CONFIDENCE INTERVALS FOR INDEPENDENT EXPONENTIAL SERIES SYSTEMS

by

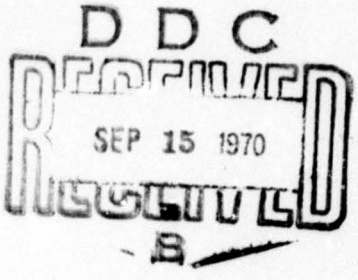
GERALD J. LIEBERMAN and SHELDON M. ROSS

TECHNICAL REPORT NO. 130
AUGUST 1970

Supported by the Army, Navy and Air Force
Under Contract N00014-67-A-0112-0052 (NR-042-002)
with the Office of Naval Research

Gerald J. Lieberman, Project Director

DEPARTMENT OF OPERATIONS RESEARCH
and
DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA



CONFIDENCE INTERVALS

FOR INDEPENDENT EXPONENTIAL SERIES SYSTEMS

by

Gerald J. Lieberman and Sheldon M. Ross

1. Introduction and Summary

Suppose $X_1, X_2, \ldots, X_n$ are independent identically distributed exponential random variables with parameter λ_1, i.e., the density function is given by

$$f_X(x) = \begin{cases} 0 & , \text{ for } x < 0 \\ \lambda_1 e^{-\lambda_1 x} & , \text{ for } x \geq 0 . \end{cases}$$

Let $Y_1, Y_2, \ldots, Y_m$ also be independent identically distributed exponential random variables but with parameter λ_2, and assume that the X's and Y's are independent. The problem is to estimate

$$R(t) = e^{-(\lambda_1+\lambda_2)t} .$$

The motivation behind this is that if one has a series system with two independent exponential components then $R(t)$ represents the reliability of the system at time t, i.e., the probability that the system survives until time t.

In Section 2 we present some point estimates. In Section 3 we present a procedure for determining an exact $(1-\alpha)$ level lower confidence bound for $e^{-(\lambda_1+\lambda_2)t}$. In doing so we also obtain an interesting characterization of the minimum of two independent gamma random variables. In Section 4 we compare our procedure with others presented in the literature. We show that our procedure always gives

smaller confidence intervals than the procedure presented by Kraemer [2] and that it dominates (in the sense of yielding stochastically smaller confidence intervals) a procedure due to Sarkar [4]. Other procedures are also considered. Section 5 is concerned with the same problem when the data is of a different form, including Type II censoring at r out of R. In the final section we point out that our results easily extend to the k-sample problem.

2. Point Estimators

Since $\bar{X} = \sum_{i=1}^{n} X_i/n$ and $\bar{Y} = \sum_{i=1}^{m} Y_i/m$ are the "usual" estimators of $1/\lambda_1$ and $1/\lambda_2$ respectively, a "reasonable" estimator (the maximum likelihood estimator) or $e^{-(\lambda_1+\lambda_2)t}$ is given by

$$\exp\left\{-\left(\frac{\bar{X}\,\bar{Y}}{\bar{X}+\bar{Y}}\right)t\right\}. \tag{1}$$

Another possible point estimator is the unique minimum variance unbiased estimator (MVUE), which can be obtained as follows:

Letting

$$Z = \begin{cases} 1, & \text{if } X_1 > t,\ Y_1 > t \\ 0, & \text{otherwise} \end{cases},$$

it follows that

$$E(Z) = e^{-(\lambda_1+\lambda_2)t},$$

i.e., Z is an unbiased estimator of $e^{-(\lambda_1+\lambda_2)t}$. Now, since the joint statistic $\left(\sum_{i=1}^{n} X_i,\ \sum_{i=1}^{m} Y_i\right)$ is both sufficient and complete, it follows from the Rao-Blackwell (also referred to as the Lehmann-

Scheffé) theorem that $E\left(Z \middle| \sum_{i=1}^{n} X_i, \sum_{i=1}^{m} Y_i\right)$ is the unique MVUE. Now,

$$E\left(Z \middle| \sum_{i=1}^{n} X_i, \sum_{i=1}^{m} Y_i\right) = P\left\{X_1 > t, Y_1 > t \middle| \sum_{i=1}^{n} X_i, \sum_{i=1}^{m} Y_i\right\}$$

$$(2) \qquad = \begin{cases} 0 & , \text{ if } \sum_{i=1}^{n} X_i \le t \text{ or } \sum_{i=1}^{m} Y_i \le t \\ \left[\dfrac{\sum_{i=1}^{n} X_i - t}{\sum_{i=1}^{n} X_i}\right]^{n-1} \left[\dfrac{\sum_{i=1}^{m} Y_i - t}{\sum_{i=1}^{m} Y_i}\right]^{m-1} & , \text{ if } \sum_{i=1}^{n} X_i > t, \ \sum_{i=1}^{m} Y_i > t , \end{cases}$$

where the last equation follows from the well known fact that given $\sum_{i=1}^{n} X_i$, then X_1 is distributed as the smallest of $(n-1)$ order statistics from a uniform distribution on $\left(0, \sum_{i=1}^{n} X_i\right)$. (A similar statement holds, of course, for the Y's.)

Thus, either (1) or (2) may be used as point estimators of $e^{-(\lambda_1+\lambda_2)t}$. However, neither estimator is useful in obtaining a confidence interval for the reliability. This difficulty arises because the distributions of (1) and (2) depend on both λ_1 and λ_2 and not only on $\lambda_1 + \lambda_2$. In other words, a nuisance parameter would have to be eliminated. In the next section, we present a method for obtaining an exact confidence interval for $e^{-(\lambda_1+\lambda_2)t}$.

3. Confidence Intervals

We shall use the notation

$$V \sim \text{Gamma } (\gamma, \beta)$$

to indicate that the random variable V is distributed as a gamma

random variable with parameters γ and β, i.e., its density function is given by

$$f_V(v) = \begin{cases} \frac{\beta^\gamma}{\Gamma(\gamma)} v^{\gamma-1} e^{-\beta v}, & \text{for } v \geq 0 \\ 0, & \text{for } v < 0, \end{cases}$$

where $\gamma, \beta > 0$. Similarly, the notation

$$V \sim \chi^2_r$$

will indicate that the random variable V is distributed as a chi-square random variable with r degrees of freedom.

Now, let

$$U = \min\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{m} Y_i \right)$$

and let

$$K = \left\{ \text{largest } j \leq n: \sum_{i=1}^{j} X_i \leq U \right\} + \left\{ \text{largest } j \leq m: \sum_{i=1}^{j} Y_i \leq U \right\}.$$

Note that

$$\min(m,n) \leq K \leq m + n - 1 .$$

We shall show that given $K = k$, then

$$U \sim \text{Gamma } (k, \lambda_1 + \lambda_2) ,$$

or equivalently, that given $K = k$, then

$$2(\lambda_1 + \lambda_2)U \sim \chi^2_{2k} . \tag{3}$$

Equation (3) enables us to determine an upper confidence bound for

$\lambda_1 + \lambda_2$, with confidence coefficient $1-\alpha$, namely

$$\chi^2_{\alpha;2k}/2U \,, \tag{4}$$

where $\chi^2_{\alpha;2k}$ is such that

$$P\{\chi^2_{2k} > \chi^2_{\alpha;2k}\} = \alpha \,.$$

From (4), we can easily obtain a lower confidence bound for $R(t) = e^{-(\lambda_1+\lambda_2)t}$, with confidence coefficient $1-\alpha$, namely

$$\exp\{-\chi^2_{\alpha;2k}t/2U\} \,. \tag{5}$$

The remainder of this section will be devoted to showing that given $K = k, U \sim \text{Gamma}\ (k, \lambda_1 + \lambda_2)$. To facilitate this we shall slightly change our notation and also make use of the equivalence between the partial sums of independent identically distributed exponential random variables and the Poisson process.

Let $\{N_1(t),\ t \geq 0\}$ and $\{N_2(t),\ t \geq 0\}$ be independent Poisson processes having respective rates λ_1 and λ_2. Also, let $N(t) = N_1(t) + N_2(t)$. It follows from known results that $\{N(t),\ t \geq 0\}$ is also a Poisson process but with rate $\lambda_1 + \lambda_2$. Now let T_i be the time between the $(i-1)^{st}$ and i^{th} event in the $N(t)$ process, and define

$$J_i = \begin{cases} 1 & \text{if the } i^{th} \text{ event in the } N(t) \text{ process comes from the } N_1(t) \text{ process,} \\ 2 & \text{if the } i^{th} \text{ event in the } N(t) \text{ process comes from the } N_2(t) \text{ process.} \end{cases}$$

Finally, consider a new Poisson process $\{N^*(t),\ t \geq 0\}$, independent of the above, and having rate $\lambda_1 + \lambda_2$. Let T_i^* be the i^{th}

interarrival time for this process and let J_i^*, $i = 1,2, \ldots$ be an independent identically distributed sequence of random variables, independent of the above, and such that

$$J_i^* = \begin{cases} 1 & \text{with probability } \dfrac{\lambda_1}{\lambda_1 + \lambda_2}, \\ 2 & \text{with probability } \dfrac{\lambda_2}{\lambda_1 + \lambda_2}. \end{cases}$$

Theorem 1: The two stochastic processes $\{(T_i,J_i),\ i \geq 1\}$ and $\{(T_i^*,J_i^*),\ i \geq 1\}$ have the same probability law.

Proof:

The theorem will be proven by using induction to show that

$$P\{T_1 \leq a_1,\ J_1 = j_1,\ T_2 \leq a_2,\ J_2 = j_2, \ldots, T_n \leq a_n,\ J_n = j_n\}$$

$$= P\{T_1^* \leq a_1,\ J_1^* = j_1,\ T_2^* \leq a_2,\ J_2^* = j_2, \ldots, T_n^* \leq a_n,\ J_n^* = j_n\}.$$

Let

$$j_c = \begin{cases} 2 & \text{if } j = 1, \\ 1 & \text{if } j = 2. \end{cases}$$

Now,

$$(6)\quad P\{T_1 \leq a,\ J_1 = j\} = \int_0^\infty P\{T_1 \leq a,\ J_1 = j \mid Y_1 = y\}\lambda_{j_c} e^{-\lambda_{j_c} y}\, dy,$$

where Y_1 is defined to be the first interarrival time of the process $\{N_{j_c}(t),\ t \geq 0\}$. Hence,

$$P\{T_1 \le a,\ J_1 = j\} = \int_0^a (1-e^{-\lambda_j y})\lambda_{j_c} e^{-\lambda_{j_c} y}\, dy + \int_a^\infty (1-e^{-\lambda_j a})\lambda_{j_c} e^{-\lambda_{j_c} y}\, dy$$

(7)

$$= \frac{\lambda_j}{\lambda_1 + \lambda_2}\,(1-e^{-(\lambda_1+\lambda_2)a})$$

$$= P\{T_1^* \le a,\ J_1^* = j\}\ .$$

Note that the last equality follows since T_1^* is just the time of the first event of a Poisson process with rate $\lambda_1 + \lambda_2$ and J_1^* is j with probability $\frac{\lambda_j}{\lambda_1 + \lambda_2}$ (independently of anything else). Now by (7) we have that

$$P\{T_1 \le a_1,\ J_1 = j_1, \ldots, T_n \le a_n,\ J_n = j_n\}$$

$$= \int_0^{a_1} P\{T_2 \le a_2,\ J_2 = j_2, \ldots, T_n \le a_n,\ J_n = j_n \mid T_1 = a,\ J_1 = j_1\} \times \lambda_{j_1} e^{-(\lambda_1+\lambda_2)a}\, da\ .$$

However, by the lack of memory of the Poisson Process it follows that at time T_1 (which is the first time that an event from either N_1 or N_2 occurs) both the processes $\{N_1(t),\ t \ge 0\}$ and $\{N_2(t),\ t \ge 0\}$ start over again, and hence (8) equals

$$\int_0^a P\{T_1 \le a_2,\ J_1 = j_2,\ \cdots,\ T_{n-1} \le a_n,\ J_{n-1} = j_n\}\lambda_{j_1} e^{-(\lambda_1+\lambda_2)a}\, da$$

which by our induction hypothesis equals

$$\int_0^{a_1} P\{T_1^* \le a_2, J_1^* = j_2, \ldots, T_{n-1}^* \le a_n, J_{n-1}^* = j_n\} \lambda_{j_1} e^{-(\lambda_1+\lambda_2)a} da$$

$$= \int_0^{a_1} \prod_{i=1}^{n-1} \left[(1-e^{-(\lambda_1+\lambda_2)a_{i+1}}) \left(\frac{\lambda_{j_{i+1}}}{(\lambda_1+\lambda_2)} \right) \right] \lambda_{j_1} e^{-(\lambda_1+\lambda_2)a} da$$

$$= \prod_{i=1}^{n} \frac{\lambda_{j_i}}{\lambda_1 + \lambda_2} (1-e^{-(\lambda_1+\lambda_2)a_i})$$

$$= P\{T_1^* \le a_1, J_1^* = j_1, \ldots, T_n^* \le a_n, J_n^* = j_n\} .$$

Q.E.D.

Now consider the first event in the $N(t)$ process such that to the left of (and including) that event there are either n events having their J's equal 1 or m events having their J's equal 2. Suppose this is the K^{th} event and let $U = T_1 + \cdots + T_K$ denote the time of this event. (Note that these are the same K and U as were defined at the beginning of this section.)

Theorem 2: Given $K = k$, U has a gamma distribution with parameters k and $\lambda_1+\lambda_2$. That is, its conditional density is given by

$$f_{U|K=k}(u) = \frac{\lambda^k}{(k-1)!} u^{k-1} e^{-\lambda u} .$$

Proof:

Define K^* and U^* in an equivalent manner for the $N^*(t)$ process. Since the values J_i are determined completely at random for the N^* process the result follows for U^* and K^*. The result must then follow for U and K by Theorem 1.

Q.E.D.

Corollary 3: Let $X \sim \text{Gamma}(n,\lambda_1)$ and $Y \sim \text{Gamma}(m,\lambda_2)$, where the X's and Y's are independent. Then

$$\min(X,Y) \sim \text{Gamma}(K,\lambda_1 + \lambda_2)$$

where

$$P\{K = k\} = \binom{k-1}{n-1}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^n\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{k-n} + \binom{k-1}{m-1}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{k-m}\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^m$$

$$\text{for } \min(m,n) \le k \le m + n - 1 .$$

The correct way of interpreting Corollary 3 is that conditional on $K = k$, $\min(X,Y) \sim \text{Gamma}(k,\lambda_1 + \lambda_2)$; i.e., letting $\Gamma_{k,\lambda}(a)$ be the probability that a gamma random variable with parameters (k,λ) is greater than a, then

$$P\{\min(X,Y) > a\} = \sum_{k=\min(m,n)}^{m+n-1} \Gamma_{k,\lambda_1+\lambda_2}(a)P\{K = k\} .$$

Proof:

The proof is immediate.

4. Comparison with Other Procedures

Sarkar [4] considered the case $m = n$ and obtained an exact confidence bound for $\lambda_1 + \lambda_2$ (and hence for $e^{-(\lambda_1+\lambda_2)t}$) by the following method. He defined $Z_i = \min(X_i,Y_i)$, $i = 1, \ldots, n$, and based his confidence bound on the fact that the Z_i are independent identically distributed exponential random variables with mean $1/(\lambda_1+\lambda_2)$. Hence, $\sum_1^n Z_i \sim \text{Gamma}(n,\lambda_1+\lambda_2)$. His upper confidence bound for $\lambda_1 + \lambda_2$, with confidence coefficient $1-\alpha$, is thus

$$\chi^2_{\alpha;2n}/2 \sum_{1}^{n} Z_i \ .$$

The weakness of Sarkar's procedure is that he is throwing away a good deal of information. For instance, if $X_1 < Y_1$ then he only uses X_1 and makes no use of Y_1. However, given that $X_1 < Y_1$ it follows (by the lack of memory of the exponential) that the excess, $Y_1 - X_1$, is exponentially distributed with mean $1/\lambda_2$. Hence by using the "excesses" one can generate more X's and Y's. This is equivalent to what the procedure of Section 3 does. Let us illustrate by an example.

Example: Independent random samples of size 4 are obtained on each of two components, X and Y. Assume that the X's are exponentially distributed with parameter λ_1 and the Y's are exponentially distributed with parameter λ_2. Suppose the data are as follows:

$$(X_1,X_2,X_3,X_4) = (2,7.1,6,3)$$

$$(Y_1,Y_2,Y_3,Y_4) = (5,4,2,3.1) \ .$$

Sarkar would thus generate a random sample of size 4, namely, (2,4,2,3) from an exponential distribution having mean $1/(\lambda_1+\lambda_2)$. Our procedure (described in Section 3) is equivalent to the following: Determine the minimum of X_1 and Y_1, thereby obtaining the value of 2. However, note that the excess, $Y_1 - X_1 = 3$, yields another Y data point. Hence, after generating the value 2 we are left with the "random samples"

X: 7.1, 6, 3

Y: 3, 4, 2, 3.1 .

Again, we determine the minimum of the first X and the first Y, i.e., min(7.1,3) = 3, thereby obtaining the "outcome" 3 and note that the excess, 7.1 - 3 = 4.1, yields another X data point. Having generated the value 3, we are left with the "random samples"

X: 4.1, 6, 3

Y: 4, 2, 3.1 .

We determine the minimum of the first X and the first Y, i.e., min (4.1,4) = 4, thereby obtaining the "outcome" 4 and note that the excess, 4.1 - 4 = 0.1, yields another X point. Having generated the value 4, we are left with the "random samples"

X: 0.1, 6, 3

Y: 2, 3.1 .

The minimum of the first X and the first Y is 0.1, thereby obtaining the "outcome" 0.1. The excess, 2 - 0.1 = 1.9, yields another Y point. Having generated the value 0.1, we are left with the "random samples"

X: 6, 3

Y: 1.9, 3.1 .

Since min(6,1.9) = 1.9, we obtain the "outcome" 1.9, and the excess of 4.1 yields another X point. Having generated the value 1.9, we are left with the "random samples"

X: 4.1, 3

Y: 3.1 .

Since $\min(4.1,3.1) = 3.1$, we obtain the "outcome" 3.1, and the excess of 1 yields another X point. Having generated the value 3.1, we are left with the "random samples"

X: 1, 3

Y: .

Since there are no more Y data points we must stop the procedure. If we treat the "outcomes" as the outcomes of a random sample of size 6 - namely 2, 3, 4, 0.1, 1.9, 3.1 - from an exponential distribution with mean $1/(\lambda_1+\lambda_2)$, then the sum should behave as a gamma distributed random variable with parameters 6 and $\lambda_1 + \lambda_2$. However, note that the sum is just the random variable $U = \min\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{m} Y_i\right)$ defined in Section 3, and 6 is just the value taken on by the random variable K also defined in Section 3. From the results of Section 3 these generated values can then, in fact, be considered as the outcomes of a random sample from an exponential distribution with mean $1/(\lambda_1+\lambda_2)$. As indicated earlier our confidence bound would be arrived at by using the fact that $U \sim \text{Gamma}\ (K = 6, \lambda_1+\lambda_2)$. Hence, our procedure yields stochastically smaller confidence intervals than does Sarkar's since we are generating a larger random sample (i.e., $n \leq K \leq (n-1)$).

The above example also points out some of the weaknesses of our procedure. First, since we had no Y points to compare them with, we had to throw out the final two X points. Also K, the number of data points used, is a function of the ordering of the X's and Y's. Thus, for instance, random samples (X_1,X_2,X_3), (Y_1,Y_2,Y_3)

might produce a different value of K than would random samples (X_2,X_1,X_3), (Y_2,Y_1,Y_3).

Another procedure is the one due to Kraemer [2]. Kraemer uses the fact that $\sum_1^n X_i \sim \text{Gamma}\ (n,\lambda_1)$ (or equivalently that $2\lambda_1 \sum_1^n X_i \sim \chi^2_{2n}$) and $\sum_1^m Y_i \sim \text{Gamma}\ (m,\lambda_2)$. Thus, since the X's and Y's are independent it follows that

$$2\left[\lambda_1 \sum_1^n X_i + \lambda_2 \sum_1^m Y_i\right] \sim \chi^2_{2(n+m)} .$$

Again, letting $U = \min\left(\sum_1^n X_i, \sum_1^m Y_i\right)$ we have that

$$2(\lambda_1+\lambda_2)U \leq 2\left[\lambda_1 \sum_1^n X_i + \lambda_2 \sum_1^m Y_i\right] .$$

However, $2\left[\lambda_1 \sum_1^n X_i + \lambda_2 \sum_1^m Y_i\right] \sim \chi^2_{2(n+m)}$ so that an upper confidence bound for $\lambda_1 + \lambda_2$, with confidence coefficient at least $1-\alpha$, is given by

$$\chi^2_{\alpha;2(n+m)}/2U . \tag{9}$$

Equation (9) expresses the confidence bound for Kraemer's procedure. However, our procedure (equation (4)) yields a smaller confidence interval since $K < 2(n+m)$. This is intuitively clear since Kramer bases her estimate on U and gets a bound on its distribution while we base our estimate on both U and K and use the exact joint distribution of U and K.

Another procedure has been suggested by Lentner and Buehler [3].

Their procedure is based on the fact that given $\sum_{1}^{n} X_i - \sum_{1}^{m} Y_i$, then the distribution of $\sum_{1}^{n} X_i$ depends on λ_1 and λ_2 only through $\lambda_1 + \lambda_2$. They then use the Lehmann-Scheffe theory to obtain a conditional (on $\sum_{1}^{n} X_i - \sum_{1}^{m} Y_i$) uniformly most powerful unbiased test for the hypothesis $\lambda_1 + \lambda_2 = c$. By inverting this test they obtain an upper confidence bound for $\lambda_1 + \lambda_2$, with confidence coefficient $1-\alpha$.

Since the conditional distribution $\sum_{1}^{n} X_i$ given $\sum_{1}^{n} X_i - \sum_{1}^{m} Y_i$ is somewhat complicated it is not theoretically clear how good the Lentner-Beuhler procedure is. It should also be pointed out that their methods are computationally difficult.

Other relevant literature that should be mentioned is a procedure given by Grubbs [1] which obtains fiducial intervals for the reliability.

Another possible approach to the problem would be to assume prior distributionson λ_1 and λ_2. It would then be straightforward to obtain the posterior distribution of $\lambda_1 + \lambda_2$ and Bayesian probability intervals can then be obtained.

5 Other Sampling Schemes

A common type of sampling used in practice is type II censoring at r out of R $(r \leq R)$ Type II censoring at r out of R refers to the situation where R items are tested simultaneously and the test is terminated when the first r out of R have failed. The time of failure of each item is recorded. This corresponds to recording the order statistics, i.e., $X_{(1)}, X_{(2)}, \cdots, X_{(r)}$. In

the notation of this paper put N units of the first component (X) on test and terminate the test when the n^{th} failure occurs. Similarly put M units of the second component (Y) on test and terminate the test when the m^{th} failure occurs. Let $X_{(i)}$ and $Y_{(j)}$ be the respective order statistics $(i = 1,2, \ldots, n$ and $j = 1,2, \ldots, m)$. Denote by

$$\delta_i = (n-i+1)(X_{(i)} - X_{(i-1)}), \quad i = 1, 2, \ldots, n \quad (X_{(0)} \equiv 0)$$

and

$$\tau_j = (m-j+1)(Y_{(j)} - Y_{(j-1)}), \quad j = 1, 2, \ldots, m \quad (Y_{(0)} \equiv 0) .$$

It is well known that the δ's and τ's are independent and identically distributed random variables with respective parameters λ_1 and λ_2. Hence, the procedure of Section 3 is applicable.

Another type of sampling can be described as follows. For the first component (X) put one item on test. When it fails replace it. When this second item fails replace it. A similar procedure is followed for the second component (Y). Terminate the test when there have been either n failures of component 1 $(X$'s$)$ or m failures of component 2 $(Y$'s$)$. Follow the techniques outlined in Section 3 to determine a confidence bound for $e^{-(\lambda_1+\lambda_2)t}$. Note that in this case our procedure uses all the available information. The data can be represented by $X_1, \ldots, X_r$ and $Y_1, \ldots, Y_s$ where either $r = n$ or $s = m$. Consider the case $r = n$, then the likelihood is given by

$$L(x_1,\ldots,x_n,y_1,\ldots,y_s) = e^{-\lambda_2\left(\sum_1^n x_i - \sum_1^s y_i\right)}\left(\prod_{i=1}^n \lambda_1 e^{-\lambda_1 x_i}\right)\left(\prod_{i=1}^s \lambda_2 e^{-\lambda_2 y_i}\right),$$

where the first term arises since the next y value - namely y_{s+1} - must be greater than $\sum_{1}^{n} x_i - \sum_{1}^{s} y_i$. Hence,

$$L(x_1, \ldots, x_n, y_1, \ldots, y_s) = \lambda_1^n \lambda_2^s e^{-(\lambda_1+\lambda_2)\sum_{1}^{n} x_i} .$$

Letting $\lambda = \lambda_1 + \lambda_2$ and $\theta = \lambda_1$ we have

$$L(x_1, \ldots, x_n, y_1, \ldots, y_s) = \theta^n (\lambda-\theta)^s e^{-\lambda \sum_{1}^{n} x_i}$$

or

$$\log L(x_1, \ldots, x_n, y_1, \ldots, y_s) = n \log \theta + s \log (\lambda-\theta) - \lambda \sum_{1}^{n} x_i .$$

Taking the partial derivatives and setting them equal to zero leads to

$$\frac{\partial}{\partial \lambda} \log L = \frac{s}{\lambda - \theta} - \sum_{1}^{n} x_i = 0$$

and

$$\frac{\partial}{\partial \theta} \log L = \frac{n}{\theta} - \frac{s}{\lambda - \theta} = 0 .$$

Solving these equations simultaneously results in the maximum likelihood estimate $\hat{\lambda}$ of λ, i.e.,

$$\hat{\lambda} = \frac{s + n}{\sum_{1}^{n} x_i} .$$

Also, in the case where $s = m$ we have that

$$\hat{\lambda} = \frac{r + m}{\sum_{1}^{m} y_i} .$$

Hence, in either case we arrive at $\hat{\lambda} = \frac{K}{U}$ where U and K are as given in Section 3.

6. Final Remarks

The results easily extend to a system consisting of C independent components connected in series. In this case U is defined as

$$U = \min \left[\sum_{i=1}^{n_1} (X_{1i}), \sum_{i=1}^{n_2} (X_{2i}), \cdots, \sum_{i=1}^{n_C} (X_{Ci}) \right]$$

and K is given by

$$K = \left\{ \text{largest } j \leq n_1 : \sum_{i=1}^{j} X_{1i} \leq U \right\} + \cdots + \left\{ \text{largest } j \leq n_C : \sum_{i=1}^{j} X_{Ci} \leq U \right\} .$$

It can be easily shown that

$$U \sim \text{Gamma } (K, \lambda_1 + \lambda_2 + \cdots + \lambda_C) ,$$

so that a confidence bound on $\lambda_1 + \lambda_2 + \cdots + \lambda_C$ is obtainable.

REFERENCES

[1] Grubbs, F., "Approximate Fiducial Bounds for the Reliability of a Series System for Which Each Component Has an Exponential Time-To-Fail Distribution," unpublished paper.

[2] Kraemer, H. C., "One-Sided Confidence Intervals for the Quality Indices of a Complex Item," Technometric, 5, 400-403 (1963).

[3] Lentner, M. M., and Buehler, R. J., "Some Inferences about Gamma Parameters with an Application to a Reliability Problem," Journal of the American Statistical Association, 58, 670-677 (1963).

[4] Sarkar, T. K., "An Exact Lower Confidence Bound for the Reliability of a Series System when Each Component Has an Exponential Time to Failure Distribution," Technical Report No. 117, Department of Operations Research and Department of Statistics, Stanford University (March 5, 1969).

UNCLASSIFIED
Security Classification

DOCUMENT CONTROL DATA · R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Dept. of Operations Research Stanford University Stanford, California	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

Confidence Intervals for Independent Exponential Series Systems

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Technical Report

5. AUTHOR(S) (Last name, first name, initial)

Lieberman, Gerald J. and Ross, Sheldon M.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
August 1970	18	4

8a. CONTRACT OR GRANT NO. N00014-67-A-0112-0052 b. PROJECT NO. NR-042-002 c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) Technical Report No. 130 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. AVAILABILITY/LIMITATION NOTICES

Distribution of this document is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Logistics & Math. Statistics Branch Office of Naval Research Washington, D. C. 20360

13. ABSTRACT

Suppose $X_1, X_2, \ldots, X_n$ are independent identically distributed exponential random variables with parameter λ_1. Let $Y_1, Y_2, \ldots, Y_m$ also be independent identically distributed exponential random variables with parameter λ_2, and assume that X's and Y's are independent. The problem is to estimate

$$R(t) = e^{-(\lambda_1+\lambda_2)t}.$$

The motivation behind this is that if one has a series sytem with two independent exponential components then $R(t)$ represents the reliability of the system at time , i.e., the probability that the system survives until time t.

A procedure for determining an exact $(1-\alpha)$ level lower confidence bound for $R(t)$ is presented. In doing so an interesting characterization of the minimum of two independent gamma random variables is obtained. The suggested procedure is then compared with others presented in the literature.

DD FORM 1 JAN 64 1473

UNCLASSIFIED
Security Classification

UNCLASSIFIED

Security Classification

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Reliability Series systems Exponential						

INSTRUCTIONS

1. **ORIGINATING ACTIVITY:** Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization *(corporate author)* issuing the report.

2a. **REPORT SECURITY CLASSIFICATION:** Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2b. **GROUP:** Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. **REPORT TITLE:** Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. **DESCRIPTIVE NOTES:** If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. **AUTHOR(S):** Enter the name(s) of author(s) as shown on or in the report. Enter last name, *first* name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. **REPORT DATE:** Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7a. **TOTAL NUMBER OF PAGES:** The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7b. **NUMBER OF REFERENCES:** Enter the total number of references cited in the report.

8a. **CONTRACT OR GRANT NUMBER:** If appropriate, enter the applicable number of the contract or grant under which the report was written.

8b, 8c, & 8d. **PROJECT NUMBER:** Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9a. **ORIGINATOR'S REPORT NUMBER(S):** Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9b. **OTHER REPORT NUMBER(S):** If the report has been assigned any other report numbers *(either by the originator or by the sponsor)*, also enter this number(s).

10. **AVAILABILITY/LIMITATION NOTICES:** Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through _____."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through _____."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through _____."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. **SUPPLEMENTARY NOTES:** Use for additional explanatory notes.

12. **SPONSORING MILITARY ACTIVITY:** Enter the name of the departmental project office or laboratory sponsoring *(paying for)* the research and development. Include address.

13. **ABSTRACT:** Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS), (S), (C), or (U)*.

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. **KEY WORDS:** Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, roles, and weights is optional.

DD FORM 1 JAN 64 1473 (BACK)

UNCLASSIFIED

Security Classification