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THEMIS SIGNAL ANALYSIS STATISTICS RESEARCH PROGRAM

CORRELATION BETWEEN TWO HOTELLING'S T^2

by

A. M. Kshirsagar* and John C. Young

Technical Report No. 79
Department of Statistics THEMIS Contract



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1. Introduction:

Let B be a $p \times p$ symmetric matrix having the Wishart distribution

$$(1.1) \quad W_p(B|I|f)dB = C_{pf} |B|^{(f-p-1)/2} e^{-1/2 \text{tr}B} dB ,$$

where

$$(1.2) \quad C_{pf}^{-1} = 2^{fp/2} \pi^{p(p-1)/4} \prod_{i=1}^p \Gamma\left(\frac{f+1-i}{2}\right) ,$$

and dB stands for the product of the differentials of the $p(p+1)/2$ distinct elements of B . Let \underline{x} and \underline{y} be two vector variables of p components, distributed independently of B , and also independently of each other, as

$$(1.3) \quad \frac{1}{(2\pi)^{p/2}} e^{-1/2 \underline{x}'\underline{x}} d\underline{x} ,$$

and

$$(1.4) \quad \frac{1}{(2\pi)^{p/2}} e^{-1/2 \underline{y}'\underline{y}} d\underline{y}$$

respectively. While considering the problem of multivariate statistical outliers, Wilks (1963) used statistics of the type,

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$$(1.5) \quad r = |B + \underline{y}\underline{y}'| / |B + \underline{x}\underline{x}' + \underline{y}\underline{y}'|, \quad s = |B + \underline{x}\underline{x}'| / |B + \underline{x}\underline{x}' + \underline{y}\underline{y}'|.$$

He has remarked that the exact distribution (joint) of r and s is complicated and has given the expected values, variances and covariance of r and s . Unfortunately, his expressions for the variance and covariance are in error. The purpose of this note is to derive the exact joint distribution of r and s and to give correct expressions for the moments.

2. Joint distribution:

In the joint distribution of B , \underline{x} and \underline{y} , make the following transformation

$$(2.1) \quad \begin{aligned} A &= B + \underline{x}\underline{x}' + \underline{y}\underline{y}', \\ \underline{u} &= A^{-1/2} \underline{x}, \\ \underline{v} &= A^{-1/2} \underline{y}, \end{aligned}$$

where $A^{-1/2}$ is any matrix such that $A^{-1/2} \cdot A^{-1/2} = A^{-1}$. The Jacobian of transformation from B to A is 1 and that from \underline{x} to \underline{u} or \underline{y} to \underline{v} is $|A|^{1/2}$ and hence, the joint distribution of A , \underline{u} and \underline{v} comes out as

$$(2.2) \quad \frac{C_{pf}}{(2\pi)^p} |A|^{\frac{(f+2)-p-1}{2}} e^{-1/2 \operatorname{tr}A} \cdot |I - \underline{u}\underline{u}' - \underline{v}\underline{v}'|^{\frac{f-p-1}{2}} dA d\underline{u} d\underline{v},$$

$$\text{as } |B| = |A - A^{1/2} \underline{u}\underline{u}' A^{1/2} - A^{1/2} \underline{v}\underline{v}' A^{1/2}| = |A| |I - \underline{u}\underline{u}' - \underline{v}\underline{v}'|.$$

This shows that A has a Wishart distribution of $f+2$ degrees of freedom and is independent of \underline{u} and \underline{v} . Splitting the constant suitably, the joint distribution of \underline{u} and \underline{v} is

$$(2.3) \quad \frac{\Gamma(f+1)}{(2\pi)^p \Gamma(f-p+1)} |I - \underline{u}\underline{u}' - \underline{v}\underline{v}'|^{\frac{f-p-1}{2}} d\underline{u} d\underline{v}.$$

Observe that the statistics r , s of Wilks are given by

$$(2.4) \quad r = \frac{|B + \underline{y}\underline{y}'|}{|B + \underline{x}\underline{x}' + \underline{y}\underline{y}'|} = \frac{|A - \underline{x}\underline{x}'|}{|A|} = |I - \underline{u}\underline{u}'| = 1 - \underline{u}'\underline{u},$$

and

$$(2.5) \quad s = \frac{|B+xx'|}{|B+xx'+yy'|} = 1 - \underline{v}'\underline{v} \quad .$$

Also observe that in (2.3)

$$(2.6) \quad \begin{aligned} |I - \underline{u}\underline{u}' - \underline{v}\underline{v}'| &= (1 - \underline{u}'\underline{u})(1 - \underline{v}'\underline{v}) - (\underline{u}'\underline{v})^2 \\ &= rs - (\underline{u}'\underline{v})^2 \quad . \end{aligned}$$

In (2.3), transform from \underline{v} to $\underline{w} = [w_1, w_2, \dots, w_p]'$, by an orthogonal transformation

$$(2.7) \quad \underline{w} = L\underline{v} \quad ,$$

where

L is a $p \times p$ orthogonal matrix, whose last row is $\underline{u}'/\sqrt{\underline{u}'\underline{u}}$. The Jacobian of this transformation is $|L| = 1$ and $\underline{v}'\underline{v} = \underline{w}'\underline{w} = 1-s$. Also

$$(2.8) \quad \underline{u}'\underline{v} = \underline{u}'L'L\underline{v} = [0 \dots 0, \sqrt{\underline{u}'\underline{u}}] \underline{w} = \sqrt{1-r} \cdot w_p \quad .$$

The joint distribution of \underline{u} and \underline{w} is, therefore,

$$(2.9) \quad \frac{\Gamma(f+1)}{(2\pi)^p \Gamma(f-p+1)} \left\{ rs - (1-r)w_p^2 \right\}^{\frac{f-p-1}{2}} \underline{d}\underline{u}\underline{d}\underline{w} \quad .$$

From \underline{u} , transform to $r = 1 - \underline{u}'\underline{u}$ and $p-1$ other variables

$\phi_1, \phi_2, \dots, \phi_{p-1}$ by

$$(2.10) \quad \begin{aligned} u_1 &= (1-r)^{1/2} \cos\phi_1 \cos\phi_2 \dots \cos\phi_{p-1} \quad , \\ u_j &= (1-r)^{1/2} \cos\phi_1 \cos\phi_2 \dots \cos\phi_{p-j} \sin\phi_{p-j+1} \\ &\quad (j=2, 3, \dots, p) \end{aligned}$$

Similarly, transform from \underline{w} to $s = 1 - \underline{w}'\underline{w}$ and $p-1$ other variables

$\theta_1, \theta_2, \dots, \theta_{p-1}$ by

$$(2.11) \quad \begin{aligned} v_1 &= (1-s)^{1/2} \cos\theta_1 \cos\theta_2 \dots \cos\theta_{p-1} \quad , \\ v_j &= (1-s)^{1/2} \cos\theta_1 \cos\theta_2 \dots \cos\theta_{p-j} \sin\theta_{p-j+1} \quad . \\ &\quad (j=2, 3, \dots, p) \end{aligned}$$

The Jacobian of transformation from \underline{u} to $r, \phi_1, \dots, \phi_{p-1}$ is

$$\frac{1}{2}(1-r)^{\frac{1}{2} p-1} \prod_{i=1}^{p-2} \cos \phi_i^{p-i-1}$$

and a similar expression in s and θ_i for the Jacobian of transformation from \underline{w} to s and the θ 's. Now θ_{p-1} and ϕ_{p-1} vary from 0 to 2π , the other θ 's and ϕ 's vary from $-\pi/2$ to $\pi/2$ while r and s vary from 0 to 1. Integrating out all the ϕ 's and all θ 's except θ_1 , we obtain the joint distribution of r, s and θ_1 as

$$(2.12) \quad \frac{\Gamma(f+1)}{4\pi\Gamma(p-1)\Gamma(f-p+1)} \{rs - (1-r)(1-s)\sin^2\theta\}^{\frac{f-p-1}{2}} \cos^{p-2}\theta \, dr ds d\theta$$

where θ_1 is replaced by θ .

The joint distribution of r, s alone can now be obtained by integrating out θ but this does not seem to yield a manageable expression, as the bracket in (2.12) will have to be expanded in a series.

3. Moments of r, s .

Only the product moment of r and s is difficult to obtain. The mean and variance of r (or s) can be very easily obtained from the marginal distribution of r , which is related to the well-known Hotelling's

T^2 by $r = \frac{1}{1 + \left(\frac{T^2}{f+1}\right)}$. In the joint distribution of \underline{u} and \underline{v} , given by

(2.3), if we transform to $\underline{z} = [z_1, \dots, z_p]'$ from \underline{v} by

$$(3.1) \quad \underline{v} = (\mathbf{I} - \underline{u}\underline{u}')^{1/2} \underline{z},$$

we shall find that \underline{u} and \underline{z} are independently distributed as

$$(3.2) \quad K(\underline{u}|f) d\underline{u} = \frac{f}{\pi^{p/2}(f-p)} \cdot \frac{\Gamma(f/2)}{\frac{\pi}{2}(f-p)} |\mathbf{I} - \underline{u}\underline{u}'|^{\frac{f-p}{2}} d\underline{u}$$

$$(3.3) \quad \text{and } K(\underline{z}|f-1) d\underline{z}, \text{ respectively.}$$

From (3.2), one can easily show that

$$(3.4) \quad E(r^h) = E(1 - \underline{u}'\underline{u})^h = E|1 - \underline{u}\underline{u}'|^h \\ = \frac{\Gamma(\frac{f-p}{2} + h)\Gamma(\frac{f}{2})}{\Gamma(\frac{f}{2} + h)\Gamma(\frac{f-p}{2})}$$

This will also be the h^{th} moment of s by symmetry. This leads to

$$(3.5) \quad E(r) = \frac{f-p+2}{f+2}, \quad v(r) = \frac{2p(f-p+2)}{(f+2)^2(f+4)}$$

$$\begin{aligned} \text{Now Cov}(r,s) &= E\{(1-\underline{u}'\underline{u})(1-\underline{v}'\underline{v})\} - E(r)E(s) \\ &= E\{(1-\underline{u}'\underline{u})[1-\underline{z}(I-\underline{u}\underline{u}')\underline{z}]\} - \{E(r)\}^2 \quad \text{by (3.1)} \\ &= E(r) - E\{r[\underline{z}'\underline{z} - (\underline{z}'\underline{u})^2]\} - \{E(r)\}^2 \\ (3.6) \quad &= E(r) - E(r)E(\underline{z}'\underline{z}) + E\{r(\underline{z}'\underline{u})^2\} - \{E(r)\}^2, \end{aligned}$$

as \underline{z} and r are independent. Since \underline{z} has the same distribution as \underline{u} with f changed $f-1$,

$$(3.7) \quad \begin{aligned} E(\underline{z}'\underline{z}) &= 1 - E(1 - \underline{u}'\underline{u}) \text{ with } f \text{ replaced by } f-1 \\ &= \frac{p}{f+1} \end{aligned}$$

Hence (3.6) reduces to

$$(3.8) \quad \text{Cov}(r,s) = \frac{-p(f-p+2)}{(f+1)(f+2)^2} + E\{r(\underline{z}'\underline{u})^2\}.$$

Now

$$(3.9) \quad E\{r(\underline{z}'\underline{u})^2\} = \int (1-\underline{u}'\underline{u})(\underline{z}'\underline{u})^2 K(\underline{u}|f)K(\underline{z}|f-1) d\underline{u}d\underline{z}$$

where the integration is over the range of values of \underline{u} and \underline{z} such that $\underline{u}'\underline{u} \leq 1$, $\underline{z}'\underline{z} \leq 1$. Transform from \underline{z} to $\underline{\xi} = [\xi_1, \dots, \xi_p]$ by the transformation

$$\underline{\xi} = L\underline{z},$$

where L is already defined to be a $p \times p$ orthogonal matrix, whose last row is $\underline{u}'/\sqrt{\underline{u}'\underline{u}}$. Then,

$$\underline{z}'\underline{u} = \underline{z}'L'Lu = \underline{\xi}'Lu = \xi_p \sqrt{\underline{u}'\underline{u}} = (1-r)^{1/2} \xi_p.$$

Hence (3.9) reduces to

(3.10) $\int r(1-r)K(\underline{u}|f)d\underline{u} \cdot \int \int_p^2 K(\underline{t}|f-1)d\underline{t} = K(r-r^2) \cdot \frac{1}{p} K(\underline{t}'\underline{t})$, due to symmetry of the distribution of \underline{t} . Now \underline{t} has the same distribution as \underline{u} with f replaced by $f-1$ and hence finally, (3.10) reduces to

$$\frac{p(f-p+2)}{(f+4)(f+2)} \cdot \frac{1}{f+1}.$$

The covariance between r and s , therefore, is (from (3.5))

(3.11) $\frac{-2p(f-p+2)}{(f+1)(f+2)^2(f+4)}.$

Remarks:

Wilks considers a sample of size n and has a Wishart matrix based on $n-1$ degrees of freedom as deviations are from the sample means. He then removes two observations as outliers and thus his $(n-1)-2$ corresponds to our f . His $E(r)$ agrees with our result, with this correspondence but the other moments are in error.

Reference

Wilks, S. S. (1963). "Multivariate Statistical Outliers," Sankhyā, Vol. 25, p. 407-426.

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14. ABSTRACT If B is a Wishart matrix and \underline{x} , \underline{y} are two vectors of p components each having a multinormal distribution and if all these quantities are independently distributed, the joint distribution of the two statistics $r = \frac{ B + \underline{y}\underline{y}' }{ B + \underline{xx}' + \underline{y}\underline{y}' } \quad \text{and} \quad s = \frac{ B + \underline{xx}' }{ B + \underline{xx}' + \underline{y}\underline{y}' }$ is derived in this paper. The correlation between r and s is also obtained. r and s are related to Hotelling's T ² and are useful in problems of testing multivariate outliers.			