$$

From equation (31), describing the longitudinal movement of center of gravity of the flight vehicle in the shaft of the silo, it follows that

$$\dot{\xi} = v = g_0\left(\int_0^t n_x(t)\,dt - t\right), \quad \ddot{\xi} = g_0(n_x - 1).$$

Placing these expressions in formulas (43) and conducting certain elementary conversions, which we will omit here, we find the following expression for components $n^0_{y_1}$ and A of transverse overload (35):

$$\left.\begin{aligned}
n^0_{y_1}(\xi) = &-\frac{2\Delta_0\pi}{g_0 L a}\left\{\left[4\pi\frac{v^2}{L}\left(x_{1\text{н}} - x_{1\text{в}}\cos 4\pi\frac{a}{L}\right) + \right.\right.\\
&\left. + g_0(n_x - 1)\,x_{1\text{в}}\sin 4\pi\frac{a}{L}\right]\cos 4\pi\frac{\xi}{L} - \\
&\left. - \left[4\pi\frac{v^2}{L}x_{1\text{в}}\sin 4\pi\frac{a}{L} + g_0(n_x - 1)\left(x_{1\text{н}} - x_{1\text{в}}\cos 4\pi\frac{a}{L}\right)\right]\sin 4\pi\frac{\xi}{L}\right\},\\
A(\xi) = &\,2\pi\frac{\Delta_0}{L}\left\{\left[g_0(n_x - 1)\sin 4\pi\frac{a}{L} + \right.\right.\\
&\left. + 4\pi\frac{v^2}{L}\left(1 - \cos 4\pi\frac{a}{L}\right)\right]\cos 4\pi\frac{\xi}{L} + \\
&\left. + \left[g_0(n_x - 1)\left(1 - \cos 4\pi\frac{a}{L}\right) - 4\pi\frac{v^2}{L}\sin 4\pi\frac{a}{L}\right]\sin 4\pi\frac{\xi}{L}\right\}.
\end{aligned}\right\} \qquad (7.44)$$

As can be seen from these formulas, the magnitude of transverse overload in the center of gravity of the flight vehicle is proportional to the longitudinal overload and square of speed. At the beginning of motion of the vehicle, when its speed is slow and longitudinal overload differs comparatively little from a unit, transverse overload and support reactions from inertial forces also will be small. Consequently, under these conditions the influence of accuracy of manufacture and assembling of the guide rail, especially in the lower section of the silo shaft, will be small.

Since longitudinal overload for the entire time that the flight vehicle is located in the silo changes in comparatively small limits from 1 to n_{x0}), then with an increase of speed of the vehicle a basic influence on values $n_{y1}(x_1)$ and $R_{\text{вн}}$, $R_{\text{нн}}$ will be exerted only by members containing v^2. Therefore with sufficient accuracy one may assume that in the upper part of the silo jacket

$$\left.\begin{aligned}
n^0_{y_1}(\xi) &= -\frac{8\pi^2}{g_0 L^2}\Delta_0 v^2\cos\left(\frac{4\pi}{L}\xi\right)\left[x_{1\text{н}} - x_{1\text{в}}\cos\left(\frac{4\pi}{L}a\right)\right],\\
A(\xi) &= \frac{8\pi^2}{L^2}v^2\left[\left(1 - \cos\frac{4\pi a}{L}\right)\cos\frac{4\pi}{L}\xi - \sin\frac{4\pi a}{L}\sin\frac{4\pi}{L}\xi\right]\Delta_0.
\end{aligned}\right\} \qquad (7.45)$$

Maximum value of n_{y_1} (35) will be observed at $\xi = n\frac{L}{2}$ $(n = 1, 2, \ldots)$:

$$\max n_{y_1}(x_1) = -\frac{8\pi^2 v^2}{L^2 a g_0}\Delta_0\left[x_{1н} - x_1\left(1 - \cos\frac{4\pi a}{L}\right) - x_{1н}\cos\frac{4\pi}{L}a\right]. \quad (7.46)$$

In the case of the rapid approach of the engine to operating conditions in first approximation it is possible to consider $n_x = n_{x0} = \text{const}$. Then

$$v = g_0(n_{x0} - 1)t.$$

From (46) it follows that transverse overload will be greater, the greater the speed of the flight vehicle in silo, i.e., the greater n_{x0} and length of the guide rails. Therefore the most radical method for decreasing it is limitation of this speed (for example, by means of introduction of an intermediate stage of the thrust force). On the other hand, the same effect can be obtained by means of decreasing Δ_0/L and increasing L/l. The graphs presented in Fig. 7.9 show how important it is to seek an increase of ratio L/l, and, consequently, a decrease of Δ_0/L, i.e., an increase of accuracy of fabrication of guide rails on the final sector of movement of the flight vehicle along the jacket of the silo.

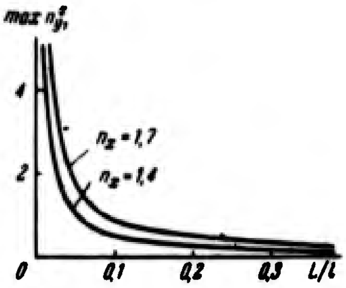

Fig. 7.9. Dependence of magnitude of overload on the accuracy of fabrication of the guide rail.

After tha upper support loses contact with the guide rail ($R_в$ becomes equal to zero) the flight vehicle under the action of disturbing wind loads, which were pointed out in the preceding section, will turn freely relative to the lower support $x_{1н}$. This turning in the case will determine the necessary gaps $\Delta\eta$ between the body of the flight vehicle and the silo guide rail. It is clear that these necessary gaps will be less, the less the time the flight vehicle is moving on one pair of lugs, i.e., the less the distance a between the lugs. Therefore when designing such a type of launcher it is natural to strive to decrease to the maximum this distance, establishing the upper support closer to the tail of the flight vehicle. However, as one may see from formulas (37), any lowering of the magnitude a will lead to an increase of support reactions, and also to an increase of transverse forces and bending moments in sections of the body of the flight vehicle. The latter in this case of load will be determined by expressions

$$\left.\begin{aligned} Q(x_1) &= S\int_0^{x_1} q\,(x_1,\xi)\frac{\partial c_y\,(x_1,\xi)}{\partial x_1}\,dx_1 + \\ &\qquad + \int_0^{x_1} q_{0_y}(x_1)\,n_{y_1}(x_1,\xi)\,dx_1 - R_н\sigma_0(x_1 - x_{1н}), \\ M(x_1) &= \int_0^{x_1} Q(x_1)\,dx_1, \end{aligned}\right\} \tag{7.47}$$

where σ_0 designates the Heaviside unit function. Since with rigid supports the angles of rotation of longitudinal axis of the flight vehicle will be small, then one should expect that the corresponding control forces also will be small. To avoid the appearance of large values of components of reactions $R_{нy}$ from any kind of probable deviations of control devices it is more expedient in this case to include in the operation of a control system after emergence of the flight vehicle from the guide rails.

Thus the projected dynamic calculation of construction of a flight vehicle in the case $S_ш$ is reduced to determination of such values of parameters a, Δ_0, L, n_{x0}, at which total support reactions

(39), transverse forces, and bending moments (47) in all sections do not exceed permissible values. In other words, at which the given case of load will not be calculated for construction of the vehicle. For certain flight vehicles fulfillment of the above condition in the case of rigid supports can turn out to be difficult. Then one should use elastic supports, which are not so sensitive to errors of fabrication and assembling of guide rails and permit the installation of additional intermediate supporting points.

§ 7.6. Dynamics of Launching Vehicles Elastic Supports

In the presence of elastic supports the layout of external forces acting on a flight vehicle in plane $\xi\eta$ has the form shown in Fig. 7.10. Equating the sum of projections of these forces on axes ξ and η of the starting system of coordinates to the corresponding forces of inertia, we obtain the unknown equations of forward motion of the flight vehicle in the form

$$m\ddot{\eta}_{\text{т}} = P\vartheta + Y(\xi, \dot{\xi}) + R_{y_1} + R_{\text{н}} + R_{\text{в}}, \tag{7.48}$$

$$m\ddot{\xi} = P - G - R_{x_1} - X. \tag{7.49}$$

Here if the combustion chamber of the main engine is used as control devices, then R_{x_1} is taken equal to zero. Putting together the sum of moments of external forces relative to the transverse axis passing through the center of gravity of the flight vehicle, and equating to its inertial moment, we find the following equations of rotation of the vehicle as a solid body (in plane $\xi\eta$):

$$\begin{aligned} J_z\ddot{\vartheta} = M_{\text{рв}} + Y(\xi, \dot{\xi})[x_{1\text{т}} - x_{1\text{д}}(\xi)] - \\ - R_{y_1}(x_{1\text{р}} - x_{1\text{т}}) - R_{\text{в}}(x_{1\text{в}} - x_{1\text{т}}) - R_{\text{н}}(x_{1\text{н}} - x_{1\text{т}}). \end{aligned} \tag{7.50}$$

We will assume for simplicity that the flight vehicle has only two belts of elastic supports, located in sections $x_{1\text{в}}$ and $x_{1\text{н}}$. The values of reactions $R_в$ and $R_н$ in such a case will depend on the rigidity of these supports and the rigidity of the elements for attaching them to the body of the flight vehicle. Let us designate the total quantities of these rigidities correspondingly through $c_в$ and $c_н$. Then in the presence of contact between supports of the flight vehicle and guide rails

$$R_в = -c_в\eta(x_{1в}), \quad R_н = -c_н\eta(x_{1н}), \tag{7.51}$$

where

$$\eta(x_{1в}) = \eta_т + \vartheta(x_{1т} - x_{1в}), \; \eta(x_{1н}) = \eta_т + \vartheta(x_{1т} - x_{1н}).$$

In the absence of such contact, for example, in the presence of gaps ε_0 between the supports and guide rails, these reactions will become equal to zero, and equations (48), (49), and (50) will describe free motion of the flight vehicle in the silo.

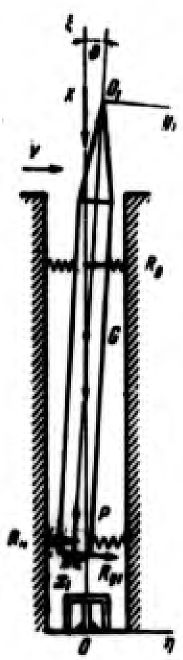

Fig. 7.10. Arrangement of forces acting on a flight vehicle during movement on a guide rail.

With $|\eta(x_{1в})| > \varepsilon_0$ and $|\eta(x_{1н})| > \varepsilon_0$

$$\left.\begin{aligned} R_в &= -c_в[\eta(x_{1в}) \mp \varepsilon_0], \\ R_н &= -c_н[\eta(x_{1н}) \mp \varepsilon_0]. \end{aligned}\right\} \tag{7.52}$$

where a minus sign before ε_0 is taken in the case of $\eta(x_i)>0$, and plus - in the case of $\eta(x_i)<0$. The influence of distortion of the guide rail or deviation of it from the vertical position (in the plane of action of disturbing forces) can be considered by means of introduction of additional shift ε, which is a certain function of coordinate ξ (see § 7.5). Since its value will be different for upper (в) and lower (н) belts of supports (41) and (42), then

$$R_{в} = -c_{в}[\eta(x_{1в}) + \varepsilon(\xi_{в})],$$
$$R_{н} = -c_{н}[\eta(x_{н}) + \varepsilon(\xi_{н})],\ \xi_{н} = \xi_{в} - a.$$

It is obvious that the influence of distortion of guide rails on the nature of load on the construction of the flight vehicle will essentially depend on the rigidity of elastic supports. Disturbing forces

$$\left.\begin{aligned} \Delta R_{в} &= -c_{в}\varepsilon(\xi_{в}), \\ \Delta R_{н} &= -c_{н}\varepsilon(\xi_{н}) \end{aligned}\right\} \tag{7.53}$$

will be lesser, the less $c_{в}$ and $c_{н}$.

Using expressions (51), (52) and (53), we write equations (48) and (50) in the form

$$\left.\begin{aligned} \ddot{\eta}_{т} + c\eta_{т} &= c_1\vartheta + f_{в} + R_{y_1}, \\ \ddot{\vartheta} + b_2\dot{\vartheta} &= b_3\eta_{т} + M_{в} - R_{y_1}(x_{1р} - x_{1т}), \end{aligned}\right\} \tag{7.54}$$

where

$$c = \frac{1}{m}(c_{в} + c_{н}),$$

$$\left.\begin{aligned}
c_1 &= \frac{1}{m}\left[P + c_{\text{в}}(x_{1\text{в}} - x_{1\text{т}}) + c_{\text{н}}(x_{1\text{н}} - x_{1\text{т}})\right],\\
b_2 &= \frac{1}{J_z}\left[c_{\text{в}}(x_{1\text{т}} - x_{1\text{в}})^2 + c_{\text{н}}(x_{1\text{т}} - x_{1\text{н}})^2\right],\\
b_3 &= \frac{1}{J_z}\left[c_{\text{в}}(x_{1\text{в}} - x_{1\text{т}}) + c_{\text{н}}(x_{1\text{н}} - x_{1\text{т}})\right],\\
f_{\text{в}} &= \frac{1}{m}\left[Y(\xi) + P\,\Delta\beta - c_{\text{в}}\varepsilon(\xi) - c_{\text{н}}\varepsilon(\xi - a)\right],\\
M_{\text{в}} &= \frac{1}{J_z}\{M_{\text{рв}} + Y(\xi)\left[x_{1\text{т}} - x_{1\text{д}}(\xi)\right] + c_{\text{в}}(x_{1\text{в}} - x_{1\text{т}})\varepsilon(\xi) +\\
&\qquad + c_{\text{н}}(x_{1\text{н}} - x_{1\text{т}})\varepsilon(\xi - a).\}
\end{aligned}\right\} \tag{7.55}$$

The solution of this system of ordinary nonlinear differential equations jointly with equation (49) and the equation of control describing the dependence of R_{y_1} on the parameters of rotation of the flight vehicle relative to the transverse axis, passing through center of gravity, are sought for by some method of approximation, for instance, the method of numerical integration. Here the beginning of countdown of time is taken not as the moment of starting the engine, but the moment of achievement of thrust force with a value equal to

$$P = R_{x_1} + G + X. \tag{7.56}$$

For comparatively large flight vehicles and especially in the case of high speed of emergence out of the silo a noticeable influence can be exerted on the value of support reactions, transverse forces, and bending moments by inertial forces, conditioned by elastic oscillations in the construction of the flight vehicle itself. Transverse shifting of any point of longitudinal axis of the vehicle x_1 in this case is determined by the expression

$$\eta(x_1, t) = \eta_{\text{т}}(t) + \vartheta(t)(x_{1\text{т}} - x_1) + \sum_{n=1}^{\infty} S_n(t) f_n(x_1). \tag{7.57}$$

Here $f_n(x_1)$ is designated the form of inherent bending oscillations of the structure of the flight vehicle of the n-th (as a beam with free ends), and S_n - unknown function of time. Here the corresponding support reactions will equal

$$\left.\begin{aligned} R_{\text{в}} &= -c_{\text{в}}\left[\eta_{\text{т}} + \vartheta(x_{1\text{т}} - x_{1\text{в}}) + \sum_{n=1}^{\infty} S_n f_n(x_{1\text{в}}) + \varepsilon(\xi_{\text{в}}) \mp \varepsilon_0\right], \\ R_{\text{н}} &= -c_{\text{н}}\left[\eta_{\text{т}} + \vartheta(x_{1\text{т}} - x_{1\text{н}}) + \sum_{n=1}^{\infty} S_n f_n(x_{1\text{н}}) + \varepsilon(\xi_{\text{н}}) \mp \varepsilon_0\right]. \end{aligned}\right\} \tag{7.58}$$

The equation of forced bending oscillations of the body of the flight vehicle in this case will be written in the form

$$\begin{aligned} \sum_{n=1}^{\infty} S_n \frac{d^2}{dx_1^2}\left[B(x_1)\frac{d^2 f_n(x_1)}{dx_1^2}\right] + m(x_1)\sum_{n=1}^{\infty} f_n(x_1)\frac{d^2 S_n}{dt^2} = \\ = \frac{\partial Y_1(x_1)}{\partial x_1} + R_{y_1}\delta(x_1 - x_{1\text{p}}) + R_{\text{в}}\delta(x_1 - x_{1\text{в}}) + \\ + R_{\text{н}}\delta(x_1 - x_{1\text{н}}) + M_{\text{pв}}\frac{d}{dx_1}\delta(x_1 - x_{1\text{п}}), \end{aligned} \tag{7.59}$$

where $\delta(x_1 - x_i)$ - Dirac-delta function $(x_i = x_{1\text{p}},\ x_{1\text{в}},\ x_{1\text{н}},\ x_{1\text{п}})$.

We will multiply the left and right sides of this equation by $f_n(x_1)$ and integrate with respect to x_1 from 0 to l. Taking into account the condition of orthogonality of functions $f_n(x_i)$ with weight $m(x_1)$ (4.62) and equalities

$$\int_0^l \frac{d^2}{dx_1^2}\left[B(x_1)\frac{d^2 f_n(x_1)}{dx_1^2}\right] f_n(x_1)\,dx_1 = M_n\omega_n^2$$

and

$$\int_0^l \frac{\partial Y_1(x_1)}{\partial x_1} f_n(x_1)\,dx_1 = \int_0^{x_1(t)} f_n(x_1)\frac{\partial Y_1(x_1)}{\partial x_1}\,dx_1,$$

we obtain the following ordinary differential equations for determination of functions S_n:

$$\begin{aligned} \ddot{S}_n + 2h_n\dot{S}_n + \omega_{n1}^2 S_n = \frac{1}{M_n}\Bigg[\int_0^{x_1(t)} \frac{\partial Y_1(x_1)}{\partial x_1} f_n(x_1)\,dx_1 + R_{y_1} f_n(x_{1\text{p}}) + \\ + M_{\text{pв}}\frac{df_n(x_{1\text{п}})}{dx_1} - \sum_{m=1}^{\infty} a_{mn}S_m + R_{\text{в}} f_n(x_{1\text{в}}) + R_{\text{н}} f_n(x_{1\text{н}})\Bigg] \end{aligned} \tag{7.60}$$

with $m \neq n$ $(n = 1, 2, \ldots)$.

where

$$
\left.\begin{aligned}
x_1(t) &\approx \xi(t),\\
a_{mn} &= f_m(x_{1\text{в}})f_n(x_{1\text{в}})c_\text{в} + f_n(x_{1\text{н}})f_m(x_{1\text{н}})c_\text{н},\\
\omega_{n1}^2 &= \frac{1}{M_n}\left\{\int_0^l B(x_1)\left[\frac{d^2 f_n(x_1)}{dx_1^2}\right]^2 dx_1 + c_\text{в} f_n^2(x_{1\text{в}}) + c_\text{н} f_n^2(x_{1\text{н}})\right\}.
\end{aligned}\right\} \qquad (7.61)
$$

The system of equations (60) is solved jointly with equations describing the motion of the flight vehicle as a solid body. Substituting in (48) and (50) in place of $R_\text{в}$ and $R_\text{н}$ the expressions (58), we will have

$$
\left.\begin{aligned}
\ddot{\eta}_\text{т} + c\eta_\text{т} &= c_1\vartheta + f_\text{в} + R_{y_1} - \sum_{n=1}^{\infty} a_{n\eta}\ddot{S}_n,\\
\ddot{\vartheta} + b_2\dot{\vartheta} &= b_3\eta + M_\text{в} - R_{y_1}(x_{1\text{p}} - x_{1\text{т}}) - \sum_{n=1}^{\infty} a_{n\vartheta}\ddot{S}_n,
\end{aligned}\right\} \qquad (7.62)
$$

where

$$
\begin{aligned}
a_{n\eta} &= \frac{1}{m}\left[c_\text{в} f_n(x_{1\text{в}}) + c_\text{н} f_n(x_{1\text{н}})\right],\\
a_{n\vartheta} &= \frac{1}{J_z}\left[c_\text{в} f_n(x_{1\text{в}})(x_{1\text{т}} - x_{1\text{в}}) + c_\text{н} f_n(x_{1\text{н}})(x_{1\text{т}} - x_{1\text{н}})\right].
\end{aligned}
$$

Knowing $\ddot{S}_n, \ddot{\vartheta}, \ddot{\eta}_\text{т}, R_\text{в}, R_\text{н}, Y(x_1)$, it is not difficult to construct a diagram of transverse forces and bending moments.

Somewhat more complex is the arrangement of load on the construction of the flight vehicle during action of disturbing forces and moments in plane $\xi\zeta$, perpendicular to the plane of location of the guide rails. If in the preceding case motion of the flight vehicle was flat, then in this case there can also be twisting of the body of the flight vehicle relative to the longitudinal axis. Let us designate by $r(x_{1\text{в}})$ and $r(x_{1\text{н}})$ the distances of supporting points of the lugs from the longitudinal axis of the vehicle x_1, and through $\varphi_0(x_{1\text{в}})$ and $\varphi_0(x_{1\text{н}})$ the corresponding angles of rotation of sections $x_{1\text{в}}$ and $x_{1\text{н}}$ of the body relative to this axis. Expressions for support reactions of the lugs, located on the left ($R'_{\text{в}\zeta}$, $R'_{\text{н}\zeta}$) and on the right ($R''_{\text{в}\zeta}$ and $R''_{\text{н}\zeta}$) from plane $\xi\zeta$ (Fig. 7.11), will have the form

$$
\left.
\begin{aligned}
R'_{\text{в}\zeta} &= -c_{\text{в}}\left[\zeta_{\text{т}} + \psi(x_{1\text{т}} - x_{1\text{в}}) + \varphi_0(x_{1\text{в}})r(x_{1\text{в}}) + e'(\xi)\right],\\
R''_{\text{в}\zeta} &= -c_{\text{в}}\left[\zeta_{\text{т}} + \psi(x_{1\text{т}} - x_{1\text{в}}) - \varphi_0(x_{1\text{в}})r(x_{1\text{в}}) + e''(\xi)\right],\\
R'_{\text{н}\zeta} &= -c_{\text{н}}\left[\zeta_{\text{т}} + \psi(x_{1\text{т}} - x_{1\text{н}}) + \varphi_0(x_{1\text{н}})r(x_{1\text{н}}) + e'(\xi - a)\right],\\
R''_{\text{н}\zeta} &= -c_{\text{н}}\left[\zeta_{\text{т}} + \psi(x_{1\text{т}} - x_{1\text{н}}) - \varphi_0(x_{1\text{н}})r(x_{1\text{н}}) + e''(\xi - a)\right],
\end{aligned}
\right\}
\tag{7.63}
$$

where ψ - angle of rotation of the flight vehicle relative to the transverse axis, passing through the center of gravity, in plane $\xi\zeta$, $\zeta_{\text{т}}$ - lateral shifting of center of gravity of the vehicle in this plane. Here the angles of rotation of support belts ϕ_0 will be equal to the sum of angles of rotation of the corresponding cross section due to twisting strain of the body $\Delta\phi$ and angle of rotation of the flight vehicle as a solid body (angle of roll ϕ). Using the method of separation of variables, we will present $\Delta\phi$ in the form of a series in terms of functions $X_{p\mu}(x_1)$, determining the normal forms of inherent torsional oscillations of the construction,

$$
\begin{gathered}
\varphi_0(x_1, t) = \varphi(t) - \Delta\varphi(x_1, t),\\
\Delta\varphi(x_1, t) = \sum_{p=1}^{\infty} \mu_p(t) X_{p\mu}(x_1).
\end{gathered}
\tag{7.64}
$$

Here μ_p is a certain function of time t. According to (3.3) the conditions of dynamic equilibrium of an element of a rod in the case of torsional oscillations are expressed by the formula

$$
-I_m(x_1)\frac{\partial^2 \Delta\varphi(x_1, t)}{\partial t^2} = \frac{\partial M_{\text{кр}}(x_1)}{\partial x_1},
\tag{7.65}
$$

where $M_{\text{кр}}(x_1)$ - torque created by elastic forces (3.1), support reactions, and control forces. Thus

$$
\begin{aligned}
\frac{\partial M_{\text{кр}}(x_1, t)}{\partial x_1} = \frac{\partial}{\partial x_1}\left[GI_p(x_1)\frac{\partial \Delta\varphi(x_1, t)}{\partial x_1}\right] + \sum_{i=1}^{4} R_{зi} r(x_{1p})\delta(x_1 - x_{1p}) -\\
-(R'_{\text{в}\zeta} - R''_{\text{в}\zeta})r(x_{1\text{в}})\delta(x_1 - x_{1\text{в}}) - (R'_{\text{н}\zeta} - R''_{\text{н}\zeta})r(x_{1\text{н}})\delta(x_1 - x_{1\text{н}}).
\end{aligned}
\tag{7.66}
$$

Here $r(x_{1p})$ designates the distance from point of application to the body of the control force to the longitudinal axis of the flight vehicle, G - shear modulus of elasticity, and $I_p(x_1)$ - polar area moment of inertia of cross section of the carrying section in the construction if the body of the flight vehicle. Placing (66) in equation (65) and using formula (64), and also corresponding conditions of orthogonality of forms of torsional oscillations with weight $I_m(x_1)$, we obtain

$$\int_0^l I_m(x_1)X_{p\mu}(x_1)X_{k\mu}(x_1)dx_1=0 \quad \text{with} \quad p\neq k;$$

after carrying out of conversions, similar to those which were executed during derivation of bending oscillations in the construction of a flight vehicle (multiplication of all members of the equation by $X_{p\mu}(x_1)$ and integration in terms of x_1), we obtain the following ordinary differential equation for function $\mu_p(t)$:

$$\ddot{\mu}_p+2h_p\dot{\mu}_p+\omega_p^2\mu_p=\frac{1}{J_{p\mu}}\Big[(R'_{\text{в}\zeta}-R''_{\text{в}\zeta})X_{p\mu}(x_{1\text{в}})r(x_{1\text{в}})+ (R'_{\text{н}\zeta}-R''_{\text{н}\zeta})X_{p\mu}(x_{1\text{н}})r(x_{1\text{н}})-\sum_{i=1}^{4}R_{zi}r(x_{1p})X_{p\mu}(x_{1p})\Big], \tag{7.67}$$

where

$$J_{p\mu}=\int_0^l I_m(x_1)X_{p\mu}^2(x_1)dx_1 \quad (p=1,\ 2,\ \ldots), \tag{7.68}$$

ω_p - frequency of inherent torsional oscillations in the construction of the flight vehicle.

Equation of oscillations of the flight vehicle as a solid body relative longitudinal axis x_1 has the form

$$J_x\ddot{\varphi}=(R'_{\text{в}\zeta}-R''_{\text{в}\zeta})r(x_{1\text{в}})+(R'_{\text{н}\zeta}-R''_{\text{н}\zeta})r(x_{1\text{н}})+M_{xy}, \tag{7.69}$$

where J_x - mass moment of inertia of the vehicle relative to longitudinal axis, M_{xy} - controlling moment based on channel of bank.

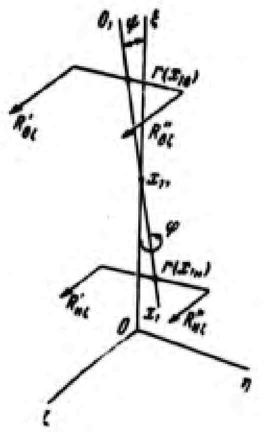

Fig. 7.11. Arrangement of load on lugs in a plane perpendicular to the location of the guide rails.

Thus, for calculation of support reactions $R'_{в\zeta}$, $R''_{в\zeta}$, $R'_{н\zeta}$, $R''_{н\zeta}$, turning and bending moments in a plane perpendicular to the plane of location of the guide rails, it is necessary to solve the system of equations (67) and (69) jointly with equations of transverse oscillations of the flight vehicle, analogous to equations (60), (48), and (50), and equations of control for channels of yawing, pitch, and bank. Moreover in equations (60), (48), and (50) instead of $R_в$ and $R_н$ it is necessary to substitute the expressions

$$R_{в\zeta} = R'_{в\zeta} + R''_{в\zeta},$$
$$R_{н\zeta} = R'_{н\zeta} + R''_{н\zeta},$$

and instead of η_T and θ - variables ζ_T and ψ correspondingly. Assuming that all disturbing forces and moments act in one plane, and namely in plane $\xi\zeta$, we obtain the largest possible values of these reactions, transverse forces turning and bending moments, and the necessary gaps between the body of the flight vehicle and the guide rails.

Torque is calculated in accordance with expression (66) by the formula

$$M_{\text{кр}}(x_1, t) = \int\limits_0^{x_1} \frac{\partial M_{\text{кр}}(x_1, t)}{\partial x_1}\, dx_1.$$

Calculation of quasi-static values of transverse forces and bending moments is done taking into account the support reactions according to the following formulas:

$$Q_\text{c}(x_1, t) = Y(x_1, t) + R_{y_1}\sigma_0(x_1 - x_{1\text{р}}) + \int\limits_0^{x_1} q_{0y}(x_1)\, n_{y_1}(x_1, \xi)\, dx_1 + \\ + R_\text{в}\sigma_0(x_1 - x_{1\text{в}}) + R_\text{н}\sigma_0(x_1 - x_{1\text{н}}), \tag{7.70}$$

$$M_\text{c}(x_1, t) = \int\limits_0^{x_1} Q_\text{c}(x_1, t)\, dx_1. \tag{7.71}$$

Dynamic components $Q_\text{д}(x_1)$ and $M_\text{д}(x_1)$ are determined by expressions (3.3), and the total values of transverse force and bending moment in plane $\xi\zeta$ - by formulas (3.2).

During the appraisal of necessary carrying capacity for the construction of the flight vehicle an analysis is made of the geometric sum of transverse forces and bending moments acting in both planes $\xi\eta$ and $\xi\zeta$:

$$\left.\begin{aligned} Q(x_1, t) &= \left[Q_\eta^2(x_1, t) + Q_\zeta^2(x_1, t)\right]^{\frac{1}{2}}, \\ M(x_1, t) &= \left[M_\eta^2(x_1, t) + M_\zeta^2(x_1, t)\right]^{\frac{1}{2}}. \end{aligned}\right\} \tag{7.72}$$

Here it is considered that maximum loads from wind can act on the flight vehicle at the same time in only one of these planes. The last remark pertains, naturally, also to the calculation of loads on support elements, which was already noted, are installed on the most powerful frames of the body of the flight vehicle, which are able to absorb large concentrated forces in radial and tangential directions.

Above we assumed that guide rails and that base to which they are joined are rigid, i.e., are not noticeably deformed under the influence of moving concentrated loads R_i and $R_{i\zeta}$ (i = в, н). If this assumption is not justified, then one should also consider the influence of elasticity of the launcher itself. This is possible to do approximately, taking instead of $c_i (c_в, c_н)$ a certain given rigidity c_i'

$$c_i' = \frac{c_i c_c(\xi)}{c_i + c_c(\xi)} \qquad (i = \text{в, н}), \tag{7.73}$$

where $c_c(\xi)$ designates the local rigidity of a guide rail jointly with the base (jacket, boom, or girder of the launcher), equal to the ratio of magnitude of concentrated force, applied at point ξ, to the radial (or tangential) shifting of the point caused by it.

The influence of general deformation of this base (jacket or boom) as a thin-walled rod can be considered by means of inclusion in system of additional equations of the type

$$\ddot{K}_p + 2h_p\dot{K}_p + \omega_{pc}^2 K_p = -\frac{1}{M_{pc}} \sum_i R_i \Phi_p(x_i) \qquad (p = 1, 2, \ldots; \; i = \text{в, н}), \tag{7.74}$$

where

$$M_{pc} = \int_0^{l_c} m_{cт}(x) \Phi_p^2(x)\, dx,$$

and ω_{pc} - angular frequency of inherent transverse elastic oscillations of the jacket, $\Phi_p(x_i)$ - form of these oscillations in current point x_i, satisfying the corresponding boundary conditions, $m_{cт}(x)$ - linear mass of jacket with guide rails, M_{pc} - reduced mass. Moreover in the expressions for support reactions (58) or (63) we additionally introduce the member

$$\Delta R_i = c_i \sum_{p=1}^{\infty} K_p(t) \Phi_p(x_i) \qquad (i = \text{в, н}). \tag{7.75}$$

If launching of the flight vehicle is done from an elastically suspended base (without preliminary turning off of shock absorbers), then it is also necessary to consider additional degrees of freedom of system caused by the motion of this base (jacket) as a solid body in the motionless system of coordinates $\xi\eta\zeta$:

$$M_c \ddot{\eta}_{cT} = -\sum_i R_i + \sum_{j=1}^{n_j} R_{cj}, \tag{7.76}$$

$$J_c \ddot{\vartheta}_c = \sum_i R_i (x_i - x_T) + \sum_{j=1}^{n_j} R_{cj} (x_{Tc} - x_{cj}) \quad (i = \text{в, н}). \tag{7.77}$$

Here M_c, J_c - mass and mass moment of inertia of base (jacket) relative to the transverse axis passing through its center of gravity x_{Tc}, R_{cj} - support reactions of shock absorbers located in sections x_{cj} of the base equal to

$$R_{cj} = -c_{cj}[\eta_{cT} + \vartheta_c (x_{Tc} - x_{cj})], \tag{7.78}$$

where η_{cT} - transverse shift of center of gravity of the base, ϑ_c - angle of rotation of its longitudinal axis x in plane $\xi\eta$, c_{cj} - rigidity of j-th support of base. Correspondingly corrections are made of values of reactions R_i for magnitude

$$\Delta R_i = c_i [\eta_{cT} + \vartheta_c (x_{cT} - x_i)]. \tag{7.79}$$

Calculation of influence of elasticity of the jacket (girder of launcher, and so forth) and their support elements on the nature of load on the construction of the flight vehicle complicates the solution of the problem. Therefore the question of composition of the dynamic system for calculation of its construction this particular case of load should be approached carefully. It is desirable preliminarily to conduct a comparative appraisal of the influence on R_1, $Q(x_1)$, $M(x_1)$ of each degree of freedon of the system. In the case when rigidities of supports of the flight vehicle are small and permissible transverse shifts of it in the launcher are large an influence of any kind of additional shifts of the supports, caused by comparatively small errors in fabricating and assembling of the guide rails, and local and general deformation of the launcher and guide rails can be disregarded.

§ 7.7. Peculiarities of Load During Launch from a Firing Tube of a Ship

Everything said in the preceding sections with respect to launching of a flight vehicle from a motionless silo remains true also for the case of launching from a firing tube located on a surface vessel or submarine. The only thing different will be the magnitudes of external forces and moments acting on the flight vehicle. In all cases it will be additionally loaded by transverse inertial forces caused by rocking of the ship on waves. Since the periods of these oscillations are great as compared to the periods of inherent bending oscillations in the construction of the flight vehicle and even exceed the time of movement of it in the shaft of a silo, then these inertial loads almost always will have a static nature. It is not difficult to consider their influence on parameters of motion of the vehicle and on the values of support reactions, transverse forces, and bending moments. For this it follows in equations (76), (77) to consider functions η_{CT} and ϑ_c, determining the motion of the actual base (jacket) of the tube, as assigned, for instance, changing according to the law

$$\eta_{cT} = \eta_{cT0} \sin \omega t,$$
$$\vartheta_c = \vartheta_{c0} \sin \omega t,$$

where η_{CT0} - amplitude of oscillations of the ship at a point coinciding with the center of gravity of the tube jacket, ϑ_{c0} - amplitude of angular oscillations of the jacket relative to the transverse axis passing through this center of gravity, and ω - frequency of oscillations of the ship.

During launching from under water the values of aerodynamic (in this case hydrodynamic) forces and moments, entering in equations (32), (32'), (32"), (49), (54), and (60) will be changed. The vehicle will be influenced by additional ejecting force $A_{\text{в}}$ (Archimedian force). Simultaneously drag will increase significantly. Increase of the latter is explained not only by an increase of impact pressure due to the high density of water, but also by the appearance of suction drag. It is obvious that these forces should be introduced in the equation of longitudinal motion of a flight vehicle in a silo (32) or (49), writing it in the form

$$m\ddot{\xi} = P - G - X + A_{\text{в}} - R_{x_1}.$$

In equations (30) and (30') instead of average wind speed the speed of the ship is taken, and in certain cases the influence of pulsation of flow rate caused by agitation of sea is considered. Here during the calculation of coefficients m and J_z, entering in the equation both of longitudinal and transverse motion of a flight vehicle, the adjoint mass of water is considered. The latter can have values, comparable with the mass of the flight vehicle itself. The magnitude of coefficient of hydrodynamic damping of transverse oscillations of the construction of the vehicle increases insignificantly. And, finally, if the calculation of forces in the body of the flight vehicle in cases $S_{\text{к}}$ and $V_{\text{к}}$ is conducted for storm conditions of load on a ship, then in that particular case the necessity can arise for calculation of additional overloads, caused by elastic oscillations in the construction of the ship itself. The value of these low-frequency vibration overloads is determined as a result of carrying out a dynamic calculation of the construction of the ship.

Footnotes

[1]Aviation Week, 9v., vol. 72, No. 19, p. 55.

[2]Voprosy raketnoy tekhniki, No. 2, 1967, str. 107.

[3]An exception are cases of crossing through an obstacle and movement over periodically repeated roughnesses with an unfavorable combination of frequencies of natural oscillations with frequency of the disturbing influence of the roughnesses.

[4]Voprosy raketnoy tekhniki, No. 2, 1967, str. 9.

CHAPTER VIII

DYNAMICS OF LAUNCHING A FLIGHT VEHICLE FROM AN ELASTIC LAUNCHING INSTALLATION

§ 8.1. Cases of Load

The launching of uncontrolled and controlled flight vehicles which are intended for time use (aircraft or ballistic type) is usually carried out with the help of a special launcher. Construction of such an installation is determined basically by the assignment of the vehicle, its geometric dimensions, launching weight, and component arrangement. Sometimes it also depends on the structure diagram of the body, in particular, on the place of location of the body of support section which is able to transmit the weight of the vehicle on the launcher. Most frequently selected as such a section is the point of application of the thrust force of the engines or forces from side-mounting boosters.

Construction of the launcher is complicated significantly when the mentioned support section is located comparatively far away or when for absorption of crosswind loads, disturbing moment from the system of boosters, and also transverse component of weight force (during inclined launching) is there the necessity of introduction of transverse supports. In such cases for determination of optimum (based on conditions of body strength of the flight vehicle) values of rigidity of these supports the necessity arises for carrying out an investigation of dynamics of the flight vehicle launching taking into account the influence of elasticity of construction of the launcher.

We will consider a flight vehicle equipped with side-mounting boosters (Б, В, Г, Д, ...). Let us assume that each booster has an independent propulsion system, the thrust force of which is transmitted to the body of the vehicle in a certain section $x_{\text{В}}$. This section, which absorbs comparatively large longitudinal loads, we will take as the support section during installation of the flight vehicle on the launcher. Certainly it is also possible to take some other section, for instance a section located on the bodies of the side-mounting booster, and so forth.

In a general case the supporting structure of a launcher can be presented schematically in the form of a system of rods, some of which perceive only vertical loads (weight force), and some - only lateral load and moments acting on the flight vehicle in the period it is in the launcher. Let us assume for simplicity that the number of support rods is equal to the number of side-mounting boosters, i.e., each booster is suspended to a separate support rod of the launcher, and the body of the flight vehicle rests on the booster.

Structural load of such a flight vehicle and launcher in the process of launching will depend on the nature of change in thrust force of the engines during starting and emergency shutdown. Scattering of time for the onset of an increase (drop) in thrust force (from the moment of issuance of the command for launching) and difference in the actual laws of change of pressure in the combustion chambers of engines of oppositely located boosters, for instance Б and Г, will lead to the appearance of large dynamic disturbing moments, transmitted through the body of the flight vehicle to the launcher. To anticipate all the possible variants of actual switching on of engines is practically impossible and it is doubtful whether it is necessary. It is possible to establish certain most probable laws of change in these moments only in the presence of a comparatively large number of corresponding experimental data. In the process of planning as a rule such data are absent. Therefore during the determination of necessary strength and rigidity for the structure of a flight vehicle and the structure of the launcher it is necessary to be oriented on limiting cases of load:

a) case of identical change of thrust force for all engines of the flight vehicle, i.e., a case of action on the flight vehicle of maximum longitudinal force (in the absence of disturbing transverse forces and moments);

b) case of identical change of thrust force in a group of side-by-side boosters on the assumption that the actual switching on of engines of the remaining boosters occurs with the highest possible delay for the given system, i.e., a case of action of maximum disturbing moment on the system flight vehicle-launcher.

In both cases both the process of approach to conditions and also the process of cutting off of the engines are considered, if of course the latter is anticipated by the starting arrangement. For the flight vehicle having a one-sided power connection with the boosters (in the direction of action of thrust force), it is expedient additionally to analyze and a case of delay of switching on the engine of one of the side-mounting boosters. The stated cases of load are conditional, since in reality during the influence of aerodynamic and control forces on a flight vehicle in the process of launching the vehicle-launcher system will accomplish oscillations in longitudinal and transverse directions simultaneously. Only in case (a) and in an additional case of load longitudinal oscillations will be predominant, and case (b) - transverse. Methods for calculation of longitudinal oscillations in the structure of a flight vehicle in the process of launching, magnitude of its jump above the launcher, and coefficient of dynamics for supporting members of the launcher (in longitudinal direction) were presented in preceding chapter. The question on longitudinal elastic oscillations of the structure of a flight vehicle with isolated side-mounting boosters will be considered more specifically in Chapter XI. In this chapter we will dwell only on transverse oscillations of the vehicle-launcher system.

§ 8.2. Equations of Free Transverse Oscillations of a Vehicle-Launcher System

For calculation of dynamic components of transverse forces and bending moments in cross sections of the body of boosters and the vehicle itself, and also transverse dynamic loads on support elements of launcher in the process of launching it is necessary first of all to determine the form and frequency of inherent transverse oscillations of the flight vehicle-launcher system. For flight vehicles which do not have side-mounting boosters this problem is solved comparatively simply. Actually if the masses of transverse support elements of the launcher are small as compared to mass of the flight vehicle, then they can be presented schematically in the form of weightless elastic supports possessing a certain rigidity E_B and E_H. Then, considering the flight vehicle as an elongated beam, located (in a transverse direction) on two elastic supports "B" and "H" it is possible to find directly the unknown values ω_m and $f_m(x)$ of this system by using the method of iteration.

In this case when it is impossible to disregard the mass of transverse support elements of the launcher of the structure of the flight vehicle consists of several in parallel and elastically connected blocks (pack of blocks) the direct calculation of normal forms of transverse oscillations of the system by the method of iterations is inexpedient. In spite of the apparent simplicity, this method is laborious for systems consisting of a cluster of beams. A change of any parameter of the system requires a repetition of the calculation from the very beginning. In the process of planning, and also in the analysis of the dynamic arrangement of the system, this has to be done very frequently. From this point of a more flexible, and consequently simpler, method is based on the use of linear transformation of generalized coordinates to normal coordinates (Chapter V). We will also use this method in this chapter.

At first we will make up equations for determining the frequencies and forms of inherent transverse oscillations of a flight vehicle (without boosters) which is suspended in a transverse direction, for example, on eight elastic support rods with a variable section of comparatively large mass (located in planes $\xi\eta$ and $\xi\zeta$). Let us assume that all the rods included in one support belt (upper "B" or lower "H") are completely identical, i.e., axis ξ is the axis of symmetry of the system. Further we will consider that all the support rods in a plane perpendicular to the plane of their positioning are rigidly fastened on one end in the stationary base of the launcher, and on the other end are connected by a joint with the body of the flight vehicle (in sections x_B and x_H). In the plane of their disposition they are able to rotate freely relative to points O_1 and O_2 (Fig. 8.1). Furthermore, for definiteness we shall accept that the weight force of the flight vehicle is transmitted on the longitudinal supports of the launcher in section x_B, i.e., in the area of disposition of the upper transverse supports.

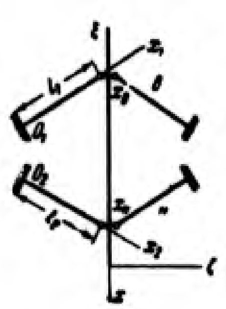

Fig. 8.1. Dynamic arrangement of the vehicle-elastic launcher system in a transverse direction.

We will examine small free transverse oscillations of the system in one of the planes of symmetry, for example, $\xi\eta$. In addition we will introduce the auxiliary systems of coordinates $O_1x_1y_1$ and $O_2x_2y_2$, connected with the nondeformed state of the support rods of the upper and lower belts. Axes x_1 and x_2 of these systems we will draw through points O_1, O_2 and the points of connection of the rods with the body of the flight vehicle, and axes y_1 and y_2 we direct parallel to axis η of the launching system of coordinates. Let us assume that displacement of any points x_1 and x_2 of longitudinal axes of rods "B" and "H" (due to bending) will equal $\eta_B(x_1, t)$ and $\eta_H(x_2, t)$. It is obvious that these shifts of the ends of these support rods ($x_1 = l_1$, $x_2 = l_2$) $\eta_B(l_1, t)$ and $\eta_H(l_2, t)$ will evoke forward and rotational motion of the flight vehicle as a solid body. If one were to designate transverse shifts of points of the longitudinal axis of the flight vehicle due to bending strain of body through y(x, t), then the transverse shift of any point x of this axis in a motionless system of coordinates in the plane of oscillations will be determined by the expression

$$\eta(x,\, t) = y(x,\, t) + \eta_{\text{в}}(l_1,\, t)\left(1 - \frac{x - x_{\text{в}}}{a}\right) + \eta_{\text{н}}(l_2,\, t)\frac{x - x_{\text{в}}}{a}, \qquad (8.1)$$

where a - distance between supports $x_{\text{в}}$ and $x_{\text{н}}$, l_1 and l_2 - lengths of rods "в" and "н."

If the transverse support rods of launcher which are located in plane ξη do not take part in transverse oscillations of the system in this plane ξη, then the kinetic energy of the system will consist only of the sum of kinetic energy of oscillations of two pairs of transverse support rods ("в" and "н") and kinetic energy of motion of the flight vehicle itself:

$$T_0(t) = \frac{1}{2}\int\limits_0^L m(x)\dot{\eta}^2(x,\, t)\,dx + \int\limits_0^{l_1} m(x_1)\dot{\eta}_{\text{в}}^2(x_1,\, t)\,dx_1 + \int\limits_0^{l_2} m(x_2)\dot{\eta}_{\text{н}}^2(x_2,\, t)\,dx_2. \qquad (8.2)$$

If, however, these support rods participate in the movement of the system, then additionally one should consider the kinetic energy of rotation of two other pairs of rods, equal to

$$\Delta T_0(t) = \frac{J_1}{l_1^2}\dot{\eta}_{\text{в}}^2(l_1,\, t) + \frac{J_2}{l_2^2}\dot{\eta}_{\text{н}}^2(l_2,\, t). \qquad (8.3)$$

Here J_1 and J_2 designate the moments of inertia of mass of rods "в" and "н" relative to transverse axes y_1 and y_2.

Potential energy U_0 of the given elastic system will consist of the sum of strain energies of the flight-vehicle body, strain energy of the rods, and energy of the position. Potential energy of the position will be equal to the work which is accomplished by gravity on the vertical displacement of the center of gravity of the flight vehicle, caused by the turning of axis x relative to the longitudinal support. Usually the share of this energy in the overall value of potential energy of the system is small and its influence in first approximation can be disregarded. For simplification of the problem we will assume that it is also possible to disregard the energy deformation of the longitudinal support rods of the launcher due to transverse oscillations of the system. In case of jump of the flight vehicle above the launcher it actually will be equal to zero. Thus,

$$U_0(t)=\frac{1}{2}\int_0^L B(x)\left[\frac{\partial^2 y(x,t)}{\partial x^2}\right]^2 dx+ \\ +\int_0^{l_1} B(x_1)\left[\frac{\partial^2 \eta_{\text{п}}(x_1,t)}{\partial x_1^2}\right] dx_1+\int_0^{l_2} B(x_2)\left[\frac{\partial^2 \eta_{\text{н}}(x_2,t)}{\partial x_2^2}\right]^2 dx_2. \tag{8.4}$$

In accordance with method of separation of variables we will present the transverse deformations of transverse support rods and the body of the flight vehicle in the form

$$\left.\begin{aligned} y(x,t)&=\sum_{n=1}^{\infty} S_n(t)f_n(x), \\ \eta_{\text{п}}(x_1,t)&=\sum_{p=1}^{\infty} K_p(t)\Phi_p(x_1), \\ \eta_{\text{н}}(x_2,t)&=\sum_{k=1}^{\infty} T_k(t)\dot{X}_k(x_2), \end{aligned}\right\} \tag{8.5}$$

where S_n, K_p, T_k - function only of time, Φ_p - function only of coordinate x_1, X_k - function only of coordinate x_2. Functions of coordinates are selected in such a way that they satisfy the corresponding boundary conditions. For a flight vehicle, both ends of which are free, they consist of the absence of forces in end sections of the body x = 0 and x = L:

$$\frac{d}{dx}\left[B(x)\frac{d^2 f_n(x)}{dx^2}\right]=0, \qquad \frac{d^2 f_n(x)}{dx^2}=0. \tag{8.6}$$

For support rods "в" and "н" - in equality to zero for transverse shifts and angles of turning of cross sections in the area of fixing and in equality to zero for bending moments and transverse forces on free ends

$$
\left.
\begin{array}{l}
\Phi_p(x_1)=0,\quad \dfrac{d\Phi_p(x_1)}{dx_1}=0 \text{ with } x_1=0,\\
X_k(x_2)=0,\quad \dfrac{dX_k(x_2)}{dx_2}=0 \text{ with } x_2=0,\\
\dfrac{d^2\Phi_p(x_1)}{dx_1^2}=0,\quad \dfrac{d}{dx_1}\left[B(x_1)\dfrac{d^2\Phi_p(x_1)}{dx_1^2}\right]=0 \text{ with } x_1=l_1,\\
\dfrac{d^2X_k(x_2)}{dx_2^2}=0,\quad \dfrac{d}{dx_2}\left[B(x_2)\dfrac{d^2X_k(x_2)}{dx_2^2}\right]=0 \text{ with } x_2=l_2.
\end{array}
\right\} \tag{8.7}
$$

In reality the values of transverse forces on ends of support rods will be determined by the magnitude of the corresponding support reactions R_B and R_H. Therefore the boundary conditions written above (7) at first glance can appear unnatural. However, as we will see subsequently, the influence of these transverse forces on function $\Phi_p(x_1)$ and $X_k(x_2)$ is more expedient to consider not in boundary conditions, but by means of introduction of the corresponding apparent additional concentrated masses on the ends of the rods.

Also imposed on function $f_n(x)$ is the additional condition of equality to zero of sags in the body of the flight vehicle in places of installation of supports "в" and "н," i.e.,

$$f_n(x_в)=f_n(x_н)=0. \tag{8.8}$$

Furthermore, for obtaining the expression of potential and kinetic energy of the flight vehicle and transverse support rods in the form of the sum of squares, we require that functions $f_n(x)$, and also functions $\Phi_p(x_1)$ and $X_k(x_2)$, be orthogonal among themselves with the corresponding weight $m(x)$, $m(x_1)$ and $m(x_2)$. In other words, these functions should constitute forms of natural oscillations of certain partial systems, obtained in a specific way from the considered system. In order to better grasp the method of obtaining these partial systems, we will follow the derivation of equations of free oscillations of the system up to the very end.

We will take as a generalized coordinate the functions $S_n(t)$, $K_p(t)$, and $T_k(t)$. Taking into account the above-indicated properties of functions $f_n(t)$, $\Phi_p(x_1)$, and $X_k(x_2)$ and using equation (4.1) and formulas (2), (4), and (5), after certain elementary conversions, which we will omit here, we obtain the following system of ordinary differential equations, describing free transverse oscillations of system

$$\left.\begin{aligned} M_n(\ddot{S}_n+\omega_n^2 S_n)+\sum_{p=1}^{\infty} a_{np}\ddot{K}_p+\sum_{k=1}^{\infty} a_{nk}\ddot{T}_k=0,\\ 2M_{np}(\ddot{K}_p+\omega_{np}^2 K_p)+\sum_{n=1}^{\infty} a_{pn}\ddot{S}_n+\sum_{k=1}^{\infty} b_{pk}\ddot{T}_k=0,\\ 2M_{nk}(\ddot{T}_k+\omega_{nk}^2 T_k)+\sum_{n=1}^{\infty} a_{kn}\ddot{S}_n+\sum_{p=1}^{\infty} b_{kp}\ddot{K}_p=0\end{aligned}\right\} \tag{8.9}$$

$$(n=1,\ 2,\ \ldots;\ p=1,\ 2,\ \ldots;\ k=1,\ 2,\ \ldots).$$

Coefficients of these equations will be equal to

$$\left.\begin{aligned} a_{np}=a_{pn}=\frac{\Phi_p(l_1)}{a}\int_0^L m(x)f_n(x)(x_{\text{н}}-x)\,dx,\\ a_{nk}=a_{kn}=\frac{1}{a}X_k(l_2)\int_0^L m(x)f_n(x)(x-x_{\text{в}})\,dx,\\ b_{pk}=b_{kp}=\frac{1}{a^2}X_k(l_2)\Phi_p(l_1)\{M[x_{\text{т}}(x_{\text{н}}+x_{\text{в}})-x_{\text{в}}x_{\text{н}}]-J(0)\},\\ M_n=\int_0^L m(x)f_n^2(x)\,dx,\\ M_{np}=\int_0^{l_1} m_{np}(x_1)\Phi_p^2(x_1)\,dx_1,\\ M_{nk}=\int_0^{l_2} m_{np}(x_2)X_k^2(x_2)\,dx_2,\end{aligned}\right\} \tag{8.10}$$

where x_T - coordinate of center of gravity of the flight vehicle, $J(0)$ - moment of inertia of its mass relative to transverse axis z, passing through the origin of coordinates x = 0 (summit of body of the vehicle), M - mass of flight vehicle, $m_{\text{пр}}$ - reduced mass of support rods, equal to

$$m_{\text{пр}}(x_1) = m(x_1) + \frac{J_z(x_н)}{2a^2} \cdot \delta(x_1 - l_1),$$
$$m_{\text{пр}}(x_2) = m(x_2) + \frac{J_z(x_в)}{2a^2} \cdot \delta(x_2 - l_2).$$

Here $J_z(x_B)$ and $J_z(x_H)$ - mass moments of inertia of flight vehicle relative to the transverse axes passing through the upper x_B and lower x_H supports. With ω_n, ω_{Bp} and ω_{Hk} we designate the expressions

$$\left.\begin{aligned}
\omega_n^2 &= \frac{1}{M_n}\int_0^L B(x)\left[\frac{d^2 f_n(x)}{dx^2}\right]^2 dx,\\
\omega_{вp}^2 &= \frac{1}{M_{вp}}\int_0^{l_1} B(x_1)\left[\frac{d^2\Phi_p(x_1)}{dx_1^2}\right]^2 dx_1,\\
\omega_{нk}^2 &= \frac{1}{M_{нk}}\int_0^{l_2} B(x_2)\left[\frac{d^2 X_k(x_2)}{dx_2^2}\right]^2 dx_2.
\end{aligned}\right\} \qquad (8.11)$$

Comparing them with (4.65) we see that they constitute partial frequencies of inherent transverse oscillations of the flight vehicle, and the upper and lower support rods correspondingly. It is not difficult to note that function $f_n(x)$ in this case should satisfy equation (4.67), boundary conditions (6), and the condition of orthogonality (4.62). Functions $\Phi_p(x_1)$ and $X_k(x_2)$ in the case of above-indicated boundary conditions (7) should represent the forms of inherent transverse oscillations of support rods with apparent additional concentrated masses $J_z(x_H)/2a^2$ and $J_z(x_B)/2a^2$ in sections $x_1 = l_1$ and $x_2 = l_2$, i.e., satisfy equations

$$\frac{d^2}{dx_1^2}\left[B(x_1)\frac{d^2\Phi_p(x_1)}{dx_1^2}\right] - \omega_{вp}^2 m_{\text{пр}}(x_1)\Phi_p(x_1) = 0, \qquad (8.12)$$

$$\frac{d^2}{dx_2^2}\left[B(x_2)\frac{d^2 X_k(x_2)}{dx_2^2}\right] - \omega_{нk}^2 m_{\text{пр}}(x_2) X_k(x_2) = 0 \qquad (8.13)$$

$$(p = 1, 2 \ldots; \quad k = 1, 2, \ldots)$$

Solution of the system of uniform differential equations in reverse form (9) can be presented in the form

$$\left.\begin{aligned} S_n(t) &= S_{nm}e^{i\omega t},\\ K_p(t) &= K_{pm}e^{i\omega t},\\ T_k(t) &= T_{km}e^{i\omega t},\end{aligned}\right\} \qquad (8.14)$$

$$(n = 1,\ 2,\ \ldots;\ p = 1,\ 2,\ \ldots;\ k = 1,\ 2,\ \ldots),$$

where ω - frequency of natural oscillations of the system on the whole, and S_{nm}, K_{pm} and T_{km} - amplitude of oscillations of points of reduction of corresponding partial systems. Substituting these solutions in equations (9) and equating to zero the determinant of the system, composed of coefficients with unknowns S_{nm}, K_{pm} and T_{km}, we obtain an algebraic equations relative to ω^2 - equation of frequencies of the system. This equation will have the simplest form in the particular case when it is possible to be limited to a calculation of only the first tones of partial frequencies of the system:

$$\begin{vmatrix} (\omega_n^2 - \omega^2)M_n & -\omega^2 a_{np} & -\omega^2 a_{nk} \\ -\omega^2 a_{pn} & 2(\omega_{\text{н}p}^2 - \omega^2)M_{\text{н}p} & -\omega^2 b_{pk} \\ -\omega^2 a_{kn} & -\omega^2 b_{kp} & 2(\omega_{\text{н}k}^2 - \omega^2)M_{\text{н}k} \end{vmatrix} = 0.$$

By expanding the determinant we have

$$\begin{aligned} &\omega^6(4M_nM_{\text{н}p}M_{\text{н}k} + a_{np}b_{pk}a_{kn} - 2M_{\text{н}p}a_{kn}^2 - M_nb_{kp}^2 - 2M_{\text{н}k}a_{np}^2) - \\ &\quad - \omega^4(4F_{\text{н}k}M_nM_{\text{н}p} + 4E_{\text{н}p}M_nM_{\text{н}k} + 4E_nM_{\text{н}p}M_{\text{н}k} - 2E_{\text{н}p}a_{nk}^2 - \\ &\quad - E_nb_{kp}^2 - 2E_{\text{н}k}a_{np}^2) + 4\omega^2(E_{\text{н}p}E_{\text{н}k}M_n + E_nE_{\text{н}k}M_{\text{н}p} + E_nE_{\text{н}p}M_{\text{н}k}) - \\ &\qquad\qquad - 4E_{\text{н}p}E_{\text{н}k}E_n = 0, \end{aligned} \qquad (8.15)$$

where

$$\begin{aligned} E_n &= \omega_n^2M_n,\\ E_{\text{н}p} &= \omega_{\text{н}p}^2M_{\text{н}p},\\ E_{\text{н}k} &= \omega_{\text{н}k}^2M_{\text{н}k}. \end{aligned}$$

§ 8.3. Transverse Oscillations of the Structure of an Apparatus with a Pack Arrangement

Now we will consider free transverse elastic oscillations of a more complex system, namely oscillations of a flight vehicle with side-mounting boosters (pack layout) on the elastic launcher described in the preceding section. If such a flight vehicle consists of a large number of blocks, connected in one or several packs, then the problem of determination of its natural frequencies and forms of oscillations will present great difficulties, basically of a calculating nature. The actual process of composition of equations of free oscillations of such a system with an increase in the number of blocks and packs is complicated comparatively little. Therefore, for simplification of the letter we will subsequently be limited to a consideration of oscillations of a hypothetical flight vehicle, representing a system which is made up of one basic block (A) and eight side-mounting boosters (Fig. 8.2) absolutely identical in their structural and weight parameters, located symmetrically around it. Here we will consider that the connection of side-mounting boosters Б, В, Г, Д with the body of block A in transverse and lateral directions is carried out with the help of identical elastic connections, **and** boosters В', В", Д', and Д" with the help of connections of equal rigidity in plane $\xi\eta(\zeta\eta)$.

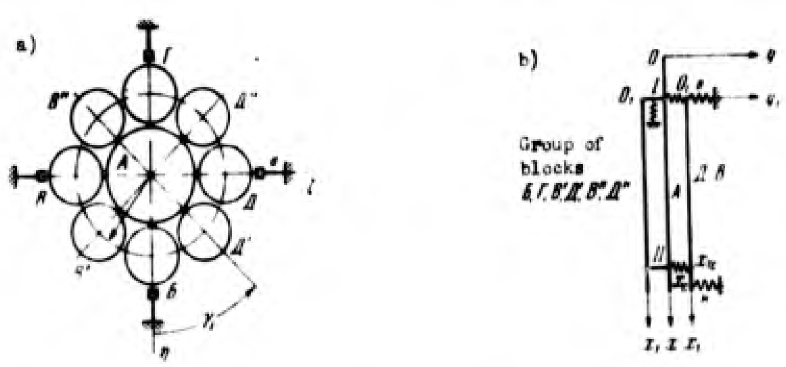

Fig. 8.2. Dynamic layout of a system of boosters of the type "pack" - elastic launcher in a transverse direction.

We will present this flight vehicle schematically in the form of a system of elastically connected thin-walled rods of variable section (Fig. 8.2b). Let us assume that the weight of transverse support elements of launcher is small as compared to weight of the side-mounting boosters. In this case the mentioned support elements can be presented in the form of weightless[1] elastic elements with given rigidities $E_{в}$ and $E_{н}$. The value of the latter in practice can be defined as the ratio of the magnitude of transverse force, applied in the end section of the support element to the transverse displacement caused by it.

Let us assume further that elastic supports "в" and "н" are fixed on ends of bodies of side-mounting boosters (in planes БГ and ВД) and that they (supports) absorb only loads acting in a direction perpendicular to the plane of their location.

In the case of complete symmetry of the considered system free oscillations of it in planes of location of longitudinal axes of boosters В, Д and Б, Г will be mutually independent, and the equations describing these oscillations will have an identical form. Therefore it is possible to limit ourselves to a consideration of free oscillations of this system only in one plane $\xi\eta$ (plane of location of boosters Б and Г), considering the correctness of the results obtained also for the other plane $\xi\zeta$.

We will make one more important simplifying assumption, concerning flight vehicles equipped with engines working on liquid fuel. Usually for such boosters the fuel tanks at launching are almost completely filled with liquid. In such case (Chapter II) the influence of its mobility on inertial characteristics of the flight vehicle, and consequently also on frequencies and forms of inherent elastic oscillations of the system, will be comparatively small, and in first approximation they can be disregarded.

---

[1]The calculation of the influence of mass of support elements does not present any fundamental difficulties.

Along with the basic system of coordinates xyz, connected with the body of block A, we will introduce an auxiliary systems of coordinates $x_1y_1z_1$, connected with the bodies of the side-mounting boosters. The origins of these auxiliary systems of coordinates we will place in the summits of the boosters, heading axis x_1 along the longitudinal axes of their bodies to the tail, and axis y_1 parallel to axis y (Fig. 8.2b). The latter we will dispose in one of the planes of symmetry of the system, for instance in plane ξη of the launching system of coordinates. Here we will consider that the angle between axes x_1 and x is small. Let us assume further that shiftings of points of longitudinal axes of blocks Б, Д", Д', В", В', Г in plane ξη are absolutely identical and are characterized completely by shifts of points of one block, for instance Б. We will also consider shift of corresponding points of blocks В and Д as identical.

The expression for kinetic energy of the considered system will consist of the sum of kinetic energies of small transverse oscillations with respect to the position of equilibrium (underformed state of the system) the basic block, and side-mounting boosters.

$$T_0 = \frac{1}{2}\int_0^L m(x)\dot{\eta}_A^2(x)\,dx + \int_0^l m(x_1)\dot{\eta}_Д^2(x_1)\,dx_1 + 3\int_0^l m(x_1)\dot{\eta}_Б^2(x_1)\,dx_1, \qquad (8.16)$$

where L - length of vehicle (block A), l - length of booster, η_A, $\eta_Д$ and $\eta_Б$ - shifts of points of longitudinal axes of the corresponding blocks (А, Д, Б) in plane ξη of the launching system of coordinates.

Transverse shifting of points of the body of side-mounting booster Д (and this also means В) will be composed of forward and rotary shifts of its longitudinal axis, caused by deformation of elastic supports "в" and "н" and shifts caused by bending of the boostor body itself $y_{1Д}(x_1, t)$

$$\eta_Д(x_1, t) = \eta_В(x_1, t) = \left(1 - \frac{x_1}{l}\right)\eta_в(t) + \frac{x_1}{l}\eta_н(t) + y_{1Д}(x_1, t). \qquad (8.17)$$

Transverse shifts of points of the longitudinal axis of the basic block $\eta_A(x, t)$ can also be presented in the form of the sum of shifts caused by its motion as a solid body and deformation of the body as beam, located on supports I and II. Forward and rotary shifting of axis x will in this case be determined by deformation of supports "в" and "н," by transverse shifts of points of longitudinal axes of boosters Д and В (in places of joining of elastic connection II), and deformations of elastic connections I and II themselves. By the latter are understood shifts of y_I and y_{II} of points x_B and x_C of blocks A relative to the corresponding points ($x_1 = 0$ and $x_1 = x_{1c} = a$) of the longitudinal axes of boosters Д and В, considering not only local deformation of elements of connection of the basic block and boosters, but also the twisting strain on bodies of side-mounting boosters. Thus we will have

$$\eta_A(x, t) = y_A(x, t) + [y_{II}(t) + \eta_Д(x_{1c}, t)]\frac{x - x_B}{a} + \\ + [\eta_B(t) + y_I(t)]\left(1 - \frac{x - x_B}{a}\right), \tag{8.18}$$

where $a = x_C - x_B$.

On the basis of formula (17)

$$\eta_A(x, t) = y_A(x, t) + \frac{x - x_B}{a} y_{II}(t) + \left(1 - \frac{x - x_B}{l}\right)\eta_B(t) + \\ + \frac{x - x_B}{l}\eta_н(t) + \frac{x - x_B}{a} y_{1Д}(x_{1c}, t). \tag{18.18'}$$

Considering further that connections I and II are absolutely rigid in a radial direction and in the direction of axis x, we obtain an analogous expression for transverse shifting of points of longitudinal axis of the booster Б (Г, В', В", Д' and Д"):

$$\eta_Б(x_1, t) = y_{1Б}(x_1, t) + \left(1 - \frac{x_1}{l}\right)\eta_B(t) + \\ + \frac{x_1}{a} y_{II}(t) + \frac{x_1}{l}\eta_н(t) + \frac{x_1}{a} y_{1Д}(x_{1c}, t), \tag{18.19}$$

where $y_{1Б}(x_1, t)$ is the magnitude of sag of the axis of booster Б as a beam located on hinged supports in section $x_1 = 0$ and $x_1 = x_{1c}$. It is necessary to note that when using the units for transmission of longitudinal load from the boosters to the body of block A as the front transverse connection I, the rigidity of this connection can turn out to be very great as compared to the rigidity of the lower force connection II. In such a case the influence of the shifting of $y_I(t)$ on the lowest forms and frequencies of inherent elastic oscillations of the system will be small. On this basis by conducting further simplification we will consider that $y_I(t) = 0$, and (for simplicity of designation) $y_{II} = y_c$.

Potential energy of this system will consist of the sum of the energy of transverse deformation of the construction of basic block A and side-mounting boosters and the energies of deformation of elastic connections and support elements of the launcher. Just as in the preceding section, we will consider that (in view of the smallness of oscillations) potential energy of the position of the system can be disregarded. Furthermore, we will not consider strain energy of longitudinal support elements of the launcher.

We will present the sags of block A and boosters Б and Д in the form of infinite series

$$\left.\begin{aligned} y_A(x, t) &= \sum_{n=1}^{\infty} S_n(t) f_n(x), \\ y_{1Б}(x_1, t) &= \sum_{p=1}^{\infty} K_{pБ}(t) \Phi_{pБ}(x_1), \\ y_{1Д}(x_1, t) &= \sum_{p=1}^{\infty} K_{pД}(t) \Phi_{pД}(x_1). \end{aligned}\right| \qquad (8.19')$$

Here f_n and Φ_{pi} (i = Б, Д) are functions only of coordinates x and x_1 correspondingly, and S_n and K_{pi} - functions only of time t.

We will take functions S_n and K_{pi} and displacements y_c, η_B and η_H also depending only on time t, as generalized coordinates q_j. We select functions $f_n(x)$ and $\Phi_{pi}(x_1)$ in such a way that the expression for potential energy of the system is a homogeneous function of the second degree from the stated generalized coordinates. For this it is sufficient that these functions are normal forms of inherent transverse oscillations of the corresponding partial systems. The method of determination of these partial forms of oscillations (taking into account the influence of transverse and longitudinal forces, inertia of rotation of sections) is presented in Chapter IV.

Omitting, for simplification of recording, the members which are dependent on Q(x) and N(x), we will present the expression for potential energy of the system in the form

$$U_0(t)=\frac{1}{2}E_c y_c^2(t)+E_{\text{н}}\eta_{\text{н}}^2(t)+E_{\text{в}}\eta_{\text{в}}^2(t)+$$
$$+\frac{1}{2}\sum_{n=1}^{\infty}S_n^2(t)\int_0^L B(x)\left[\frac{d^2f_n(x)}{dx^2}\right]^2dx+$$
$$+\sum_i\sum_{p=1}^{\infty}K_{pi}^2(t)\int_0^l B_i(x_1)\left[\frac{d^2\Phi_{pi}(x_1)}{dx_1}\right]^2dx_1. \qquad (8.20)$$
$$(i=\text{Б},\ \text{В}',\ \text{Д},\ \text{Д}'.)$$

Here E_c designates the rigidity of the lower elastic connection (II), equal to the proportionality factor between force, applied at point of crossing of axis of connection with asix x of the flight vehicle, and the corresponding deformation y_c. the latter is found on the assumption that points x_{1c} of boosters Д and В are motionless.

Substituting (16) and (20) in (4.1), we obtain a system of ordinary differential equations with constant coefficients in a reverse form, which also will describe free transverse oscillations of the particular flight vehicle on an elastic launcher

$$\sum_{j=1}^{\infty}a_{ij}\ddot{q}_j+b_{ii}q_i=0 \quad (i=1,\ 2,\ \ldots). \qquad (8.21)$$

If the flight vehicle does not jump above the launcher, i.e., its support section x_B has constant contact with the longitudinal support elements of the launcher, the transverse oscillations of the system will be accompanied also by longitudinal oscillations. In this case in the expression for potential energy (20) it is also necessary to consider strain energy of longitudinal supports, and in the expressions for kinetic energy of the system - the corresponding kinetic energy of longitudinal oscillations of blocks which are suspended on these longitudinal supports.

We will designate the rigidity of one longitudinal support through E_x, and its distance from the longitudinal axis of block through $r(x_B)$. Since the angle of rotation of support section x_B of the body of block A relative to the neutral axis during transverse oscillations of the system is equal to $\partial\eta_A(x_B, t)/\partial x$, then the corresponding longitudinal shifts of side-mounting boosters of group Б will be determined by the expression

$$r(x_B)\frac{\partial\eta_A(x,t)}{\partial x}\bigg|_{x=x_B}\cos\gamma_i,$$

and their kinetic energy - by formula

$$\Delta T_0(t)=\frac{1}{2}\sum_{i=1}^{6}m_{бi}r^2(x_B)\left[\frac{\partial^2\eta_A(x,t)}{\partial x\,\partial t}\bigg|_{x=x_B}\right]^2\cos^2\gamma_i,$$

where $m_{бi}$ - mass of side-mounting boosters, γ_i - angle between planes $\xi\eta$ and xx_{ii} (Fig. 8.2a). The corresponding increase of potential energy of the system will be equal to

$$\Delta U_0(t)=\frac{1}{2}E_xr^2(x_B)\sum_{i=1}^{6}\left[\frac{\partial}{\partial x}\eta_A(x,t)\big|_{x=x_B}\right]^2\cos^2\gamma_i.$$

In the case of a more strict investigation of this case of load one should consider, obviously, the longitudinal and transverse elastic oscillations of the system jointly. A separate investigation of them is permissible only if the lowest partial frequencies of inherent longitudinal elastic oscillations of the bodies of the boosters are consiberably greater than the lowest partial frequencies of inherent transverse oscillations, i.e., when it is possible to hope that the interconnection of transverse and longitudinal oscillations will be weak.

If we take into account the above-indicated additional values of potential and kinetic energies, then we obtain equations of free oscillations of the system in the form

$$\sum_{j=1}^{k} a_{ij}\ddot{q}_j + \sum_{j=k}^{k} b_{ij}q_j = 0 \qquad (i = 1,\ 2,\ \ldots,\ k). \tag{8.22}$$

In accordance with the accepted designations for shifts of points of system (17), (18), and (19), subsequently when writing expressions for coefficients a_{ij} and b_{ij}, and also when writing equations (22) in expanded form, for greater clarity we will use, along with numbers (1, 2, ...), the corresponding letter designations for subscripts i and j, namely: n, н, в, с, рБ, рД.

Thus, for instance, the equation for i = н will have the form

$$\begin{aligned} & a_{\text{нн}}\ddot{q}_{\text{н}} + a_{\text{нв}}\ddot{q}_{\text{в}} + a_{\text{нс}}\ddot{q}_{\text{с}} + \sum_{p=1}^{\infty} a_{\text{нрБ}}\ddot{q}_{\text{рБ}} + \\ & \quad + \sum_{p=1}^{\infty} a_{\text{нрД}}\ddot{q}_{\text{рД}} + \sum_{p=1}^{\infty} a_{\text{нрВ}}\ddot{q}_{\text{рВ}} + \sum_{p=1}^{\infty} a_{\text{нрГ}}\ddot{q}_{\text{рГ}} + \sum_{n=1}^{\infty} a_{\text{н}n}\ddot{q}_{n} + \\ & \quad + b_{\text{нн}}q_{\text{н}} + b_{\text{нв}}q_{\text{в}} + b_{\text{нс}}q_{\text{с}} + \sum_{p=1}^{\infty} b_{\text{нрД}}q_{\text{рД}} + \sum_{p=1}^{\infty} b_{\text{нрВ}}q_{\text{рВ}} + \\ & \quad\quad + \sum_{p=1}^{\infty} b_{\text{нрГ}}q_{\text{рГ}} + \sum_{p=1}^{\infty} b_{\text{нрБ}}q_{\text{рБ}} + \sum_{n=1}^{\infty} b_{\text{н}n}q_{n} = 0. \end{aligned}$$

Expressions for coefficients of these equations and methods of determination of partial forms and frequencies of natural oscillations of a system are given in the following section.

§ 8.4. Calculation of Coefficients for Equations of Free Transverse Oscillations of a System

Equations (21) and (22) contain coefficients of three forms:

1) coefficients a_{ij} and b_{ij}, for which both subscripts are identical. The first of these coefficients designates reduced mass of the i-th partial system, and the second - reduced rigidity;

2) coefficients a_{ij} and a_{ji}, which are obtained in the presence in the expression for kinetic energy of a system of members with products of derivative from generalized coordinates. They characterize the dynamic (inertial) connection of the corresponding (i-th and j-th) partial systems between each other ($i \neq j$);

3) coefficients b_{ij} and b_{ji} ($i \neq j$), the presence of which in the stated equations is conditioned by the presence of members with products of generalized coordinates in the expression for potential energy of the system. These coefficients determine the static (power) connection of i-th and j-th partial systems between each other.

Values of all the above-indicated coefficients depend on structure of the system, its dynamic arrangement. It is easy to check this if one were to compare the expressions for the corresponding coefficients of equations (21) and (22) which were given in the preceding section (describing free transverse oscillations of essentially similar dynamic systems having the same partial systems). In the first case, i.e., in the presence of jump of the flight vehicle above the support plane of the launcher, the reduced masses of partial systems will be determined by the formulas:

$$\left.\begin{aligned}
&a_{\text{пп}} = M_{\text{п}},\\
&a_{\text{ии}} = M_{\text{и}} = \frac{1}{l^2} J(x_{\text{в}}),\\
&a_{\text{cc}} = M_{\text{c}} = \frac{1}{a^2}\left[J_{\text{А}}(x_{\text{в}}) + 6J_{\text{б}}\right],\\
&a_{\text{вв}} = M_{\text{в}} = M + \frac{J(x_{\text{в}})}{l^2} - \frac{2}{l} M(x_{\text{т}} - x_{\text{в}}),\\
&a_{\text{ДД}p} = M_{p\text{Д}} = M_{p\text{В}} = \int_0^l m_{\text{пр}}(x_1)\Phi^2_{p\text{Д}}(x_1)\,dx_1,\\
&a_{\text{ББ}p} = M_{p\text{Б}} = M_{p\text{Г}} = \int_0^l m(x_1)\Phi^2_{p\text{B}}(x_1)\,dx_1,
\end{aligned}\right| \qquad (8.23)$$

where

$$\left.\begin{aligned}
&M = m_{\text{А}} + 8m_{\text{б}},\\
&J(x_{\text{в}}) = J_{\text{А}}(x_{\text{в}}) + 8J_{\text{б}}, \qquad m_{\text{пр}}(x_1) = m(x_1) + \Delta m \cdot \delta(x_1 - x_{1c}),\\
&\Delta m = \frac{1}{2a^2}\left[J_{\text{А}}(x_{\text{в}}) + 6J_{\text{б}}\right]
\end{aligned}\right\} \qquad (8.24)$$

In the second case, in the absence of the possibility for free turning of support section $x_{\text{в}}$ of the body of the flight vehicle, these reduced masses will equal

$$\left.\begin{aligned}
&a'_{\text{пп}} = a_{\text{пп}} + e_0\left[\frac{df_{\text{п}}(x)}{dx}\Big|_{x=x_{\text{в}}}\cdot a\right]^2, \quad a'_{\text{ББ}\rho} = a_{\text{ББ}\rho},\\
&a'_{\text{сс}} = a_{\text{сс}} + e_0, \qquad a'_{\text{вв}} = a_{\text{вв}} + \frac{a^2}{l^2}e_0,\\
&a'_{\text{нн}} = a_{\text{нн}} + \frac{a^2}{l^2}e_0, \quad a'_{\text{ДД}\rho} = a_{\text{ДД}\rho} + e_0\Phi^2_{\rho\text{Д}}(x_{1c}),\\
&e_0 = \frac{1}{a^2}m_0 r^2(x_{\text{в}})\sum_{i=1}^{8}\cos^2\gamma_i.
\end{aligned}\right\} \qquad (8.25)$$

Approximate expressions of reduced rigidity of these particular partial systems in the first case will have the form:

$$\left.\begin{aligned}
&b_{\text{сс}} = E_{\text{с}}, \quad b_{\text{вв}} = E_{\text{в}}, \quad b_{\text{нн}} = E_{\text{н}},\\
&b_{\text{пп}} = \int_0^L B(x)\left[\frac{d^2 f_{\text{п}}(x)}{dx^2}\right]^2 dx,\\
&b_{\rho\text{ДД}} = b_{\rho\text{ВВ}} = \int_0^l B(x_1)\left[\frac{d^2\Phi_{\rho\text{Д}}(x_1)}{dx_1^2}\right]^2 dx_1,\\
&b_{\rho\text{ББ}} = b_{\rho\text{ГГ}} = \int_0^l B(x_1)\left[\frac{d^2\Phi_{\rho\text{Б}}(x_1)}{dx_1^2}\right]^2 dx_1.
\end{aligned}\right\} \qquad (8.26)$$

In the second case:

$$\left.\begin{aligned}
&b'_{\text{пп}} = b_{\text{пп}} + e_x\left[\frac{df_{\text{п}}(x)}{dx}\Big|_{x=x_{\text{в}}}\right]^2,\\
&e_x = r^2(x_{\text{в}})\sum_{i=1}^{8}E_{xi}\cos^2\gamma_i,\\
&b'_{\text{сс}} = b_{\text{сс}} + e_x\frac{1}{a^2},\\
&b'_{\text{вв}} = b_{\text{вв}} + e_x\frac{1}{l^2},\\
&b'_{\text{нн}} = b_{\text{нн}} + e_x\frac{1}{l^2},\\
&b'_{\rho\text{ДД}} = b'_{\rho\text{ВВ}} = b_{\rho\text{ДД}} + e_x\frac{1}{a^2}\Phi^2_{\rho\text{Д}}(x_{1c}),\\
&b'_{\rho\text{ББ}} = b'_{\rho\text{ГГ}} = b_{\rho\text{ГГ}}.
\end{aligned}\right\} \qquad (8.27)$$

Connection coefficients a_{ij} and b_{ij} (with $i \neq j$) for conservative systems satisfy the condition of symmetry, i.e., $a_{ij} = a_{ji}$, $b_{ij} = b_{ji}$.

Their values in the second case will be determined by formulas (28), and in the first - (29):

$$
\left.
\begin{aligned}
&a'_{\text{пн}} = a'_{\text{нп}} = a_{\text{пн}} + e_0 \frac{a^2}{l} \frac{df_n(x)}{dx}\Big|_{x=x_{\text{в}}},\\
&a'_{\text{пв}} = a'_{\text{вп}} = a_{\text{пв}} - e_0 \frac{a^2}{l} \frac{df_n(x)}{dx}\Big|_{x=x_{\text{в}}},\\
&a'_{\text{пс}} = a'_{\text{сп}} = a_{\text{пс}} + e_0 a \frac{df_n(x)}{dx}\Big|_{x=x_{\text{в}}},\\
&a'_{\text{пД}p} = a'_{p\text{Дп}} = a_{\text{пД}p} + e_0 a \Phi_{p\text{Д}}(x_{1\text{c}}) \frac{df_n(x)}{dx}\Big|_{x=x_{\text{в}}},\\
&a'_{p\text{Дв}} = a'_{\text{вД}p} = a_{p\text{Дв}} - e_0 \frac{a}{l} \Phi_{p\text{Д}}(x_{1\text{c}}),\\
&a'_{p\text{Дс}} = a'_{\text{сД}p} = a_{p\text{Дс}} + e_0 \Phi_{p\text{Д}}(x_{1\text{c}}),\\
&a'_{p\text{Дн}} = a'_{\text{нД}p} = a_{p\text{Дн}} + e_0 \frac{a}{l} \Phi_{p\text{Д}}(x_{1\text{c}}),\\
&a'_{\text{св}} = a'_{\text{вс}} = a_{\text{св}} - e_0 \frac{a}{l},\\
&a'_{\text{сн}} = a'_{\text{нс}} = a_{\text{сн}} + e_0 \frac{a}{l},\\
&a'_{\text{нв}} = a'_{\text{вн}} = a_{\text{нв}} - e_0 \frac{a^2}{l^2},\\
&e_0 = \frac{1}{a_3} m_6 r^2(x_{\text{в}}) \sum_{i=1}^{8} \cos^2 \gamma_i,
\end{aligned}
\right\}
\qquad (8.28)
$$

$$
\left.
\begin{aligned}
&a_{\text{нп}} = \frac{1}{l} \int_0^L m(x) f_n(x)(x - x_{\text{в}})\, dx,\\
&a_{\text{вп}} = \int_0^L m(x) f_n(x)\, dx - a_{\text{нп}},\\
&a_{\text{сп}} = a_{\text{нп}} \frac{l}{a},\\
&a_{\text{в}p\text{Д}} = \left(\beta_{p\text{Д}} - \frac{\alpha_{p\text{Д}}}{l}\right) + \frac{1}{2a} \Phi_{p\text{Д}}(x_{1\text{c}}) \left[m_{\text{A}}(x_{\text{тА}} - x_{\text{в}}) + \right.\\
&\qquad\qquad \left. + 6 m_{\text{б}} x_{\text{тб}} - a M_{\text{c}}\right] = a_{\text{в}p\text{В}},
\end{aligned}
\right\}
\qquad (8.29)
$$

$$
\left.
\begin{aligned}
&a_{\rho Дп} = a_{\rho Вп} = a_{cп}\Phi_{\rho Д}(x_{1c}),\\
&a_{н\rho Д} = a_{н\rho В} = \frac{1}{l}\,\alpha_{\rho Д},\\
&a_{c\rho Д} = a_{c\rho В} = M_c\Phi_{\rho Д}(x_{1c}),\\
&a_{в\rho В} = a_{в\rho Г} = \beta_{\rho Б} - \frac{\alpha_{\rho Б}}{l},\\
&a_{н\rho В} = a_{н\rho Г} = \frac{\alpha_{\rho Б}}{l},\\
&a_{c\rho В} = a_{c\rho Г} = \frac{\alpha_{\rho Б}}{a},\\
&a_{вв} = \frac{1}{l^2}\,[lM(x_т - x_в) - J(x_в)],\\
&a_{cв} = \frac{1}{al}\,[m_A l(x_{тA} - x_в) + 2m_б x_{т_б} - J_A(x_в) - 6J_б],\\
&a_{cв} = M_c\frac{a}{l},\\
&a_{\rho в Д} = \Phi_{\rho Д}(x_{1c})\,\frac{\alpha_{\rho Д}}{a},
\end{aligned}
\right| \qquad (8.29\ \text{Cont'd})
$$

$$
\left.
\begin{aligned}
&\alpha_{\rho Б} = \int_0^l m(x_1)x_1\Phi_{\rho В}(x_1)\,dx_1,\\
&\alpha_{\rho Д} = \int_0^l m_{нр}(x_1)x_1\Phi_{\rho Д}(x_1)\,dx_1,\\
&\beta_{\rho Б} = \int_0^l m(x_1)\Phi_{\rho В}(x_1)\,dx_1,\\
&\beta_{\rho Д} = \int_0^l m_{нр}(x_1)\Phi_{\rho Д}(x_1)\,dx_1,
\end{aligned}
\right\} \qquad (8.30)
$$

where m_A - mass of central block A, $J_A(x_B)$ - its mass moment of inertia relative to the transverse axis passing through x_B, $J_б$ - mass moment of inertia of booster relative to transverse axis z_1, passing through the summit of its body, x_{TA} and $x_{TБ}$ - coordinates of centers of gravity of isolated blocks A and Б (or Д) in inherent connected systems of coordinates (oxy) and $(o_1x_1y_1)$ correspondingly:

$$
\left.\begin{aligned}
b'_{n\text{с}} &= b'_{\text{с}n} = e_x \frac{1}{a} \left.\frac{df_n(x)}{dx}\right|_{x=x_{\text{в}}},\\
b'_{n\text{Д}p} &= b'_{p\text{Д}n} = e_x \frac{1}{a} \Phi_{p\text{Д}}(x_{1\text{с}}) \left.\frac{df_n(x)}{dx}\right|_{x=x_{\text{н}}},\\
b'_{n\text{н}} &= b'_{\text{н}n} = e_x \frac{1}{l} \left.\frac{df_n(x)}{dx}\right|_{x=x_{\text{н}}},\\
b'_{n\text{в}} &= b'_{\text{в}n} = -\, b'_{n\text{н}},\\
b'_{\text{с}\text{Д}p} &= b'_{p\text{Дс}} = e_x \frac{1}{a^2} \Phi_{p\text{Д}}(x_{1\text{с}}),\\
b'_{\text{сн}} &= b'_{\text{нс}} = e_x \frac{1}{al},\\
b'_{\text{св}} &= b'_{\text{вс}} = -\, b'_{\text{сн}},\\
b'_{p\text{Дн}} &= b'_{\text{нД}p} = e_x \frac{1}{al} \Phi_{p\text{Д}}(x_{1\text{с}}),\\
b'_{p\text{Дв}} &= b'_{\text{вД}p} = -\, e_x \frac{1}{al} \Phi_{p\text{Д}}(x_{1\text{с}}),\\
b'_{\text{нв}} &= b'_{\text{вн}} = -\, e_x \frac{1}{l^2}.
\end{aligned}\right\} \qquad (8.31)
$$

According to (4.65) the ratios of reduced rigidities of partial systems to their reduced masses determine the values of squares of circular frequencies of natural oscillations of these partial systems

$$
\omega_{ii}^2 = \frac{b_{ii}}{a_{ii}} \qquad (8.32)
$$

$$
(i = \text{н, в, с}, n, p\text{Б}, p\text{Д}; \quad p = 1, 2, \ldots; \; n = 1, 2, \ldots)
$$

Considering what was said about it is not difficult to establish those conditions which have to be satisfied by functions $f_n(x)$, $\Phi_{p\text{Б}}(x_1)$, $\Phi_{p\text{Д}}(x_1)$, describing the forms of inherent transverse oscillations of the corresponding partial systems. In the first case (in the presence of vehicle jump) function $f_n(x)$ is found exactly as in the example considered in the preceding section, i.e., as the solution of equation (4.67) with boundary conditions (6) and conditions of connection (8). In the determination of functions $f_n(x)$ in the second case (in the absence of jump) one should consider the presence in section $x_{\text{в}}$ of the concentrated moment of inertia of rotation from apparent additional masses of side-mounting boosters a^2e_0, and also elastic fixing of this section, limiting its angle of rotation (Fig. 8.3, "A").

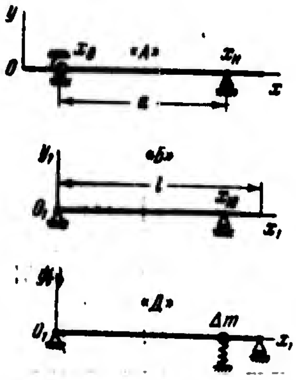

Fig. 8.3. Arrangements of partial systems.

During the calculation of forms and frequencies of inherent bending oscillations of side-mounting boosters Б and Г ($\Phi_{p\text{Б}}(x_1)$ and $\Phi_{p\Gamma}(x_1)$) the latter in both cases are considered as freely supported beams (Fig. 8.3, "Б"). Here the boundary conditions for them will have the form:

$$\left.\begin{array}{c} \Phi_{p\text{Б}}(x_1)=0,\ \dfrac{d^2\Phi_{p\text{Б}}(x_1)}{dx_1^2}=0 \text{ with } x_1=0, \\ \dfrac{d^2\Phi_{p\text{Б}}(x_1)}{dx_1^2}=0,\ \dfrac{d}{dx_1}\left[B(x_1)\dfrac{d^2\Phi_{p\text{Б}}(x_1)}{dx_1^2}\right]=0 \\ \text{with } x_1=l, \end{array}\right\} \tag{8.33}$$

and condition of coupling

$$\Phi_{p\text{Б}}(x_1)=0 \text{ with } x_1=x_{1c}.$$

During calculation of partial frequencies and forms of inherent transverse oscillations of side-mounting boosters Д and В to which are directly joined the support elements "в" and "н" of the launcher, a model of a freely supported beam is used, which in section $x_1=x_{1c}$ is carrying concentrated mass Δm. In the first case of load this mass is determined by formula (24), and in the second by formula

$$\Delta m=\frac{1}{2a^2}\left[J_A(x_н)+6J_б+m_б r^2(x_н)\sum_{i=1}^{8}\cos^2\gamma_i\right]. \tag{8.33'}$$

In certain cases in the deduction of equations, and consequently also in the determination of partial frequencies and forms of inherent transverse oscillations of side-mounting boosters, it is more convenient to start from the presence in section $x_1=x_{1c}$ of the additional elastic support shown in Fig. 8.3, Д conditionally in the form of a spring.

Boundary conditions for this side-mounting booster (Д) are taken in the form

$$\Phi_{p\text{Д}}(x_1)=0,\ \frac{d^2}{dx_1^2}\Phi_{p\text{Д}}(x_1)=0 \text{ with } x_1=0,\ x_1=l.$$

In the deduction of equations of free oscillations of the systems mentioned above we used approximate expressions for strain energy in the construction of a flight vehicle (with side-mounting boosters). This was done basically for preserving the letter. Accuracy of description of this potential energy for deduction of equations does not have any fundamental value. It is important only in the determination of partial forms and frequencies of natural oscillations of a flight vehicle. Therefore it is recommended that the calculation of the latter in all cases regardless of the form of equations be carried out according to the method presented in Chapter IV, i.e., taking into account influence of shift, longitudinal forces, and inertia of rotation of cross sections. Calculation of frequencies and forms of natural oscillations of a vehicle-launcher system on the whole should be executed by the method given in Chapter V.

As an illustration in Fig. 8.4 the structure is shown of the spectrum of natural frequencies of a similar system (for one of the hypothetical flight vehicles with a pack arrangement), and in Fig. 8.5 - the normal forms of natural oscillations of the system $f_m^*(x_1)$ corresponding to these frequencies. The values of these forms are determined by the totality of amplitudes $A_{1m}(S_{nm}$, $K_{p\text{Д}m}$, $\eta_{\text{в}m}$, $\eta_{\text{н}m}$, etc.) and depend on the characteristics and arrangement of the system on the whole, and the forms of natural oscillations of partial systems $f_n(x)$, $\Phi_{p\text{Б}}(x_1)$, ... According to (18'), (19), and (17)

$$
\left.
\begin{aligned}
f^{\circ}_{m}(x) &= \sum_{n=1}^{n_n} S_{nm} f_n(x) + \eta_{\text{в}m}\left(1 - \frac{x - x_{\text{в}}}{l}\right) + \\
&+ \left[\sum_{p=1}^{n_p} K_{p\text{Д}m} \Phi_{p\text{Д}}(x_{1c}) + y_{cm} + \frac{a}{l}\eta_{\text{н}m}\right] \frac{x - x_{\text{в}}}{a}, \\
f^{\circ}_{m\text{Б}}(x_1) &= \sum_{p=1}^{n_p} K_{p\text{Б}m} \Phi_{p\text{Б}}(x_1) + \frac{x_1}{a}\left[\sum_{p=1}^{n_p} K_{p\text{Д}m} \Phi_{p\text{Д}}(x_{1c}) + \right. \\
&\left. + y_{cm} + \eta_{\text{н}m}\frac{a}{l}\right] + \eta_{\text{в}m}\left(1 - \frac{x_1}{l}\right), \\
f^{\circ}_{m\text{Д}}(x_1) &= \sum_{p=1}^{n_p} K_{p\text{Д}m} \Phi_{p\text{Д}}(x_1) + \eta_{\text{н}m}\frac{x_1}{l} + \eta_{\text{в}m}\left(1 - \frac{x_1}{l}\right).
\end{aligned}
\right\} \qquad (8.34)
$$

The graphs are constructed for a case of calculation of only four tones ($n_n = 4$) of partial frequencies of bending oscillations of central block A, three tones ($n_p = 3$) of partial frequencies of inherent bending oscillations of side-mounting boosters Д, В, and two tones of partial frequencies of inherent bending oscillations of side-mounting boosters Б and Г. Here we show only the symmetric forms of natural oscillations of a system, determined on the assumption that blocks Д, В, В', ..., and also blocks Б and Г oscillate synchronously.

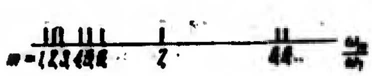

Fig. 8.4. Spectrum of natural frequencies of transverse symmetric oscillations of the system: vehicle with pack layout - elastic launcher.

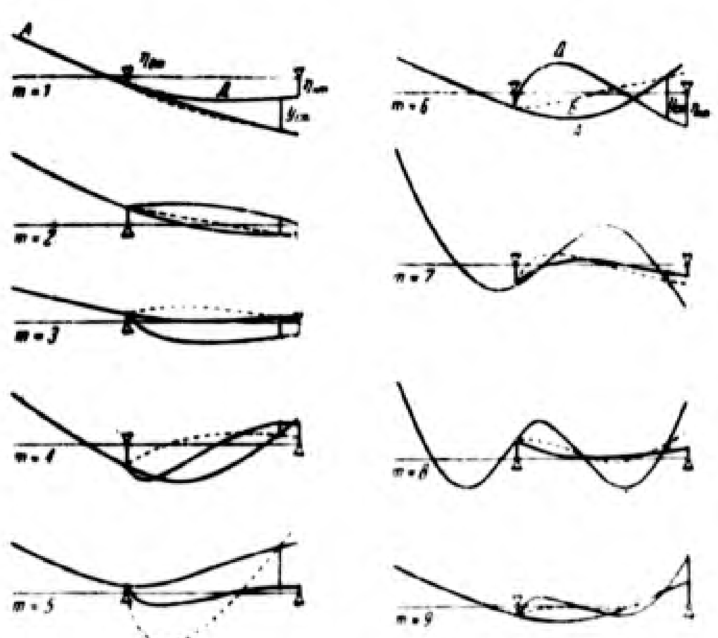

Fig. 8.5. Normal forms of oscilaltions of a system.

§ 8.5. Equations of Forced Oscillations of the Vehicle-Launcher System

Equations of forced transverse oscillations of the flight vehicle-launcher system in one of the planes of symmetry (plane $\xi\eta$ or $\xi\zeta$) in general in generalized coordinates will have the form

$$\sum_{j=1}^{\infty} a_{ij}\ddot{q}_j + \sum_{j=1}^{\infty} b_{ij}q_j = Q_i \qquad (i = 1,\ 2,\ \ldots), \tag{8.35}$$

where Q_i designates the generalized force, corresponding to generalized coordinate q_i. The value of this generalized force depends basically on the power structure of the system, in particular on the place of location of transverse support elements of the body of the flight vehicle, and also on the method of transmission of thrust force to the structure of the vehicle. The simplest expression for generalized forces will be in the case of distribution of transverse supports directly on the body of the carrier central block of the flight vehicle or on the side-mounting boosters (in places where transverse force connections are attached to them).

For the assembly layout of the flight vehicle-launcher system considered in the preceding sections the generalized forces will be determined by the expression

$$Q_j = \frac{d}{dq_j}\left[\frac{d\eta_A(x)}{dx}\bigg|_{x=x_\text{п}}\right] \cdot r(x_\text{п}) \sum_{i=1}^{8} P_{\text{б}i} \cos\gamma_i. \tag{8.36}$$

Using formulas (18), (19') we obtain

$$Q_\text{п} = r(x_\text{п}) \frac{df_\text{п}(x)}{dx}\bigg|_{x=x_\text{п}} \sum_{i=1}^{8} P_{\text{б}i} \cos\gamma_i, \tag{8.37}$$

$$Q_{\rho\text{Д}} = \frac{1}{a} r(x_\text{п}) \Phi_{\rho\text{Д}}(x_{1c}) \sum_{i=1}^{8} P_{\text{б}i} \cos\gamma_i, \tag{8.38}$$

$$Q_{\rho\text{Б}} = 0, \tag{8.39}$$

$$Q_c = r(x_\text{п}) \frac{1}{a} \sum_{i=1}^{8} P_{\text{б}i} \cos\gamma_i. \tag{8.40}$$

$$Q_{\text{к}} = \frac{a}{l} Q_{c},\qquad (8.41)$$

$$Q_{\text{в}} = -Q_{\text{к}}.\qquad (8.42)$$

The determination of parameters of forced oscillations of any system with a large number of degrees of freedom by means of solution of a system of equations in generalized coordinates of the type (35) presents specific difficulties and requires the application of computers (analog or digital). This problem is essentially simplified the normal forms and frequencies of natural oscillations of this system are known. In this case the equations of forced oscillations can be reduced to a system n_0 of independent ordinary second order differential equations with constant coefficients of the form

$$\ddot{p}_m + \omega_m^2 p_m = Q_m \quad (m = 1, 2, \ldots, n_0).\qquad (8.43)$$

Using the formula of expansion of transverse shift $\eta_A(x, t)$ in normal forms of the system

$$\eta_A(x, t) = \sum_{m=1}^{n_0} p_m(t) f_m^{\circ}(x),\qquad (8.44)$$

we obtain for Q_m the following expression:

$$Q_m = r(x_0) \frac{1}{M_m} \frac{df_m^{\circ}(x)}{dx} \bigg|_{x=x_0} \sum_{i=1}^{8} P_{0i} \cos \gamma_i\qquad (8.45)$$

or, using formulas (34), (37)-(42), the expression

$$Q_m = \frac{1}{M_m} \sum_{i=1}^{n_0} Q_i A_{im},\qquad (8.46)$$

where

$$M_m = \frac{\sum_{i=1}^{n_0} \sum_{j=1}^{n_0} b_{ij} A_{im} A_{jm}}{\omega_m^2} \quad (m = 1, 2, \ldots, n_0).\qquad (8.47)$$

In the case of necessity for calculation of the influence of control forces, i.e., consideration of the system of equations (43) jointly with the equation for the system of control, generalization force Q_m will have the form

$$Q_m = \frac{1}{M_m}\left[r(x_*) \frac{df_m^*(x)}{dx}\Big|_{x=x_*} \sum_{i=1}^{8} P_{\text{б}i} \cos\gamma_i + 2\sum_{i=1}^{4} R_{yi}^{\delta} \delta_i f_m^*(x_{pi}) \sin\gamma_i \right], \qquad (8.48)$$

where x_{pi} - abscissa of point of application of control force to the flight vehicle, and δ_i - angle of deviation of automatic control device, for example, the combustion chamber in the engine of a side-mounting booster.

For a flight vehicle consisting of one central block and eight side-mounting boosters, having their own independent engine systems, disturbing forces will be the greatest in the presence of delay in approach to operating conditions in engines of oppositely located side-mounting boosters. Substituting expression (34) in formula (48), we obtain for this particular case (in the absence of control devices on boosters B', D', B", D" and block A) with $x_{1p} = l$

$$Q_m = \frac{1}{M_m}\Bigg\{ r(x_*)\left[P_{\text{Б}} + \frac{\sqrt{2}}{2}(P_{\text{Б}'} + P_{\text{Д}'})\right]\Bigg[\sum_{n=1}^{n_n} S_{nm} \frac{df_n(x)}{dx}\Big|_{x=x_*} +$$

$$+ y_{cm}\frac{1}{a} + \sum_{p=1}^{n_p} K_{p\text{Д}m} \Phi_{p\text{Д}}(x_{1c})\frac{1}{a} + \frac{1}{l}(\eta_{\text{н}m} - \eta_{\text{в}m})\Bigg] +$$

$$+ 2R_{y\text{Д}}^{\delta} \cdot \delta_{\text{Д}}\left[\sum_{p=1}^{n_p} K_{p\text{Д}m} \Phi_{p\text{Д}}(x_{1p}) + (\eta_{\text{н}m} - \eta_{\text{в}m})\frac{x_{1p}}{l} + \eta_{\text{в}m}\right]\Bigg\}. \qquad (8.49)$$

Angle δ_1 is a function of angle of rotation ϑ and angular velocity $\dot{\vartheta}$ of that section of the body of the central block of the flight vehicle, in which the sensing devices for automatic stabilization are installed. Since ϑ is a function of pm

$$\vartheta(x_0, t) = \frac{\partial\eta_{\text{А}}(x, t)}{\partial x}\Big|_{x=x_0} = \sum_{m=1}^{n_0} p_m(t)\frac{df_m^*(x)}{dx}\Big|_{x=x_0}, \qquad (8.50)$$

then in general equations (43), describing forced transverse oscillations in the construction of a flight vehicle with a pack arrangement on an elastic launcher, turn out to be mutually connected by means of the equation for the control system. Here the values of control forces will depend not only on magnitude of thrust and the order of engines arriving at operating conditions, but also on the rigidity of transverse support elements of the launcher which essentially influence the magnitude of angle $\vartheta(x_0)$.

Both during the determination of partial forms and frequencies of natural oscillations and also during the calculation of normal forms and frequencies of natural oscillations of a system on the whole we absolutely disregarded the force of damping. The influence of these forces on partial forms and frequencies of oscillations of bodies of flight vehicles indeed is very small. Therefore during the investigation of forced oscillations of beams and rods usually the calculation of damping forces is done by means of formal introduction in equations of motion of the corresponding members, the form of which is determined by the nature of these dissipative forces. From this it follows that there is no necessity of the calculation of the influence of these forces both on forms and frequencies of natural oscillations of a system on the whole (since the energy of damping forces is minute as compared to potential and kinetic energy).

The influence of resisting forces of the parameters of free and forced oscillations of a system can be estimated approximately by means of introduction of the corresponding generalized damping forces. In this case the system of equations (35) will have the form

$$\sum_{j=1}^{n_0} a_{ij}\ddot{q}_j+\sum_{j=1}^{n_0} b_{ij}q_j=Q_i-2h_i\dot{q}_i a_{ii} \qquad (i=1,\ 2,\ \ldots,\ n_0). \tag{8.51}$$

where $2h_i$ - drag coefficient of the i-th partial system.

On the basis of formulas (46) and (5.56) the right sides of the corresponding equations in normal coordinates will be equal to

$$Q'_m = \frac{1}{M_m}\sum_{i=1}^{n_0}\left[Q_i - 2h_i a_{ii}\sum_{k=1}^{n_0} A_{ik}\dot{p}_k\right] A_{im}. \tag{8.52}$$

As a result of such a transformation instead of (43) we obtain the following system of equations:

$$\ddot{p}_m + 2h_m\dot{p}_m + \omega_n^2 p_m = Q_m - \sum_{k=1}^{n_0} Q_{mk}\dot{p}_k$$
$$(m = 1, 2, \ldots, n_0), \tag{8.53}$$

where

$$h_m = \frac{1}{M_m}\sum_{i=1}^{n_0} h_i A_{im}^2 a_{ii}, \tag{8.54}$$

$$Q_{mk} = \frac{2}{M_m}\sum_{i=1}^{n_0} h_i A_{ik} A_{im} a_{ii} \text{ with } k \neq m. \tag{8.55}$$

As can be seen, this system of equations is already connected. Fortunately, in most cases the magnitudes of coefficients Q_{mk} turn out to be small, and the interconnection of equations in normal coordinates at the expense of damping forces can be disregarded. It is natural that the influence of separate sections of the system on overall energy dissipation during various forms of oscillations of a system will be different. Therefore in that case when it is possible to separate the determining links the calculation of coefficients (54) is noticeably simplified.

§ 8.6. Transverse Forces and Bending Moments in Sections of Bodies of Blocks

In accordance with the accepted method for calculating the forces of load on support elements of a launcher, transverse forces and bending moments in cross sections of bodies of blocks of a flight vehicle with a pack arrangement will be equal to the sum of the corresponding static and dynamic components. The calculation of static components $Q_c(x)$ and $M_c(x)$, and also of static reactions in places of connection of blocks with each other and with carrier elements of the launcher is conducted by known methods of structural mechanics of rod systems.

The system of hinged united blocks considered by us as an example (Fig. 8.2) during launching (in case (b) § 8.1) will be acted on (in plane $\xi\eta$) by a dynamic disturbing moment equal to

$$M_{вр} = r(x_в)\sum_{i=1}^{8} P_{бi}\cos\gamma_i. \tag{8.56}$$

In the case of static application of this moment and in the absence of contact of the flight vehicle with longitudinal support elements of the launcher (for instance, during jumping of the vehicle above the launcher) the diagram of transverse forces and bending moments for block A will have the form shown in Fig. 8.6. With $x_B < x < x_C$:

$$Q_{cA}(x) = 2R_B, \tag{8.57}$$

$$M_{cA}(x) = M_{вр} - 2R_B(x - x_B). \tag{8.58}$$

Having compiled the equations of static equilibrium of a system on the whole and each block separately, we obtain the corresponding reactions to upper support R_B and lower connection R_C

$$2R_B = -R_c = M_{вр}\frac{1}{x_c - x_B}. \tag{8.59}$$

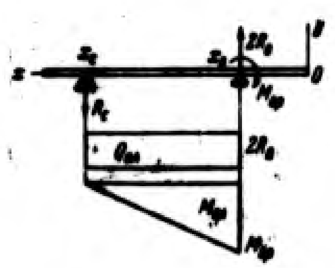

Fig. 8.6. Diagram of transverse forces and bending moments for the central block of a flight vehicle with side-mounting boosters.

In the case of distribution of lower support "н" of the launcher in section x_c the static values $Q_c(x_1)$ and $M_c(x_1)$ for all side-mounting boosters will be equal to zero, and $R_н = R_c/2$. In the case of installation of support "н" on blocks Д and В in section $x_{1н} = x_{1c}$.

$$\left.\begin{aligned} -R_н = R_в = \frac{M_{вр}}{2x_{1н}}, \\ M_{cД}(x) = M_{вр}\left(1 - \frac{x - x_в}{x_{1н}}\right). \end{aligned}\right\} \tag{8.60}$$

Corresponding diagrams of $Q_{сД}(x_1)$ and $M_{сД}(x_1)$ for $x_{1н} > (x_c - x_в)$ are given in Fig. 8.7.

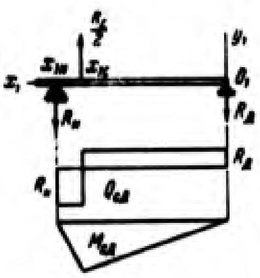

Fig. 8.7. Diagram of transverse force and bending moments for a side-mounting booster.

Within the limits of change of x_1 from 0 to $x_{1c} - a$

$$\left.\begin{aligned} Q_{cД}(x_1) = R_Д, \\ M_{cД}(x_1) = -R_Д \cdot x_1, \end{aligned}\right\} \tag{8.61}$$

and on section

$$\left.\begin{aligned} x_{1c} < x_1 < x_{1н}, \\ Q_{cД}(x_1) = -R_н, \\ M_{cД}(x_1) = \frac{1}{2} R_c (x_1 - x_{1c}) - x_1 R_Д. \end{aligned}\right\} \tag{8.62}$$

In case of absence of flight-vehicle jump disturbing moment $M_{вр}$ which is acting on the system will be absorbed not only by transverse, but also by longitudinal support elements of the launcher. Moreover, in the case of distribution of lower supports "н" in sections $x_{1н} \neq x_{1c}$

the static support reactions will depend on the rigidity of all the support elements, and also on the rigidity of force connections and rigidity of bodies of blocks of the side-mounting boosters.

We will introduce a certain equivalent rigidity c, equal to the ratio of force $R_C = 2R_H$ to the shifting caused by it of point x_C of basic block A at $c_B = \infty$. In a particular case during installation of support "H" on the body of this block in section $x = x_C$ c will be equal to c_H. Then the arrangement of forces acting on the basic block of the flight vehicle in the case of static application of disturbing moment M_{Bp} can be presented in the form shown in Fig. 8.8, in which for clarity the sites of location of longitudinal and upper transverse supports are conditionally displaced relative to each other. From the conditions of static equilibrium we obtain

$$\left.\begin{aligned} R_H &= -R_B,\\ M_{Bp} &= aR_H + r(x_B)\sum_{i=1}^{8} R_{x_i}\cos\gamma_i. \end{aligned}\right\} \tag{8.63}$$

Here

$$R_{x_i} = r(x_B)\,\vartheta(x_B)\,E_{x_i}\cos\gamma_i, \tag{8.64}$$

where $\vartheta(x_B)$ - angle of rotation of section x_B, and E_{x_i} - rigidity of i-th longitudinal support. Approximately, considering the body of block A absolutely rigid, it is possible to write that

$$\vartheta(x_B) = \frac{\eta(x_B)-\eta(x_C)}{x_C - x_B} = \frac{R_H}{a}\left(\frac{1}{E_H}+\frac{1}{E_B}\right).$$

Then

$$R_{x_i} = r(x_B)\,R_H\,\frac{E_{x_i}}{a}\left(\frac{1}{E_H}+\frac{1}{E_B}\right)\cos\gamma_i. \tag{8.65}$$

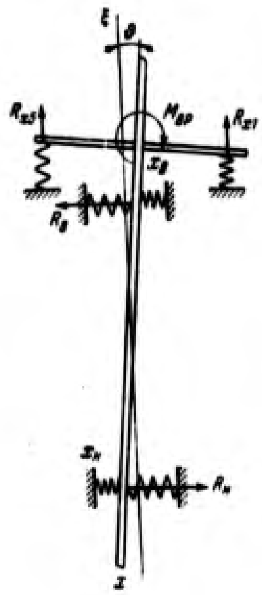

Fig. 8.8. Scheme of forces acting on the central block of a flight vehicle during static application of external disturbing moment.

Substituting this expression in formula (63), we have

$$R_{\text{н}} = \frac{M_{\text{вр}}}{a(1+k)}, \qquad (8.66)$$

where

$$k = \frac{r^2(x_{\text{в}})}{a^2}\left(\frac{1}{E_{\text{н}}} + \frac{1}{E_{\text{в}}}\right)\sum_{i=1}^{8} E_{x_i}\cos^2\gamma_i.$$

Thus the additional static load on the longitudinal support of the lanucher will be equal to

$$R_{x_i} = \frac{r^2(x_{\text{в}})}{a^2(1+k)}\left(\frac{1}{E_{\text{н}}} + \frac{1}{E_{\text{в}}}\right)E_{x_i}\cos\gamma_i\sum_{j=1}^{8} P_{\text{б}j}\cos\gamma_j,$$

and static values of transverse force and bending moment on section $x_B < x < x_C$ of the body of the basic block of the vehicle will be determined by expressions

$$Q_{cA}(x) = R_{\text{н}}, \qquad (8.67)$$

$$M_{cA}(x) = (x_c - x)\, M_{\text{вр}}\, \frac{1}{a(1+k)}. \qquad (8.68)$$

In comparing (58), (59), and (68) we see that in the absence of flight-vehicle jump above the support surface of the launcher the static values of bending moment in sections of the body of the central block turn out to be $(1 + k)$ times less than in the presence of such jumping. Here the stated decrease of static moments will be even greater, the greater the relative rigidity of longitudinal support elements of the launcher E_{x1}/E_{H} and E_{x1}/E_{B}. An especially large influence on magnitude of disturbing moment $M_{Bp}(x_{B})$ is exerted by the ratio $r(x_{B})/a$. Usually values $r(x_{B})$ is determined by the diameter of the body of the flight vehicle, and the distance between supports x_{B} and x_{H} by the overall arrangement and place of location on body of frames which are able to absorb large concentrated forces - transverse support reactions. Therefore it is possible to establish some requirement for the ratio $r(x_{B})/a$ only when this case of load becomes determining for the strength of the body of the flight vehicle at all possible values of E_{x1}, E_{H} and E_{B} .

From what has been said it follows that the selection of magnitude E_{x1} (with assigned $n_{xп}$) should be conducted in such a manner that jumping of the flight vehicle while the engines are approaching operating conditions is absent. Having E_{x1}, it is not difficult to establish those necessary values for E_{B} and E_{H}, at which bending moments will not exceed permissible values. However, for a noticeable decrease of static component of bending moment the values of E_{B} and E_{H} have to be small as compared to E_{x1}. Thus, for example, for decreasing $M_{cA}(x)$, let us assume by 30% in the case of $r(x_{B})/a =$ $= 0.1$ and $E_{H} = E_{B}$, it is required that E_{x1}/E_{H} be of the order of 7.5. For appraisal of the influence of magnitude E_{H} and E_{B} on values of the dynamic component of bending moment $M_{Д}(x)$ one should consider the forced oscillations of system described by equations (53).

On the basis of formula (5.56) the dynamic values of support reactions will equal

$$
\left.
\begin{aligned}
R_{\text{н}} &= \sum_{m=1}^{n_0} p_m \eta_{\text{н}m} \cdot E_{\text{н}},\\
R_{\text{в}} &= \sum_{m=1}^{n_0} p_m \eta_{\text{в}m} \cdot E_{\text{в}},\\
R_{\text{c}} &= \sum_{m=1}^{n_0} p_m y_{\text{c}m} E_{\text{c}}.
\end{aligned}
\right\}
\qquad (8.69)
$$

Dynamic components of transverse forces and bending moments in cross sections of the body of the central block will be determined by expressions

$$
\left.
\begin{aligned}
Q_{\text{дл}}(x) = &- \sum_{m=1}^{n_0} \tilde{p}_m \Big\{ \eta_{\text{в}m} \Big[Q^0(x) - m_A (x_{\text{c}} - x_{\text{тл}}) \frac{1}{a} \Big] + \\
&+ \frac{1}{a} \Big[\frac{a}{l} (\eta_{\text{н}m} - \eta_{\text{в}m}) + y_{\text{c}m} + \sum_{p=1}^{n_p} K_{p\text{Д}m} \Phi_{p\text{Д}}(x_{1\text{c}}) \Big] \times \\
&\times \Big\{ [Q^x(x) - x_{\text{в}} Q^0(x)] - \Big[m_{\text{л}} (x_{\text{тл}} - x_{\text{в}}) - \frac{1}{a} J_{\text{л}}(x_{\text{в}}) \Big] \Big\} + \\
&+ \sum_{n=1}^{n_n} S_{nm} \Big[Q_{nx}(x) - \int_0^L m(x) f_n(x)\, dx + \frac{1}{a} \int_0^L m(x) \times \\
&\times (x - x_{\text{в}}) f_n(x)\, dx \Big] \Big\},\\
M_{\text{дл}}(x) = &\int_0^x Q_{\text{дл}}(x)\, dx \text{ with } x_{\text{в}} < x < x_{\text{c}}.
\end{aligned}
\right\}
\qquad (8.70)
$$

$$
\left.
\begin{aligned}
Q_{\text{дл}}(x) = &- \sum_{m=1}^{n_0} \tilde{p}_m \Big\{ \eta_{\text{в}m} Q^0(x) + \frac{1}{a} \Big[\frac{a}{l} (\eta_{\text{н}m} - \eta_{\text{в}m}) + \\
&+ y_{\text{c}m} + \sum_{p=1}^{n_p} K_{p\text{Д}m} \Phi_{p\text{Д}}(x_{1\text{c}}) \Big] [Q^x(x) - x_{\text{в}} Q^0(x)] + \\
&+ \sum_{n=1}^{n_n} S_{nm} Q_{nx}(x) \Big\},\\
M_{\text{дл}}(x) = &- \sum_{m=1}^{n_0} \tilde{p}_m \Big\{ \eta_{\text{в}m} M^0(x) + \frac{1}{a} \Big[\frac{a}{l} (\eta_{\text{н}m} - \eta_{\text{в}m}) + \\
&+ y_{\text{c}m} + \sum_{p=1}^{n_p} K_{p\text{Д}m} \Phi_{p\text{Д}}(x_{1\text{c}}) \Big] [M^x(x) - x_{\text{в}} M^0(x)] + \\
&+ \sum_{n=1}^{n_n} S_{nm} M_{nx}(x) \Big\} \text{ with } x < x_{\text{в}}.
\end{aligned}
\right\}
\qquad (8.71)
$$

The values of dynamic components of transverse forces and bending moments in cross sections of bodies of side-mounting boosters Б and Д without a calculation of support reaction are equal to

$$
\left.
\begin{aligned}
Q_{\text{дБ}}(x_1) = &-\sum_{m=1}^{n_0}\ddot{p}_m\left\{\sum_{p=1}^{n_p}K_{p\text{Б}m}Q_{px}(x_1)+\eta_{\text{в}m}Q^0(x_1)+\right.\\
&+\frac{1}{l}(\eta_{\text{н}m}-\eta_{\text{в}m})Q^x(x_1)+\left[\sum_{p=1}^{n_p}K_{p\text{Д}m}\Phi_{p\text{Д}}(x_{1c})+y_{cm}\right]\times\\
&\left.\times Q^x(x_1)\frac{1}{a}\right\},\\
M_{\text{дБ}}(x_1) = &-\sum_{m=1}^{n_0}\ddot{p}_m\left\{\sum_{p=1}^{n_p}K_{p\text{Б}m}M_{px}(x_1)+\eta_{\text{в}m}M^0(x_1)+\right.\\
&+\frac{1}{l}(\eta_{\text{н}m}-\eta_{\text{в}m})M^x(x_1)+\left[\sum_{p=1}^{n_p}K_{p\text{Д}m}\Phi_{p\text{Д}}(x_{1c})+y_{cm}\right]\times\\
&\left.\times\frac{1}{a}M^x(x_1)\right\}\text{ with } x_1<a,
\end{aligned}
\right|
\tag{8.72}
$$

$$
\left.
\begin{aligned}
Q_{\text{дД}}(x_1) = &-\sum_{m=1}^{n_0}\ddot{p}_m\left[\sum_{p=1}^{n_p}K_{p\text{Д}m}Q_{px}(x_1)+\eta_{\text{в}m}Q^0(x_1)+\right.\\
&\left.+\frac{1}{l}(\eta_{\text{н}m}-\eta_{\text{в}m})Q^x(x_1)\right],\\
M_{\text{дД}}(x_1) = &-\sum_{m=1}^{n_0}\ddot{p}_m\left[\sum_{p=1}^{n_p}K_{p\text{Д}m}M_{px}(x_1)+\eta_{\text{в}m}M^0(x_1)+\right.\\
&\left.+\frac{1}{l}(\eta_{\text{н}m}-\eta_{\text{в}m})M^x(x_1)\right]\text{ with } x_1<a.
\end{aligned}
\right|
\tag{8.73}
$$

The coefficient (transverse component) vibration overload in points of longitudinal axis of central block A will be determined by the formula

$$
\begin{aligned}
n_{y1\text{A}}(x) = &-\frac{1}{g_0}\sum_{m=1}^{n_0}\ddot{p}_m\left[\sum_{n=1}^{n_0}S_{nm}f_n(x)+\right.\\
&+\frac{1}{a}(x-x_{\text{в}})\sum_{p=1}^{n_p}K_{p\text{Д}m}\Phi_{p\text{Д}}(x_{1c})+\frac{1}{a}(x-x_{\text{в}})\left(y_{cm}+\frac{a}{l}\eta_{\text{н}m}\right)+\\
&\left.+\eta_{\text{в}m}\left(1-\frac{x-x_{\text{в}}}{l}\right)\right],
\end{aligned}
\tag{8.74}
$$

and in points of the longitudinal axis of a side-mounting booster, for example Д, by expression

$$n_{y,\text{Д}}(x_1) = -\frac{1}{g_0}\sum_{m=1}^{n_0}\ddot{p}_m\Bigg[\sum_{p=1}^{n_p}K_{p\text{Д}m}\Phi_{p\text{Д}}(x_1) + \\ + \eta_{\text{в}m}\left(1-\frac{x_1}{l}\right)+\eta_{\text{н}m}\frac{x_1}{l}\Bigg]. \qquad (8.75)$$

In the presence of constant contact (in a longitudinal direction) of the central block with side-mounting boosters the bending oscillations of this block will cause longitudinal oscillations of the boosters with overloads

$$\Delta n_x = \frac{r(x_{\text{в}})}{g_0}\sum_{m=1}^{n_0}\ddot{p}_m\sum_{n=1}^{n_n}S_{nm}\frac{df_n(x)}{dx}\bigg|_{x=x_{\text{в}}}\cos\gamma_i. \qquad (8.76)$$

The values of vibration overloads will obviously depend on the system for starting the engines, dynamic and inertial characteristics of the flight vehicle, and the rigidity of carrier elements of launcher. It is necessary to stress that the dynamic components of transverse force and bending moment (70), (71), (72), and (73) are complex functions of coordinates x and x_1, and therefore it is impossible to estimate the effect of the influence on such a system of dynamic disturbing forces and moments by some unique coefficient of dynamics.

§ 8.7. Concluding Remarks

We will consider as an example the forced transverse oscillations of a system flight vehicle - launcher in the process of switching on and emergency shutdown of the engines of side-mounting boosters which are located in plane ξη. Let us assume that the nature of build-up and drop of thrust force of all the boosters is absolutely identical and is described by those formulas which are given in Chapter I, namely (1.13) and (1.12) correspondingly. Let us assume further that switching on of the engine of one of the side-mounting boosters, for instance Б, occurs with a delay equal to $\Delta t_{\text{В}}$, and the command for turning off of all the engines of the flight vehicle is given at a certain moment of time t_c. Here its actual realization occurs also with a certain scattering Δt_c.

Thus in the period of starting the engines of the side-mounting boosters on the flight vehicle in section x_B a disturbing moment (56) will be acting which is equal to

$$\left.\begin{aligned} M_{Bp} &= P_0 f(x_B)(1-e^{-\tau t}) \text{ with } 0<t<\Delta t_B, \\ M_{Bp} &= P_0\left[e^{-\tau(t-\Delta t_B)} - e^{-\tau t}\right] \text{ with } t>\Delta t_B, \end{aligned}\right\} \tag{8.77}$$

and in period of turning off - moment

$$\left.\begin{aligned} M_{BC} &= P_0 f(x_B)\frac{t-t_c}{T_c} \text{ with } t_c<t<(t_c+\Delta t_c), \\ M_{BC} &= P_0 f(x_B)\frac{\Delta t_c}{T_c} \text{ with} (t_c+\Delta t_c)<t<(t_c+T_c), \\ M_{BC} &= -P_0 f(x_B)\frac{(t-t_c-T_c)}{T_c} \text{ with} (t_c+T_c)<t< \\ &\qquad <(t_c+T_c+\Delta t_c), \\ M_{BC} &= 0 \qquad \text{ with } t>(t_c+\Delta t_c+T_c). \end{aligned}\right| \tag{8.78}$$

It is absolutely clear that the direction of section M_{BC} can be any, and, consequently, also one sign with moment (77).

As a result of solution of the system n_0 of equations (53) with zero initial conditions (for the particular case of influence of moments M_{Bp} and M_{BC}) we obtain the following approximate expressions for $\ddot{p}_m$:

$$\left.\begin{aligned} \ddot{p}_m(t) &= \ddot{p}_{m_1}(t) = \frac{C_m\tau^2}{\omega_m^2+\tau^2}\left\{e^{-\tau t} + \left[\frac{\omega_m}{\tau}\sin\omega_m t + \frac{\omega_m^2}{\tau^2}\cos\omega_m t\right]\right\} - \\ &- C_m\cos\omega_m t, \qquad \text{with } 0<t<\Delta t_B, \\ \ddot{p}_m(t) &= \ddot{p}_{m_1}(t) - \ddot{p}_{m_1}(t-\Delta t_B) \text{ with } t>\Delta t_B \\ &\qquad (m=1,\ 2,\ \ldots,\ n_0). \end{aligned}\right| \tag{8.79}$$

An analogus solution of these equations for a case of turning off of engines (at zero initial conditions) will have the form:

$$
\left.\begin{aligned}
&\ddot{p}'_m(t)=\frac{C_m}{\omega_m T_c}\sin\omega_m(t-t_c)\ \text{with}\ t_c<t<(t_c+\Delta t_c),\\
&\ddot{p}'_m(t)=\frac{C_m}{\omega_m T_c}\left[e^{-h_m(t-t_c)}\sin\omega_m(t-t_c)-e^{-h_m(t-t_c-\Delta t_c)}\times\right.\\
&\qquad\left.\times\sin\omega_m(t-t_c-\Delta t_c)\right]\text{with}\ \Delta t_c<(t-t_c)<T_c,\\
&\ddot{p}'_m(t)=\frac{C_m}{\omega_m T_c}\left\{e^{-h_m(t-t_c)}\left[\sin\omega_m(t-t_c)-e^{h_m T_c}\times\right.\right.\\
&\times\sin\omega_n(t-t_c-T_c)\left.\right]-e^{h_m(t-t_c-\Delta t_c)}\sin\omega_m(t-t_c-\Delta t_c)\left.\right\}\\
&\qquad\qquad\text{with}\ T_c<(t-t_c)<(T_c+\Delta t_c),\\
&\ddot{p}'_m(t)=\frac{C_m}{\omega_m T_c}\left\{e^{-h_m(t-t_c)}\left[\sin\omega_m(t-t_c)-e^{h_m T_c}\times\right.\right.\\
&\quad\times\sin\omega_m(t-t_c-T_c)\left.\right]-e^{-h_m(t-t_c-\Delta t_c)}\left[\sin\omega_m\times\right.\\
&\quad\times(t-t_c-\Delta t_c)-\sin\omega_m(t-t_c-\Delta t_c-T_c)\left.\right]\left.\right\}\\
&\qquad\qquad\text{with}\ (t-t_c)>(T_c+\Delta t_c).
\end{aligned}\right\}\qquad(8.80)
$$

We will designate bending moments, caused by the action of disturbing moment M_{BP} (77), by subscript 1, and additional bending moments, appearing as a result of action on system by disturbing moment M_{BC} (78), by subscript 2. Values $M_{Д2}(x, t)$ will be determined by the same formulas as values $M_{д1}(x, t)$, but with replacement of variable $\ddot{p}_m(t)$ (79) by $\ddot{p}'_m(t)$ (80). Then

$$
\left.\begin{aligned}
&\text{with}\ t<t_c\\
&\qquad M(x,t)=M_{c1}(x,t)+M_{д1}(x,t)=M_1(x,t),\\
&\text{with}\ t>t_c\\
&M(x,t)=[M_{c1}(x,t)+M_{д1}(x,t)]\pm\\
&\qquad\pm[M_{c2}(x,t-t_c)+M_{д2}(x,t-t_c)].
\end{aligned}\right\}\qquad(8.81)
$$

Since in most cases the magnitude of Δt_c is small as compared to Δt_B, then it is possible to expect that M_2 will be less than
The nature of change of M_1 by t for a system, the normal forms of oscillations of which are shown in Fig. 8.5, for three values of Δt_B is given in Fig. 8.9. Calculations show that values of M_1 increase with an increase of τ and with an increase (up to definite limits) of delay time Δt_B. In the absence of contact of the flight vehicle with longitudinal support elements of the launcher M_1 is increased with an increase of rigidity of the upper transverse support beam E_B and decreases with an increase of rigidity of the lower support beam E_H (Fig. 8.10). Figures 8.11 and 8.12 show the dependence of relative

values of transverse overload and support reaction $\bar{R}_B$ on the rigidity of these beams, and also on the rigidity of the lower force connection. As can be seen, R_B increases with an increase of E_B and E_H and almost does not depend on E_c, and n_{y1} (in the particular section) decreases with an increase of E_H. The nature of change of n_{y1} by t for two sections of the central block (at $t < t_c$) is shown in Fig. 8.13).

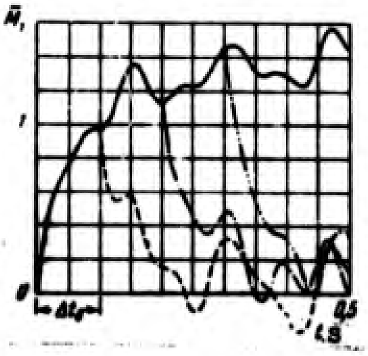

Fig. 8.9. Influence of delay time on values of bending moments.

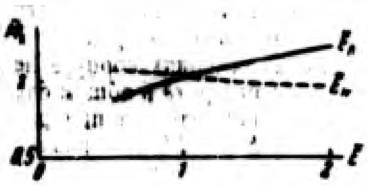

Fig. 8.10.

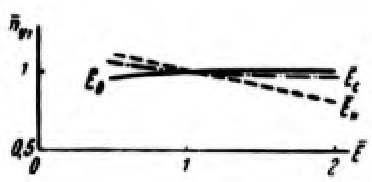

Fig. 8.11.

Fig. 8.10. Influence of rigidity of transverse beams of the launcher on magnitude of bending moment.

Fig. 8.11. Influence of rigidity of transverse beams of the launcher and rigidity of lower force connection of a side-mounting booster of a flight vehicle on the magnitude of transverse overload.

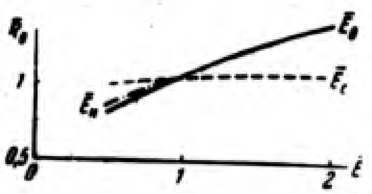

Fig. 8.12. Dependence of support reaction on the rigidity of beams of the launcher and rigidity of the force connection.

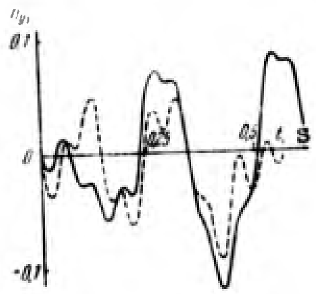

Fig. 8.13. Approximate nature of change of transverse overload in various sections of the central block of a flight vehicle in the process of launching.

The values of dynamic bending moments $M_{Д1}$ and $M_{Д2}$ depend on a shift of phases of emposed oscillations. Modifying within the limits of fixed allowances of the magnitudes of $\Delta t_{В}$ and Δt_{C}, it is possible to find for each cross section of bodies of blocks the highest possible values for $Q_{Д1}$, $Q_{Д2}$ and $M_{Д1}$, $M_{Д2}$. In this case max M_1 and max M_2 will correspond to max $\Delta t_{В}$ and max Δt_{C}. If a certain range of time (counted off from the beginning of actual switching on of engines) is set, during which the command can be given for emergency shutdown of all engines, then for determination of the largest values of total bending moments (81) one should also vary the magnitude of t_c. This can be done graphically, by means of imposition of curve $M_2(x, t - t_c)$ on $M_1(x, t)$. These operations must be repeated for each value of x, since the largest magnitudes of bending moments in various cross sections of the body of a flight vehicle in general can be observed at various t_c, Δt_c and $\Delta t_{В}$ and in various moments of time t.

The solution of the problem is complicated considerably if the necessity appears for calculation of the influence of control forces by means of introduction of equations of control. The influence of these forces on M(x, t) can be disregarded only in that case when they are small or when automatic control devices are joined to the body of the flight vehicle in places close to place of distribution of supports of the lower transverse beams of the launcher.

In absolutely the same way M(x, t) for a case of the action of disturbing moments in plane $\xi\zeta$. By adding them geometrically it is possible to find the values of total bending moments for any combination of external influences. If the disturbing moments acting in these planes are independent, then in principle it is possible to also add their maximum values. However, one should consider that the moment of appearance of the largest magnitude of total bending moment in general still does not determine the heaviest case of load on elements of construction of the flight vehicle. In process of starting and shutting off of engines the longitudinal forces in sections of all the side-mounting boosters, and also in certain sections of the central block are changed differently. Furthermore, there is the possibility of the significant influence of those disturbing forces which appear in the process of bringing the engines up to the main stage. Thus the layout of load on a flight vehicle with a pack arrangement in the case of launching is very complex. Much depends on the structural arrangement of the flight vehicle itself and the launch procedure. Therefore for finding the most dangerous, from the point of view of strength of construction, moment of load one should look for the largest values of total stresses from longitudinal and transverse loads. Only for those sections, in which longitudinal forces remain constant in the period of launching, it is possible to be limited to calculations of max M(x, t).

For the given cross section of the body of a flight vehicle the probability of realization of a dangerous combination of values of parameters t_C, Δt_C, Δt_B, and others can also in reality be small. This should be considered in the selection of calculation cases of load and assignment of safety factors. In practice thrust forces of different engines in transitional regimes are changed differently; also different are the actual values of magnitudes Δt_B, Δt_C. Therefore in the determination of dynamic response for this case of load it is expedient to be guided by the mean values of those parameters which are random functions of time.

For decreasing the magnitude of transverse forces, bending moments, and support reactions which appear in the process of launching the flight vehicles which have a propulsion system made up of a packet of engines, one should aspire to decreasing the gradient of change is the thrust force of engines both in the process of starting and also in the process of cutoff. In all cases it is necessary to limit delay in the time of onset of actual build-up and the beginning of the actural decrease of pressures in the combustion chambers of all the engines of a given stage. In the case of stepped approach of flight-vehicle engines to operating conditions it is necessary to select the magnitude of the intermediate stage with a consideration of requirements of strength of construction of the vehicle itself and also the launcher.

The account presented in this chapter of a method for calculation of the dynamic reaction of a flight vehicle with side-mounting boosters to the influence of disturbing forces which appear in the process of launching can readily be extended to other group systems of flight vehicles: to simpler ones than those considered, for example to arrangements of vehicles with suspended fuel tanks, located inside or on the outside of the carrier central block, and to more complicated ones, consisting of several serially connected packs of blocks with carrier or suspension tanks.

SECTION IV

FLIGHT LOADING CASES

C H A P T E R IX

REACTION OF THE VEHICLE CONSTRUCTION TO THE INFLUENCE OF WIND IN FLIGHT

§ 9.1. Reaction of Gust of Wind

One of the most important cases of loading for all flight vehicles is the case of flight in a restless atmosphere (case W). It determines the required bearing capacity of many elements of construction of nonmaneuverable flight vehicles, having comparatively small programmed values of lateral overloads, and has an essential influence on the fatigue strength of the construction of maneuvering flight vehicles. Therefore, in the process of designing to the question of investigation of the reaction of flight vehicles to the influence of wind there is always allotted considerable attention.

As already was noted, the motion of air in the atmosphere can be schematically represented in the form of the sum of two flows: steady horizontal wind, characterized by small vertical shifts of velocities, and gusts of wind, velocity vector of which is a random function of time and coordinates. Pulsation of wind velocity in the free atmosphere, as in the surface layer, can in turn be represented in the form of small continuous turbulence, on which discrete gusts of great intensity are superimposed. These gusts and steady wind for all practical purposes determine the required bearing capacity of the construction of flight vehicle on a given case of loading.

The influence of small atmospheric turbulence is of interest mainly in connection with questions of fatigue strength, which are not considered in this book.

Since from a mathematical point of view the reaction of the flight vehicle to the influence of steady wind is a particular case of the reaction of the construction to a discrete gust of wind, investigation of the latter will comprise the basic content of this chapter.

Let us designate the components of wind velocity in a continuous system of coordinates xyz through u_x, u_y, and u_z. Component u_x, directed along the tangent to undisturbed trajectory of the flight vehicle, will directly affect only the magnitude of relative velocities of the vehicles. As a result with nonzero angle of attack the corresponding change of impact pressure will lead to some change of lateral overload of the center of gravity of the flight vehicle. According to § 2.1 this overload will obtain an increase equal to

$$\Delta n_{y_1}^0 = \frac{\rho_0 c_y^\alpha v S \Delta_0}{2G_0(1-\bar{l})}\left(2+\frac{u_x}{v}\right)\alpha u_x; \qquad (9.1)$$

where

$$\Delta_0 = \frac{\rho}{\rho_0} \approx \frac{T_0^0}{T^0} e^{-\int_0^h \frac{g\,dh}{RT^0}}, \qquad (9.1')$$

T^0 – temperature of air at altitude h, T_0^0 – at the ground; R – gas constant, equal to 287.05 $m^2/(s^2 \cdot deg)$.

Normal and binormal velocity components will basically affect angle of attack of the flight vehicle. Change of this angle is equivalent to the influence of additional lateral aerodynamic forces on the construction

$$\left.\begin{aligned} \Delta Y &= (q+\Delta q) c_y^\alpha S \frac{u_y}{v}, \\ \Delta Z &= (q+\Delta q) c_z^\alpha S \frac{u_z}{v}. \end{aligned}\right\} \qquad (9.2)$$

Lateral motion of the center of gravity of the flight vehicle under action of such forces, for instance, force ΔY, is described by the following ordinary differential second order equation:

$$\ddot{y} = -(\dot{y} - u_y)\frac{1}{v}\frac{qSc_y^{\alpha}g_0}{G_0(1-\bar{t})k_n^0}, \qquad (9.3)$$

where k_n^0 – coefficient considering the influence of mobility of liquid in tanks. By replacing in this equation variable $\dot{y}=\beta_0$ and using expression (1'), we will have

$$\dot{\beta}_0 = -\frac{vc_y^{\alpha}(\beta_0 - u_y)\Delta_0}{l\lambda(1-\bar{t})k_n^0},$$

where λ – is relative density of the flight vehicle, equal to the ratio of its initial weight to the weight of air located in volume $lS/2$:

$$\lambda = \frac{2G_0}{lS\rho_0 g_0}. \qquad (9.4)$$

Let us assume that at initial moment $t_1=0$ (corresponding to moment of beginning of the action of gust of wind on the flight vehicle $\bar{t}=\bar{t}_0$) $y(t_1)=\dot{y}(t_1)=0$. Then as a result of integration we obtain

$$\beta_0(t_1) = e^{-\int_0^{t_1} a_1\,dt}\int_0^{t_1} a_1 e^{\int_0^{t_1} a_1\,dt} u_y(t_1)\,dt_1.$$

Here

$$a_1 = \frac{vc_y^{\alpha}\Delta_0}{l\lambda(1-\bar{t})k_n^0},$$

and t_1 – time counted off from the moment of beginning of action of a gust of wind. By differentiating this expression with respect to t_1, we will find the increase of normal overload of the center of gravity of the flight vehicle in the form

$$\Delta n_y^0(t_1) = \frac{a_1}{g_0}\left[e^{-\int_0^{t_1} a_1\,dt}\int_0^{t_1} a_1 u_y(t_1)\,e^{\int_0^{t_1} a_1\,dt}\,dt_1 - u_y(t_1)\right].$$

Since function v(t) keeps the sign on the entire interval of change t_1, then, by using mean value theorem, we can write

$$\int_0^{t_1} a_1\,dt = \frac{1}{\lambda_1}\int_0^{t_1} \frac{v}{l}\,dt, \tag{9.4'}$$

where λ_1 – value of $\lambda\left[k_n^0(1-\bar{l})\frac{1}{c_y^{\alpha}\cdot\Delta_0}\right]$ in interval $(0, t_1)$.

If t_1 is small as compared to $T=\frac{G_0}{\dot{G}}$, then in the first approximation it is possible to disregard the change of magnitude of λ in interval $(\bar{t}_0, \bar{t}_0+\bar{t}_1)$. It is necessary to note that during calculation of λ_1 it is possible to use the value of derivative of the coefficient of lateral aerodynamic force c_y^{α}, determined by steady-state theory or by means of wind tunnel tests of a model, only for large values of relative density λ. With small λ the influence of transient aerodynamic phenomena turns out to be essential, and it is impossible to disregard them.

Instead of time t_1 it is expedient to introduce new dimensionless variable

$$s=\int_0^{t_1} \frac{v}{l}\,dt, \tag{9.5}$$

expressing shift of the flight vehicle in units of length of its body (or chord of the wing). Then

$$\Delta n_y^0 = \frac{la_1}{g_0}\left(e^{-\frac{s}{\lambda_1}}\int_0^s \frac{a_1 u_y}{v} e^{\frac{s}{\lambda_1}}\,ds - \frac{u_y}{l}\right), \tag{9.6}$$

or

$$\Delta n_y^0 = \frac{v}{l\lambda_1^2 g_0}\int_0^s u_y e^{-(s-\sigma)\frac{1}{\lambda_1}}\,d\sigma - \frac{v u_y}{l g_0 \lambda_1}.$$

In case of instantaneous (sudden) change of wind velocity (sharply limited gust of wind, instantly encompassing the whole flight vehicle)

$$u_y=\begin{cases} 0 & \text{when} \quad s<0, \\ \frac{u_m}{\sqrt{\Delta_0}} & \text{when} \quad s>0, \end{cases}$$

corresponding increase of normal overload will be equal to

$$\Delta n_y^0 = -\frac{v u_m}{g_0 \lambda_1 l \sqrt{\Delta_0}} e^{-\frac{s}{\lambda_1}}. \tag{9.7}$$

It is simple to note that with such gust of wind overload Δn_y^0 will be maximum at initial moment, i.e., when s = 0

$$\max \Delta n_y^0 = -\frac{u_m c_y^\alpha v \sqrt{\Delta_0}}{g_0 l \lambda (1 - l) k_n^0}. \tag{9.8}$$

By comparing expressions (8) and (1), we obtain that at high speeds of vehicle flight, when u_y/v is small as compared to one, relation $\frac{\Delta n_{y_1}^0(u_x)}{\max \Delta n_y^0}$ will be of order α. Consequently, at small programmed angles of attack of the flight vehicle the normal component of the velocity of the gust of wind has decisive value in loading of the construction.

In the beginning of flight (at low velocities of the flight vehicle) and at high altitudes (with small relative air density) the values of $\max \Delta n_y^0$ are comparatively small. Therefore, during investigation of lateral overloads and loads affecting the flight vehicle during motion in restless air, we subsequently will pay basic attention only to the phase of flight with relatively high impact pressures, in particular on the region of motion of the vehicle at supersonic speeds.

The rotary component of lateral overload for the case of instantaneous action of a gust of wind on an unguided flight vehicle can be found, having equated the perturbing aerodynamic moment to inertial moment. Having assumed

$$-\Delta\ddot{\vartheta} = \frac{v u_y \Delta_0 \rho_0}{2 J_z k_n^l} S c_y^\alpha (x_{1\text{д}} - x_{1\text{т}}),$$

we will have

$$-\Delta n_y^\varepsilon = \frac{v u_y}{r^2 l g_0 \lambda_1} (x_{1\text{д}} - x_{1\text{т}})(x_1 - x_{1\text{т}}), \tag{9.9}$$

where $r^2 = \frac{J_z k_n^l}{m k_n^0}$, and k_n^l and k_n^0 – coefficients considering the influence of mobility of liquid in tanks (see Chapter II).

Strictly speaking, overloads found in such a way are fictitious, since gusts of wind with sudden change of velocity do not exist in nature. The wind velocity is changed continuously by s, reaching its maximum value u_m on certain segment H.

Let us see how the magnitude of path of increase of the velocity of a gust of wind affects the amount of overload in the center of gravity of the flight vehicle. For an example let us take a gust with linear law of change of wind velocity

$$u_y = \frac{u_m}{\sqrt{\Delta_0}} \frac{s}{s_m} \text{ when } 0 < s \leqslant s_m, \text{ where } s_m = \frac{H}{l}. \tag{9.10}$$

Having placed this value u_y in expression (6) and taking into account formula (7), after integration we obtain

$$-\Delta n_y^0 = \frac{v u_m}{\lambda_1 g_0 l s_m \sqrt{\Delta_0}} \left(1 - e^{-\frac{s}{\lambda_1}}\right). \tag{9.11}$$

It is clear that magnitude Δn_y^0 (with respect to modulus) will be maximum when $s = s_m$. The ratio of the biggest value of this overload to $\max \Delta n_y^0$ (8) depends only on one parameter $\frac{s_m}{\lambda_1}$:

$$k_s = \frac{1 - e^{-\frac{s_m}{\lambda_1}}}{\frac{s_m}{\lambda_1}}. \tag{9.12}$$

The magnitude of parameter $\frac{s_m}{\lambda_1}$ is determined by the velocity and altitude of flight of the vehicle, specific load on the midsection and c_y^α. Graphic representation of the relationship of coefficient k_s, characterizing weakening (damping) of the influence of a gust of wind on a flight vehicle, to $\frac{s_m}{\lambda_1}$ is given on Fig. 9.1. It is necessary to note that the character of change of the velocity of a gust of wind $u_y(s)$ at small values of this parameter does not essentially affect the magnitude of coefficient k_s. Thus, for sinusoidal law of change of u_y

$$u_y = \frac{u_m}{\sqrt{\Delta_0}} \sin\left(\frac{\pi s}{2s_m}\right) \quad \text{when} \quad 0 < s \leqslant s_m, \tag{9.13}$$

$$k'_a = \left(\frac{\lambda_1}{s_m}\right)^2 \frac{1 - \frac{2s_m}{\pi\lambda_1} e^{-\frac{s_m}{\lambda_1}}}{0{,}4 + \left(\frac{\lambda_1}{s_m}\right)^2}.$$

Graph k'_a is shown on Fig. 9.1 by a dotted line. It is clear that k'_a is less than k_a. When $\frac{s_m}{\lambda_1} < 0{,}5$ this difference does not exceed 5%, i.e., lies within limits of accuracy of calculations. For large values of s_m large overloads are given by linear law of change of u with respect to s.

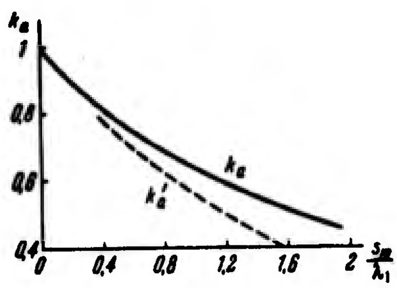

Fig. 9.1. Relationship of coefficient k_a to parameter $\frac{s_m}{\lambda_1}$.

Magnitude of overload for section $s > s_m$ can be found by means of imposition of back directed gust of wind on $u_y(s)$, i.e., by subtraction of the value of $\Delta n_y^0(s - s_m)$, obtained by formula (6) by means of replacement of s in it by $s - s_m$, from $\Delta n_y^0(s)$. The overload is also found entirely for a triangular form of gust. For this it is sufficient, starting with $s = s_m$, to impose a reverse gust of wind with doubled gradient $\frac{2u_m}{s_m\sqrt{\Delta_0}}$ on $u_y(s)$. It is obvious that if the velocity of a gust of wind remains constant when $s > s_m$, then due to the growth of $\dot{y}$ a decrease of the angle of attack of the flight vehicle will occur. As a result with increase of s the value of Δn_y^0 will be decreased. For determination of the biggest value of overload it is possible to be limited by consideration of only the initial transition section $s < s_m$, not being interested in subsequent change of the velocity of wind (under the condition, of course, that it remains less than $\frac{u_m}{\sqrt{\Delta_0}}$).

If for the flight vehicle λ_l is great ($\lambda_l > 300$), then the influence of path of increase of the velocity of a gust of wind H on the magnitude of the biggest transverse (or lateral) overload will show noticeably only at large values of s_m and basically on the initial section of the trajectory. Calculations show that this influence can practically be disregarded at all $H < 10l$. This means that with small H the action of "gradient" wind on a rigid flight vehicle is actually equivalent to the action of a sharply limited gust of wind. For large values of H one should consider the corresponding decrease of the magnitude of normal overload, which will be more intense, the large c_y^α of the vehicle. The true angle of attack of the flight vehicle will be changed analogously, equal to

$$\alpha = -\frac{\Delta n_y^0}{A}, \tag{9.14}$$

where

$$A = \frac{qSc_y^\alpha}{G_0(1-\bar{t})},$$

Δn_y^0 – real value of overload. For linear relationship of u to s, according to (11),

$$\alpha = \frac{\bar{u}}{v} k_s. \tag{9.15}$$

§ 9.2. Action of a Gust of Wind on an Unguided Vehicle

In preceding the paragraph we examined the influence of the gradient of wind velocity on the magnitude of normal overload of the center of gravity of an aerodynamically neutral flight vehicle. In cases when the center of pressure does not coincide with the center of gravity, transverse forward motion of the flight vehicle is accompanied by rotation with respect to the center of gravity. As a result an additional angle of attack appears. Not only the magnitude of forward component of lateral overload $n_{y_1}^{n}$ is changed, but the rotary component of overload $n_{y_1}^{r}$. Values of these overload components will depend on the degree of aerodynamic (static) stability of the flight vehicle and on its mass moment of inertia.

Let us consider first of all the reaction of an aerodynamically stable unguided flight vehicle to a gust of wind, parameters of undisturbed motion of which are changed comparatively slowly. Values of components of lateral overload in this case will be determined by solution of the system of equations (2.24), (2.25) with variable coefficients. Having assumed $\delta = 0$, let us write them in the form

$$\Delta\ddot{\vartheta} + b'_{\lambda}\Delta\dot{\vartheta} + b'_{\alpha}\Delta\alpha = f_1, \qquad (9.16)$$

$$\ddot{y} + c'_{\lambda}\frac{\dot{y}}{v} + c'_{\alpha}\Delta\vartheta = f_2, \qquad (9.17)$$

where

$$\left.\begin{aligned} & f_1 = -b'_{\alpha}\frac{u}{v}, \quad f_2 = \frac{uv}{l\lambda_1}, \\ & b'_{\lambda} = b_{\lambda}\frac{1}{J_z}, \quad c'_{\lambda} = \frac{c_{\lambda}}{m}, \quad c'_{\alpha} = \frac{c_{\alpha}}{m}, \\ & b'_{\alpha} = \frac{b_{\alpha}}{J_z} = \omega_0^2. \end{aligned}\right| \qquad (9.17')$$

Let us assume that instantaneous wind velocity is constant at all points, located perpendicular to the longitudinal axis of the flight vehicle and is changed linearly along the length of the vehicle (more exact in the direction of its flight). A similar change of wind velocity is equivalent to change of local angles of attack by magnitude

$$-\frac{u_m(x_{1T} - x_1)}{lvs_m}.$$

In other words, the corresponding change of the moment of perturbing aerodynamic forces can be represented in the form of a damping moment, acting on the flight vehicle during rotation with respect to the center of gravity at constant angular velocity $\frac{u_m}{ls_m}$.

Consequently, it is possible to write that

$$\left.\begin{aligned} f_1 &= B_1 s - B_2, \\ f_2 &= B_3 s + B_4, \end{aligned}\right\} \tag{9.18}$$

where

$$B_1 = -\frac{u_m b'_a}{v s_m}, \quad B_2 = \frac{u_m b'_\lambda}{l s_m},$$
$$B_3 = -\frac{u_m}{l s_m \lambda_1},$$
$$B_4 = u_m \left[x_{1T} - \frac{1}{c_y^a} \int_0^l \frac{\partial c_y^a(x_1)}{\partial x_1} x_1 \, dx_1 \right] \frac{1}{l^2 \lambda_1 s_m}.$$

Since in this case the perturbing force is considered as a function of space coordinates s, then in the shown equations (16) and (17) it is expedient to replace independent variable t by s, using formulas

$$\begin{aligned} \Delta\dot{\vartheta} &= \frac{v}{l}\frac{d\Delta\vartheta}{ds}, \\ \Delta\ddot{\vartheta} &= \left(\frac{v}{l}\right)^2 \frac{d^2\Delta\vartheta}{ds^2} + \frac{\dot{v}}{l}\frac{d\Delta\vartheta}{ds}. \end{aligned} \tag{9.19}$$

Taking into account that the obtained system of equations is solvable relative to leading derivatives of functions entering it, let us represent it in the form of a system of linear first order differential equations in normal Cauchy form. For this let us introduce the following phase coordinates:

$$x_1 = \Delta\vartheta, \quad x_2 = \frac{d\Delta\vartheta}{ds}, \quad x_3 = \frac{dy}{ds},$$

$$\left.\begin{aligned} \frac{dx_1}{ds} &= x_2, \\ \frac{dx_2}{ds} &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + F_2, \\ \frac{dx_3}{ds} &= a_{31}x_1 + a_{33}x_3 + F_3, \end{aligned}\right\} \tag{9.20}$$

where

$$\left.\begin{aligned} &a_{21} = -b'_{\alpha}\frac{l^2}{v^2}, \quad a_{22} = -\frac{l}{v}\left(b'_{\lambda} + \frac{\dot{v}}{v}\right), \\ &a_{23} = -a_{21}\frac{1}{l}, \quad a_{31} = -c'_{\alpha}\frac{l^2}{v^2}, \\ &a_{33} = -\frac{l}{v^2}(c'_{\lambda} + \dot{v}), \\ &\qquad F_2 = f_1\frac{l^2}{v^2}, \quad F_3 = f_2\frac{l^2}{v^2}. \end{aligned}\right\} \tag{9.21}$$

Calculations show that in most cases coefficients a_{kn} ($k = 2, 3$; $n = 1, 2, 3$) are changed very little with respect to s. When $s < 20$ they can practically (with accuracy up to 5%) be considered constant.

The solution of this system of equations will be sought with the help of Laplace transformation [38], i.e., with the help of transformation of the function from variable s into function of variable p by formula

$$f(p) = \int_0^\infty f(s)e^{-ps}\,ds.$$

Application of a similar operation to all terms of the left and right sides of equations (20) permits reducing the linear differential equations to algebraic equations. Having obtained the solution of these algebraic equations in the form of some functions of p, let us then apply to it the operation of inverse transformation to variable s.

Having designated the representation of sought functions $x_k(s)$ through $x_k(p)$, let us write the corresponding system of representing equations at zero initial conditions in the form

$$px_k(p) = \sum_{n=1}^{3} a_{kn}x_n(p) + F_k(p) \quad (k = 1, 2, 3).$$

Its solution is determined by expression

$$x_n(p) = -\sum_{k=1}^{3} F_k(p)\frac{\Delta_{kn}(p)}{\Delta(p)}. \tag{9.22}$$

in which $F_k(p) \xrightarrow{\cdot} F_k(s)$ is the representation of function $F_k(s)$, and $\Delta(p)$ – determinate of this system, equal to

$$\Delta(p) = \begin{vmatrix} -p & 1 & 0 \\ a_{21} & a_{22}-p & a_{23} \\ a_{31} & 0 & a_{33}-p \end{vmatrix},$$

$\Delta_{kn}(p)$ – adjoint of the element of k-th line and n-th column of the given determinant. Let us formulate characteristic equation

$$-\Delta(p) = p^3 + e_1 p^2 + e_2 p + e_3 = 0, \tag{9.23}$$

where

$$\begin{aligned} e_1 &= -(a_{33} + a_{22}), \\ e_2 &= a_{22}a_{33} - a_{21}, \\ e_3 &= a_{21}a_{33} - a_{23}a_{31}. \end{aligned}$$

Let us assume that the roots of this equation are equal to

$$\begin{aligned} \lambda_1 &= -p_0, \\ \lambda_2 &= -p_1 + p_2 i, \\ \lambda_3 &= -p_1 - p_2 i. \end{aligned}$$

Representation of the angle of deflection of the axis of the flight vehicle will have the form

$$\Phi(p) = -\frac{1}{p\Delta(p)}(\bar{B}_1 + \bar{B}_2 p + \bar{B}_3 p^2).$$

Here

$$\left.\begin{aligned} &\bar{B}_1 = -\frac{l^2}{v^2}(a_{33}B_1 + a_{23}B_3), \\ &\bar{B}_3 = -\left(\frac{l}{v}\right)^2 B_2. \quad \bar{B}_2 = \frac{l^2}{v^2}(B_1 + a_{33}B_2 - a_{23}B_4). \end{aligned}\right\} \tag{9.24}$$

By using the second expansion theorem [38], we obtain that

$$\Delta\vartheta(p)=\frac{D_0}{p_0}+\frac{D_1}{p+p_0}+\frac{D_2 p+D_3}{p^2+2p_1 p+p_3^2}=\frac{\bar{B}_1+p\bar{B}_2+p^2\bar{B}_3}{p(p+p_0)(p^2+2p_1 p+p_3^2)}. \tag{9.25}$$

Values of constants D_k in this expression are determined by formulas (k = 0, 1, 2, 3)

$$\left.\begin{gathered} D_0=\frac{\bar{B}_1}{p_0 p_3^2},\quad D_1=\frac{p_0\bar{B}_2+\bar{B}_1+p_0^2\bar{B}_3}{p_0(p_0^2-2p_1p_0+p_3^2)},\\ D_2=-(D_0+D_1),\quad D_3=D_1(p_0-2p_1)+\bar{B}_3-2p_1D_0,\\ p_3^2=p_1^2+p_2^2. \end{gathered}\right\} \tag{9.26}$$

Hence

$$\begin{gathered}\Delta\vartheta(p)\div\!\!\to\Delta\vartheta(s)=\frac{D_1}{p_0}(1-e^{-p_0 s})+D_0 s+\\ +\frac{D_3}{p_3^2}(1-e^{-p_1 s}\cos p_2 s)+\frac{1}{p_2}e^{-p_1 s}\left(D_2-p_1\frac{D_3}{p_3^2}\right)\sin p_2 s. \end{gathered} \tag{9.27}$$

On the basis of equation (17) and formulas (18) and (19) the representation of additional angle of attack of the flight vehicle will be equal to

$$\Delta\alpha(p)=\frac{1}{p}(n_0-pn_1)-pn_3(p+n_2)\Delta\vartheta(p).$$

By placing expression $\Delta\vartheta(p)$, we obtain that

$$\Delta\alpha(p)=\frac{1}{p}(n_0-pn_1)+\frac{E_0+pE}{p+p_0}-\frac{E_1p+E_2}{p^2+2pp_1+p_3^2},$$

where

$$\begin{gathered} E_0=p_0(E_3+n_3\bar{B}_3),\\ E_1=\bar{B}_2n_3-E_0+n_2\bar{B}_3-2n_3p_1\bar{B}_3,\\ E_2=p_3^2E+\bar{B}_1n_2n_3\frac{1}{p_0}-n_3\bar{B}_3p_3^2,\quad E=n_1\bar{B}_1,\\ n_0=\frac{B_1}{b'_a},\quad n_1=\frac{B_2}{b'_a}=\frac{b_\text{д}u_m}{b_a l s_m},\quad n_2=\frac{v}{l}\left(b'_\text{д}+\frac{v}{v}\right)\frac{1}{b'_a},\\ n_3=\left(\frac{v}{l}\right)^2\frac{1}{b'_a},\quad E_3=n_3D_1(p_0-n_2). \end{gathered}$$

Thus,

$$-\Delta\alpha(s) = n_0 s - E(e^{-p_0 s} - 1) + \\ + \frac{1}{p_2} e^{-p_1 s}\left(E_1 - \frac{p_1 E_2}{p_3^2}\right)\sin p_2 s + \frac{E_2}{p_3^2}(1 - e^{-p_1 s}\cos p_2 s). \tag{9.28}$$

By using formulas (2.14) and (2.8) and equation (17), the coefficient of lateral overload of the center of gravity of the flight vehicle can be represented in the form

$$\Delta n^0_{y_1} = -\left(\frac{1}{g_0}\frac{d^2 y}{dt^2} + n_x \Delta\theta\right) \tag{9.29}$$

or through surface forces

$$\Delta n^0_{y_1}(s) = -\frac{1}{g_0} f_2(s) + \Delta\alpha(s)\left(\frac{c'_\alpha}{g_0} - n_x\right). \tag{9.30}$$

Rotary component of lateral overload will be equal to

$$\Delta n^z_{y_1}(x_1, s) = \frac{x_1 - x_{1T}}{g_0}\left[f_1 - b'_\alpha \Delta\alpha(s) - b'_z \frac{v}{l}\frac{d\Delta\theta(s)}{ds}\right], \tag{9.31}$$

where

$$\frac{d\Delta\theta(s)}{ds} = D_0 + D_1 e^{-p_0 s} + \frac{1}{p_1} e^{-p_1 s}(D_3 - p_1 D_2)\sin p_2 s + D_2 e^{-p_1 s}\cos p_2 s.$$

As an example let us consider loading of a ballistic type flight vehicle, which has a comparatively large reserve of aerodynamic stability. Graph of $\Delta\alpha(s)$ for this vehicle is presented on Fig. 9.2. In the same place for comparison there is given a graph of function $\Delta\theta(s)$. As can be seen, the difference of $\Delta\alpha$ from $\Delta\theta$ at small s is comparatively small. The influence of rotation of the body relative to the lateral axis, passing through the center of gravity, shows on the magnitude of $\Delta n^0_{y_1}$ only at comparatively large values of s_m.

Figure 9.3 shows the change with respect to s of the coefficient of lateral overload of the center of gravity of the flight vehicle. Maximum amount of this overload starts to decrease noticeably with increase of the gradient distance only when $s_m > 8$. When $s_m < 15$ $\Delta n^0_{y_1}$ reaches its maximum value when $s = s_m$ even when the wind velocity is constant at $s > s_m$. When $s_m > 15$ there is observed displacement of the position of $\max \Delta n^0_{y_1}$ relative to $s = s_m$ toward smaller values of s. The relationship of $\Delta n^0_{y_1}$ to s_m is determined mainly by the value of coefficient b'_α. With decrease of its magnitude the influence of lateral rotation of the flight vehicle on lateral overload $\Delta n^0_{y_1}$ drops and when $b'_\alpha = 0$ completely disappears.

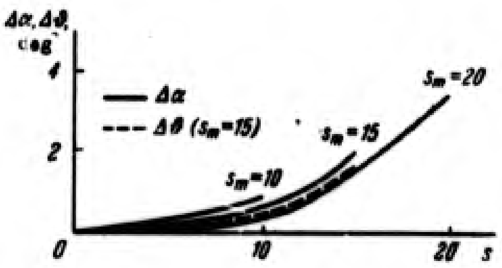

Fig. 9.2. Influence of transverse dimension of a gust of wind on the change of angle of attack of a flight vehicle.

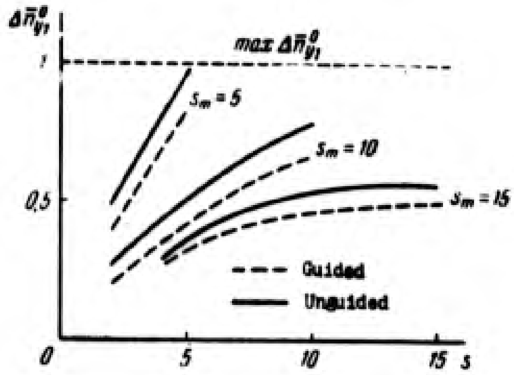

Fig. 9.3. Change of lateral overload of the center of gravity of aerodynamically stable guided and unguided flight vehicles in the process of passage through a gust of wind.

Thus, from this example we can make the conclusion that the effect of decrease of lateral overloads due to rotation of an unguided aerodynamically stable flight vehicle should be considered only when H/l is great. The considered case of loading is of practical interest for gliders, descending vehicles, separable nose cones and others.

§ 9.3. Influence of Automatic Control System

Let us now investigate the influence of parameters of an automatic control system (automatic pilot) on the reaction of the flight vehicle in case of the effect of a discrete gust of wind. In the preceding paragraph it was shown that at comparatively small values of s(s < 20) the effect of angle of inclination of the tangent to the trajectory on the magnitude of angle of attack of the flight vehicle in the first approximation can be disregarded. By using this result, let us represent the system of perturbation equations of a guided flight vehicle as a solid body in somewhat simplified form, namely without taking into account the interconnection with equation (17)

$$\Delta\ddot{\vartheta} + b'_{\text{д}}\Delta\dot{\vartheta} + b'_{\alpha}\Delta\vartheta + b'_{\delta}\delta = f_1, \tag{9.32}$$

$$\delta = F(\Delta\vartheta,\ \Delta\dot{\vartheta},\ \Delta\ddot{\vartheta},\ \delta,\ \dot{\delta},\ \ldots). \tag{9.32'}$$

The character of influence of the automatic pilot will naturally depend basically on the accepted law of control of motion of the vehicles. In this case the structure of the equation of control will determine in some sense the method of solution of the given problem. If this equation has the form

$$\delta = a_0\Delta\vartheta + a_1\Delta\dot{\vartheta} + a_2\Delta\ddot{\vartheta},$$

motion of the system will be described by one second order equation

$$\Delta\ddot{\vartheta} + b_{\text{д}0}\Delta\dot{\vartheta} + b_{\alpha 0}\Delta\vartheta = f_{10},$$

where

$$b_{\text{д}0} = \frac{b'_{\text{д}} + a_1 b'_{\delta}}{1 + a_2 b'_{\delta}},\quad b_{\alpha 0} = \frac{b'_{\alpha} + a_0 b'_{\delta}}{1 + a_2 b'_{\delta}},\quad f_{10} = \frac{f_1}{1 + a_2 b'_{\delta}},\quad b'_{\delta} = \frac{b_{\delta}}{I_z}.$$

The solution of this equation at zero initial conditions will be equal to

$$\Delta\vartheta = \frac{1}{\sqrt{b_{\alpha 0}}}\int_0^t f_{10}(\tau)\, e^{-\frac{b_{\text{д}0}}{2}(t-\tau)} \sin\sqrt{b_{\alpha 0}}\,(t-\tau)\, d\tau.$$

If the equation of control can be represented in the form

$$\delta = a_0 \Delta\vartheta + b_1 \dot{y},$$

for finding the reaction of the flight vehicle to a gust of wind it is possible to directly use the solution obtained in § 9.2, having changed only the expression of certain coefficients (21), and namely instead of a_{21}, a_{23}, a_{31} and a_{33} to take accordingly

$$a'_{21} = -\frac{l^2}{v^2}\left(b'_\alpha + a_0 \frac{b_\delta}{I_z}\right), \quad a'_{23} = \frac{l}{v^2}\left(b'_\alpha - \frac{b_1 b_\delta v}{I_z}\right),$$
$$a'_{31} = -\frac{l}{v}\left(c'_\alpha + a_0 \frac{c_\delta}{m}\right), \quad a'_{33} = -\frac{l}{v^2}\left(c'_\alpha + c_\delta \frac{v b_1}{m} + \dot{v}\right).$$

If in the equation of control it is impossible to disregard the terms containing derivatives of δ, for instance in case

$$a_2 \Delta\ddot{\vartheta} + a_1 \Delta\dot{\vartheta} + a_0 \vartheta - \delta - \tau_1 \dot{\delta} - \tau_2 \ddot{\delta} = 0, \tag{9.33}$$

it is necessary to solve a system of two equations (32) and (33). Assuming that this system is solvable relative to leading derivatives $\Delta\ddot{\vartheta}$ and $\ddot{\delta}$, let us reduce it to an equivalent system of first order equations. For this in these equations after carrying out the operation of replacement of independent variable t by s, let us introduce a new system of unknown functions:

$$x_1 = \Delta\vartheta, \quad x_2 = \frac{d\,\Delta\vartheta}{ds}, \quad x_3 = \delta, \quad x_4 = \frac{d\delta}{ds}.$$

Then the corresponding system of representing equations in expanded form will be

$$\left.\begin{aligned} -x_1(p)p + x_2(p) &= 0, \\ x_1(p)a_{21} + x_2(p)(a_{22} - p) + x_3(p)a_{23} &= -F_2(p), \\ -x_3(p)p + x_4(p) &= 0, \\ x_1(p)a_{41} + x_2(p)a_{42} + x_3(p)a_{43} + x_4(p)(a_{44} - p) &= -F_4(p), \end{aligned}\right\} \tag{9.34}$$

where

$$F_2(p) = \frac{l^2}{v^2} f_1(p),$$
$$F_4(p) = \frac{l^2}{v^2}\frac{a^2}{\tau_2} f_1(p),$$
$$a_{21} = -b'_\alpha \frac{l_2}{v^2},$$
$$a_{22} = -\frac{l}{v}\left(b'_A + \frac{\dot{v}}{v}\right), \quad a_{42} = -\frac{l}{v\tau_2}(a_2 b'_A - \rho_1),$$
$$a_{23} = -b'_\delta \frac{l^2}{v^2}, \quad a_{43} = -\frac{l^2}{v^2\tau_2}(1 + a_2 b'_\delta),$$
$$a_{41} = \frac{l^2}{v^2\tau_2}(a_0 - a_2 b'_\alpha), \quad a_{44} = -\frac{l}{v}\left(\frac{\tau_1}{\tau_2} + \frac{\dot{v}}{v}\right).$$

The approximate solution of it at zero initial conditions will be determined by expression

$$x_i(p) = -\sum_{k=1}^{4} F_k(p)\frac{\Delta_{ki}(p)}{\Delta(p)}.$$

Here $\Delta(p)$ – determinant of system (34).

By λ_i $(i = 1, 2, 3, 4)$ let us designate the roots of characteristic equation

$$\Delta(p) = p^4 + T_1 p^3 + T_2 p^2 + T_3 p + T_4 = 0, \tag{9.35}$$

coefficients of which are equal to

$$T_1 = \frac{l}{v}\left(b'_A + \frac{\tau_1}{\tau_2} + 2\frac{\dot{v}}{v}\right),$$
$$T_2 = \frac{l^2}{v^2}\left[\left(b'_A + \frac{\dot{v}}{v}\right)\left(\frac{\tau_1}{\tau_2} + \frac{\dot{v}}{v}\right) + b'_\alpha + \frac{1}{\tau_2}(1 + a_2 b'_\delta)\right],$$
$$T_3 = \frac{l^3}{v^3}\left[b'_\alpha\left(\frac{\tau_1}{\tau_2} + \frac{\dot{v}}{v}\right) + \left(b'_A + \frac{\dot{v}}{v}\right)\frac{1}{\tau_2} + \frac{b'_\delta}{\tau_2}\left(a_1 + a_2\frac{\dot{v}}{v}\right)\right],$$
$$T_4 = \frac{l^4}{v^4}(b'_\alpha + a_0 b'_\delta)\frac{1}{\tau_2}.$$

Since in this case the motion of flight vehicle is stable, equation (35) will have roots, the real parts of which are negative. According to Routh criterion this is fulfilled if the signs of all coefficients of equation (35) and discriminant $R = T_1T_2T_3 - T_1^2T_4 - T_3^2$ are identical. Let us assume that λ_1 and λ_2 – negative real roots, and λ_3 and λ_4 – conjugate complex roots:

$$\left.\begin{aligned} \lambda_3 &= -\alpha_0 - i\beta, \\ \lambda_4 &= -\alpha_0 + i\beta. \end{aligned}\right\} \tag{9.36}$$

Representation of the angle of displacement $\Delta\vartheta$ of longitudinal axis of the flight vehicle in this case will have the form

$$\Delta\vartheta(p)=\frac{l^2}{v^2\Delta(p)}f_1(p)(p^2+S_1p+S_2)=$$
$$=\frac{l^2}{v^2}\left[\frac{P_1p^2+P_2p+P_3}{p(p-\lambda_1)(p-\lambda_2)}-\frac{P_4p+P_5}{p^2+2\alpha_0p+k^2}\right].$$

Hence on the basis of the theorem of convolution

$$\Delta\vartheta(s)=\frac{l^2}{v^2}\left\{\frac{1}{\lambda_1^2\lambda_2^2}\left[P_2\lambda_1\lambda_2+P_3(\lambda_1+\lambda_2)\right]+\right.$$
$$+s\frac{P_3}{\lambda_1\lambda_2}+\frac{P_1\lambda_1^2+P_2\lambda_1+P_3}{\lambda_1^2(\lambda_1-\lambda_2)}e^{\lambda_1 s}+\frac{P_1\lambda_2^2+P_2\lambda_2+P_3}{\lambda_2^2(\lambda_2-\lambda_1)}e^{\lambda_2 s}-$$
$$\left.-\frac{e^{-\alpha_0 s}}{\beta}\left(P_4-\frac{\alpha_0}{k^2}P_5\right)\sin\beta s-\frac{P_5}{k^2}(1-e^{-\alpha_0 s}\cos\beta s)\right\}, \tag{9.37}$$

where

$$\left.\begin{aligned}
&P_1=P_4,\ P_3=B_1S_2\frac{1}{k^2},\\
&S_2=\frac{l^2}{v^2}\tau_2,\ k^2=\alpha_0^2+\beta^2,\\
&P_5=P_1\beta_1+P_2+B_2,\ \beta_1=2\alpha_0+\lambda_1+\lambda_2,\\
&P_2=\frac{1}{k^2-\lambda_1\lambda_2}\left[B_2(\lambda_1\lambda_2-S_2)+B_1\left(S_1-2\alpha_0\frac{S_2}{k^2}\right)+P_1\lambda_1\lambda_2\beta_1\right],\\
&S_1=\frac{l}{v}\left(\frac{\tau_1}{\tau_2}+\frac{\dot{v}}{v}\right)=-a_{44},\\
&P_1=\frac{1}{D}\left[B_1-\frac{\beta_1}{k^2-\lambda_1\lambda_2}\left(S_1-2\alpha_0\frac{S_2}{k^2}\right)\right]-\\
&\qquad-\frac{1}{D}B_2\left[S_1+\lambda_1+\lambda_2+\frac{\beta_1}{k^2-\lambda_1\lambda_2}(\lambda_1\lambda_2-S_2)\right],\\
&D=k^2-\lambda_1\lambda_2+(\lambda_1+\lambda_2)\beta_1+\frac{\lambda_1\lambda_2}{k^2-\lambda_1\lambda_2}\beta_1^2.
\end{aligned}\right\} \tag{9.38}$$

The angle of displacement of vanes δ in function s is determined analogously:

$$\delta(p)=\frac{l^2}{v^2\tau_2}f_1(p)\frac{1}{\Delta(p)}(p^2+S_3p+S_4)a_2,$$
$$\delta(s)=\frac{a_2}{\tau_2}\Delta\vartheta(s). \tag{9.39}$$

Only in this case during calculation of all coefficients P_i $(i=1,2,\ldots,5)$ should we replace S_1 and S_2 by S_3, S_4 respectively

$$S_3 = \frac{l}{v}\left(\frac{a_1}{a_2} + \frac{\dot{v}}{v}\right),$$

$$S_4 = \frac{l^2}{v^2}\frac{a_0}{a_2}. \qquad (9.40)$$

The magnitude of lateral overload in the center of gravity of the flight vehicle in the considered case (i.e., taking into account the reaction of automaton of stabilization) is found just as in the preceding paragraph. On the basis of equation

$$\ddot{y} + c'_g\frac{\dot{y}}{v} + c'_\alpha\Delta\vartheta + c'_\delta\delta = f_2$$

and formula

$$\Delta n^0_{y_1} = -\left(\frac{\ddot{y}}{g_0} + n_x\Delta\alpha\right)$$

we obtain that approximately (when $\Delta\alpha \approx \Delta\vartheta$)

$$\Delta n^0_{y_1} = \frac{1}{g_0}(c'_\delta\cdot\delta - f_2) + \left(\frac{c'_\alpha}{g_0} - n_x\right)\Delta\alpha. \qquad (9.41)$$

Thus, lateral loads, caused by a gust of wind for a guided flight vehicle are functions of not only its inertial and aerodynamic properties, but also parameters of stabilization automaton. In this case the influence of coefficients of equation of control on $\Delta n^0_{y_1}$ basically depends on the aerodynamic and weight arrangement of the flight vehicle, and namely on the relative location of the center of pressure and center of gravity. For an aerodynamically stable guided flight vehicle the lateral overload in the center of gravity will be less than for unguided. This decrease of the magnitude of $\Delta n^0_{y_1}$ is explained basically by the presence of control force R_{y1}, directed opposite the direction of action of perturbing force. Therefore, in many cases it can turn out to be comparatively small, especially for heavy flight vehicles. Calculations conducted for ballistic type "Fau-2" flight vehicle (when $a_0=2{,}5$; $a_1=0{,}49\,a_0$; $a_2=0{,}34\,a_0$; $\tau=0{,}4$) show that for all $s_m < 20$ the magnitude of $\Delta n^0_{y_1}$ is reduced 10-17%.

For aerodynamically unstable flight vehicles the opposite picture is observed, i.e., the actual value of lateral overload of the center of gravity is larger than fictitious (8), corresponding to the instantaneous action of a gust of wind. This occurs not only due to the composition of control and perturbing forces directed to one side, but also due to some increase of the angle of attack of the flight vehicle relative to incident air flow. For illustration on Fig. 9.4 there are graphs of the change of lateral overload with respect to s and s_m for hypothetical aerodynamically unstable ballistic type flight vehicle similar in characteristics to the "Fau-2." In the same place there are graphs of function $\Delta\vartheta(s)$ for $s_m = 10$ and $s_m = 20$. All calculations are carried out for $\bar{t} = 0.43$ when $u_m = 30$ m/s, and $a_0 = 10,\ a_1 = a_0,\ a_2 = \frac{1}{2}a_0,\ \tau_1 = 0{,}8,\ \tau_2 = 0{,}1$. As can be seen, due to the increase of δ and $\Delta\vartheta$ the magnitude of $\Delta n^0_{y_1}$ increases with growth of s_m, where coefficient τ_2 has a noticeable influence on growth. From this it follows that it is possible to disregard the term proportional to $\ddot{\delta}$ in equation (33) only at small values of τ_2 and small s_m (when $\tau_2 = 0.14$ and $s_m = 10$ the error will be on the order of 10%, and when $s_m = 20$ – around 40%). Coefficients a_1 and a_2 comparatively weakly affect the magnitude of lateral overload of the center of gravity of the flight vehicle. At small s_m the increase of a_2 is accompanied by some decrease of $\Delta n^0_{y_1}$. When $s_m > 10$ for aerodynamically stable vehicles, conversely, there is observed increase of $\Delta n^0_{y_1}$ with growth of a_2. Increase of the static amplification factor of the control system can lead to some decrease of overload (for aerodynamically stable flight vehicles at small s_m, and for aerodynamically unstable – at large s_m). It is obvious that decrease of the degree of aerodynamic instability (with preservation of constancy of value of c^α_y) will always be accompanied by a lowering of lateral overloads.

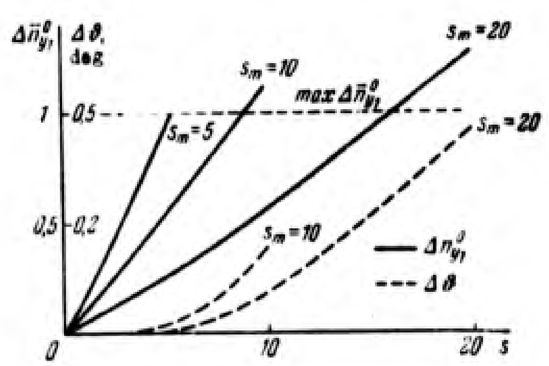

Fig. 9.4. Change of lateral overload in the center of gravity and increase of pitch angle of aerodynamically unstable guided flight vehicle in the process of passage through a gust of wind.

As can be seen from Fig. 9.5, the change of lateral overload of the center of gravity of the flight vehicle with respect to s is analogous to change of the velocity of a gust of wind. Change of control forces (when the system is dynamically stable) occurs with some delay, and usually max δ does not correspond to max u, and consequently also max $\Delta n^0_{y_1}$.

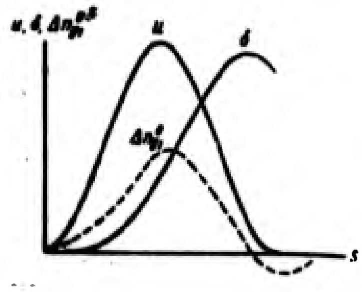

Fig. 9.5. Phase shift between control force and external perturbation (gust of wind).

It is necessary once again to emphasize that equation (33) only approximately describes the work of the control system. If it is necessary to calculate the derivatives of δ of higher orders, then for solution of the considered problem one should use other methods, for instance the method of numerical integration or method of modeling. Conversely, in examining the static influence of wind on a flight vehicle with comparatively small vertical shifts of velocities (large H) it is possible in the first approximation to replace the differential equation of control (33) by linear algebraic equation

$$\delta = a_0 \Delta\vartheta. \tag{9.42}$$

In this case, according to formula (15), the true wind load on the flight vehicle will be determined by expression $k_a f_2$. By introduction of coefficient k_a (12) we as if compensate those errors which are caused by disregarding the influence of lateral motion of the center of gravity of the flight vehicle. Having placed $\Delta\dot{\vartheta} = \Delta\ddot{\vartheta} = 0$ in equation (32) and multiplied F_2 by k_a, we will find

$$\Delta\vartheta = \frac{k_a f_1}{b'_a + a_0 b'_\delta}, \quad \delta = \frac{a_0 k_a f_1}{b'_a + a_0 b'_\delta}. \tag{9.43}$$

Having substituted these values of $\Delta\vartheta$ and δ, and also the values of coefficients $c_{\alpha}^{\prime}, c_{\delta}^{\prime}, f_1$ and n_x in expression (41) and conducting certain transformations, we obtain the following approximate formula for lateral overload of the center of gravity of a flight vehicle (for considered case of static loading):

$$\Delta n_{y_1}^{0}=\frac{a_0 R_{y_1}^{\delta} u k_{\text{в}}\,(x_{1\text{р}}-x_{1\text{д}})}{G_0\,(1-\bar{t})\,(x_{1\text{д}}-x_{1\text{т}})\,v\,(1+b)}, \tag{9.44}$$

where

$$b=\frac{a_0 R_{y_1}^{\delta}\,(x_{1\text{р}}-x_{1\text{т}})}{qSc_y^{\alpha}\,(x_{1\text{д}}-x_{1\text{т}})}. \tag{9.45}$$

From this formula it is clear that for an aerodynamically unstable flight vehicle ($b < 0$) the magnitude of coefficient of lateral overload will be larger than for aerodynamically stable, for which $b > 0$.

Figure 9.6 shows graphs of envelope values max $\Delta n_{y_1}^{0}$, calculated for the averaged wind velocities shown in Fig. 1.4, when $\frac{du}{dh}=0{,}02$ for two values of static amplification factor a_0 and $2a_0$ when $b < 0$ and $b > 0$. These graphs show that increase of a_0 when $b > 0$ leads to an insignificant increase of $\Delta n_{y_1}^{0}$, and when $b < 0$ conversely.

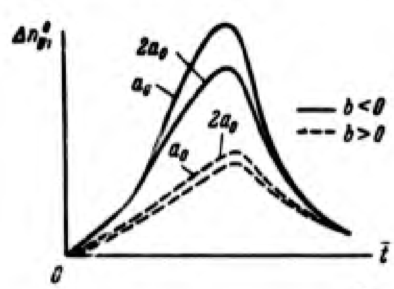

Fig. 9.6. Envelope of maximum values of lateral overload of the center of gravity of aerodynamically unstable flight vehicles from the influence of steady wind.

§ 9.4. Appraisal of the Influence of Elastic Oscillations of the Construction

Since usually the periods of natural elastic oscillations of the constructions of flight vehicles are comparatively small, the role of dynamic components of forces will be noticeable only with the influence of gusts of wind with large gradients of velocity (small H) on the construction. In a similar case for appraising the influence of elastic oscillations it is possible to be limited by a simplified formulation of the problem, in particular it is possible to consider all coefficients of equations of elastic oscillations of the construction as constants (for almost all types of flight vehicles). It is possible in the first approximation to disregard the influence of control forces, and the change of angles of attack due to rotation of the flight vehicle relative to the lateral axis, passing through its center of gravity. In principle the influence of these factors is simple to consider, if we preliminarily solve the static problem of the reaction of the flight vehicle as a solid body to the examined gust of wind.

The shown assumptions usually do not introduce large errors to results of calculation of dynamic forces, but noticeably simplify computations, which has a definite value at the stage of sketch designing. In many cases the change of control force in the process of impulsive load of the flight vehicle occurs with comparatively large delay. Thus, this force reaches maximum value already upon exiting the vehicle from the zone of action of the gust of wind. Angles of deflection of axis of the vehicle are analogously changed. Usually the influence of R_y, and $\Delta\alpha$ increases with increase of gradient distance H, i.e., with decrease of the influence of elastic oscillations of the construction.

Thus, in the considered case of loading the forced transverse oscillations of the construction of the flight vehicle can be approximately described (depending upon its assembly diagram) either by one differential equation (4.23), or a system of differential equations of form (4.43)-(4.47). Solution of these equations with known normal forms of natural transverse oscillations of the construction is reduced to solution of ordinary second order differential equation

$$\ddot{S}_n + 2h_n\dot{S}_n + \omega_n^2 S_n = H_n \qquad (9.46)$$
$$(n = 1, 2, \ldots, n_0)$$

with corresponding initial conditions. The latter are found by means of expansion of initial values of functions of deflections and velocities of points of construction of the flight vehicle $y_1(x_1, t_0)$ and $\dot{y}_1(x_1, t_0)$ according to the forms of natural transverse oscillations with weight $m(x_1)$

$$\left.\begin{aligned} y_1(x_1, t_0) &= \sum_{n=1}^{\infty} S_{n0}(t_0) f_n(x_1) m(x_1), \\ \dot{y}_1(x_1, t_0) &= \sum_{n=1}^{\infty} \dot{S}_{n0}(t_0) f_n(x_1) m(x_1). \end{aligned}\right\} \tag{9.47}$$

As a result of fulfillment of this procedure, we obtain

$$S_{n0}(t_0) = \frac{1}{M_n} \int_0^l y_1(x_1, t_0) f_n(x_1) dx_1, \tag{9.48}$$

$$\dot{S}_{n0}(t_0) = \frac{1}{M_n} \int_0^l \dot{y}_1(x_1, t_0) f_n(x_1) dx_1 \tag{9.49}$$

$$(n = 1, 2, \ldots).$$

Solution of the equation describing free oscillations of the construction

$$\ddot{S}_{n0} + 2h_n \dot{S}_{n0} + \omega_n^2 S_{n0} = 0 \tag{9.50}$$

in this case will have the form

$$S_{n0}(t) = e^{-h_n t} \left[S_{n0}(t_0) \cos \omega_n t + \frac{\dot{S}_{n0}(t_0)}{\omega_n} \sin \omega_n t \right]. \tag{9.51}$$

For finding the function of S_n, determining forced elastic oscillations of the construction, let us decompose the external lateral load, affecting the flight vehicle in the considered case of loading:

$$\begin{aligned} q_{\text{в}}(x_1, t) = qS \frac{\partial c_y^{\alpha}(x_1)}{\partial x_1} \left[k_{\text{в}} \frac{u(x_1, t)}{v} + \Delta\vartheta(t) \right] + \\ + m(x_1) g_0 \Delta n_{y_1}(x_1, t) + R_{y_1}(t) \delta(x_1 - x_{1\text{p}}), \end{aligned} \tag{9.52}$$

where $\delta(x_1 - x_{1p})$ – Dirac-delta function, also with respect to normal forms of natural oscillations $f_n(x_1)$ with weight $m(x_1)$:

$$q_a(x_1, t) = \sum_{n=1}^{\infty} H_n(t) f_n(x_1) m(x_1), \tag{9.53}$$

where

$$H_n(t) = \frac{1}{M_n} \int_0^l q_a(x_1, t) f_n(x_1)\, dx_1. \tag{9.54}$$

Since

$$\int_0^l m(x_1) f_n(x_1)\, dx_1 = 0,$$

then, by substituting (52) in (54), we obtain

$$H_n(t) = \frac{1}{M_n} qS \left[a_n(t) \frac{k_n}{v} + \Delta\Phi(t) \int_0^l f_n(x_1) \frac{\partial c_y^a(x_1)}{\partial x_1}\, dx_1 \right] + \frac{1}{M_n} R_{y_1}(t) f_n(x_{1p}). \tag{9.55}$$

At small s_m, i.e., large l, the wind velocity is impossible to consider constant along the length of body of the flight vehicle and during calculation of integral

$$a_n(t) = \int_0^l u(z) \frac{\partial c_y^a(x_1)}{\partial x_1} f_n(x_1)\, dx_1, \tag{9.56}$$

where $z = vt - x_1$, one should consider the change of local angle of attack $u(x_1)/v$ in the function of time or path vt of the vehicle through a gust of wind.

In many cases the relationship of integral (56) to t for the lowest tones (n = 1; 2) can be approximately approximated by equation

$$a_n(t) = \frac{a_{n0}}{2}\left(1 - \cos\frac{\pi v t}{H}\right), \tag{9.57}$$

where by a_{n0} there is designated the biggest value of this integral on section $0<t<\frac{2H}{v}$. In the same form it is possible to represent the change of static bending moment with respect to t

$$\Delta M_c(x_1, t)=\frac{1}{2}\max\Delta M_c(x_1)\left(1-\cos\frac{\pi vt}{H}\right). \tag{9.58}$$

The latter in this case is determined by expression

$$\Delta M_c(x_1, s)=\frac{qS}{v}\left[\int_0^{x_1}\int_0^{x_1}u(sl-x_1)\frac{\partial c_{y_1}^{\alpha}(x_1)}{\partial x_1}dx_1\,dx_1-\right.$$
$$\left.-c_{y_0}M^0(x_1)\frac{1}{G}+\frac{1}{J_z}(x_{1\text{т}}-x_{1\text{д}})\int_0^{x_1}\int_0^{x_1}m(x_1)(x_1-x_{1\text{т}})dx_1\,dx_1\right], \tag{9.59}$$

where

$$x_{1\text{т}}-x_{1\text{д}}=\int_0^l(x_{1\text{т}}-x_1)u(sl-x_1)\frac{\partial c_{y_1}^{\alpha}(x_1)}{\partial x_1}dx_1,$$

$$c_{y_0}=\int_0^l u(sl-x_1)\frac{\partial c_{y_1}^{\alpha}(x_1)}{\partial x_1}dx_1,$$

and $u(sl-x_1)$ – value of velocity of gust in the current section of the flight vehicle body. Calculation of change of wind velocity along the length of the flight vehicle leads in some cases to noticeable lowering of bending moments, since not only the value of coefficient $a_n(t)$ is decreased, but the magnitude of aerodynamic perturbing moment, affecting the vehicle is essentially changed. For example on Fig. 9.7 there are presented graphs of functions a_n when n = 1 and $x_{1\text{д}}$ from $\left(\frac{vt}{H}\right)$ when s_m = 1.2. It is clear that by intersecting the gust of wind, the flight vehicle at first becomes statically unstable ($\bar{x}_{1\text{д}}<1$), and then statically stable.

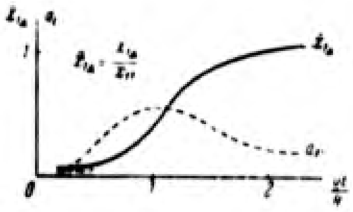

Fig. 9.7. Change of position of the center of pressure and coefficient of aerodynamic load of the flight vehicle in the process of passage through a gust of wind.

By using formulas (57) and (55), from equations (50) and (46) we can find the values of components of accelerations of the point of reduction $\ddot{S}_n$ and $\ddot{S}_{n0}$, and by formulas (4.31) and (4.32) calculate the dynamic components of lateral force and bending moment.

For obtaining a general concept of the influence of elastic oscillations of the construction on the character of loading of a flight vehicle during flight in restless air it is possible in the first approximation to be limited by consideration of only upper appraisals of $\ddot{S}_n$, obtained without taking into account damping terms, namely

$$\ddot{S}_n = \frac{qSa_{n0}k_g}{2vM_n}\left\{\cos\omega_n t + \frac{1}{1-\left(\frac{\omega_n}{\omega}\right)^2}\left[\left(\frac{\omega_n}{\omega}\right)^2\cos\omega_n t - \cos\omega t\right]\right\}, \tag{9.60}$$

where

$$\omega = \frac{\pi v}{2H}.$$

Moreover, in the majority of cases it is possible to take only the first terms of series (47). Usually with an increase of the order of index n the values of $\ddot{S}_n$ and $\ddot{S}_{n0}$ are decreased.

In a number of cases, especially during fulfillment of design calculations of various modifications of the same construction of flight vehicle, it is convenient to use the idea of dynamic coefficient. Although it is a function of coordinate x_1, its determination in similar cases has some practical value. The magnitude of this coefficient, equal to the ratio of total bending moment to the biggest value of static bending moment

$$\eta(x_1, t) = \frac{\Delta M_c(x_1, t) + M_g(x_1, t)}{\max \Delta M_c(x_1, t)}, \tag{9.61}$$

graphically characterizes the degree of the influence of elastic oscillations of the construction. Its value depends on the gradient distant H and frequence ω_n of natural transverse oscillations of the construction. The magnitude of gradient distance, corresponding to

$$H_{кр} = \frac{3\pi v}{4\omega_1}, \qquad (9.62)$$

will be critical for the construction of the flight vehicle. Figure 9.8 shows graphs of functions η, M, $M_д$, ΔM_c of $\frac{\omega}{\omega_n}$ for n = 1. It is clear that M and η reach maximum values with ω, close to ω_1. The character of change of dynamic coefficient with respect to vt/H is similar to change of function $M(vt/H)$. It reaches the biggest value in the region close to max u.

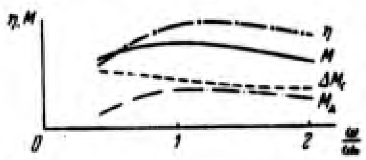

Fig. 9.8. Change of dynamic coefficient and components of bending moment from $\omega = \frac{\pi v}{2H}$.

§ 9.5. Selection of Calculation Scheme

Thus, we clarified that the values of lateral forces, bending moments, angles of attack and lateral overloads during flight of the flight vehicle in the zone of restless air on the powered-flight phase depend on very many factors: on atmospheric composition, parameters of the trajectory, inertial and dynamic characteristics of the vehicle, parameters and type of control system, aerodynamic arrangement of the flight vehicle and its purpose. If the flight vehicle is equipped with a control system of lateral motion of the center of gravity, then it is possible to expect an increase of lateral loads due to increase of the damping factor of gust of wind $k_в$.

If the control system reacts to overload of the center of gravity of the flight vehicle, but not to angles of rotation of sections of its body, then, conversely, there can be obtained substantial lowering of static values of lateral forces and bending moments, corresponding to the influence of steady flows of wind on the flight vehicle. From this point of view a system carrying out control of motion of the flight vehicle by means of mismatch of thrust force in opposite located combustion chambers of combined engines, possesses a certain advantage. When the flight vehicle is statically unstable, its application leads to some lowering of lateral overloads (basically due to the absence of lateral component of control force). Many of the above-indicated factors are continuously changed during the entire time of flight of the flight vehicle in dense layers of the atmosphere. On one phase of the trajectory the flight vehicle can be statically unstable, on another – neutral, and on a third – statically stable.

The effect of the influence of atmospheric turbulence, gusts of wind and steady horizontal flows of wind on the construction in many respects depends on the magnitude of vertical velocity component of the center of gravity of the flight vehicle. Thus, for aircraft, the velocities of vertical motion of which are low, vertical gusts of wind, parameters of which (u_m and H) are normalized, are dangerous from the point of view of static strength of the construction. Although the time the aircraft stay in the zone of restless air is calculated in tens and hundreds of hours, the influence of small atmospheric turbulence is of interest mainly for fatigue strength.

Usually the calculations of reaction of the construction to the influence of turbulence is reduced to determination of transfer functions of the flight vehicle to input signal, represented by a sinusoidal gust of wind of unit amplitude (see Chapter VI). For comparatively low horizontal velocities of flight these transfer functions are found taking into account the delay of increase of lift (caused by aerodynamic inertial effects), which leads to a lowering of lateral overloads by 10-20%. In proportion to increase of flight speed of the vehicle the influence of the shown unsteadiness of aerodynamic forces drops and when M > 1.6 it becomes insignificant. Functions of the spectral density of energy and for the considered case of flight are described by formulas such as (6.4) and (6.5), only other mean square values of pulsation of flow rate σ_u and scales of turbulence L. As in case V, these parameters depend on the flight altitude and especially on local meteorological conditions. Many researchers recommend taking minimum values of turbulence scale, on the order of 300 m, for flight cases of loading. It is natural that with such large values of L, considerably exceeding the characteristic dimensions of flight vehicles (length of wing chord, length of the body), it is possible to disregard space nonuniformity of turbulence and consider the spectral density of its energy only in a function of the frequency of oscillations and speed of the flight vehicle.

For flight vehicles possessing large vertical components of velocity, the time of flight in the zone of turbulence, thickness of which does not exceed several kilometers, is measured in seconds. Therefore, the dynamic reaction of the construction in this case will have a clearly expressed unsteady character and will not essentially affect the static and fatigue strength. Since usually for flight vehicles of the shown type the region of high impact pressures coincides in altitude to the region of existence of jet streams, then the influence of the latter will determine the necessary bearing capacity of their construction in the considered case of loading. There are two approaches to the solution of the given problem of such flight vehicles. The first is based on the use of actual wind profiles, obtained experimentally for the assumed region of launch. Calculation of the possible reaction of the construction in this case is conducted by means of numerical integration of the total system of differential equations, describing undisturbed and disturbed motion of the flight vehicle on the guidance section (from the moment of launch), for series of realizations of function u(h), corresponding to the most windy days. The second approach is reduced to the use of conditional (normalized) profiles of wind flow, i.e., to normalization of the required bearing capacity of the construction. In this case the possibility of safe flight of the flight vehicle is determined by comparison of basic parameters (maximum average wind velocity max u_0 and gradient $\frac{du_0}{dh}$) of true wind flow in the zone of launch with normalized. It is obvious that the necessity of preliminary sounding of the atmosphere and necessary accuracy of measurement of the shown parameters will depend on the value of u_0 and $\frac{du_0}{dh}$, accepted as calculated.

Calculation of the reaction of the construction of any flight vehicle to normalized profile of external influence can be performed with sufficient accuracy (considering the conditional character of initial data) by the above-stated method. For nonmaneuvering flight vehicles of ballistic type the calculation scheme will have the following form.

At each point of the trajectory the values of $M(x_1)$ and $Q(x_1)$ are represented in the form of the sum of static lateral forces and bending moments, caused by the action on the flight vehicle of steady wind (with maximum average velocity max u_0 and vertical shift $\left[\frac{du_0}{dh}\right]_{\text{средн}}$), and dynamic components of lateral forces and bending moments, caused by deviation of the true wind velocity Δu from average u_0 (Fig. 9.9). Parameters of this pulsation of wind velocity are characterized by coefficient of gustiness k and magnitude of shift $\frac{d\Delta u}{dh}$. Magnitudes of the latter depend on the purpose of the flight vehicle and meteorological conditions in the proposed regions of flight. Tentatively

$$\left[\frac{du_0}{dh}\right]_{\text{средн}} = 0{,}01 + 0{,}0003u_0,$$

$$\max\left(\frac{d\Delta u}{dh}\right) \leqslant 0{,}1\ \frac{1}{s},$$

$$\max \Delta u = (k-1)\,u_0 = \frac{u_m}{\sqrt{\Delta_0}}.$$

Moreover in the first approximation one may assume that isolated profiles of steady wind and "gust of wind" are described by one law, but with different H:

$$H = \frac{\max u_0(h)}{\frac{du_0}{dh}}, \qquad H = \frac{\Delta u}{\left(\frac{du}{dh}\right)_{\max}},$$

and that the axes of these profiles coincide (Fig. 9.9), although similar coincidence does not always give maximums of bending moments. Initial conditions in this case of loading, in particular, can be determined by the reaction of the flight vehicle to the influence of atmospheric turbulence for the lower boundary of jet stream.

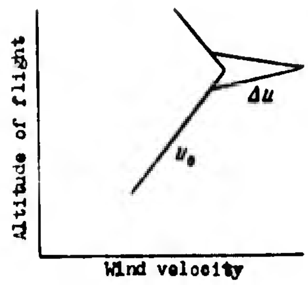

Fig. 9.9. Calculated profile of wind flow.

During flight outside the jet stream the parameters of steady wind max $u_0(h)$ and $\frac{du_0}{dh}$ can be taken in accordance with normalized law of change of wind velocity with respect to altitude, and parameters of the gust of wind in accordance with the accepted coefficient of gustiness, but not less than u_m = 10-15 m/s,

$$H=\begin{cases} H_{\text{кр}} & \text{when} \quad H_{\text{кр}} \geqslant \frac{3u_m}{\sqrt{\Delta_0}}, \\ 3\frac{u_m}{\sqrt{\Delta_0}} & \text{when} \quad H_{\text{кр}} < \frac{3u_m}{\sqrt{\Delta_0}}. \end{cases}$$

In both cases the crew of static values of $Q_c(x_1)$ and $M_c(x_1)$ is performed in the following sequence:

1. In accordance with specified graph of distribution of u_0 with respect to h there is selected (for specific flight altitude) the maximum value of average velocity of wind, then by specified amount of vertical shift of this velocity there are determined H and s_m.

2. By formula (12) there is found the damping factor of velocity of wind $k_{\text{в}}$.

3. There are determined (when k_u^0 = 1 and $k_{\text{н}}^l = 1$) coefficient of lateral overload of the center of gravity of the flight vehicle $n_{y_1}^0$ and angle of attack α;

4. There are calculated unit values of lateral forces and bending moments $Q_a(x_1)$, $M_a(x_1)$, $Q^0(x_1)$, $M^0(x_1)$, $Q^x(x_1)$, $M^x(x_1)$;

5. And, finally, by formulas (3.15) there are found $Q_c(x_1)$ and $M_c(x_1)$.

Calculation of lateral forces and bending moments, caused by the action of a gust of wind, is performed in two stages: first there are calculated static components of lateral force and bending moment, corresponding to maximum velocity of a gust of wind and selected gradient distance H, and then dynamic. Calculation of $\Delta Q_c(x_1)$ and $\Delta M_c(x_1)$ is performed by the above-mentioned scheme, but taking into account the influence of mobility of liquid in fuel tanks.

Calculation of $Q_{\text{д}}(x_1, t)$ and $M_{\text{д}}(x_1, t)$ is performed in the following order

1) frequencies and forms of natural bending oscillations of the construction of lowest tones are calculated;

2) from equations (46) and (50) the values of accelerations $\ddot{S}_n$ and $\ddot{S}_{n0}$;

3) then, by using expressions (4.31) and (4.32) functions $Q_{\text{д}}(x_1, t)$ and $M_{\text{д}}(x_1, t)$ are determined.

If the length of the body of the flight vehicle is great, then calculation of static and dynamic components of bending moment is recommended by taking into account the change of velocity of the gust of wind along the length of the body.

During investigation of loading of the construction of the flight vehicle with side-mounting boosters $Q_c(x_1)$ and $M_c(x_1)$ are calculated for each unit separately (taking into account the corresponding support reactions).

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Generally the vertical profile of wind has a complicated form. Therefore, the calculation scheme, providing for calculation of only one gust of wind, is very simplified. For all practical purposes on an averaged wind profile there is imposed a series of [illegible] of various intensity, located at distances on the order [illegible] km (along the vertical) from each other. In certain cases the effect of the influence of a series of such gusts of wind on the construction of the flight vehicle is considered by the introduction of a conditional cyclic gust of wind into the calculation scheme.

In conclusion let us note that for nonmaneuvering flight vehicles the wind loads in the region of high impact pressures usually determine the required values of controlling moments, i.e., required maximum angles of deflection of control devices. The values of these moments for maneuvering flight vehicles are often established by the required magnitudes of lateral overloads. With use of air vanes as basic control devices the required maximums of angles of their deflection will be determined by maneuvers in the region of low impact pressures, if, of course, according to the purpose of the vehicle at large pressures there are not required "sharper" maneuvers. As a result the available lateral overloads for such flight vehicles in the zone of high impact pressures can considerably exceed required and for all practical purposes will be limited only by the power of control drive. It is obvious that the influence of wind loads on the required bearing capacity of the construction of such class of flight vehicles will not have a decisive value. Therefore, during investigation of the dynamic reaction of their construction in the case of a sharp programmed maneuver (with specified law of change of angle of deflection of vanes to maximum value) it is possible to be limited by the calculation of the influence of only steady wind flows.

C H A P T E R X

EQUATIONS OF TRANSVERSE OSCILLATIONS OF THE CONSTRUCTION OF A FLIGHT VEHICLE

§ 10.1. Equations of Transverse Oscillations of a Vehicle Taking into Account the Elasticity of Its Construction and the Mobility of Liquid in Fuel Tanks

If the parameters of motion of a flight vehicle are rapidly changed in time, for determination of the reaction of its construction to an external influence it is necessary to use the method of numerical integration of a system of perturbation equations with variable coefficients. In this case there sometimes appears the necessity of more precise definition of the equations of motion themselves, in particular the calculation of the mutual influence of elastic oscillations of the construction and oscillations of liquid in fuel tanks of the lowest tones. Similar more precise definition is simple by using the method presented in Chapter VI.

Let us write the conditions of dynamic equilibrium of the construction of a flight vehicle in form of Lagrange equations, representing the influence of liquid on parameters of its motion in the form of an external surface load. In this case the lateral component of velocity of any point of the construction of the flight vehicle can be expressed by formula (Fig. 10.1)

$$\dot{y}(x_1, t) = \dot{y}_{\text{т}}(t) + \Delta\dot{\vartheta}(x_{1\text{т}} - x_1) + \sum_{n=1}^{n_{\text{в}}} \dot{S}_n(t) f_n(x_1), \qquad (10.1)$$

where y_T – lateral displacement of the flight vehicle center of gravity, and $f_n(x_1)$ – normal form of natural bending oscillations of the flight vehicle construction with solidified fuel.

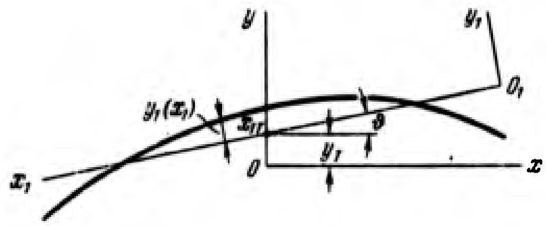

Fig. 10.1. Components of lateral shift of points of the longitudinal axis of flight vehicle during plane perturbed motion.

According to Chapter IV the kinetic energy of the vehicle construction without taking into account liquid filling is determined by expression

$$T_0(t)=\frac{1}{2}\Big\{M_{\text{к}}\dot{y}_T^2(t)+J_{\text{к}z}\Delta\dot{\vartheta}^2(t)+\int_0^l m_{\text{к}}(x_1)\Big[\sum_{n=1}^{n_n}\dot{S}_n(t)f_n(x_1)\Big]^2dx_1+$$
$$+2\dot{y}_T(t)\sum_{n=1}^{n_n}\dot{S}_n(t)\int_0^l m_{\text{к}}(x_1)f_n(x_1)\,dx_1+$$
$$+2\Delta\dot{\vartheta}(t)\sum_{n=1}^{n_n}\dot{S}_n(t)\int_0^l m_{\text{к}}(x_1)(x_{1T}-x_1)f_n(x_1)\,dx_1+$$
$$+2\dot{y}_T(t)\Delta\dot{\vartheta}(t)\int_0^l m_{\text{к}}(x_1)(x_{1T}-x_1)\,dx_1\Big\}, \qquad (10.2)$$

and potential – by expression (4.6). Here $m_{\text{к}}(x_1)$ – linear mass of the construction (without fuel),

$$M_{\text{к}}=\int_0^l m_{\text{к}}(x_1)\,dx_1,$$

$$J_{\text{к}z}=\int_0^l m_{\text{к}}(x_1)(x_{1T}-x_1)^2\,dx_1.$$

Having taken functions $S_n(t)$, $y_{\text{т}}(t)$ and $\Delta\vartheta(t)$ as generalized coordinates, we obtain the following system of equations, describing transverse oscillations of an elastic flight vehicle with liquid filling in flight:

$$M_{\text{к}}\ddot{y}_{\text{т}} + a_{\vartheta\text{к}}\Delta\ddot{\vartheta} + \sum_{n=1}^{n_n} a_{n\text{к}}\ddot{S}_n = Q_y, \tag{10.3}$$

$$J_{\text{к}}\Delta\ddot{\vartheta} + a_{\vartheta\text{к}}\ddot{y}_{\text{т}} + \sum_{n=1}^{n_n} b_{n\text{к}}\ddot{S}_n = Q_{\vartheta}, \tag{10.4}$$

$$\sum_{p=1}^{n_n} c_{pn\text{к}}\ddot{S}_p + a_{n\text{к}}\ddot{y}_{\text{т}} + b_{n\text{к}}\Delta\ddot{\vartheta} + \frac{\partial U_0}{\partial S_n} = Q_{Sn}, \tag{10.5}$$

where

$$a_{n\text{к}} = \int_0^l m_{\text{к}}(x_1) f_n(x_1)\, dx_1, \quad a_{\vartheta\text{к}} = \int_0^l m_{\text{к}}(x_1)(x_{1\text{т}} - x_1)\, dx_1,$$

$$b_{n\text{к}} = \int_0^l m_{\text{к}}(x_1) f_n(x_1)(x_{1\text{т}} - x_1)\, dx_1,$$

$$c_{pn\text{к}} = \int_0^l m_{\text{к}}(x_1) f_n(x_1) f_p(x_1)\, dx_1.$$

In this case generalized forces Q_y, Q_{ϑ} and Q_{Sn} (see Chapters II and VI) will have the form

$$\left.\begin{aligned} Q_y &= \Delta Y_{\text{в}} + R_y^{\delta}\delta - c_{\alpha}\Delta\vartheta - c_{\text{д}}\dot{y}_{\text{т}}\frac{1}{v} - \sum_{n=1}^{n_n} c_{nc}S_n + Y_{\text{в}}^{\bullet}, \\ Q_{\vartheta} &= \Delta M_{z\text{в}} - b_{\delta}\delta - b_{\alpha}\Delta\alpha - b_{\text{д}}\Delta\vartheta - \sum_{n=1}^{n_n} b_{nc}S_n + M_{\text{в}}^{\bullet}, \\ Q_{Sn} &= B_{\text{в}n} + t_{\delta n}\cdot\delta - \sum_{p=1}^{n_n} t_{pn}\dot{S}_p - \sum_{p=1}^{\infty} t_{pnc}S_p + B_{\text{в}n}^{\bullet}, \end{aligned}\right\} \tag{10.6}$$

where

$$\left.\begin{aligned} -Y_{\text{в}}^{\bullet} &= \sum_{j=1}^{n_j}\int_0^{-h_j} q_{\text{т}}(x_j)\, dx_j, \\ -M_{\text{в}}^{\bullet} &= \sum_{j=1}^{n_j}\int_0^{-h_j} q_{\text{т}}(x_j)(x_{1\text{т}} - x_{\text{д}j} + x_j)\, dx_j + \sum_{j=1}^{n_j}\Delta M_{\text{д}j}, \\ c_{nc} &= P\left.\frac{df_n(x_1)}{dx_1}\right|_{x_1 = x_{1\text{в}}}, \end{aligned}\right\} \tag{10.7}$$

$$
\left.\begin{aligned}
b_{nc} &= P\left[f_n(x_{1\text{п}}) + (x_{1\text{т}} - x_{1\text{п}})\frac{df_n(x_1)}{dx_1}\bigg|_{x_1=x_{1\text{п}}}\right],\\
t_{pnc} &= Pf_n(x_{1\text{п}})\cdot\frac{df_n(x_1)}{dx_1}\bigg|_{x_1=x_{1\text{п}}}, \quad t_{\delta n} = R_y^{\delta}\cdot f_n(x_{1\text{p}}),\\
-B^{*}_{\text{в}n} &= \sum_{j=1}^{n_j}\int_0^{-h_j} q_{\text{т}}(x_j)f_n(x_j)\,dx_j + \sum_{j=1}^{n_j}\Delta M_{\text{д}j}\frac{df_n(x_j)}{dx_j},\\
B_{\text{в}n} &= qS\left(\frac{u}{v}+\Delta\alpha\right)\int_0^{l}\frac{\partial c_y^{\alpha}(x_1)}{\partial x_1}f_n(x_1)\,dx_1.
\end{aligned}\right\} \qquad (10.7)\ \text{Cont'd.}
$$

$\Delta M_{\text{д}j}$ – moment concentrated in section $x_{1\text{д}j}$ (at place of attachment of the lower end of j-th tank), caused by the presence of nonuniform pressure of liquid on the bottom of the tank (2.57) with (6.19), (6.19').

For cylindrical tanks, partially filled with an ideal liquid, the value of potential Φ_j, according to formulas (2.41), (2.48), (2.50), (1), (6.20), taking into account the influence of only the first terms of series (6.18), i.e., $n_0 < n_n$, will be equal to

$$
\Phi_j = (F_j + x_j y_j)\left[\Delta\dot{\vartheta} + \sum_{n=1}^{n_0}\dot{S}_n\frac{df_n(x_j)}{dx_j}\right] + v_{0yj}y_j +
+ y_j\sum_{n=1}^{n_0}\dot{S}_n f_n(x_{0j}) + 2a_j\beta_{kj}\frac{J_1(\xi_k r_j)\,\mathrm{ch}\,[\xi_k(x_j+h_j)]}{(\xi_k^2-1)\,J_1(\xi_k)\,\mathrm{ch}\,\mu_{kj}}\sin\eta. \qquad (10.8)
$$

where

$$
v_{0yj} = \Delta\dot{\vartheta}(x_{1\text{т}} - x_{0j}) + \dot{y}_{\text{т}} \quad (n_0 = 1 \text{ or } 2).
$$

Equations for functions β_{kj}, determining the amplitude of oscillations of liquid in tanks, are found from boundary conditions on the free surface of liquid (2.40") when $x_j = 0$. After carrying out the transformations, which we omit here, we will have

$$
\ddot{\beta}_{kj} + \omega_{kj}^2\beta_{kj} + \Delta\ddot{\vartheta}(x_{1\text{т}} - x_{0j}) + \sum_{n=1}^{n_0}\ddot{S}_n f_n(x_{0j}) +
+ \ddot{y}_{\text{т}} + \sigma_{kj}^2\left[\Delta\vartheta + \sum_{n=1}^{n_0}S_n\frac{df_n(x_j)}{dx_j}\right] = 0. \qquad (10.9)
$$

where

$$\sigma_{kj}^2 = n_{x_1} g_0 \left(1 + \frac{2}{\operatorname{ch} \mu_{kj}}\right),$$
$$\frac{df_n(x_j)}{dx_j} = -\frac{1}{h_j}[f_n(x_{kj}) - f_n(x_{0j})] \approx -\frac{df_n(x_1)}{dx_1}.$$

According to (2.32)

$$\begin{aligned} q_\text{т}(x_j) = {} & -\Delta\ddot{\vartheta} m_\text{т}(x_j)(x_{1\text{т}} - x_{0j} + x_j) - \\ & - \frac{4\pi\rho_j a_j^3 \operatorname{sh}(\xi_k \bar{x}_j)}{\xi_k(\xi_k^2 - 1)\operatorname{ch}\mu_{kj}} \left[\Delta\ddot{\vartheta} + \sum_{n=1}^{n_0} \ddot{S}_n \frac{df_n(x_j)}{dx_j}\right] - \\ & - m_\text{т}(x_j)\ddot{y}_\text{т} - \sum_{n=1}^{n_0} \ddot{S}_n f_n(x_j) m_\text{т}(x_j) - 2\beta_{kj} m_\text{т}(x_j) \frac{\operatorname{ch}[\xi_k(\bar{x}_j + h_j)]}{(\xi_k^2 - 1)\operatorname{ch}\mu_{kj}}, \end{aligned} \qquad (10.10)$$

where $m_\text{т}(x_j)$ – intensity of distribution of the mass of fuel along the length of the tank.

Having placed (10) in (7) and in equations (3), (4), (5) and performing certain simplifications, we obtain the sought system of approximate differential equations of transverse oscillations of the flight vehicle in the form

$$\begin{aligned} & m\ddot{y}_\text{т} + c_\alpha \dot{y}_\text{т} \frac{1}{v} + c_\alpha \Delta\vartheta + c_\vartheta \Delta\ddot{\vartheta} + \sum_{n=1}^{n_0} c_{Sn} \ddot{S}_n = \\ & \qquad = \Delta Y_\text{в} + \sum_{j=1}^{n_j} c_{kj} \beta_{kj} + R_y^\delta \cdot \delta - \sum_{n=1}^{n_n} c_{nc} S_n, \end{aligned} \qquad (10.11)$$

$$\begin{aligned} & J_z^\circ \Delta\ddot{\vartheta} + b_\alpha \Delta\vartheta + \dot{b}_\alpha \Delta\dot{\vartheta} - \frac{\dot{y}_\text{т}}{v} b_\alpha + \sum_{n=1}^{n_0} b_{n\vartheta} \ddot{S}_n = \\ & \qquad = \Delta M_{z\text{в}} + \sum_{j=1}^{n_j} b_{kj} \ddot{\beta}_{kj} - b_\delta \cdot \delta - \sum_{n=1}^{n_n} b_{nc} S_n + \sum_{j=1}^{n_j} b_{k0j} \beta_{kj}, \end{aligned} \qquad (10.12)$$

$$\begin{aligned} & M_n^\circ(\ddot{S}_n + 2h_n \dot{S}_n + \omega_n^{\circ 2} S_n) + b_{n\vartheta}^\circ \Delta\ddot{\vartheta} + \sum_{p=1}^{n_0} t_{pn} \ddot{S}_p + c_{Sn} \ddot{y}_\text{т} = \\ & \qquad = B_{\text{в}n} + \sum_{j=1}^{n_j} t_{kjn} \beta_{kj} + t_{\delta n} \cdot \delta + \sum_{p=1}^{n_0} t_{pnc} S_p + \sum_{j=1}^{n_j} t_{kn j0} \beta_{kj}. \end{aligned} \qquad (10.13)$$

$$n \neq p.$$

To these equations are joined: equation (9), equations of control and equation of sensing device, connecting the angles of rotation of the section of the flight vehicle body with angles of deflection of control devices,

$$u_{\theta} = k_{\theta}(\Delta\vartheta - \Delta\vartheta_{y}), \tag{10.14}$$

where u_v – voltage of command current, k_v – amplification factor, and

$$\Delta\vartheta_{y} = \sum_{n=1}^{n_n} S_n \frac{df_n(x_1)}{dx_1}\Big|_{x_1 = x_\theta}.$$

During formulation of control equations it is recommended in this case to use experimental values of amplitude-frequency and phase-frequency characteristics of stabilization automaton, measured in the range of frequencies close to ω_n (n = 1 and n = 2). Coefficients of equations (11), (12), (13) are determined by (2.27), (2.61), (6.23') (6.23") and (7) taking into account (4.122')

$$\left.\begin{aligned}
c_{Sn} &= \sum_{j=1}^{n_1} \frac{\pi\rho_j a_j^3}{2} \frac{df_n(x_j)}{dx_j}\left(\frac{1}{\operatorname{ch}\mu_{kj}} - 1\right),\\
b_{n0} &= \sum_{j=1}^{n_1} \frac{4\pi\rho_j a_j^4}{\xi_k^2(\xi_k^2-1)}\left[L_j\left(\frac{1}{\operatorname{ch}\mu_{kj}} - 1\right) + h_j\right]\frac{df_n(x_j)}{dx_j},\\
\dot{b}_{n0} &= \sum_{j=1}^{n_1} \frac{a_j\xi_k}{2\operatorname{ch}\mu_{kj}} \int_{-h_j}^{0} m_\tau(x_j) f_n(x_j)\operatorname{sh}(\xi_k \bar{x}_j)\,dx_j,\\
t_{\delta n} &= R_{y_1}^{\delta} f_n(x_{1p}),\\
t_{pnc} &= -\sum_{j=1}^{n_1} \frac{2\pi\rho_j a_j^4}{\xi_k^2(\xi_k^2-1)}\sigma_{kj}^2 \frac{df_p(x_j)}{dx_j}\cdot\frac{df_n(x_j)}{dx_j},\\
t_{knj0} &= -\frac{2\pi\rho_j a_j^4}{\xi_k^2(\xi_k^2-1)}\omega_{kj}^2 \frac{df_n(x_j)}{dx_j},\\
t_{pn} &= \sum_{j=1}^{n_1} \frac{2a_j^2}{\xi_k^2(\xi_k^2-1)}\left[\pi\rho_j a_j^2\left(\frac{2a_j}{\xi_k}\operatorname{th}\mu_{kj} + h_j\right)\frac{df_p(x_j)}{dx_j}\frac{df_n(x_j)}{dx_j} +\right.\\
&\left. + \frac{2\xi_k}{a_j\operatorname{ch}\mu_{kj}}\frac{df_n(x_j)}{dx_j}\int_{-h_j}^{0} m_\tau(x_j) f_p(x_j)\operatorname{sh}(\xi_k\bar{x}_j)\,dx_j\right],\\
\omega_n^{*2} &= \frac{1}{M_n^*}\left\{M_n\omega_n^2 + \sum_{j=1}^{n_j}\frac{2\pi\rho_j a_j^4}{\xi_k^2(\xi_k^2-1)}\sigma_{kj}^2\left[\frac{df_n(x_j)}{dx_j}\right]^2\right\}.
\end{aligned}\right\} \tag{10.15}$$

In accordance with (3.15), (3.19) the lateral force and bending moment are equal to

$$Q(x_1)=Q_a(x_1)-\ddot{y}_T Q^0_{ж}(x_1)-\Delta\ddot{\delta}Q^x_{ж}(x_1)+\Delta Q_{Tj}(x_1)-\sum_{n=1}^{n_a}\ddot{S}_n Q_{nx}(x_1). \quad (10.16)$$

$$M(x_1)=M_a(x_1)-\ddot{y}_T M^0_{ж}(x_1)-\Delta\ddot{\vartheta}M^x_{ж}(x_1)+\Delta M_{Tj}(x_1)-\sum_{n=1}^{n_a}\ddot{S}_n M_{nx}(x_1). \quad (10.17)$$

For illustration of the effect of mobility of liquid on elastic oscillations of the construction Fig. 10.2 shows graphs of the change of bending moment in the process of action of a gust of wind ($s_m = 3$) on the flight vehicle, calculated by taking into account and without taking into account the mobility of liquid in tanks. Figure 10.3 shows values of lateral overload of the center of gravity of the vehicle, angle of deflection of control devices and angular acceleration corresponding to these cases.

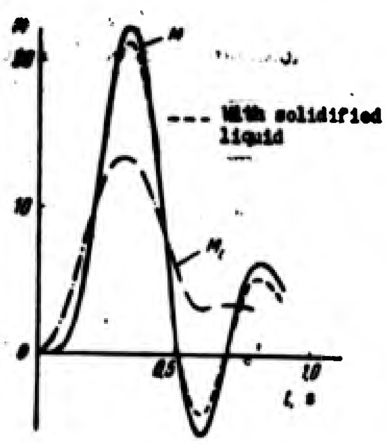

Fig. 10.2. The effect of mobility of liquid on the magnitude of bending moment with the effect of a gust of wind on the flight vehicle.

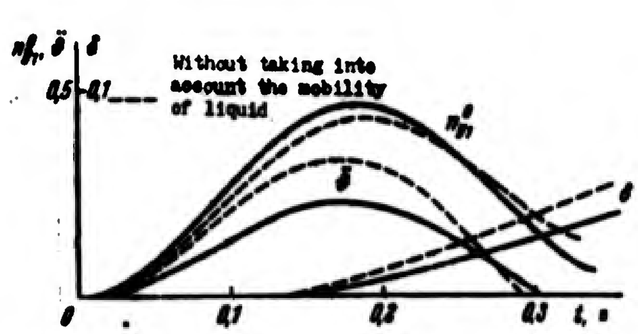

Fig. 10.3. The effect of mobility of liquid in fuel tanks on the values of overload of center of gravity, angle of deflection of control devices and angular acceleration during the action of a gust of wind on the flight vehicle.

§ 10.2. Equations of Transverse Oscillations of the Vehicle of Pack Scheme

Let us represent a flight vehicle of pack scheme (with side-mounting boosters) in the form of a system of rods of large elongation, connected together with rigid connections "в" and "н" (Fig. 10.4). Let us place the origin of the basic connected system of coordinates xyz at a point coinciding in the initial moment of oscillations with the apex of the flight vehicle, and the origin of auxiliary moving coordinate system $x_1y_1z_1$ – at the apex of side-mounting booster. Let us direct axis x along the longitudinal axis of an undeformed body of a vehicle (considered as a rigid body), and axis x_1 – through points of intersection of longitudinal axis of the undeformed body of the side-mounting booster with cross connections "в" and "н." The direction of these axes will be kept the same as in case V, and axes y, y_1 will be placed in the pitching plane. Continuous system of coordinates, rigidly connected with undisturbed trajectory, will be designated in this case by $\xi\eta\zeta$.

Fig. 10.4. Connected systems of coordinates for flight vehicle of pack configuration.

Let us assume that oscillations of the flight vehicle as a solid body with respect to the center of gravity and oscillations of the center of gravity itself, and also elastic oscillations of the construction are small. As in the preceding paragraph, we will consider that deformation of the body does not affect the values of aerodynamic forces and moments which enter equations of motion of the flight vehicle as a solid body, and that energy dissipation in the construction is small.

Let us designate lateral deflection of the center of gravity of the flight vehicle in system of coordinates $\xi\eta\zeta$ by $\eta_T(t)$, and increase of angles of pitch and bank by $\Delta\vartheta(t)$ and $\phi(t)$ respectively. Further let us represent elastic deformation of the construction of a flight vehicle and side-mounting boosters in the form of series

$$y(x,\ t)=\sum_{n=1}^{\infty} S_n(t) f_n(x), \quad y_{1i}(x_1,\ t)=\sum_{p=1}^{\infty} K_{pi}(t)\Phi_{pi}(x_1),$$

where $f_n(x)$ and $\Phi_{pi}(x_1)$ – functions describing partial forms of oscillations of units. Values of these functions should satisfy boundary conditions (8.6), (8.7) and equations

$$\left.\begin{aligned} &\frac{d^2}{dx^2}\left[B(x)\frac{d^2 f_n(x)}{dx^2}\right]-m_{\text{np}}(x) f_n(x)\omega_n^2=0, \\ &\frac{d^2}{dx_1^2}\left[B(x_1)\frac{d^2\Phi_{pi}(x_1)}{dx_1^2}\right]-m(x_1)\Phi_{pi}(x_1)\omega_{pi}^2=0. \end{aligned}\right\} \qquad (10.18)$$

Here

$$m_{\text{np}}(x)=m(x)+\left[\Delta m\frac{f_n(x_{\text{н}})}{f_n(x_{\text{в}})}+m(x_{\text{в}})\right]\delta(x-x_{\text{в}})+$$
$$+\left[\Delta m\frac{f_n(x_{\text{в}})}{f_n(x_{\text{н}})}+m(x_{\text{н}})\right]\delta(x-x_{\text{н}}),$$

and

$$m(x_{\text{в}})=\sum_{i=1}^{k_i}\left[m_{\text{б}i}\left(1\mp\frac{2x_{\text{тб}i}}{a}\right)+\frac{J_{\text{б}i}}{a^2}\right],$$
$$m(x_{\text{н}})=\sum_{i=1}^{k_i}\frac{J_{\text{б}i}}{a^2}$$

– corresponding apparent masses of side-mounting boosters. In this case the condition of orthogonality for functions $f_n(x)$ will have the form

$$\int_0^L m_{\text{np}}(x) f_n(x) f_k(x)\,dx=0 \quad \text{when} \quad n\neq k,$$
$$\Delta m=\sum_{i=1}^{k_i}\left(\frac{m_{\text{б}i}x_{\text{т}\text{б}i}}{a}-\frac{J_{\text{б}i}}{a^2}\right).$$

For simplification of computations let us assume that twisting strain of bodies can be disregarded.[1] Then total lateral shift of arbitrary point x of the longitudinal axis of the flight vehicle in plane $\xi\eta$ (taking into account bending of its body) will have the form

$$\eta(x, t)=\eta_{\text{т}}(t)+\Delta\vartheta(t)(x_{\text{т}}-x)+\sum_{n=1}^{\infty} S_n(t) f_n(x). \tag{10.19}$$

Analogous shift of some point x_i of the longitudinal axis of side-mounting booster (taking into account rotation of the flight vehicle relative to longitudinal axis) will be equal to

$$\begin{aligned}\eta_i(x_1, t)&=\eta_{\text{т}}(t)+\Delta\vartheta(t)(x_{\text{т}}-x_{\text{в}}-x_1)+\varphi(t) r_i(x_{\text{в}}) \sin\gamma_i+\\&+\sum_{p=1}^{\infty} K_{pi}(t)\Phi_{pi}(x_1)+\sum_{n=1}^{\infty} S_n(t)\left[f_n(x_n)\frac{x_1}{a}+f_n(x_{\text{в}})\left(1-\frac{x_1}{a}\right)\right],\end{aligned} \tag{10.20}$$

where γ_i – angle between the plane passing through longitudinal axes x and x_i (of considered i-th side-mounting booster) and pitching plane $(\xi\eta)$, $r_i(x_i)$ – distance of point x_i of longitudinal axis of side-mounting booster from the longitudinal axis x of the flight vehicle, and $x_{\text{т}}$ – abscissa of center of gravity of the flight vehicle on the whole in system of coordinates xyz, k_i – number of accelerators.

In this case as generalized coordinates let us take functions $\eta_{\text{т}}(t)$, $\Delta\vartheta(t)$, $\varphi(t)$ and set of functions $K_{pi}(t)$, $S_n(t)$ $(p=1, 2, \ldots;\ n=1, 2, \ldots)$. Generalized forces corresponding to these generalized coordinates will be equal to

$$\begin{aligned}Q_\eta=R_y^\delta\delta+\sum_{i=1}^{k_i} R_{yi}^\delta\delta_i\sin\gamma_i-c_x\frac{1}{v}\dot{\eta}_{\text{т}}-c_\alpha\alpha+\\+\Delta Y_{\text{в}}-\sum_{n=1}^{\infty} c_{nc}S_n-\sum_{i=1}^{k_i}\sum_{p=1}^{\infty} c_{pci}K_{pi},\end{aligned} \tag{10.21}$$

[1]Questions connected with the account of twisting strain of the flight vehicle body are considered in Chapter XIII and in [93].

$$Q_{\vartheta} = -b_{\delta}\delta - \sum_{i=1}^{r_i} b_{\delta i}\delta_i \sin\gamma_i - b_{\lambda}\Delta\dot{\vartheta} - b_{\alpha}\alpha + \\ + \Delta M_{zв} - \sum_{n=1}^{\infty} b_{nc}S_n - \sum_{i=1}^{k_i}\sum_{p=1}^{\infty} b_{pci}K_{pi}, \tag{10.22}$$

$$Q_{\varphi} = M_{x\delta}^{\delta}\delta + \sum_{i=1}^{k_i} a_{\delta i}\delta_i + \sum_{i=1}^{k_i} a_{вi}\sin\gamma_i - \sum_{i=1}^{k_i}\sum_{p=1}^{\infty} a_{pci}K_{pi}, \tag{10.23}$$

$$Q_{Sn} = t_{\delta n}\delta + \sum_{i=1}^{k_i} t_{\delta ni}\delta_i \sin\gamma_i + B_{вn} - \sum_{p=1}^{\infty} t_{pnc}S_p - \sum_{i=1}^{k_i}\sum_{p=1}^{\infty} t_{pnic}K_{pi} - \psi_{\cdot n}, \tag{10.24}$$

$$Q_{Kpi} = \tau_{p\delta i}\delta_i \sin\gamma_i + B_{вpi} - \sum_{n=1}^{\infty} \tau_{pnci}K_{ni}, \tag{10.25}$$

where

$$\left.\begin{aligned}
c_{pci} &= P_{\delta i}\left.\frac{d\Phi_{pi}(x_1)}{dx_1}\right|_{x_1 = x_{1п}}, \\
c_{\alpha} &= -(P_c + qc_y^{\alpha}S - R_x), \\
P_c &= P + \sum_{i=1}^{k_i} P_{\delta i}, \\
b_{\delta i} &= -R_{yi}^{\delta}(x_т - x_д - x_{1pi}), \\
b_{\alpha} &= qSc_y^{\alpha}(x_{д} - x_т), \\
b_{pci} &= P_{\delta i}\left[\Phi_{pi}(x_{1п}) + (x_т - x_д - x_{1п})\left.\frac{d\Phi_{pi}(x_1)}{dx_1}\right|_{x_1 = x_{1п}}\right], \\
a_{\delta i} &= R_{yi}^{\delta} r_i(x_{1p}), \\
a_{pci} &= P_{\delta i} r_i(x_{1п})\left.\frac{d\Phi_{pi}(x_1)}{dx_1}\right|_{x_1 = x_{1п}}, \\
a_{вi} &= qSc_{yi}^{\alpha} r_i(x_{1дi})\left(\alpha + \frac{u}{v}\right), \\
t_{\delta ni} &= R_{yi}^{\delta}\left[f_n(x_д)\left(1 - \frac{x_{1p}}{a}\right) + f_n(x_д)\frac{x_{1p}}{a}\right], \\
B_{вn} &= qS\left(\alpha + \frac{u}{v}\right)\int_0^L f_n(x)\left[\frac{\partial c_y^{\alpha}(x)}{\partial x}\right]_{\text{изр}} dx,
\end{aligned}\right\} \tag{10.26}$$

$$\left[\frac{\partial c_y^{\alpha}(x)}{\partial x}\right]_{\text{пр}} = \frac{\partial c_y^{\alpha}(x)}{\partial x} + \frac{1}{a}\sum_{i=1}^{k_i}\int_0^{l_i} x_1 \frac{\partial c_y^{\alpha}(x_1)}{\partial x_1}dx_1\delta(x-x_{\text{п}}) +$$

$$+\sum_{i=1}^{k_i}\int_0^{l_i}\left(1-\frac{x_1}{a}\right)\frac{\partial c_y^{\alpha}(x_1)}{\partial x_1}dx_1\delta(x-x_{\text{з}}),$$

$$t_{pnc} = Pf_n(x_{\text{п}})\left.\frac{df_p(x)}{dx}\right|_{x=x_{\text{п}}},$$

$$t_{pnic} = P_{6i}\left\{\left.\frac{d\Phi_{pi}(x_1)}{dx_1}\right|_{x_1=x_{1\text{п}}}\left[f_n(x_{\text{п}})\frac{x_{1\text{п}}}{a}+f_n(x_{\text{з}})\left(1-\frac{x_{1\text{п}}}{a}\right)\right] -\right.$$

$$\left.-\Phi_{pi}(x_{1\text{п}})\frac{1}{a}\left[f_n(x_{\text{п}})-f_n(x_{\text{з}})\right]\right\},$$

$$\tau_{p\delta i} = R_{yi}^{\delta}\Phi_{pi}(x_{1p}),$$

$$B_{\text{в}pi} = qS\beta'_{pi},$$

$$\tau_{pnci} = P_{6i}\Phi_{pi}(x_{1\text{п}})\left.\frac{d\Phi_{pi}(x_1)}{dx_1}\right|_{x_1=x_{1\text{п}}},$$

$$\psi_{\text{в}n} = -\frac{qS}{b}\left\{\sum_{p=1}^{\infty}m_{\text{п}p}^{\omega}\dot{S}_p+\alpha'_n\Delta\dot{\theta}+\right.$$

$$+\sum_{i=1}^{k_i}\sum_{p=1}^{\infty}\dot{K}_{pi}\left[f_n(x_{\text{з}})\left(\beta'_{pi}-\frac{\alpha'_{pi}}{a}\right)+f_n(x_{\text{п}})\frac{\alpha'_{pi}}{a}\right]+$$

$$\left.+\dot{\varphi}\sum_{i=1}^{k_i}c_{yi}^{\alpha}r_i(x_{1\text{д}i})\left[f_n(x_{\text{з}})\left(1-\frac{x_{1\text{д}i}}{a}\right)+f_n(x_{\text{п}})\frac{x_{1\text{д}i}}{a}\right]\sin\gamma_i\right\}. \qquad (10.26)\ \text{Cont'd.}$$

Expressions for the remaining coefficients are given in § 10.1.

Here R_y^{δ} – gradient of control force, δ – angle of deflection of control devices of the central unit of the vehicle, and R_{yi}^{δ}, δ_i – the same for side-mounting booster; q – impact pressure, S – characteristic area, x_p, $x_{\text{п}}$ – coordinates of points of application of control force and thrust force to the body of the flight vehicle, $x_{1\text{д}i}$ – coordinate of the center of pressure of i-th side-mounting booster, c_y^{α} and m_z^{ω} – aerodynamic coefficients of lift and damping moment, u – instantaneous wind velocity at a given flight altitude, P_{6i} – thrust of one side-mounting booster, P – thrust force of central unit of the vehicle,

$$\alpha'_{pl} = \int_0^{l_l} x_1 \Phi_{pl}(x_1) \frac{\partial c^{\alpha}_{yl}(x_1)}{\partial x_1} dx_1,$$

$$\beta'_{pl} = \int_0^{l_l} \Phi_{pl}(x_1) \frac{\partial c^{\alpha}_{yl}(x_1)}{\partial x_1} dx_1,$$

$$\alpha'_n = \int_0^L f_n(x)(x - x_\tau) \left[\frac{\partial c^{\alpha}_y(x)}{\partial x}\right]_{\text{np}} dx,$$

$$m^{\omega}_{np} \approx \int_0^L f_n(x) f_p(x) \frac{\partial c^{\alpha}_y(x)}{\partial x} dx.$$

When determining internal forces the effect of generalized damping forces (ψ_{Sn}) can be disregarded. It is expedient to consider them only during investigation of stability of motion and establishment of amplitudes of natural oscillations of the system. Terms proportional to S_n and K_{pl}, which consider displacement and turn of the line of thrust force due to elastic oscillations of the body, are comparatively small.

Having determined the kinetic and potential energies of the system and using Lagrange equations, we obtain approximate equations of transverse oscillations of the flight vehicle of pack configuration (in the pitching plane) in the form (see Chapter [redacted])

$$m\ddot{\eta}_\tau + \sum_{n=1}^{n_n} c_n \ddot{S}_n + \sum_{l=1}^{k_l} \sum_{p=1}^{n_p} \beta_{pl} \ddot{K}_{pl} + \ddot{\varphi} \sum_{l=1}^{k_l} m_{\text{б}l} r_l(x_{\tau\text{б}l}) \sin\gamma_l = Q_\eta, \qquad (10.27)$$

$$J_z \Delta\ddot{\vartheta} + b_\varphi \ddot{\varphi} + \sum_{n=1}^{n_n} b_n \ddot{S}_n + \sum_{l=1}^{k_l} \sum_{p=1}^{n_p} b_{kpl} \ddot{K}_{pl} = Q_\vartheta, \qquad (10.28)$$

$$J_x \ddot{\varphi} + b_\varphi \Delta\ddot{\vartheta} + \sum_{l=1}^{k_l} \sum_{p=1}^{n_p} a_{pl} \ddot{K}_{pl} \sin\gamma_l +$$
$$+ \ddot{\eta}_\tau \sum_{l=1}^{k_l} m_{\text{б}l} \dot{r}_l(x_{\tau\text{б}l}) \sin\gamma_l + \sum_{l=1}^{k_l} \sum_{n=1}^{n_n} a_{nl} \ddot{S}_n \sin\gamma_l = Q_\varphi, \qquad (10.29)$$

$$M_n(\ddot{S}_n + 2h_n \dot{S}_n + \omega_n^2 S_n) + c_n \ddot{\eta}_\tau + b_n \Delta\ddot{\vartheta} +$$
$$+ \sum_{l=1}^{k_l} \sum_{p=1}^{n_p} t_{npl} \ddot{K}_{pl} + \ddot{\varphi} \sum_{l=1}^{k_l} a_{nl} \sin\gamma_l = Q_{Sn}, \qquad (10.30)$$

$$M_{pi}(\ddot{K}_{pi} + 2h_{pi}\dot{K}_{pi} + \omega_{pi}^2 K_{pi}) + \beta_{pi}\ddot{\eta}_{\text{т}} + b_{kpi}\Delta\ddot{\vartheta} + \sum_{n=1}^{n_n} t_{npi}\ddot{S}_n + \ddot{\varphi} a_{pi}\sin\gamma_i = Q_{kpi} \quad (10.31)$$

$$(n = 1,\ 2,\ \ldots,\ n_n;\ p = 1,\ 2,\ \ldots,\ n_p;\ i = 1,\ 2,\ \ldots,\ k_i),$$

where

$$\left.\begin{aligned}
c_n &= \Delta m\,[f_n(x_{\text{в}}) + f_n(x_{\text{н}})],\\
\alpha_{pi} &= \int_0^{l_i} m(x_1)\,x_1\Phi_{pi}(x_1)\,dx_1,\\
b_n &= \Delta m\,[f_n(x_{\text{н}})(x_{\text{т}} - x_{\text{в}}) + f_n(x_{\text{в}})(x_{\text{т}} - x_{\text{н}})],\\
b_{kpi} &= \beta_{pi}(x_{\text{т}} - x_{\text{в}}) - \alpha_{pi},\\
\beta_{pi} &= \int_0^{l_i} m(x_1)\,\Phi_{pi}(x_1)\,dx_1,\\
b_{\varphi} &= \sum_{i=1}^{k_i} m_{6i} r_i(x_{\text{т}6i})(x_{\text{т}} - x_{\text{в}} - x_{\text{т}6i})\sin\gamma_i,\\
a_{pi} &= r_i(x_{\text{в}})\left(\beta_{pi} - \frac{\alpha_{pi}}{a}\right) + r_i(x_{\text{н}})\frac{\alpha_{pi}}{a},\\
a_{ni} &= m_{6i} r_i(x_{\text{т}6i})\left[f_n(x_{\text{н}})\frac{x_{\text{т}6i}}{a} + f_n(x_{\text{в}})\left(1 - \frac{x_{\text{т}6i}}{a}\right)\right],\\
t_{npi} &= f_n(x_{\text{н}})\frac{\alpha_{pi}}{a} + f_n(x_{\text{в}})\left(\beta_{pi} - \frac{\alpha_{pi}}{a}\right),\\
\Delta m &= \sum_{i=1}^{k_i}\left(m_{6i}\frac{x_{\text{т}6i}}{a} - \frac{J_{6i}}{a^2}\right),\\
M_n &= \int_0^L m_{\text{пр}}(x)\,f_n^2(x)\,dx.
\end{aligned}\right\} \quad (10.32)$$

Equations of transverse oscillations of the flight vehicle in the yawing plane will have the same form. Only instead of sin γ_i cos γ_i will stand everywhere. With this sin γ_i (cos γ_i) is taken according to modulus in all equations of this chapter. By considering jointly the equations for ϕ in pitching plane (29) and in yawing plane, we obtain one equation, describing rotation of the flight vehicle relative the longitudinal axis. This equation will connect oscillations of the flight vehicle in the pitching plane with oscillations in the yawing plane.

Thus, generally the system of equations of transverse oscillations of an elastic flight vehicle of pack configuration cannot be subdivided into independent subsystems of equations, as it is accepted to do during investigation of the stability of motion of the vehicle as a solid body. In a particular case, when the flight vehicle is a combination of one central and 4 – x absolutely identical lateral bodies (boosters) symmetrically located around it, such separation of the system of equations of motion (in case of the presence of an independent stabilization system along the channel of rotation) is possible in principle. In this case (having designated angle of deflection of control devices by $\delta_{\varphi i}$, and by $N_{pi}(t)$ – function determining the amount of sags of side-mounting boosters only due to oscillations of the flight vehicle in the plane of bank) it is possible to separate the following subsystem of equations of oscillations of the flight vehicle with respect to axis x:

$$I_x\ddot{\varphi}+\sum_{i=1}^{k_i}\sum_{p=1}^{k_p}a_{pi}\ddot{N}_{pi}-\sum_{i=1}^{k_i}R^{\delta}_{pi}r_i(x_{1p})\delta_{\varphi i}-\Psi_{\varphi}+$$

$$+\sum_{i=1}^{k_i}a_{xi}(\sin\gamma_i+\cos\gamma_i)+\Delta M_{xs}+M^{\delta}_{x0}\delta_{\varphi}, \qquad (10.33)$$

$$M_{pi}(\ddot{N}_{pi}+2h_{pi}\dot{N}_{pi}+\omega^2_{pi}N_{pi})=\tau_{p\delta i}\delta_{\varphi i}-a_{pi}\ddot{\varphi}+B_{xnpi} \qquad (10.34)$$

$$(p=1,\ 2,\ \ldots;\ i=1,\ 2,\ \ldots,\ k_i)$$

and equation of sensing device

$$u_{\varphi}=k_{\varphi}\varphi. \qquad (10.34')$$

Here k_{ϕ} – amplification factor of the transducer of control system, u_{ϕ} – command current, Ψ_{ϕ} – damping moment, ΔM_{xs} – perturbing moment and $M^{\delta}_{x0}\delta_{\varphi}$ – controlling moment with respect to bank for the central unit, equal to

$$\delta_{\varphi}M^{\delta}_{x0}=n_{\delta}r(x_p)R^{\delta}_{p}\delta_{\varphi},$$

and n_{δ} – number of vanes, working in the plane of bank.

With parallel location of axes of side-mounting boosters of the flight vehicle

$$a_{pl} = \beta_{pl} r\left(x_{\text{т}6l}\right).$$

In the pitching plane the subsystem of equations of oscillations of such flight vehicle at small ϕ will consist of:

equation of motion of center of masses

$$m\ddot{\eta}_{\text{т}} + \sum_{n=1}^{n_n} c_n \ddot{S}_n + \sum_{l=1}^{k_l} \sum_{p=1}^{n_p} \beta_{pl} \ddot{K}_{pl} = R_y^{\delta} \delta + \\ + \sum_{l=1}^{k_l} R_{yl}^{\delta} \delta_l \sin \gamma_l + \Delta Y_{\text{в}} - c_{\text{д}} \frac{1}{v} \dot{\eta}_{\text{т}} - c_{\alpha} \alpha + G \dot{\eta}_{\text{т}} \frac{1}{v} \sin \theta_0 - \\ - \sum_{n=1}^{n_n} c_{nc} S_n - \sum_{l=1}^{k_l} \sum_{p=1}^{n_p} c_{pcl} K_{pl}; \qquad (10.35)$$

equation of motion of flight vehicle with respect to center of masses

$$J_z \Delta\ddot{\vartheta} + \sum_{n=1}^{n_n} b_n \ddot{S}_n + \sum_{l=1}^{k_l} \sum_{p=1}^{n_p} b_{\text{к}pl} \ddot{K}_{pl} = -b_{\delta} \delta - \sum_{l=1}^{k_l} b_{\delta l} \delta_l \sin \gamma_l - \\ - b_{\text{д}} \Delta\dot{\vartheta} - b_{\alpha} \alpha + \Delta M_{z\text{в}} - \sum_{n=1}^{n_n} b_{nc} S_n - \sum_{l=1}^{k_l} \sum_{p=1}^{n_p} b_{pcl} K_{pl}; \qquad (10.36)$$

system n_n of equations of elastic oscillations of the construction of central unit

$$M_n(\ddot{S}_n + 2h_n \dot{S}_n + \omega_n^2 S_n) + c_n \ddot{\eta}_{\text{т}} + b_n \Delta\ddot{\vartheta} + \\ + \sum_{l=1}^{k_l} \sum_{p=1}^{n_p} t_{npl} \ddot{K}_{pl} = t_{\delta n} \delta + B_{\text{в}n} + \sum_{l=1}^{k_l} t_{\delta nl} \delta_l \sin \gamma_l - \\ - \sum_{p=1}^{n_n} t_{pnc} S_p - \sum_{l=1}^{k_l} \sum_{p=1}^{n_p} t_{pnlc} K_{pl}; \qquad (10.37)$$

system n_p of equations of bending oscillations of bodies of side-mounting boosters

$$M_{\partial l}(\ddot{K}_{pl} + 2h_{\partial l}\dot{K}_{pl} + \omega_{pl}^2 K_{pl}) + \beta_{pl}\ddot{\eta}_{\tau} + b_{kpl}\Delta\vartheta + \sum_{n=1}^{n_n} t_{npl}\ddot{S}_n =$$
$$= \tau_{p\delta l}\delta_l \sin\gamma_l + B_{\text{в}pl} - \sum_{n=1}^{n_p} \tau_{pncl}K_{nl} \qquad (10.38)$$

and equation of sensing device of the follow-up system (stabilization automaton)

$$u_{\vartheta} = k_{\vartheta}\left[\Delta\vartheta - \sum_{\eta=1}^{n_n} S_n \frac{df_n(x)}{dx}\bigg|_{x=x_{\vartheta}}\right].$$

The subsystem of equations of oscillations of the flight vehicle for the yawing plane (with replacement sin γ_l by cos γ_l) will have analogous form.

To the three shown subsystems of equations, describing oscillations of the flight vehicle, there are joined corresponding equations of the control (stabilization) system along channels of rotation, pitch and yawing, connecting the angles of rotation of cross section x_{ϑ} of the body of central unit of the flight vehicle with angles of rotation of steering elements.

§ 10.3. Influence of Elasticity of Cross Connections

Let us very briefly consider what influence the elasticity of cross connections "в" and "н" (Fig. 10.4) has on parameters of transverse oscillations of the construction of a flight vehicle of pack configuration. With elastic connections the shifts of points of longitudinal axis of side-mounting booster (unit) in the case of transverse oscillations of the flight vehicle will be comprised of shifts caused by bending of the body of this unit $y_{li}(x_l, t)$, and shifts caused by sags y(x, t) of points $x_{\text{в}}$ and $x_{\text{н}}$ of central block of the flight vehicle and flexibility of connections "в" and "н" ($y_{\text{в}l}$ and $y_{\text{н}l}$). Having formulated the expressions for kinetic and potential energies of the flight vehicle and taking functions $\eta_{\tau}(t)$, $\Delta\vartheta(t)$, $y_{\text{в}l}(t)$, $y_{\text{н}l}(t)$, $S_n(t)$ and $K_{pl}(t)$, as generalized coordinates, instead of (35)-(38) we obtain following system of ordinary differential equations, approximately describing its transverse oscillations in the pitching plane on the powered flight phase (at small ϕ):

$$m\ddot{\eta}_{\text{т}} + c_{\text{д}}\frac{1}{V}\dot{\eta}_{\text{т}} + c_{\alpha}\alpha + \sum_{n=1}^{n_n} c_n\ddot{S}_n + \sum_{l=1}^{k_l}\sum_{p=1}^{n_p}\beta_{pl}\ddot{K}_{pl} + \sum_{l=1}^{k_l} c_{\text{в}l}\ddot{y}_{\text{в}l} + \sum_{l=1}^{k_l} c_{\text{н}l}\ddot{y}_{\text{н}l} = R_y^{\delta}\delta + \sum_{l=1}^{k_l} R_{yl}^{\delta}\delta_l \sin\gamma_l + \Delta Y_{\text{в}}, \tag{10.39}$$

$$J_z\Delta\ddot{\vartheta} + b_{\text{д}}\Delta\dot{\vartheta} + b_{\alpha}\alpha + \sum_{n=1}^{n_n} b_n\ddot{S}_n + \sum_{l=1}^{k_l}\sum_{p=1}^{n_p} b_{kpl}\ddot{K}_{pl} + \sum_{l=1}^{k_l} b_{\text{в}l}\ddot{y}_{\text{в}l} + \sum_{l=1}^{k_l} b_{\text{н}l}\ddot{y}_{\text{н}l} = -b_{\delta}\delta - \sum_{l=1}^{k_l} b_{\delta l}\delta_l \sin\gamma_l + \Delta M_{z\text{в}}, \tag{10.40}$$

$$M_n(\ddot{S}_n + 2h_n\dot{S}_n + \omega_n^2 S_n) + c_n\ddot{\eta}_{\text{т}} + b_n\Delta\ddot{\vartheta} + \sum_{l=1}^{k_l}\sum_{p=1}^{n_p} t_{npl}\ddot{K}_{pl} + \sum_{l=1}^{k_l} t_{\text{в}ln}\ddot{y}_{\text{в}l} + \sum_{l=1}^{k_l} t_{\text{н}ln}\ddot{y}_{\text{н}l} = t_{\delta n}\delta + \sum_{l=1}^{k_l} t_{\delta nl}\delta_l \sin\gamma_l + B_{\varphi\eta}, \tag{10.41}$$

$$M_{pl}(\ddot{K}_{pl} + 2h_{pl}\dot{K}_{pl} + \omega_{pl}^2 K_{pl}) + \beta_{pl}\ddot{\eta}_{\text{т}} + b_{kpl}\Delta\ddot{\vartheta} + \sum_{n=1}^{n_n} t_{npl}\ddot{S}_n + \frac{\alpha_{pl}}{a}\ddot{y}_{\text{н}l} + \tau_{\text{в}lp}\ddot{y}_{\text{в}l} = \tau_{p\delta l}\delta_l \sin\gamma_l + B_{\text{в}pl}, \tag{10.42}$$

$$m_l(x_{\text{в}})(\ddot{y}_{\text{в}l} + \omega_{\text{в}l}^2 y_{\text{в}l}) + c_{\text{в}l}\ddot{\eta}_{\text{т}} + b_{\text{в}l}\Delta\ddot{\vartheta} + \sum_{n=1}^{n_n} t_{\text{в}ln}\ddot{S}_n + \sum_{p=1}^{n_p}\tau_{\text{в}lp}\ddot{K}_{pl} + \Delta m_{\text{б}l}\ddot{y}_{\text{н}l} = R_{yl}^{\delta}\left(1 - \frac{x_{1p}}{a}\right)\delta_l + B_{\text{в}\text{в}l}, \tag{10.43}$$

$$m_l(x_{\text{н}})(\ddot{y}_{\text{н}l} + \omega_{\text{н}l}^2 y_{\text{н}l}) + c_{\text{н}l}\ddot{\eta}_{\text{т}} + b_{\text{н}l}\Delta\ddot{\vartheta} + \sum_{n=1}^{n_n} t_{\text{н}ln}\ddot{S}_n + \sum_{p=1}^{n_p}\frac{\alpha_{pl}}{a}\ddot{K}_{pl} + \Delta m_{\text{б}l}\ddot{y}_{\text{в}l} = R_{yl}^{\delta}\frac{x_{1p}}{a}\delta_l + B_{\text{в}\text{н}l} \tag{10.44}$$

$$(n = 1, 2, \ldots, n_n;\quad n = 1, 2, \ldots, n_p;\quad l = 1, 2, \ldots, k_l),$$

where

$$\left.\begin{aligned}
&c_{\text{в}l} = m_{\text{б}l}\left(1 - \frac{x_{\text{т}\text{б}l}}{a}\right),\quad c_{\text{н}l} = \frac{1}{a}m_{\text{б}l}x_{\text{т}\text{б}l},\\
&b_{\text{в}l} = m_{\text{б}l}(x_{\text{т}} - x_{\text{в}} - x_{\text{т}\text{б}l}) - b_{\text{н}l},\quad b_{\text{н}l} = \frac{1}{a}m_{\text{б}l}x_{\text{т}\text{б}l}(x_{\text{т}} - x_{\text{в}}) - \frac{J_{\text{б}l}}{a^2},\\
&t_{\text{в}ln} = f_n(x_{\text{в}})\,m_l(x_{\text{в}}) + f_n(x_{\text{н}})\,\Delta m_{\text{б}l},\quad \tau_{\text{в}lp} = \beta_{pl} - \frac{\alpha_{pl}}{a},\\
&t_{\text{н}ln} = f_n(x_{\text{н}})\,m_l(x_{\text{н}}) + f_n(x_{\text{в}})\,\Delta m_{\text{б}l},\quad m_l(x_{\text{н}}) = \frac{J_{\text{б}l}}{a^2},\\
&m_l(x_{\text{в}}) = m_{\text{б}l}\left(1 - \frac{2x_{\text{т}\text{б}l}}{a}\right) + \frac{J_{\text{б}l}}{a^2},\quad \omega_{\text{в}l}^2 = \frac{E_{\text{в}l}}{m_l(x_{\text{в}})},\quad \omega_{\text{н}l}^2 = \frac{E_{\text{н}l}}{m_l(x_{\text{н}})},\\
&B_{\text{в}\text{в}l} = qSc_{yl}^{\alpha}\left(\alpha + \frac{u}{v}\right)\left(1 - \frac{x_{1\text{в}l}}{v}\right),\quad B_{\text{в}\text{н}l} = qSc_{yl}^{\alpha}\left(\alpha + \frac{u}{v}\right)\frac{x_{1\text{в}l}}{a},
\end{aligned}\right\} \tag{10.45}$$

and $E_{\text{в}l}, E_{\text{н}l}$ – stiffness coefficients of connections "в" and "н."

The vibration component of lateral overload for the central unit will be determined by formula

$$\Delta n_{y\text{вл}}(x,\ t)=-\frac{1}{g_0}\sum_{n=1}^{n_n}\ddot{S}_n(t)f_n(x), \qquad (10.46)$$

and for i-th side-mounting booster – by expression

$$\Delta n_{y\text{в}i}(x_1,\ t)=-\frac{1}{g_0}\left\{\sum_{p=1}^{n_p}\ddot{K}_{pi}(t)\Phi_{pi}(x_1)+\right.$$
$$\left.+\sum_{n=1}^{n_n}\ddot{S}_n(t)\left[\left(1-\frac{x_1}{a}\right)f_n(x_\text{в})+\frac{x_1}{a}f_n(x_\text{н})\right]+\ddot{y}_{\text{в}i}\left(1-\frac{x_1}{a}\right)+\ddot{y}_{\text{н}i}\frac{x_1}{a}\right\}. \qquad (10.47)$$

Total values of coefficients of lateral overload for these parts of the flight vehicle construction (pack configuration) will be equal to

$$\left.\begin{aligned}n_y(x,\ t)&=n_y^0(t)+n_y^x(x,\ t)+\Delta n_{y\text{вл}}(x,\ t),\\ n_{yi}(x_1,\ t)&=n_{y_1}^0(t)+n_{y_1}^x(x_1,\ t)+\Delta n_{y\text{в}i}(x_1,\ t).\end{aligned}\right\} \qquad (10.48)$$

Solution of the given system of equations in generalized coordinates can be obtained with an electronic digital computer. For example let us consider the behavior of the construction described in § 8.3 and consisting of one central unit and eight side-mounting boosters, in the case of action of a gust of wind of the following form on it:

$$\begin{aligned}&u=u_m\frac{vt}{H} &&\text{when}\quad 0<t<t_1,\\ &u=u_m\left[1-\frac{v}{H}(t-t_1)\right] &&\text{when}\quad t_1<t<2t_1,\\ &u=0 &&\text{when}\quad t>2t_1,\end{aligned}$$

where $t_1=\frac{H}{v}$, u_m – maximum value of wind velocity, H – distance during which the velocity of a gust of wind is changed from zero to maximum magnitude u_m. In this case we will use a simplified system of control equations for the channel of pitch, considering derivatives of δ and t to the seventh order inclusively, and equations of motion of the flight vehicle (39)-(44).

The character of change of generalized coordinates $S_n(t)$ (for n = 1 and n = 2) and $y_{mi}(t)$ in the process of action of the shown gust of wind ($2t_1 = 0{,}4$ s) is given on Fig. 10.5. As can be seen, in this case the values of S_2 are considerably less than S_1. The influence of S_2 on magnitudes of bending moments turned out to be noticeable only for certain sections of the body of the central unit. In particular, in the middle of this body the component of dynamic bending moment, proportional to S_2, did not exceed 10% of the component of dynamic bending moment, proportional to S_1. Of the side-mounting boosters the most loaded turned out to be those which are located in plane ξζ (Fig. 10.6), i.e., Д(В).

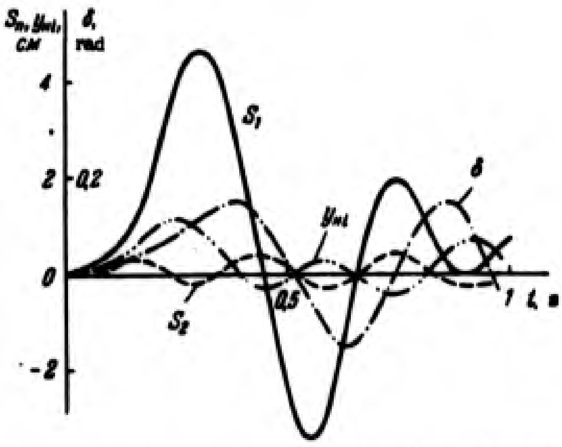

Fig. 10.5. Character of change of generalized coordinates S_1, S_2, y_{mi}, δ in the process of passage of the vehicle through a gust of wind.

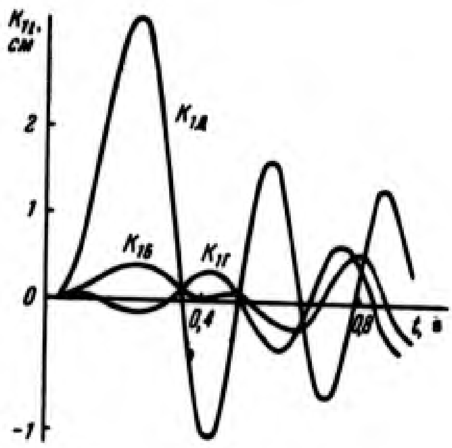

Fig. 10.6. Character of change of generalized coordinates K_{yi} (i = Б, Г, Д) in the process of passage of the flight vehicle through a gust of wind.

Dynamic components of bending moments in certain sections of the body of the central unit exceeded the values of corresponding static moments by 50%, and for side-mounting boosters by 30%. In this case we noted the insignificant influence of elastic oscillations of the system on the magnitude of rotary component of lateral overload (n_y^x). Furthermore, calculation showed that the introduction of at least one elastic cross connection of comparatively small rigidity in the system can be noticeably reflected on the character of change of almost all parameters, determining perturbed motion of the considered flight vehicle.

In conclusion let us note that the above-mentioned differential equations of motion of the flight vehicle of pack configuration can be easily spread to more complicated assembly diagrams of constructions, in particular to diagrams having underslung fuel tanks and loads or constituting one or several groups of parallel located (inside or on the outside of the flight vehicle body) and different method of connected units.

§ 10.4. Transverse Oscillations of a Vehicle of Pack Configuration Taking into Account the Mobility of Liquid in Tanks

By relating the hydrodynamic pressure of liquid on walls of tanks to the category of external surface forces, it is possible to formulate approximate equations of transverse oscillations (similar to the equations given in § 10.1) and for a flight vehicle of pack configuration (with longitudinal and transverse division of stages). In this case the value of potential Φ_j for tanks of the central unit (cylindrical shape) will be determined by the same formulas as for fuel tanks of a flight vehicle without side-mounting boosters (8), and value of functions β_{kj} by equation (9).

For fuel tanks of side-mounting boosters the velocity of the center of free surface of liquid can be determined on the basis of formula (20), and taking into account the elasticity of connections – by formula

$$v_{0yj} = \dot{\eta}_{\text{т}} + \Delta\dot{\vartheta}(x_{\text{т}} - x_{\text{в}} - x_{10j}) + \sum_{p=1}^{n_p} \dot{K}_{pi}\Phi_{pi}(x_{10j}) + \dot{y}_{\text{в}i}\left(1 - \frac{x_{10j}}{a}\right) + \\ + \dot{y}_{\text{н}i}\frac{x_{10j}}{a} + \sum_{n=1}^{n_n} \dot{S}_n\left[f_n(x_{\text{в}})\left(1 - \frac{x_{10j}}{a}\right) + f_n(x_{\text{н}})\frac{x_{10j}}{a}\right]. \qquad (10.49)$$

In this case the angle of rotation of longitudinal axis of j-th tank on the section filled with liquid will be

$$\Delta\vartheta_{ij} = \Delta\vartheta - \sum_{n=1}^{n_n} S_n[f_n(x_{\text{н}}) - f_n(x_{\text{в}})]\frac{1}{a} - \frac{1}{a}(y_{\text{н}i} - y_{\text{в}i}) - \\ - \sum_{p=1}^{n_p} K_{pi}[\Phi_{pi}(x_{\text{н}j}) - \Phi_{pi}(x_{10j})]\frac{1}{h_{ij}}. \qquad (10.50)$$

Having placed (49) and (50) in (2.41) and using formula (2.48) and boundary condition on the free surface of liquid (2.40"), we obtain the following approximate equation for finding the coefficients of decomposition of oscillations of the free surface with respect to forms of natural oscillations of liquid in j-th tank of i-th side-mounting booster:

$$\ddot{\beta}_{kij} + \omega_{kij}^2\beta_{kij} + \Delta\vartheta_{ij}(x_{\text{т}} - x_{\text{в}} - x_{10j}) + \ddot{\eta}_{\text{т}} + \ddot{y}_{\text{н}i}\frac{x_{10j}}{a} + \\ + \sum_{p=1}^{n_p} \ddot{K}_{pi}\Phi_{pi}(x_{10j}) + \sum_{n=1}^{n_n} \ddot{S}_n\left[f_n(x_{\text{в}})\left(1 - \frac{x_{10j}}{a}\right) + f_n(x_{\text{н}})\frac{x_{10j}}{a}\right] + \\ + \sigma_{kj}^2\Delta\vartheta_{ij} + \ddot{y}_{\text{в}}\left(1 - \frac{x_{10j}}{a}\right) = 0 \qquad (10.51)$$

$$(i = 1, 2, \ldots, k_i;\ j = 1, 2, \ldots, n_j).$$

Having formulated expressions for potential and kinetic energies of the flight vehicle construction on the whole without taking into account the liquid and expressions for generalized forces, corresponding to generalized coordinates $\eta_{\text{т}}$, $\Delta\vartheta$, S_n, K_{pi}, $y_{\text{н}i}$ and $y_{\text{в}i}$ taking into account hydrodynamic loads, it is simple to find approximate equations of small oscillations of a flight vehicle of pack configuration in the pitching (yawing) plane, considering the influence of both elasticity of the construction and mobility of liquid in tanks. With the presence of fuel tanks of complex shape the parameters of oscillations of liquid, utilized in corresponding equations, can be determined experimentally (for instance, by means of modeling).

Computation of the shown approximate equations is connected with carrying out bulky transformations. In this case the matrix of coefficients of the system of these equations will be asymmetric. However, this asymmetry is small and will not practically affect the results of calculation. In particular, it permits the application of linear transformation of coordinates, presented in Chapter V.

In conclusion let us note that transverse oscillations of a guided flight vehicle on the powered flight phase generally are described by an infinite number of ordinary second order differential equations connected together. During solution of practical problems, and namely when determining internal power factors and investigation of motion stability of the flight vehicle can be limited by consideration of only a finite number of these equations. In particular, it is possible to consider, as was already emphasized, only the fundamental tone of oscillations of liquid in fuel tanks (i.e., to take $k = 1$) and only the lowest tones of natural elastic oscillations of the construction of central and lateral units (for instance, $n_n = 3$ and $n_p = 2$). Selection of the number of degrees of freedom (i.e., values of n_n and n_p) and other simplifications in the shown equations of transverse oscillations should be performed in every concrete case depending upon the peculiarities of the constructive layout of the flight vehicle, its dynamic characteristics and parameters of motion. Thus, for instance, when the frequencies of free bending oscillations of the construction of units of the flight vehicle are higher than partial frequencies of free oscillations of liquid in tanks $\left(\frac{\omega_n}{\omega_{bj}} > 2\right)$, it is possible to disregard the mutual influence of elastic oscillations of liquid in tanks. Furthermore, with symmetric location of side-mounting boosters relative to the central unit it is possible to decrease the number of equations describing elastic oscillations of the construction of the vehicles themselves.

§ 10.5. Equations of Transverse Oscillations of a Vehicle of Pack Configuration in Normal Coordinates

Solution of systems of equations in generalized coordinates, describing transverse oscillations of a flight vehicle of pack configuration in flight, even for a limited number of degrees of freedom is a time-consuming operation, especially when the mobility of liquid in fuel tanks is considered. For simplification of the problem it is possible to use, as was already noted, the method of linear transformation of systems of equations in generalized coordinates to a system of equations in normal coordinates (§ 5.5). For this it is necessary only to preliminarily reduce the considered system of equations to corresponding form (5.55) or (5.68).

In particular, in order to reduce system of equations (41)-(44), describing transverse elastic oscillations of the flight vehicle in flight without taking into account the mobility of liquid in tanks, to the required form, it is necessary to exclude from it the terms containing variables $\ddot{\eta}_T$ and $\Delta\ddot{\vartheta}$. For this it is sufficient in these equations to substitute expressions $\ddot{\eta}_T$ and $\Delta\ddot{\vartheta}$, found from equations (39) and (40) respectively. As a result we obtain the sought system of equations with modified coefficients. By designating the latter with a prime above (a'_{ij} and b'_{ij}), we will have

$$\sum_{j=1}^{m_0-2} a'_{ij}\ddot{q}_j + \sum_{j=1}^{m_0-2} b'_{ij}q_j = Q_i \qquad (i = 1, 2, \ldots, m_0 - 2).$$

This system is easily converted to a system of independent equations of form

$$\ddot{p}_m + 2h_m\dot{p}_m + \omega_m^2 p_m = Q_m, \tag{10.52}$$

where ω_m – angular frequency of natural elastic oscillations of the construction of a flight vehicle in flight, and Q_m – corresponding generalized force, determined by formula

$$Q_m = \frac{1}{M_m}\sum_{i=1}^{m_0-2} Q_i A_{im}. \tag{10.53}$$

Here by Q_i we designate the right sides of corrected equations (41), (42), (43) and (44), m_0 – the number of degrees of freedom.

Thus, equations of transverse oscillations of the construction of a flight vehicle of pack configuration (without taking into account the mobility of liquid in fuel tanks) in normal coordinates in the pitching (yawing) plane will have the following form:

$$m\ddot{\eta}_{\tau} + c_{\text{д}} \frac{1}{v} \dot{\eta}_{\tau} + c_{\alpha}\alpha = R_y^{\delta}\delta + \sum_{l=1}^{k_l} R_{yl}^{\delta}\delta_l \sin\gamma_l + \Delta Y_{\text{в}}, \tag{10.54}$$

$$J_z \Delta\ddot{\vartheta} + b_{\text{д}} \Delta\dot{\vartheta} + b_{\alpha}\alpha = -b_{\delta}\delta - \sum_{l=1}^{k_l} b_{\delta l}\delta_l \sin\gamma_l + \Delta M_{z\text{в}}, \tag{10.55}$$

$$\ddot{p}_m + 2h_m \dot{p}_m + \omega_m^2 p_m = Q_m. \tag{10.56}$$

In this case

$$u_{\vartheta} = k_{\vartheta}\left(\Delta\vartheta + \sum_{m=1}^{m_0-2} p_m d_m\right),$$

$$d_m = \Delta\vartheta_m - \sum_{n=1}^{n_n} S_{nm} \frac{df_n(x)}{dx}\bigg|_{x=x_g},$$

$$h_m = \frac{1}{M_m} \sum_{l=1}^{m_0-2} h_l A_{lm}^2 M_l.$$

The form of transverse oscillations of the construction, for instance, of the central unit, corresponding to m-th tone of natural oscillations of the system, will be determined by an expression similar to (19)

$$f_m(x) = \eta_m + \sum_{n=1}^{n_n} S_{nm} f_n(x) + \Delta\vartheta_m (x_{\tau} - x), \tag{10.57}$$

and side-mounting booster – by an expression similar to (20). Coefficients $\Delta\vartheta_m$ and η_m are found from equations (40) and (39)

$$\left.\begin{aligned} \Delta\vartheta_m &= -\left[\sum_{n=1}^{n_n} b_n S_{nm} + \sum_{l=1}^{k_l}\left(\sum_{p=1}^{n_p} b_{kpl} K_{plm} + b_{\alpha l} y_{\alpha lm} + b_{\kappa l} y_{\kappa lm}\right)\right]\frac{1}{I_z}, \\ \eta_m &= -\left[\sum_{n=1}^{n_n} c_n S_{nm} + \sum_{l=1}^{k_l}\left(\sum_{p=1}^{n_p} \beta_{pl} K_{plm} + c_{\alpha l} y_{\alpha lm} + c_{\kappa l} y_{\kappa lm}\right)\right]\frac{1}{m} \end{aligned}\right\} \tag{10.58}$$

and establish the position of the axis of oscillations in the system of coordinates connected with initial undeformed state of the flight vehicle construction. Equation of this axis is the following:

$$\eta_0(x) = \eta_m + \Delta\theta_m(x_\tau - x). \qquad (10.59)$$

Before proceeding to transformation of the system of equations of transverse oscillations of the flight vehicle in generalized coordinates of the flight vehicle, considering the mobility of liquid in fuel tanks, we should separate it into two subsystems, namely, separate

1) equations describing only elastic oscillations,

2) equations describing only oscillations of liquid in fuel tanks.

In both subsystems of equations (in generalized coordinates) let us exclude terms with $\ddot{\eta}_\tau$ and $\Delta\ddot{\theta}$ as is done in the preceding case. Then let us convert the first subsystem of equations to normal coordinates, relating terms containing $\ddot{\beta}_{kj}$ and $\ddot{\beta}_{kjl}$ to generalized forces. As a result we obtain a system of equations of form

$$M_m(\ddot{p}_m + \omega_m^2 p_m) = \sum_{j=1}^{n_1} c_{mj}\ddot{\beta}_{kj} + \sum_{l=1}^{k_l}\sum_{j=1}^{n_2} c_{mjl}\ddot{\beta}_{kjl} + Q_m M_m, \qquad (10.60)$$

where

$$c_{mj} = \sum_{p=1}^{n_p} A_{pm} t'_{kjp}, \qquad c_{mjl} = \sum_{p=1}^{n_p} A_{pm} t'_{kjlp},$$

n_1 – number of fuel tanks in central unit, n_2 – in side-mounting booster.

Let us connect the second subsystem of equations to it

$$c_{kj}(\ddot{\beta}_{kj} + \omega_{kj}^2 \beta_{kj}) = \sum_{m=1}^{m_0-2} \ddot{p}_m f_m(x_{0j}) c_{kj}, \qquad (10.61)$$

$$c_{kjl}(\ddot{\beta}_{kjl}+\omega_{kjl}^2\beta_{kjl})=\sum_{m=1}^{m_0-2}\ddot{p}_m f_m(x_{10l})c_{kjl}. \quad (10.62)$$

Then to system of equations (60), (61), (62) in new generalized coordinates p_m, β_{kj} and β_{kjl} $(m=1, 2, \ldots, m_0-2;\ j=1, 2, \ldots, n_1(n_2);\ l=1, 2, \ldots, k_1)$ coordinates

$$\left.\begin{aligned} p_m(t)&=\sum_{p=1}^{n_3} q_p(t)D_{pm},\\ \beta_{kj}(t)&=\sum_{p=1}^{n_3} q_p(t)D_{pj},\\ \beta_{kjl}(t)&=\sum_{p=1}^{n_3} q_p(t)D_{pjl},\end{aligned}\right\} \quad (10.63)$$

where n_3 – number of degrees of freedom of the system on the whole, equal to $n_3=m_0-2+n_1+k_1n_2$, and q_p – normal coordinate of p-th tone. Coefficients t_{kjp} (for central unit) and t_{kjlp} (for lateral units) are found by formulas similar to the formula for t_{kjn} (6.23').

As a result we obtain the sought equations of transverse oscillations of an elastic flight vehicle of pack configuration with liquid filling in normal coordinates.

C H A P T E R XI

DYNAMICS OF THE CONSTRUCTION IN THE PROCESS OF SEPARATION OF STAGES

§ 11.1. Cases of Loading

In this chapter there is investigated loading of the construction of a flight vehicle by external perturbing forces and moments in the period of shutdown of engines of one stage and switching on engines of another (following) stage. We consider the transient process from one steady state of loading (case B_1) to another state (case L_1), accompanied by abrupt change of inertial, dynamic, aerodynamic and constructive characteristics of the flight vehicle.

The character of loading of the construction of a flight vehicle in the process of separation stage basically depends on the assembly diagram of the vehicle itself and the diagram of switching on the engine of the subsequent stage. Selection of the latter is produced in such a way that:

1) at the moment of starting of liquid-propellant engines there was ensured entry of fuel into pumps, i.e., longitudinal overload n_{x_1} was larger than zero;

2) time of free flight of the flight vehicle (when P = 0) was short;

3) as far as possible this case of loading was not rated for basic elements of the flight vehicle construction;

4) stage separation and operation of all mechanisms occurred reliably, i.e., there was no collision of separated parts with the flight vehicle;

5) perturbations received by the flight vehicle in the process of separation were minimum.

The simplest form of the scheme of separation is for flight vehicles with lateral division of stages and with solid-propellant engines, for which fulfillment of the first condition $(n_{x_1}>0)$ is not required. It appears so: the operating engine is shut down, locks or bolts connecting the separable unit with the flight vehicle are broken, then the engine of the following stage is turned on (Fig. 11.1). Separation by such a scheme is called "cold." For flight vehicles with liquid-propellant engines a more complicated scheme of "hot" separation is applied, at which a drop of thrust force of the separable unit (Fig. 11.2).

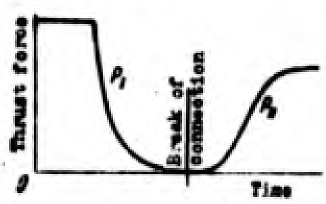

Fig. 11.1. Change of thrust force of units of the flight vehicle with "cold" separation stage.

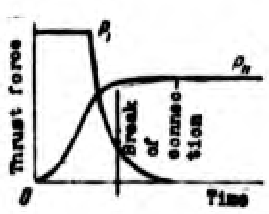

Fig. 11.2. Change of thrust force of units of the flight vehicle with "hot" separation stage.

Depending upon the parameters of transient processes of the shutdown and turned on engines, much variety is possible in the scheme of loading of the construction of the flight vehicle by external longitudinal forces. In particular, with rapid decrease of thrust force $P_I(t)$ and slow build-up of thrust force $P_{II}(t)$ at first there is turned on the engine of unit II (second stage), then, after achievement of a definite value by force $P_{II}(t)$, the engine of unit I (first stage) is turned off. At the moment when $n_{x\,II} = \frac{P_{II}}{G_{II}}$ takes a value equal to or larger than specified, and thrust force of unit I becomes

$$P_I(t) < G_I \frac{P_{II}(t)}{G_{II}}, \qquad (11.1)$$

where G_{II} – weight of flight vehicle without separable unit I, G_I – weight of unit I, mechanical separation stage is produced. If, conversely, the drop of thrust force of unit I is slow as compared to the growth of thrust force of unit II, then at first there is given the command for shutdown of the engine of the first stage, and then the command for turning on the engine of the second stage.

A peculiarity of "hot" separation is the presence of the influence of a stream of gases, flowing from the combustion chamber of the engine of unit II, on the body of the separable unit I. As a result at the place of attachment (to the body of this unit) of deflecting device x_0 to the flight vehicle (from the moment of beginning of growth of thrust force $P_{II}(t)$ to the moment of separation of the unit) additional longitudinal force R_{II} will act (Fig. 11.3). The magnitude of this force can be comparable with the magnitude of force P_{II}, and its character of change with respect to t can be similar to the character of change of thrust force $P_{II}(t)$.

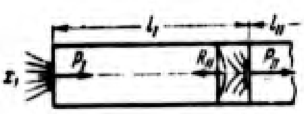

Fig. 11.3. Diagram of external longitudinal forces, affecting the flight vehicle in the process of "hot" stage separation.

Separation of side-mounting boosters from a flight vehicle of pack configuration can occur by the following program. All engines of side-mounting boosters are turned off and all power connections between the bodies of boosters and the body of the central unit are broken. The latter continues flight under the action of thrust force of its engines and departs from the boosters. In this case for elimination of the possibility of collision of boosters with the central unit (in the process of uncoupling) forced withdrawal of boosters in a lateral direction can be applied.

The class of "cold" separation stage can include the case of separation of a nose cone from the flight vehicle (during normal or emergency flight) with the aid of a special device (spring, pneumatic or pyrotechnic) or jet engine. The basic requirement, which is imposed on this case of loading (case K_{Γ}), consists of limitation of disturbances of the parameters of motion of the separated nose cone (descent vehicle). Fulfillment of the shown requirement is important, of course, for separation of all stages of the vehicle, since perturbation, which the flight vehicle receives in the process of stage separation, can noticeably affect the character of loading of the following stage.

In many cases the magnitudes of longitudinal and lateral forces and bending moments in the period of stage separation essentially depend on the initial values of these forces at the moment of beginning of the separation process. In connection with this below, along with case K_1, the following cases of loading are considered:

1) case $B_1(\max n_{x1})$, coinciding with the moment of beginning of the process of separation of i-th stage;

2) case L_1, corresponding to loading of the flight vehicle after separation of k-th stage $(\min n_{x,i+1})$.

In case K_1 the flight vehicle, besides longitudinal, can be affected by various types of lateral perturbing forces and moments. The source of these forces can be the external medium, and also operation of the engine and control system. Thus, for instance, if separation occurs at comparatively high impact pressures, then the flight vehicle will be affected by lateral aerodynamic loads, caused by the presence of angles of attack (programmed or from the influence of the wind). With the use of a compound propulsion system (cluster of engines), due to inevitable scattering of actual laws of change of thrust force of separate engines (during shutdown and starting), the construction will be affected by concentrated perturbing moments, the value of which will depend on the power of engines and lateral dimensions of the vehicle. Values of lateral control forces will in turn depend on magnitudes of the above-indicated perturbing moments and on those angular oscillations of the flight vehicle, which it had at the moment of beginning of stage separation.

Almost all external lateral forces and moments, affecting the flight vehicle in the process of stage separation, are random functions of time. If at the stage of sketch designing of the vehicle there is no information about estimators of these functions, then it is necessary to be oriented mainly to the possible limiting particular cases of loading, analogous to those which were considered in the case of launch of a flight vehicle, namely:

1) case of synchronous change of lateral force for all engines of the vehicle;

2) case of appearance of limiting values of lateral perturbing forces and moments.

§ 11.2. Equations of Longitudinal Oscillations

By the method, discussed in detail in Chapters V and X, equations of longitudinal oscillations for any flight vehicle can be obtained. According to this method the construction is divided into a series of partial systems, dynamic properties of which are characterized by generalized mass and generalized rigidity (partial frequency of natural oscillations). The method of determination of these generalized parameters of partial systems depends on peculiarities of the dynamic layout of particular flight vehicles. The latter can be represented in the form of a beam of variable section with apparent masses or in the form of rigid bodies elastically connected together. When comparatively large concentrated loads (underslung fuel tanks, propulsion system, pods with instruments and others) are applied to the carrier body of a flight vehicle with the aid of elastic connections, having low partial frequencies of natural oscillations, or when this body is degenerated into the construction, slightly resembling a thin-walled rod (for instance, into frame, short shell), replacement of the partial distributed system by a system with a finite number of elastically connected rigid bodies is natural.

For many flight vehicles, especially ballistic type, longitudinal oscillations of the construction in case K_1 are described well by the dynamic scheme, representing a beam (or system of such beams) with arbitrarily distributed mass and rigidity, to which loads are elastically suspended in different sections. Longitudinal shifts u of centers of gravity of such loads during elastic oscillations will be composed of shifts caused by general deformations of the flight vehicle body and deformation of the elastic connection itself (taking into account local deformation of the construction of the body at the point of connection), i.e.,

$$u(x_i, t)=\sum_{n=1}^{\infty} T_n(t)X_n(x_i)+\Delta x_i.$$

Potential energy of such a system will consist of potential strain energy of the flight vehicle body and strain energy of elastic connections, and kinetic energy — the sum of kinetic energies of small oscillations of elements of the body and loads. Lagrange equations for a similar system will take the form

$$\left.\begin{aligned} &\Delta\ddot{x}_i + 2h_i\Delta\dot{x}_i + \omega_i^2\Delta x_i + \sum_{n=1}^{\infty}\ddot{T}_n X_n(x_i) = 0, \\ &M_n(\ddot{T}_n + 2h_n\dot{T}_n + \omega_n^2 T_n) + \sum_{i=1}^{k}\Delta\ddot{x}_i m_i X_n(x_i) = F_{xn}, \end{aligned}\right\} \tag{11.2}$$

where F_{xn} — some function describing the external influence on the system in longitudinal direction, x_i — coordinate of attachment of load of mass m_i to the body of the flight vehicle, Δx_i — relative longitudinal shift of load, and ω_i — partial frequency of oscillations of the load·

$$\omega_i^2 = \frac{c_i}{m_i}.$$

Here c_i is the generalized rigidity of connection, equal to the magnitude of longitudinal force applied to the center of gravity of the load causing unit relative longitudinal shift of this center of gravity (due to deformation of the connection, local deformation of the flight vehicle body and structural elements of the load itself). Of course the values of c_i can be determined by another method, for instance by means of dynamic tests of the construction or its model.

By reducing system of equations (2) to normal coordinates, we obtain a system of independent second order equations, coinciding in form with equations of longitudinal oscillations of a beam. In particular cases the above-indicated system of equations is degenerated into a system of equations of form (7.20) or into a system of equations describing motion of a branched chain of elastically connected rigid bodies.

The dynamic layout of any construction of a flight vehicle, represented in the form of a system of rigid bodies, connected together by springs, can be composed of typical links. In such a layout it is possible to separate: a) end link, representing a rigid body of mass m_{1j} (Fig. 11.4), suspended on a spring, working on tension-compression, bending or torsion; b) simple intermediate link, representing a rigid body of mass m_i (Fig. 11.4), connected by elastic elements to adjacent bodies m_{ij} and m_{i-1}; c) complicated intermediate link m_i (Fig. 11.5), differing from the simple intermediate link by the presence of a series of joined end links, and finally; d) basic link, to which is joined a number of complicated intermediate links (Fig. 11.6).

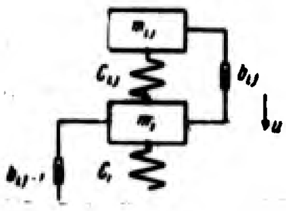

Fig. 11.4. Simple intermediate link.

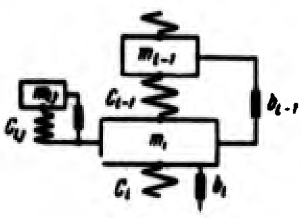

Fig. 11.5. Complicated intermediate link.

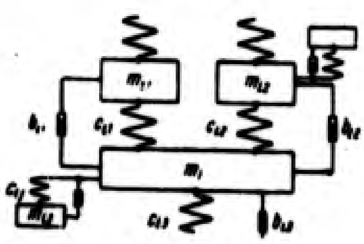

Fig. 11.6. Basic link.

The motion of each typical link occurs under action of forces caused by deformations of the corresponding elastic connections, damping forces and external forces, applied to the part of the construction of the flight vehicle, oscillations of which are simulated by the given link. Thus, equations of longitudinal oscillations of end links will have the form

$$m_{ij}\ddot{u}_{ij} + c_{ij}(u_{ij} - u_i) + b_{ij}(\dot{u}_{ij} - \dot{u}_i) = F_{ij}, \qquad (11.3)$$

where c_{ij} – generalized rigidity of j-th elastic connection, equal to force applied to the center of gravity of the part of the construction simulated by given body m_{ij}, and causing its unit shift (in the direction of action of force) with respect to the center of gravity of i-th part of the construction (m_i), F_{ij} – external force, u_{ij} and u_j – longitudinal shifts of centers of gravity of the shown bodies, b_{ij} – damping factor.

Longitudinal oscillations of simple and complicated intermediate links of mass m_i (taking into account only the basic component of damping forces) are described by the following differential equations:

$$m_i\ddot{u}_i + c_i(u_i - u_{i+1}) + c_{ij}(u_i - u_{ij}) + b_i(\dot{u}_i - \dot{u}_{i+1}) = F_i, \quad (11.4)$$

$$m_i\ddot{u}_i + c_i(u_i - u_{i+1}) + c_{i-1}(u_i - u_{i-1}) + b_i(\dot{u}_i - \dot{u}_{i-1}) + \sum_{j=1}^{k_i} m_{ij}\ddot{u}_{ij} = F_i. \quad (11.4')$$

Equation of motion of the basic link is presented in the form

$$m_i\ddot{u}_i + \sum_{p=1}^{k_p} c_{ip}(u_i - u_{ip}) + \sum_{j=1}^{k_i} m_{ij}\ddot{u}_{ij} + \sum_{p=1}^{k_p} b_{ip}(\dot{u}_i - \dot{u}_{ip}) = F_i. \quad (11.5)$$

In these formulas c_{ip} – generalized rigidity of elastic connection of p-th intermediate link, joined to the basic link, k_i – number of end links joined to i-th intermediate or basic link, k_p – number of complicated intermediate links joined to basic link.

By using the above-mentioned equations of separate links it is simple to formulate equations of longitudinal oscillations of the construction of any flight vehicle. The quantity of such equations will be equal to the overall number of links in the system, and consequently, will be basically determined by the assembly diagram of the vehicle and requirements of the problem. It is obvious that the smaller the quantity of these links, the simpler the solution of the problem, but the lower the accuracy of description of real oscillations of the system. Therefore, during formulation of a similar dynamic layout it is preliminarily expedient to estimate the degree of influence of parameters of separate links on the reaction of the system on the whole. Solution of systems of equations (3), (4) and (5), and also (2) can be found directly by the method of numerical integration or by means of conversion to normal coordinates.

§ 11.3. Calculation of Longitudinal Forces

Longitudinal forces, appearing in the process of "hot" stage separation in structural elements of the flight vehicle, equations of oscillations of which are represented in normal coordinates, can be determined in the following way.

Let us designate all parameters of the flight vehicle, pertaining to the first stage of flight (before separation), by index I, and to the second – index II. In accordance with the accepted method of calculation the instantaneous value of longitudinal forces will be equal to the sum of static and dynamic components

$$N(x_1, t) = N_c(x_1, t) - \sum_{n=1}^{\infty} \ddot{T}_n(t_1) N_{nx}(x_1)$$

The static component is proportional to static longitudinal overload, which in this case will be determined by expressions

$$n_{x_{\rm I}}(t_1) = \frac{P_{\rm I}(t_1)}{G_{\rm I}(t_1)} \quad \text{when } 0 < t_1 < t_{\rm II},$$

$$n_{x_{\rm I}}(t_1) = \frac{1}{G_{\rm I}(t_1)} [P_{\rm I}(t_1) + P_{\rm II}(t_1 - t_{\rm II}) - R_{\rm II}(t_1 - t_{\rm II})] \quad \text{when } t_{\rm II} < t_1 < t_0,$$

$$n_{x_{\rm II}}(t_1) = \frac{P_{\rm II}(t_1)}{G_{\rm II}(t_1)} \quad \text{when } t_1 > t_0,$$

where t_1 – time counted from the beginning of the actual drop of thrust force of the first stage, t_{II} – time of actual switch-on of engines of the second stage, and t_0 – moment of mechanical separation of units of the construction.

Differential equation for determination of acceleration of the point of reduction $\ddot{T}_n(t_1)$ (for section $0<t_1<t_0$) will have the form

$$\ddot{T}_n(t_1)+2h_n\dot{T}_n(t_1)+\omega_n^2T_n(t_1)=$$
$$=-\frac{1}{M_n}[P_{II}(t_1)X_n(x_{1nII})-R_{II}(t_1)\cdot X_n(x_{1n2})]\,\sigma(t_1-t_{II})-$$
$$-\frac{P_I(t_1)}{M_n}X_n(x_{1nI}), \qquad (11.6)$$

where

$$\sigma(t_1-t_{II})=\begin{cases}0 & \text{when } t_1<t_{II},\\ 1 & \text{when } t_1>t_{II},\end{cases}$$

ω_n, X_n, M_n — frequency, form and reduced mass of n-th tone of natural longitudinal oscillations of the flight vehicle on the whole (with free ends), x_{1nI}, x_{1nII} and x_{1n2} — coordinates of those sections of the body, in which there are applied forces P_I, P_{II} and R_{II} respectively.

Solution of this system of equations is sought with initial conditions

$$T_n(t_1)=T_{n0},\quad \dot{T}_n(t_1)=\dot{T}_{n0} \quad \text{when } t_1=0,$$

which are determined by means of expansion in the forms of natural oscillations with weight $m(x_1)$ of actual values of initial longitudinal shifts $u(x_1,t)$ and rates $\dot{u}(x_1,t)$ of cross sections of the body of the flight vehicle at moment $\bar{t}=\bar{t}_k$

$$\left.\begin{aligned}T_{n0}&=\frac{1}{M_n}\int\limits_0^{l_1}u(x_1,\bar{t}_k)\,m(x_1)X_{n1}(x_1)\,dx_1,\\ \dot{T}_{n0}&=\frac{1}{M_n}\int\limits_0^{l_1}\dot{u}(x_1,\bar{t}_k)\,m(x_1)X_{n1}(x_1)\,dx_1.\end{aligned}\right\}\qquad (11.7)$$

When $t_1 > t_0$, i.e., after separation of part of the construction from the flight vehicle, all the dynamic characteristics of the vehicle are changed.

In this case the values of T_{mII} will be determined from equation

$$\ddot{T}_{mII}(t_1) + 2h_{mII}\dot{T}_{mII}(t_1) + \omega^2_{mII}T_{mII}(t_1) = -\frac{X_{mII}(x_{1eII})}{M_{mII}}P_{II}(t_1) \qquad (11.8)$$

with initial conditions

$$T_{mII}(t_1) = T_{mII0}, \quad \dot{T}_{mII}(t_1) = \dot{T}_{mII0} \text{ when } t_1 = t_0.$$

In accordance with formulas (7)

$$\left.\begin{aligned} T_{mII0} &= \frac{1}{M_{mII}}\sum_{n=1}^{\infty} T_{nI}(t_0)\int_0^{l_{II}} m(x_1)X_{nI}(x_1)X_{mII}(x_1)\,dx_1, \\ \dot{T}_{mII0} &= \frac{1}{M_{mII}}\sum_{n=1}^{\infty} \dot{T}_{nI}(t_0)\int_0^{l_{II}} m(x_1)X_{nI}(x_1)X_{mII}(x_1)\,dx_1, \end{aligned}\right\} \qquad (11.9)$$

where $T_{nI}(t_0)$, $\dot{T}_{nI}(t_0)$ — values of function $T_{nI}(t_1)$ and its derivative at moment $t_1 = t_0$; ω_{mII}, X_{mII}, M_{mII} — frequency, form and reduced mass of m-th tone of natural longitudinal oscillations of the flight vehicle construction at the beginning of the second phase of flight respectively, l_{II} – length of the second stage of the vehicle. In this case

$$N_{xII}(x_1, t_1) = -\sum_{m=1}^{\infty} \ddot{T}_{mII}(t_1)N_{mx}(x_1), \qquad (11.10)$$

where

$$N_{mx}(x_1) = \int_0^{x_1} m(x_1)X_{mII}(x_1)\,dx_1.$$

At the beginning of the second phase of flight the total longitudinal force will be approximately equal to

$$N(x_1, t) = P_{11}(t)\frac{G(x_1)}{G_{11}} - \sum_{m=1}^{\infty} N_{mx}(x_1) \times$$
$$\times\left\{\ddot{T}_{m11}(t) - \omega_{m11}^2\left[T_{m110}\cos\omega_{m11}t + \frac{1}{\omega_{m11}}\dot{T}_{m110}\sin\omega_{m11}t\right]e^{-h_{m11}t}\right\}, \qquad (11.10')$$

where t — time counted from the moment of mechanical stage separation.

For illustration Fig. 11.7 shows a graph of the change of longitudinal force in one of the sections of the flight vehicle body in the process of "hot" stage separation. In this case it was assumed that at first the engine of the second stage is switched on, and the engine of the first stage will be shutdown with some delay.

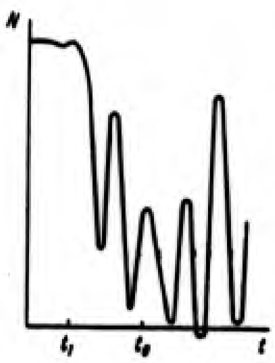

Fig. 11.7. Change of longitudinal force in the cross section of the flight vehicle body in the process of "hot" stage separation.

Values of dynamic components of longitudinal forces depend basically on the speed of change of external forces. The greater it is, the bigger is $N_{\text{д}}(x_1, t)$ and the greater the role the oscillations of construction of the highest tones will play. However, as it is known, the accuracy of determination of the forms and frequencies of natural elastic oscillations drops with increase of the number of tone. Therefore, for complicated assembly diagrams of flight vehicles, conditions of elastic oscillations of which in the required range of frequencies can essentially depend on the local rigidity of separate structural elements, it is better to use a system of equations in generalized coordinates of the form (3)-(5). Moreover, in the case of very rapid (sudden) removal of thrust force the values of longitudinal accelerations of links of the system can be obtained by means of solving the shown system of equations with initial conditions corresponding to static loading of the construction by longitudinal mass forces at the moment of beginning of drop of thrust force, i.e., when $t_1=0$ and $F_i=0$, $F_{ij}=0$. Thus, for instance, with sudden removal of thrust force of the first stage (in particular, with "cold" separation of stages) these initial conditions (in the absence of longitudinal oscillations of the construction in case B) for the above-indicated typical links of the system lead to equality of the longitudinal rates of all links to zero ($\dot{u}_i=\dot{u}_{ij}=0$) and to the following values of initial shifts:

for end links

$$u_{ij}(0)=u_i(0)+g_0 m_{ij}\frac{1}{c_{ij}}\max n_{x1},$$

for simple intermediate links

$$u_i(0)=u_{i+1}(0)+g_0\frac{1}{c_i}\max n_{x1}\sum_{j=1}^{i} m_j,$$

for complicated intermediate links

$$u_i(0)=u_{i+1}(0)+g_0\frac{1}{c_i}\max n_{x1}\left(\sum_{j=1}^{i} m_j+\sum_{j=1}^{k_i} m_{ij}\right)$$

and for basic links

$$u_i(0)=u_{i+1}(0)+g_0\frac{1}{c_i}\max n_{x1}(m_i+m_{0i}),$$

where m_{01} – mass of construction, joined to this link (from above). For n-th link, to which thrust force of engines of the separable stage is applied, $u_n(0)=0$.

In the conclusion of this paragraph let us note that the above-mentioned solution can also be used for determination of parameters of longitudinal oscillations of the construction of the central unit of a flight vehicle of pack configuration in the case of separation of side-mounting boosters. For this it is necessary only to place

$$P_{11}=P_A,\quad R_{11}=0,\quad P_1=\sum_{l=1}^{k_l}(P_{\text{б}l}-n_{x1}G_{\text{б}l}),$$

$$n_{x1}=\frac{P_A+\sum_{l=1}^{k_l}P_{\text{б}l}}{G_A+\sum_{l=1}^{k_l}G_{\text{б}l}}.$$

An important feature of loading of the flight vehicle construction in case K_1 is the possibility of change of the sign of longitudinal forces and overloads in the process of separation of stages, i.e., the appearance of longitudinal tensile loads in the sections of the body and especially in mounting lugs of loads and at places where units are connected together.

§ 11.4. Transverse Oscillations of the Construction in the Process of Separation of Stages

The influence of transverse elastic oscillations of the construction on the required bearing capacity in this case of loading turns out to be essential only for flight vehicles with parallel arrangement of stages, i.e., equipped with side-mounting boosters (for instance, Б and В).

Let us assume that side-mounting boosters are separated from the flight vehicle so that the possibility of their rotation around upper hinged connection "в" is preserved. For determinacy let us assume that commands for shutdown of engines of boosters and for break of lower lateral force connections "н" are given simultaneously. Due to the variation of time of operation of the system turning off the engines, and namely the time from the moment of issue of command to the moment of termination of operation of the corresponding fuel shutoff valves, there will always occur a delay of the beginning of actual drop of thrust force t_0 of one engine with respect to the other, and also variation of the magnitudes (and laws of drop) themselves of these forces. In other words, real values of longitudinal forces (reactions), transmitted to the flight vehicle from each booster, will be different at each moment of time. As a result during separation of stages the flight vehicle will always be affected by unbalanced dynamic moment, equal to

$$M_{\text{нр}} = r(x_{\text{в}})[P_{\text{Б}}(t_1) - P_{\text{В}}(t_1)]. \qquad (11.11)$$

With rotation of the side-mounting booster to small angle μ_1 (Fig. 11.8) it will additionally be affected by concentrated (in section $x_в$) lateral forces: on the part of booster Б

$$R_{y\text{Б}} = P_{\text{Б}}(t_1)\mu_{\text{Б}} - m_{б\text{Б}}\ddot{\mu}_{\text{Б}}$$

and on the part of booster В

$$R_{y\text{В}} = -P_{\text{В}}(t_1)\mu_{\text{В}} + m_{б\text{В}}\ddot{\mu}_{\text{В}},$$

where $\dot{\mu}_i$ $(i=\text{Б}, \text{В})$ — initial value of angle of rotation of longitudinal axis of i-th booster, t_1 — current time, counted from the moment of beginning of drop of thrust force of one of the boosters, for instance Б.

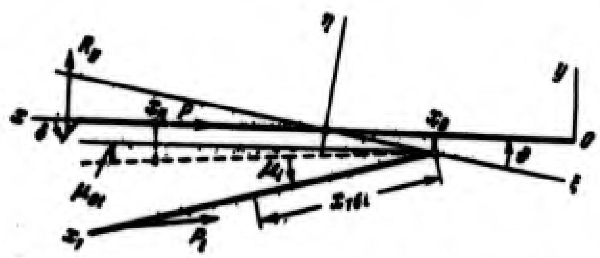

Fig. 11.8. Calculated scheme of separation of side-mounting booster.

With sufficient accuracy it is possible to express the changes of thrust force of these boosters in the process of engine shutdown by formulas

$$\left.\begin{aligned} &P_{\text{Б}}(t_1) = P_{\text{к}}\left(1 - \frac{t_1}{\tau_{\text{Б}}}\right), \\ &P_{\text{В}}(t_1) = P_{\text{к}} \end{aligned}\right\} \text{ when } t_1 < \tau_0,$$

$$\left.\begin{aligned} &P_{\text{Б}}(t_1) = P_{\text{к}}\left(1 - \frac{t_1}{\tau_{\text{Б}}}\right), \\ &P_{\text{В}}(t_1) = P_{\text{к}}\left(1 - \frac{t_1 - \tau_0}{\tau_{\text{В}} - \tau_0}\right) \end{aligned}\right\} \text{ when } \tau_0 < t_1 < \tau_{\text{Б}}^{\circ},$$

$$\left.\begin{aligned} &P_{\text{Б}}(t_1) \leqslant n_{x_1} G_{б\text{Б}}, \\ &P_{\text{В}}(t_1) = P_{\text{к}}\left(1 - \frac{t_1 - \tau_0}{\tau_{\text{В}} - \tau_0}\right) \end{aligned}\right\} \text{ when } \tau_{\text{Б}}^{\circ} < t_1 < \tau_{\text{В}}^{\circ},$$

$$\left.\begin{aligned} &P_{\text{Б}}(t_1) < n_{x_1} G_{б\text{Б}}, \\ &P_{\text{В}}(t_1) \leqslant n_{x_1} G_{б\text{В}} \end{aligned}\right\} \text{ when } t_1 > \tau_{\text{В}}^{\circ},$$

where $\tau_Б^*, \tau_В^*$ – times corresponding to moments of separation of boosters Б and В from the flight vehicle, i.e., to moments when longitudinal components of forces, transmitted on the part of the boosters to the body of the flight vehicle, become equal to zero (Fig. 11.9a). This is observed when

$$P_Б(\tau_Б^*) = n_{x_1}(\tau_Б^*) G_{бБ}, \quad P_В(\tau_В^*) = n_{x_1}(\tau_В^*) G_{бВ}.$$

Thus, the character of change of perturbing moment with respect to t_1 will have the form shown on Fig. 11.9b. It is possible, of course, to approximate function $P_i(t_1)$ by more complicated formulas. This question is answered in every concrete case depending upon the required accuracy of calculation.

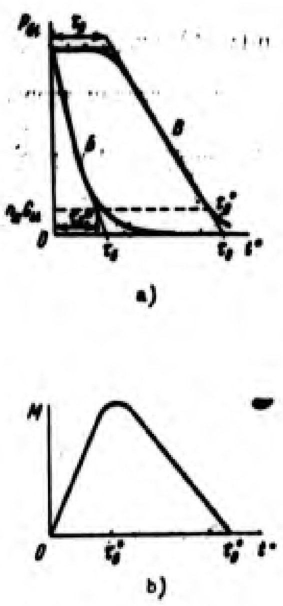

Fig. 11.9. Change of thrust force of boosters Б and В (a) and disturbing moment (b) in the process of separation of side-mounting boosters.

Longitudinal overload $n_{x_1}(t_1)$ in the process of separation of boosters is determined by expression

$$n_{x_1}(t_1) = \frac{P_A + \sum_{i=1}^{k} P_i}{G_A + \sum_{i=1}^{k} G_{бi}}. \tag{11.12}$$

Here k designates the quantity of boosters, remaining at the considered moment of time t_1 in contact with the flight vehicle, and $G_{\text{б}i}$ – weight of boosters (particularly Б and В) at the moment of separation. It is necessary to note that generally due to deviation of weight of the construction of boosters (due to allowances) their weights will be different.

Since we are interested in force only in the cross sections of the central body of the flight vehicle, and not side-mounting boosters, then in the first approximation it is possible to consider not the influence of elastic oscillations of the construction of bodies of these boosters on the values of reactions $R_{\text{Б}}$ and $R_{\text{В}}$, but to consider them absolutely rigid. If times $\tau^*_{\text{Б}}$ and $\tau^*_{\text{В}}$ are comparatively short (for instance on the order of τ_0), then angles of rotation of boosters μ_i for these time intervals (at small μ_{0i}) will be small, and it is possible to approximately consider

$$\sin(\mu_{0i}+\mu_i)\approx\mu_{0i}+\mu_i.$$

Lateral shift (in a continuous system of coordinates) of points of longitudinal axis x^* of the flight vehicle in the plane of location of boosters Б and В and in the considered case of loading will be determined by expression (10.19)

$$\eta(x,t_1)=y_{\text{т}}(t_1)+\Delta\vartheta(t_1)(x_{\text{т}}-x)+\sum_{n=1}^{n_n}S_n(t_1)f_n(x).$$

Lateral shifts of points of longitudinal axes of side-mounting boosters can be approximately represented in the form

$$\eta_i(x_1,t_1)=y_{\text{т}}(t_1)+\Delta\vartheta(t_1)(x_{\text{т}}-x_{\text{в}})+\sum_{n=1}^{n_n}S_n(t_1)f_n(x_{\text{в}})-x_1\mu_i(t_1)\cos\gamma_i.$$

Kinetic energy of the given system will be equal to

$$2T_0=m_{\text{c}}\dot{y}_{\text{т}}^2+J_{\text{c}}\Delta\dot{\vartheta}^2+\sum_{i=1}^{k}J_{\text{б}i}\dot{\mu}_i^2+\sum_{n=1}^{n_n}M_n\dot{S}_n^2-$$
$$-2\left[\dot{y}_{\text{т}}+\Delta\dot{\vartheta}(x_{\text{т}}-x_{\text{в}})+\sum_{n=1}^{n_n}\dot{S}_nf_n(x_{\text{в}})\right]\sum_{i=1}^{k}m_{\text{б}i}x_{\text{т}\text{б}i}\dot{\mu}_i\cos\gamma_i,$$

where m_c, J_c – mass and moment of inertia of the system relative to lateral axis, passing through its center of gravity,

$$m_c = m_A + \sum_{i=1}^{k} m_{бi},$$

$$x_т = \frac{1}{m_c}\left(m_A x_{тA} + \sum_{i=1}^{k} m_{бi} x_в\right),$$

$$J_c = \int_0^L m(x)(x_т - x)^2 dx + \sum_{i=1}^{k} m_{бi}(x_т - x_в)^2,$$

$$M_n = \int_0^L m_{np}(x) f_n^2(x) dx \quad (n = 1, 2, \ldots, n_n),$$

$$m_{np}(x) = m(x) + \sum_{i=1}^{k} m_{бi}\delta(x - x_в) \quad (k = 1, 2), \; \gamma_i = 0, \pi.$$

Potential energy of the system (with accepted assumptions) will be determined by the strain energy of only the central body of the flight vehicle.

With small impact pressures the generalized forces corresponding to generalized coordinates $y(t_1)$, $\Delta\theta(t_1)$, $\mu_i(t_1)$, $S_n(t_1)$, will be approximately equal to

$$Q_y = P_A \Delta\theta + \sum_{i=1}^{k} P_i \mu_i \cos\gamma_i + R_y^{\delta}\delta, \tag{11.13}$$

$$Q_{\mu i} = x_{1n}\mu_{0i} P_i - n_{x_1} G_{бi} r'_i x_{тбi}, \tag{11.14}$$

$$Q_{\theta} = (x_т - x_p) R_y^{\delta}\delta + \sum_{i=1}^{k} (P_i - n_{x_1} G_{бi})[r_i(x_в) + (\mu_{0i} + \mu_i)(x_т - x_в)] \cos\gamma_i, \tag{11.15}$$

$$Q_{Sn} = R_y^{\delta}\delta f_n(x_p) + \sum_{i=1}^{k} (P_i - n_{x_1} G_{бi})\mu_i f_n(x_в)\cos\gamma_i - \sum_{i=1}^{k} (P_i - n_{x_1} G_{бi}) r_i(x_в) \frac{df_n(x_в)}{dx} \cos\gamma_i \tag{11.16}$$

$$(i = 1, 2, \ldots, k), \quad (n = 1, 2, \ldots, n_n).$$

When it is necessary to calculate the influence of mobility of liquid in the fuel tanks of central unit (10.9), to expressions (13), (15) and (16) we should add terms

$$\Delta Q_y = \sum_{j=1}^{k_j} c_{kj}\beta_{kj}, \quad \Delta Q_\vartheta = \sum_{j=1}^{k_j} (b_{kj}\beta_{kj} + b_{k0j}\beta_{kj}),$$

$$\Delta Q_{Sn} = \sum_{j=1}^{k_j} l_{kjn}\beta_{kj}.$$

The values of coefficients c_{kj}, b_{kj}, b_{k0j} and l_{kjn} in this case will be determined by formulas (2.60), (10.15). By the same method it is possible to consider the influence of lateral aerodynamic forces.

The sought approximate equations of lateral oscillations of the flight vehicle construction in the process of separation of side-mounting boosters will have the following form:

$$m_c\ddot{y}_\text{т} - \sum_{i=1}^{k} m_{\text{б}i}x_{\text{тб}i}\ddot{\mu}_i \cos\gamma_i = Q_y, \tag{11.17}$$

$$J_c\Delta\ddot{\vartheta} - (x_\text{т} - x_\text{в}) \sum_{i=1}^{k} m_{\text{б}i}x_{\text{тб}i}\ddot{\mu}_i \cos\gamma_i = Q_\vartheta, \tag{11.18}$$

$$M_n(\ddot{S}_n + 2h_n\dot{S}_n + \omega_n^2 S_n) - \sum_{i=1}^{k} m_{\text{б}i}x_{\text{тб}i}f_n(x_\text{в})\ddot{\mu}_i \cos\gamma_i = Q_{Sn}, \tag{11.19}$$

$$J_{\text{б}i}\ddot{\mu}_i - \left[\ddot{y}_\text{т} + \Delta\ddot{\vartheta}(x_\text{т} - x_\text{в}) + \sum_{n=1}^{n_n} \ddot{S}_n f_n(x_\text{в})\right] m_{\text{б}i}x_{\text{тб}i} \cos\gamma_i = Q_{\mu_i}, \tag{11.20}$$

$$\ddot{\beta}_{kj} + \omega_{kj}^2\beta_{kj} + \Delta\ddot{\vartheta}(x_\text{т} - x_{0j}) + \ddot{y}_\text{т} + \sum_{n=1}^{n_n} \ddot{S}_n f_n(x_{0j}) + \\ + \sigma_{kj}^2\left[\Delta\vartheta + \sum_{n=1}^{n_n} S_n \frac{df_n(x_j)}{dx_j}\right] = 0, \tag{11.21}$$

$$u_\vartheta = k_\vartheta\left[\Delta\vartheta - \sum_{n=1}^{n_n} S_n \frac{df_n(x)}{dx}\bigg|_{x=x_\vartheta}\right]. \tag{11.22}$$

In this case the corresponding forms $f_n(x)$ and frequencies ω_n of natural transverse oscillations of the body of the central unit of the flight vehicle are determined by taking into account apparent (in section $x_{\text{в}}$) masses of side-mounting boosters with uniform boundary conditions (4.24). As the boosters separate from the flight vehicle, i.e., as number k changes, the apparent mass and consequently also the dynamic characteristics of the central unit of the flight vehicle will be changed. This can substantially complicate the solution of the problem (especially in the presence of side-mounting boosters of comparatively great mass). However, if mass $m_{\text{б}1}$ is small as compared to m_c, then in the first approximation the effect of change of number k (in the process of separation) on ω_n and $f_n(x)$ can be desregarded.

Having determined accelerations $\ddot{y}_T, \Delta\ddot{\vartheta}, \ddot{S}_n$ and $\ddot{\beta}_{kj}$, it is simple to find the values of lateral forces and bending moments in all cross sections of the body of the central unit of the flight vehicle of interest to us.

From equations (17), (18), (19) and (21) we can obtain equations of lateral oscillations for a flight vehicle with lateral division of stages. For this it is sufficient to discard all terms containing variable μ_1 and its derivatives.

The character of change of components of lateral overload $n_{y_1}^{0}$, $n_{y_1}^{x}$ and n_{y_1} in the process of separation of side-mounting boosters for some hypothetical flight vehicle is shown on Fig. 11.10. In this case for clarity the values of total lateral overload are given for two points of the longitudinal axis of the central unit of a flight vehicle $x=\frac{L}{4}$ and $x = L$. As can be seen, in this case of loading the vibration component, caused by elastic oscillations of the construction of the unit, has the basic influence on the magnitude of lateral overload n_{y_1}.

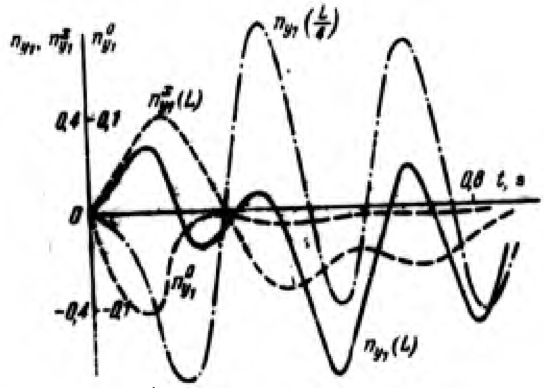

Fig. 11.10. Change of components of lateral overload in the process of separation of stages.

With a small damping factor the elastic oscillations will attenuate comparatively slowly, and consequently, their effect on the character of loading of the flight vehicle can be sensed in case L_1 (after separation of boosters). Therefore, we should pay attention to the process of drop of thrust force of engines, in particular to values of parameters τ_0 and τ_1^*. Decrease of the time of delay and slowing of the change of thrust force $\frac{dP}{dt}$ always leads to decrease of vibration overloads and stresses in the structural elements of the flight vehicle. Since in most cases the drop of thrust force of engines is an unregulated process, then an analogous effect is attained by introduction of step-by-step shutdown of engines. Corresponding selection of the intermediate stage or sequence of pairwise shutdown of engines can noticeably decrease the magnitude of perturbing moment affecting the flight vehicle in the process of separation of stages or separation of side-mounting boosters. Angles of rotation μ_1 of side-mounting boosters in the process of separation magnitudes of which depend on the time of delay of beginning of the drop of thrust force of engines of these boosters with respect to the moment of break of cross connections essentially affect the state of stress of the construction. The longer this time is, the larger is μ_1.

§ 11.5. Loading of Construction in Cases K_1 and B_1

Case K_1. Values of static and dynamic components of lateral forces and bending moments are determined by the magnitude of perturbing moment $M_{вр}$ (11). They will be maximum when delay τ_0 and gradients will be the greatest. On the other hand, longitudinal forced (static and dynamic) will be maximum in the absence of delay, i.e., with small values of perturbing moment $M_{вр}$. Therefore, it is difficult to establish directly which of the considered limiting cases of loading is dangerous for some section of the flight vehicle body or another. This can be done only by investigating the character of change of total stresses separately in compressed and stretched fibers of cross sections of the body in the process of separation of stages. External values of these stresses will be determined by the magnitude of equivalent longitudinal force, equal to

$$N_{э}(x,\ t_1)=N(x,\ t_1)\pm M(x,\ t_1)E(x)\frac{a(x)F_c(x)}{B(x)}, \qquad (11.23)$$

where B(x) – rigidity of construction of the body during bending, a(x) – distance of examined fiber from the neutral axis, $F_c(x)$ – area of cross section of the supporting part of the construction, and $M(x, t_1)$ – total bending moment, equal to

$$M(x,\ t_1)=[M_y^2(x,\ t_1)+M_z^2(x,\ t_1)]^{\frac{1}{2}}. \qquad (11.24)$$

Components of this moment $M_y(x, t_1)$, $M_z(x, t_1)$ are calculated on the basis of values of accelerations $\ddot{y}_T$, $\Delta\ddot{\theta}$, $\ddot{S}_n$ obtained from solution of equations of transverse oscillations of the flight vehicle (17), (18) and (19). If separation of stages occurs at comparatively low altitudes, then to moment $M_z(x, t_1)$ one should add bending moments, caused by the influence of corresponding wind perturbations on the flight vehicle.

Generally the external values of equivalent longitudinal forces (23) for different sections of the body will be observed at different moments of loading flight vehicle. Every part of the construction can have its own unfavorable combination of loading. For its manifestation one should consider a number of particular cases of loading of a vehicle taking into account the relationship of values of $N_э(x, t_i)$ to the prehistory of loading of the flight vehicle on the whole, which was mentioned above. As already was noted, the values of longitudinal and lateral forces and bending moments in the period of separation of stages will depend on the character of loading of construction in case B_1 (at the moment of beginning of the process of separation). In turn the state of construction of the flight vehicle at the moment of mechanical separation of stages determines the character of its loading in case L_1.

Case B_1. With shutdown of the engine in one stage or through an intermediate stage with short delay time one may assume that longitudinal forces in case B_1 are static, i.e., are determined by the formulas given in Chapter III. If the engine before separation stage is preliminarily transferred to an intermediate stage and is maintained at these conditions a comparatively short time Δt_3 (on the order of several seconds), then longitudinal force at the moment of beginning of the process of separation itself will contain a dynamic component. The value of this dynamic longitudinal force can be found by solution of equations (2) with corresponding initial conditions and assigned law of transition of lateral force into intermediate stage. When necessary one should select delay time Δt_3 in such a manner that the magnitude of this component of longitudinal force was decreased (due to the influence of resisting forces). Sometimes in case B_1 it is necessary to consider the dynamic components of longitudinal force, appearing as a result of pulsation of thrust force with a frequency close to the frequency of natural longitudinal oscillations of the flight vehicle construction. Methods of their calculation are expounded in the following chapter.

Values of lateral forces and bending moments at the considered moment of flight of the vehicle depend on parameters of its trajectory and characteristics of the control system. It is obvious that with realization of separation of stages in the zone of comparatively high impact pressures lateral aerodynamic loads both programmed, proportional to programmed angle of attack of the flight vehicle, and random, caused, for instance, by action of the wind, will be essential. A similar picture of loading can take place for many flight vehicles. To equations describing transverse oscillations of similar flight vehicles in the process of separation of stages one should introduce the corresponding terms, proportional to impact pressure. When programming the trajectory of such flight vehicles it is necessary to see that in case B_1 the angle of attack is close to zero.

If separation of stages occurs at comparatively high altitudes (at low impact pressures), then the basic source of lateral loads in case B_1 is the control system. The appearance of control forces on this section of the trajectory is caused by two factors: first, by the presence of perturbing moment $M_{\text{вр}}$, caused by misalignment and noncoincidence of the line of thrust force with longitudinal axis of the flight vehicle, passing through the center of gravity; secondly, due to the dynamic instability of the system vehicle-stabilization automation, as a result of which there is possible the appearance of oscillations of both control devices and construction of the flight vehicle at frequencies close to that of natural bending oscillations of the body or natural oscillations of liquid in fuel tanks. Similar oscillations frequently determine the computed tanks. Similar oscillations frequently determine the computed value of lateral forces and bending moments in case B_1 with low impact pressures, and also initial conditions for functions $S_n(t)$ (in case of separation of stages). Methods of approximation of calculation of parameters of these oscillations are listed in Chapter XIII.

Possible additional lateral loads on the flight vehicle construction in case B_1 due to random deflections of control devices are usually considered by a safety factor.

§ 11.6. Case L_I

This generalized case of loading encompasses initial sections on motion of any stages of a flight vehicle. It determines the state of stress of the construction during transient processes, caused by initial perturbations, received at the moment of liftoff of the flight vehicle from the launcher or during mechanical separation of stages.

Values of static and dynamic components of forces for this case of loading are found by solution of the corresponding equations of undisturbed and perturbed motion at nonuniform initial conditions. The latter are comparatively easily established in each concrete case, proceeding from operating peculiarities of the flight vehicle. With abrupt change of dynamic characteristics of the flight vehicle only the selection of initial conditions for generalized coordinates, characterizing elastic oscillations of the construction, may cause some difficulty. In this case it is expedient to use the method of expansion of deformations in normal forms of oscillations, given in § 11.4. Thus, for instance, with an open above ground launch the initial values of amplitude of transverse elastic oscillations of the construction at the point of reduction (for case L_I) will be determined by the following formula:

$$S_{n10}=\frac{1}{M_{n1}}S_{n0}\int_{0}^{l} m(x_1)f_{n1}(x_1)f_n(x_1)\,dx_1, \qquad (11.25)$$

where S_{n0} – amplitude of transverse oscillations of the point of reduction of the flight vehicle in case V, $f_n(x_1)$ – form of transverse oscillations of its construction in the same case, and $f_{n1}(x_1)$ – form of transverse oscillations of the construction of the vehicle in case L_I.

Linearized equations of transverse oscillations of the construction of the flight vehicle for the boost phase of flight are given in Chapter X. They can also be used for description of motion of nonmaneuvering flight vehicles in case B_1.

For maneuvering flight vehicles, separation of stages of which can occur at large angles of attack, it is necessary to apply nonlinear equations. A good illustration of the method of solution of a similar problem for such type of flight vehicles is determination of the character of loading of the construction of the recoverable part of these vehicles in the process of its withdrawal from the body with the aid of a special emergency rescue system (special propulsion system, operating on solid propellant). Usually in this case (case K_1) withdrawal of the recoverable part of the vehicle is carried out with an almost limiting initial value of longitudinal overload for the purpose of guarantee of the impact pressures required for subsequent opening of parachutes (during flight vehicle emergency at low altitudes of flight) and protection of the construction of the recoverable part from the effect of fragments during a possible explosion of the fuel compartment of the flight vehicle.

Typical for the recoverable part of a flight vehicle is a design scheme consisting[1] of the descent part (cabin or capsule with the crew), located on elastic supports inside the part of the cowl separable from the flight vehicle, and solid-propellant engine, installed in the apex of this cowl. Simplified dynamic layout of the described recoverable part of the vehicle construction (in longitudinal direction) can be represented in the form depicted on Fig. 11.11.

[1]Miller E. S., Bloom H. L., Design consideration for boost phase abort of manned space flights, IAS Paper No. 33, 1962.

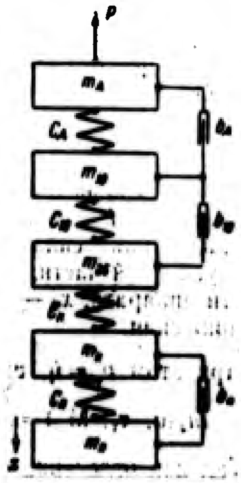

Fig. 11.11. Simplified dynamic layout of the recoverable part of a flight vehicle.

We will consider its motion (as a solid body) in a fixed system of coordinates with origin coinciding with the position of center of gravity of the descent part at the moment of breakaway from the body of the flight vehicle. Axes ξ and η of the given system of coordinates will be directed parallel to axes of the starting system of coordinates x_0 and y_0 (Fig. 1.1) respectively. The origin of the connected system of coordinates xy will be placed, as usual, at the apex of the body of the descent part of the flight vehicle. In the case when perturbing forces act in plane $\xi\eta$, the corresponding differential equations of motion of the center of gravity of the descent part of the vehicle will have the form

$$m\ddot{\xi} = -qS[c_x(\alpha, M_\infty)\cos\vartheta + c_y(\alpha, M_\infty)\sin\vartheta] + P\cos\vartheta - R_y\sin\vartheta, \qquad (11.26)$$

$$m\ddot{\eta} = -qS[c_x(\alpha, M_\infty)\sin\vartheta - c_y(\alpha, M_\infty)\cos\vartheta] + P\sin\vartheta + R_y\cos\vartheta. \qquad (11.27)$$

Equation of rotation of the aerodynamically stable descent part around the center of gravity will be

$$I_z\ddot{\vartheta} = -qS\left[m_z^{\omega} l^2 \frac{\dot{\vartheta}}{v} + m_z(\alpha, x_T, M_\infty)\right] - R_y(x_p - x_T). \qquad (11.28)$$

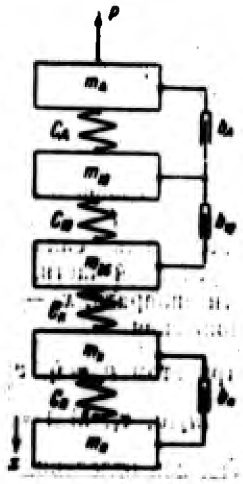

Fig. 11.13. Simplified dynamic layout of the recoverable part of a flight vehicle.

We will consider its motion (as a solid body) in a fixed system of coordinates with origin coinciding with the position of center of gravity of the descent part at the moment of breakaway from the body of the flight vehicle. Axes ξ and η of the given system of coordinates will be directed parallel to axes of the starting system of coordinates x_0 and y_0 (Fig. 1.1) respectively. The origin of the connected system of coordinates xy will be placed, as usual, at the apex of the body of the descent part of the flight vehicle. In the case when perturbing forces act in plane $\xi\eta$, the corresponding differential equations of motion of the center of gravity of the descent part of the vehicle will have the form

$$m\ddot{\xi} = -qS[c_x(\alpha, M_\infty)\cos\vartheta + c_y(\alpha, M_\infty)\sin\vartheta] + P\cos\vartheta - R_y\sin\vartheta, \qquad (11.26)$$

$$m\ddot{\eta} = -qS[c_x(\alpha, M_\infty)\sin\vartheta - c_y(\alpha, M_\infty)\cos\vartheta] + P\sin\vartheta + R_y\cos\vartheta. \qquad (11.27)$$

Equation of rotation of the aerodynamically stable descent part around the center of gravity will be

$$I_z\ddot{\vartheta} = -qS\left[m_z^{\omega}l^2\frac{\dot{\vartheta}}{v} + m_z(\alpha, x_T, M_\infty)\right] - R_y(x_p - x_T). \qquad (11.28)$$

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In these equations m – mass of descent part of the flight vehicle, P – thrust force of the engine of escape system, R_y – control force, c_x and c_y – aerodynamic coefficients of descent part, ϑ – pitch angle, l – length of descent part of the vehicle, $x_\text{т}$ – coordinate of its center of gravity, v – speed.

In this case

$$\left.\begin{aligned} v_\eta = \dot{\eta},\ v_\xi = \dot{\xi},\ v^2 = v_\eta^2 + v_\xi^2,\ \theta = \operatorname{arctg}\frac{v_\eta}{v_\xi}, \\ m_z(\alpha,\ x_\text{т},\ \mathrm{M}_\infty) = c_y(\alpha,\ \mathrm{M}_\infty)[x_\text{д}(\alpha,\ \mathrm{M}_\infty) - x_\text{т}(t)]. \end{aligned}\right\} \qquad (11.29)$$

Since separation of the descent part from the flight vehicle can occur at comparatively large angles of attack, then the aerodynamic coefficients entering these equations generally will be nonlinear functions of α. In connection with this, the influence of wind on the descent part should be considered by means of change of the magnitude of angle of attack, i.e., take

$$\alpha(t) = \vartheta(t) - \theta(t) + \frac{u(t)}{v(t)}. \qquad (11.30)$$

System of equations of elastic longitudinal oscillations of the given construction (Fig. 11.11) will consist of equations of type (3) and (4).

$$\left.\begin{aligned} m_\text{д}\ddot{x}_\text{д} + c_\text{д}(x_\text{д} - x_{10}) + b_\text{д}(\dot{x}_\text{д} - \dot{x}_{10}) = -P, \\ m_{10}\ddot{x}_{10} - c_\text{д}(x_\text{д} - x_{10}) + c_{10}(x_{10} - x_{20}) + b_{10}(\dot{x}_{10} - \dot{x}_{20}) = 0, \\ m_{20}\ddot{x}_{20} - c_{10}(x_{10} - x_{20}) + c_\text{к}(x_{20} - x_\text{к}) = 0, \\ m_\text{к}\ddot{x}_\text{к} - c_\text{к}(x_{20} - x_\text{к}) + c_\text{п}(x_\text{к} - x_\text{п}) = 0, \\ m_\text{п}\ddot{x}_\text{п} - c_\text{п}(x_\text{к} - x_\text{п}) + b_\text{п}(\dot{x}_\text{п} - \dot{x}_\text{к}) = 0, \end{aligned}\right\} \qquad (11.31)$$

where $m_\text{д}$ – mass of engine with fuel, m_{10}, m_{20} – mass of parts of construction of cowl, $m_\text{к}$ – mass of recoverable cabin, and $m_\text{п}$ – mass of seat with pilot. The letter c with subscripts designates corresponding generalized rigidity of the shown sections of the system (Fig. 11.11), b_i – damping factors of oscillations ($i = \text{д},\ 10,\ \text{п}$).

Usually from aerodynamic considerations of engine casing of the system of emergency rescue is made in the form of an elongated cylinder. Therefore, during investigation of transverse elastic oscillations of the construction of the descent part this body together with the separable part of the cowl can be approximated by a thin-walled rod of variable section with arbitrarily distributed mass. In this case the simplified dynamic layout of the system (in transverse direction) will consist of a beam of variable section, to which (for instance, in sections x_I and x_{II}) there is joined (with the aid of elastic connections) a rigid body, simulating the cabin with pilot. Transverse elastic oscillations of such a system in generalized coordinates can be described approximately by equations of form (2), and namely

$$M_n(\ddot{S}_n + 2h_n\dot{S}_n + \omega_n^2 S_n) + c_n\ddot{y}_T + t_{In}\ddot{y}_I + t_{IIn}\ddot{y}_{II} = F_n, \tag{11.32}$$

$$\frac{J_2}{a^2}(\ddot{y}_I + \omega_I^2 y_I) + b_I\ddot{y}_T + \Delta m\ddot{y}_{II} + \sum_{n=1}^{\infty} t_{In}\ddot{S}_n = 0, \tag{11.33}$$

$$\frac{J_1}{a^2}(\ddot{y}_{II} + \omega_{II}^2 y_{II}) + b_{II}\ddot{y}_T + \Delta m\ddot{y}_I + \sum_{n=1}^{\infty} t_{IIn}\ddot{S}_n = 0, \tag{11.34}$$

$$m\ddot{y}_T + b_I\ddot{y}_I + b_{II}\ddot{y}_{II} + \sum_{n=1}^{\infty} c_n\ddot{S}_n = B, \tag{11.35}$$

where

$$F_n = qS\int_0^l f_n(x)\frac{\partial c_y(\alpha, M_\infty)}{\partial x}dx, \quad B = qSc_y(\alpha, M_\infty),$$

$$c_n = [f_n(x_I) + f_n(x_{II})]\Delta m, \quad \omega_I^2 = c_I\frac{a^2}{J_1},$$

$$t_{In} = \frac{J_2}{a^2}f_n(x_I) + \Delta m f_n(x_{II}),$$

$$t_{IIn} = \frac{J_1}{a^2}f_n(x_{II}) + \Delta m f_n(x_I),$$

$$b_I = m_\kappa\left(1 - \frac{x_{T\kappa} - x_I}{a}\right), \quad b_{II} = \frac{m_\kappa}{a}(x_{T\kappa} - x_I),$$

$$a = x_2 - x_1 = x_{II} - x_I, \quad \omega_{II}^2 = c_{II}\frac{a^2}{J_2},$$

$$M_n = \int_0^l m_{\text{пр}}(x)f_n^2(x)dx, \quad m = m_0 + m_\kappa,$$

$$m_{\text{пр}}(x) = m_0(x) + \frac{J_2}{a^2}\delta(x - x_I) + \frac{J_1}{a^2}\delta(x - x_{II}),$$

$$\Delta m = \frac{m_\kappa}{a}(x_{T\kappa} - x_I) - \frac{J_1}{a^2}.$$

In these formulas J_1, J_2 – mass moments of inertia of the cabin (with pilot) relative to lateral axes, located in cross sections with coordinates x_1 and x_2 respectively, m_K – mass of cabin (with pilot), x_1 and x_2 – coordinates of its lateral supports in the connected (with cabin) system of coordinates, x_I and x_{II} – coordinates of the same supports in the system of coordinates connected with cowl, x_{TK} – coordinate of center of gravity of the recoverable part of the vehicle (cabin with pilot) in system of coordinates xy, ω_n – partial frequency of oscillations of a beam with apparent masses in sections x_I and x_{II} and uniform boundary conditions, c_I and c_{II} – generalized rigidities of cross connections, y_I – additional lateral shift of the center of gravity of the descent part of the flight vehicle.

A distinctive feature of the given problem is not only the presence of large angles of attack, but also a sharp change of mass characteristics of the system and impact pressures in the process of very brief (on the order of several seconds) operation of the escape system engine. Therefore, solution of the given nonlinear differential equations with variable coefficients can be found only by the method of numerical integration.

For illustration of the character of loading of the considered descent part of the flight vehicle in the shown period of time, Fig. 11.12 shows graphs of the possible change of longitudinal and lateral components of overload of the center of gravity of the descent part, and on Fig. 11.13 – graph of the relationship of dynamic coefficient for elements of attachment of the cabin to the rigidity of its longitudinal connection.

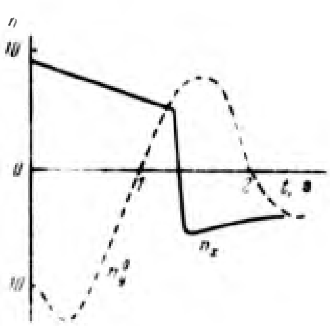

Fig. 11.12. Change of overload components for the recoverable part in case L_1.

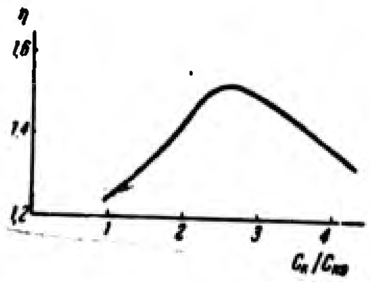

Fig. 11.13. Relationship of dynamic coefficient to the rigidity of longitudinal connection.

§ 11.7. Loading of Points of Connection of Stages

Stages of the flight vehicle are connected together with special quick-opening locks or explosive bolts. Their quantity is established in each particular case depending upon magnitude of tensile load affecting the point and the bearing capacity of the structure of the lock itself. Usually for such points of connection the cases of ground operation, case W and case K_{i-1} (separation of preceding stages) are calculated. If there are limitations on angular perturbations of the flight vehicle in the process of separation of stages (at initial conditions in case K_1), then cases K_1 can be calculated for them.

The points of attachment of separable stages (section $x_1 = l_{II}$, Fig. 11.3) in the process of "hot" separation are affected by longitudinal forces, total quantity of which is determined by expression

$$N(l_{II},\ t) = N_c(l_{II},\ t) + N_x(l_{II},\ t). \qquad (11.36)$$

In this case the value of $N_x(l_{II},\ t)$ is calculated by formula (10), and

$$N_c(l_{II},\ t) = -P_{II}(t)\left[1 - \frac{G_{0II}}{G(t)}\right] + [P_I(t) - R_{II}(t)]\,\frac{G_{0II}}{G(t)}.$$

While force $N(l_{II}, t)$ remains compressive, these points of connection are not loaded. As P_I decreases and P_{II} increases longitudinal load

$N_c(l_{II}, t)$ — changes sign and the shown point of connection are subjected to the influence of tensile forces right up to the moment corresponding to the beginning of their operation. Due to the presence of variation of parameters, characterizing the response time of locks (explosive bolts), their break naturally will not occur simultaneously, but with some delay (Δt). Such a limiting case is possible (with a small number of locks) when for all locks, with the exception of one, the time lag will be practically close to zero. In other words, a large dynamic load will act on the remaining unopened lock. For guarantee of the necessary reliability of operation of such a lock and sufficient strength of the point of its attachment to the body of the flight vehicle, it is required to know not only the actual value of maximum force R, affecting this point of connection, but also its relationship to time of delay Δt and generalized rigidity of the point of connection on the whole (lock, elements for attaching it to bodies of units (stages) and part of the shell of the bodies of units themselves, absorbing concentrated force R).

Let us designate this rigidity through c_6. Let us assume that the considered point of connection works only under tension. Then the diagram of its loading will have the form shown on Fig. 11.14. With the designations accepted there the magnitude of force R (without taking into the static component, corresponding to t = 0) will be equal to

$$R = c_6(x_2 - x_1) \quad \text{when} \quad 0 < t < \Delta t,$$
$$R = 0 \quad \text{when} \quad t > \Delta t,$$

where

$$x_2 = x_{20} + y\vartheta_2,$$
$$x_1 = x_{10} + y\vartheta_1,$$

x_{20} — relative longitudinal shift of the center of gravity of the second stage of the flight vehicle, x_{10} — relative longitudinal shift of the center of gravity of the separable part of the construction; ϑ_2, ϑ_1 — angles of rotation of these parts of the vehicle relative to their centers of gravity. Approximate values of the shown shifts of separable parts of the flight vehicle are determined by equations

$$\left.\begin{aligned} m_{II}\ddot{x}_{20} &= P_{II} - R, & m_{I}\ddot{x}_{10} &= R - R_{II} + P_{I}, \\ -J_{II}\ddot{\vartheta}_2 &= yR, & J_{I}\ddot{\vartheta}_1 &= yR, \end{aligned}\right\} \qquad (11.37)$$

where m_I, m_{II} – masses of divided parts, J_I, J_{II} – mass moments of inertia of these parts relative to their centers of gravity, y – coordinate of the place of installation of lock (in this case $y=-r$). From the last two equations it follows that

$$\ddot{\vartheta}_1 = -\ddot{\vartheta}_2 \frac{J_{II}}{J_I}.$$

Let us introduce designations:

$$x_{20} - x_{10} = x,$$
$$\vartheta_2 - \vartheta_1 = \vartheta.$$

Then equations (37) will take the following form:

$$\left.\begin{aligned} \ddot{x} + \omega^2 x &= P_0 + r\omega^2\vartheta, \\ \ddot{\vartheta} - a_0 r\vartheta + a_0 x &= 0, \end{aligned}\right\} \qquad (11.38)$$

where

$$\omega^2 = \frac{c_6(m_I + m_{II})}{m_I m_{II}}, \qquad -a_0 = c_6 r\left(\frac{1}{J_I} + \frac{1}{J_{II}}\right),$$
$$P_0 = \frac{1}{m_I m_{II}}\left[P_{II} m_I + m_{II}(R_{II} - P_I)\right].$$

Having calculated x, ϑ we obtain

$$R = c_6(x - r\vartheta).$$

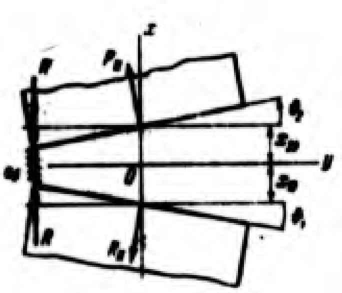

Fig. 11.14. Diagram of forces affecting the point of connection of stages.

As can be seen from these formulas, values of x and θ are also functions of rigidity c_6, and consequently, the relationship of R to c_6 will be nonlinear.

The probability of appearance of the considered limiting case of loading of locks depends on their quantity. The fewer the locks, the greater the probability of realization of such case. With a large number of such locks it is possible to find the total load affecting the group of locks or to use methods of the theory of probability and mathematical statistics.

CHAPTER XII

STEADY-STATE LONGITUDINAL OSCILLATIONS OF THE CONSTRUCTION OF A FLIGHT VEHICLE

§ 12.1. Introduction

In the preceding chapter we considered damped longitudinal oscillations of the construction of a flight vehicle, caused by a change of thrust force in the process of transient operating conditions of propulsion systems. In this chapter we investigate steady longitudinal oscillations of ballistic type flight vehicles on the boost phase of flight with steady state of operation on the propulsion system.

The source of such oscillations is the propulsion system itself and partially the control system. Oscillations of effectors of the control system can lead to pulsation of the longitudinal component of control force $\overline{R}_{x_1}$ (resisting force of air and gas current vanes, longitudinal component of thrust force of various types of controlling engines). However, in most cases the values of these disturbing forces turn out to be comparatively small and their influence on the parameters of longitudinal oscillations of the construction can be disregarded (with the exception of resonance case, when the frequency of change of $\overline{R}_{x_1}$ becomes close to the frequency of natural longitudinal oscillations of the construction).

More important from this point of view are longitudinal disturbing forces, produced by pulsation of thrust force of the basic propulsion system. This pulsation can be caused by random fluctuations of pressure in the combustion chamber, instability of the process of combustion, oscillations of some system, locking the engine (for instance, oscillations in the system "fuel line to engine-pump," appearing with the presence of cavitational phenomena), operation of control system of vehicle speed, operation of the control system, built on the principle of throttling of thrust force and so forth. For propulsion systems with annular arrangement of combustion chambers the pulsation of total thrust force can appear due to acoustic loads on the bottom of the body, and also base pressure fluctuation caused by the interaction of the engine jets.

Especially dangerous for the construction of flight vehicles of ballistic type, engines of which operation on liquid propellant, are oscillations of thrust force, caused by dynamic instability of the closed system vehicle (with fuel feed system) - propulsion system. If such a system is unstable, then any random pressure fluctuation of liquid at the fuel feed pump inlet or pressure fluctuation in the combustion chamber may cause sustained longitudinal oscillations of the entire system with frequency close to one of the lowest frequencies of natural longitudinal oscillations of the construction. Thus, for instance, similar oscillations with frequency on the order of 11 Hz were observed in the process of flight of "Titan" class ballistic type flight vehicles.

The reaction of the construction to steady pulsation of thrust force can be determined by solution of system of equations (11.2) with harmonic change of generalized force (see Chapter IV). Reaction to pulsation of thrust force or base pressure, being a random process, is characterized by spectral density

$$\Phi_N(\omega)=\Phi_{дд}(\omega)\,|G_N(i\omega)|^2, \qquad (12.1)$$

where $\Phi_{ДД}(\omega)$ - spectral density of the energy of fluctuation of force of base pressure, and $G_N(i\omega)$ - complex transfer function. With the same assumptions which were made in the case of transverse oscillations (see § 6.3)

$$|G_N(i\omega)|^2 \approx \sum_{n=1}^{n_n} \frac{\omega^4 N_{nx}^2(x_1)\, X_n^2(x_{1n})}{\omega_n^4 M_n^2 \left[\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \frac{4h_n^2\omega^2}{\omega_n^4}\right]}.$$

In this case, if the main part of energy of the process will occur at the region of comparatively high frequencies, instead of formula (1) it is possible to use the method of approximation of solution of equations of longitudinal oscillations of the construction in form (11.2)-(11.5), simulating random continuous process of pulsation of external force by the sum of sinusoidal components with small pitch of change of frequencies.

More complicated is the question of determination of parameters of steady-state oscillations (natural oscillations) of nonlinear vehicle-propulsion system. It is possible to say that the basic problem of dynamic calculation of the construction for the considered case of loading consists namely of guarantee of stability of the shown system and establishment of corresponding limitations for amplitudes of its oscillations. Its solution is reduced to determination of transfer functions of longitudinal oscillations of the flight vehicle taking into account the influence of liquid filling and to finding the frequencies and amplitudes of steady-state oscillations, in particular the amplitudes of oscillations of longitudinal dynamic forces.

Below we will pause only on certain questions of the given problem, having a direct relationship to normalization of the required strength of flight vehicle construction. In this case we will not touch upon questions connected with the dynamic stability of elastic systems, about which there is extensive literature.

§ 12.2. Approximate Equations of Longitudinal Oscillations of the Vehicle Construction

Let us consider longitudinal low-frequency elastic oscillations of the flight vehicle body, having m consecutively located supporting cylindrical tanks partially filled with liquid.

Let us designate forward displacements of the system of coordinates x_1y_1 relative to system xy (in the direction of axis x) through ξ_0. Let us represent displacements of points of cross sections of the flight vehicle body (perpendicular to axis x_1) at small longitudinal elastic oscillations (compression-tension) in the form of finite series

$$u(x_1, t) = \sum_{n=1}^{n_n} T_n(t) X_n(x_1),$$

where X_n - some selected linearly independent functions of x_1, satisfying uniform boundary conditions at ends of the flight vehicle body, and T_n - indeterminate functions of time. Then the kinetic energy of longitudinal oscillations of the vehicle structure (without taking into account liquid filling) will be determined by expression

$$T_0(t) = \frac{1}{2}\int_0^l m_к(x_1)\left[\dot{\xi}_0(t) - \sum_{n=1}^{n_n} \dot{T}_n(t) X_n(x_1)\right]^2 dx_1, \qquad (12.2)$$

and potential energy, accumulated by the body in the process of elastic deformation, by formula

$$U_0(t) = \frac{1}{2}\int_0^l E(x_1) F_c(x_1)\left[\sum_{n=1}^{n_n} T_n(t)\frac{dX_n(x_1)}{dx_1}\right]^2 dx_1, \qquad (12.3)$$

where $m_к(x_1)$ - intensity of mass of the structure without taking into account liquid filling, l - length of the flight vehicle body, $F_c(x_1)$ - area of cross section of the supporting (power) part of the body, $E(x_1)$ - modulus of longitudinal elasticity.

Functions $\xi_0(t)$ and $T_n(t)$ will be taken as generalized coordinates q_m. Generalized forces Q_{ξ_0} and Q_{Tn} corresponding to them will be represented in the form of the sum of three components, namely: force, considering the effect of liquid on the flight vehicle, generalized external perturbing force and generalized force of internal nonelastic resistance of the body Q_{cn}:

$$Q_{\xi_0} = P + \sum_{j=1}^{n_1} P_j, \qquad (12.4)$$

$$Q_{\tau n} = -PX_n(x_{1n}) - \sum_{j=1}^{n_1} P_j X_n(x_{\text{д}j}) - Q_{cn}. \tag{12.5}$$

Here by P_j there is designated the load from liquid on the bottom of j-th tank, equal to

$$P_j = \int_0^{2\pi}\int_0^{a_j} p_j(r, \varphi)\, r\, dr\, d\varphi, \tag{12.6}$$

where a_j - radius of tank, $X_n(x_{\text{д}j})$ - value of function of shape for the bottom of the tank, p_j - dynamic pressure of liquid on the bottom, determined by formula

$$p_j = -\rho_j \frac{\partial \Phi_j}{\partial t}, \tag{12.7}$$

and ρ_j - mass density of liquid, Φ_j - velocity potential. With longitudinal oscillations of the tank this potential should satisfy Laplace equation (2.40)

$$\frac{\partial^2 \Phi_j}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi_j}{\partial r} + \frac{\partial^2 \Phi_j}{\partial x_1^2} = 0 \tag{12.8}$$

and corresponding boundary conditions, and namely the condition on free surface of the liquid

$$p_j(r, \varphi) = 0 \text{ with } x_{1j} = 0 \tag{12.9}$$

and conditions at the wall of the cavity. The latter consists of equality of radial velocity of liquid particles (with $r = a_j$) to radial rate of wall $\dot{w}_j$

$$v_r = \frac{\partial \Phi_j}{\partial r} = \dot{w}_j \tag{12.10}$$

and of equality of longitudinal rates of liquid for the bottom of the tanks (when $x_j = h_j$) to rates of oscillations of points of the bottom

$$v_x = \frac{\partial \Phi_j}{\partial x} = \xi(x_{\text{д}j}). \tag{12.11}$$

By using the method of separation of variables, particular solution of equation (8) can be found in the form

$$\Phi_j(x_j, t, r) = C_n R(r) X(x_j) \dot{\xi}(t). \tag{12.12}$$

With steady-state oscillations

$$\xi(t) = u_0 e^{i\omega t} = \xi_0(t) - u(x_{kj}, t). \tag{12.13}$$

Value of function $X(x_j)$ will be selected so that boundary conditions were satisfied on the free surface of liquid (9)

$$X(x_j) = \sin \mu_j x_j, \tag{12.14}$$

where μ_j - some unknown function. Fulfillment of boundary condition on the tank bottom is ensured by corresponding selection of coefficients C_n.

The simplest form of function R(r) is in the case when the supporting tank is an unreinforced thin-walled shell. Radial deformation of the wall of such a tank with steady-state oscillations is determined by an expression analogous to (14),

$$w_j = B_j \sin(\mu_j x_j) e^{i\omega t}, \tag{12.15}$$

in which coefficient B_j is a function of the frequency of oscillations of liquid pressure in the tank and frequency of natural radial oscillations of the shell of the tank ω_{cj}. Such oscillations can be described by equation

$$\ddot{w}_j + \omega_{cj}^2 w_j = \frac{p_j}{\delta_{cj} \rho_{cj}}, \tag{12.16}$$

where ρ_{cj} - mass density of material of the wall of the tank, and δ_{cj} - the thickness of this wall. Having substituted expression (15) and the value of p_j, determined by formulas (7), (12), (13), in this equation, let us obtain the value of coefficient B_j in the form

$$B_j = C_n \frac{R(r) \rho_j u_0 \omega^2}{(\omega_{cj}^2 - \omega^2) \delta_{cj} \rho_{cj}}. \tag{12.17}$$

Let us assume

$$R(r) = I_0(\mu_j, \bar{r}), \tag{12.18}$$

where I_0 - modified Bessel function of the first type of zero order.

By equating the expression for $\dot{w}_j$ obtained in such a way to value v_r (10), we find the equation for calculation of unknown function u_j

$$\frac{I_0(\mu_j)}{\mu_j I_1(\mu_j)} = \frac{\delta_{cj}\rho_{cj}}{\rho_j a_j}\left(\frac{\omega_{cj}^2}{\omega^2} - 1\right), \tag{12.19}$$

where

$$I_1(\mu_j) = \frac{a_j}{\mu_j}\frac{\partial I_0(\mu_j r)}{\partial r}.$$

Here $I_1(\mu_j)$ - modified Bessel function of the first type of the first order.

When $\omega < \omega_{cj}$ this equation will have one real root. If $\omega > \omega_{cj}$, then all roots will be imaginary. In this case expression Φ_j should be sought in the form (2.49) with use of hyperbolic Bessel functions.

Thus, approximately

$$\Phi_j(x_j, r, t) = \Big[C_n I_0(\mu_j \bar{r}_j)\sin(\mu_j \bar{x}_j) + \sum_{k=1}^{\infty} C_k J_0(\mu_j \bar{r}_j)\,\mathrm{sh}(\mu_j \bar{x}_j)\Big]\,\xi(t). \tag{12.20}$$

In most cases (especially at comparatively large values of the level of liquid in the tank h_j) with sufficient accuracy for practical calculations it is possible to be limited only by the first term of this expression, i.e., to take Φ_j in the form [84]

$$\Phi_j(x_j, r, t) = \frac{a_j I_0(\mu_j \bar{r}_j)\sin(\mu_j \bar{x}_j)}{2\alpha_j I_1(\mu_j)\cos(\mu_j \bar{h}_j)}\,\xi(x_{aj}, t). \tag{12.21}$$

where

$$\bar{r}_j = \frac{r}{a_j}, \quad \bar{h}_j = \frac{h_j}{a_j}.$$

In this case the boundary condition on the bottom can be fulfilled only approximately, by means of selection of coefficient α_j in such a way that the difference between average longitudinal velocity of particles of liquid in the plane of attachment of the tank bottom $x_j = x_{\text{д}j}$ and the rate of the tank itself is equal to change (due to deformation of the volume of elastic bottom Δv_j, filled with liquid, divided by area of cross section of the cylinder πa_j^2. For absolute rigid bottom $\alpha_j = 1$.

Bleich proposes calculating the influence of compressibility of liquid (for large h_j) by means of substitution in (21) instead of roots μ_j of equation (19) magnitudes μ_j'

$$\mu_j' = \left[\mu_j^2 + \omega^2 \frac{a_j^2}{c_j^2}\right]^{\frac{1}{2}},$$

where c_j - speed of sound in liquid.

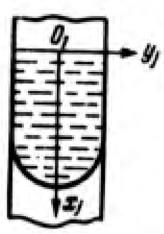

Fig. 12.1. Auxiliary system of coordinates.

The above-mentioned expression of velocity potential of liquid is written in system of coordinates $x_j y_j$ (Fig. 12.1), the origin of which is located in the center of free surface of liquid.

By placing in formula (6) the approximate value of liquid pressure on the bottom of tank p_j, obtained in such a way, we find that

$$P_j(t) = -m_j^* M_j \left[\ddot{\xi}_0(t) - \sum_{n=1}^{n_n} \ddot{T}_n(t) X_n(x_{\text{д}j})\right]. \quad (12.22)$$

where M_j - mass of liquid located in the considered j-th cavity, and $\bar{m}_j^*$ - relative reduced mass of liquid, equal to

$$\bar{m}_j^* = \frac{\bar{m}_j^0}{\alpha_j},$$

where

$$\bar{m}_j^0 = \frac{\operatorname{tg}(\mu \bar{h}_j)}{\mu \bar{h}_j}. \qquad (12.23)$$

Coefficient α_j, considering the effect of elasticity of the bottom of the tank, is determined by expression

$$\alpha_j = 1 - \bar{m}_j^0 \left(\frac{\omega}{\omega_{дj}}\right)^2. \qquad (12.24)$$

Here $\omega_{дj}$ - angular frequency of natural elastic oscillations of the bottom with apparent mass of liquid M_j

$$\omega_{дj} = \sqrt{\frac{E_{дj}}{M_{дj}}}, \qquad (12.25)$$

and $E_{дj}$ - rigidity of bottom:

$$E_{дj} = P_j^0 \frac{\pi a_j^2}{\Delta v_{дj}}, \qquad (12.26)$$

where P_j^0 is a certain fixed surface load on the bottom, and $\Delta v_{дj}$ - change of the volume of bottom corresponding to this load.

The value of relative reduced mass depends basically on the frequency of longitudinal oscillations of the flight vehicle structure. Its magnitude is close to one at small ω and sharply increases in proportion to approximation of ω to some "resonance" value. In this case the load on the bottom P_j will increase (which can be considerable even at very small amplitudes of oscillations of the tank). With further increase of frequency the magnitude of $\bar{m}_j^*$ even becomes negative. Elasticity of the bottom essentially affects the value of this "resonance" frequency ω_{rj}. For illustration Fig. 12.2 contains a graph of function $\bar{m}_j^*(\omega)$, and Fig. 12.3 - graph of the dependence of ω_{rj} to $\omega_{дj}$ (or $E_{дj}$). Figure 12.2 also shows a graph of function $\bar{m}_j^*(\omega)$ when $\alpha_j = 1$ and $\omega_{\text{л}j} = \omega_{rj0}$. As can be seen, the introduction of α_j shifts ω_{rj} into the region of lower frequencies (dotted curve).

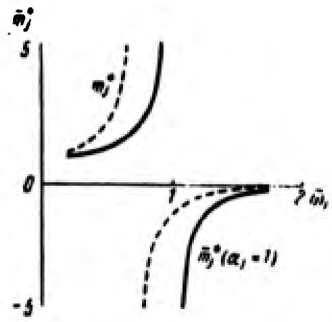

Fig. 12.2.

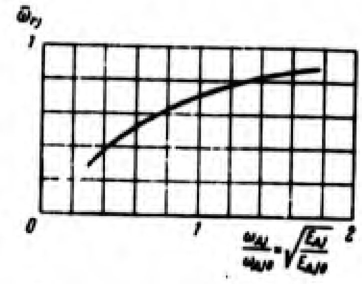

Fig. 12.3.

Fig. 12.2. Influence of the frequency of longitudinal oscillations of the tank on the magnitude of relative reduced mass of liquid.

Fig. 12.3. Dependence of "resonance" frequency to rigidity of tank bottom.

Having placed expressions (2), (3), (4) and (5) in Lagrange equation

$$\frac{\partial}{\partial t}\left(\frac{\partial T_0}{\partial \dot{q}_m}\right) + \frac{\partial U_0}{\partial q_m} = Q_m \tag{12.27}$$

and using formula (22), we obtain the following system of ordinary differential equations, approximately describing longitudinal steady-state oscillations of an elastic flight vehicle with cylindrical tanks, partially filled with liquid:

$$M\ddot{\xi}_0 + \sum_{n=1}^{n_n} c_n \ddot{T}_n = P, \tag{12.28}$$

$$\sum_{m=1}^{n_n} m_{nm}\ddot{T}_m + \sum_{m=1}^{n_n} k_{nm} T_m + c_n\ddot{\xi}_0 = c_{np}P - Q_{rn}. \tag{12.29}$$

Coefficients of these equations depend on the frequency of longitudinal oscillations of the flight vehicle and are determined by formulas

$$
\left.
\begin{aligned}
&M = M_{c} + \sum_{j=1}^{n_1} \bar{m}_j^{*} M_j,\\
&M_{c} = \int_0^l m_{\text{к}}(x_1)\,dx_1,\\
&-c_{\text{пр}} = X_n(x_{1n}),\\
&-c_n = \int_0^l m_{\text{к}}(x_1) X_n(x_1)\,dx_1 + \sum_{j=1}^{n_1} \bar{m}_j^{*} M_j X_n(x_{\text{д}j}),\\
&k_{nm} = \int_0^l E(x_1) F_{c}(x_1) \frac{dX_n(x_1)}{dx_1} \frac{dX_m(x_1)}{dx_1}\,dx_1,\\
&m_{nm} = \int_0^l m_{\text{к}}(x_1) X_n(x_1) X_m(x_1)\,dx_1 +\\
&\qquad + \sum_{j=1}^{n_1} \bar{m}_j^{*} M_j X_n(x_{\text{д}j}) X_m(x_{\text{д}j}).
\end{aligned}
\right| \qquad (12.30)
$$

§ 12.3. Natural Frequencies of Longitudinal Oscillations of the Construction of the Vehicle Taking Into Account the Mobility of Liquid in Tanks

During formulation of approximate equations of steady longitudinal oscillations of the system vehicle body - liquid, longitudinal deformation of the body was represented in the form of a series by arbitrarily selected functions of coordinate x_1. It is natural as these functions to take fundamental functions $X_n(x_1)$ of the problem about natural longitudinal elastic oscillations of the flight vehicle structure with solidified liquid, i.e., functions satisfying equation of form

$$
\frac{d}{dx_1}\left[E(x_1) F_{c}(x_1) \frac{dX_n(x_1)}{dx_1}\right] + m(x_1) X_n(x_1) \omega_n^2 = 0 \qquad (12.31)
$$

and corresponding boundary conditions. For a body with free ends the latter, as it is known, are equal to

$$
\frac{dX_n(x_1)}{dx_1} = 0 \text{ when } x_1 = 0,\ x_1 = l.
$$

Here by ω_n there is designated the angular frequency of natural longidutinal elastic oscillations of the flight vehicle, by $m(x_1)$ - mass per unit of its length (taking into account masses of liquid

M_j, concentrated in sections $x_1 = x_{1j}$). Shown functions are calculated taking into account peculiarities of the dynamic scheme of every flight vehicle, for instance taking into account the elasticity of suspension of propulsion system, pumps and so forth.

Due to the orthogonality of given functions $X_n(x_1)$ with weight $m(x_1)$, expressions for coefficients (30) are noticeably simplified and will have the form

$$\left.\begin{aligned}
&m_{nm} = -\sum_{j=1}^{n_1} M_j(1-\bar{m}_j^*) X_n(x_{1j}) X_m(x_{1j}) \text{ when } m \neq n,\\
&m_{nn} = M_n - \sum_{j=1}^{n_1} M_j(1-\bar{m}_j^*) X_n^2(x_{1j}) \text{ when } m = n,\\
&k_{nn} = \omega_n^2 M_n,\quad M_n = \int_0^l m(x_1) X_n^2(x_1)\,dx_1,\\
&c_n = \sum_{j=1}^{n_1} M_j(1-\bar{m}_j^*) X_n(x_{1j}).
\end{aligned}\right\} \tag{12.32}$$

Having placed P = 0, Q_{cn} = 0 and excluded $\ddot{\xi}_0$ from (28) and (29) we obtain the following equation, describing free elastic longitudinal oscillations of the considered system:

$$\ddot{T}_n\left(m_{nn} - \frac{c_n^2}{M}\right) + \omega_n^2 M_n T_n + \sum_{m=1}^{n_n}\left(m_{nm} - \frac{c_n c_m}{M}\right)\ddot{T}_m = 0 \tag{12.33}$$
$$\text{when } m \neq n \qquad (n = 1, 2, \ldots, n_n).$$

Having selected the solution of these equations in the form

$$T_n = T_{np} e^{i\omega t},$$

where T_{np} - amplitude, and ω - frequency of oscillations, and having equated the determinant, comprised of their coefficients, to zero, we obtain transcendental equation for determination of the frequencies of system ω_p. Having calculated these frequencies, let us find the values of amplitudes T_{np} determining (with accuracy to a constant factor) the degree of effect of each function $X_n(x_1)$ on the normalized normal form of longitudinal oscillations $f_p(x_1)$, corresponding to the given frequency of the system,

$$f_p(x_1) = \sum_{n=1}^{n_n}\left[X_n(x_1) - \frac{c_n}{M}\right] T_{np}, \tag{12.34}$$

and consequently, on the form of radial oscillations of the shell of the tank

$$w_{jp}(x_1) = \frac{\mu_j f_p(x_{\text{д}j})}{2a_j \cos(\mu_j h_j)} \sin(\mu_j \bar{x}_j). \tag{12.35}$$

The solution of frequency equation, taking into account the complicated dependence of $\bar{m}_j^*$ to ω, is expediently sought by the trial-and-error method. Accuracy of calculation of functions $f_p(x_1)$ and ω_p in such a way depends on the number of taken functions $X_n(x_1)$. For all practical purposes it is undesirable to take n_n large. The influence of higher harmonics ω_n and $X_n(x_1)$ on the lowest tones ω_p and $f_p(x_1)$ frequently turns out to be insignificant, so that in many cases for determination of the frequency and form of natural longitudinal oscillations of the system of the first (sometimes the second) tone it is possible to be limited by $n_n = 1$, having applied the method of successive approximations for more precise determination of values of ω_p and $f_p(x_1)$ (corresponding to $p = 1$).

Before changing to the account of this method, let us consider how the elasticity of walls and bottom of the tank affects natural frequencies of the system and the form of longitudinal oscillations of the flight vehicle, how much values of ω_p and $f_p(x_1)$ differ from frequencies ω_n and forms $X_n(x_1)$ of longitudinal elastic oscillations of the construction on the whole, calculated on the assumption of solidified liquid. For example let us take $n_n = 2$. Frequency equation in this case will have the form

$$F(\omega) = -1, \tag{12.36}$$

where

$$\left.\begin{aligned} &F(\omega) = \omega^4(a_1a_2 - b_1b_2) - \omega^2(a_1 + a_2),\\ &a_1 = \frac{1}{\omega_1^2 M_1}\left(m_{11} - \frac{c_1^2}{M}\right), \quad a_2 = \frac{1}{\omega_2^2 M_2}\left(m_{22} - \frac{c_2^2}{M}\right),\\ &b_1 = \frac{1}{\omega_1^2 M_1}\left(m_{12} - \frac{c_1c_2}{M}\right), \quad b_2 = \frac{1}{\omega_2^2 M_2}\left(m_{21} - \frac{c_1c_2}{M}\right). \end{aligned}\right\} \tag{12.37}$$

Graph of function $F(\omega)$ for $n_1=3$ and $\frac{1}{M}\sum_{j=1}^{3} M_j = 0.76$ is represented on Fig. 12.4. Points of intersection of this curve with straight line F = -1 determine the sought values of natural frequencies of the system (ω_{p1}, ω_{p2}, ω_{p3}). Normal forms of oscillations f_1, f_2, f_3 corresponding to them are shown on Fig. 12.5.

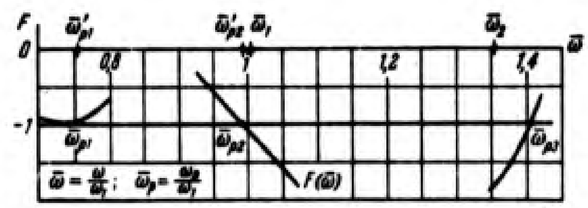

Fig. 12.4. For determination of natural frequencies of the system.

On these graphs for comparison there are given values of ω_n and $X_n(x_1)$ for n = 1 and n = 2, and also natural frequencies of system (ω'_{p1} and ω'_{p2}) for case n_n = 1.

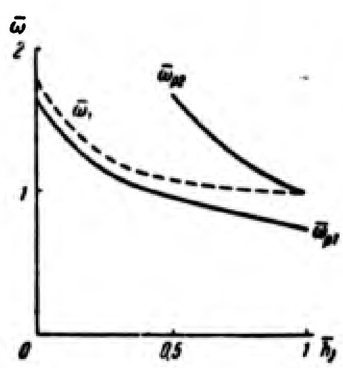

Fig. 12.5. Influence of the level of liquid in tanks on values of natural frequencies on the system.

As can be seen, the influence of function ω_2 on ω_{p1} on ω_{p2} in this case is comparatively small. It shows up only on f_3 and ω_{p3} and disappears with decrease of the mass of liquid in these tanks. In this case the interconnection of equations of system (33) is weakened, and with sufficient accuracy for practical calculation it is possible (when determining the lowest frequencies of natural longitudinal oscillations of such flight vehicles) to be limited by case n_n = 1, i.e., to take the frequency equation in the form

$$m_{11} - \frac{c_1^2}{M} = M_1 \frac{\omega_1^2}{\omega^2}. \qquad (12.38)$$

The quantity of liquid in tanks essentially affects the general type of functions $f_p(x_1)$ and value of frequencies of natural oscillations of the system ω_p.

Figure 12.5 contains graphs of change of ω_{p1}, ω_{p2} and ω_1 depending upon the level of liquid h_j in tanks $j = 2$ and $j = 3$ at constant value of h_j for tank $j = 1$. It is clear that with decrease of h_j, i.e., mass of liquid, the frequency of the first tone of natural oscillations of the system approaches ω_1, and the frequency of second tone (which when $h_j = h_{j0}$ almost coincides with ω_1), conversely, departs from ω_1. In this case there is observed increase of the values and amplitudes of radial oscillations of the wall of tank (Fig. 12.6) in the part filled with liquid. Distinction of ω_{p1} from ω_1 when $h_j = 0$ is explained by the influence of mobility of liquid located in the first tank ($j = 1$).

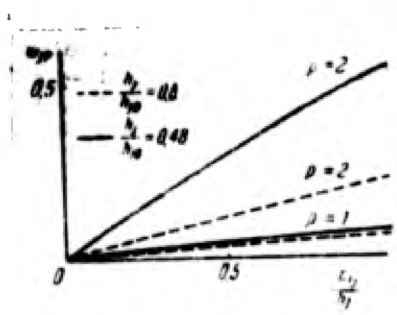

Fig. 12.6. The effect of the level of liquid on amplitudes of radial oscillations of the shell of the tank.

By analyzing the structure of coefficients (30) or (32), it is easy to notice that system of equations (28) and (29) is divided ($c_n = 0$) when the form of natural longitudinal oscillations of the considered structure of flight vehicle satisfies (at shown boundary conditions) not equation (31), but the nonlinear equation, which is obtained after replacement of $m(x_1)$ by

$$m_{np}(x_1) = m_\kappa(x_1) + \sum_{j=1}^{n_1} M_j m_j^*(\omega_n)\,\delta(x_1 - x_{1j}),$$

where $\delta(x_1 - x_{1j})$ - Dirac delta function.

For finding the solution of this equation, for instance, obtaining values of frequency and form of natural longitudinal oscillations of the first tone, it is possible to apply the method of successive approximations, essence of which consists of the following. Having assumed $\omega = \omega_1$, by formula (23) we find the values of $\bar{m}_j^*$ for every tank. Then we determine the corresponding reduced masses of liquid $\bar{m}_j^* M_j$. Having placed their values in equation (31), we calculate the first approximation for function $f_1(x_1)$ and frequency ω_1. After that we definitize $\bar{m}_j^*$ and repeat the calculation. This process will converge faster if after each approximation we correct the frequency of natural oscillations of the system, by using modified equation (38).

These changes lead to replacement of factor $(1 - \bar{m}_j^*)$ by factor $(m_{jk-1}^* - m_{jk}^*)$ in all coefficients, in which by k there is designated the number of approximation. In this case it is taken that when k = 1. The value of function $f_1(x_1)$, obtained by the shown method for the considered case of loading of flight vehicle little differs from values of function (34). Frequencies ω_1 and ω_{p1} are close. However, in many cases the influence of radial oscillations of the shell of tanks and mobility of liquid leads only to the appearance of additional frequencies of natural oscillations of the system.

It is necessary to note that the method of successive approximations gives good results only with ω_{p1}, differing from ω_{rj}, i.e., when $|\bar{m}_j^*|$ moderately differs from one. It is possible to judge actual values of $\bar{m}_j^*$ by Fig. 12.7, on which there are constructed graph of functions $\bar{m}_j^*(h_j)$ when $\omega = \omega_{p1}$ for all three tanks. With $h_j/h_{j0} = 0.48$, when $m_j^*(\omega_{p1}) = 2.15$, for obtaining $f_1(x_1)$ several approximations in all were required. With this ω_p was found from the following equation:

$$m_{11}\omega^2 = \omega_1^2 M_1. \qquad (12.39)$$

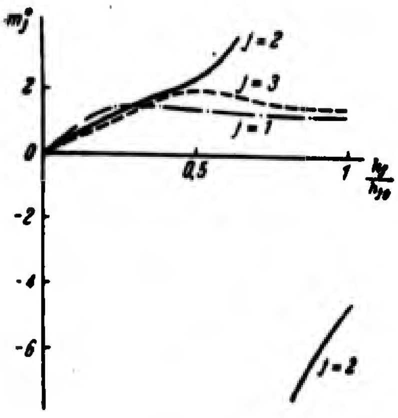

Fig. 12.7. Value of relative reduced mass of liquid in the tanks when

After establishing the fact of the influence of elasticity of bottoms and walls of tanks on the frequencies of longitudinal oscillations of the flight vehicle with liquid filling, the question appears about what are the requirements for $\omega_{Дj}$ and ω_{cj}. Calculations show that with increase of rigidity of bottoms the frequencies of the system increase (Fig. 12.8). As can be seen, when $\omega_{Дj}/\omega_{10} > 1$ the influence of $\omega_{Дj}$ on frequency ω_{p1}, corresponding to the form of longitudinal oscillations of the flight vehicle close to $f_1(x_1)$ becomes small. The magnitude of ω_{cj} has an analogous effect. With increase of the frequency of natural radial oscillations of the shell, the frequency of natural longitudinal oscillations of the construction of the flight vehicle is increased (Fig. 12.9).

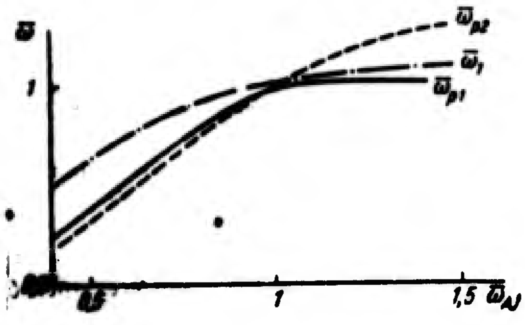

Fig. 12.8. The effect of rigidity of bottoms of tanks on the frequencies of natural longitudinal oscillations of the flight vehicle.

Equations of longitudinal oscillations of the flight vehicle taking into account forces of internal nonelastic drag will have the form

$$\ddot{T}_p + 2h_p\dot{T}_p + \omega_p^2 T_p = -\frac{P}{M_p} f_p(x_{1n}). \qquad (12.40)$$

where

$$2h_p = \delta_p \frac{\omega_p}{\pi}, \qquad (12.41)$$

and δ_p - logarithmic damping decrement of longitudinal elastic oscillations of structure of the flight vehicle of p-th tone.

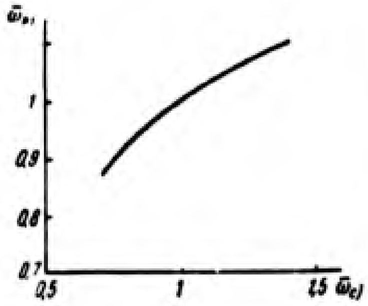

Fig. 12.9. Dependence of frequencies of natural longitudinal oscillations of the flight vehicle to partial frequency of natural radial oscillations of the shell of tanks.

In the first approximation it is possible to accept (Chapter IV) that

$$\delta_p = \delta + T_{p0}\delta_{p0}, \qquad (12.42)$$

where T_{p0} - amplitude of oscillations of the point of reduction ($f_p = 1$).

§ 12.4. Equations of Longitudinal Oscillations of a Vehicle with Side-Mounting Boosters

In § 12.2 there were listed equations of steady-state oscillations of an elastic flight vehicle, having m consecutively arranged cylindrical tanks, partially filled with liquid. By using the same method, it is possible to formulate analogous equations of longitudinal oscillations for a system of such vehicles (units), connected together with the aid of elastic connections in a cluster.

Let us assume that λ_{ks} is deformation of elastic connection of s-th unit of k-th group, and $x_к$ - coordinate of the place of connection of this k-th group parallel to arranged units to a certain frame or to the same elastic flight vehicle with cavities partially filled with liquid. If we represent deformation of each s-th unit in longitudinal direction (in the connected system of coordinates $x_{ks}y_{ks}$) in the form of series

$$u_{ks}(x_{ks},\ t) = \sum_{m=1}^{l_m} T_{ksm}(t)\, X_{ksm}(x_{ks}), \tag{12.43}$$

then displacement of any section of s-th unit of k-th group in system of coordinates $\xi\eta$ will be equal to

$$\begin{aligned} -\xi_{ks}(x_{ks},\ t) = {} & -\xi_0(t) + \lambda_{ks}(t) + \\ & + \sum_{n=1}^{l_n} T_n(t)\, X_n(x_n) + \sum_{m=1}^{l_m} T_{ksm}(t)\, X_{ksm}(x_{ks}). \end{aligned} \tag{12.44}$$

Having determined kinetic and potential energies of this system of units and generalized forces, corresponding to generalized coordinates ξ_0, λ_{ks}, T_n and T_{ksm}, we obtain the following system of equations, describing longitudinal oscillations of the totality of elastic units (each of which has k_j consecutively arranged tanks, partially filled with liquid):

$$\left.\begin{aligned} & M^*\ddot{\xi}_0 + \sum_{n=1}^{l_n} c_n \ddot{T}_n + \sum_{k=1}^{l_k}\sum_{s=1}^{l_s} c_{ks}\ddot{\lambda}_{ks} + \\ & \quad + \sum_{k=1}^{l_k}\sum_{s=1}^{l_s}\sum_{m=1}^{l_m} c_{ksm}\ddot{T}_{ksm} = P + \sum_{k=1}^{l_k}\sum_{s=1}^{l_s} P_{ks}, \\ & M_n^*\ddot{T}_n + H_n\dot{T}_n + k_{nn}T_n + \sum_{m=1}^{l_m} a_{nm}\ddot{T}_m + c_n\ddot{\xi}_0 + \\ & \quad + \sum_{k=1}^{l_k}\sum_{s=1}^{l_s} a_{ksn}\ddot{\lambda}_{ks} + \sum_{m=1}^{l_m} k_{nm}T_m + \\ & \quad + \sum_{k=1}^{l_k}\sum_{s=1}^{l_s}\sum_{m=1}^{l_m} a_{ksmn}\ddot{T}_{ksm} = -PX_n(x_n). \end{aligned}\right| \tag{12.45}$$

$$\left.\begin{aligned} & M_{ksm}^*\ddot{T}_{ksm} + H_{ksm}\dot{T}_{ksm} + k_{ksmm}T_{ksm} + \sum_{n=1}^{l_n} c_{ksmn} \times \\ & \quad \times \ddot{T}_{ksm} + c_{ksm}\ddot{\xi}_0 + d_{ksm}\ddot{\lambda}_{ks} + \sum_{n=1}^{l_n} k_{ksmn}T_{ksn} + \\ & \quad + \sum_{n=1}^{l_n} a_{ksmn}\ddot{T}_n = -P_{ks}X_{ksm}(x_{ksm}), \\ & M_{ks}^*\ddot{\lambda}_{ks} + e_{ks}\lambda_{ks} + c_{ks}\ddot{\xi}_0 + \sum_{n=1}^{l_n} a_{ksn}\ddot{T}_n + \\ & \quad + \sum_{m=1}^{l_m} d_{ksm}\ddot{T}_{ksm} = -P_{ks} \text{ when } m \neq n. \end{aligned}\right| \tag{12.46}$$

If each group of units has a common connection, then λ_{ks} can be represented also in the form of series

$$\lambda_{ks}(y_{ks}, t) = \sum_{r=1}^{i_r} \lambda_{kr}(t)\Phi_{kr}(y_s), \tag{12.47}$$

having taken $i_r = 1$ during calculation of the lowest frequencies of the system. In the last case the coefficients of these equations will be equal to

$$\left.\begin{aligned}
a_{ksn} &= M_{ks}X_n(x_\kappa)\Phi_k(y_s),\\
a_{ksmn} &= X_n(x_\kappa)\left[\int_0^{l_{ks}} m_{\kappa ks}(x)X_{ksm}(x)\,dx + \right.\\
&\qquad \left. + \sum_{j=1}^{k_j} M_{jks}\bar{m}^{*}_{jks}X_{ksm}(x_{\kappa s\pi j})\right],\\
d_{ksm} &= a_{ksmn}\Phi_k(y_s)\frac{1}{X_n(x_\kappa)},\\
k_{nm} &= \int_0^{l} E(x)F_c(x)\frac{dX_n(x)}{dx}\,\frac{dX_m(x)}{dx}\,dx,\\
k_{ksmn} &= \int_0^{l_{ks}} E(x)F_c(x)\frac{dX_{ksm}(x)}{dx}\,\frac{dX_{ksn}(x)}{dx}\,dx,\\
k_{nn} &= k_{nm},\ k_{ksmn} = k_{ksmm} \text{ when } n = m,\\
M^{*} &= M - \sum_{j=1}^{k_j} M_j(1-\bar{m}^{*}_j) - \sum_{j=1}^{k_j}\sum_{k=1}^{i_k}\sum_{s=1}^{i_s} M_{jks}(1-\bar{m}^{*}_{jks}),\\
M^{*}_n &= \int_0^{l} m_\kappa(x)X_n^2(x)\,dx + \sum_{j=1}^{k_j} M_j\bar{m}^{*}_jX_n^2(x_{\pi j}) + \\
&\qquad + \sum_{k=1}^{i_k}\sum_{s=1}^{i_s} M_{ks}X_n^2(x_\kappa),\\
M_{ks} &= M^{c}_{ks} + \sum_{j=1}^{k_j} M_{jks}\bar{m}^{*}_{jks},\\
M^{*}_{ks} &= M_{ks}\Phi^2_k(y_s),\\
M^{*}_{ksm} &= \int_0^{l} m_{\kappa ks}(x)X^2_{ksm}(x)\,dx + \sum_{j=1}^{k_j} M_{jks}\bar{m}^{*}_{jks}X^2_{ksm}(x_{ks\pi j}).
\end{aligned}\right| \tag{12.48}$$

$$\left.\begin{aligned}
-c'_n &= \int_0^l m_{\text{к}}(x) X_n(x)\,dx + \\
&\qquad + \sum_{k=1}^{l_k}\sum_{s=1}^{l_s} M_{ks} X_n(x_{\text{к}}) + \sum_{j=1}^{k_j} M_j \bar{m}_j^{*} X_n(x_{\lambda j}),\\
-c_{ks} &= M_{ks}\Phi_k(y_s),\\
-c_{ksmn} &= \int_0^{l_{ks}} m_{\text{к}ks}(x) X_{ksm}(x) X_{ksn}(x)\,dx + \\
&\qquad + \sum_{j=1}^{k_j} M_{jks} \bar{m}^{*}_{jks} X_{ksm}(x_{ks\lambda j}) X_{ksn}(x_{ks\lambda j}),\\
a_{nm} &= \int_0^l m_{\text{к}}(x) X_n(x) X_m(x)\,dx + \sum_{k=1}^{l_k}\sum_{s=1}^{l_s} M_{ks} X_n(x_{\text{к}}) X_m(x_{\text{к}}) + \\
&\qquad + \sum_{j=1}^{k_j} M_j \bar{m}_j^{*} X_n(x_{\lambda j}) X_m(x_{\lambda j}),
\end{aligned}\right\} \qquad \text{(12.48 Cont'd)}$$

where M - mass of system on the whole, M^c_{ks} - mass of the s-th unit of k-th group without liquid, $m_{\text{к}}$ - mass of frame or body of carrier (without liquid), P_{ks} - external force acting on s-th unit of k-th group.

As $X_{ksm}(x_{ks})$ it is possible to take the forms of natural longitudinal oscillations of s-th unit of k-th group (with solidified liquid), rigidly attached at the place of joining of elastic connection, and as $X_n(x)$ - forms of natural elastic longitudinal oscillations of the frame (or basic carrier unit) with free ends, calculated taking into account the masses of all units joining it (through elastic connections) in sections $x_{\text{к}}$. Partial frequencies corresponding to these forms of oscillations will be designated by ω_{ksm} and ω_n. It is necessary to note that in the presence of separate (isolated) connections for each s-th unit it is expedient to determine ω_n, ω_{ksm} and $X_n(x)$, $X_{ksm}(x_{ks})$ taking into account the elasticity of these connections, assuming the unit not rigid, but elastically attached (at the place of connection to the frame). In this case in equations (45) and (46) all terms with λ_{ks} disappear and the equations for λ_{ks} vanish. If we are limited by consideration of only the first tones of natural oscillations of joining units, then as $X_{ks1}(x_{ks})$ it is possible to take the normal form of natural longitudinal oscillations of the system: body of vehicle-liquid $f_i(x_{k_i})$, methods of calculation of which are given in § 12.2. By using the method discussed in Chapter X, system of equations (45) and (46) can be written in the normal coordinates, i.e., reduced to the form considered in the following paragraph.

§ 12.5. Parameters of Steady-State Longitudinal Oscillations of the System "Vehicle-Propulsion System"

In the case when external longitudinal forces are changed according to harmonic law with assigned frequency $\omega \neq \omega_{cj}$, equations of steady longitudinal oscillations of the flight vehicle with cylindrical tanks, partially filled with liquid, are reduced to a system of ordinary differential equations with constant coefficients, solution of which does not present special difficulties.

The problem is complicated when there is a dependence of thrust force P to parameters of motion of the flight vehicle, and namely to pressure of liquid in the tanks and lines. In this case feedback appears and the action diagram will have the form of a closed circuit. Random longitudinal oscillations of the flight vehicle body $\xi(x_i)$ will be called pressure fluctuation p_j on the bottom of the tank and in the pipe, supplying liquid from the tank to the propulsion system. Pressure fluctuation of liquid in the line will lead to pulsation of thrust force P. Oscillations of the latter depending upon phase shift in the system will either strengthen or weaken the longitudinal oscillations of the flight vehicle body. With strengthening the system will swing and in the presence of nonlinearity can enter conditions of steady-state oscillations (natural oscillations). For solution of problems of strength it is important to know how to find at least the approximate values of frequencies and amplitudes of these oscillations and to determine the conditions of their appearance.

For simplification of computations let us consider steady-state oscillations of such a system on the assumption that thrust force is a function of liquid pressure of only one of the fuel tanks, namely the tank with oxidizer. Let us represent the dynamic properties of the feed system of liquid to the propulsion system (taking into account elastic properties of the main line, hydraulic losses and inertness of flow in the line) in the form of operational equation

$$L(s)p_{в}=E(s)k_{m}p_{j}. \tag{12.49}$$

Here L and E - polynomials with respect to s, s - variable of Laplace transformation, $p_{в}$ - outlet pressure of the feed system of liquid and

$$p_{j}=p_{\tau j}+p_{\text{д}j}, \tag{12.50}$$

where $p_{\text{д}j}$ - liquid pressure on the bottom of the tank at the pipe entrance, equal to (in connected system of coordinates)

$$p_{\text{д}j}=\rho_{j}d_{j}h_{j}\dot{m}_{j}^{\circ}\xi(x_{\text{д}j}), \tag{12.51}$$

and d_j - correction factor, considering the influence of the place of location of the entrance section of the pipe on the bottom of the tank, i.e., its distance from the axis of cavity ($r_{\tau j}$). When $\mu_j < 0.5$ it is possible to approximately consider $d_j = 1$, and when $\mu_j \geq 0.5$.

$$d_{j}=\mu_{j}\frac{I_{0}\left(\mu_{j}\frac{r_{\tau j}}{a_{j}}\right)}{2I_{1}(\mu_{j})}.$$

By $p_{\tau j}$ there is designated additional inertial pressure of liquid, located in the pipe itself (with length $h_{\tau j}$):

$$p_{\tau j}=\rho_{j}h_{\tau j}\xi(x_{1в}); \tag{12.52}$$

Let us assume further that the equation determining the dependence of trust force to liquid pressure $p_{в}$ during oscillations of the system is linear. In this case the values of thrust force can lag a certain interval of time τ with respect to $p_{в}$. Then

$$R(s)P = D(s)p_{\text{в}}k_c e^{-\tau s}. \tag{12.53}$$

Equations (49) and (53) can be reduced to one equation of form

$$W(s)P = k_p p_j Q(s) e^{-\tau s}, \tag{12.54}$$

where

$$W(s) = R(s)L(s),$$
$$Q(s) = D(s)E(s),$$
$$k_p = k_c k_m.$$

The system of equations describing longitudinal oscillations of the flight vehicle will have the form

$$s^2 \xi_0 c_p + S(s) T_p = a_p P, \tag{12.55}$$
$$s^2(\xi_0 + c_p T_p) = cP, \tag{12.56}$$

where

$$S(s) = s^2 + 2h_p s + \omega_p^2,$$
$$a_p = -\frac{f_p(x_{1n})}{M_p}, \quad c = \frac{1}{M}.$$

In the case when equations of longitudinal elastic oscillations of flight vehicle are represented in normal coordinates, c_p will be equal to zero, and the dependence of p_j to T_p will be determined by expression

$$p_j = e_0 P + e_p s^2 T_p, \tag{12.57}$$

where

$$e_0 = c\rho_j(h_{\tau j} + h_j d_j \bar{m}_j^{\bullet}),$$
$$e_p = -\rho_j\left[h_j d_j \bar{m}_j^{\bullet} f_p(x_{\text{л}j}) + h_{\tau j} f_p(x_{1n})\right].$$

For finding the approximate values of frequency Ω and amplitude T_{p0} of steady-state oscillations (natural oscillations) of the considered closed system, it is necessary to formulate a characteristic equation of this system and to write conditions at which it will have a pair of purely imaginary roots. In this case it is assumed, of course, that the remaining roots of the characteristic equation of the system have negative real parts. Having substituted $s = i\omega$ and equated the imaginary and real parts of the obtained equality to zero, we find the following equations for determination of Ω and T_{p0}:

$$\begin{aligned} &W_0(\omega) S_0(\omega) - W_1(\omega) S_1(\omega,\ T_{p0}) + \{k\omega^2 Q_0(\omega) + k_\tau [Q_1(\omega) S_1(\omega,\ T_{p0}) - \\ &\quad - S_0(\omega) Q_0(\omega)]\} \cos\omega\tau + \{k\omega^2 Q_1(\omega) - k_\tau [Q_1(\omega) S_0(\omega) + \\ &\quad + S_1(\omega,\ T_{p0}) Q_0(\omega)]\} \sin\omega\tau = 0, \end{aligned} \tag{12.58}$$

$$\begin{aligned} &W_0(\omega) S_1(\omega,\ T_{p0}) + S_0(\omega) W_1(\omega) + \{k\omega^2 Q_1(\omega) - k_\tau [Q_0(\omega) S_1(\omega,\ T_{p0}) + \\ &\quad + Q_1(\omega) S_0(\omega)]\} \cos\omega\tau - \{k\omega^2 Q_0(\omega) - k_\tau [Q_0(\omega) S_0(\omega) - \\ &\quad - Q_1(\omega) S_1(\omega,\ T_{p0})]\} \sin\omega\tau = 0, \end{aligned} \tag{12.59}$$

where $k = k_p a_p e_p$ - amplification factor of the system elastic body of the flight vehicle-propulsion system, and $k_T = k_p e_0$ - amplification factor of the system flight vehicle as an absolutely rigid body-propulsion system. By $W_0(\omega)$ there is designated the polynomial relative to ω composed of terms of polynomial $W(s)$, containing zero and even degrees s (with replacement of s^2 by $-\omega^2$); and by $W_1(\omega)$ - polynomial composed from terms containing odd degrees of variable s (with replacement of s by ω, s^3 by $-\omega^3$, etc.). Analogous designations are accepted for real and imaginary parts of polynomials $Q(\omega)$ and $S(\omega)$. For instance,

$$S_0(\omega) = \omega_p^2 - \omega^2, \tag{12.60}$$

$$S_1(\omega,\ T_{p0}) = 2h_p\omega. \tag{12.61}$$

Having excluded $S_1(\omega, T_{p0})$ from equations (58) and (59), we obtain the following equation for finding the frequency of steady-state oscillations of system $\omega = \Omega$:

$$\omega^2 = \frac{\omega_p^2}{1 + k\dfrac{A(\omega)}{B(\omega)}}, \tag{12.62}$$

where

$$A(\omega)=k_{\tau}\left[Q_0^2(\omega)+Q_1^2(\omega)\right]-\left[W_0(\omega)\,Q_0(\omega)+\right.$$
$$\left.+W_1(\omega)\,Q_1(\omega)\right]\cos\omega\tau-\left[Q_1(\omega)\,W_0(\omega)-Q_0(\omega)\,W_1(\omega)\right]\sin\omega\tau,$$
$$B(\omega)=W_0^2(\omega)+W_1^2(\omega)+k_{\tau}^2\left[Q_0^2(\omega)+Q_1^2(\omega)\right]-$$
$$-2k_{\tau}\left[W_0(\omega)\,Q_0(\omega)+W_1(\omega)\,Q_1(\omega)\right]\cos\omega\tau-$$
$$-2k_{\tau}\left[W_0(\omega)\,Q_1(\omega)-W_1(\omega)\,Q_0(\omega)\right]\sin\omega\tau.$$

From this formula it follows that Ω depends on amplification factors k and k_T, parameters of the liquid feed system and the propulsion system itself. However, as calculations show, in most cases, having practical value, the influence of these parameters on the frequency of oscillations is small, and one may approximately assume that $\Omega = \omega_p$. This important conclusion was used for an experimental check of the accuracy of determination of natural frequencies of elastic longitudinal oscillations of a flight vehicle with cylindrical tanks, partially filled with liquid, by the proposed method of approximation. The experimental installation, consisting of two tanks and engine, was suspended by the upper and on an elastic support. By means of change of the amplification factor of the system in the latter were excited natural oscillations with different levels of liquid in the tanks. With the aid of accelerometers and strain gauges we measured the frequencies of oscillations of the system. Experimental values of these frequencies are shown by points on Fig. 12.10. ı the same place for comparison there are given computed values of the first two natural frequencies of the given system (elastic body-liquid), obtained from equation (39), i.e., when $n_n = 1$. The dotted line shows the value of partial frequency ω_n when $n = 1$ (for "solidified" liquid taking into account the elasticity of bottoms and elasticity of support). As can be seen, calculated data will agree with experimental for almost all values of levels of liquid in the tanks.

The mobility of liquid in tanks due to radial oscillations of walls and elasticity of bottoms leads not only to some lowering of the natural frequency of longitudinal oscillations of the flight vehicle construction of the first tone, but also the appearance of additional frequency ω_{p2}. Calculation of the effect of function

$X_2(x_1)$, corresponding to the second tone of natural longitudinal oscillations of the construction of the flight vehicle with solidified liquid, in this case leads only to a small change of computed values of ω_{p2} and does not affect ω_{p1}. The value of frequency of natural oscillations of the system of first tone, obtained by the method of successive approximations discussed in § 12.2, little differs from ω_1.

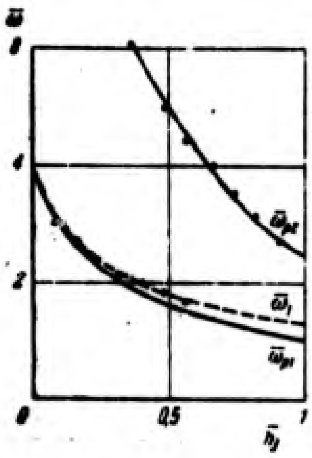

Fig. 12.10. Effect of the level of liquid in tanks on frequencies of natural oscillations of the system (points show experimental values of frequencies).

By considering the depencence of S_1 to T_{p0}, by knowing Ω it is possible from equations (58) and (59) to obtain expression for determination of amplitude of steady-state oscillations of the system[1]

$$2h_p(T_{p0}) = k\Omega\frac{C(\Omega)}{B(\Omega)}, \tag{12.63}$$

where

$$C(\Omega) = [W_1(\Omega)Q_0(\Omega) - W_0(\Omega)Q_1(\Omega)]\cos\Omega\tau + \\ + [W_1(\Omega)Q_1(\Omega) + W_0(\Omega)Q_0(\Omega)]\sin\omega\tau, \\ B(\Omega) = B(\omega) \text{ when } \omega = \Omega.$$

The value of this amplitude will depend on the form of function $\delta_p(T_{p0})$. In our case, according to formula (42),

$$T_{p0} = k\frac{\pi\Omega C(\Omega)}{\delta_{p0}\omega_p B(\Omega)} - \frac{\delta}{\delta_{p0}}. \tag{12.64}$$

When necessary, by using the method of harmonic linearization [54], besides internal nonelastic drag of material it is possible to consider the force of dry friction. By knowing T_{p0}, it is possible to calculate the amplitudes of oscillations of bottoms, shells of tanks and pulsations of thrust force.

Thus, T_{p0} is a positive magnitude, then a necessary condition of the existence of steady-state oscillations of the system is inequality

$$k > \frac{\delta\omega_p B(\Omega)}{\pi\Omega C(\Omega)}.$$

The system gets on boundary of stability when

$$k = k_{\text{гр}} = \frac{\delta\omega_p B(\Omega)}{\pi\Omega C(\Omega)}. \qquad (12.65)$$

The boundary value of the amplification factor of the system can be determined approximately by formula

$$k_{\text{гр}} = \frac{\delta}{\pi}\,\frac{B(\omega_p)}{C(\omega_p)}. \qquad (12.66)$$

From (64) and (65) it follows that

$$T_{p0} = \frac{\delta}{\delta_{p0}}\left(\frac{k}{k_{\text{гр}}} - 1\right). \qquad (12.67)$$

Longitudinal forces appearing with such oscillations in cross sections of the flight vehicle body with amplitude equal to

$$\Delta N_x(x_1) = \omega_p^2 T_{p0} \int_0^{x_1} \left[m_k(x_1) f_p(x_1) + \sum_{l=1}^{n_1} M_l m_l^{*} f_p(x_{\text{д}l})\,\delta(x_1 - x_{\text{д}l}) \right] dx_1, \qquad (12.68)$$

where $\delta(x_1 - x_{\text{д}l})$ is Dirac delta function, can reach large values.

By using formula (65), we can determine the boundary values of not only the amplification factor, but any parameter of the considered system. It is possible to construct the regions of existence of steady-state oscillations of the system (or region of stability of the system) depending upon parameters of the liquid feed system and characteristics of the propulsion system. For example on Fig. 12.11 there is shown the effect of time of delay τ on the boundary values of amplification factor $k_{гр}$ (for experimental installation-vehicle suspended on an elastic support) depending upon h_j (when $\Omega = \omega_{p1}$). In this case we used equation (53) in the form

$$(T_3s^3 + T_2s^2 + T_1s + 1)p = k_c(T_0s + 1)p_в e^{-\tau s},$$

and $p_в$ was determined by formula (50), i.e., $p_в = p_j$.

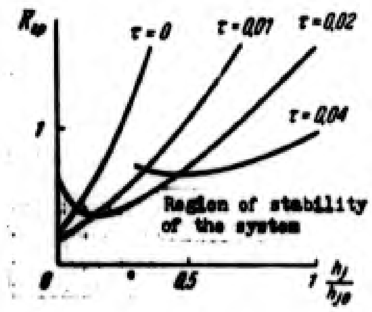

Fig. 12.11. The effect of time of delay on the boundary values of amplification factor of the system.

Investigations show that parameters of the pipe for supply of liquid fuel from the tank to propulsion system can essentially effect the stability of the given system. It is easy to see this if one considers the equation of this main line (49) even in simplified form

$$(s^2 + h_т s + \omega_т^2)p_в = c_т p_j.$$

For guarantee of stability of the system in a similar case, i.e., in the presence of a comparatively long pipe for fuel feed, it is necessary that $\omega_т$ be considerably lower than ω_p. This can be attained by corresponding selection of the dynamic characteristics of separate sections of the main line.

In a number of cases natural oscillations of the shown system with limited amplitudes can be permissible on separate sections of the trajectory. It is natural that additional limiting forces appearing in this case should be appropriately considered when determining the safety factors of the construction. Moreover, the magnitudes of these forces can serve as one of the criteria for appraisal of the technical stability of the system.

Footnotes

[1]In certain cases nonlinearity of the characteristic of fuel feed pump, for instance in the presence of cavitation, can have essental value.

[2]K. S. Kolesnikov, Low-frequency instability of nominal conditions of a liquid-fuel rocket engine, Academy of Sciences of USSR, Journal of applied mechanics and tech. physics, 2, 1965.

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CHAPTER XIII

STEADY TRANSVERSE OSCILLATIONS OF GUIDED FLIGHT VEHICLES

§ 13.1. Formulation of the Problem

The motion stability of many flight vehicles on the boost phase of flight is provided by an automatic system of angular stabilization (automatic stabilizer, automatic pilot). This system can be self-adjusting (with limitations of various types) or with constant coefficients. The first type of control system permits ensuring the necessary motion stability of the flight vehicle with change of its dynamic characteristics in comparatively wide limits. The second type places essential limitations on the accuracy of determination of these characteristics and frequently does not ensure the total absence of natural oscillations in the system in the required range of frequencies.

Block diagram of the system flight vehicle-automatic stabilization control (with constant amplification factor) in outline has the form shown on Fig. 13.1. The sensing device of the automatic pilot (free gyroscope) is installed in one of the sections x_0 of the flight vehicle body. A signal, proportional to deflection of the angle of rotation of this section $\Delta\vartheta(x_0)$, is fed to the input of amplifier-converter. As a result there is formed a command, proportional to derivative of $\Delta\vartheta$, which after amplification is fed to actuators. For the purpose of obtaining the required high-speed operation of control system and coordination of fed control signals with actual results of their realization, feedback from control elements is introduced to one of the amplifier stages. An electro-hydraulic machine is used as drive of control devices. Usually the structure of the control system in pitching and yawing planes is identical. When the connection between equations of oscillations of the construction of the flight vehicle in pitching, yawing and rolling planes is small, the stability of system vehicle-automatic stabilization control can be considered separately for each of the named planes.

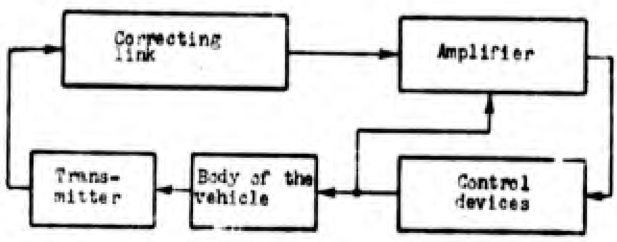


Fig. 13.1. Block diagram of system flight vehicle-automatic stabilization control.

Oscillations of the construction of the flight vehicle are the source of additional signals, proceeding from gyro-instruments to automatic stabilization control. These signals will be proportional to the angle of rotation of the cross section due to bending of the body and to its time derivatives. In other words, the sensing device will perceive elastic oscillations of the flight vehicle construction as angular oscillations of a solid body, issuing the corresponding signals, magnitudes of which will depend on the place of installation of sensing transmitters on the body, and also the rigidity of their bracing elements. The control system converts into these signals control forces. Depending upon the phase, they will either damp oscillations of the body or, conversely, amplify them. Values of function $S_n(t)$, in the final analysis determining stresses in structural elements of the flight vehicle, will either decrease or, conversely, grow. In the first case the motion of the flight vehicle will be stable and its oscillations will be comparatively rapidly damped even in the absence of damping forces. In the second case the system turns out to be unstable and only because of nonlinearity of characteristics of the real system its oscillations in the end will have a steady character. The appearance on the separate phase of guided flight of sustained oscillations in the system vehicle-automatic stabilization control (automatic pilot) in pitching and yawing planes with frequencies close to that of natural transverse elastic oscillations of the body, or frequencies of natural oscillations of liquid in the tanks at the stage of flying-design tests is usual phenomenon. Sometimes (because of variation of parameters of automatic stabilization control) similar oscillations appear even for experimentally developed versions of flight vehicles. Therefore, the given case of loading (case G) of the construction has important value for flight vehicles and especially for ballistic type vehicles, the possibilities of natural development of which are very limited.

In the preceding chapters it was noted that the connection of bending oscillations of the construction with oscillations of the flight vehicle as a solid body for all practical purposes is absent, and the interconnection of elastic oscillations of the construction and oscillations of liquid in fuel tanks is small. Only the connection of oscillations of liquid with oscillations of the flight vehicle as a solid body is essential. Proceeding from this, during investigation of motion stability of the system vehicle-automatic stabilization control we prefer to consider the following two subsystems separately:

a) vehicle as a solid body-automatic stabilization control,
b) elastic body of vehicle-automatic stabilization control.

It is natural that such an approach to the solution of the problem of stability is simplified and serves as one of the sources of deviation of calculated dynamic layout from real. The presence of noncorrespondence of real and calculated dynamic layout of the system because of errors of its description or unaccounted for variance of parameters in a number of cases can be the cause of appearance of steady-state oscillations of the system.

So that the motion of a guided flight vehicle with liquid filling would be stable, the automatic stabilization control should provide phase lead at frequencies corresponding to the frequencies of natural oscillations of liquid in fuel tanks ω_{kj}, and delay at frequencies corresponding to the frequencies of natural elastic oscillations of the construction ω_n of beam type. Critical frequency of automatic stabilization control should be larger than ω_{kj} and less than ω_n. Since these requirements are contradictory, they often cannot be fulfilled in the same range of frequencies simultaneously. Therefore, in similar cases for guarantee of stability we try to scatter the mentioned frequencies by different constructive measures: reduce the mass of fluctuating part of liquid, increase damping, since it is obvious that it is very difficult to change noticeably the frequencies of natural elastic oscillations of the construction of the flight vehicle.

The shown requirements for adjustment of automatic stabilization control pertain basically to the lowest frequencies (less than 8-15 Hz). Higher frequencies do not affect the processes of adjustment. The spectrum of action of control forces on the flight vehicle construction is almost always low-frequency and the amplitude of its harmonics is decreased with increase of their number according to equality

$$\delta_n = \frac{\max \delta}{\omega_n} \quad (n = 1, 2, \ldots).$$

If the transmission band of the control system is less than frequencies of natural transverse elastic oscillations of the flight vehicle construction, then amplitudes of these oscillations are considerably decreased with passage through automatic stabilization control. In this case the smaller the amplification factor of the system, the greater is their weakening. Smallness of the magnitude of amplification factor of the system is ensured by installation of stabilizers, air vanes. However, minimum value of the amplification factor is limited by conditions of providing the necessary high-speed operation of the control system. Therefore, lowering of the amplification factors turns out to be unsuitable, and for decrease of amplitudes of high-frequency oscillations of the flight vehicle special electrical filters are often used. Application of the latter in turn affects the phase angles on low frequencies, i.e., in the range in which phase stabilization is applied. Usually the frequency of adjustment of automatic stabilization control lies in the region 0.5 Hz [6]. Convergence of frequencies of adjustment and elastic transverse oscillations of the construction always lowers the operating reliability of the system of control and requires stabilization.[1] When it is not possible to scatter the frequencies of natural oscillations of the construction and frequencies of natural oscillations of liquid, steady-state oscillations of the system vehicle-automatic tabilization control can be observed at one of these frequencies.

First of all the question arises about how to consider the influence of oscillations of control forces on the state of stress of the construction. Should these oscillations pertain to exploitational operating conditions of the system and their influence be considered during determination of the required bearing capacity of the construction and magnitude of safety factor or pertain to emergency cases of loading? Usually when designing flight vehicles we try to select parameters of the system vehicle-automatic stabilization control in such a way that it is dynamically stable, i.e., that sustained oscillations of control devices are practically absent. However, the denser the spectrum of natural frequencies of transverse elastic oscillations of the flight vehicle construction and the more complicated the dynamic layout, the more difficult it is to provide its stability in the above-indicated sense. Consequently, the greater the danger that the considered case of loading will affect the required bearing capacity of the construction and even the assembly diagram of the vehicle, the more rigid the requirements must be for operating conditions of the control system, one of the basic problems of which is decrease of overall and local overloads, i.e., amplitudes of elastic oscillations.

[1]With such stabilization the control forces will be changed with frequency of elastic oscillations in antiphase with oscillations of the construction of the vehicle, i.e., additional damping of elastic oscillations of the body will take place.

Transverse oscillations of the flight vehicle construction not only cause additional forces in cross sections of the body, but also harmfully affect the operation of certain instruments. They can lead to saturation of the control system and loss of stability, i.e., to the appearance in the system of impermissibly strong additional lateral loads on the construction. Furthermore, they are the source of additional lateral perturbations for the vehicle in the process of stage separation, etc.

Thus, dynamic calculation of the construction for the considered case of loading G is reduced to solution of two problems:

1) appraisal of the effect of steady-state oscillations of the system vehicle-automatic stabilization control on the required strength of construction,

2) establishment of limitations for frequency and amplitude of transverse oscillations of the system on the part of the strength of construction of the flight vehicle.

The first problem appears when there is information about parameters of oscillations of the system, obtained by calculation, by means of natural tests of the flight vehicle or by means of modeling. Basic advantage of modeling consists of the possibility of using real equipment of automatic stabilization control (on an analog model there is collected only equations of transverse oscillations of the flight vehicle). In this case there is reproduced the variability of coefficients of equations and peculiarity of current-collecting transmitters of sensing devices of the control system, and namely their gradation. This gradation of the transmitter may cause impulsive loading of the construction by control forces (creating jerks of control devices), and consequently, can excite its elastic oscillations. With small angles of deflection, conversely, it can (by periodically disturbing the continuity of entry of signals from gyro-instruments into automatic stabilization control) damp oscillations of the system. During theoretical solution of this problem one should use experimental amplitude-phase characteristics of automatic stabilization control. The presence of such characteristics permits calculating the parameters of steady-state oscillations of the system by the same method of harmonic balance [53] that was applied in Chapter XII during the investigation of longitudinal oscillations of the system vehicle-propulsion system.

It is most often necessary to solve the second problem in the process of flying-design tests of the vehicle. Less often - in the period of development of the construction, when serious difficulties are revealed in the guarantee of stability of the system at low frequencies.

§ 13.2. Determination of Conditions of Steady-State Oscillations of the System Vehicle-Automatic Stabilization Control

Closed system vehicle-automatic stabilization control is a nonlinear system. Nonlinear functions are contained both in equations of the object of control itself (nonelastic and aerodynamic drag, oscillations of liquid), and in equations of the control system (amplifier, control actuator and so forth). At definite values of parameters and definite initial conditions the shown system will be in a position of stable equilibrium. With other initial conditions or other values of parameters of automatic stabilization control and the flight vehicle itself, it can be unstable or accomplish steady-state oscillations, i.e., be in stable self-oscillation condition. Usually the basic nonlinear element of the given system, limiting the amplitude of oscillations, is the control actuator (servodrive).

In simplified form the characteristic of such actuator, namely the relationship of $d\delta/dt$ to input current i, is depicted by the graph shown on Fig. 13.2. Nonlinearity of the amplifier, in particular the relationship of its amplification factor to the amplitude of input signal, usually has small effect on conditions of natural oscillations of the system. Therefore, subsequently for simplification of the problem we will consider this link linear.

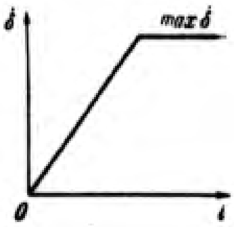


Fig. 13.2. Determination of conditions of steady-state oscillations of the system vehicle-automatic stabilization control.

Bearing in mind the comparatively slow change of all parameters of the system in time, in this case for finding the frequencies and amplitude of steady-state oscillations it is possible to use the method of harmonic balance [53]. By ϕ_{AC}, ϕ_c and ϕ_n let us designate phase angles of automatic stabilization control, the flight vehicle as a solid body and an elastic flight vehicle, and by W_{AC}, W_c, W_n - amplitude-frequency characteristics of these links respectively. Typical graphs of these functions are presented on Fig. 13.3.

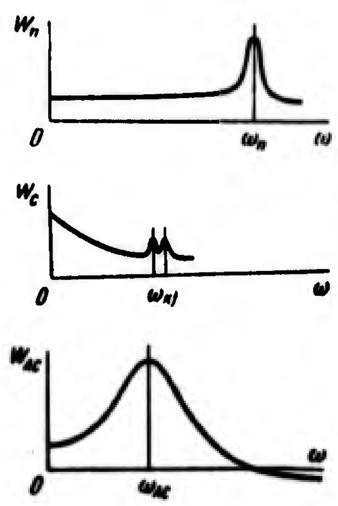


Fig. 13.3. Typical amplitude-frequency characteristics of links of the system vehicle-automatic stabilization control.

The type of frequency characteristic of automatic stabilization control depends on the magnitude of static component of input signal, possible, limits of deviation of characteristics of real automatic stabilization control from nominal values, and also on possible variation of parameters of the controlled system.

Characteristic equation for closed subsystem flight vehicle as a solid body with liquid filling-automatic stabilization control will have the form

$$1 + W_{AC} W_c e^{i\varphi_{AC}} e^{i\varphi_c} = 0. \qquad (13.1)$$

Analogously there is written the equation for the subsystem elastic flight vehicle-automatic stabilization control

$$1 + W_{AC} e^{i\varphi_{AC}} \sum_{n=1}^{n_n} W_n e^{i\varphi_n} = 0. \qquad (13.2)$$

The latter is essentially simplified in those cases when frequencies of natural bending oscillations of the construction are spread, and consequently, it is possible to be limited by consideration of only one tone (n = 1 or n = 2) of elastic oscillations of the construction. As can be seen from Fig. 13.3, W_n is great only with ω close to ω_n.

Steady-state oscillations of the mentioned nonlinear subsystem are possible when they are unstable in linear setting. Determination of the values of parameters of automatic stabilization control, at which all real roots and real parts of complex-conjugate roots of the characteristic equation of the system will be negative, is the object of investigation of motion stability of a guided flight vehicle. Therefore, here we will not elaborate on this question in detail, but will give only a general idea of the method of approximation for determination of conditions of such steady-state oscillations (natural oscillations) on an example of the subsystem elastic body of the flight vehicle-automatic stabilization control.

Natural oscillations of the shown subsystem usually occur with frequency very close to one of the natural frequencies of bending oscillations of the construction in the pitching (yawing) plane. In this case, as already was mentioned above, it is possible in the first approximation to take equations of transverse oscillations of the flight vehicle in simplified form, without taking into account its motion as a solid body of without taking into account the influence of aerodynamic forces, i.e., in the form

$$(T_2p^2 + T_1p + 1)S_n = k_0\delta, \qquad (13.3)$$

where

$$k_0 = \frac{f_n(x_{1p})}{\omega_n^2 M_n} R^{\delta}_{y_1},$$

$$T_2 = \frac{1}{\omega_n^2}, \quad T_1 = \frac{2h_n}{\omega_n^2},$$

$$p = \frac{d}{dt}, \quad a\ n = 1 \text{ or } 2, \text{ or } 3, \ldots$$

Linearized equation of amplifier-converter in the region of frequencies close to ω_n can be approximately represented in the following form:

$$(\tau_2p^2 + \tau_1p + 1)i_c = (a_0 + a_1p)\Delta\vartheta_n, \qquad (13.4)$$

where i_c - current in servodrive, a_i and τ_i - coefficient of equation (i = 0, 1, 2). For accepted direction of axis x_1

$$\Delta\vartheta_n(t) = -S_n(t)\frac{df_n(x)}{dx_1}\bigg|_{x_1=x_0}. \qquad (13.4')$$

Motion of servodrive (taking into account feedback) is described by nonlinear equation

$$(Tp + 1)\delta = F(i_c). \qquad (13.5)$$

In accordance with the method of harmonic linearization with symmetric oscillations the nonlinearities of saturation type with respect to velocity (Fig. 13.2) can be approximately (retaining only the lowest harmonics) represented in the form [54]

$$F(i_c) = q(a)i_c,$$

where

$$\left.\begin{aligned} &q(a) = k_y \text{ when } a < b, \\ &q(a) = \frac{2k_y}{\pi}\left(\arcsin\frac{b}{a} + \frac{b}{a}\sqrt{1 - \left(\frac{b}{a}\right)^2}\right) \text{ when } a \geq b, \end{aligned}\right| \qquad (13.6)$$

T - time constant, k_y - amplification factor, a - amplitude, and $b = \frac{\max \dot{\delta}}{k_y}$.

Let us write the linear part of the considered system (3), (4) and (4') in the following way:

$$Q_{\text{л}}(p) l_c = R_{\text{л}}(p)\delta, \tag{13.7}$$

where $Q_{\text{л}}(p)$ and $R_{\text{л}}(p)$ - operational polynomials.

Then the characteristic equation of the given closed (linearized by method of harmonic balance) system will have the form

$$(Tp+1) Q_{\text{л}}(p) - q(a) R_{\text{л}}(p) = 0. \tag{13.8}$$

So that this system would be stable, it is necessary that all coefficients of equation (8) be positive. In particular, the following coefficient with zero degree p should be positive

$$c_0 = -\left[\omega_n^2 + k_0 a_0 q(a) \frac{dl_n(x_0)}{dx_1}\right] > 0.$$

From this condition it follows that for increase of reserve of stability it is necessary to try to install sensing devices of automatic stabilization control so that $k_0 \frac{dl_n(x_0)}{dx_1} > 0$. If this cannot be done, then it is possible to decrease the amplification factor of system

$$k = k_0 a_0 q(a).$$

By p = iω in equation (8) and dividing the real and imaginary parts, we obtain equation of Mikhaylov curve

$$F(i\omega) = X(\omega) + iY(\omega). \tag{13.9}$$

Assuming

$$X(\omega)=0 \text{ and } J(\omega)=0,$$

we find the sought equations for determination of ω and a. Solution of these equations is best of all by graphic means. The scheme of calculation of conditions of steady-state oscillations and subsystems vehicle with liquid filling-automatic stabilization control will be the same.

Results of calculations of amplitudes and frequencies of steady-state oscillations of the shown subsystems for various types of flight vehicles permit making the following conclusions:

1. Parameters of automatic stabilization control in most cases weakly affect the frequency of steady-state oscillations of the system. Therefore, it differs little from corresponding partial frequencies ω_n and ω_{kj}.

2. Amplitude of steady-state oscillations of the angle of deflection of control devices δ_m is determined with sufficient accuracy for practical purposes by equality

$$\delta_m = \frac{\max \dot{\delta}}{\omega}. \tag{13.10}$$

By using these conclusions, it is possible for the given case of loading to construct a very simple scheme of approximation calculation of the limiting values of forces in structural elements of the flight vehicle. Since with small difference of ω from ω_n the corresponding dynamic coefficient differs comparatively little from its limiting value, in the first approximation it is possible to use limiting values of lateral forces and bending moments, corresponding to $\omega = \omega_n$ or $\omega = \omega_{kj}$, i.e., to consider forced steady-state oscillations of the flight vehicle construction under the action of force

$$R_{y_i} = R_{y_i}^{\delta} \frac{\max \dot{\delta}}{\omega} \sin \omega t. \tag{13.11}$$

Bearing in mind that the values of these limiting forces depend on values of damping factor, gradient of control force and speed of its change, in all cases we should try to decrease the values of $R^{0}_{\psi_1}$ and δ and to increase coefficient h_n.

§ 13.3. Determination of Static Values of Lateral Forces and Bending Moments

In the preceding paragraph there was given the method of approximation for determination of conditions of steady-state oscillations of the subsystem elastic body-automatic stabilization control. Frequencies and amplitudes of steady-state oscillations of subsystem vehicle-liquid-automatic stabilization control are found absolutely the same way. As already was noted, separate consideration of these subsystems makes sense only when the frequencies of partial oscillations ω_n and ω_{kj} are spaced.

Calculation of the amplitude of steady-state oscillations of the system on the whole by the presented method presents definite difficulties, not even mentioning that it is unrealizable at the design stage of the flight vehicle. In connection with this, approximate methods of calculation of lateral forces and bending moments are of great value.

By knowing the frequency and amplitude of steady-state oscillations of some parameter of the system, for instance angle δ, it is possible to reduce the solution of the above-indicated nonlinear problem to solution of the linear problem of forced transverse oscillations of the vehicle under action of control forces, variable according to harmonic law [71].

Let us first examine calculation quasi-static loads, affecting the construction during steady-state oscillations of guided flight vehicle (as a solid body) on the boost phase of flight. The influence of mobility of liquid in tanks of parameters of motion of the vehicle will be considered by the use of corresponding listed inertial characteristics (Chapter II). Equations of transverse oscillations of the flight vehicle in the pitching (yawing) plane in this case of loading will differ from equations (2.25) and (9.33) by magnitudes coefficients of the effect of k^0, k^1 and by right parts:

$$\left.\begin{aligned} \Delta\ddot{\vartheta} + b^*_{д}\Delta\dot{\vartheta} + b^*_{\alpha}\left(\Delta\vartheta - \frac{\dot{y}_т}{v}\right) + b^*_{\delta}\delta = 0, \\ \ddot{y}_т + c^*_{д}y_т + c^*_{\alpha}\Delta\vartheta + c^*_{\delta}\delta = 0, \end{aligned}\right\} \tag{13.12}$$

where

$$c^*_{\alpha} = \frac{c_{\alpha}}{mk^0}, \quad c^*_{\delta} = \frac{c_{\delta}}{mk^0}, \quad b^*_{\alpha} = \frac{b_{\alpha}(x_{1д} - x^*_{1т})}{k^1 J_z (x_{1д} - x_{1т})}, \quad b^*_{д} = \frac{b_{д}(m_z^{\omega})^*}{k^1 J_z m_z^{\omega}},$$

$$b^*_{\delta} = \frac{b_{\delta}}{k^1 J_z}\frac{(x_{1p} - x^*_{1т})}{(x_{1p} - x_{1т})}, \quad c^*_{д} = \frac{c_{д}}{mk^0 v}.$$

Particular solution of this system of equations when $\delta = \delta_m \sin \omega t$ will have the form

$$\Delta\vartheta(t) = \Delta\vartheta_m \sin(\omega t + \beta), \tag{13.13}$$

where

$$\overline{\Delta\vartheta}_m = \frac{\Delta\vartheta_m}{\delta_m} = \frac{b^*_{\delta}}{r_0^2}\left(\frac{\omega^2 + a_0^2}{\omega^2 + \mu_0^2}\right)^{\frac{1}{2}}\left[\left(1 - \frac{\omega^2}{r_0^2}\right)^2 + 4\alpha_1^2\frac{\omega^2}{r_0^4}\right]^{-\frac{1}{2}}, \tag{13.14}$$

$$r_0^2 = \omega_0^2 + \alpha_1^2, \quad a_0^2 = \frac{c^*_{\delta}b^*_{\alpha}}{b^*_{\delta}v} + c^*_{д},$$

and μ_0, $(-\alpha_1 - i\omega_0)$, $(-\alpha_1 + i\omega_0)$ - roots of characteristic equation

$$p^3 + (b^*_{д} + c^*_{д})p^2 + (b^*_{д}c^*_{д} + b^*_{\alpha})p + b^*_{\alpha}\left(c^*_{д} + \frac{c^*_{\alpha}}{v}\right) = 0,$$

$$\beta = \operatorname{arctg}\frac{\omega}{\omega_0} - \operatorname{arctg}\frac{\omega}{-\mu_0} - \operatorname{arctg}\frac{2\alpha_1\omega}{r_0^2 - \omega^2}. \tag{13.15}$$

Having placed the values of $\Delta\vartheta$ and its derivative in equation (12), we obtain the following expression for angle of attack

$$\Delta\alpha(t) = \Delta\alpha_m \sin(\omega t + \beta + \beta_1), \tag{13.16}$$

where

$$\Delta\bar{\alpha}_m = \frac{\Delta\alpha_m}{\delta_m} = \frac{1}{b_\alpha^\bullet}\left[(\omega^2\,\overline{\Delta\vartheta}_m - b_\delta^\bullet\cos\beta)^2 + (\omega b_\lambda^\bullet\,\overline{\Delta\vartheta}_m - b_\delta^\bullet\sin\beta)^2\right]^{\frac{1}{2}},$$

$$\beta_1 = \operatorname{arctg}\frac{b_\delta^\bullet\sin\beta - \omega b_\lambda^\bullet\,\overline{\Delta\vartheta}_m}{\omega^2\,\overline{\Delta\vartheta}_m - b_\delta^\bullet\cos\beta}. \tag{13.17}$$

In case of small transverse oscillations of the flight vehicle and liquid (with k^0 and k^1, close to one) the difference of $\Delta\alpha$ from $\Delta\vartheta$ is small and one may approximately assume that

$$\overline{\Delta\alpha}_m = \overline{\Delta\vartheta}_m = b_\delta^\bullet\left[(b_\alpha^\bullet - \omega^2)^2 + \omega^2 b_\lambda^{\bullet 2}\right]^{-\frac{1}{2}} = b_\delta^\bullet\eta_0,$$

$$\beta = \operatorname{arctg}\beta_\lambda^\bullet\frac{\omega}{\omega^2 - b_\alpha^\bullet}. \tag{13.18}$$

Values of amplitude of oscillations $\Delta\vartheta_m$ are basically determined by dynamic coefficient η_0, magnitude of which is a function of the ratio of frequencies ω/r_0 and degree of damping of oscillations.

While not here analyzing the relationship of parameters of transverse oscillations of the flight vehicle to the frequency of oscillations of the system, let us note only that with approach of k^0 to zero, and also with large values of k^0 and k^1 the magnitudes of amplitudes $\overline{\Delta\vartheta}_m$ and $\overline{\Delta\alpha}_m$ will rapidly decrease, approaching zero.

Coefficient of lateral overload of the center of gravity of the flight vehicle, according to (12), in this case will be equal to

$$\Delta n_{y_1}^0(t) = \frac{c_\delta\delta(t)}{k^0 G} + \frac{\Delta\alpha(t)}{k^0}\left(\frac{c_\alpha}{G} - n_x\right). \tag{13.19}$$

After substitution of the value of $\Delta\alpha$ in it we find that

$$\Delta n_{y_1}^0 = \Delta n_{y_1 m}^0\sin(\omega t + \varphi), \tag{13.20}$$

where

$$\frac{\Delta n^0_{y_1 m}}{\delta_m} = \frac{c_\delta}{Gk^0}\left[1 + \left(\frac{qSc^{\alpha}_{y_1}}{c_\delta}\frac{\Delta\alpha_m}{\delta_m}\right)^2 + 2\left(\frac{\Delta\alpha_m}{\delta_m}\right)\frac{qSc^{\alpha}_{y_1}}{c_\delta}\cos(\beta+\beta_1)\right]^{\frac{1}{2}},$$

$$\varphi = \operatorname{arctg}\frac{\frac{\Delta\alpha_m}{\delta_m}qSc^{\alpha}_{y_1}\sin(\beta+\beta_1)}{c_\delta + \frac{\Delta\alpha_m}{\delta_m}qSc^{\alpha}_{y_1}\cos(\beta+\beta_1)}.$$

Rotary component of lateral overload will be

$$\left.\begin{aligned} \Delta n^x_{y_1}(x_1, t) &= \Delta n^x_{y_1 m}(x_1)\sin(\omega t+\beta), \\ \Delta n^x_{y_1 m}(x_1) &= -\frac{\omega^2}{g_0}\Delta\vartheta_m(x_1 - x^*_{1\text{т}}), \\ \Delta n_{x_1}(x_1) &= \omega^2\Delta\vartheta^2_m\frac{1}{g_0}(x_1 - x^*_{1\text{т}})\sin^2(\omega t+\beta). \end{aligned}\right\} \quad (13.21)$$

By knowing $\Delta n^0_{y_1 m}$ and $\Delta n^x_{y_1 m}$, it is simple to obtain the expression for amplitude of oscillations of total lateral overload at any point of the body of the flight vehicle and the corresponding phase shift ψ_0 between Δn_{y_1} and δ:

$$\left.\begin{aligned} \Delta n_{y_1 m} &= \left[\Delta n^{02}_{y_1 m} + \Delta n^{x2}_{y_1 m} + 2\Delta n^0_{y_1 m}\Delta n^x_{y_1 m}\cos(\varphi-\beta)\right]^{\frac{1}{2}}, \\ \psi_0 &= \operatorname{arc\,tg}\frac{\Delta n^0_{y_1 m}\sin\varphi + \Delta n^x_{y_1 m}\sin\beta}{\Delta n^0_{y_1 m}\cos\varphi + \Delta n^x_{y_1 m}\cos\beta}. \end{aligned}\right\} \quad (13.22)$$

Total values of lateral force and bending moment (from inertial and aerodynamic forces) can be determined by formulas similar to (3.19)

$$\left.\begin{aligned} Q_{\text{c}}(x_1) &= Q^{\alpha}_{\text{a}}(x_1)\Delta\alpha + Q^0_{\text{к}}(x_1)\Delta n^0_{y_1} + Q^x_{\text{к}}(x_1)\frac{\Delta\vartheta}{g_0} + \sum_{j=1}^{n_1} Q_{\text{т}j}(x_1), \\ M_{\text{c}}(x_1) &= M^{\alpha}_{\text{a}}(x_1)\Delta\alpha + M^0_{\text{к}}(x_1)\Delta n^0_{y_1} + \\ &\quad + M^x_{\text{к}}(x_1)\frac{\Delta\vartheta}{g_0} + \sum_{j=1}^{n_1}[M_{\text{т}j}(x_j) + \Delta M_{\text{д}j}(x_{\text{д}j})]. \end{aligned}\right\} \quad (13.23)$$

It is necessary to pay attention to the fact that when $\omega > 2\pi/T_0$ (2.72) and $\omega < \omega_n$ for statically stable flight vehicle the aerodynamic moment, proportional to $\Delta\alpha$, will be arithmetically added to the disturbing moment from control forces. Thus, with steady-state oscillations the lateral overloads and bending moments for such type of flight vehicles can be (other things being equal) larger than for statically unstable. In this case the values of Δn_{y_1} and $M_c(x_1)$ will increase with growth of the reserve of static stability and c_y^{α}. In region $2\pi/T_0 > \omega > \omega_n$, where $k^0 < 0$, the aerodynamic moment is directed opposite the direction of action of perturbing moment.

If we express M_{Tj} by components proportional to $\Delta n_{y_1}^0$, $\Delta n_{y_1}^x$, and unite them with M_k^0 and M_k^x, then we can obtain the following expression for $M_c(x_1)$:

$$M_c(x_1,\ t) = M_{cm}(x_1)\sin(\omega t + \psi_1),$$

where

$$M_{cm} = (M_{am}^2 + M_m^{02} + M_m^{x2} + 2M_m^0 M_m^x \cos\beta_2 + \\ + 2M_m^0 M_{am}\cos\beta_3 + 2M_{am} M_m^x \cos\beta_1)^{\frac{1}{2}}, \qquad (13.24)$$

$$M_m^0 = M_{km}^0 + M_{Tm}^0, \quad M_m^x = M_{km}^x + M_{Tm}^x, \quad M_{km}^0 = M_k^0 \times \Delta n_{y_1 m}^0,$$

$$M_{km}^x = M_k^x \Delta\theta_m \omega^2 \frac{1}{g_0}, \quad M_{am} = M_a^\alpha \Delta\alpha_m.$$

If the frequency of oscillations of the system is close to the frequency of natural oscillations of liquid in the tank, then into the equation describing oscillations of liquid (10.9) one should introduce damping terms (viscous resistance), values of which are usually found experimentally (taking into account the particular construction of the tank, the presence of partitions, the influence of frames and so forth).

It is necessary to note that the source of oscillations of control forces can be in the drive itself. In particular, they can be caused by the action of friction forces in the drives of controls, discrete entry of pulses from the program mechanism of pitch angle. Forced transverse oscillations of the construction can take place in the case of use of self-adjusting control system, characteristics of adjustment of which are determined by the reaction of the system to an artificial external influence.

§ 13.4. Approximate Calculation of Dynamic Components of Lateral Forces and Bending Moments

Dynamic components of lateral forces and bending moments and in this case of loading are found by formulas (4.31) and (4.32), in which there are substituted expressions $\ddot{S}_n$, obtained from solution of equation (10.13) when

$$\begin{gathered} \delta=\delta_m \sin \omega_m t, \\ B_{вn}=0. \end{gathered} \tag{13.25}$$

With a very small difference of the frequency of oscillations of the system from one of the partial frequencies of its bending oscillations of the construction, the shape of its elastic line during forced oscillations will practically coincide with the partial form of natural oscillations. Therefore, in the first approximation one may assume that

$$\left.\begin{aligned} Q_д(x_1, t) &= -Q_{nx}(x_1)\ddot{S}_n(t), \\ M_д(x_1, t) &= -M_{nx}(x_1)\ddot{S}_n(t). \end{aligned}\right\} \tag{13.26}$$

The vibration component of the coefficient of lateral overload is determined by expression (when $x_1=x_д$)

$$\Delta n_{y,в}(x_д, t) = -\frac{1}{g_0}\ddot{S}(t) f_n(x_д). \tag{13.27}$$

Having excluded $\ddot{S}_n$ from formulas (26) and (27), we obtain

$$\left.\begin{aligned} Q_д(x_1) &= \Delta n_{y,в}(x_д) g_0 Q_{nx}(x_1)\frac{1}{f_n(x_д)}, \\ M_д(x_1) &= \Delta n_{y,в}(x_д) g_0 M_{nx}(x_1)\frac{1}{f_n(x_д)}. \end{aligned}\right\} \tag{13.28}$$

If the frequency of oscillations of the system differs from partial frequencies ω_n so that elastic oscillations of the flight vehicle construction cannot be described by one term of series (9.55) without large errors, then for calculation of $M_{Д}(x_1, t)$ it is possible to use the method of numerical integration. In the absence of large concentrated masses the application of this method permits avoiding preliminary finding of frequencies and forms of its bending oscillations of the construction of different tones.

Since perturbing force R_{y_1} is applied to the body usually at the end section, forced transverse oscillations of the flight vehicle can be described by homogeneous equation

$$\frac{\partial^2}{\partial x_1^2}\left[B(x_1)\frac{\partial^2 y_1(x_1, t)}{\partial x_1^2}\right]+m(x_1)\frac{\partial^2 y_1(x_1, t)}{\partial t^2}=0, \tag{13.29}$$

having presented boundary conditions in the form

$$\left.\begin{array}{l}\frac{\partial^2 y_1(x_1, t)}{\partial x_1^2}=0, \quad \frac{\partial}{\partial x_1}\left[B(x_1)\frac{\partial^2 y_1(x_1, t)}{\partial x_1^2}\right]=0 \text{ when } x_1=0, \\ \frac{\partial^2 y_1(x_1, t)}{\partial x_1^2}=0, \quad \frac{\partial}{\partial x_1}\left[B(x_1)\frac{\partial^2 y_1(x_1, t)}{\partial x_1^2}\right]=R_{y_1}^{\delta}\delta \\ \qquad\qquad\qquad\qquad\qquad \text{when } x_1=x_{1p}=l.\end{array}\right\} \tag{13.30}$$

Solution of this equation is sought in the form (with zero initial conditions)

$$y_1(x_1, t)=u(x_1)\sin\omega t, \tag{13.31}$$

where ω - frequency of oscillations of control force.

Having placed values of (31) and (25) in (30) and in equation (29), for $u(x_1)$ we obtain the following ordinary linear differential equation:

$$\frac{d^2}{dx_1^2}\left[B(x_1)\frac{d^2 u(x_1)}{dx_1^2}\right]=\omega^2 m(x_1)\,u(x_1) \tag{13.32}$$

with boundary conditions

$$\left.\begin{aligned}&\frac{d^2u(x_1)}{dx_1^2}=0,\quad \frac{d}{dx_1}\left[B(x_1)\frac{d^2u(x_1)}{dx_1^2}\right]=0 \quad \text{when} \quad x_1=0,\\&\frac{d^2u(x_1)}{dx_1^2}=0,\quad \frac{d^3u(x_1)}{dx_1^3}=\frac{R_{y_1}^{\delta}\delta_m}{B(x_{1p})} \quad \text{when}\ x_1=x_{1p}=l,\ B(x_{1p})\neq 0.\end{aligned}\right\} \tag{13.32'}$$

It is simple to note that equation (32) is similar to equation (4.67), and consequently, for its solution it is possible to apply the method of approximate numerical integration, discussed in § 5 of Chapter IV. Arbitrary constants U_1 and U_2 in this case can be determined from boundary conditions (32') when $x_1=x_{1p}=l$

$$\left.\begin{aligned}&U_1\left.\frac{d^2p(x_1)}{dx_1^2}\right|_{x_1=l}+U_2\left.\frac{d^2\xi(x_1)}{dx_1^2}\right|_{x_1=l}=0,\\&U_1\left.\frac{d^3p(x_1)}{dx_1^3}\right|_{x_1=l}+U_2\left.\frac{d^3\xi(x_1)}{dx_1^3}\right|_{x_1=l}=\frac{R_{y_1}^{\delta}\delta_m}{B(l)}.\end{aligned}\right\} \tag{13.33}$$

Since outside the zone of resonance the determinant of this system of equations will not be equal to zero, then, considering equation (4.82), we obtain

$$\left.\begin{aligned}&U_1=-\frac{R_{y_1}^{\delta}d(l)\delta_m}{e(l)\left.\frac{dd(x_1)}{dx_1}\right|_{x_1=l}-d(l)\left.\frac{de(x_1)}{dx_1}\right|_{x_1=l}},\\&U_2=\frac{R_{y_1}^{\delta}e(l)\delta_m}{e(l)\left.\frac{dd(x_1)}{dx_1}\right|_{x_1=l}-d(l)\left.\frac{de(x_1)}{dx_1}\right|_{x_1=l}}.\end{aligned}\right\} \tag{13.34}$$

In this case for calculation of derivatives $\frac{de(x_1)}{dx_1}$ and $\frac{dd(x_1)}{dx_1}$ it is possible to use approximate extrapolation formula

$$\frac{de_n}{dx_1}=\left(\Delta e_{n-1}-\frac{1}{6}\Delta^3 e_{n-3}\right)\frac{1}{\Delta h},$$

where Δh - integration step.

While preserving the designations shown in Chapter IV, we obtain the following expression for lateral force and bending moment:

$$Q_{\text{д}}(x_1, t) = \frac{d}{dx_1}\left[B(x_1)\frac{d^2u(x_1)}{dx_1^2}\right]\sin\omega t = \left[U_1\frac{de(x_1)}{dx_1} + U_2\frac{dd(x_1)}{dx_1}\right]\sin\omega t,$$
$$M_{\text{д}}(x_1, t) = B(x_1)\frac{d^2u(x_1)}{dx_1^2}\sin\omega t = [U_1e(x_1) + U_2d(x_1)]\sin\omega t. \quad (13.35)$$

Vibration component of lateral overload in this case will be equal to

$$\Delta n_{y,\text{в}}(x_1, t) = \frac{\omega^2}{g_0}u(x_1)\sin\omega t = \frac{\omega^2}{g_0}[U_1p(x_1) + U_2\xi(x_1)]\sin\omega t. \quad (13.35')$$

Displacement and angle of rotation of the tangent to elastic line in any section x_1 will be determined by formulas

$$y_1(x_1, t) = [U_1p(x_1) + U_2\xi(x_1)]\sin\omega t = \frac{g_0}{\omega^2}\Delta n_{y,\text{в}}(x_1, t),$$
$$\frac{\partial y_1(x_1, t)}{\partial x_1} = \left[U_1\frac{dp(x_1)}{dx_1} + U_2\frac{d\xi(x_1)}{dx_1}\right]\sin\omega t. \quad (13.36)$$

By using expression $M_{\text{см}}(x_1)$ (24), we can find the value of dynamic coefficient in any section x_1

$$\eta(x_1) = 1 + \frac{U_1e(x_1) + U_2d(x_1)}{M_{\text{см}}(x_1)}. \quad (13.37)$$

Magnitude of this coefficient will depend on relation

$$\frac{e(l)\left.\frac{dd(x_1)}{dx_1}\right|_{x_1=l}}{d(l)\left.\frac{de(x_1)}{dx_1}\right|_{x_1=l}},$$

i.e., on the relationship of frequencies of forced and natural oscillations of the flight vehicle construction.

By using the given method of calculation of forced elastic oscillations of the vehicle, it is possible to establish certain recommendations on the selection of rigidity of separate sections of construction of the body, and also to estimate the effect of elastic oscillations on motion parameters of the flight vehicle. For instance, calculations show that change of flexual rigidity of the body of the tail section does not practically affect magnitude of $M_{Д}(x_1, t)$ in rather wide limits. The value of the dynamic component of bending moment is determined mainly by rigidity of the middle part of the flight vehicle body. On the basis of this it is possible in the first approximation to take $B(x_1) = B = \text{const}$, having taken rigidity of the middle part of the body as B. Having assumed $m(x_1) = m/l = \text{const}$, for $M_{Д}(x_1)$ we obtain the following expression:

$$M_{am}(x_1) = \frac{R_{y_1}^{\delta}\delta_m}{2\alpha_0 \operatorname{ch}\alpha_0 l \cos\alpha_0 l}[(\cos\alpha_0 l - \operatorname{ch}\alpha_0 l)\operatorname{sh}\alpha_0 x_1 + (\operatorname{sh}\alpha_0 l - \sin\alpha_0 l)\operatorname{ch}\alpha_0 x_1 + (\operatorname{ch}\alpha_0 l - \cos\alpha_0 l)\sin\alpha_0 x_1 + (\sin\alpha_0 l - \operatorname{sh}\alpha_0 l)\cos\alpha_0 x_1], \tag{13.38}$$

where

$$\alpha_0 = \left(\frac{m\omega^2}{lB}\right)^{\frac{1}{4}}.$$

In many cases this approximate value of bending moment differs comparatively little from (35), especially at the beginning of flight, when there is much liquid in fuel tanks of the flight vehicle. The degree of their coordination essentially depends, of course, on the selection of magnitude B.

§ 13.5. Torsional Bending Oscillations of the Construction

Natural oscillations of the system vehicle-automatic stabilization control can occur in the rolling plane at frequencies close to the frequencies of its torsional or torsional bending oscillations of the construction. Forced oscillations of the flight vehicle are possible relative to the longitudinal axis with frequencies corresponding to the frequencies of natural oscillations of the system in yawing pitching planes, caused by variation of values of lateral forces of oppositely located control devices.

Besides the control system itself, a source of oscillations of the flight vehicle in the rolling plane can be aerodynamic moments caused by a symmetry of the shape of its body relative to the longitudinal axis, and also moments caused by rotation of the liquid in the fuel tanks. Parameters of these oscillations are calculated exactly as transverse oscillations. Naturally, there are only used corresponding system of equation of torsional or torsional bending oscillations of the construction.

Since similar oscillations can have essential value only for systems possessing low frequencies of its torsional bending oscillations, and primarily for flight vehicles of pack configuration, we will not pause here in detail on this question. We are limited only by reduction of corresponding equations of torsional and torsional bending oscillations of the construction.

Equations of torsional oscillations of the flight vehicle body are analogous in form to equations of longitudinal elastic oscillations of the construction

$$\ddot{\mu}_n + 2h_n\dot{\mu}_n + \omega^2_{n\mu}\mu_n = \frac{R^{\phi}_{\mu 1}}{J_{n\mu}}\delta_{\phi}X_{n\mu}(x_{1p}). \qquad (13.39)$$

Here $X_{n\mu}(x_1)$ - function characterizing the form of torsional oscillation of the body of n-th tone, $\mu_n(t)$ - function determining the portion of this form of oscillations in values of angles of rotation of cross sections of the body during torsional oscillations, $\omega_{n\mu}$ - frequency of its torsional oscillations of n-th tone, and $J_{n\mu}$ - corresponding reduced mass moment of inertia of the flight vehicle, equal to

$$J_{n\mu} = \int_0^l i_p(x_1)X^2_{n\mu}(x_1)\,dx_1,$$

where $i_p(x_1)$ - mass polar moment of inertia of the cross section of the body. With the presence of longitudinal partitions in the tank, the polar mass moment of inertia of the section will be equal to the sum of moments of inertia of the construction of the tank and mass of apparent liquid.

Equations for calculation of functions $X_{n\mu}(x_1)$ and natural frequencies $\omega_{n\mu}$ have the form

$$\frac{d}{dx_1}\left[G(x_1)I_{pc}(x_1)\frac{dX_{n\mu}(x_1)}{dx_1}\right] = -\omega^2_{n\mu}i_p(x_1)X_{n\mu}(x_1) - \\ -\sum_{\nu=1}^{l}\omega^2_{n\mu}i_{p\nu}X_{n\mu}(x_\nu)\delta(x_1 - x_\nu) \quad (n = 1, 2, \ldots), \tag{13.40}$$

where index ν pertains to concentrated loads, I_{pc} - polar moment of inertia of cross-sectional area, G - shear modulus. Their solutions are sought with boundary conditions

$$\frac{dX_{n\mu}(x_1)}{dx_1} = 0 \text{ when } x_1 = 0, \quad x_1 = l.$$

These equations can be used for calculation of partial forms and frequencies of natural torsional oscillations of the central unit of a flight vehicle with side-mounting boosters. For this to the right side of equation (40) one should additionally introduce terms considering apparent masses of side-mounting boosters in the form of concentrated loads (10.18),

$$\left.\begin{aligned} i_{p1}(x_{в}) &= m(x_{в})r^2(x_{в}) + \Delta m r(x_{н}) r(x_{в})\frac{X_{n\mu}(x_{н})}{X_{n\mu}(x_{в})}, \\ i_{p2}(x_{н}) &= m(x_{н})r^2(x_{н}) + \Delta m r(x_{н}) r(x_{в})\frac{X_{n\mu}(x_{в})}{X_{n\mu}(x_{н})}. \end{aligned}\right\} \tag{13.41}$$

In this case

$$J_{n\mu} = \int_0^l i_p(x_1)X^2_{n\mu}(x_1)dx_1 + i_{p\nu}(x_{в})X^2_{n\mu}(x_{в}) + i_{p\nu}(x_{н})X^2_{n\mu}(x_{н})$$

and

$$\omega^2_{n\mu} = \frac{1}{I_{n\mu}} \int_0^l G(x_1) I_{pc}(x_1) \left[\frac{dX_{n\mu}(x_1)}{dx_1} \right]^2 dx_1 \tag{13.42}$$

or

$$\omega^2_{n\mu} = \frac{I_{n\mu}}{\int\limits_0^l \frac{M^2_{n\mu x}(x_1)}{G(x_1) I_{pc}(x_1)}},$$

where

$$M_{n\mu x}(x_1) = \int_0^{x_1} i_p(x_1) X_{n\mu}(x_1) dx_1 +$$
$$+ i_{pv}(x_a) X_{n\mu}(x_a) \int_0^{x_1} \delta(x_1 - x_a) dx_1 + i_{pv}(x_n) X_{n\mu}(x_n) \int_0^{x_1} \delta(x_1 - x_n) dx_1.$$

The integral in formula (42) is obtained by integration of the following expression by parts

$$\int_0^l \frac{d}{dx_1} \left[G(x_1) I_{pc}(x_1) \frac{dX_{n\mu}(x_1)}{dx_1} \right] X_{n\mu}(x_1) dx_1 =$$
$$= G(x_1) I_{pc}(x_1) \frac{dX_{n\mu}(x_1)}{dx_1} X_{n\mu}(x_1) \Big|_0^l - \int_0^l G(x_1) I_{pc}(x_1) \left[\frac{dX_{n\mu}(x_1)}{dx_1} \right]^2 dx_1$$

with boundary conditions

$$\frac{dX_{n\mu}(x_1)}{dx_1} = 0 \text{ when } x_1 = 0, \; x_1 = l.$$

Equations (39) together with equation $J_x \ddot{\phi} = M^{\delta}_{x\delta} \delta_{\phi}$ will describe small oscillations of the flight vehicle construction in the rolling plane.

The connection between angle of rotation of the cross section of the flight vehicle body and automatic stabilization control is determined by equation of the transmitter of sensing device of the control system (10.34')

$$u_\varphi = k_\varphi\left[\varphi + \sum_{n=1}^{n\mu} \mu_n X_{n\mu}(x_0)\right], \tag{13.43}$$

where k_ϕ - corresponding amplification factor, x_0 - abscissa of point of attachment of transmitter to the body of the flight vehicle.

Analogous equations in generalized coordinates for a flight vehicle of pack configuration, having k_i side-mounting boosters, will consist of equations of form (10.33) and (10.34), namely:

equation of moments relative to longitudinal axis of the flight vehicle

$$J_x\ddot{\varphi} + \sum_{i=1}^{k_i}\sum_{p=1}^{n_p} a_{pi}\ddot{N}_{pi} + \sum_{i=1}^{k_i} a_{ni}\ddot{y}_{ni} = M_{x\delta}^{\delta}\delta_\varphi + \sum_{i=1}^{k_i} M_{x\delta i}^{\delta}\delta_{\varphi i}, \tag{13.44}$$

equation of bending oscillations of i-th side-mounting boosters

$$M_{pi}(\ddot{N}_{pi} + 2h_{pi}\dot{N}_{pi} + \omega_{pi}^2 N_{pi}) + \frac{a_{pi}}{a}\ddot{y}_{ni} + a_{pi}\ddot{\varphi} + \\ + \sum_{n=1}^{n_\mu} \tau_{n\mu p}\ddot{\mu}_n = \tau_{p\delta i}\delta_{\varphi i} + B_{xpi}, \tag{13.45}$$

equation of elastic oscillations of power connections of side-mounting boosters with the body of central unit

$$m_i(x_n)\ddot{y}_{ni} + E_{ni}y_{ni} + a_{ni}\ddot{\varphi} + \\ + \sum_{p=1}^{n_p}\ddot{N}_{pi}\frac{a_{pi}}{a} + \sum_{n=1}^{n_\mu} t_{\mu ni}\ddot{\mu}_n = R_{y,i}^{\delta}\frac{x_{1p}}{a}\delta_{\varphi i} + B_{ni}, \tag{13.46}$$

equation of torsional oscillations of the central unit

$$J_{n\mu}(\ddot{\mu}_n + 2h_{n\mu}\dot{\mu}_n + \omega_{n\mu}^2\mu_n) + \sum_{i=1}^{k_i}\sum_{p=1}^{n_p}\tau_{n\mu p}\ddot{N}_{pi} + \sum_{i=1}^{k_i} t_{\mu ni}\ddot{y}_{ni} = \\ = M_{x\delta}^{\delta}X_{n\mu}(x_{1p})\delta_\varphi + \sum_{i=1}^{k_i} R_{y,i}^{\delta}\left[X_{n\mu}(x_n)\left(1 - \frac{x_{1p}}{a}\right)r(x_n) + \right. \\ \left. + r(x_n)X_{n\mu}(x_n)\right]\delta_{\varphi i}. \tag{13.47}$$

It is expedient to reduce the shown equations to normal coordinates, i.e., to apply to them the method, presented in Chapter V of linear transformation of amplitudes of oscillations of generalized coordinates q_i to amplitudes of normal oscillations by formula

$$q_i=\sum_{m=1}^{n_0} A_{im} p_m.$$

As a result we obtain the equation of oscillations of the flight vehicle as a solid body relative to the longitudinal axis in the form

$$J_x\ddot{\varphi}=M_{x\delta}^{\delta}\delta_{\varphi}+\sum_{l=1}^{k_l} M_{x\delta l}^{\delta}\delta_{\varphi l} \tag{13.48}$$

and system of equations of torsional bending oscillations of the construction

$$\ddot{p}_m+2h_m\dot{p}_m+\omega_m^2 p_m=B_{\delta m}\delta_{\varphi}+\sum_{l=1}^{k_l} B_{\delta lm}\delta_{\varphi l}, \tag{13.49}$$

where

$$B_{\delta m}=\frac{\omega_m^2}{D_m} M_{x\delta}^{\delta}\left[\varphi_m+\sum_{n=1}^{n_\mu}\mu_{nm}X_{n\mu}(x_{1p})\right],$$

$$B_{\delta lm}=\frac{\omega_m^2}{D_m}\left\{M_{x\delta l}^{\delta}\varphi_m+\sum_{p=1}^{n_p}\tau_{p\delta l}N_{plm}+R_{y_1 l}^{\delta}\frac{x_{1p}}{a}y_{\text{н}ml}+\right.$$
$$\left.+\mu_{nm}\left[r(x_{\text{н}})X_{n\mu}(x_{\text{н}})+X_{n\mu}(x_{\text{в}})\left(1-\frac{x_{1p}}{a}\right)r(x_{\text{в}})\right]R_{y_1 l}^{\delta}\right\},$$

$$\varphi_m=-\frac{1}{J_x}\sum_{l=1}^{k_l}\left(\sum_{p=1}^{n_p}a_{pl}N_{plm}+a_{\text{н}l}y_{\text{н}lm}\right),$$

$$D_m=\sum_{j=1}^{n_0}\omega_j^2A_{jm}^2M_j \qquad (m=1,\ 2,\ \ldots,\ n_0).$$

Equation (43) in this case will have the form

$$u_{\varphi}=k_{\varphi}\left\{\sum_{m=1}^{n_0} p_m\left[\varphi_m+\sum_{n=1}^{n_\mu}\mu_{nm}X_{n\mu}(x_0)\right]+\varphi\right\}. \tag{13.50}$$

Dynamic components of the bending moment in sections of the body of side-mounting boosters, located at distance r(x) from the axis of central unit, will be equal to

$$\begin{aligned}-M_{xl}(x_1)=&\left[M^0(x_1)\,r(x_{\text{в}})+M^x(x_1)\frac{r(x_{\text{н}})-r(x_{\text{в}})}{a}\right]\ddot{\varphi}+\\&+\sum_{m=1}^{n_0}\ddot{p}_m\left\{M^x(x_1)\frac{y_{\text{н}lm}}{a}+\sum_{p=1}^{n_p}M_{px}(x_1)N_{plm}+\sum_{n=1}^{n_\mu}\left[r(x_{\text{в}})X_{n\mu}(x_{\text{в}})M^0(x_1)+\right.\right.\\&\left.\left.+M^x(x_1)\frac{r(x_{\text{н}})}{a}X_{n\mu}(x_{\text{н}})-\frac{r(x_{\text{в}})}{a}X_{n\mu}(x_{\text{в}})M^x(x_1)\right]\mu_{nm}\right\}.\end{aligned} \tag{13.51}$$

Lateral overload

$$\begin{aligned}n_{y_1}(x_1)=&-\frac{1}{g_0}\sum_{m=1}^{n_0}\ddot{p}_m\left\{\frac{x_1}{a}y_{\text{н}lm}+\sum_{p=1}^{n_p}N_{plm}\Phi_{pl}(x_1)+\right.\\&\left.+\sum_{n=1}^{n_\mu}\mu_{nm}\left[r(x_{\text{в}})X_{n\mu}(x_{\text{в}})\left(1-\frac{x_1}{a}\right)+r(x_{\text{н}})X_{n\mu}(x_{\text{н}})\frac{x_1}{a}\right]\right\}-\ddot{\varphi}\frac{r(x_1)}{g_0}.\end{aligned} \tag{13.52}$$

In conclusion let us note that torsional oscillations of the flight vehicle body can affect the operating conditions of different gyroscopic systems. Let us illustrate this by the following example. Sometimes the axis of rotation of the turbine for fuel feed pumps to the combustion chamber of the engine is placed perpendicular to the longitudinal axis of the flight vehicle. With torsional oscillations of the body of the vehicle the turbine shaft will turn relative to this longitudinal axis with angular velocity ω_T. As a result its supports will be affected (in the direction of axis x_1) by additional loads, characterized by moment

$$M_{\text{т}}=J_{\text{т}}\omega_{\text{т}}\omega,$$

where J_T - mass moment of inertia of all revolving parts of the turbopump unit with respect to the axis of rotation, ω_T - angular velocity of rotation of the turbine. If we conditionally consider that the distance between supports (groups of supports) is equal to a_T, then the total load on support A or B (Fig. 13.4) will be equal to

$$N_{A,B} = G_T(0.5n_{x_1} \mp \bar{\omega} n_T i_T^2).$$

Here G_T - weight of revolving parts of the turbopump unit, i_T - relative radius of inertia

$$i_T = \frac{1}{a_T}\sqrt{\frac{g_0 J_T}{G_T}}, \quad \bar{\omega} = \frac{\omega}{\omega_T}, \quad n_T = \frac{1}{g_0} a_T \omega_T^2.$$

As can be seen, the magnitude of the shown load will be proportional to the frequency of torsional oscillations of the vehicle construction, and consequently, in certain cases can considerably exceed the calculated static loads.

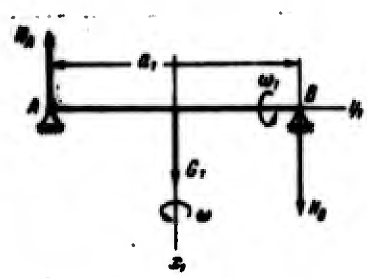


Fig. 13.4. Diagram of forces affecting the elements of attachment of a turbine with rotation of the vehicle relative to longitudinal axis.

CHAPTER XIV

DYNAMICS OF DESCENT VEHICLES

§ 14.1. Cases of Loading of the Construction of Descent Vehicles

In this chapter are considered certain questions of the dynamics of descent vehicles (aircraft, descent capsules of spaceships, recoverable parts of various flight vehicles, separable nose cones and so forth), in particular questions of loading of their construction in the process of free flight in dense layers of the atmosphere (cases C and D), when landing on the ground (case E) and on water (case E').

Characteristic for the shown cases of loading is the essential relationship of values of external forces, affecting the construction of descent vehicles, to initial values of motion parameters, and namely to speed of the center of gravity v_0 and angular parameters of motion with respect to the center of gravity (such as $\theta_0, \vartheta_0, \dot{\theta}_0, \psi_0, \dot{\psi}_0$).

Initial magnitudes of ϑ_0 and θ_0 at the moment of separation of the descent part from the flight vehicle is dense layers of the atmosphere or at the moment of passage of a certain fixed altitude h_0 (conditionally taken as the boundary of the dense layer of atmosphere) are determined by the trajectory of preceding motion of the vehicle and in many respects depend on its purpose. For existing types of flight vehicles the values of velocities v_0 are measured both in meters per second and kilometers per second. Initial angular disturbances are usually determined by motion parameters of the vehicle at the moment of beginning of free flight (for instance, at the moment of separation from the carrier). However, during prolonged flight of the separated descent part outside dense layers of the atmosphere (and especially in the presence of large variations of the duration of this flight) the relationship of angles of slope of its axes at the moment of entry into dense layers of the atmosphere to initial angular disturbances is not perceived for all practical purposes. This is explained by the fact that at high altitudes in the absence of aerodynamic damping forces and moments the descent vehicle (with any nonuniform initial angular disturbances) can revolve about the center of gravity. As a result at specified altitude h_0 the longitudinal axis of the body of this vehicle can occupy any position with respect to the velocity vector. In other words, angles of attack of such unguided flight vehicles during entry into dense layers of the atmosphere can be from 0 to π with equal probability.

Since at low angular velocities the current values of angles of attack of the descent vehicle (during motion in dense layers of the atmosphere) are the function of values of these angles at altitude h_0, then it is simple to establish that from the point of view of strength of their construction the most dangerous will be those limiting cases of loading for which

$$\alpha(h)=0, \quad \dot{\alpha}(h)=0 \qquad (14.1)$$

or

$$\alpha(h)=\alpha_{0\max}, \quad \dot{\alpha}(h)=0 \qquad (14.2)$$

when $h = h_0$.

The first limiting case (case C) corresponds to undisturbed motion of the descent vehicle with small (zero) angles of attack and limiting possible impact pressures on the entire atmosphere section of the trajectory. For all practical purposes, if damping of angular oscillations of the vehicle in the zone of comparatively small impact pressures occurs, this case can encompass a considerable range of initial values of nonzero angles of attack.

The second limiting case (case D) corresponds to perturbed motion of the unguided descent vehicle with limiting possible angles of attack on the entire atmospheric section of the trajectory. As it was shown, in a particular case the values of $\alpha_{0\ max}$ can be equal to π. When $\dot{\alpha}(h_0) = 0$ a similar combination of initial conditions will correspond to undisturbed motion of the vehicle with tail forward. However, for all practical purposes the probability of realization of such a case of flight is minute. Motion of the descent vehicle in such a position is, as a rule, unstable. Any external disturbance or inaccuracy of observance of the shown conditions will move it from this position. Therefore, for determinacy as limiting value it is recommended to take $\alpha_{0\ max} = 178\text{-}179°$ (when $\dot{\alpha}(h_0) = 0$).

Initial values of all parameters for guided descent vehicles are determined by the flight program, and in emergency cases of loading - by motion parameters of the flight vehicle at the moment of separation of the recoverable part.

The character of loading of the construction of the descent vehicle during landing depends on initial values of parameters of its motion at the moment of contact with the surface. Although real magnitudes of overloads in cases E and E', because of the presence of deviations of actual mechanical properties of the surface of local landing strips from calculated, are random, their maximums always will be limited from above either by the efficiency of the equipment, that should function after landing, or the endurance of the human body to short-duration overloads. These limitations are attained, on the one hand, by limitation of the speed of the vehicle at the moment of landing, and on the other hand, by introduction of local or overall shock absorption, absorbing the main part of the kinetic energy of the vehicle. An exception is only when before the descent vehicle there are placed no problems besides hitting a planet.

Limitation of the landing speed requires the application of special brake facilities, constructed on the principle of increase of drag of the descent vehicle (by means of introduction of a parachute system, aerody[illegible]c dive brakes and so forth) or the creation of active brake force with jet engines.

Values of external loads affecting the construction of the descent vehicle in all cases of deceleration (cases F_j) are determined by the program of switching on these brake facilities and are characterized usually by the magnitude of longitudinal (and for gliding vehicles - lateral) overload. In this case the place and character of application of external (braking) load to the body of the vehicle is determined in each particular case depending upon the accepted scheme of deceleration and its constructive design. Thus, for instance, with the use of parachute systems (with impact pressures lying within 100-1000 kg/m^2) braking force $X_{\text{п}}$ is applied to the vehicle construction in the form of concentrated forces at points of attachment of shroud lines. The character of change of its magnitude in time depends on duration of filling of the canopy, which in turn depends on the area of the latter. If this area is great, then build up of drag occurs comparatively slowly and max $X_{\text{п}}$ is attained in the process of braking. If the indicated area is small, then the time of filling of parachute will also be small and for all practical purposes $X_{\text{п}}$ will be observed at the moment of introduction of the parachute. The amount of parachute drag is determined by a formula of the form

$$X_{\text{п}} = qc_{\text{п}}F_{\text{п}},$$

where $F_{\text{п}}$ - characteristic area of the parachute, and $c_{\text{п}}$ - coefficient of aerodynamic drag of the canopy (on the order of 0.5-0.65). In this case usually the magnitudes of $F_{\text{п}}$ are selected by proceeding from required finite uniform velocity of motion of the descent vehicle, namely from equality $X_{\text{п}} = 0$.

§ 14.2. Loading of the Vehicle on the Descent Phase

Generally the motion of the flight vehicle during descent is three-dimensional and is described by a system of nonlinear differential equations with variable coefficients with random initial conditions. For unguided descent vehicles this system of equations consists of equations of form (2.6) and relationships

$$\dot{x}_0 = v\cos\theta, \quad -\dot{y}_0 = v\sin\theta.$$

For guided descent vehicles to the right sides of the shown equations there are additionally introduced corresponding control forces and moments. As was noted in the preceding paragraph, an important feature of loading of the construction of many flight vehicles on the descent phase is the large relationship of their motion parameters to initial conditions at the moment of entry into dense layers of the atmosphere. Possible realizations of these initial conditions in many respects are determined by the assignment of the descent vehicle. They will be different for separable parts of ballistic type flight vehicles, for piloted vehicles, aircraft, for recoverable parts separable during an accident and so forth. Initial conditions, in particular initial angles of attack, especially greatly affect the character of loading of the construction of unguided descent vehicles, possessing small aerodynamic damping. The duration of damping of initial angular perturbations for similar vehicles in certain cases can be comparable with the duration of their flight in dense layers of the atmosphere. Under such conditions, naturally, the effect of the influence of other perturbations, including wind, is reduced and consequently, the required bearing capacity of the construction of the shown vehicles in essence will be determined by limiting values of initial conditions (1) and (2).

Taking into account the random character of initial conditions, and also the magnitudes of all aerodynamic and inertial coefficients, it is possible to use the statistical methods of solution of the given nonlinear problem (for instance, Monte Carlo method). However, for all practical purposes, considering the large labor input of calculation of distribution functions of internal forces in the region of too small probabilities, it is more expedient to use deterministic methods, transfering calculation of the statistical nature of the given problem to the region of normalization of safety factors. With orientation to limiting values of initial conditions, the necessity of using an exact system of equations of motion of the descent vehicle drops. In other words, in this case it is possible to essentially simplify formulation of the problem itself, particularly to be limited by consideration of only two-dimensional perturbed motion of the vehicle, described by a system of nonlinear equations of form

$$\Delta\ddot{\vartheta} + b_{\text{д}}\Delta\dot{\vartheta} + b_{\alpha}(\alpha) = 0, \tag{14.3}$$

$$v\Delta\dot{\theta} + g\cos\theta + c_{\alpha}(\alpha) = 0, \tag{14.4}$$

$$m\dot{v} - G\sin\theta + X(\alpha) = 0, \tag{14.5}$$

where

$$b_{\text{д}} = qSl^2 m_z^{\omega}(\alpha, \mathrm{M}_\infty)\frac{1}{vI_z},$$

$$b_{\alpha}(\alpha) = qSm_z(\alpha, \mathrm{M}_\infty)\frac{1}{I_z},$$

$$c_{\alpha}(\alpha) = -qSc_y(\alpha, \mathrm{M}_\infty)\frac{1}{m},$$

$$h = h_0 - \int_0^t v\sin\theta\, dt.$$

In this case the effect of various types of wind influences is calculated by corresponding change of the angle of attack of the flight vehicle

$$\alpha = \Delta\vartheta - \Delta\theta + \frac{u}{v}.$$

In a particular case with comparatively small changes of angles of pitch $\Delta\vartheta$, when equations (3) and (4) can be considered linear, calculation of the reaction of the construction of the descent vehicle to the influence of wind loads and calculation of aerodynamic loads, caused by the presence of initial disturbances of angles of attack (2), can be executed independently from each other.

Solution of the above-mentioned system of nonlinear differential equations with initial conditions (2) is usually found by the method of numerical integration. There exist a number of approximate methods of calculation of envelope of angles of attack (when the altitude or time of flight) and instantaneous values of the frequencies of oscillations of such vehicles. All of them are based on the assumption of smallness of change of the angle of inclination of the tangent to the trajectory, which permits considering not two equations (3) and (4), but one equation of form

$$\ddot{\alpha} + b_z\dot{\alpha} + b_\alpha(\alpha) = 0. \qquad (14.6)$$

Solution of the latter can be obtained by the method of successive approximations. Having taken the trajectory of undisturbed motion of the descent vehicle, corresponding to zero initial values of α and $\dot{\alpha}$, as initial, by approximation formula (for instance, from [85]) we find the envelope of angles of attack of the first approximation

$$\alpha(t) = \frac{\alpha_0(t_0)}{\sqrt[4]{\frac{\partial b_\alpha(\alpha)}{\partial\alpha}}} e^{-\frac{1}{2}\int_{t_0}^{t} b_z dt} \sin\left(\int_{t_0}^{t} \sqrt{\frac{\partial b_\alpha(\alpha)}{\partial\alpha}}\, dt\right). \qquad (14.7)$$

Then, by using this enveloping, let us calculate the mean values of drag coefficients of the descent vehicle of the first approximation (with perturbed motion) in the function of number M_∞

$$c_{x_{cp}}(\alpha, M_\infty) = \frac{1}{2}\left[c_x(\alpha_{max}, M_\infty) + c_x(\alpha = 0, M_\infty)\right].$$

Having substituted them in equation (5), let us more accurately calculate the trajectory of undisturbed motion of the center of gravity of the vehicle on the atmospheric section of flight. The values of trajectory parameters (v, h, q and M_{∞}) obtained in such a way are taken as initial for calculation of envelope of angles of attack and mean values of drag coefficients of the vehicle of the second approximation and so forth. In most cases it is possible to be limited by three-four such approximations. By knowing the parameters of perturbed motion of the descent vehicle, it is simple to find the values of corresponding components of acceleration factors

$$\left.\begin{aligned} n_{y_1}(x_1) &= n_{y_1}^0 + n_{y_1}^x(x_1), \\ n_{x_1}(x_1) &= n_{x_1}^0 + n_{x_1}^x(x_1). \end{aligned}\right\} \qquad (14.8)$$

Here

$$n_{y_1}^0 = -qSc_{y_1}(\alpha)\frac{1}{G},$$
$$n_{y_1}^x = -qSc_{y_1}(\alpha)\frac{1}{g_0 J_z}(x_{1\text{д}} - x_{1\text{т}})(x_1 - x_{1\text{т}}),$$
$$n_{x_1}^0 = -qSc_{x_1}(\alpha)\frac{1}{G},$$
$$n_{x_1}^x = \frac{\dot{\alpha}^2}{g_0}(x_1 - x_{1\text{т}}).$$

Typical graphs of the change of envelopes of values of α, $n_{y_1}^0$, n_{x_1} and $n_{x_1}^0$ are presented in Figs. 14.1 and 14.2. As can be seen, maxima of overloads $n_{x_1}^0$ and $n_{y_1}^0$ are observed at different altitudes, while maximum $n_{y_1}^0$ is usually attained outside the zone of action of jet streams. In similar cases the angles of attack of the vehicle at altitude less than 10-12 km are obtained comparatively small and the corresponding system of equations of motion can be linearized with sufficient accuracy, and consequently, additional lateral overloads from the influence of the wind can be calculated by approximate formula of the form (see Chapter IX)

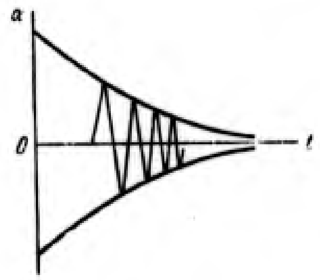
$$\Delta n_{y_1}(x_1) = \Delta n_{y_1}^0 + \Delta n_{y_1}^x(x_1), \qquad (14.9)$$

where

$$\Delta n_{y_1}^0 = \frac{uv}{g_0 \lambda_1 l s_m}\left(1 - e^{-\frac{s_m}{\lambda_1}}\right),$$
$$\Delta n_{y_1}^x(x_1) = -\frac{uv\rho S}{2 g_0 J_z} c_{y_1}(\alpha)(x_{1\text{д}} - x_{1\text{т}})(x_1 - x_{1\text{т}}) k_s.$$

Total overload in any section x_1 of the body in this case will be equal to

$$n(x_1) = \sqrt{[n_{y_1}(x_1) + \Delta n_{y_1}(x_1)]^2 + n_{x_1}^2(x_1)}.$$



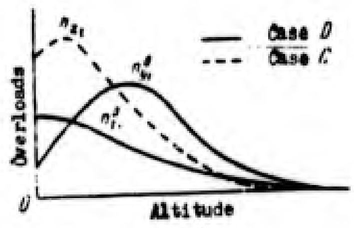


Fig. 14.1. Fig. 14.2.

Fig. 14.1. Envelope of maximum values of angles of attack of descent vehicle in case D.

Fig. 14.2. Typical change of maximum values of overload components according to the altitude of flight in the center of gravity of the descent vehicle in cases C and D.

For ballistic type descent vehicles the values of n_{x_1} often considerably exceed the longitudinal overloads which are observed on the boost phase of flight. In this case its value (in case D) can essentially depend on coordinate x_1 of the considered point, which should be considered at least during selection of the place of location of instruments, sensitive to overload. For all practical purposes it is possible to noticeably decrease the limiting values of overloads, in particular, lateral $n_{y_1}(x_1)$, by means of application of active stabilization of motion of the vehicle on the descent phase.

Characteristic for the considered case of loading (case D) is asymmetric distribution of external aerodynamic pressure along the contour of the cross section of the body of the descent vehicle.

The magnitude of this pressure at numbers $M_\infty > 5$ can be calculated with sufficient accuracy by approximate Newtonian theory. For conical shape of the body

$$p = 2q\beta(\sin\theta_s\cos\alpha - \cos\theta_s\sin\alpha\sin\varphi)^2,$$

where θ_s - half-angle of cone, β - correction factor, introduced for coordination of results of calculations by this formula with Kopall tables (when $\alpha = 0$).

Values of corresponding static components of longitudinal and lateral forces and bending moments are determined by expressions

$$\left.\begin{aligned} N_c(x_1) &= X_1(x_1) + \int_0^{x_1} q_{Gx}(x_1)n_{x_1}(x_1)dx_1, \\ Q_c(x_1) &= qS\int_0^{x_1}\frac{\partial c_{y_1}(\alpha, x_1)}{\partial x_1}dx_1 + \int_0^{x_1} q_{Gy}(x_1)n_{y_1}(x_1)dx_1, \\ M_c(x_1) &= \int_0^{x_1} Q_c(x_1)dx_1. \end{aligned}\right\} \quad (14.10)$$

Dynamic components of these forces and moments $Q_д(x_1)$ and $M_д(x_1)$ are calculated only when the instantaneous frequency of oscillations of the descent vehicle as a solid body in the flow of air, approximately equal to

$$\omega = 0.71v\left(\rho m_z^\alpha S\frac{l}{I_z}\right)^{\frac{1}{2}},$$

turns out to be close to the frequency of natural bending oscillations of its construction ω_n. In a similar case the equation for calculation of acceleration of the point reduction $\ddot{S}_n$ will have the form

$$\ddot{S}_n + 2h_n\dot{S}_n + \omega_n^2 S_n = B_{n\alpha}\cdot\alpha, \quad (14.11)$$

where

$$B_{nn}=\frac{qS}{M_n}\int_0^l \frac{\partial c_{y_1}^{\alpha}(x_1)}{\partial x_1} f_n(x_1)\,dx_1.$$

The magnitude of ω is determined basically by the amount of impact pressure and margin of aerodynamic stability of the descent vehicle. The greater the impact pressure and the distance between the center of pressure and center of gravity, the greater is the instantaneous frequency of natural oscillations of the descent vehicle in the flow of air, and consequently the frequency of change of external lateral aerodynamic load affecting its body. Since limits of change of margins of aerodynamic (static) stability are small, then with comparatively high values of frequencies of natural elastic transverse oscillations of the construction of the descent vehicle the proximity of frequencies ω and ω_n can be practically observed only in the region of great impact pressures.

In case of the action of wind on the descent vehicle the influence of transverse elastic oscillations of its construction will be noticeable only with ω_n close to $\pi v/2H$, where v - mean values of the speed of flight of the vehicle on the considered phase of the trajectory $0 < \Delta t \leq 2H/v$, H - gradient distance (see Chapter IX).

In conclusion let us note that the state of stress of the construction of ballistic type descent vehicle in case C is characterized, in essence by one parameter - maximum value of longitudinal overload. From equations (2.6) it follows that with high speeds of undisturbed motion of such a vehicle the speeds of change of the angle of slope of the tangent to trajectory ($\dot{\theta} = 1/v\ \ddot{y}$)

$$\theta=\frac{g_0}{v}\cos\theta$$

will be small. Thus, with comparatively short flight times of the descent vehicle in dense layers of the atmosphere its trajectory will be almost rectilinear, and one may consider with sufficient accuracy that $\theta = \theta_0$. With large numbers $\mathbf{M}_\infty$ for all practical purposes the drag coefficient of the descent vehicle remains constant. With the enumerated conditions the integration of the first equation (2.6) is essentially simplified and it is possible to estimate maximums of longitudinal loads on the construction of the descent flight vehicle analytically.

Having accepted that the air density is changed with altitude in the case of exponential law, and having introduced certain empirical corrections into the solution of the problem, we obtain (for $45° > \theta_0 > 10°$) the following approximate formulas for maximum magnitudes of longitudinal overload:

$$\max n_{x_1\,max} = 3{,}7 v_0^2 \sin\theta_0 \text{ for } v_0 < 4 \text{ km/s}, \tag{14.12}$$

$$\max n_{x_1\,max} = (13 + 2{,}9 v_0^2) \sin\theta_0 \text{ for } v_0 > 4 \text{ km/s}, \tag{14.13}$$

where v_0 - initial speed of entry of the descent vehicle into dense layers of the atmosphere (at altitude h_0) in km/s. Corresponding values of distributed aerodynamic loads in this case will be determined by the magnitude of limiting impact pressure equal to

$$\max q_{max} = \frac{G}{Sc_{x_1}} \max n_{x_1\,max}. \tag{14.14}$$

Thus, maximums of longitudinal forces in any cross section of the body of the descent vehicle in case C will be approximately equal to

$$\max N_c(x_1) = G\left[\frac{1}{c_{x_1}} \int_0^{x_1} \frac{\partial c_{x_1}(x_1)}{\partial x_1} dx_1 - \frac{1}{G} \int_0^{x_1} q_{Gx}(x_1) dx_1\right] \max n_{x_1\,max}. \tag{14.15}$$

The given formulas remain valid until analytic maximum of longitudinal overload is attained in flight or on the level of the surface of earth, i.e., when $h \geq 0$.

§ 14.3. Dynamics of Landing the Vehicle

In case E the descent vehicle, besides mass and aerodynamic forces, is affected by reaction forces of the surface. The latter are applied to the parts of construction of the body or wing of the flight vehicle, which directly contacts the earth's surface (landing gear, special landing devices, front end of descent module and so forth). Values of components of these reactions in fixed system of coordinates depend on very many factors: assignment and dimensions of the flight vehicle, speed of landing, mechanical characteristics and terrain of the landing surface, shock absorption and character of the landing itself.

Usually the horizontal component of reaction R_{x_0}, proportional to the friction force between the landing mechanism and surface, does not noticeably affect the character of loading of the construction of the vehicle. The values of internal power factors are determined primarily by the magnitude and speed of change of vertical component of reaction R_{y_0}. Side reaction, appearing when landing with a cant, affects only the required strength of certain elements of the landing mechanism itself.

Value of vertical component of the reaction depends mainly on the vertical component of velocity of the descent vehicle at the moment of contact with the surface. Therefore, the greatest magnitudes of this velocity are usually used for appraisal of the quality of landing. When $v_{y_0} \leq 2$ m/s the landing is considered <u>soft</u>, when $v_{y_0} = 2$-4 m/s - <u>rough</u>, when $v_{y_0} = 5$-10 m/s - <u>crash</u>, and when $v_{y_0} > 10$ m/s - <u>hard</u>.

Hard landing is applied only for pilotless descent vehicles of one-time use. The remaining types of landing pertain basically to piloted flight vehicles or pilotless descent vehicles of repeated use. Case of crash landing, corresponding to breakdown of part of the brake landing systems or shock absorption system, is usually taken as calculated only for those elements of the construction of the descent vehicle, failure of which can threaten the safety of pilots (for instance, for points of attachment of different loads, instruments and so forth).

If we disregard the effect of lateral component of reaction of the ground R_{z0j}, then motion of the flight vehicle (for instance, an aircraft) as a solid body in case E can be approximately described in fixed system of coordinates x_0y_0 (§ 1.1) by the following system of differential equations:

$$\left.\begin{aligned} m\Delta\ddot{y}_0 - Y + G - \sum_{j=1}^{k_j} R_{y_0 j} = 0,\\ m\Delta\ddot{x}_0 + X - P_{x_0} + \sum_{j=1}^{k_j} R_{x_0 j} = 0,\\ J\vartheta + Y(x_{1д} - x_{1т}) + P_{x_0}(y_{1p} - y_{1т}) -\\ -\sum_{j=1}^{k_j}[R_{y_0 j}(x_{1т} - x_{1j}) - R_{x_0 j}(y_{1т} - y_{1j})] = 0 \end{aligned}\right| \qquad (14.16)$$

with initial conditions

$$\Delta y_0 = 0,\ \Delta x_0 = 0,\ \vartheta = \vartheta_0,\ \dot{\vartheta} = \dot{\vartheta}_0,\ \Delta\dot{y}_0 = -v_{y_0},\ \Delta\dot{x}_0 = v_{x_0} \text{ when } t = 0,$$

where Y, X - lift and drag of the flight vehicle, $R_{x_0 j}$, $R_{y_0 j}$ - horizontal and vertical components of the reaction of landing strip surface, Δy_0 - vertical displacement of the center of gravity of the vehicle due to deformation of the ground and corresponding shifts of shock absorbers (compression of pneumatic tires, elastic and nonelastic deformations of elements of the landing mechanism), Δx_0 - horizontal displacement of the center of gravity of the vehicle, $x_{1т}$ and $y_{1т}$ - coordinates of the center of gravity, $x_{1д}$ - abscissa of the center of pressure of the flight vehicle is connected system of coordinates x_1y_1, k_j - number of elements of landing mechanism, being in contact with the earth's surface at the considered moment of time.

Values of support reactions in this case of loading can be nonlinear functions of displacements of supporting points of the vehicle construction. Magnitude of lift will be determined by angle ϑ and the speed of the flight vehicle at the moment of beginning of touchdown. With use of a parachute system of deceleration or special retrorockets in Y there is included only the vertical component of corresponding braking forces. The horizontal component of the thrust force of the retrorocket (for descent vehicle) and basic engine (for aircraft) is introduced into equations (16) in the form of force P_{x_0}, applied at distance y_{1p} from the center of gravity of the vehicle.

For descent vehicles, not having special external shock-absorbing landing aids, but touching the surface of the earth with the forward end of the flight vehicle body, equations of motion in the plane of action of the horizontal components of aerodynamic (wind) load, i.e., in the plane of drift of the descent vehicle, will have the form

$$\left.\begin{aligned} m\Delta\ddot{y}_0 + G - R_{y_0} - Y &= 0,\\ m\Delta\ddot{x}_0 + R_{x_0} - X &= 0,\\ J_a\ddot{\vartheta} + b_a X + b_y Y - b_a G &= 0, \end{aligned}\right\} \qquad (14.17)$$

where

$$\left.\begin{aligned} J_a &= J + m(x_{1a}^2 + y_{1a}^2),\\ b_a &= x_{1a}\cos\vartheta - y_{1a}\sin\vartheta,\\ b_y &= x_{1a}\sin\vartheta + y_{1a}\cos\vartheta,\\ b_a &= y_{1a}\cos\vartheta + x_{1\text{т}}\sin\vartheta, \end{aligned}\right\} \qquad (14.18)$$

$x_{1\text{д}}$ - abscissa of application of force Y to the body of the vehicle, J - moment of inertia of the flight vehicle with respect to the center of gravity, ϑ - angle of deflection of longitudinal axis x_1 of the vehicle body from axis y_0 of fixed system of coordinates, read counterclockwise, x_{1a} and y_{1a} - coordinates of contact point of the front bottom with the ground in connected system of coordinates x_1y_1 with origin in the center of this bottom and with axis x_1, directed to the aft bottom of the vehicle. In this case it is assumed that the signs of ϑ and y_{1a} are opposite.

The position of the mentioned point of contact of the bottom with the ground is basically determined by initial angle ϑ_0. In particular, when the bottom of the descent vehicle has spherical shape of radius R,

$$x_{1a} = R(1-\cos\vartheta_0),$$
$$y_{1a} = -R\sin\vartheta_0$$

when $y_{1a} < r$, where r - radius of body of the vehicle at place of connection of the bottom.

Solution of system of equations (17) is sought with initial conditions:

$$\Delta y_0 = 0, \quad \Delta x_0 = 0, \quad \vartheta = \vartheta_0, \quad \dot{\vartheta} = \dot{\vartheta}_0,$$
$$\Delta \dot{y}_0 = -v_{y_0}, \quad \Delta \dot{x}_0 = -v_{x_0} \text{ when } t = 0$$

and with assigned relationships of the magnitude of components of support reaction to components of displacement of point (x_{1a}, y_{1a}).

The form of equations of elastic oscillations of the construction in this case of loading in many respects depends on the assembly scheme of the descent vehicle and the required accuracy of calculation. For aircraft type flight vehicle it is expedient to consider these equations in normal coordinates (see § 4.7). In this case it is possible to disregard the influence of elastic displacements of the construction by magnitudes of components of support reactions, and also the relationship of its oscillation conditions to values of aerodynamic forces. With attachment of landing gear to the body of the fuselage it is possible to take

$$\ddot{S}_n + 2h_n\dot{S}_n + \omega_n^2 S_n = Q_n, \qquad (14.19)$$

and

$$
\begin{gathered}
Q_n = \sum_{i=1}^{k_0} Q_{ni} = \\
= \frac{1}{M_n} \sum_{i=1}^{k_0} \left[R_{y_1 i} f_n(x_{ri}) - e_{\text{ш}} \frac{df_n(x_{ri})}{dx_1} \left(R_{x_1 i} \sin\gamma_i - R_{y_1 i} \cos\gamma_i \right) \right].
\end{gathered} \tag{14.20}
$$

With **attachment** of part of the landing gear struts on the wing Q_n will be determined by expression

$$
\begin{gathered}
Q_n = \sum_{i=1}^{k_0 - m} Q_{ni} + \frac{1}{M_n} \Big[R_{y_1 i} \Big\{ \Phi_n(\xi_{\kappa i}) - \\
- \sigma_{\text{ш}i}(\xi_{\kappa i}) \Big[\varphi_n(\xi_{\kappa i}) \cos\varkappa_\kappa - \frac{d\Phi_n(\xi_{\kappa i})}{d\xi_\kappa} \sin\varkappa_\kappa \Big] \Big\} - \\
- R_{x_1 i} \Big[\varphi_n(\xi_{\kappa i}) \cos\varkappa_\kappa - \frac{d\Phi_n(\xi_{\kappa i})}{d\xi_\kappa} \sin\varkappa_\kappa \Big] e_{\text{ш}} \cos\gamma_i \Big],
\end{gathered} \tag{14.21}
$$

where

$$
\begin{gathered}
R_{y_1 i} = R_{y_0 i} \cos\vartheta + R_{x_0 i} \sin\vartheta, \\
R_{x_1 i} = R_{y_0 i} \sin\vartheta - R_{x_0 i} \cos\vartheta, \quad \sigma_{\text{ш}i} = \zeta_{\text{ш}i} - e_{\text{ш}} \cos\gamma_i,
\end{gathered}
$$

k_0 - total number of landing gear struts, m - number of landing gear struts connected to the wing, x_{ri}, $\xi_{\kappa i}$ - coordinates of attachment of struts to the fuselage and wing respectively, $e_{\text{ш}}$ - distance from the axis of rigidity of the wing to the point of the attachement of landing gear strut (in the plane of location of the strut), γ_i - angle of inclination of the strut to longitudinal axis of the flight vehicle, ϑ - angle between axes x_1 and x_0, S_n - elastic lateral displacements of the point of reduction, corresponding to n-th tone. The remaining designations are the same as in § 4.7.

Equations (19) are connected with equations (16), (17) and (18) by relationship of components of reactions $R_{x_0 i}$ and $R_{y_0 i}$ to generalized coordinates Δy_0, Δx_0, ϑ. As a result of the joint solution of these equations all the necessary data are found for calculation of the dynamic and static forces (lateral forces and bending moments) in parts of the aircraft construction. Considering the essential effect of the condition of the landing strip and initial values of angular parameters of the flight vehicle at the moment of contact with the surface on $R_{x_0 1}$, $R_{y_0 1}$, during establishment of the required bearing capacity of the landing gear structure and certain elements of the flight vehicle itselt it is necessary to consider a number of particular cases of loading such as landing with front impact, with side impact, with braking, landing with no spin-up of wheels and so forth. All these particular cases of loading are listed in corresponding norms of strength of aircraft.

If the assembly diagram of the descent vehicle is comparatively simple, it is expedient to represent the dynamic layout of its construction in the given generalized case of loading in the form of a system of elastically connected masses (see Chapter XI). It is natural that the quantity of these masses and the character of their interconnection are established in each particular case depending upon peculiarities of the structure diagram of some descent vehicle, and also requirements of the problem to be solved. Thus for instance, if the construction of the descent vehicle is a supporting spherical or short conical shell, or, finally, a certain frame system (to which are attached payloads, pilot seat and life-support equipment), then the simplified dynamic layout of it can be represented in the form of a system of elastically connected masses (see § 11.2).

Generally a similar system will consist of mass m_0 (simulating the mass of the body of the descent vehicle and all loads rigidly connected to it) and connected to it (with elastic connections) masses of different loads m_1 and masses of shock-absorbing parts of seats (with pilots) m_k. Let us designate the number of these seats and loads through k_k and k_1 respectively.

Equations of two-dimensional perturbed motion of such a system in the process of landing (in projections to axes of system of coordinates x_1y_1 connected with the descent vehicle with origin at the center of the front bottom) can be represented in the following form:

$$m_0\ddot{u}_a+\sum_{k=1}^{k_k} m_k\ddot{u}_k+\sum_{i=1}^{k_i} m_i\ddot{u}_i+\Delta\ddot{\vartheta}m_0(y_{1\text{т}}-y_{1a})=R_x, \qquad (14.22)$$

$$m_0\ddot{v}_a+\sum_{k=1}^{k_k} m_k\ddot{v}_k+\sum_{i=1}^{k_i} m_i\ddot{v}_i-\Delta\ddot{\vartheta}m_0(x_{1\text{т}}-x_{1a})=R_y, \qquad (14.23)$$

$$J_a\Delta\ddot{\vartheta}+\sum_{k=1}^{k_i} m_i\left[(y_{1i}-y_{1a})\ddot{u}_i-(x_{1i}-x_{1a})\ddot{v}_i\right]+$$
$$+\sum_{k=1}^{k_k}\left[R_{\text{л}k}(y_{1\text{л}k}-y_{1a})+R_{\text{п}k}(y_{1\text{п}k}-y_{1a})\right]=eY, \qquad (14.24)$$

$$m_k\ddot{u}_k=R_{\text{л}k}+R_{\text{п}k}, \qquad (14.25)$$

$$m_k\ddot{v}_k=R_{yk}, \qquad (14.26)$$

$$J_k\ddot{\psi}_k=R_{\text{п}k}y_{1\text{п}k}-R_{\text{л}k}y_{1\text{л}k}+(R_{\text{п}k}-R_{\text{л}k})y_{1\text{т}k}, \qquad (14.27)$$

$$m_i\ddot{u}_i+b_{xi}\dot{u}_i+c_{xi}u_i=c_{xi}\left[u_a+\Delta\vartheta(y_{1i}-y_{1a})\right], \qquad (14.28)$$

$$m_i\ddot{v}_i+b_{yi}\dot{v}_i+c_{yi}v_i=c_{yi}\left[v_a-\Delta\vartheta(x_{1i}-x_{1a})\right]. \qquad (14.29)$$

In these equations by u and v with subscripts (k, i) there are designated longitudinal and lateral components of displacement of centers of gravity of separate masses of the system in the above-indicated system of coordinates, and by u_a and v_a - components of displacement of the point of contact of the front bottom of the descent vehicle with the ground (x_{1a}, y_{1a}). Coordinates of points of attachment of longitudinal shock absorbers of seats (left "л" and right "п") to the body of the vehicle are designated by $y_{1\text{л}k}$ and $y_{1\text{п}k}$ respectively, coordinates of centers of gravity of masses m_k by $x_{1\text{т}k}$ and $y_{1\text{т}k}$, and mass m_0 by $x_{1\text{т}}$ and $y_{1\text{т}}$. Longitudinal and lateral displacement of the supporting points of shock absorbers of seats are designated by $u_{0\text{л}k}$, $u_{0\text{п}k}$, v_{0k}, and displacements of supporting points of moving parts of the seats themselves corresponding to them by $u_{\text{л}k}$ and $u_{\text{п}k}$. Thus, the angle of rotation of the seat relative to its center of gravity will be determined by expression

$$\psi_k=\frac{u_{\text{л}k}-u_{\text{п}k}}{y_{1\text{л}k}-y_{1\text{п}k}}.$$

Angle of rotation of the body of the descent vehicle with respect to the point of contact of the bottom with the ground (in plane x_1y_1) is equal to

$$\vartheta = \vartheta_0 + \Delta\vartheta.$$

Furthermore, the following designations are used: J_a - moment of inertia of mass m_0 relative to point (x_{1a}, y_{1a}), x_{1i}, y_{1i} - coordinates of points of the i-th load to the body of the descent vehicle, c_{xi} and c_{yi} - generalized rigidities of points of attachment of this load in the direction of axes x_i and y_i, determined taking into account the local rigidity of the body of the vehicle itself R_{yk} and $R_{лk}$, $R_{пk}$ - lateral and longitudinal (left and right) support reactions of shock absorbers of masses m_k, calculated by formulas:

$$\left.\begin{array}{l} R_{yk} = R_{yky} = c_{yk}(v_{0k} - v_k) \text{ when } R_{yky} < R_{ykk}, \\ R_{лk} = R_{лky} = c_{лk}(u_{0лk} - u_{лk}) \text{ when } R_{лky} < R_{лkk}, \\ R_{пk} = R_{пky} = c_{пk}(u_{0пk} - u_{пk}) \text{ when } R_{пky} < R_{пkk}, \\ R_{yk} = R_{ykk} \text{ when } R_{yky} \geqslant R_{ykk}, \\ R_{лk} = R_{лkk} \text{ when } R_{лky} \geqslant R_{лkk}, \\ R_{пk} = R_{пkk} \text{ when } R_{пky} \geqslant R_{пkk} \end{array}\right\} \quad (14.30)$$

or by some other formulas. Corresponding displacements of supporting points of seats will be equal to:

$$\left.\begin{array}{r} u_{лk} = u_k + \psi_k(y_{1лk} - y_{1тk}), \\ u_{пk} = u_k + \psi_k(y_{1пk} - y_{1тk}), \\ u_{0лk} = u_a + \Delta\vartheta(y_{1лk} - y_{1a}), \\ u_{0пk} = u_a + \Delta\vartheta(y_{1пk} - y_{1a}), \\ v_{0k} = v_a + \Delta\vartheta(x_{1a} - x_{1yk}). \end{array}\right\} \quad (14.31)$$

Solution of the given simplified system of equations is sought with initial conditions:

$$\begin{array}{c} \vartheta = \vartheta_0, \ u_i = 0, \quad v_i = 0, \quad u_k = 0, \quad v_k = 0, \quad u_a = 0, \quad v_a = 0, \\ \dot\vartheta = \dot\vartheta_0, \quad \psi_k = 0, \quad \dot\psi_k = 0, \quad \dot u_i = 0, \quad \dot v_i = 0, \quad \dot v_k = 0, \\ \dot v_a = -v_{y_0}\sin\vartheta_0 - v_{x_0}\cos\vartheta_0, \quad \dot u_a = v_{x_0}\sin\vartheta_0 - v_{y_0}\cos\vartheta_0. \end{array}$$

Usually the greatest interest for strength of the construction is presented by maximum magnitudes of dynamic forces and overloads, which in this case of loading are observed in the process of the first impact of the vehicle against the ground. Therefore, if values of reactions

$$R_x = R_{y_0}\cos\vartheta - R_{x_0}\sin\vartheta,$$
$$R_y = R_{y_0}\sin\vartheta + R_{x_0}\cos\vartheta$$

nonlinearly depend on Δy_0 and Δx_0 (i.e., u_a and v_a), then for simplification of the solution of the problem it is possible to consider them assigned function of time. In particular, approximate values of these reactions can be determined by solution of system of equations (17), describing the motion of descent vehicle as a solid body. Some idea of the character of change of the value of reaction R_x in time is given by the graph of change of longitudinal overload of the center of gravity of the vehicle (descent module) in the process of landing on the ground (with initial vertical velocity 9 m/s), presented on Fig. 14.3. As can be seen from this figure, maximum magnitudes of longitudinal overload are measured in tens of units, and the time of its action - only hundredths of a second. It is obvious that in similar cases with sufficient accuracy for practical calculations it is possible to disregard (with calculation of reaction and coefficient e) the change of angle ϑ with respect to t, assuming

$$R_x = R_{y_0}\cos\vartheta_0 - R_{x_0}\sin\vartheta_0,$$
$$R_y = R_{y_0}\sin\vartheta_0 + R_{x_0}\cos\vartheta_0,$$
$$e = x_{1a}\cos\vartheta_0 - y_{1a}\sin\vartheta_0.$$

Values of reactions R_{x_0} and R_{y_0} and consequently, the magnitudes of dynamic loads on structural elements of the descent vehicle have a random character and can be changed in very wide limits depending upon the place of landing and the state of the landing strip. In particular, when landing on comparatively soft ground these loads will be determined basically by the mechanical properties of the ground (dynamic coefficient of elasticity, shearing strength), when landing on sufficiently hard (rock) surface - by elastic properties of the construction of the bottom of the vehicle. The probability of realization of similar limiting cases of loading will depend on many factors and can be somehow estimated only for particular flight vehicles. In conformity with values of these probabilities a calculated model of the ground of a landing strip is selected.

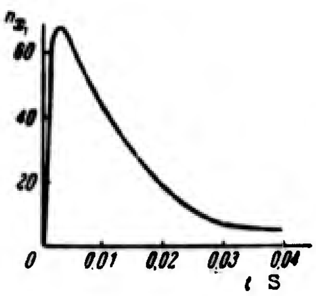


Fig. 14.3. Typical change of longitudinal overload in the process of landing of a descent vehicle on solid ground.

Expressions for longitudinal and lateral components of overload at different points of the construction of the descent vehicle will have the form:

a) for points of the vehicle body with coordinates

$$n_{x_1}(y_1) = 1 - \frac{1}{g_0}\left[\ddot{u}_a + \Delta\ddot{\vartheta}(y_1 - y_{1a})\right],$$
$$n_{y_1}(x_1) = -\frac{1}{g_0}\left[\ddot{v}_a - \Delta\ddot{\vartheta}(x_1 - x_{1a})\right];$$

b) at centers of gravity of loads elastically suspended from the body

$$n_{x,i} = 1 - \frac{\ddot{u}_i}{g_0}, \quad n_{y,i} = -\frac{\ddot{v}_i}{g_0}.$$

c) at points of the cushioned part of seats

$$n_{x,k} = 1 - \frac{\ddot{u}_k}{g_0} + \frac{\ddot{\psi}_k}{g_0}(y_{1\tau k} - y_{1k}), \quad n_{y,k} = -\frac{\ddot{v}_{0k}}{g_0} + \ddot{\psi}_k \frac{1}{g_0}(x_{1k} - x_{1yk}).$$

For decrease or limitation of the maximum values of total or local overloads there is usually applied external or internal (local) shock absorption. The type of shock absorption and permissible magnitudes of overload components are selected in each concrete case depending upon the assignment of the descent vehicle.

§ 14.4. "Soft" Landing of Flight Vehicle

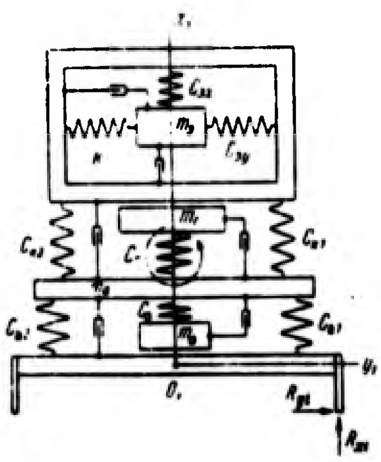
With realization of a "soft" landing of the flight vehicle, possessing low horizontal velocity and limited maneuverablity, i.e., limited possibility of selection of a landing strip, required for subsequent launch, the initial position of the vehicle is assured basically by the construction of the landing mechanism. Typical for the considered class of vehicles are landing mechanisms consisting of four isolated or kinematically connected support struts can be freely changed in certain limits, adjusting to the local terrain of the landing strip. The requirement for preservation of the stability of the descent vehicle is the process of landing imposes definite limitations on the general configuration of such craft. The tendency to achieve a low position of the center of gravity of similar systems usually leads to increase of lateral dimensions of their construction, and consequently also to increase of its flexural and shear rigidity. Therefore, the essential influence of local rigidity of points and elements of their connection with each other on the state of stress of the construction becomes characteristic for such vehicles.

The dynamic layout of flight vehicles of the considered class will depend not only on overall formulation of the problem, but also on concrete structural designing of their typical[1] joints: cabin (indes "к"), which accomodates the crew (index "э") and necessary life-support facilities for the crews and return to earth; special engine unit (index "a") (with "α" and "г" tanks), with the aid of which landing of the vehicle is executed, and, finally, the landing mechanism itself (index "пу").

Since the problem of dynamic calculation for the given case of loading leads essentially to selection of characteristics of external

[1]Khilderman, Muller, Mantus, Dynamics of lunar impact of the expeditionary section of "Apollo," Questions of rocket technology, No. 2, 1967.

shock absorption, it is possible to assume that the question of accuracy of description of elastic oscillations of the system in this case does not have decisi.e values, and it is possible to be limited by consideration of the simplified dynamic model of the descent vehicle. Such a simplified model in the form of a system of elastically connected solid bodies for a typical assembly diagram of descent vehicle is depicted on Fig. 14.4.



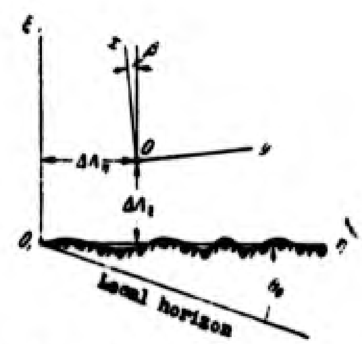


Fig. 14.4. Fig. 14.5.

Fig. 14.4. Simplified dynamic model of descent vehicle, designed for "soft" landing.

Fig. 14.5. Fixed system of coordinates.

Let us formulate equations of motion of this vehicle in one of the planes of symmetry, using the Lagrange method. For this we introduce the following systems of coordinates: a) fixed $O_1\xi\eta$, axis η of which coincides with the tangent to averaged profile of the surface at the landing place of the vehicle (Fig. 14.5); b) connected $O_1x_1y_1$ (Fig. 14.4), axis x_1 of which is directed along the undeformed longitudinal axis of the descent vehicle upwards, and axis y_1 - perpendicular to axis x_1 along the line of intersection of the plane of symmetry of the vehicle with some conditional reference plane of the landing mechanism; c) moving coordinate system Oxy, axes of which coincide with instantaneous position of axes x_1 and y_1.

Motion of system of coordinates Oxy relative to the fixed system will be characterized by projections $\Delta\lambda_{\xi}$, $\Delta\lambda_{\eta}$ of forward displacement of the origin of coordinates (point O) and angle of rotation β (relative to point O). In projections to axes, instantly coinciding with axes of the moving coordinate system, components of displacement of this point will be equal to

$$\left.\begin{aligned} \Delta\lambda_x &= \Delta\lambda_\eta \sin\beta + \Delta\lambda_\xi \cos\beta, \\ \Delta\lambda_y &= \Delta\lambda_\eta \cos\beta - \Delta\lambda_\xi \sin\beta. \end{aligned}\right\} \qquad (14.32)$$

Corresponding components of velocities of points of separate parts of the vehicle in system of coordinates Oxy will be determined by expressions

$$\left.\begin{aligned} v_{x\text{o}} &= \Delta\dot\lambda_x + \Delta\dot x_{\text{а}} + \dot\beta y_{10} + \Delta\dot x_{\text{o}} + \dot\psi_{\text{а}} y_{10}, \\ v_{y\text{o}} &= \Delta\dot\lambda_y - \dot\beta x_1 - \dot\psi_{\text{а}}(x_1 - x_{10}) \end{aligned}\right\}. \qquad (14.33)$$

for points of tank "o,"

$$\left.\begin{aligned} v_{x\text{г}} &= \Delta\dot\lambda_x + \Delta\dot x_{\text{а}} + (\dot\beta + \dot\psi_{\text{а}} + \dot\psi_{\text{г}})\, y_{1\text{г}} + \Delta\dot x_{\text{г}}, \\ v_{y\text{г}} &= \Delta\dot\lambda_y - \dot\beta x_1 - (\dot\psi_{\text{а}} + \dot\psi_{\text{г}})(x_1 - x_{1\text{г}}) \end{aligned}\right\} \qquad (14.34)$$

for points of tank "Г,"

$$\left.\begin{aligned} v_{x\text{к}} &= \Delta\dot\lambda_x + (\dot\beta + \dot\psi_{\text{а}} + \dot\psi_{\text{к}})\, y_1 + \Delta\dot x_{\text{а}} + \Delta\dot x_{\text{к}}, \\ v_{y\text{к}} &= \Delta\dot\lambda_y - \dot\beta x_1 - (x_1 - x_{1\text{к}})(\dot\psi_{\text{а}} + \dot\psi_{\text{к}}) \end{aligned}\right\} \qquad (14.35)$$

for points of cabin,

$$\left.\begin{aligned} v_{x\text{а}} &= \Delta\dot\lambda_x + \dot\beta y_1 + \Delta\dot x_{\text{а}} + \dot\psi_{\text{а}} y_1, \\ v_{y\text{а}} &= \Delta\dot\lambda_y - \dot\beta x_1 - \dot\psi_{\text{а}}(x_1 - x_{1\text{а}}) \end{aligned}\right\} \qquad (14.36)$$

for points of support section "a" and

$$\left.\begin{aligned} v_{x\text{э}} &= v_{x\text{к}} + \Delta\dot x_{\text{э}}, \\ v_{y\text{э}} &= v_{y\text{к}} + \Delta\dot y_{\text{э}} \end{aligned}\right\} \qquad (14.37)$$

for elastically suspended loads inside the cabin.

Components of velocities of points of undeformed parts of the landing mechanism will be equal to

$$\left.\begin{aligned} v_{xy} &= \Delta\dot{\lambda}_x + \beta y_i, \\ v_{yy} &= \Delta\dot{\lambda}_y - \beta x_i. \end{aligned}\right\} \tag{14.38}$$

In these formulas $\psi_\text{г}$, $\psi_\text{к}$, $\psi_\text{а}$ - angles of rotation of longitudinal axes of tank "г," cabin "к" and section "a" due to torsional strain of support elements of the tank and unequal deformation of support brackets of the cabin and section "a" respectively, $x_{i\text{г}}$ - coordinate of support frame of tank "г," and $x_{i\text{к}}$, $x_{i\text{а}}$ - support frame of the cabin of the craft and section "a," Δx_j and Δy_j - displacement of parts of the craft due to deformation of corresponding elements (j = a, o, г, к).

Having formulated expressions for kinetic and potential energy and using equation (4.1), we obtain the sought system of differential equations describing the process of "soft" landing (symmetric relative to longitudinal axis) in the form

$$\left.\begin{aligned} &\Delta\ddot{\lambda}_x m + \Delta\ddot{x}_\text{г} m_\text{г} + \Delta\ddot{x}_\text{о} m_\text{о} + \Delta\ddot{x}_\text{а}(m - m_\text{пу}) + \\ &\qquad + \Delta\ddot{x}_\text{э} m_\text{э} + \Delta\ddot{x}_\text{к} m_\text{к} = \sum_{i=1}^{n} R_{xi}, \\ &\Delta\ddot{\lambda}_y m + \Delta\ddot{y}_\text{э} m_\text{э} - \dot{\beta} S - \ddot{\psi}_\text{а} S_\text{а} - \ddot{\psi}_\text{к} S_\text{к} - \ddot{\psi}_\text{г} S_\text{г} = \sum_{i=1}^{n} R_{yi}, \\ &\ddot{\beta} I_z + \ddot{\psi}_\text{а} a_\text{а} + \ddot{\psi}_\text{к} a_\text{к} + \ddot{\psi}_\text{г} a_\text{г} - \Delta\ddot{\lambda}_y S - \Delta\ddot{y}_\text{э} S_\text{э} = \\ &\qquad = \sum_{i=1}^{n} (R_{xi} y_i + R_{yi} x_i). \end{aligned}\right| \tag{14.39}$$

$$\left.\begin{aligned} &\Delta\ddot{x}_\text{г} + 2h_\text{г}\Delta\dot{x}_\text{г} + \Delta\ddot{\lambda}_x + \Delta\ddot{x}_\text{а} + \omega_\text{г}^2\Delta x_\text{г} = 0, \\ &\Delta\ddot{x}_\text{о} + 2h_\text{о}\Delta\dot{x}_\text{о} + \Delta\ddot{\lambda}_x + \Delta\ddot{x}_\text{а} + \omega_\text{о}^2\Delta x_\text{о} = 0, \\ &\Delta\ddot{x}_\text{к} + 2h_\text{к}\Delta\dot{x}_\text{к} + \Delta\ddot{\lambda}_x + \Delta\ddot{x}_\text{а} + \Delta\ddot{x}_\text{э}\frac{m_\text{э}}{m_\text{к}} + \omega_\text{к}^2\Delta x_\text{к} = 0, \\ &\Delta\ddot{x}_\text{э} + 2h_\text{э}\Delta\dot{x}_\text{э} + \Delta\ddot{\lambda}_x + \Delta\ddot{x}_\text{а} + \Delta\ddot{x}_\text{к} + \omega_\text{э}^2\Delta x_\text{э} = 0, \\ &\Delta\ddot{x}_\text{а} + \Delta\ddot{\lambda}_x + (\Delta\ddot{x}_\text{о} m_\text{о} + \Delta\ddot{x}_\text{г} m_\text{г} + \Delta\ddot{x}_\text{к} m_\text{к} + \\ &\qquad + \Delta\ddot{x}_\text{э} m_\text{э})\frac{1}{(m - m_\text{пу})} + 2h_\text{а}\Delta\dot{x}_\text{а} + \omega_\text{а}^2\Delta x_\text{а} = 0, \\ &\Delta\ddot{y}_\text{э} + 2h_\text{э}\Delta\dot{y}_\text{э} + \Delta\ddot{\lambda}_y - \ddot{\beta} S_\text{э} - \ddot{\psi}_\text{к} S_\text{аэ} - \ddot{\psi}_\text{а} S_\text{аэ} + \omega_{y\text{э}}^2\Delta y_\text{э} = 0, \end{aligned}\right\} \tag{14.40}$$

$$
\left.\begin{aligned}
&\ddot{\psi}_{\text{а}} J_{\text{а}} + \dot{\psi}_{\text{а}} b_{\text{а}} + \beta a_{\text{а}} - \Delta\ddot{\lambda}_y S_{\text{а}} + \ddot{\psi}_{\text{к}} J_{\text{к}} + \\
&\qquad + \ddot{\psi}_{\text{г}} J_{\text{г}} - \Delta\ddot{y}_{\text{э}} S_{\text{аэ}} + \omega^2_{\psi\text{а}} J_{\text{а}} \psi_{\text{а}} = 0, \\
&\ddot{\psi}_{\text{к}} J_{\text{к}} + \dot{\psi}_{\text{к}} b_{\text{к}} + \beta a_{\text{к}} - \Delta\ddot{\lambda}_y S_{\text{к}} + \ddot{\psi}_{\text{а}} J_{\text{к}} + \omega^2_{\psi\text{к}} J_{\text{к}} \psi_{\text{к}} = 0, \\
&\ddot{\psi}_{\text{г}} J_{\text{г}} + \dot{\psi}_{\text{г}} b_{\text{г}} + \beta a_{\text{г}} - \Delta\ddot{\lambda}_y S_{\text{г}} + \ddot{\psi}_{\text{а}} J_{\text{г}} + \omega^2_{\psi\text{г}} J_{\text{г}} \psi_{\text{г}} = 0,
\end{aligned}\right\} \qquad (14.41)
$$

where n - number of support struts of the landing mechanism in contact with the surface at the considered moment of time, x_i and y_i - coordinates of the point of application of reaction of i-th support, m - mass of descent vehicle, m_j (j = о, г, к, э, г.у) - mass of loads, J_z - mass moment of inertia of the descent vehicle with respect to axis z_1; J_a, $J_г$, and $J_к$ - moments of inertia of masses m_a, $m_г$ and $m_к$ with respect to axes z_1, passing through x_{1j}; S_j - static moments of j-th mass with respect to axis z_1. In this case partial frequencies of natural elastic oscillations of the parts of construction of the descent vehicle are calculated by formulas

$$
\left.\begin{aligned}
&\omega_j^2 = \frac{c_j}{m_j}, \quad c_j = \sum_{p=1}^{n_p} c_{jp} \qquad (j = \text{а, о, г, к, э}), \\
&\omega^2_{y\text{э}} = \frac{c_{y\text{э}}}{m_{\text{э}}}, \quad \omega^2_{\psi i} = \frac{c_i'}{J_i} \\
&c_i' = \sum_{p=1}^{n_p} c_{ip} y_{ip}^2 \cos^2 \gamma_{pi} \qquad (i = \text{а, к, г}).
\end{aligned}\right\} \qquad (14.42)
$$

Remaining coefficients are equal to

$$
\left.\begin{aligned}
&a_{\text{а}} = J_z - m_{\text{а}} x_{1\text{а}} x_{\text{ат}} - m_{\text{к}} x_{1\text{к}} x_{\text{кт}}, \\
&a_{\text{к}} = J_{\text{к}z} - m_{\text{к}} x_{1\text{к}} x_{\text{кт}}, \qquad a_{\text{г}} = J_{\text{г}z} - m_{\text{г}} x_{1\text{г}} x_{\text{гт}}, \\
&S = m_{\text{а}} x_{\text{ат}} + m_{\text{к}} x_{\text{кт}}, \qquad S_{\text{к}} = m_{\text{к}} (x_{\text{кт}} - x_{1\text{к}}), \\
&S_{\text{а}} = S - m_{\text{а}} x_{1\text{а}} - m_{\text{к}} x_{1\text{к}}, \; S_{\text{аэ}} = m_{\text{э}} (x_{\text{эт}} - x_{1\text{к}}), \; S_{\text{э}} = m_{\text{э}} x_{\text{эт}}.
\end{aligned}\right\} \qquad (14.43)
$$

By the given method it is simple to obtain equations for another number and location of masses of the system, and also equations of three-dimensional motion of the descent vehicle.

The solution of the above-mentioned system of equations is found by the method of numerical integration with zero initial values of all parameters characterizing elastic oscillations of the parts of the construction, and assigned initial values of motion parameters of the descent vehicle as a solid body

$$\beta=\beta_0=\vartheta_0-\theta_0,\quad \dot{\beta}=\dot{\beta}_0,\quad \lambda_x=\lambda_y=0,$$
$$\dot{\lambda}_x=-v_{y_0}\cos\vartheta_0-v_{x_0}\sin\vartheta_0,$$
$$\dot{\lambda}_y=-v_{y_0}\sin\vartheta_0+v_{x_0}\cos\vartheta_0+\dot{\beta}_0x_{1\text{т}}\text{ when } t=0,$$

where ϑ_0 - angle of slope of longitudinal axis of the vehicle to vertical, v_{y0} and v_{x0} - vertical and horizontal velocity components of the center of gravity of the vehicle at the moment of contact with the surface.

Let us present schematically support elements of the landing mechanism in the form of struts, free ends of which (supporting points, touching the surface of the landing strip) in system of coordinates Oxy can obtain displacements in the form

$$\Delta y_i=(l_0+\Delta l_i)\sin(\alpha_0+\Delta\alpha_i)-l_0\sin\alpha_0,$$
$$\Delta x_i=-(l_0+\Delta l_i)\cos(\alpha_0+\Delta\alpha_i)+l_0\cos\alpha_0,$$

where l_0 - initial length of strut, α_0 - initial angle of its slope of axis x_1 (in plane x_1y_1), Δl_i and $\Delta\alpha_i$ - increase of length and angle of slope of i-th strut. Generally the change of length of the strut can occur both due to shortening of its separate elements (for instance, shock absorbers) under load, and due to free displacements of the supporting points in the process of their attachment to the local surface terrain (if this is provided by construction of the landing mechanism). In fixed system of coordinates the position of supporting points is characterized by functions

$$\xi_i(t)=\xi_{i0}+\int_0^t v_{\xi i}\,dt,$$
$$\eta_i(t)=\eta_{i0}+\int_0^t v_{\eta i}\,dt,$$

where $v_{\xi i}$ and $v_{\eta i}$ - velocity components, and ξ_{i0} and η_{i0} - initial coordinates of supporting point of i-th strut. Condition $\xi_i(t) = 0$ in combination with additional conditions, determining the moment of cessation of free motion of the strut in the process of attachment to local terrain, establish the fact of application of support reaction (R_{xi}, R_{yi}) to the given strut.

It is possible to formulate many different combinations of inclusion of different struts in the operation of shock absorbers. Each such combination will determine the possible particular case of loading of the construction of the descent vehicle. The most important of all these particular cases are cases corresponding to the biggest values of longitudinal and lateral components of overload, particularly cases of simultaneous landing of the descent vehicle on all the support struts taking into account ($v_{\eta i} = v_{\eta i\ max}$) and without taking into account ($\eta_i = 0$) lateral motion (with specified profiles of the landing strip). The shown cases of loading actually will establish the allowable maximum magnitudes of support reactions (forces of shock absorbers) for any vertical velocities of landing, since

$$\max R_{xi} = \max n_x \frac{G}{n}, \tag{14.44}$$

where max n_x - allowable longitudinal overload in the center of gravity of the vehicle, and n - total number of struts.

Components of overload reduced to terrestrial conditions at any point of the vehicle are calculated by known formulas

$$n_{xj} = -\frac{1}{g_0}\dot{v}_{xj}, \quad n_{yj} = -\frac{1}{g_0}\dot{v}_{yj}$$
$$(j = \text{о, г, к, э, а}),$$

where g_0 - acceleration due to gravity on the earth; v_{xj}, v_{yj} - velocity components of the considered point of descent vehicle, calculated by formulas (33)-(37). Thus, for instance, for points of support section

$$n_{x\text{в}} = -\frac{1}{g_0}(\Delta\ddot{\lambda}_x + \ddot{\beta}y_1 + \Delta\ddot{x}_\text{в} + \ddot{\psi}_\text{к}y_{1\text{в}}),$$
$$n_{y\text{в}} = -\frac{1}{g_0}[\Delta\ddot{\lambda}_y - \ddot{\beta}x_1 - \ddot{\psi}(x_1 - x_{1\text{в}})].$$

Magnitudes of longitudinal forces in structural elements of the descent vehicle will be determined by relative elastic displacements of separate joints. Particularly, in places of connection of tanks with the body of the vehicle they will be equal to

$$N_{x0} = c_0\Delta x_0,$$
$$N_{x\text{г}} = c_\text{г}\Delta x_\text{г},$$

and in cabin supports

$$N_{x1} = c_{\text{к}1}(\Delta x_\text{к} + \psi_\text{к}y_{\text{к}1}),$$
$$N_{x3} = c_{\text{к}3}(\Delta x_\text{к} - \psi_\text{к}y_{\text{к}3}).$$

Values of dynamic components of internal forces in structural elements and vibration components of local overloads in the process of "soft" landing depend on the gradient of change of support reactions of the land mechanism. By selection of the magnitude of $dR_{xi}/d\Delta_{ei}$ for the initial phase of operation of the shock absorber (up to the moment of achievement of maximum value of support reaction) it is possible to regulate the values of these forces in rather wide limits.

§ 14.5. Loading of Descent Vehicle when Landing on Water

With unsteady motion of the descent vehicle in an incompressible ideal infinite volume of liquid, its inertia increases at the expense of inertia of the part of liquid which is attracted by the body of the vehicle. The shown effect is usually considered by the introduction of apparent masses and apparent moments of inertia of liquid into equations of motion. Values of these apparent inertial characteristics depend on the shape and dimensions of the vehicle and especially on the density of liquid. With motion of the descent vehicle in water they can be comparable with its mass and moment of inertia.

The problem of determination of apparent masses m_i^* is reduced to finding the velocity potentials of liquid ϕ_i during motion of the vehicle along axes x_1, y_1 and z_1 with unit velocities

$$m_i^* = \rho_j \int_S \int \varphi_i \frac{\partial \varphi_i}{\partial n} ds, \qquad (14.45)$$

where s - surface limiting the construction of the vehicle, ρ_j - density of liquid. These functions ϕ_i must satisfy Laplace equation (2.31), condition of the absence of liquid motion at infinity and corresponding boundary conditions of the moistened surface of the vehicle, which for forward motion take the form $\partial\phi_i/\partial n = \cos \alpha_i$, where α_i - angles of slope of the normal $\bar{n}$ to axes x_1, y_1 and z_1.

In case of partial submersion of the vehicle in liquid the velocity potential should additionally satisfy the boundary condition on the free surface of liquid

$$\varphi = \sum_i \varphi_i v_i = 0 \qquad (i = x_1, y_1, z_1),$$

where v_i - velocity components of the vehicle in connected system of coordinates. By using this condition, it is possible [64] to analytically continue potential ϕ through the free surface, assuming $\varphi(x_1, y_1, z_1) = -\varphi(-x_1, y_1, z_1)$.

In other words, it is possible to replace the problem of impact of the vehicle against the free surface of liquid by the problem of unsteady motion of a symmetric body in liquid, obtained by mirror image of the part of the body of the descent vehicle immersed in water. Values of apparent masses for symmetric bodies of different shape are determined by a number of authors analytically. When necessary they can be found experimentally - by direct measurement of acceleration of the body, moving in liquid under the action of specified force, or by means of comparison of natural frequencies of oscillations of the body in liquid and in air.

In the case of landing of the descent vehicle on water of interest to us apparent masses will be the function of depth of submersion h. In proportion to increase of h the shape and dimensions of the immersed part of the vehicle will be changed, and consequently, m*. With increase of apparent mass the velocity and acceleration of the descent vehicle will decrease. This is well illustrated by a graph of change of longitudinal overload of the center of gravity of the descent module in time (Fig. 14.6), obtained with vertical velocity of landing 9 m/s. As can be seen, n_{x_1} reaches maximum values at the initial moment of submersion of the vehicle in water with small h. Forward motion of the descent vehicle on this section can be described by equations

$$(m+m_x^*)\ddot{h}+G-(X+A)=0, \tag{14.46}$$

$$-(m+m_y^*)\ddot{y}+Y=0, \tag{14.47}$$

with initial conditions $h(t)=0,\ -\dot{h}(t)=v_{y_0},\ y(t)=0,\ \dot{y}(t)=v_{x_0}$ when t = 0, where m - mass of the vehicle, A - Archimedean force, X and Y - components of external aerodynamic forces (wind and braking). With small initial values of angular velocities of the body of descent vehicle the influence of its rotation with respect to the center of gravity in this case of loading can be disregarded.

If components of initial velocity of the center of gravity of the descent vehicle are comparatively great (on the order of 5-10 m/s), then with small h the above-mentioned equations can be essentially simplifed, namely are represented in the form

$$(m + m_x^*)\ddot{h} = 0, \tag{14.48}$$

$$(m + m_y^*)\ddot{y} = 0. \tag{14.49}$$

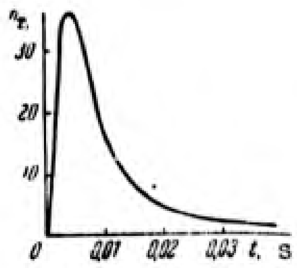


Fig. 14.6. Character of change of longitudinal overload in case of landing the descent vehicle in water.

The basic difficulty of solution of the given system of nonlinear differential equations consists of determination of the relationship of apparent masses to the depth of submersion of the body of the vehicle. When the lower part of such vehicles has elliptical or spherical shape, it is possible in the first approximation to approximate the body, obtained by mirror image of this part of the vehicle immersed in water, by a flattened ellipsold of rotation. Then [64]

$$m_x^* = \frac{2}{3}\pi\rho_l r^2 h \frac{a_0}{2 - a_0}, \tag{14.50}$$

$$m_y^* = m_z^* = \frac{2}{3}\pi\rho_l r^2 h \frac{b_0}{2 - b_0}, \tag{14.51}$$

where

$$a_0 = \frac{2\bar{h}}{(1-\bar{h}^2)^{3/2}} \frac{\sqrt{1-\bar{h}^2}}{\bar{h}} - \arcsin\sqrt{1-\bar{h}^2},$$

$$b_0 = \frac{\bar{h}}{(1-\bar{h})^{3/2}}\left(\arcsin\sqrt{1-\bar{h}} - \bar{h}\sqrt{1-\bar{h}^2}\right),$$

$$\bar{h} = \frac{h}{r},$$

and r - radius of cross section of the bottom at the level of free surface of liquid. For a spherical bottom of radius R $r = \sqrt{2Rh - h^2}$.

When desired these expressions can be definitized experimentally by measurement of longitudinal and lateral overloads with releases of mockups of vehicles on the water modelling their weight and shape of the lower part.

Usually in the considered case of loading on the construction of the descent vehicles there are imposed only requirements for preservation of airtightness of their bodies and sufficient strength of points of suspension of different loads. So that corresponding loads (average hydrodynamic pressure on the bottom of the vehicle and inertial forces) will be completely determined by values of overload components

$$n_x = -\frac{\dot{h}}{g_0}, \quad n_y = -\frac{\dot{y}}{g_0}$$

and by depth of submersion h of the vehicle in water. Moreover, in the first approximation one may assume that

$$\frac{\max n_x}{v_{y_0}^2} = \text{const.}$$

For strength of certain elements of the bottom the law of distribution of hydrodynamic pressure can be of definite interest. It is recommended to find it experimentally in view of the complexity of calculation of hydrodynamic interaction of the vehicle with waves.

S E C T I O N V

ESTABLISHMENT OF REQUIRED BEARING CAPACITY OF THE CONSTRUCTION OF A FLIGHT VEHICLE

C H A P T E R XV

NORMALIZATION OF STRENGTH OF THE CONSTRUCTION

§ 15.1. Concerning the Question of Normalization of the Strength of the Flight Vehicle Construction

In the preceding sections methods of determination of external loads and internal forces in the construction of a flight vehicle body were given for all ground and flight generalized cases of loading. In this concluding section certain questions are considered, connected with normalization of strength, i.e., with establishment of required bearing capacity of the construction.

A necessary condition of any normalization of flight vehicle strength is preliminary revealing of calculated cases of loading for every part of its construction. The presence of calculated cases considerably reduces the overall volume of calculations on strength, simplifies gravimetric analysis of the structure diagram of constructions and experimental treatment of their bearing capacity.

As already was noted, as design there is taken the case of loading which is the most dangerous for strength of the construction. Comparative appraisal of the measure of danger of the different states of stress can be performed with the help of corresponding theories of strength. In this case one should only consider the peculiarities of the utilized methods of calculations of constructions for strength. As it is known, two such methods exist: method of allowable stresses (loads) or deformations and the method of limiting states (breaking loads) with respect to strength or stability. In machine building the method of allowable stresses is usually used. Method of limiting states, allowing more complete use of the bearing capacity of the construction, is applied primarily in those regions of technology, in which questions of economy of weight play a decisive role.

According to the first method maximum stress or deformation, appearing at some point of the construction, are compared with permissible values. In this case the loading, in which these stresses (deformations) or load corresponding to them turn out to be the greatest, is considered the most dangerous. By the second method the maximum effective load is compared with breaking. The latter determines the bearing capacity of the construction, i.e., the limiting deformed state of the construction with respect to strength or stability, which corresponds to the beginning of its failure. Thus, according to the method of limiting states, the case of loading which gives the biggest value of required bearing capacity of the construction both with respect to strength and stability will be design.

Basic difficulty of determination of allowable stresses (deformations) and required bearing capacity of the construction of any flight vehicle consists of coordination of calculated data with actual. Actual values of stresses, deformations, internal forces in structural elements in all cases of loading are random functions of time, and the actual bearing capacity of the construction - random variable. Therefore, the question of strength of the construction should be considered from a probability point of view. With great changeability of real external loads and parameters, characterizing the bearing capacity of the construction, the condition of strength is reduced to obtaining a guarantee that the limiting state will not once be exceeded for the entire time of operation of the flight vehicle.

In case of representation of computed values of required bearing capacity in deterministic form, such a guarantee is provided by the introduction of so-called safety factor f. This factor is used both when determining allowable stresses (deformations)

$$\sigma_{д}=\frac{\sigma_{np}}{f}, \tag{15.1}$$

where σ_{np} - limiting stress (yield point for plastic materials, ultimate strength for brittle materials, critical stress of loss of local or overall stability), and when calculating breaking loads

$$N^{p}(x_1)=fN(x_1). \tag{15.2}$$

Values of the shown factor depend first of all on the accuracy of calculation of external loads, internal power factors and bearing capacity of the construction, on conformity of calculated diagrams to real. It is obvious that when counting on the safety factor of functions for compensation of errors of the mathematical description of real processes of loading, dynamic layout and bearing capacity of the flight vehicle the design load (2) in essence becomes fictitious. Consequently, calculated cases, established according to such loads, cannot correspond to calculated cases determined by operational loads. To this one should add that in a number of cases the change of geometric shape of the flight vehicle construction due to great deformation (elastic and residual) can lead to change of the magnitude and especially the distribution of external loads. In similar cases the actual breaking load will differ from calculated, obtained by multiplication of N by f. Even when the shown proportionality takes place right up to failure, it is not always possible to use a single safety factor for all internal power factors. Thus, for instance with combined loading of the carrier fuel tank of a ballistic type flight vehicle its bearing capacity under compression will essentially depend on the magnitude of internal excess gas boost pressure. Therefore, the operation of multiplication of the latter by the safety factor (especially with the presence of nonlinear relationship of critical stress of loss of longitudinal stability of the tank shell to the magnitude of this pressure) does not have practical meaning.

Difficulties of theoretical solution of all questions connected with establishment of the required bearing capacity of flight vehicle construction are so great that normalization of their strength becomes necessary. This normalization is usually reduced to establishment of design cases of loading and safety factors for all main junction points of the construction, and also to indication of the methods of calculation of external loads and internal power factors. Sometimes strength norms limit maximums of overloads, angles of attack, impact pressures and other parameters. Furthermore, normalization is not infrequently thought of as an experimental check of the actual bearing capacity of the construction of all design cases of loading, particularly by means of static tests.

For flight vehicle, manufactured in small series, norms of strength can be attached to a particular vehicle, i.e., they can normalize the required bearing capacity of separate parts of the designed construction. For reusable vehicles this required bearing capacity of the construction is established taking into account the effect of recurrence of loads and specified service life.

For flight vehicles having low frequencies of natural elastic oscillations, it is recommended to normalize also the rigidity of basic structural elements. The required values of these regidities can be established by dynamic calculation of the construction at the stabe of normalization of its required bearing capacity.

§ 15.2. Selection of Safety Factor

The structure of the safety factor, expressed through the relationship of real breaking load to some design load (constituting the possible value of operating load), is very complicated and is polyhedral. The value of this coefficient depends on many factors, many of which carry a random character. Since it is theoretically impossible to establish maximums of internal forces and minimum values of bearing capacity of the construction without indication of the probability of their realization, the values of safety factor can be determined only from a probability point of view. As was mentioned in the preceding paragraph, values of f essentially depend on the method of calculation of the construction for strength.

With use of standard methods of establishment of the required bearing capacity of the flight vehicle construction the role of safety factors leads basically to compensation of:

1) discrepency between the deterministic form of representation of results of calculation for strength and the random character of actual values of external loads, internal forces and the bearing capacity of the construction;

2) deviation of calculated dynamic layout of the flight vehicle and design conditions of loading from real.

It is natural that normalized safety factors are also statistical in their nature, since their selection is substantiated by numerous experimental data, obtained in the process of many years of operation of different types of flight vehicles. It is clear that the application of similar experimental safety factors to new flight vehicles or to new design cases of loading can lead both to excess of the required bearing capacity and to lowering of the reliability of the construction. Therefore, recently considerable attention has been paid to development of theoretical bases of normalization of safety factors, especially in the region of structural mechanics and machine building. In this connection one should note the work of V. V. Bolotin [9], A. R. Rzhanistsyn [60], N. S. Streletskiy [68], [69] and other authors.

The experimental character and probability nature of these factors uncover means of their dismemberment into a finite number of statistically independent components, characterizing the changeability of operating conditions of the construction and its strength. In the first approximation the safety factor can be represented in the form of the product of two coefficients f_1 and f_2.

The first coefficient is a certain generalized characteristic of changeability of true values of external loads, internal power factors and bearing capacity of the construction. Its value is estimated by variation of actual values of generalized load N and bearing capacity of the construction $N_н$. Thus, for instance, with normal laws of distribution N and $N_н$ coefficient f_1 establishes the required mutual location of their mean values $\bar{N}$ and $\bar{N}_н$ (Fig. 15.1) at known coefficients of variation

$$w = \frac{\sigma_N}{\bar{N}} \quad \text{and} \quad w_н = \frac{\sigma_н}{\bar{N}_н} \tag{15.3}$$

and specified probability of failure of the construction

$$f_1 = \frac{\bar{N}_н}{\bar{N}}. \tag{15.4}$$

If the condition of failure of the construction $N_н - N \leqslant 0$ is presented in the form

$$(\overline{N_н - N}) - p\sigma = 0, \tag{15.5}$$

where

$$\sigma = [\sigma_N^2 + \sigma_н^2]^{\frac{1}{2}},$$

where p - the number of standards corresponding to specified probability of exceeding the limiting state, then the following equation can be obtained for appraisal of the minimum magnitude of coefficient f_1:

$$f_1^2(1-p^2w_{\text{н}}^2)-2f_1+(1-p^2w^2)=0. \tag{15.6}$$

Since f_1 should be larger than one,

$$\min f_1=\frac{1+\left[1-(1-p^2w_{\text{н}}^2)(1-p^2w^2)\right]^{\frac{1}{2}}}{1-p^2w_{\text{н}}^2}. \tag{15.7}$$

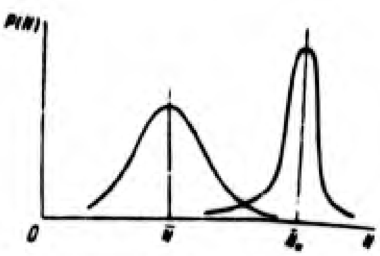


Fig. 15.1. Distribution curves of functions characterizing external load and bearing capacity of the construction.

Values of this coefficient will be different for various cases of loading and even for different junction points of the flight vehicle construction. It is obvious that generally the probability of failure P_{1m}, allowable for separate parts of the construction, should be less than the failure probability of the construction on the whole. Therefore, proceeding from the smallness of magnitudes of P_{1m}, it is possible to approximately take

$$P_{im}=\frac{P_i}{m_i},$$

where m_1 - number of junction points of the construction, for which the considered i-th case of loading is design, and P_1 failure probability of the vehicle construction on the whole in the same case of loading, determined from condition

$$P=\sum_{i=1}^{n} P_i,$$

in which n - overall number of calculated cases for the given construction, and P - allowable (specified) failure probability of the flight vehicle in the process of operation.

A necessary condition of application of methods of the theory of probability and mathematical statistics to appraisal of values of coefficient f_1 for each design case of loading is the presence of a sufficiently large amount of corresponding experimental data for determination of coefficients of variation w and w_H, obtained in uniform conditions. In the absence of such data they can be used only for qualitative analysis of certain results.

With determination of coefficient f_1 for reusable flight vehicles in certain cases of loading one should take into account the influence of such factor as the possible accumulation of damages in the process of prolonged operation.

The second coefficient f_2 characterizes deviation of calculated mean values of $\bar{N}$ and $\bar{N}_H$ from true, caused by errors of mathematical description of the processes of loading and failure of the construction, and also inaccuracy of determination of coefficients of variation. It is obvious that its values can be found only experimentally, by means of comparison of corresponding calculated data with experimental. The presence of this coefficient explains the experimental way of establishing the safety factors and the absence of the possibility of solution of the problem of its selection by the purely theoretical method. If the values of coefficient f_1 is always greater than one, the magnitude of coefficient f_2 in an ideal case can be equal to one.

The question of selection of safety factor is connected with reliability of the construction (only in rare cases is the latter ensured by means of duplicating the separate supporting members). Therefore, it is natural that it should be different for various cases of loading and should depend on the sequence of their realization. Formula (7) can be used for compariative appraisal of the effect of the magnitude of coefficient f_1 (with specified values of w_H and w) on the failure probability of the construction.

Since coefficient f_2 is introduced for covering the possible systematic deviations of actual values of external loads, the real temperature rate of operation of structural elements and actual values of critical (limiting) stresses from design, then it is obvious that any normalization of its magnitude should presuppose the use of fully defined and checked methods of calculation. Loads can be represented in the form of mean, and nominal, and limiting values. Therefore, the magnitudes of safety factors for every case of loading must be determined in accordance with the accepted method of calculation of the required bearing capacity of the flight vehicle construction. Without indication of such a method it is impossible to judge the degree of reliability of the construction by the magnitude of safety factor. Usually the normalized values of these coefficients are changed [11, 17] during calculation by the method of limiting states from 1.5 to 2 for different ground cases of loading from 1.4 to 2 - for flight cases of loading. In this case there often is allowed for the necessity of their increase (by approximately 25%) for especially critical junction points (in particular, for points of attachment of loads, joints of compartments and so forth) and for junction points, the technology of production of which allows larger variance of bearing capacity than for the majority of supporting members. Most often this is connected with unequal variance of strength characteristics of applied structural materials.

The application of comparatively exact methods of calculation thorough investigation of the operating conditions of all structural elements of the flight vehicle in all cases of loading, comprehensive experimental check of the actual bearing capacity and rigidity of the construction, realization of all measures, excluding the possibility of appearance of unforseen cases of loading, are necessary conditions for the application of small coefficients f_2. For all practical purposes this is realizable only with a high overall culture of designing and production or with creation of modification of flight vehicles on the basis of developed models.

During sketch designing of new flight vehicles it is usually necessary to use a priori values of safety factors. At the latter there are taken coefficients substantiated by the practice of designing of similar types of flight vehicles. In this case it is recommended to apply the upper values of these coefficients. Small magnitudes of f not only require high accuracy of corresponding initial data and perfected methods of calculation, but for all practical purposes often do not give the proper effect. This is explained by the fact that comparatively small changes of initial data on distribution of masses, rigidity, aerodynamic forces, and also variations of parameters of different flight vehicle systems, which are inevitable when developing a new construction, can (when $f = f_{min}$) subsequently lead to continuous further improvements of the construction (or other systems, for instance the control system) or to a lowering of the reliability of the flight vehicle.

Frequently the excess strength of the construction is characterized by so-called safety factor of strength $n_{\text{з}}$, constituting the ratio of the actual bearing capacity of the construction to required.[1] This coefficient in contrast to safety factors determines namely the reserve of strength, with which the designer can have at his disposal. Its actual values, equal to the ratio of actual bearing capacity to actual load, divided by coefficient $n_0 = 1.05$

[1]Sometimes the coefficient of excess strength $n_{\text{н}} = n_{\text{з}} - 1$ is also applied

to 1.1 are random variables, which should always be considered especially during analysis and treatment of results of natural experiments.

In conclusion let us note that one of the problems of dynamic calculation of the construction of a flight vehicle, closely connected with the question considered in this paragraph, is the establishment of allowances for possible variance of those parameters which directly or indirectly affect the required and actual bearing capacity of the construction. Its solution is usually reduced to calculation and comparative analysis of the limiting values of external perturbations and internal forces, realization probability of which is in specified limits. Such limit loads in some cases of loading can be used for appraisal of the required bearing capacity of the construction. They can be obtained either by corresponding calculations by means of introduction of correction factors (variance coefficients) for each component of internal force and external load. The application of the method of limit loads leads to lowering of magnitudes of coefficient f, and in the case of simultaneous use of limiting values of bearing capacity the necessity of its introduction in essence, disappears. Selection of safety factors for additional cases of loading, corresponding to an emergency condition of the piloted flight vehicle, is performed depending upon requirements imposed on strength of structural elements on the part of the crew emergency rescue system. Usually these requirements lead to the guarantee of safe ejection of the seat or cabin with the pilot.

§ 15.3. Methods of Finding Design Cases of Loading

With establishment of design cases of loading it is necessary to consider both the character of the state of stress of the construction and the external conditions of its operation, which directly affect the strength, i.e., the ability of the construction to absorb, while not being destroyed, specified external loads. There should basically be remembered the effect of magnitude and speed of heating on the mechanical and elastic properties of the material of the construction, which determine the values of allowable stresses or its bearing capacity and rigidity, particularly yield points and strength and elastic modulus.

The degree of heating of the construction of flight vehicle is determined by the speed of their motion in dense layers of the atmosphere, aerodynamic configuration, thickness of body skin and certain characteristics of its material, namely specific heat c and specific gravity γ. The larger that c and γ are, the greater and more rapidly the material of the body shell is heated. For ballistic flight vehicles type, made from aluminum alloys, the heating rate composes 2-3°C per second, and in separate cases (for thin shells) it can reach up to 5°C per second and more.

It is possible to indicate several methods of revealing design cases of loading, allowing consideration of the loading features of particular constructions. The heaviest operating conditions of the separate parts and sections of flight vehicles can be determined by maximum values of external load or internal force.

The first method (method of maximum load) is applied when the considered structural element is loaded by some external force only (with constant temperature conditions), for instance by only external or internal pressure, only thrust force and so forth.

The essence of the second method (method of determining or dominating load) consists of the fact that as design there is considered the state of stress of the construction (at constant temperature rate of operation) from the combined influence of all external loads at these moments of operation when one of the components of these loads (or internal forces) reaches its biggest value. This method in principle scarcely differs from the method of maximum load, and therefore gives good results only when all components of the external loads, expect those considered remain constant.

Generally with the combined action of arbitrarily variable loads and with unsteady temperature operating conditions of the construction it is expedient to use an improved method, which we will call the method of conditional load. The essence of this method is that as design we consider the case of loading of the construction at the moment of time of flight when some fictitious (conditional) load reaches its biggest value. Magnitude of the latter is determined in such a way that we considered both the effect of the combined influence of external forces and the effect of the relationship of the bearing capacity of the construction to its temperature.

The effect of the combined action of longitudinal force, bending moment and excess pressure on the body shell of the flight vehicle can be approximately estimated by the magnitude of reduced force. For finding this force it is possible to use the theory of the greatest tangential stresses, which gives the simplest criterion of danger of combined state of stress for plastic materials the most frequently applied during manufacture of bodies of flight vehicles. Although Coulomb-St. Venant conditions are not criteria of failure of the construction, but determine only the moment of transition of the material of shell from elastic to plastic state, they can serve as the basis for comparative appraisal of danger of plane state of stress. According to this condition the more severe case of loading will be that which corresponds the greatest amount of reduced stress

$$-\sigma_{э}(x_1) = \sigma_{max}(x_1) - \sigma_{min}(x_1), \tag{15.8}$$

where σ_{max} - tensile stress, and σ_{min} - compression stress.

Although the validity of this theory is proven only for some constant temperature operating conditions of the material, it can be used with sufficient accuracy in the case of the short-duration influence of heating. It is possible, of course, to use more exact formulas for calculation of reduced stresses. However, considering that the shown reduced stress is used here not for establishment of the magnitude of strength margin of the construction, but only for revealing the design case of loading, it is doubtful whether the application of complicated formulas is expedient. While complicating the calculation, they actually do not lead to an essential refinement of the position of the calculated case itself.

For all practical purposes it is more convenient to use not reduced stress (8), but reduced or <u>equivalent</u> longitudinal force, equal to the product of this stress by area of cross section of the supporting part of the vehicle construction

$$N_{э}^{\sigma}(x_1) = \sigma_{э}(x_1) F_c(x_1). \tag{15.9}$$

by means of corresponding correction of the value of this reduced longitudinal load it is possible to approximately consider the effect of change of allowable stresses or bearing capacity of the construction in time caused by change of its temperature operating conditions,

$$N_v^{\sigma}(x_1) = k_t(x_1, t^0) N_{э}^{\sigma}(x_1),$$

where $k_t(x_1, t^0)$ - some correction factor. Since the temperature affects elastic constants of material and its mechanical properties differently, then during a similar relative appraisal of the danger of the state of stress one should consider the possible character of loss of bearing capacity of the construction. Thus, for instance, if failure occurs due to disturbance of the strength of material, then the shown correction factor can be presented in the form of relationship of values of ultimate strength, corresponding to normal (t_0^0) and elevated temperature (t^0).

Bearing in mind that the skin temperature of the body depends not only on motion parameters of the flight vehicle (speed and altitude), boundary layer characteristics and specific heat and specific gravity of the material, but also on the time of flight, into this relationship one should introduce an additional correction factor, considering the effect of the history of heating of the considered part of the construction. In certain cases it is necessary to consider the relationship of limiting stress ($\sigma_{в}$ or σ_s) to the speed of change of stress $\sigma_э(x_1)$, especially in the region of high temperatures.

Consequently, with failure of the construction because of exceeding the ultimate strength (or yield point) of material this correction factor will have the form

$$k_{\sigma_{в}}(x_1, t^0) = k_v(\bar{t})\,\frac{\sigma_{в}(x_1, t_0^0)}{\sigma_{в}(x_1, t^0)}$$

or

$$k_{\sigma}(x_1, t^0) = k_v(\bar{t})\,\frac{\sigma_s(x_1, t_0^0)}{\sigma_s(x_1, t_0)}, \qquad (15.10)$$

where $k_v(\bar{t})$ - coefficient considering the effect of the speed of heating and speed of loading of the construction on the magnitude of limiting stress.

If failure of the structural element occurs due to overall or local loss of equilibrium stability, then a similar correction factor will be determined by the relationship of corresponding critical stresses (at normal and operating temperatures)

$$k_E(x_1, t^0) = k_v(\bar{t})\,\frac{\sigma_{кр}(x_1, t_0^0)}{\sigma_{кр}(x_1, t^0)}. \qquad (15.11)$$

In this case the magnitude of critical stress of local loss of stability of shells reinforced by stringers will be affected not only by change of elastic modulus E with respect to t^0, but also by change of mechanical properties of the material of stringers

σ_s, σ_p. Thus, in examining the limiting state of the construction with respect to strength, conditional load will be determined by expression

$$N_y^{\sigma}(x_1, \bar{t}) = k_i\left[x_1, t^0(\bar{t})\right] N_y^{\sigma}(x_1, \bar{t}) \quad (i = \sigma, \sigma_s), \tag{15.12}$$

and with respect to stability - by expression

$$N^E(x_1, \bar{t}) = k_E\left[x_1, t^0(\bar{t})\right] N_y^E(x_1, \bar{t}) \tag{15.12'}$$

Thus, during appraisal of the strength of construction by the method of allowable stresses the design case of loading will be revealed by maximum value of conditional load (12) or (12'). With use of the method of limiting states the design case of loading will be that case or the moment of operation of flight vehicle (with imposition of several cases of loading), in which there is observed maximum value of design conditional load

$$N_{yp}^{\sigma}(x_1, \bar{t}) = fN_y^{\sigma}(x_1, \bar{t}) \tag{15.13}$$

or

$$N_{yp}^E(x_1, \bar{t}) = fN_y^E(x_1, \bar{t}). \tag{15.13'}$$

For illustration of the effect of operating conditions of the construction on the position of design case of loading, Fig. 15.2 shows graphs of change of equivalent and conditional loads with respect to time of flight for one section of the carrier fuel tank of hypothetical single-stage ballistic type flight vehicle. Point PC marks the design case, corresponding to max N_y^E. As can be seen, its position does not coincide with the biggest value of equivalent longitudinal load $N_э$, not yet indicating the coincidence of separate components of forces N, $N_н$, ... with maximum values.

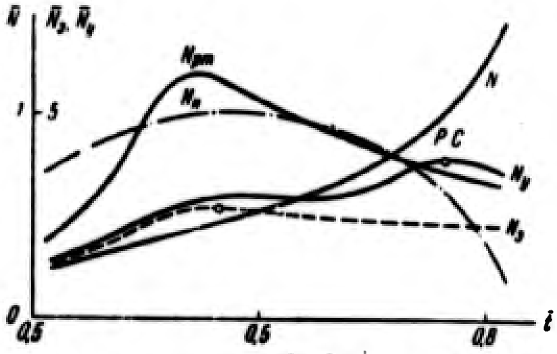


Fig. 15.2. Concerning the question of establishment of design case of loading for carrier fuel tank.

For finding the design cases by the method of conditional load, besides diagrams of internal forces and values of external and internal pressures, in all sections of the flight vehicle body for all cases of loading and in principle for all points of the trajectory it is necessary to have:

1) results of calculation of temperature operating conditions for all supporting elements of the construction;

2) experimental data on the relationship of σ_s, $\sigma_в$, E to temperature for different speeds of heating and loading.

On the basis of the shown data we computed conditional forces N_y^{σ}, $N_y^{\sigma в}$ and N_y^{E} for all designated sections and for all cases of loading, and found their biggest values. The place of location of calculated sections is determined by the construction of sections, and their number - by required accuracy of construction of diagrams of longitudinal and thrust forces and bending moments.

For many parts of the construction of flight vehicle on the basis of accumulated experience it is possible to separate a certain limited number of cases of loading, containing the sought design case. It is possible to thereby considerably reduce (especially at the sketch designing stage) the volume of required calculations, since the design case will be revealed from comparison of values of conditional loads, calculated for only a few cases of loading. A list of similar design cases of loading for the main part of the construction of ballistic type flight vehicles is presented in the following chapter.

§ 15.4. Equivalent Longitudinal Force for a Reinforced Cylindrical Shell

Let us consider a case when a thin cylindrical shell, reinforced by longitudinal and transverse bracing, is under the action of compression force $N_1(x)$ and internal excess pressure p(x).

Since critical stress of longitudinal bending of reinforced panels increases with decrease of the distance between stringers, then we usually try to place a large number of stringers of comparatively small section. In a similar case the effect of stringers on the magnitude of longitudinal compression stress in a shell with thickness δ can be considered with sufficient accuracy by means of introduction of additional reduced thickness of the shell, equal to $\delta_c = S_c/b_c$, where S_c - area of cross section of one stringer, and b_c - distance between stringers along the circumference. In other words, the relationship of linear longitudinal force to longitudinal deformation of such a shell, loaded relative to the axis, can be represented in the form

$$T_1 = \frac{E\delta}{1-\mu^2}\left(\alpha\frac{du}{dx} - \mu\frac{w}{R}\right) = \frac{N_1}{2\pi R}, \tag{15.14}$$

where

$$\alpha = 1 + \frac{E_c\delta_c}{E\delta}(1-\mu^2),$$

E, E_c - moduli of longitudinal elasticity of the shell and stringer, μ - Poisson's ratio, u, w, R - longitudinal and radial displacement and radius of the middle surface of shell respectively (Fig. 15.3).

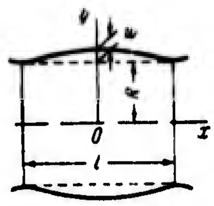


Fig. 15.3. Auxiliary system of coordinates for a reinforced shell.

The effect of a lateral reinforcing assembly on annular stresses in the shell cannot be considered in such a way, since usually the frames are installed comparatively farther apart. Therefore, it is necessary to preliminarily calculate radial deformation of the shell. As it is known [72], the value of linear annular force is determined by expression

$$T_2 = \frac{E\delta}{1-\mu^2}\left(\frac{w}{R} - \mu\frac{du}{dx}\right). \qquad (15.15)$$

Having excluded du/dx, from formulas (14) and (15) we obtain

$$T_2 = \frac{wE\delta}{R(1-\mu^2)}\left(1 - \frac{\mu^2}{\alpha}\right) - \frac{\mu}{\alpha}T_1. \qquad (15.16)$$

Having taken into account relationship

$$\frac{dQ_1}{dx} - \frac{T_2}{R} + p = 0, \qquad (15.17)$$

where by Q_1 there is designated linear lateral force, equal to

$$Q_1 = -\frac{dM}{dx} = -D\frac{d^3w}{dx^3},$$

We find the following equation for determination of radial displacement of the shell between two frames located side by side:

$$\frac{d^4w}{dx^4} + 4k^4w = \frac{p}{D}\beta, \tag{15.18}$$

where

$$4k^4 = \frac{E\delta}{1-\mu^2}\left(1 - \frac{\mu^2}{\alpha}\right)\frac{1}{DR^2}, \tag{15.19}$$

$$\beta = 1 + \frac{\mu T_1}{\alpha pR} = 1 + \frac{\mu N_1}{2\alpha N_p},$$
$$N_p = \pi pR^2, \tag{15.20}$$

D - flexural rigidity of shell, and M - linear bending moment.

With a comparatively small distance between frames l the excess pressure of liquid p on the shell (on length l) in the first approximation can be considered constant. If this distance is great, then the influence of frames on annular stresses in the shell will carry basically a local character, i.e., will be noticeable only near the frames themselves. In such a case the annular force in the basic shell can be calculated with sufficient accuracy by formula

$$T_2 = \frac{N_{p\kappa}}{2\pi R} = pR. \tag{15.21}$$

When p = const the solution of equation (18) will have the form

$$w = w_0 + C_1 \cos kx \operatorname{ch} kx + C_2 \sin kx \operatorname{sh} kx, \tag{15.22}$$

where

$$w_0 = \frac{p\beta}{4k^4D},$$

and x - abscissa of the considered cross section of the shell in the auxiliary system of coordinates shown on Fig. 15.3. Arbitrary constants C_1 and C_2 are determined from boundary conditions when $x = l/2$, and namely from condition of equilibrium of radial forces affecting the frame with tensile forces in the cross section of frame

$$S_ш E_ш \frac{w}{R} = -2RQ_1 = 2RD\frac{d^3 w}{dx^3} \tag{15.23}$$

and from condition of equality of the derivative of deflection curve of the shell to zero

$$\frac{dw}{dx} = 0 \quad \text{when} \quad x = \frac{l}{2}. \tag{15.24}$$

The remaining two arbitrary constants are equal to zero by virtue of symmetry of the system relative to the origin of coordinates. Having placed the expression for w (22) in formulas (23) and (24) we obtain the following algebraic equations for computing the sought constants:

$$\begin{gathered}
(C_1 + C_2)\cos\frac{kl}{2}\,\mathrm{sh}\,\frac{kl}{2} + (C_2 - C_1)\sin\frac{kl}{2}\,\mathrm{ch}\,\frac{kl}{2} = 0, \\
2k^3\left[(C_2 - C_1)\cos\frac{kl}{2}\,\mathrm{sh}\,\frac{kl}{2} - (C_1 + C_2)\sin\frac{kl}{2}\,\mathrm{ch}\,\frac{kl}{2}\right] = \\
= \frac{S_ш E_ш}{2R^3 D}\left[C_1\cos\frac{kl}{2}\,\mathrm{ch}\,\frac{kl}{2} + C_2\sin\frac{kl}{2}\,\mathrm{sh}\,\frac{kl}{2} + w_0\right].
\end{gathered}$$

From these equations it follows that

$$C_1 = -w_0 B_1, \quad C_2 = w_0 B_2, \tag{15.25}$$

where

$$\begin{gathered}
B_1 = \frac{1}{A}\left(\sin\frac{kl}{2}\,\mathrm{ch}\,\frac{kl}{2} + \cos\frac{kl}{2}\,\mathrm{sh}\,\frac{kl}{2}\right), \\
B_2 = -\frac{1}{A}\left(\sin\frac{kl}{2}\,\mathrm{ch}\,\frac{kl}{2} - \cos\frac{kl}{2}\,\mathrm{sh}\,\frac{kl}{2}\right), \\
A = \frac{1}{2}(\sin kl + \mathrm{sh}\,kl) + 2\cdot\frac{E\delta}{E_ш S_ш}\left(1 - \frac{\mu^2}{\alpha}\right)\frac{\left(\mathrm{sh}^2\frac{kl}{2} + \sin^2\frac{kl}{2}\right)}{k(1-\mu^2)}.
\end{gathered} \tag{15.25'}$$

Thus, on the basis of formulas (14), (16), (21), (22) and (25) annular force $N_{рк}$ for a reinforced shell, being under internal pressure, for instance for carrier cylindrical tank, will be equal to

$$N_{рк} = 2\pi p\beta R^2(1 - B_1\cos kx\,\mathrm{ch}\,kx + B_2\sin kx\,\mathrm{sh}\,kx) - \frac{\mu}{\alpha}N_1.$$

It is easy to see that the amount of this force will be maximum between frames, i.e., when x = 0,

$$N_{рк} = 2\pi p\beta R^2(1 - B_1) - \frac{\mu}{\alpha}N_1. \qquad (15.26)$$

With the presence of meridional compression stresses in the shell $(N_1 > 0)$, the equivalent longitudinal force is determined by expression

$$N^{\sigma} = N_{рк} + N_1. \qquad (15.27)$$

Since

$$N_1 = N - N_m, \quad \text{and} \quad p = p_н(1 + \bar{p}_н). \qquad (15.28)$$

then, by using formulas (26) and (27), it is possible to represent N^{σ} in the form

$$N^{\sigma} = N\left(1 - \frac{\mu}{\alpha}\right) + N_m\left[2\beta(1 - B_1)(1 + \bar{p}_н) - \left(1 - \frac{\mu}{\alpha}\right)\right], \qquad (15.29)$$

where

$$N_m = \pi p_н R^2, \quad \bar{p}_н = \frac{p_и}{p_н},$$

and N - longitudinal force from external forces, $p_н$ - gas boost pressure (here excess), $p_и$ - inertial pressure of liquid. From (20) it follows that

$$\beta = 1 + \frac{\mu}{2\alpha}\left(\frac{N}{N_m} - 1\right)\frac{1}{(1+\bar{p}_м)}. \qquad (15.30)$$

Having placed this value of β in (29), we finally will have

$$N^{\sigma} = N\left(1 - \frac{\mu}{\alpha}B_1\right) + N_m\left[1 + 2\bar{p}_м - B_1\left(1 + 2\bar{p}_м - \frac{\mu}{\alpha}\right)\right]. \qquad (15.31)$$

For a stringer-free tank $\alpha = 1$, and

$$D = \frac{E\delta^3}{12(1-\mu^2)} \text{ and } k = \sqrt[4]{\frac{3(1-\mu^2)}{R^2\delta^2}}.$$

The effect of frames on N^{σ} is considered by coefficient B_1 (25'). It is essentially only when the shell is comparatively thin, frames are powerful and are arranged comparatively close together. The larger $S_ш$ and the smaller δ, the smaller is A and the larger is B_1. For illustration Fig. 15.4 contains graphs of B_1 as a function of kl for different values of parameter

$$a_0 = \frac{R^2\delta^3}{S_ш l^3}.$$

It is clear that even when $kl \geq 4$ B_1 hardly differs from zero. In this case (when $\alpha = 1$) approximately

$$N^{\sigma} = N + N_m(1 + 2p_м). \qquad (15.32)$$

Increase of the inertial pressure of liquid $\bar{p}_м$ on walls of the tank leads to increase of force N^{σ}. The greatest amount of this force (when taking into account the effect of temperature of the shell of conditional force $N_y^{\sigma} = N^{\sigma}k_{\sigma}$) will determine the design case (during calculation for strength). When determining the design case, corresponding to loss of longitudinal stability of the shell of carrier cylindrical tank, partially or completely filled with liquid, one should consider the relationship of critical stresses for this shell to the magnitude of internal excess pressure p.

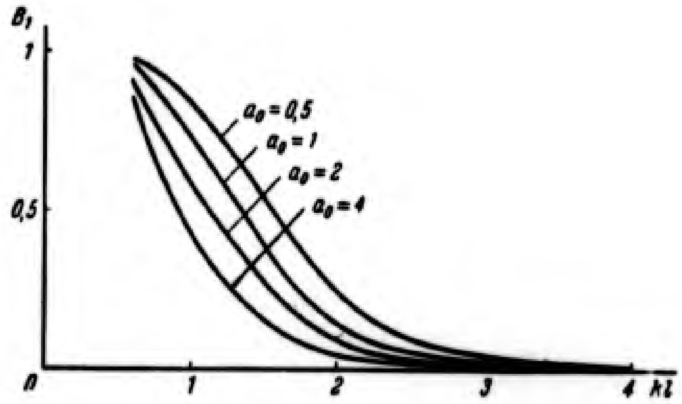


Fig. 15.4. Relationship of coefficient B_1 to kl and to parameter a_0.

Increase of boost pressure in the tanks increases not only the tensile component of longitudinal force N_m, but affects amounts of critical forces of longitudinal loss of stability of the shell $N_{кр}$. The inertial pressure of liquid on walls of the tank does not affect the magnitude of N_m, but just as boost pressure, increases $N_{кр}$. Since inertial pressure is changed in the process of motion of the flight vehicle, due to the relationship of $N_{кр}$ to p in a number of cases it can turn out that the largest value of equivalent (or conditional) longitudinal load does not determine the design case of loading.

As it is known, critical stress of longitudinal bending of an unreinforced shell and critical longitudinal force corresponding to it are determined by formulas

$$\sigma_{кр} = k_0 E \frac{\delta}{R}, \quad N_{кр} = 2\pi k_0 E \delta^2, \tag{15.33}$$

where k_0 - coefficient, depending on the precision of manufacture of the shell and magnitude of internal excess pressure. There are works in which this relationship of k_0 to p is given various magnitudes of initial roughness of the shell. However, the practical value of similar graphs is limited, since it is difficult to quantitatively estimate the possible irregularities of the shell of the body, which carry a random character beforehand (in process of designing). This random character of local initial distortions of shells explains the great variance of values of coefficient k_0, which is observed during experimental determination of longitudinal critical forces. From this it follows that the relationship of $N_{кр}$ to p can be described only by statistical methods [9]. With establishment of design cases of loading it is better to use values of coefficient k_0, obtained experimentally not on models, but on natural samples, if only when p = 0. Such experimental data will somehow reflect the quality of manufacture of shells at a given enterprise (with accepted technology). For illustration of the degree of effect of internal excess pressure on critical stress, Fig. 15.5 contains the approximate relationship of relation

$$k_p = \frac{k_0(p \neq 0)}{k_0(p = 0)} \tag{15.34}$$

to parameter $p/E\,(R/\delta)^2$ for cylindrical unreinforced shell at several values of coefficient k(p = 0) [90]. This relation shows how many times critical force $N_{кр}$ is increased with increase of excess pressure p. From inequality

$$N_1 f \leqslant N^0_{кр} k_p, \tag{15.35}$$

where $N^0_{кр}$ - longitudinal critical load, corresponding to p = 0, it follows that the moment of loading when function N_1/k_p, and not N_1 reaches the biggest value will be dangerous for the considered type of construction. In other words, during investigation of the stability of carrier fuel tank under longitudinal compression as calculated case we should take the one, corresponding to the biggest value of conditional load

$$\max N_y^E = \max (N - N_m) \frac{k_E}{k_p}. \tag{15.36}$$

In this case the effect of the bending moment can be approximately considered by means of introducing equivalent longitudinal force $N_\text{и}(x_1)$, equal to the product of area of cross section of the carrier part of the tank $F_c(x_1)$ by the greatest compression stress from bending of the construction into the expression of longitudinal load N. For a shell unreinforced in longitudinal direction

$$N_\text{и}(x_1) = M(x_1) \frac{2k_\text{и}}{a(x_1)}, \tag{15.37}$$

and for reinforced

$$N_\text{и}(x_1) = M(x_1) \frac{a(x_1) F_c(x_1) k_\text{и}}{J_c(x_1)}, \tag{15.38}$$

where $k_\text{и}$ - correction factor, considering the difference of critical stresses, corresponding to pure bending and uniform longitudinal compression.

In conclusion let us note that the calculated case for a shell, being under the action of longitudinal compression force and external excess pressure, will be determined by maximum value of expression

$$F(x_1) = \frac{f N_1(x_1)}{N_\text{кр}^0(x_1)} + \frac{f \Delta p^\text{и}(x_1)}{p_\text{кр}^\text{и}(x_1)},$$

which, according to conditions of stability, remains ≤ 1. In this case the magnitude of critical external pressure $p^{H}_{\kappa p}$ is found without taking into account the effect of force N_1.

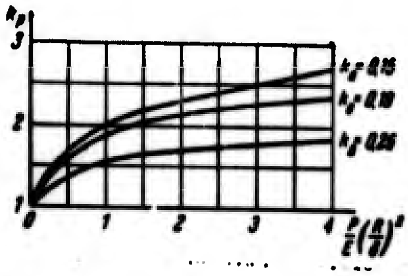


Fig. 15.5. Approximate effect of internal excess pressure on the relative critical stress of longitudinal bending of a cylindrical shell.

§ 15.5. Limitations of Operating Conditions of Different Systems According to Conditions of Strength

Selection of optimum values of dynamic characteristics of all systems: propulsion system, control system, ground equipment units and the construction of the flight vehicle itself, considered combined, is closely connected with the question of creation of rational construction of a flight vehicle, possessing minimum weight and required technical and economic indices. Usually the construction is considered rationally designed when it is of uniform strength, i.e., has identical safety factors of strength in all sections and for most cases of loading. It is obvious that this condition is necessary, but insufficient, since the weight of the construction essentially depends on the assembly diagram of the flight vehicle.

A comparative appraisal of the quality of design of separate parts of the flight vehicle construction can be obtained by analyzing the relationship of actual bearing capacity (N_H) to the weight per unit length of the supporting part of the construction

$$\varkappa(x_1) = \frac{N_H(x_1)}{q_0(x_1)}. \tag{15.39}$$

The shown coefficient - load factor of the construction considers the peculiarities of arrangement of separate sections, the influence of temperature operating conditions on the mechanical and elastic properties of utilized materials and the uniform strength of the given construction, since all these factors in the final analysis find their reflection in weight of the construction. The load factors will be high for those constructions which are made from materials possessing good thermal characteristics and high specific strength σ_{np}/γ (where γ - specific gravity of material, and σ_{np} - limiting stress), and for which the safety factors of the majority of elements are close to one.

It is obvious that a radical means of reducing weight of the construction is lowering of design loads. In many cases this lowering can be attained by means of establishment of definite limitations on those parameters and operating conditions of the above-mentioned systems, which directly affect the character of loading of the flight vehicle in the process of operation.

The demand for establishment of limitations on conditions of ground operation and operating conditions of the control system, on transient processes of propulsion systems, on stability of systems vehicle-automatic stabilization control, vehicle-propulsion system (with fuel feed system), on values of parameters of the trajectory and wind conditions already appears at the sketch designing stage for many flight vehicles.

Usually design conditions of loading, determining the required bearing capacity of the considered element, is observed only on some limited phase of the flight trajectory of the vehicle or in some definite case of loading. Consequently, only in this design case of loading the safety factor of strength of the construction will be equal to a specified value. On other phases of the trajectory and in all the remaining cases of loading the safety factor of strength will exceed the given value. In other words, in all uncalculated cases of loading the construction will possess excess bearing capacity. Thus, in uncalculated cases of loading there is the possibility of facilitating the operating conditions of the mentioned systems by expansion of allowable limits of change of their parameters, proceeding from conditions of uniform strength of the construction.

A necessary prerequisite of determination of minimum necessary limitations is knowledge of the actual safety factors of the construction on all cases of loading. With use of a beam diagram of calculation of the construction for strength the required information about these safety factors can be obtained by means of comparison of the corresponding values of longitudinal conditional loads. The difference of conditional forces

$$\Delta N_y(x_1) = N_y^p(x_1) - N_y(x_1) \tag{15.40}$$

(where N_y^p - value of conditional force for calculated case of loading, and N_y - value of conditional force for the considered case of loading) determines the excess strength (bearing stability) of the construction, which is available in the considered case of loading. By knowing ΔN_y, it is simple to find the permissible deviation of longitudinal forces, bending moments, longitudinal and lateral overloads and pressures in fuel tanks from the computed values accepted for permissible conditions in operation.

Such an approach to flight vehicle designing gives the possibility of rationally solving many questions of dynamics of the construction, in particular questions connected with the dynamic stability of motion of the flight vehicle and stability of system body - propulsion system, with selection of the scheme of stage separation and parameters of boost phase of flight, descent and landing phase, selection of required rigidity of launcher, with establishment of permissible values of longitudinal and lateral overloads for different ground cases of loading.

With such formulation all these problems are solved by the methods presented in Sections III and IV. As an example let us consider the question of selection of gas boost pressure for carrier fuel tanks of a ballistic type flight vehicle. As already was noted, for guarantee of cavitation-free operation of pumps for supplying liquid fuel into the combustion chamber of the engine, the boost pressure in the tanks should satisfy condition $p_{нj} \geq p_{дj}$. However, considering the influence of this pressure on the bearing capacity of the construction, one should introduce an additional limitation on its magnitude. Minimum value of $p_{нj}$ should satisfy the condition of preservation of stability of the shape of the shell of the tank on external excess pressure, i.e., condition

$$p_{нj} \geqslant p^{н}. \qquad (5.41)$$

For underslung tanks the magnitude of pressure $p^{н}$ will be determined by air pressure air inside the fuel compartment (in which tanks are placed), for carrier tanks - the magnitude of external aerodynamic pressure in the considered section of the body. Maximum value of $p_{нj}$ is limited by strength conditions of the tank shell. With plastic state of material

$$\left[p_{нj}(t) + 2p_{нj}(x_1, t) - p^{н}(x_1, t) + \frac{N_э(x_1, t)}{\pi a_j^2(x_1)}\right] \leqslant 2\frac{\delta_c(x_1)}{f a_j(x_1)}\sigma_s(x_1, t), \qquad (15.42)$$

where $\sigma_s(x_1, t)$ - yield point of material, considered as a function of shell temperature, and $\delta_c(x_1)$ - thickness of shell. For carrier tanks it is expedient to select $p_{нj}$ so that condition of preservation of local stability of the tank shell is satisfied in the case of action of longitudinal compression forces and bending moments

$$p_{нj}(t) \geqslant p^{н}(x_1, t) + \frac{N_э(x_1, t)}{\pi a_j^2(x_1)} - \frac{2\delta_c(x_1)}{f a_j(x_1)}\sigma_{кр}(x_1, t), \qquad (15.43)$$

where $\sigma_{кр}/f$ - permissible compression stress of the shell.

By equating the left sides of expressions (42) and (43) to the right, we obtain a system of equations for determination of optimum (according to condition of uniform strength of the construction) values of δ_c and p_{HJ}.

An example of weakening of limitations can be the following case. If the dynamic motion stability of a flight vehicle, equipped with automatic stabilization control (automatic pilot), cannot be provided under all conditions of flight or simultaneously at frequencies of natural oscillations of liquid in the fuel tanks and bending oscillations of the construction, then it is possible to set certain limitations on the amplitudes of these oscillations. Usually similar limitations are the strictest for the system elastic body of the vehicle-automatic stabilization control. Allowable magnitudes of the amplitudes of steady-state oscillations of such a system (at the point of reduction) are determined for every frequency ω, close to ω_n, and at every point of the flight trajectory with minimum value (according to length of body of the flight vehicle)

$$S_{nm}=a(x_1)\frac{N_{\text{н}}(x_1)-N_y(x_1)f}{2\omega_n^2M_{nx}(x_1)f}. \tag{15.44}$$

In this case the allowable amplitudes of oscillations of lateral overload in any section x_1 will be determined by formula

$$\Delta n_{y_1m}(x_1)=\frac{1}{g_0}\omega_n^2f_n(x_1)\min S_{nm}. \tag{15.45}$$

With simultaneous excitation of two tones of elastic oscillations of the construction (for instance, n = 1 and n = 2) their amplitudes are limited by minimum value of expression

$$S_{1m}=a(x_1)\frac{N_{\text{н}}(x_1)-N_y(x_1)f}{2f\left[\omega_1^2M_{1x}(x_1)+\omega_2^2M_{2x}(x_1)\frac{S_{2m}}{S_{1m}}\right]} \tag{15.46}$$

at specified relationship S_{2m}/S_{1m}.

Basically all limitations and recommendations for the selection of values of certain parameters of different flight vehicle systems introduced according to strength conditions, are obtained as a result of dynamic calculation of its construction at corresponding cases of loading by the method presented in Sections III and IV. They can be established only for a particular vehicle, proceeding from specific features of its construction and operating conditions. Therefore, here it is possible to bring forth only a list of such limitations. Thus, for instance, requirements imposed on parameters of powered flight of the flight vehicle can be reduced to limitation of maximum values of longitudinal and lateral overloads, impact pressures, programmed angles of attack, and also to limitation of the magnitude of these parameters on separate sections of the trajectory (for instance, in the region of maximum impact pressures, transonic speeds, in the process of stage separation and so forth). The shown type of limitations can be realized both by change of the flight program and by control of thrust force (on separate sections of the trajectory). If the possibilities of change of flight speed and altitude of the flight vehicle are small, then in a number of cases it is possible to go to limitation of operating conditions of the vehicle, in particular to limitation of average wind rates. For all practical purposes this means that before every flight of the vehicle it is necessary to analyze the picture of distribution of average wind rates along the altitude in the launch region or along the route. For flight vehicles, the region of operation of which is limited, calculated mean wind rates along the altitude can be established in accordance with local meteorological data. Some limitation on parameters of atmospheric turbulence or wind gusts should not be introduced, since these processes a little controllable and forecastable.

§ 15.6. Experimental Methods of Solution of Problems of Structural Dynamics of a Flight Vehicle

Many problems of dynamics and strength of the construction of a flight vehicle are solved experimentally. In this case the more complicated the problem, the greater the value given to these methods of investigation. They are widely used both in the process of structure designing and in the period of its manufacture and flight development.

There are several types of experimental development of dynamic and static strength of flight vehicle structures. These are, first of all, laboratory tests of structurally and dynamically similar models of vehicles and bench tests of natural junction points, compartments and the entire construction of the whole under conditions close to real, and, secondly, flight tests.

Bench (hot and cold) and laboratory tests are used for development of:

- method of calculation of loads, dynamic reaction of the construction and the bearing capacity of its parts,

- actual strength and rigidity of structural elements.

- determination of certain dynamic characteristics of the flight vehicle and reaction of its construction to an external influence, and also quality control of manufacture.

Flight tests are used basically for checking the correctness of establishment of operating conditions of the flight vehicle construction, selection of dynamic layout and design cases of loading, and also the validity of accepted safety factors.

Hot bench tests of the flight vehicle on the whole or its separate units (stages) are conducted for checking the operation of propulsion systems and fuel feed system and are one of the main preliminary sources of information about vibration conditions of the construction.

Cold bench tests are sometimes specially designed for checking the bearing capacity and rigidity of junction points of the construction. The basic difficulty of carrying out such tests consists of the necessity of sufficiently accurate simulation of the real scheme of loading and real operating conditions of the construction (magnitude and speed of heating of supporting elements of the construction and speed of change of external forces). Depending upon the realizable speed of loading, the shown tests are subdivided into static and dynamic. Usually during static tests the construction is loaded very slowly (continuously or in steps). Therefore, it is possible to use similar tests only when external loads are static. Replacement of dynamic tests by static with dynamic impact of external forces on the flight vehicle requires the preliminary reduction of the dynamic problem to static. The shown procedure in the final analysis leads to establishment of a certain conditional diagram of loading of the construction.

When the construction of a flight vehicle is of uniform strength and its bearing capacity is used completely in many cases of loading, the idea of calculated cases for static tests loses meaning. Under such conditions the static tests on only one of the cases of loading will only partially characterize the actual bearing capacity of the construction.

If various cases of loading are calculated for different sections of the body, then for all practical purposes it is inexpedient to conduct static tests of the flight vehicle construction on the whole. It is possible to determine its bearing capacity by parts (sections), simulating corresponding boundary conditions. If it is necessary to test the same compartment on several cases of loading, we are sometimes limited by tests only on the limit load. In this case we frequently deviate from the real scheme of loading, determining the bearing capacity of the construction with respect to equivalent or even conditional load. Similar cold static tests somehow characterize the bearing capacity of those structural elements which are not subjected to heating in the process of operation, and give only a general idea of the actual strength of those elements, for which the conditional longitudinal load differs from equivalent.

For revealing the real bearing capacity of the construction during bench tests it is necessary to simulate not only the actual diagram of loading, corresponding to the considered calculated case of loading, but also its actual operating conditions, and namely the magnitude and speed of heating of structural elements, and speed of change of external forces.

The nearness of simulation of real conditions of loading and random character of results of unit static tests essentially limit their value. Generally the results of the shown tests cannot be directly used for the characteristic of bearing capacity of the construction, particularly for determination of actual safety factors $n_з$ in the considered case of loading. It is necessary to correct them by means of introduction of some correction factor (analogous in structure to the safety factor), considering the noncorrespondence of loading diagrams of the construction during static tests and in the process of operation of the flight vehicle, and also possible variance of results of the experiment.

It is natural that this remark pertains to results of dynamic tests. It is necessary to note that setting up dynamic tests of natural constructions in connection with refinement of their dynamic characteristics in many respects is determined by the applied methods of solution of dynamic problems. In this case, when the dynamic layout of the flight vehicle is described by elasto-mass system, for all practical purposes the tests which give the possibility of obtaining values of generalized rigidity and damping factors of separate junction points and parts of the construction present the greatest interest. With use of the method of expansion of the solution in normal forms of natural oscillations, basically dynamic tests of the flight vehicle on the whole are of value.

Both static and dynamic tests of natural samples of flight vehicles carry basically a checking character, since they are conducted, as a rule, after completion of the design operations. In the process of development of the construction it is recommended to conduct static and vibration tests of separate junction points for revealing weak places and removal of calculation errors.

For solution of a number of problems of aeroelasticity and dynamic reaction of the construction it is expedient to apply dynamically similar models of different types. Structural design of these models and their scales are determined by requirements of the problem. They can be similar dynamically (with respect to mass and rigidity), aerodynamically (geometrically similar) and structurally similar.

Structurally similar flight vehicle models in a certain scale reproduce (from material having the same Poisson's ratio as the natural construction) all the basic supporting members of the constructions (shell, frames, stringers, mounting lugs of large loads and so forth). Therefore such models are used for solution of certain problems connected with establishment of the dynamic layout of the construction. It is obvious that conformity of results, obtained by tests of models, to natural in many respects will depend on the quality of manufacture of models and observance of the necessary criteria of similarity.

§ 15.7. Flight Tests of Vehicles

Basic and in many cases the only methods of appraisal of the correctness of establishment of design cases of loading for flight vehicles (especially ballistic type) are flight tests. In the process of these tests there are measured the actual values of all those parameters which characterize the reaction of the construction to external influence, and there are also determined the values of external forces themselves, in particular:

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- values of longitudinal and lateral overloads at different points of the construction;

- angles of deflection of controls, angles of pitch, yaw and bank of the flight vehicle;

- pressures in the combustion chambers of the propulsion system, in fuel tanks and in fuel feed lines;

- temperature of all basic supporting elements of the construction.

Certain special measurements are taken, for instance external and internal pressure on the shell of the body, deformations of separate structural elements, parameters of the trajectory and others.

For obtaining an idea of the character of loading of the construction of the descent part of the vehicle on the free-flight phase in dense layers of the atmosphere it is sufficient to measure the values of longitudinal, transverse and lateral components of overload in two cross sections of the body, located forward and aft of the center of gravity. For appraisal of the accuracy of the method of calculation of loads in cases C and D it is expedient to additionally measure the angle velocities of rotation of the flight vehicle relative to three lateral axes, passing through the center of gravity, and also such parameters of the trajectory as rate and altitude of flight.

Since certain external loads, affecting the flight vehicle in flight, carry a random character, then it is natural that not all data of flight tests can be directly used for appraisal of real safety factors of the construction. Even with the presence of a comparatively large number of realizations the actual values of reaction parameters of the construction can essentially differ from design due to the influence of particular conditions of tests (time of year, place of location of launch, etc.). This should always be considered during analysis of results of tests of flight vehicles, especially ballistic type, designed for use in different regions of the earth.

Let us examine very briefly how the shown data of measurements can be used for determination of real values of longitudinal and lateral loads, dynamic characteristics of the construction and for appraisal of the correctness of selection of dynamic layout.

a) Appraisal of longitudinal forces.

Since the accuracy of determining the weight of the dry construction and the propellant weight of a flight vehicle is comparatively high, real values of static longitudinal forces will be characterized by the magnitude of static component of longitudinal overload. By comparing calculated and experimental values of n_{x_1} for all sections of the trajectory, we obtain a rather complete idea of the correctness of determination of static longitudinal forces.

For appraisal of the accuracy of calculation of the dynamic component of longitudinal force (in cases of launch and stage separation) it is necessary to measure, besides the dynamic component of longitudinal overload, the actual value of thrust force (force in rods of the frame, pressure in the combustion chamber) in transient conditions. While using the real law of change of force P according to t, one should compute the corresponding values of longitudinal overload (at places of installation of transmitters) and compare them with measured. In case of steady-state longitudinal oscillations the amplitude of dynamic force will be approximately equal to

$$\Delta N(x_1)=g_0N_{nx}(x_1)\frac{\Delta n_{x_1}(x_1)}{X_n(x_1)}, \qquad (15.47)$$

where Δn_{x_1} - amplitude of oscillations of longitudinal overload, $N_{nx}(x_1)$ - unit longitudinal force, corresponding to the given frequency of oscillations of the flight vehicle ω_n, $x_д$ - coordinate of the place of installation of overload transmitter. The value of frequency ω_n is found directly from the graph of function $n_{x_1}(t)$ by reading off the period of oscillations. In this case it is recommended to use experimental forms of natural longitudinal oscillations of the flight vehicle and computed values of $N_{nx}(x_1)$. The degree of conformity of computed values of the function of form $X_n(x_1)$ to experimental is checked by comparison of actual relative values of the amplitudes of oscillations of longitudinal overload (in several sections of the body) with corresponding calculated relative values of functions $X_n(x_1)$, and also by comparison of phases of oscillations of longitudinal overload.

For establishment of the correctness of description of the dynamic layout of system vehicle-propulsion system it is necessary to measure pressure fluctuation in the combustion chamber and lines (before the pump and after), and when possible the pressure fluctuation of liquid on bottoms of the fuel tanks.

b) Appraisal of lateral loads.

Direct measurement of stresses in structural elements for appraisal of the actual values of bending moments in flight is performed on aircraft type flight vehicles. For ballistic type vehicles similar measurements are rarely taken. This is explained by the difficulty of deciphering the results because of distortion of recording by vibrations and insufficient accuracy of measurements, caused by the influence of temperature deformations.

Usually for such flight vehicles the accuracy of calculation of lateral forces and bending moments can be judged by the degree of conformity of calculated dynamic layout to actual. This conformity can be appraised by comparison of the actual reaction of the system to an external influence with calculated, corresponding to actual values of input parameters. For the shown analysis we select only those phases of flight, on which the investigated case of loading is observed in pure form without substantial distortions. For instance, for checking the correctness of description of the dynamic layout of system vehicle - atomatic stabilization control (in the pitching plane) it is necessary to take the section where oscillations of angles of deflection of controls, command currents, and lateral overload are almost harmonic. According to such measurements there are determined the frequency of steady-state oscillations of the system ω, transfer functions

$$\frac{\delta}{\Delta n_y(x_A)},\quad \frac{\delta}{\Delta\theta}$$

and functions characterizing the form of oscillations

$$\frac{\Delta n_{y_1}(x_{A1})}{\Delta n_{y_1}(x_{A0})},\quad \frac{\Delta n_{y_1}(x_{A2})}{\Delta n_{y_1}(x_{A0})},\ \ldots\ .$$

which are compared with corresponding computed values. With poor coincidence of results there is investigated the effect of probable deviations of parameters of the system on the characteristics of the process or the calculation scheme is refined. Appraisal of static values of lateral loads in flight is conducted on the basis of measurement of static values of lateral overloads (at not less than two points of the longitudinal axis, located on both sides of the center of gravity of the flight vehicle) and control forces.

c) Appraisal of dynamic characteristics.

During experimental investigation of the process of loading of the flight vehicle construction overload transmitters are used the most often. With their help in certain cases one can determine the real dynamic characteristics of the flight vehicle construction on the whole: frequency and form of natural elastic oscillations and logarithmic damping decrement of oscillations. For this it is necessary that on certain phases of flight there be observed either steady-state oscillations of the flight vehicle (natural oscillations) with frequency, close to one of the natural frequencies of elastic oscillations, or clearly expressed damped natural oscillations of the construction.

Since the frequencies of oscillations of systems vehicle-automatic stabilization control, vehicle-propulsion system usually differ very little from corresponding partial natural frequencies of elastic oscillations of the construction, then with sufficient accuracy for practical calculations of the latter can be obtained directly from graphs of change of overloads n_{x_1}, n_{y_1} or angles δ and so forth. For determination of the form of elastic oscillations it is necessary to know the values of overloads at not less than three-five points (on half-wave length). In this case, bearing in mind the possible distortion of phases of oscillations by these transmitters, it is possible to be limited by consideration of only the relationships of amplitudes of overloads.

The magnitude of logarithmic damping decrement of oscillations can be roughly calculated if we consider steady-state oscillations of system as forced oscillations of the construction under the action of periodic control force or pulsation of thrust force. For this it is necessary in the formula for the amplitude of similar oscillations (see Chapter XII and XIII) to place experimental values of frequency, functions of the form (at the place of installation of overload transmitter and at the place of attachment of controls) and amplitude of oscillations of control force (thrust force). Such a method gives somewhat overstated magnitudes of logarithmic decrements, especially we disregard the difference of frequency of oscillations of the system from the frequency of natural elastic oscillations of the construction of the flight vehicle. More exact is the method of determination of these decrements by the curve of damping of oscillations of lateral (longitudinal) overload with the presence, of course, of confidence in the fact that these oscillations fully correspond to a definite form of natural oscillations of the construction, and are not a result of imposition of oscillations of several forms. In this case it is sometimes possible to reveal the relationship of magnitude of logarithmic damping decrement to the amplitude of oscillations (point of reduction) of the flight vehicle construction. During analysis of magnitudes of logarithmic damping decrements of oscillations obtained by the shown methods, one should take into account the relationship of their values to the temperature of supporting elements of the construction (especially with unsteady conditions of heating of the body of the vehicle).

CHAPTER XVI

CALCULATION CASES FOR BALLISTIC TYPE FLIGHT VEHICLE

§ 16.1. General Information About the Structure Diagram of the Construction of Ballistic Type Flight Vehicles

The construction of a ballistic type flight vehicle [2, 78] in many respects depend on the type of utilized fuel and assignment of the vehicle, particularly the weight of payload and the flying range. According to the type of fuel we distinguish vehicles with engines operating on solid fuel (solid propellant) and on liquid fuel (liquid propellant). For the first the fuel is stored in the combustion chamber itself, for the second - in special reservoirs (fuel tanks).

The weight of transferable payload and flying range are in essence the basic characteristics of the considered type of flight vehicles, on which depend not only its dimensions, but also the assembly diagram. Vehicles designed for transfer of comparatively large loads at long distances (for instance, intercontinental) or for achievement of space rates, are made, as a rule, multistage in the form of a cluster of several units.

Consecutive arrangement of these units is applied the most often. An arrangement with parallel location of units (in the form of "a cluster") is encountered less often. Side units can be made in the form of boosters (independent single-stage vehicles, having independent propulsion systems) or in the form of fuel sections connected in a cluster (hinged or rigid), supplying one engine (or a cluster of engines, located on a common power frame) with fuel. Figure 16.1 shows a combined multistage vehicle, consisting of two series connected units A and B and two side-mounting boosters C and D, located on each side of unit A.

Fig. 16.1. "Titan-IIIC" multistage rocket.

Usually the quantity of units in clusters configuration depends on the initial weight of the flight vehicle, and their construction - on the method of transmission of thrust force of engines can be applied to the body of only one unit of the cluster (for instance, the central unit) or to a special frame, representing a monocoque shell. In this case the side units will in essence constitute suspended fuel compartments, located around the carrier central unit in the first case and inside the frame in the second case.

The construction of each unit of a multistage liquid-propellant flight vehicle, as the construction of the body of a single-stage vehicle, consists of three main parts (Fig. 16.2): fuel compartment, in which tanks with fuel and oxidizer are placed, engine (tail) section and nose section. For flight vehicles with engines operating on solid propellant, the separation into fuel and engine compartments will be absent. In the forward section (nose cone) of the single-stage flight vehicle and in the forward section of the unit of the last stage of a sectional flight vehicle the payload is placed. On other units of a multistage flight vehicle instead of the forward section there are installed transition frames or other elements, ensuring power connection between the bodies of the units. For certain types of flight vehicles there are special instrument compartments for accomodation of control equipment. The structure diagram of almost all compartments are determined by the system of effective external forces and depend on the overall arrangement of the flight vehicle.

Fig. 16.2. Assembly diagram of "Saturn" ballistic type missile.

Fuel compartment. There are two types of constructions of fuel compartments: with suspended tanks and with so-called carrier tanks. Suspended tanks are usually applied for flight vehicles equipped with propulsion systems with a pump system for fuel feed and with comparatively small aspect ratios of tanks (small ratio of length to diameter).

An example of construction of the fuel compartment with such tanks is the fuel compartment of a ballistic type flight vehicle shown on Fig. 3.1. Its body is a thin-walled shell, reinforced in longitudinal (along the longitudinal axis of the tank) and transverse directions by a system of stringers and frames. To attachment rings 1 of this body on rods 2 are suspended tanks Г and O. At places of connection of these tanks, i.e., application of comparatively large concentrated forces, there are sometimes installed longerons or stringers are locally strengthened by fittings. Attachment of tanks to the body in transverse direction is carried out at several points 3 with the help of special junction points not perceiving longitudinal forces. The shape of suspended tanks depends on dimensions of the body of the fuel compartment. They can be cylindrical, conical, spherical and so forth. With large diameter of the body of the flight vehicle the application, instead of one large suspended tank, of several tanks of comparatively small diameters arranged in parallel can be expedient. Maximums of these diameters are often limited by conditions of transportation of the vehicles by railroads.

For flight vehicles with carrier tanks the shell of the tanks is simultaneously the shell of the body of the fuel compartment (Fig. 16.2). With low boost pressure such tanks can be equipped with longitudinal and lateral reinforcing (for absorption of corresponding compression forces and for maintaining the required shape of the body).

The bottoms of fuel tanks can have different shape: ellipsoidal, spherical and conical. At the place of their connection to the shell of the body there are usually installed special reinforcing frames. Sometimes for improvement of the cg of the flight vehicle in the tanks we can additionally install intermediate bottoms. In most cases the fuel tanks are made from light (aluminum) alloys. With comparatively great operating boost pressures on the order of 30-40 kg/cm^2 (particularly with pressure fuel feed system) we apply steel and high-strength nonmetallic materials.

§ 16.2. Calculated Cases for Fuel Tanks

A calculated case for the shell and bottoms of suspended fuel tanks is the case of loading of a flight vehicle in which internal excess pressure on these parts of the construction reaches its highest value

$$\max p_j(x_1, \bar{t}) = [\Delta p_{\text{н}j}(\bar{t}) + p_{j\text{н}}(x_1, \bar{t})]_{\max}. \tag{16.1}$$

Since $\Delta p_{\text{н}j}$ and $p_{j\text{н}}(x_1)$ are independent functions of $\bar{t}$, then it is obvious that for various compartments of the tank shell different moments of flight of the vehicle can be calculated. In a particular case (with constant excess gas boost pressure) max p_j will be observed at moment $\bar{t} = \bar{t}_1$ (see formula (3.9)), when the inertial pressure of liquid on the walls and bottom of the tank becomes maximum

$$p_{j\text{н}}(x_1, \bar{t}) = \Delta p_{\text{т}j} + g_0\rho_j(x_1 - x_{0j})n_{x_1}(\bar{t}) + \\ + 2g_0\rho_j a_j(x_j)n_{y_1}(x_j, \bar{t}) \quad \text{при} \quad x_1 < x_{1\text{л}j}. \tag{16.2}$$

The hydrostatic component of this pressure, proportional to n_{x_1}, is a function of the time of flight. For a cylindrical tank its maximum will be observed also when $\bar{t} = \bar{t}_1$. The hydrodynamic component of pressure $\Delta p_{\text{т}j}$ is usually small as compared to $n_{x_1} \cdot h_j$, and in most cases its influence can be disregarded. Hydrostatic pressure in the pipes for fuel feed to pumps will be equal to

$$p_{\text{тр}j} = g_0\rho_j h_{\text{тр}j} n_{x_1} + p_{\text{б}j},$$

where $h_{\text{тр}j}$ - height of pipe, $p_{\text{б}j}$ - total pressure on the bottom of the tank (at pipe inlet). During start-stop operation of the engine in these pipes an increase of hydrodynamic pressure can be observed, determined by known formulas for hydraulic shock

$$\left.\begin{aligned} \Delta p_{\text{тр}j} &= \rho_j c_j v_j \quad \text{when} \quad t^* < \frac{2h_{\text{тр}j}}{c_j}, \\ \Delta p_{\text{тр}j} &= \rho_j c_j v_j \frac{h_{\text{тр}j}}{c_j t^* - h_{\text{тр}j}} \quad \text{when} \quad t^* > \frac{2h_{\text{тр}j}}{c_j}, \end{aligned}\right\} \qquad (16.3)$$

where v_j - rate of liquid in the pipeline, c_j - propagation rate of shock wave, t^* - time of covering the main line. It is necessary to bear in mind that at the moment of opening of the liquid oxygen flow valve (due to the motion of column of liquid in a long pipe at first upwards under the action of vapors and then downwards during their condensation) there also can be observed an increase of pressure, similar to hydraulic shock.

For elements of attachment of suspended tanks to the body of a flight vehicle longitudinal direction the calculated cases are also $\bar{t} = \bar{t}_1$, and in transverse - the case corresponding to $(n_y, G_{\text{т}j})_{\max}$. For ballistic type flight vehicles with comparatively short flying range, i.e., with nonseparable nose cone, the strength of attachment elements of suspended tanks should be checked for case C and D. In case C there is checked the strength of the upper bottoms of tanks which are loaded on the powered phase of flight only by boost pressure. For a number of sections, for instance for the sections located in the upper part of the tank, and also for the upper bottom, case $O_{\text{г}}$ can be calculated. Analogous case of loading can take place during static firing (bench) tests of the vehicle, if the required pressure of liquid at the pump inlet (when $n_{x_1} = 1$) is ensured by increase of boost pressure in the tanks. All sealed capacities are checked in the case of the influence of external excess atmospheric pressure (particularly with a sharp change of temperature conditions during storage and transportation of the flight vehicle).

<u>Fuel tanks of carrier configuration</u>. Calculated cases for carrier fuel tanks should be determined by the method of conditional load, i.e., by finding the biggest values of functions $N_y^{\sigma}(\bar{t})$ and $N_y^{E}(\bar{t})$. In a particular case, when the bearing capacity of the tank shell is little changed in time (for instance, for a tank with

liquid oxygen), the calculated case is established by the greatest magnitude of equivalent longitudinal force

$$N_{э}^{a}(x_1, \bar{t}) = N(x_1, \bar{t}) + N_{и}(x_1, \bar{t}) + \pi a^2(x_1)[\Delta p_{н}(\bar{t}) + 2p_{н}(x_1, \bar{t})] \quad (16.4)$$

when

$$|N(x_1, \bar{t}) + N_{и}(x_1, \bar{t})| > N_{m}(x_1, \bar{t})$$

and

$$N_{э}^{a}(x_1, \bar{t}) = N_{pк}(x_1, \bar{t}) \quad (16.5)$$

when

$$|N(x_1, \bar{t}) + N_{и}(x_1, \bar{t})| < N_{m}(x_1, \bar{t}).$$

For carrier tanks located in the lower part of the body of a ballistic type flight vehicle, dangerous cases can be prelaunch servicing (case $V_з$), filling the tanks with fuel (in the absence of excess boost pressure in the tanks), and also cases of transportation (for instance, for frames utilized as support elements and so forth).

Excess gas boost pressure in the tank, determined by the required pressure of liquid at the fuel pump inlet, partially or completely unloads its shell from compression forces. If this unloading force N_m is close to equivalent compression force $N + N_и$, then in process of operation it is necessary to check the minimum value of this pressure. For limitation of the maximum value of shown pressure usually in tanks there are installed the corresponding safety valves, dumping surpluses of gas. With pressure fuel feed system in the tanks there is created high gas pressure, which for all practical purposes determines the required strength of their construction.

Minimum boost pressure is determined by the system of fuel feed to combustion chamber. Only for a thoroughly developed construction may one assume that the lower calculated value of boost pressure corresponds to the minimum pressure of valve operation. With a pump system of fuel supply the minimum boost pressure is sometimes selected by proceeding from the condition of guarantee of normal cavitation-free operation of the pumps:

$$p_{\text{н}j}(t) \geqslant p_{\text{пот}} - n_{x_1}(h_j + h_{\text{тр}j}) g_0 \rho_j, \tag{16.6}$$

where $p_{\text{пот}}$ - the required pressure of liquid at the pump inlet. In this case its value will depend on the place of location of the tank with respect to pumps of the propulsion system. If $h_{\text{т}pj}$ is great, i.e., the tank is located in the forward part of the body of the flight vehicle, then the required boost pressure can turn out to be very small and even equal to zero. Therefore, from the point of view of strength it is expedient to introduce limitation on the minimum value of boost pressure. Since $p_{\text{н}j}$ unloads the shell of the tank from compression forces and increases its critical forces during longitudinal bending, we usually try to increase the boost pressure, but, of course, to known limits.

§ 16.3. Calculated Cases for the Main Parts of the Body of a Vehicle

Separable nose cone. Since usually the values of max n_x, max n_y^0 (corresponding to powered-flight phase) are considerably less than values of max $n_{x\ max}$, max $n^0_{y\ max}$, the required bearing capacity of the body of the separable nose cone of a flight vehicle is determined mainly by those loads which affect it on the free-flight phase. Cases C and D, considered together with case W will be calculated for such a nose cone. Considering the opposite direction of longitudinal overloads max n_x and max $n_{x\ max}$, the attachment elements of all loads inside the nose cone should be checked in case B. Calculated cases for the body of a separable nose cone, not equipped with special heat protection, are determined by maximum of corresponding conditional load.

Body of fuel compartment. Destruction of the body of the fuel compartment of the flight vehicle, with suspended tanks and parts of the construction of the body of a vehicle with carrier tanks, utilized for connecting the tanks together and to adjacent compartments, usually occurs due to loss of stability under the action of longitudinal compression force and bending moment. Their calculated cases are established by the method of conditional load

$$N_y^E(x_1, \bar{t}) = N(x_1, \bar{t}) + |N_{\text{н}}(x_1, \bar{t})| k_E(\bar{t}). \qquad (16.7)$$

In this case the conditional load of the latter is produced for all basic cases of loading: A, B, K, L, S and V.

If the value of longitudinal forces in some section of the body of the fuel compartment is slightly changed in time, then in the first approximation it is possible to take case A as calculated. For points of attachment of all loads, located inside the body of the compartment (suspended tanks, instruments and so forth) cases A, B, L (when $\bar{t} = \bar{t}_1$) can be calculated.

Body of engine compartment. The loading diagram of the body of the engine (tail) section depends on many factors: its arrangement, dimensions of the propulsion system, method of application of thrust force and control forces, the presence of stabilizers and, finally, on the diagram of installation of the flight vehicle on the launching pad. Therefore, considering some indeterminancy of the temperature conditions of operation of its construction, we should very thoroughly approach analysis of all those cases of loading, which can turn out to be calculated for elements of this compartment. If the engine section is used as support during launch, then to similar cases there should primarily pertain cases V_3 and S. In case V_3 it is loaded by longitudinal compression forces (initial weight of fueled flight vehicle) and crosswind loads (with launch from an open position), and in case $S_ш$ - additionally by external excess pressure. If the launch support elements are not placed on the body of the motor section, then cases A and K turn out to be dangerous for its construction, and namely cases of action of stabilizing aerodynamic forces (with the presence of stabilizers) or control forces (moments).

For ballistic type flight vehicles with nonseparable nose cone case D will be calculated for the body of engine (tail) section and supporting stabilizer. For engine sections of units of a sectional flight vehicle one of the cases of loading of series B_1, corresponding to the period of operation of engines of the preceding stages, can be the calculated case.

Frame of propulsion system. For elements transmitting thrust force to the body of flight vehicle the case of engines arriving at operating conditions (cases S, L_1) is usually the calculated. If in flight the longitudinal load in the frame of the propulsion system and its mounting lugs to the body of the flight vehicle is little changed in time, then it is necessary to additionally consider the cases of action of the largest magnitude of local lateral overload.

Stabilizers. For stabilizers and cowls of different type, installed on the body of the flight vehicle, the case of flight with maximum angles of attack in the zone of high impact pressures (case A) is the calculated. For cowls - the case of motion at high subsonic speeds is also calculated.

For intermediate frames, connecting separate stages of the flight vehicle together the cases of stage separation can be calculated.

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13. ABSTRACT

The book is devoted to the problem of determining the necessary carrying capacity and rigidity of design of a flight vehicle. Discussed in it are thoretical bases and practical methods of the calculation of internal force factors from external forces acting on the vehicle in the process of operation, and methods of development of calculation cases of loading are given. In this case the main attention is given to problems of the dynamics of design, in particular the selection of design configurations, the formulation of equations of dynamic equilibrium and determination of the dynamic reaction of design on the effect of external perturbations. We consider those limitations which are imposed by conditions of strength of the design on values of certain parameters of the propulsion system, automatic control system, complex of ground equipment, and also on conditions of operation of flight vehicles of different types. Orig. art. has: 118 figures. [AM9032190]

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