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# THE MOLIERE APPROXIMATION FOR WAVE PROPAGATION IN TURBULENT MEDIA

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# THE MOLIERE APPROXIMATION FOR WAVE PROPAGATION IN TURBULENT MEDIA

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PREFACE

This Memorandum is part of Rand's study for the Advanced Research Projects Agency of those phenomena which affect the performance of optical reconnaissance and guidance equipment. The objective of these studies is to provide sufficient understanding to permit the systems analyst to compute performance estimates under various operational conditions.

In optical communications and related devices, the random variations in the received signal due to atmospheric turbulence can represent a severe limitation to system performance. Theoretical studies of these fluctuations have been based on solutions to the wave propagation equation which are correct only to first order in the refractive index deviation. It is shown in this Memorandum that a solution correct to all orders is readily obtained by direct analogy with a method commonly adopted to solve the Schrödinger equation for high energy potential scattering. Study of the more highly developed scattering theory provides useful insights into the nature and limitations of the various approximate solutions of propagation theory; in particular, the Memorandum points out that the standard approximation adopted in propagation studies is based on an approach known to be invalid in scattering theory. The results should be of interest to those concerned with the use of lasers in the atmosphere.

SUMMARY

By direct analogy with a method developed by Schiff to solve the Schrödinger equation for high energy potential scattering, it is possible to solve the equation for wave propagation in a turbulent medium in a manner which explicitly demonstrates that the solution so obtained is correct to all orders in the refractive index deviation and to lowest order in the stationary phase approximation. Although the solution is readily extended to next order in stationary phase, such an extension is recognized in scattering theory as unwarranted since it neglects terms of the same order from outside the region of stationary phase. The conventional "Born" and "Rytov" solutions widely adopted in propagation theory are of questionable validity since they represent approximations (first order in refractive index deviation) to the extended solution.

THE MOLIÈRE APPROXIMATION FOR WAVE PROPAGATION IN TURBULENT MEDIA

It is the purpose of this note to point out the applicability of the Molière approximation, used in high energy potential scattering theory, to treatments of wave propagation in random or turbulent media. In its usual formulation, the Molière approximation to the solution of the Schrödinger equation

$$[\nabla^2 + k^2 - U(\vec{r})] \psi(\vec{r}) = 0 \quad (1)$$

is (1-4)

$$\psi \sim \psi_M = \exp \left[ ikz - \frac{i}{2k} \int_{-\infty}^z U(x,y,z') dz' \right] \quad (2)$$

where  $e^{ikz}$  represents the incident wave function. The integration indicated in Eq. (2) is along a straight line parallel to the z-axis.\* With the condition  $ka \gg 1$ , where the length  $a$  represents the characteristic distance over which  $U(\vec{r})$  changes by a significant fraction of itself, the approximation applies throughout and near the region of the scattering potential, and may be used in the standard integral expression to determine the scattering amplitude. Although limited to determination of small angle scattering,\*\* it may be used for strong potentials where the Born approximation is of little value.

The wave equation for a scalar wave propagating in a medium whose index of refraction is  $n(\vec{r})$ ,

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\*The WKB approximation is  $\psi \sim \psi_W = \exp(i \int \sqrt{k^2 - U} ds)$ , with the integration along the classical trajectory.

\*\*Schiff(3) has extended the Molière approximation for use in large angle scattering.

$$(\nabla^2 + n^2 k^2) \psi = 0 \quad (3)$$

may be made formally equivalent to Eq. (1) with the substitution

$$U(\vec{r}) = -\epsilon k^2, \text{ where } \epsilon(\vec{r}) = n^2 - 1.$$

With the assumption that  $n = 1$  when  $z < 0$ , the Moliere approximation for the wave in the "near zone" ( $0 < z \lesssim ka^2$ ) may be expressed in the form

$$\psi_M = \psi_0 e^{iS_0} \quad (4)$$

where  $\psi_0 = e^{ikz}$  is the incident wave, and\*

$$S_0 = \frac{k}{2} \int_0^z \epsilon(x, y, z') dz' \quad (5)$$

Expansion of Eq. (4) gives the Born approximation,

$$\psi_B = \psi_0 (1 + iS_0) \quad (6)$$

In treatments of propagation in random media, the index of refraction is usually expressed in the form  $n(\vec{r}) = 1 + n_1(\vec{r})$ ,\*\* hence  $\epsilon = 2n_1 + n_1^2$ , and  $S_0$  may be expressed in the form

$$S_0 = k \int_0^z n_1(x, y, z') dz' + \frac{k}{2} \int_0^z n_1^2(x, y, z') dz' \quad (7)$$

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\* In the WKB approximation,  $\psi_W = \exp(ik \int n ds)$ , where the integration is along the ray path.

\*\* It will be assumed that the mean value of  $n_1$  satisfies  $\langle n_1 \rangle = 0$ , and that  $\langle n_1^2 \rangle$  is constant.

If, as is often assumed,  $|n_1| \ll 1$ , the Born approximation may be expressed as

$$\psi_B = \psi_0 (1 + iS'_0) \quad (8)$$

where

$$S'_0 = k \int_0^z n_1(x, y, z') dz' \quad (9)$$

The corresponding "Rytov approximation"

$$\psi_R = \psi_0 e^{iS'_0} \quad (10)$$

is somewhat ambiguous in that its exponential form implies a higher order (in  $n_1$ ) approximation.

Schiff<sup>(3)</sup> has shown explicitly how the infinite Born expansion may be summed, with each term in the expansion evaluated to lowest order in the stationary phase approximation, to give the Moliere approximation. Thus, Eq. (4) gives the wave in the zeroth-order stationary phase approximation correct to all orders in  $n_1$ ; the Born and Rytov approximations, Eqs. (8) and (10), are zeroth-order stationary phase approximations correct to first order in  $n_1$ . Schiff points out that the next order stationary phase approximation, given by

$$\psi_M^{(1)} = \psi_0 e^{iS'_0 + S_1} \quad (11)$$

with

$$S_1 = -\frac{1}{4} \int_0^z (z - z') h(x, y, z') dz' \quad (12)$$

where  $h = \nabla^2 \epsilon$ , is inconsistent in that it neglects terms from outside the stationary phase region which, in general, are of the same order. The stationary phase expansion is an expansion in powers of  $(ka)^{-1}$ ; Gol'dman and Migdal<sup>(2)</sup> have emphasized that such "classical" expansions are asymptotic and can never give diffraction effects correctly.

Similarly, the first-order (in  $n_1$ ) approximation to Eq. (11), in either the Born or Rytov form,

$$\psi_B^{(1)} = \psi_0 (1 + iS'_0 + S'_1) \quad (13)$$

or

$$\psi_R^{(1)} = \psi_0 e^{iS'_0 + S'_1} \quad (14)$$

where

$$S'_1 = -\frac{1}{4} \int_0^z (z - z') h'(x, y, z') dz' \quad (15)$$

with  $h' = 2\nabla^2 n_1$ , is also inconsistent.

Equations (13) and (14) have been widely adopted in the propagation literature as approximations to the wave in the near zone. Their inconsistency is not manifest in the conventional development<sup>(5,6)</sup> where the wave equation is expanded in powers of  $n_1$  with  $S_0$  and  $S_1$



treated as quantities of the same order. Only after the first order, or Rytov, equation is reformulated as an integral equation is the stationary phase approximation introduced, essentially by restriction of the region of integration to the "Fresnel cone."\* This cone has its vertex at the field point and opens in the negative z-direction with angular spread  $\sim 1/ka$ ; the same angle characterizes the "diffraction cone," which has its vertex at the scattering inhomogeneity, or "turbule," and opens in the positive z-direction. The Fresnel approximation is introduced as a consequence of the well-established "forward scattering" approximation (the assumption that essentially all the diffracted radiation is confined to the diffraction cone) and is conjectured to be of general validity provided  $ka \gg 1$ . This view contrasts strongly with that adopted in potential scattering theory, where the use of the Fresnel approximation is restricted to the determination of  $S_0$ .

It is of interest in this regard to consider a simple example, the scattering of a plane wave by a single spherical turbule with radius  $a$  and constant index deviation  $n_1$  with the condition  $|n_1 ka| \ll 1$ . As is well known, the scattered field may, with reasonable unambiguity, be separated into a widespread (i.e., throughout distances of order  $a$  in the x,y-plane), weak coherent "refractive" field and a narrow, intense diffraction beam. If a field point very near (but not necessarily on the axis of) the turbule is considered, that portion of the turbule within the Fresnel cone gives the refraction field with excellent accuracy, but gives no diffraction field. At a larger distance,

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\* More precisely, the Fresnel paraboloid of revolution<sup>(3,7)</sup> (the region of stationary phase).

the Fresnel cone intersects a greater section of the turbule, and implies the existence of a weak, wide-angle diffraction beam, in addition to the refraction field. As the distance is increased, the diffraction pattern as determined by the material in the pertinent Fresnel cone increases in intensity (and total scattered energy) and decreases in angular opening; the refraction fields remain accurately determined. For distances exceeding  $ka^2$ , the Fresnel cones associated with points in the scattered field intercept essentially the entire turbule, and the approximation is obviously valid.

The errors associated with this initial elimination and gradual mixing-in of the diffraction field depend on the quantities being determined. For example, the refractive and diffractive phase shifts are of the same order, and the narrowness of the diffraction beam insures that the near-zone transverse mean-square phase deviation associated with a collection of independent spheres with randomly distributed values of  $n_1$  will have negligible error in the Fresnel approximation. In contrast, the amplitude perturbation in the diffraction field is considerably larger than that in the refraction field, and it is easy to show that, despite the narrowness of the diffraction field, the amplitude fluctuation associated with scattering by a densely packed collection of random spheres is everywhere dominated by the diffractive effects. The example, while highly artificial, is in agreement with the general conclusions of potential scattering where the Fresnel approximation is used to determine  $S_0$  (a phase perturbation) and where  $S_1$  (the Fresnel amplitude perturbation) is regarded as inconsistent.

Nevertheless, the admonition of Schiff and others against the extended use of the Fresnel approximation is of necessity somewhat imprecise in that it is based on the observation that the contributions from outside the Fresnel cone are likely to be of the same order as  $S_1$ ; it remains possible that, under certain circumstances, the outside contributions may be considerably smaller than those from inside. Although use of the general wave equation to estimate conditions for which the extended Fresnel approximation may be valid presents a formidable task, it is possible to show<sup>(8)</sup> that when  $z/\langle n_1^2 \rangle k^2 a^5 > 1$ , the conventional near-zone Fresnel solution (Eq. (14)) of the Rytov equation agrees with the assumptions underlying the derivation of the Rytov equation as an approximation to the wave equation. While this "consistency" condition, based on a solution of the approximate equation, is neither necessary nor sufficient for the validity of the extended Fresnel approximation, such conditions are commonly adopted when more rigorous conditions (those based on solutions of the general wave equation) are unavailable.

A more familiar consistency condition on the Rytov approximation is the Born condition  $|S_0| < 1$ , which implies  $\langle S_0^2 \rangle < 1$ ; and hence (note the Fresnel approximation is valid for the determination of  $S_0$ ) the "upper limit" condition  $\langle n_1^2 \rangle k^2 az < 1$ .<sup>\*</sup> Thus, values of  $z$  for which the conventional Fresnel-Rytov solutions are consistent may be determined from the parameter  $\gamma = \langle n_1^2 \rangle (ka)^3$ : When  $\gamma < 1$  the solutions

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<sup>\*</sup>This condition is neither necessary nor sufficient for the validity of the Rytov equation and has been derived many times with various approaches, some of which are discussed in Ref. 8.

are consistent (although not necessarily valid) in the interval  $\gamma \lesssim z/ka^2 \lesssim \gamma^{-1}$ ; when  $\gamma > 1$  the solutions are inconsistent (although not necessarily invalid) for all  $z$ .<sup>\*</sup> Reasonable estimates for the various parameters indicate  $\gamma \gg 1$  for optical propagation in the atmosphere, hence, in this application, the Fresnel-Rytov solutions should be used with caution.

### CONCLUSIONS

Schiff's approach to the solution of the Schrödinger equation for high energy potential scattering<sup>(3)</sup> may be used to solve the scalar wave equation for propagation in a turbulent medium in a manner which explicitly demonstrates that the Moliere solution (Eq. (4)) is correct to all orders in the refractive index deviation. The method is readily extended to higher order in stationary phase, but the extended solution (Eq. (11)) is invalid<sup>(2,3)</sup> since it neglects terms of the same order from outside the stationary phase region. The conventional near-zone Born and Rytov solutions are approximations (of order  $n_1$ ) to the extended solution, and their use is unjustified.

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\* A closely related inconsistency is that which arises from the assertion (Refs. 5 and 6) that, for  $z \lesssim ka^2$ , the Rytov equation may be further approximated and render valid the equations of linear (in  $n_1$ ) geometric optics. This view, when combined with the widely adopted upper limit given above ( $\gamma z < ka^2$ ) implies, inconsistently, the existence when  $\gamma > 1$  of an interval  $\gamma^{-1} < z/ka^2 < 1$  in which the Rytov equation is invalid and the less general geometric equation is valid.

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