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The Open-Circuit Sensitivity Of Radially Polarized Ferroelectric Ceramic Cylinders

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ABSTRACT

The open-circuit sensitivity of radially polarized ferroelectric ceramic cylinders is derived by treating the ceramic as an anisotropic material. The analysis demonstrates that if the anisotropic properties of the materials employed in the design of cylindrical hydrophone elements are considered, the resulting sensitivities are not significantly different from those derived by Langevin, who considered only stress distributions that obey isotropic criteria. Graphs of the open-circuit sensitivity versus the ratio of wall thickness to outside diameter are presented for the two dissimilar approaches that define the regions over which the differences may be neglected.

ADMINISTRATIVE INFORMATION

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The technical reviewer for this report was Dr. R. S. Woollett, consultant to the Sonar Transducer Division.

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INTRODUCTION

The primary investigation[1] of the open-circuit sensitivity of ceramic cylinders neglects the fact that the radial and tangential stress distributions depend on the anisotropic properties of the ceramic when it is radially polarized. This additional dependence is not serious when the active material is barium titanate ceramic, but it becomes more influential with the newer type hard lead zirconate-titanate (PZT) ceramics presently being employed as underwater acoustic transducers. This report presents a derivation of the open-circuit sensitivity (M_o) for a cylinder polarized radially. The values of sensitivity calculated herein are valid only for frequencies below and reasonably well removed from the lowest resonance frequency of the structure being considered.

THEORETICAL CONSIDERATIONS

Figure 1 depicts a right circul   cylinder of polarized ferroelectric ceramic. By convention,[2] the principal directions of stress in the ceramic are designated by the axes 1, 2, and 3 and are chosen to coincide with the set of cylindrical coordinates θ, z, and r, respectively. Axis 3 is parallel to the polarization vector (radial) in the ceramic.

At the outset, circular symmetry will be assumed such that the tangential particle displacement (u_θ) and any differentiations with respect to the angular coordinate (θ) can be neglected. One way of expressing the linear equations* of state of the material is as follows:

$$S_1 = s^D_{11} T_1 + s^D_{12} T_2 + s^D_{13} T_3 + g_{31} D_3 \quad , \qquad (1)$$

$$S_2 = s^D_{12} T_1 + s^D_{11} T_2 + s^D_{13} T_3 + g_{31} D_3 \quad , \qquad (2)$$

$$S_3 = s_{13} T_1 + s^D_{13} T_2 + s^D_{33} T_3 + g_{33} D_3 \quad , \qquad (3)$$

$$S_4 = s^D_{44} T_4 \; , \text{ and} \qquad (4)$$

$$\mathcal{E}_3 = - g_{31} T_1 - g_{31} T_2 - g_{33} T_3 + \beta^T_{33} D_3 \; , \qquad (5)$$

*There appears to be some discrepancy or doubt that these constitutive equations are valid for the ceramic geometrical shape considered herein, where the symmetry of the material does not correspond identically to the generally accepted Cartesian symmetry. This doubt comes about primarily because the conditions imposed on the structure can not be rigorously satisfied in a mathematical sense for certain polarization orientations, in particular, radially polarized cylindrical specimens.

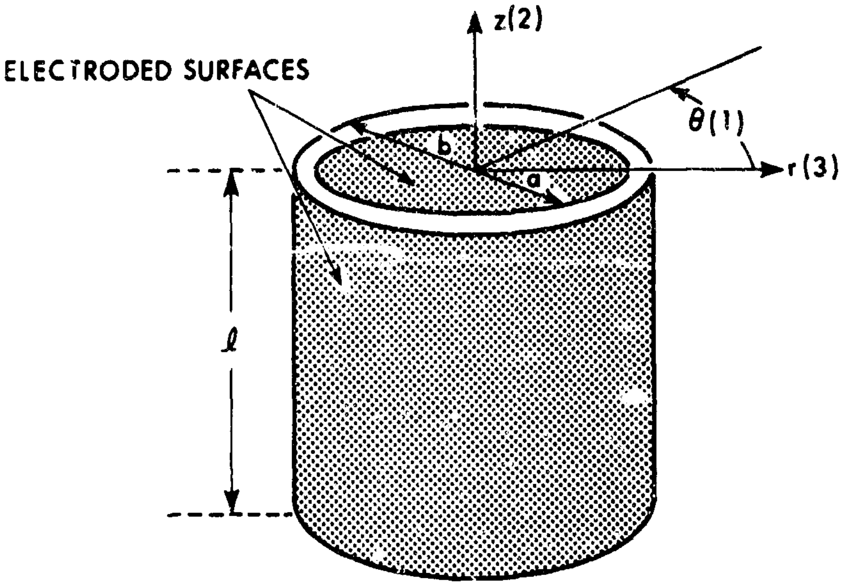

Fig. 1 - A Right Circular Cylinder of Radially Polarized Ferroelectric Ceramic

where s^D is the elastic compliance coefficient at constant electric displacement, g is the piezoelectric coefficient, and β_{33}^T is the incremental impermeability at constant stress (T). In cylindrical coordinates, the strains are expressed as

$$S_1 = \frac{u_r}{r}, \quad S_2 = \frac{\partial u_z}{\partial z}, \quad S_3 = \frac{\partial u_r}{\partial r}, \quad \text{and} \quad S_4 = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}, \tag{6}$$

where u_r and u_z are the particle displacements in the radial and axial directions, respectively.

Before proceeding further, let us look at the electric displacement ($\vec{D}$). Gauss' law requires that the divergence of the electric displacement be zero throughout the body of the ceramic because of the absence of free charge in that region. This is written mathematically as

$$\text{div } \vec{D} = \frac{\partial D_3}{\partial r} + \frac{D_3}{r} = 0 , \tag{7}$$

which is satisfied if $D_3 = C_o / r$.

Since both the inside and outside lateral surfaces of the tube are fully electroded, the total charge (q) can be found from

$$q = \int_A D_3 \, dA = \int_0^{\ell} \int_0^{2\pi} \frac{C_o}{r} r \, d\theta \, dz = 2\pi \ell C_o . \tag{8}$$

Therefore, the electric displacement can be written as

$$D_3 = \frac{q}{2\pi \ell r} . \tag{9}$$

In all the cases treated in this report, it will be assumed that the electrical terminals are open-circuited (q = 0), which implies that the electric displacement may be neglected to a first-order approximation. Thus, the equations of state (Eqs. (1) through (5)) may be rewritten as

$$S_1 = s^D_{11} T_1 + s^D_{12} T_2 + s^D_{13} T_3 = \frac{u_r}{r} , \tag{10}$$

$$S_2 = s^D_{12} T_1 + s^D_{11} T_2 + s^D_{13} T_3 = \frac{\partial u_z}{\partial z} , \tag{11}$$

$$S_3 = s^D_{13} \left(T_1 + T_2\right) + s^D_{33} T_3 = \frac{\partial u_r}{\partial r} , \tag{12}$$

$$S_4 = s^D_{44} T_4 = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} , \text{ and} \tag{13}$$

$$\mathcal{E}_3 = - g_{31} \left(T_1 + T_2\right) - g_{33} T_3 . \tag{14}$$

If Eq. (10) is used to define the radial displacement (u_r) and is differentiated once with respect to the radial coordinate, then

$$\frac{\partial u_r}{\partial r} = s^D_{11} T_1 + s^D_{12} T_2 + s^D_{13} T_3 + r s^D_{11} T_1' + r s^D_{12} T_2' + r s^D_{13} T_3' , \tag{15}$$

where the prime means the $\partial / \partial r$.

Equating the above result with Eq. (12) yields

$$\left(s^D_{11} - s^D_{13}\right) T_1 + \left(s^D_{12} - s^D_{13}\right) T_2 + \left(s^D_{13} - s^D_{33}\right) T_3 + r s^D_{11} T_1' + r s^D_{12} T_2' + r s^D_{13} T_3' = 0 . \tag{16}$$

The analysis can be kept general by introducing the equations of static equilibrium for a cylinder with circular symmetry [3] and zero-body forces:

$$\frac{\partial T_2}{\partial z} + \frac{\partial T_4}{\partial r} + \frac{T_4}{r} = 0 \text{ and } \frac{\partial T_3}{\partial r} + \frac{(T_3 - T_1)}{r} + \frac{\partial T_4}{\partial z} = 0 . \tag{17}$$

Equation (17) can be more conveniently manipulated if it is assumed that the shear stress (T_4) is zero. Therefore, the equations of static equilibrium reduce to

$$\frac{\partial T_2}{\partial z} = 0 \quad \text{and} \quad \frac{\partial T_3}{\partial r} + \frac{1}{r}\left(T_3 - T_1\right) = 0 \; . \tag{18}$$

The fact that $\partial T_2/\partial z = 0$ requires that the axial stress T_2 be constant in z. If constant boundary conditions are specified for T_2 at the cylinder ends, then $\partial T_2/\partial r = 0$ throughout the body. The last equation in the above set can be used to determine the tangential stress T_1:

$$T_1 = rT_3' + T_3 \; , \tag{19}$$

or

$$T_1' = rT_3'' + 2T_3' \tag{20}$$

Using the above reasoning and employing Eqs. (19) and (20) in Eq. (16) yield

$$T_3'' + \frac{3}{r}T_3' + \frac{(1 - s_{33}^D/s_{11}^D)}{r^2}T_3 + \frac{(s_{12}^D - s_{13}^D)}{r^2\, s_{11}^D}T_2 = 0 \; . \tag{21}$$

Inspection of Eq. (21) reveals that the radial stress is solely a function of the radial coordinate if T_2 is a constant, which will be a specified condition for the cases treated herein. The tangential stress, as defined by Eq. (19), then will also be a function of r only. If the solutions to any of the problems proposed in this report are to be rigorous, the equations of state for the ceramic cylinder must be reevaluated. More explicitly, if the shear stress (T_4) is to be identically zero, as was assumed, then Eq. (13) requires that the shear strain (S_4) be zero also. In order to achieve this constraint, the following criteria must be satisfied, namely,

$$\frac{\partial u_r}{\partial z} = -\frac{\partial u_z}{\partial r} \; . \tag{22}$$

Since T_1 and T_3 are functions of r only (for T_2 = constant), Eq. (10) requires that $\partial u_r/\partial z = 0$. Thus, in order to satisfy Eq. (22), the axial particle displacement must be such that $\partial u_z/\partial r = 0$. Using Eq. (11) to solve for the axial displacement yields

$$u_z = (s_{12}^D\, T_1 + s_{11}^D\, T_2 + s_{13}^D\, T_3)\, z + f(r) \; , \tag{23}$$

or

$$\frac{\partial u_z}{\partial r} = (s_{12}^D\, T_1' + s_{13}^D\, T_3')\, z + \frac{\partial f(r)}{\partial r} = 0 \; . \tag{24}$$

In order for Eq. (24) to be zero for all values of z, then $f(r) =$ constant and the first term in parentheses must be identically zero. This result is the same as the one that would be obtained if the conditions of compatibility were imposed upon the system. More explicitly, if Eq. (11) is differentiated twice with respect to r, Eq. (12) is differentiated twice with respect to z, and Eq. (13) is differentiated once with respect to r and once with respect to z, then

$$\frac{\partial^2}{\partial z^2} S_3 + \frac{\partial^2}{\partial r^2} S_2 = \frac{\partial^2}{\partial z\, \partial r} S_4 = 0 , \tag{25}$$

or

$$\frac{\partial^2}{\partial z^2} \left[s^D_{13}(T_1 + T_2) + s^D_{33} T_3 \right] + \frac{\partial^2}{\partial r^2} \left[s^D_{12} T_1 + s^D_{11} T_2 + s^D_{13} T_3 \right] = 0 . \tag{26}$$

Since T_1 and T_3 are functions of r only (for T_2 = constant), Eq. (26) requires that

$$\frac{\partial}{\partial r} \left[s^D_{12} T_1' + s^D_{13} T_3' \right] = 0 , \tag{27}$$

which is the same as the condition derived from Eq. (24). Incorporating Eq. (20) into Eq. (27) gives an additional condition that must be satisfied by the radial stress if it is to be a rigorous solution to any of the problems that will be considered here; namely,

$$T_1'' + \frac{T_1'}{r} (2 + s^D_{13}/s^D_{12}) = 0 . \tag{28}$$

It will be recognized that when the material is isotropic ($s^D_{13} = s^D_{12}$ and $s^D_{33} = s^D_{11}$), Eq. (28) reverts to Eq. (21). Therefore, in order to obtain a solution for the radial stress distribution, at least to a first-order approximation, which accounts for the anisotropic properties of the materials, the first two terms in Eq. (21) will be replaced by the relationship specified by Eq. (28). The resulting equation for the radial stress distribution becomes

$$T_1'' + \frac{1}{r}(2 + s^D_{13}/s^D_{12}) T_1' + \frac{1}{r^2}(1 - s^D_{13}/s^D_{11}) T_1 + \frac{(s^D_{12} - s^D_{13})}{r^2 s^D_{11}} T_2 = 0 . \tag{29}$$

When T_3 is equal to a constant ($-T^1$), Eq. (29) may be considered as an inhomogeneous, second-order differential equation:

$$T_1'' + \frac{(2 + s^D_{13}/s^D_{12})}{r} T_1' + \frac{(1 - s^D_{13}/s^D_{11})}{r^2} T_1 = \frac{(s^D_{12} - s^D_{13})}{r^2 s^D_{11}} T^1 . \tag{30}$$

The solution of the above equation is composed of two parts, namely, a complementary solution and a particular solution. The complementary solution can be obtained by choosing

$$T_c = r^{\alpha} ,$$

such that

$$T_c' = \alpha r^{\alpha - 1} \text{ and } T_c'' = (\alpha - 1) \alpha r^{\alpha - 2} .$$

Incorporating these definitions into Eq. (30) when $T^L = 0$ yields

$$\left\{ (\alpha - 1) \alpha + (2 + s_{13}^D / s_{12}^D) \alpha + (1 - s_{33}^D / s_{11}^D) \right\} r^{\alpha - 2} = 0 . \tag{31}$$

Since the above equation must hold for all values of r, the proper complementary solution can be obtained only if the bracketed term is set equal to zero, with the result that

$$\alpha_{1,2} = -\frac{1}{2}\left(1 + \frac{s_{13}^D}{s_{12}^D}\right) \pm \sqrt{\frac{(1 + s_{13}^D/s_{12}^D)^2}{4} - \left(1 - \frac{s_{33}^D}{s_{11}^D}\right)}, \tag{32}$$

or

$$\alpha_{1,2} = -\beta \pm \sqrt{\beta^2 - (1 - \gamma)} , \tag{33}$$

where

$$\gamma = \frac{s_{33}^D}{s_{11}^D} \text{ and } \beta = ½\left(1 + \frac{s_{13}^D}{s_{12}^D}\right) .$$

The complementary solution of Eq. (30) can be written, if

$\tau = \beta^2 - (1 - \gamma)$, as

$$T_c = a_1 r^{-\beta + \sqrt{\tau}} + a_2 r^{-\beta - \sqrt{\tau}} . \tag{34}$$

The particular solution of Eq. (30) is readily obtained by letting $T_p = a_o$, where

$$a_o = \frac{(s_{12}^D - s_{13}^D)}{s_{11}^D (1 - \gamma)} T^L . \tag{35}$$

The complete solution of Eq. (30) may now be written as

$$T_3 = a_o + a_1 r^{-\beta + \sqrt{\tau}} + a_2 r^{-\beta - \sqrt{\tau}} , \tag{36}$$

and the tangential stress distribution from Eq. (19) may be written as

$$T_1 = a_o + a_1\left(1-\beta+\sqrt{\tau}\right)r^{-\beta+\sqrt{\tau}} + a_2\left(1-\beta-\sqrt{\tau}\right)r^{-\beta-\sqrt{\tau}} \ . \tag{37}$$

At this point in the analysis it becomes convenient to consider specific cases, of which there are three of practical interest. The first case to be considered will be the most general since the other cases are simply specializations of this one. Consider the case of an end-capped cylinder where the lateral surfaces perpendicular to the cylinder axis at $z = 0, \ell$ are closed by rigid seals so that the cross section between $r = a$ and $r = b$ is subjected to an axial load of

$$\frac{-P_o}{[1-(a/b)^2]} \quad (= -T^L) \ ,$$

where the acoustic pressure is $-P_o$, in dynes per square centimeter. The constant (a_o) defined by Eq. (35) now becomes

$$a_o = \frac{(s_{12}^D - s_{13}^D)\,P_o}{s_{11}^D(1-\gamma)\,[1-(a/b)^2]} \ . \tag{38}$$

Boundary conditions must now be imposed on the radial stress distribution. These will be established in the following manner. The inside lateral surface will be assumed to be completely shielded from the acoustic field such that $T_3 = 0$ at $r = a$. The outside lateral surface, however, is assumed to be exposed to a uniform acoustic pressure $(-P_o)$ such that $T_3 = -P_a$ at $r = b$. When the above conditions are applied to the radial stress, Eq. (36), it is necessary that

$$a_2 = -a_o\, a^{\beta+\sqrt{\tau}} - a_1\, a^{2\sqrt{\tau}} \tag{39}$$

and

$$a_1 = -P_o\, b^{\beta-\sqrt{\tau}}\, \frac{\left\{1 + \frac{a_o}{P_o}[1-(a/b)^{\beta+\sqrt{\tau}}]\right\}}{[1-(a/b)^{2\sqrt{\tau}}]} \ . \tag{40}$$

The principal stresses can now be written in terms of the acoustic pressure as

$$T_3 = -P_o\left(\frac{b}{r}\right)^{\beta-\sqrt{\tau}} \frac{\left[1-\left(\frac{a}{r}\right)^{2\sqrt{\tau}}\right]}{\left[1-\left(\frac{a}{b}\right)^{2\sqrt{\tau}}\right]} \left\{1 + \frac{a_o}{P_o}\left[1-\left(\frac{a}{b}\right)^{\beta+\sqrt{\tau}}\right]\right\} + a_o\left[1-\left(\frac{a}{r}\right)^{\beta+\sqrt{\tau}}\right] \ , \tag{41}$$

$$T_1 = -P_o \left(\frac{b}{r}\right)^{\beta-\sqrt{\tau}} \frac{\left\{[1-\beta+\sqrt{\tau}] - \left(\frac{a}{r}\right)^{2\sqrt{\tau}}[1-\beta-\sqrt{\tau}]\right\}}{\left[1-\left(\frac{a}{b}\right)^{2\sqrt{\tau}}\right]} \left\{1 + \frac{a_o}{P_o}\left[1-\left(\frac{a}{b}\right)^{\beta+\sqrt{\tau}}\right]\right\}$$

$$+ a_o \left\{1 - \left(\frac{a}{r}\right)^{\beta+\sqrt{\tau}}\left[1-\beta-\sqrt{\tau}\right]\right\}, \qquad (42)$$

and

$$T_2 = -T^L = -\frac{P_o}{\left[1-\left(\frac{a}{b}\right)^2\right]} . \qquad (43)$$

Note that when the material is isotropic, $s_{12}^D = s_{13}^D$ and $s_{33}^D = s_{11}^D$, such that $\tau = \beta^2 = 1$, the radial and tangential stresses revert to

$$T_3 = -P_o \frac{\left[1-\left(\frac{a}{r}\right)^2\right]}{\left[1-\left(\frac{a}{b}\right)^2\right]} \quad \text{and} \quad T_1 = -P_o \frac{\left[1+\left(\frac{a}{r}\right)^2\right]}{\left[1-\left(\frac{a}{b}\right)^2\right]} ,$$

which are the conventional stress distributions for an isotropic body. [1]

However, when the cylinder ends are constrained, an additional compressive stress is generated in the radial direction. The same is true for the tangential stress. A similar type of situation exists when piezoelectric cylinders are prestressed two-dimensionally.[4]

The open-circuit voltage between the electroded surfaces can now [illegible] obtained from Eq. (14) through the following formula:

$$V_{oc} = \int_a^b \mathcal{E}_3 \, dr = \int_a^b [-g_{31}(T_1 + T_2) - g_{33} T_3] \, dr . \qquad (44)$$

Substituting Eqs. (41) through (43) into Eq. (44) and defining the open-circuit sensitivity (M_o) as the ratio of the open-circuit voltage to the magnitude of the acoustic pressure (P_o) yield

$$M_o = b\left\{ g_{31} \frac{\left[2+\left(\frac{a}{b}\right)\right]}{\left[1+\left(\frac{a}{b}\right)\right]} + \frac{g_{33}}{\left[1-\left(\frac{a}{b}\right)^{2\sqrt{\tau}}\right]} \left\{ \frac{\left[1-\left(\frac{a}{b}\right)^{1-\beta+\sqrt{\tau}}\right]}{(1-\beta+\sqrt{\tau})} - \left(\frac{a}{b}\right)^{2\sqrt{\tau}} \frac{\left[1-\left(\frac{a}{b}\right)^{1-\beta-\sqrt{\tau}}\right]}{(1-\beta-\sqrt{\tau})} \right\} \right\}$$

$$+ \frac{g_{33}\,(s^D_{12} - s^D_{13})}{\left[1-\left(\frac{a}{b}\right)^{2\sqrt{r}}\right] s^D_{11}\,(1-\gamma)\left[1-\left(\frac{a}{b}\right)^{2}\right]} \left\{ \frac{\left[1-\left(\frac{a}{b}\right)^{\beta+\sqrt{r}}\right]\left[1-\left(\frac{a}{b}\right)^{1-\beta+\sqrt{r}}\right]}{(1-\beta+\sqrt{r})} \right.$$

$$\left. - \left(\frac{a}{b}\right)^{2\sqrt{r}} \frac{\left[1-\left(\frac{a}{b}\right)^{\beta-\sqrt{r}}\right]\left[1-\left(\frac{a}{b}\right)^{1-\beta-\sqrt{r}}\right]}{(1-\beta-\sqrt{r})} \right\} - \frac{g_{33}\,(s^D_{12} - s^D_{13})}{s^D_{11}(1-\gamma)\left[1+\left(\frac{a}{b}\right)\right]} \Bigg\} . \tag{45}$$

Again it should be noted that when the material is isotropic Eq. (45) reverts to

$$M_o = b \left\{ g_{31} \frac{\left[2+\left(\frac{a}{b}\right)\right]}{\left[1+\left(\frac{a}{b}\right)\right]} + g_{33} \frac{\left[1-\left(\frac{a}{b}\right)\right]}{\left[1+\left(\frac{a}{b}\right)\right]} \right\} , \tag{46}$$

which is the same as the one given by Langevin.[1]

The second case that will be considered is that of a cylinder where the lateral surfaces at $z = 0, \ell$ are exposed to the acoustic field, such that $T_2 = -P_o$ at these extreme points. This case is simply a special circumstance of the more general situation, which was treated in the previous section of this report. In other words, Eqs. (41) through (43) can be used to represent the radial, tangential, and axial stress distributions for this case if the constant a_o is redefined to be

$$a_o = \frac{(s^D_{12} - s^D_{13})}{s^D_{11}(1-\gamma)} P_o ,$$

and the axial load is defined to be $T^L = P_o$. Therefore, the open-circuit sensitivity for an end-exposed ceramic cylinder can be written as

$$M_o = b \left\{ g_{31}\left[2-\left(\frac{a}{b}\right)\right] + \frac{g_{33}}{\left[1-\left(\frac{a}{b}\right)^{2\sqrt{r}}\right]} \left\{ \frac{\left[1-\left(\frac{a}{b}\right)^{1-\beta+\sqrt{r}}\right]}{(1-\beta+\sqrt{r})} - \left(\frac{a}{b}\right)^{2\sqrt{r}} \frac{\left[1-\left(\frac{a}{b}\right)^{1-\beta-\sqrt{r}}\right]}{(1-\beta-\sqrt{r})} \right\} \right.$$

$$
+ \frac{g_{33}\left(s_{12}^{D} - s_{13}^{D}\right)}{\left[1 - \left(\frac{a}{b}\right)^{2\sqrt{\tau}}\right] s_{11}^{D}(1-\gamma)} \left\{ \frac{\left[1 - \left(\frac{a}{b}\right)^{\beta+\sqrt{\tau}}\right]\left[1 - \left(\frac{a}{b}\right)^{1-\beta+\sqrt{\tau}}\right]}{(1-\beta+\sqrt{\tau})} \right.
$$

$$
\left. - \left(\frac{a}{b}\right)^{2\sqrt{\tau}} \frac{\left[1 - \left(\frac{a}{b}\right)^{\beta-\sqrt{\tau}}\right]\left[1 - \left(\frac{a}{b}\right)^{1-\beta-\sqrt{\tau}}\right]}{(1-\beta-\sqrt{\tau})} \right\} - \frac{g_{33}\left(s_{12}^{D} - s_{13}^{D}\right)}{s_{11}^{D}(1-\gamma)} \left[1 - \left(\frac{a}{b}\right)\right] \Bigg\} . \quad (47)
$$

The final case that will be considered is that of a cylinder where the lateral surfaces at $z = 0, \ell$ are completely shielded from the acoustic field. Therefore, the axial load (T^{L}) is zero and, in turn, a_{o} is zero also. The principal stresses for this case become

$$
T_{3} = - P_{o} \left(\frac{b}{r}\right)^{\beta-\sqrt{\tau}} \frac{\left[1 - \left(\frac{a}{r}\right)^{2\sqrt{\tau}}\right]}{\left[1 - \left(\frac{a}{b}\right)^{2\sqrt{\tau}}\right]} , \quad (48)
$$

$$
T_{1} = - P_{o} \left(\frac{b}{r}\right)^{\beta-\sqrt{\tau}} \frac{\left[1 - \beta + \sqrt{\tau}\right] - \left(\frac{a}{r}\right)^{2\sqrt{\tau}}\left[1 - \beta - \sqrt{\tau}\right]}{\left[1 - \left(\frac{a}{b}\right)^{2\sqrt{\tau}}\right]} , \quad (49)
$$

and

$$
T_{2} = 0 , \quad (50)
$$

and the open-circuit sensitivity is given as

$$
M_{o} = b \left\{ g_{31} + \frac{g_{33}}{\left[1 - \left(\frac{a}{b}\right)^{2\sqrt{\tau}}\right]} \left\{ \frac{\left[1 - \left(\frac{a}{b}\right)^{1-\beta+\sqrt{\tau}}\right]}{(1-\beta+\sqrt{\tau})} - \left(\frac{a}{b}\right)^{2\sqrt{\tau}} \frac{\left[1 - \left(\frac{a}{b}\right)^{1-\beta-\sqrt{\tau}}\right]}{(1-\beta-\sqrt{\tau})} \right\} \right\} . \quad (51)
$$

Equations (45), (47), and (51) can be used to evaluate the open-circuit sensitivities of radially polarized ferroelectric ceramic cylinders, at least to a first-order approximation, which accounts for the anisotropic behavior of the ceramic materials. The information implicit in the sensitivity equations is presented graphically in Figs. 2 through 4. Included for comparison are plots of Langevin's formulas that use the material parameters listed in Table 1. The values were taken from a Clevite Corporation publication.[5]

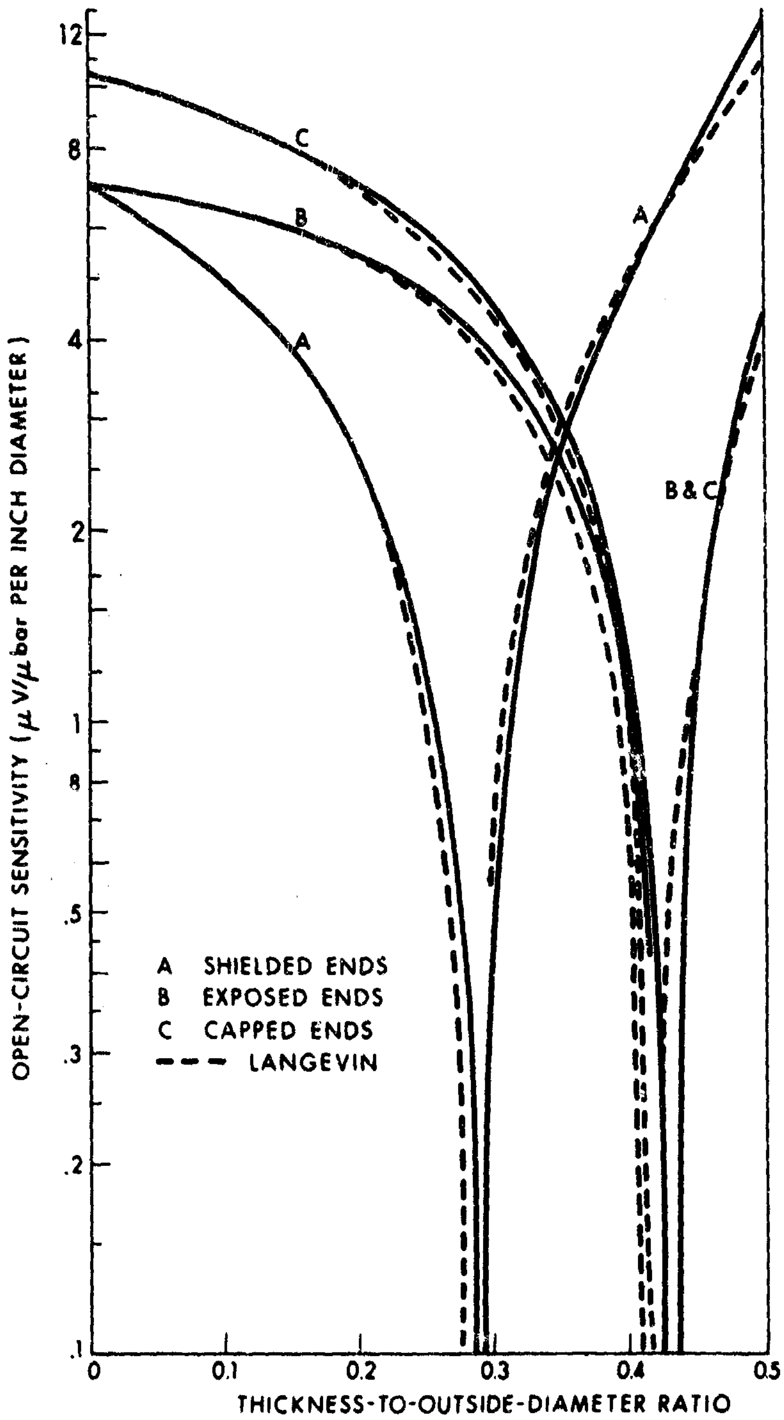

Fig. 2 - Open-Circuit Sensitivity versus Thickness-to-Outside-Diameter Ratio of a Radially Polarized Ceramic B Cylinder

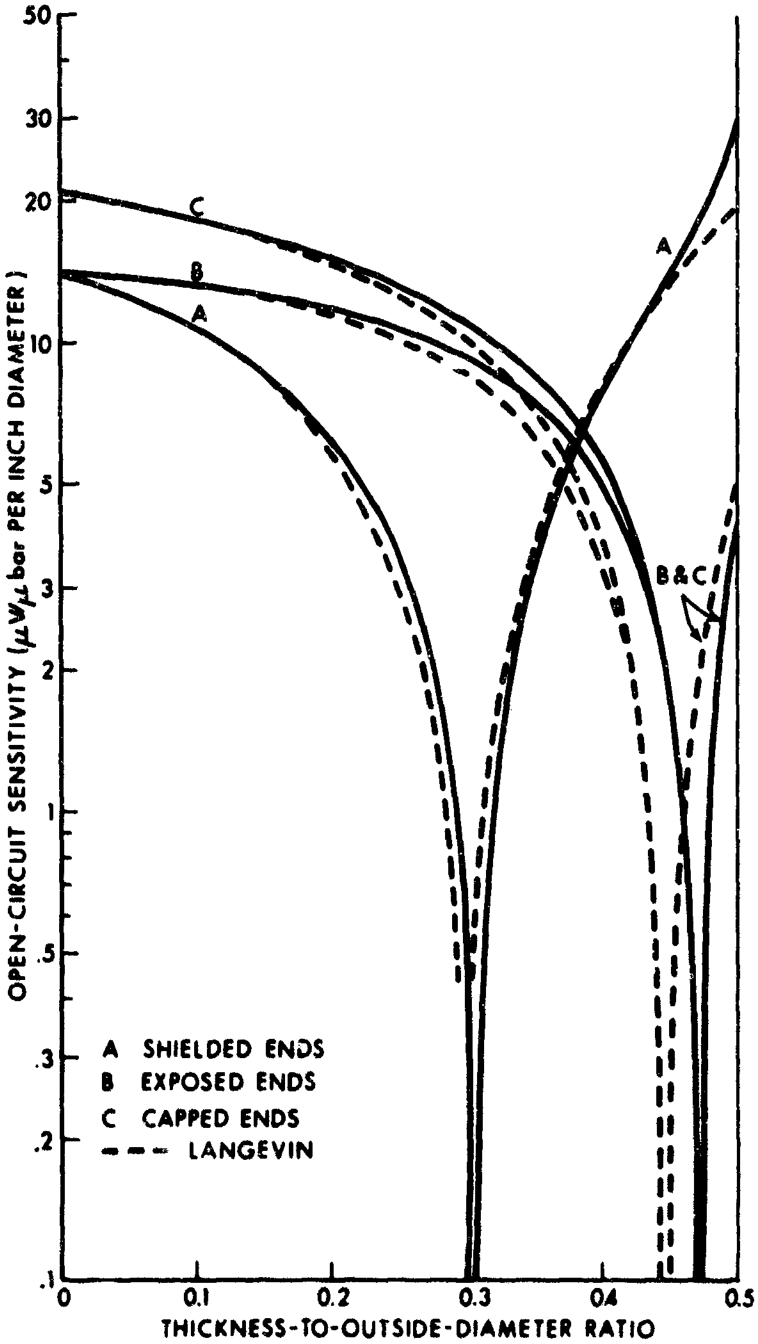

Fig. 3 - Open-Circuit Sensitivity versus Thickness-to-Outside-Diameter Ratio of a Radially Polarized PZT-4 Cylinder

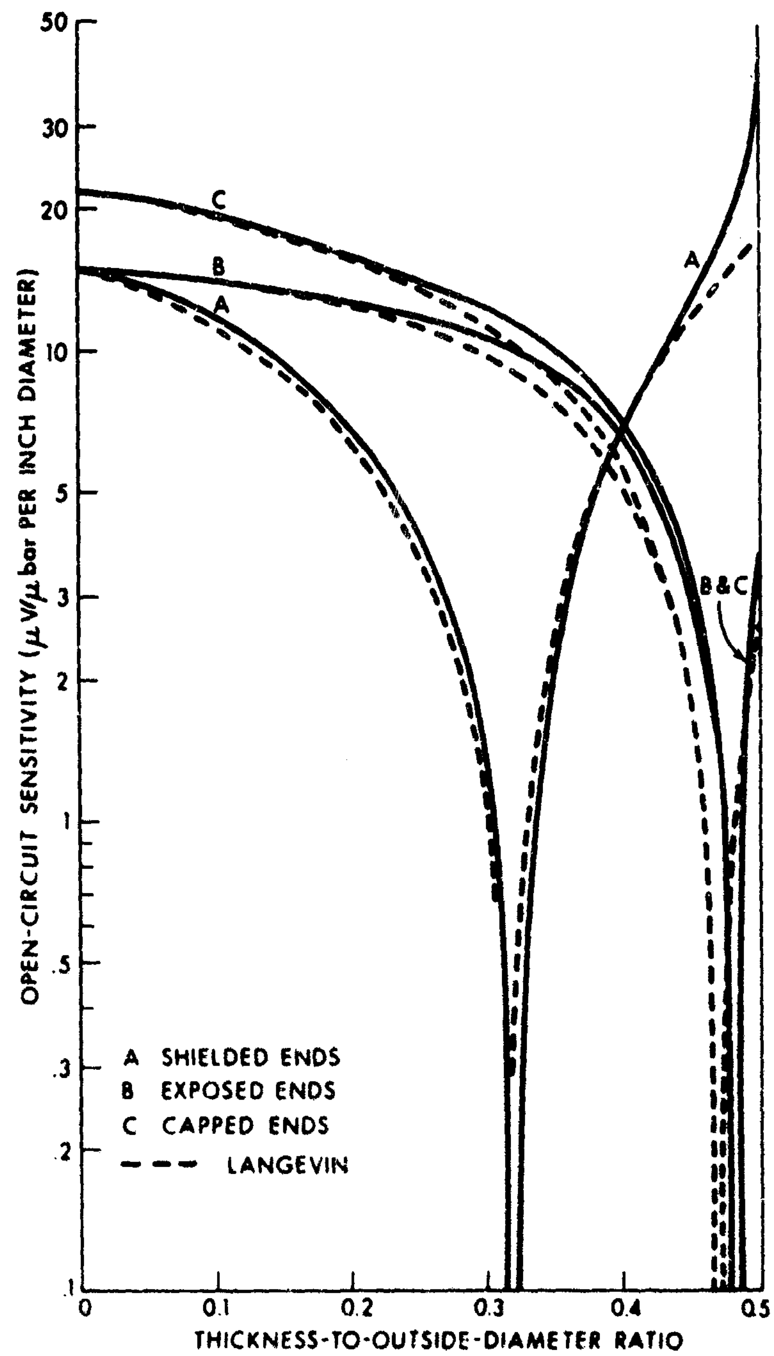

Fig. 4 - Open-Circuit Sensitivity versus Thickness-to-Outside-Diameter Ratio of a Radially Polarized PZT-5A Cylinder

Table 1

MATERIAL PROPERTIES OF CERAMICS

Parameters	Ceramic Materials		
	Ceramic B	PZT-4	PZT-5A
$s_{11}^D \left[\frac{m^2}{N}\right]$	8.3×10^{-12}	10.9×10^{-12}	14.4×10^{-12}
$g_{31} \left[\frac{\text{volt-m}}{N}\right]$	-5.5×10^{-3}	-11.1×10^{-3}	-11.4×10^{-3}
$g_{33} \left[\frac{\text{volt-m}}{N}\right]$	14.1×10^{-3}	26.1×10^{-3}	24.8×10^{-3}
$\gamma = s_{33}^D/s_{11}^D$	.843	.725	.657
$1 - \gamma$	.157	.275	.343
$(s_{12}^D - s_{13}^D) \left[\frac{m^2}{N}\right]$	-1.0×10^{-12}	-3.32×10^{-12}	-4.73×10^{-12}
$\beta = \frac{1}{2}(1 + s_{13}^D/s_{12}^D)$	.827	.693	.693
$r = \beta^2 - (1 - \gamma)$	.527	.205	.137
$2\sqrt{r}$	1.452	.906	.740
$1 - \beta + \sqrt{r}$	.899	.760	.677
$1 - \beta - \sqrt{r}$	-.553	-.146	-.063
$\beta + \sqrt{r}$	1.553	1.146	1.063
$\beta - \sqrt{r}$	.101	.240	.323
$(s_{12}^D - s_{13}^D)/s_{11}^D(1 - \gamma)$	-.767	-1.108	-.958

As is readily seen from the depicted data, the variations between the two representations are negligible as long as the wall thickness of the cylinders is kept relatively thin. The greatest variations occur at the nodes of the sensitivity curves. However, these regions are not of particular interest because they fall outside the realm of practical cylindrical hydrophone designs. If it is assumed that the maximum value of thickness-to-outside-diameter ratio (R) is approximately one quarter (0.25) for practical designs, then the greatest variation at this specified ratio occurs for an end-shielded Ceramic B ceramic cylinder and amounts to roughly 19 percent. For values of $R \leq 0.25$, the anisotropy of the material is not very influential and Langevin's formulas, which are easier to work with, can be used to predict the sensitivities of radially polarized ceramic cylinders with little error. This is especially true since the variations in the theoretical curves are well within the variations that might be expected from measured values of the open-circuit sensitivities.

The data also indicate that the nodes, positions of zero sensitivity, are altered by the degree of anisotropy of the material being used. Most noticeable is the shift in zero sensitivity of the PZT-4 ceramic cylinders for the capped- and exposed-end cases.

There is an alternative approach that can be used to determine the open-circuit sensitivities of radially polarized ceramic cylinders. This approach is based on the assumption that all shear effects can be completely ignored. In effect, this means deleting Eq. (13) from the set of constitutive equations presented earlier without any recourse to justifying the assumption. If this is done and the remaining assumptions that were previously documented are retained, then the radial-stress distribution must satisfy Eq. (21), which is repeated here for convenience:

$$T_3'' + \frac{3}{r} T_3' + \frac{(1 - s_{33}^D / s_{11}^D)}{r^2} T_3 + \frac{(s_{12}^D - s_{13}^D)}{r^2 s_{11}^D} T_2 = 0 . \qquad (52)$$

The complete solution of the above equation is the same as Eq. (36) when $\beta = 1$ and $\tau = \gamma = s_{33}^D / s_{11}^D$, or

$$T_3 = a_0 + a_1 r^{-1+\sqrt{\gamma}} + a_2 r^{-1-\sqrt{\gamma}} . \qquad (53)$$

Again it should be noted that when $s_{33}^D = s_{11}^D$ and $s_{12}^D = s_{13}^D$, Eq. (53) reverts back to the conventional stress distribution for an isotropic body.

Evaluation of the open-circuit sensitivities per inch diameter for the three cases discussed previously is straightforward and is most easily expressed by rewriting Eqs. (45), (47), and (51) with $\beta = 1$ and $\tau = \gamma$:

$$\frac{M_o}{2b}\,(\text{capped ends}) = \frac{g_{31}}{2}\,\frac{\left[2+\left(\frac{a}{b}\right)\right]}{\left[1+\left(\frac{a}{b}\right)\right]} - \frac{g_{33}\,(s^D_{12} - s^D_{13})}{2\,s^D_{11}(1-\nu)\left[1+\left(\frac{a}{b}\right)\right]}$$

$$+\;\frac{g_{33}}{2\sqrt{\gamma}}\,\frac{\left[1-\left(\frac{a}{b}\right)^{\sqrt{\gamma}}\right]}{\left[1+\left(\frac{a}{b}\right)^{\sqrt{\gamma}}\right]}\left\{1+\frac{(s^D_{12}-s^D_{13})}{s^D_{11}(1-\nu)\left[1-\left(\frac{a}{b}\right)\right]}\right\} \qquad (54)$$

$$\frac{M_o}{2b}\,(\text{exposed ends}) = \frac{g_{31}}{2}\left[2-\left(\frac{a}{b}\right)\right] - \frac{g_{33}\,(s^D_{12}-s^D_{13})}{2\,s^D_{11}\,(1-\nu)}\left[1-\left(\frac{a}{b}\right)\right]$$

$$+\;\frac{g_{33}}{2\sqrt{\gamma}}\,\frac{\left[1-\left(\frac{a}{b}\right)^{\sqrt{\gamma}}\right]}{\left[1+\left(\frac{a}{b}\right)^{\sqrt{\gamma}}\right]}\left\{1+\frac{(s^D_{12}-s^D_{13})}{s^D_{11}\,(1-\nu)}\left[1+\left(\frac{a}{b}\right)\right]\right\}, \qquad (55)$$

and

$$\frac{M_o}{2b}\,(\text{shielded ends}) = \frac{g_{31}}{2} + \frac{g_{31}}{2\sqrt{\gamma}}\,\frac{\left[1-\left(\frac{a}{b}\right)^{\sqrt{\gamma}}\right]}{\left[1+\left(\frac{a}{b}\right)^{\sqrt{\gamma}}\right]}. \qquad (56)$$

Figures 5 through 7 can be used to interpret the information contained in Eqs. (54), (55), and (56). It is obvious from the graphs that there is very little difference between the newer approach and the one introduced by Langevin, especially in the region $R \leq 0.25$. The major differences appear when the cylinder walls become excessively thick, approaching that of a solid rod. (The rod is still considered to be radially polarized.) However, this thick cylinder region is not of practical importance.

COMMENTS

The preceding analysis has demonstrated that there exists some doubt as to the rigorousness of the solutions that have been proposed for the open-circuit sensitivities of radially polarized ferroelectric ceramic cylinders. The primary paradox that evolved was that all the initial assumptions that were made could not be completely satisfied. More specifically, the shear stress and strain in the plane parallel to the cylinder axis could not be rigorously developed. It

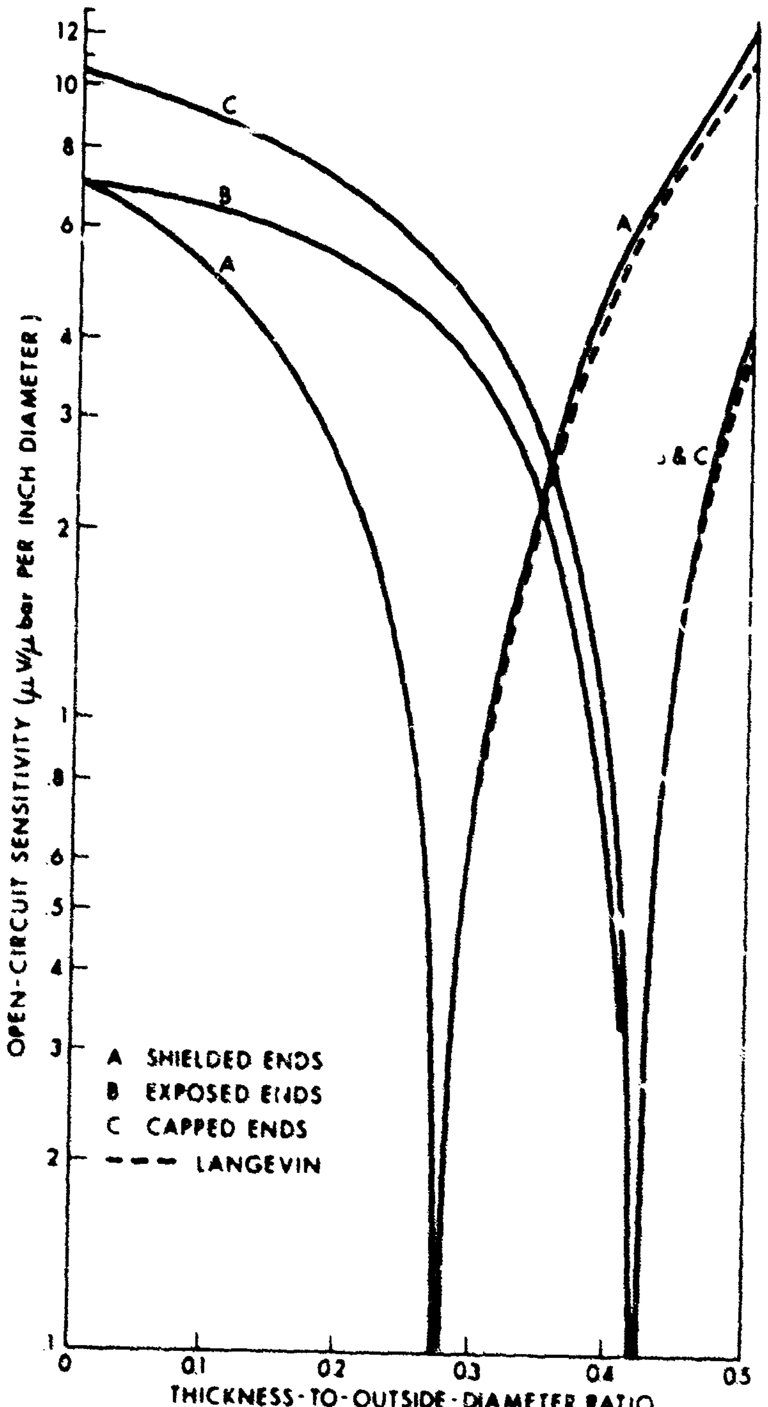

Fig. 5 - Open-Circuit Sensitivity versus Thickness-to-Outside-Diameter Ratio of a Radially Polarized Ceramic B Cylinder (Neglecting All Shear Effects)

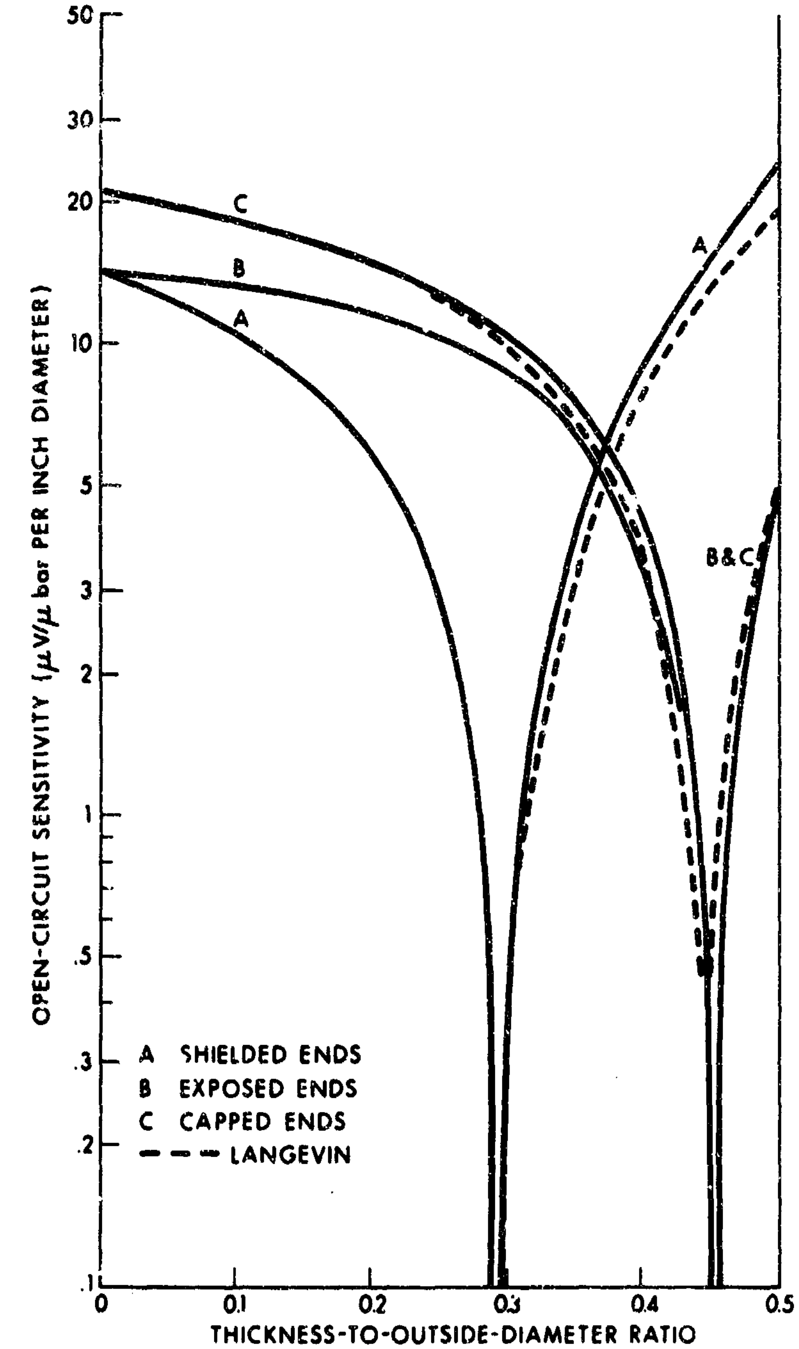

Fig. 6 - Open-Circuit Sensitivity versus Thickness-to-Outside-Diameter Ratio of a Radially Polarized PZT-4 Cylinder (Neglecting All Shear Effects)

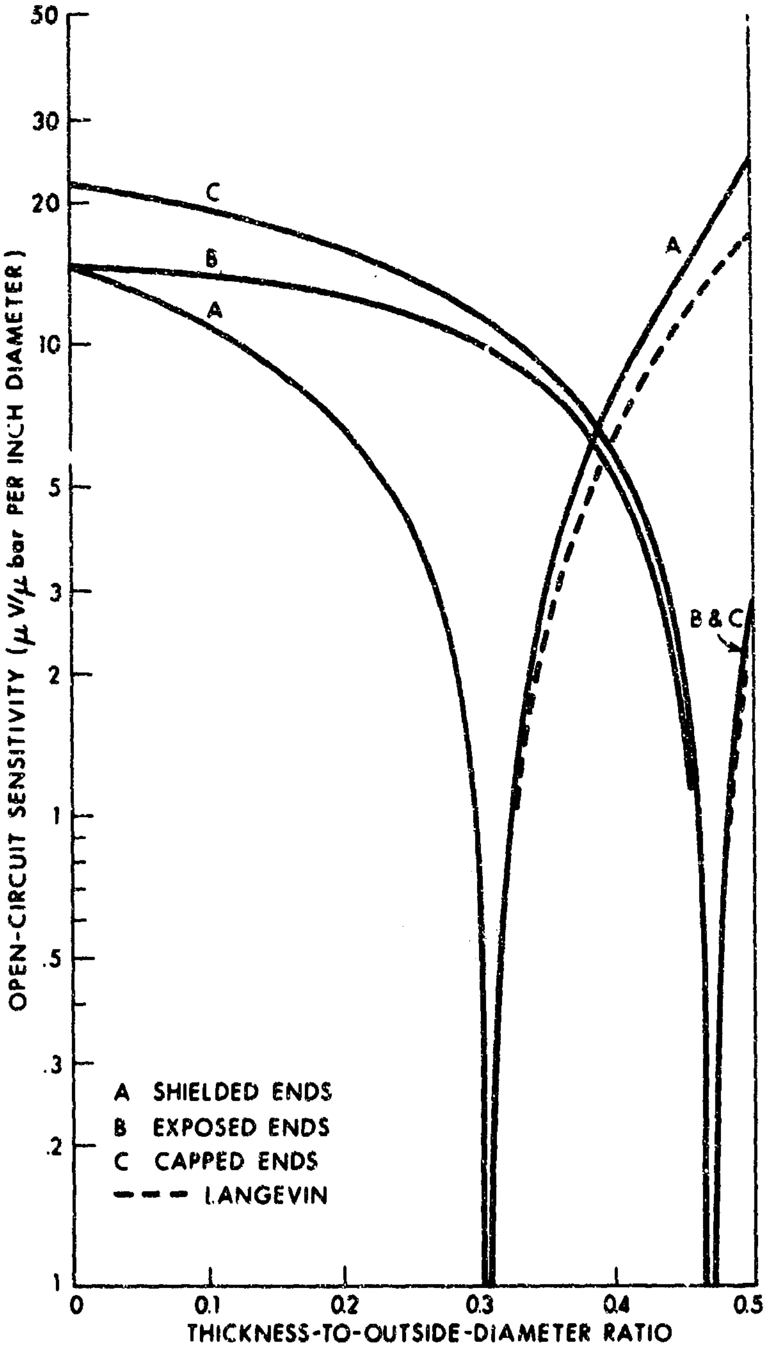

Fig. 7 - Open-Circuit Sensitivity versus Thickness-to-Outside-Diameter Ratio of a Radially Polarized PZT-5A Cylinder (Neglecting All Shear Effects)

appears as though this problem is restricted to radially polarized cylinders since it is not encountered in axially (longitudinally) polarized cylinders or in radially polarized hollow spheres. One plausible reason for the apparent paradox is the inadvertent use of the Cartesian matrix representation for the compliance (or stiffness) coefficients of polarized ferroelectric ceramics without regard to the different types of symmetry imposed by the geometrical shapes employed. Although the paradox stemmed from a quasi-static investigation, an equivalent paradox can be established for the dynamic behavior of cylinders of a similar type.[6]

In addition, this analysis shows that if the anisotropic properties of the materials employed in the design of hydrophone elements are considered, and if the conventional approach of neglecting shear effects is maintained, the results are not significantly different from those obtained by considering stress distributions that obey isotropic criteria. It should be mentioned that the preceding statement pertains to the currently available active ceramics. If newer materials that exhibit gross anisotropy are developed, then the theory will have to be altered to account for the anisotropy.

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13. ABSTRACT

The open-circuit sensitivity of radially polarized ferroelectric ceramic cylinders is derived by treating the ceramic as an anisotropic material. The analysis demonstrates that if the anisotropic properties of the materials employed in the design of cylindrical hydrophone elements are considered, the resulting sensitivities are not significantly different from those derived by Langevin, who considered only stress distributions that obey isotropic criteria. Graphs of the open-circuit sensitivity versus the ratio of wall thickness to outside diameter are presented for the two dissimilar approaches that define the regions over which the differences may be neglected.

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