# SELECTION OF WEIGHTING MATRICES FOR THE DESIGN OF LINEAR REGULATOR SYSTEMS

by

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# THESIS

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ABSTRACT

The relationship between the weighting matrices and the design objectives for finite-final-time linear regulator systems is considered. An iterative algorithm is presented for selecting a weighting matrix that reduces the absolute difference between the actual and desired values of a vector design measure. The algorithm utilizes the sensitivities of the vector design measure to determine changes for the weighting matrix. These sensitivities are approximated by finite-difference perturbations of the weighting matrix elements. Examples are presented that illustrate the design procedure.

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# I. INTRODUCTION

The linear regulator is one of the most extensively studied and well known problems of optimal control theory, [1,2,3,4,5,6,7,8]. This importance stems from the fact that many practical control problems can be formulated in the linear regulator form. Its most desirable feature is that the optimal control law employs linear state-variable feedback.

## A. THE GENERAL LINEAR REGULATOR PROBLEM

A linear regulator problem may be formulated as follows: Consider a completely controllable, completely observable multivariable time-varying dynamic system

$$\dot{\underline{x}}(t) = A(t)\underline{x}(t)+B(t)\underline{u}(t) \qquad \text{(I-1)}$$

and the performance index

$$J = \tfrac{1}{2}\underline{x}(t_f)^T H\underline{x}(t_f)+\tfrac{1}{2}\int_0^{t_f}(\underline{x}(t)^T Q(t)\underline{x}(t)+\underline{u}^T(t)R(t)\underline{u}(t))dt, \qquad \text{(I-2)}$$

where the real symmetric nxn weighting matrices Q(t) and H are non-negative definite, and the real symmetric mxm weighting matrix R(t) is positive definite. $\underline{x}(t)$ is the nxl state vector, $\underline{u}(t)$ is the mxl vector of controls, B(t) is an nxm matrix, A(t) is an nxn matrix and the final time, $t_f$, is fixed. The optimal control $\underline{u}^*(t)$ minimizes J, and is given by [1,2]

$$\begin{aligned} \underline{u}^*(t) &= -R^{-1}(t)B^T(t)K(t)\underline{x}(t), \\ \underline{u}^*(t) &\triangleq -F^*(t)\underline{x}(t). \end{aligned} \qquad \text{(I-3)}$$

$K(t)$ is the nxn positive definite symmetric matrix solution to the Riccati equation

$$\dot{K}(t) = -A^T(t)K(t)-K(t)A(t)-Q(t)+K(t)B(t)R^{-1}(t)B^T(t)K(t) \quad \text{(I-4)}$$

with the boundary condition

$$K(t_f) = H. \quad \text{(I-5)}$$

Because the system is assumed to be controllable and observable, it is possible to achieve arbitrary dynamics with state-variable feedback [7]. The practical application of optimal linear regulator theory to system design leads to the problem of selecting the weighting matrices. The state trajectory of the optimal system given by the solution of

$$\dot{\underline{x}}(t) = [A(t)-B(t)F^*(t)]\underline{x}(t) \quad \text{(I-6)}$$

is dependent upon the weighting matrices chosen. It is therefore desirable to establish a relationship between the weighting matrices and the design objectives (such as percent overshoot, time to overshoot and settling time) with the objective of developing a design procedure.

## B. A RESTRICTED LINEAR REGULATOR PROBLEM

This thesis considers only the time-invariant, single-input, finite final-time ($t_f < \infty$) linear regulator problem. For this problem A, R, and Q are constant matrices and B is the constant nxl vector $\underline{b}$.

## C. LITERATURE SURVEY

None of the literature surveyed proposes a systematic procedure for finding a suitable weighting matrix Q for the finite final-time problem ($t_f < \infty$). The infinite final-time problem has been considered by several authors [5,7,8].

The most extensively used method for selecting a suitable weighting matrix Q is to make an educated guess, observe the result, and update the guess. The major fault here is that there is no guarantee that an acceptable Q can be found in a feasible number of trials.

The most definitive procedure for finding Q in the infinite final-time problem has been developed by Chen and Shen [7]. In their procedure a weighting matrix is selected to yield a set of desired closed-loop eigenvalues. A direct relationship between an incremental change in the closed-loop eigenvalues and the corresponding change in the weighting matrix Q is established using Jacobi's sensitivity formula and perturbation of the steady-state solution of the Riccati equations ($\dot{K} = 0$). Two iterative algorithms are given for the determination of the elements of the weighting matrix Q.

This procedure would be difficult to extend to the finite final-time problem because a desired set of time-varying eigenvalues would have to be formulated. It was felt that this extension would not be beneficial.

A different approach was taken by Wakeland in his doctoral dissertation [8]. He considers the system given by

$$\dot{\underline{e}} = A\underline{e} \tag{I-7}$$

where A is an nxn matrix in phase-variable form [9] and $\underline{e}$ is the vector difference between the system output $\underline{c}$ and the desired system output $\underline{r}$. The performance criterion used is

$$J = \int_0^\infty \underline{e}^T Q \underline{e} dt \tag{I-8}$$

where Q is an nxn diagonal matrix. Wakeland's empirical results show that the elements of the weighting matrix

$$Q = \text{diag}\,[1, q_{22}, q_{33}, \ldots, q_{nn}] \tag{I-9}$$

are directly related to the slopes of the elements of the cost functional with respect to the parameters of the system matrix, where the cost functional is expressed as

$$J = \int_0^\infty e_1^2 dt + q_{22} \int_0^\infty e_2^2 dt + \ldots + q_{nn} \int_0^\infty e_n^2 dt. \tag{I-10}$$

The weighting matrix elements are defined in terms of these slopes for second-,third-and fourth-order systems. Analytic relationships between weighting matrix elements and system parameters are developed, and graphs and tables that represent these relationships for second-,third-and fourth-order systems are given. These graphs and tables present the relationships between weighting matrix elements and certain time-response performance characteristics such as overshoot and settling time.

Schultz and Melsa, [5], consider the infinite-time problem ($t_f = \infty$) where the system is assumed to be in phase-variable form and show that the cost functional given by

$$J = \int_0^\infty (x^T(t)Qx(t) + ru(t)^2)\,dt \qquad \text{(I-11)}$$

may be reduced to

$$J = \int_0^\infty \left[ (\underline{\gamma}^T \underline{x}(t))^2 + ru(t)^2 \right] dt + \underline{x}^T(\infty) S \underline{x}(\infty) - \underline{x}^T(0) S \underline{x}(0). \qquad \text{(I-12)}$$

S is an nxn symmetric constant matrix whose $n^{th}$ row and $n^{th}$ columns are set to zero, in order to make the reduction independent of the system $(A, \underline{b})$. Since the term, $\underline{x}^T(0) S \underline{x}(0)$, depends only on the initial conditions, $\underline{x}(0)$, it has no effect in the minimization of J. For the term, $\underline{x}^T(\infty) S \underline{x}(\infty)$, to have no effect, the system must be asymptotically stable.

The reduced performance index has only n weighting factors to be considered which can be related to the concept of modeling, [5]. The elements of the weighting vector, $\underline{\gamma}$, form the coefficients of an (n-1)st-order differential equation

$$\gamma_1 y + \gamma_2 \dot{y} + \gamma_3 \ddot{y} + \ldots + \gamma_n \frac{d^{(n_y - 1)}}{dt^{(n-1)}} = 0 \qquad \text{(I-13)}$$

which must be satisfied by the output of the system under consideration to minimize J. Therefore, equation (I-13) is regarded as a model for which a desired response is formulated; the model is adjusted to satisfy classical response characteristics, such as rise-time, overshoot, phase margin, and so forth. The adjusted coefficient vector, $\underline{\gamma}$, is the weighting vector for the performance index.

## II. FORMULATION OF THE WEIGHTING-MATRIX SELECTION PROBLEM

### A. INTRODUCTION

The basic system considered is a controllable, observable, multivariable, time-invariant, single-input, linear dynamic system given by

$$\dot{\underline{x}}(t) = A\underline{x}(t) + \underline{b}u(t). \qquad \text{(II-1)}$$

The performance index to be minimized is given by

$$J = \tfrac{1}{2}\int_0^{t_f} (\underline{x}^T(t)Q\underline{x}(t) + u^2(t))dt. \qquad \text{(II-2)}$$

The optimal control that minimizes equation (II-2) is given by

$$u^*(t) = -\underline{b}^T K(t)\underline{x}(t). \qquad \text{(II-3)}$$

$K(t)$ is the solution to the reduced Riccati equation

$$\dot{K}(t) = -A^T K(t) - K(t)A - Q + K(t)bb^T K(t), \qquad \text{(II-4)}$$

which has the boundary condition

$$K(t_f) = 0. \qquad \text{(II-5)}.$$

### B. TRANSFORMATION TO CANONICAL FORMS

It was decided, from a computational standpoint, that the number of non-zero elements of the Q matrix should be small. For this reason, transformation of the system to two canonical forms was examined; the diagonal state-matrix form and the phase-variable form.

A system that has no multiple eigenvalues may be readily transformed into one that has a diagonal A matrix by the following procedure. The system has a column-partioned matrix of eigenvectors given by

$$E = \left[\underline{e}_1 \underline{e}_2 \cdots \underline{e}_n\right] \tag{II-6}$$

where each eigenvector $\underline{e}_i$ is associated with its eigenvalue $\lambda_i$. This matrix may be normalized to

$$\hat{E} = \left[\hat{\underline{e}}_1 \hat{\underline{e}}_2 \cdots \hat{\underline{e}}_n\right] \tag{II-7}$$

by normalizing the individual eigenvectors to unit magnitude. By letting $\underline{x}(t) = \hat{E}\underline{z}(t)$, the new system is given by

$$\dot{\underline{z}}(t) = A_1 \underline{z}(t) + \underline{b}_1 u(t), \tag{II-8}$$

where

$$A_1 = \hat{E}^{-1} A \hat{E} \tag{II-9}$$

and

$$\underline{b}_1 = \hat{E}^{-1} \underline{b}. \tag{II-10}$$

The matrix $A_1$ has the form

$$A_1 = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & & \vdots \\ 0 & 0 & \lambda_3 & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & & 0 & \lambda_n \end{bmatrix} . \tag{II-11}$$

With the system in this form, a diagonal Q matrix may be used and still have complete influence on feedback; but the elements of the matrix $A_1$ may be conjugate complex. This requires that all computation be done using complex arithmetic.

Since the system under consideration is completely controllable, it can be transformed into the phase-variable form with a linear non-singular transformation matrix T, by letting $\underline{x} = T\underline{z}$ as in Rane [9]. This leads to the phase-variable system

$$\dot{\underline{z}} = A_0\underline{z} + \underline{b}_0 u \qquad \text{(II-12)}$$

where the system matrix has the form

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & 0 & & & & . \\ 0 & 0 & 0 & 1 & & & & . \\ . & & & & . & & & . \\ . & & & & & . & & . \\ 0 \ldots & & & & & 0 & 1 & 0 \\ 0 \ldots & & & & & & 0 & 1 \\ -a_1 & -a_2 & & & & & & -a_n \end{bmatrix} \qquad \text{(II-13)}$$

and $\underline{b}_0^T = [00....01]$. The elements in the last row of $A_0$ ($a_i$, $i = 1,\ldots,n$) are the coefficients of the characteristic polynomial for the original system matrix A. This polynomial is given by

$$s^n + a_n s^{n-1} + a_{n-1} s^{n-2} + \ldots\ldots + a_1. \qquad \text{(II-14)}$$

The transformation matrix T is given by

$$T = [\underline{t}_1, \underline{t}_2, \ldots\ldots, \underline{t}_n] \qquad \text{(II-15)}$$

where the element vectors may be formed by the recursive relations

$$\left.\begin{aligned} \underline{t}_n &= \underline{b} \\ \underline{t}_{n-1} &= A\underline{t}_n + \underline{t}_n a_n \\ \underline{t}_{n-2} &= A\underline{t}_{n-1} + \underline{t}_n a_{n-1} \\ &\vdots \\ \underline{t}_1 &= A\underline{t}_2 + \underline{t}_n a_2 \end{aligned}\right\} . \qquad \text{(II-16)}$$

C. DIAGONALIZATION OF THE Q MATRIX

Kriendler, [4], states and proves the following theorem. If a linear time-invariant plant is in the phase-variable form, then for any positive semidefinite matrix Q in the quadratic performance index

$$J = \tfrac{1}{2}\int_0^\infty (\underline{x}^T Q \underline{x} + u^2)\, dt, \qquad \text{(II-17)}$$

there exists a unique diagonal matrix

$$Q^* = \text{diag}[q_{11}, q_{22}^*, \ldots, q_{n-1,n-1}^*, q_{n,n}] \qquad \text{(II-18)}$$

which yields the same optimal control

$$\underline{u}^*(t) = -\underline{b}^T K(t) \underline{x}(t) \qquad \text{(II-19)}$$

and is related to Q by the formula

$$q_{ii}^* = q_{ii} - 2q_{i-1,i+1} + 2q_{i-2,i+2} - \cdots \qquad \text{(II-20)}$$

where the alternating sum is continued until all of the available q's are exhausted. Even though Q is required to be positive semidefinite, $Q^*$ needed not be so restricted. Let Q be the positive definite matrix

$$Q = \begin{bmatrix} 10 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} . \qquad \text{(II-21)}$$

Then

$$Q^* = \text{diag}\,[10,-3,1] \qquad \text{(II-22)}$$

where $Q^*$ is unique.

It should be noted from equation (II-20) that $Q^*$ does not depend on the elements $q_{ij}$ of Q where i+j is an odd number. Therefore when the system is in the phase-variable form, an equivalent matrix $Q^1$ (simpler than Q) may be considered, where $q^1_{ij} = q_{ij}$ if i+j is even and $q^1_{ij} = 0$ if is i+j odd. The matrix $Q^1$ retains the positive definiteness or semi-definiteness of Q.

The optimal feedback control law, $\underline{p}$, defined by

$$\underline{p} \triangleq \underline{b}^T K(t), \qquad \text{(II-23)}$$

is a time-varying row vector consisting of the $n^{th}$ column of the Riccati solution, K, for a plant in phase-variable form. For the infinite-time problem, the diagonal weighting matrix $Q^*$ is unique and yields the same feedback $\underline{p}$ as that corresponding to the original matrix Q, where

$$\underline{p} = b^T K_\infty = \underline{b}^T K^*_\infty . \qquad \text{(II-24)}$$

## D. INITIAL CONDITION CONSIDERATIONS

Designing a system for a specific initial condition does not necessarily produce the desired time response for other initial conditions. A conservative design approach is adopted

here by assuming that worst-case initial conditions occur. Worst-case initial conditions are defined as those which maximize the minimum value of the performance index as given by

$$J_{min} = \underline{x}^T(0)K(0)\underline{x}(0). \qquad \text{(II-25)}$$

$J_{min}$ is maximized when $\underline{x}(0)$ is colinear with the eigenvector associated with the largest eigenvalue of $K(0)$,[2].

## III. PROBLEM SOLUTION

### A. SEARCH IN Q SPACE

The most direct approach to selecting the weighting matrix, Q, would be to perform an exhaustive search, evaluating the state trajectory for each of several selections of Q. Even for a low-order system, this is a formidable task.

A more reasonable approach is to make an estimate for Q, evaluate the state trajectory and then change Q based on the closeness of the state trajectory to the design objectives. This procedure is continued until the state trajectory is acceptable. As this requires a subjective decision, time-shared computer operation is a necessity.

### B. STATE-TRAJECTORY SENSITIVITY

The sensitivity of the state trajectory with respect to Q, $\frac{\partial \underline{x}}{\partial \underline{Q}}$, can be evaluated by using the sensitivity function approach. This involves solving the Riccati equation

$$\dot{K} = - A^T K - KA + K\underline{bb}^T K - Q \qquad \text{(III-1)}$$

with the boundary condition

$$K(t_f) = 0 \qquad \text{(III-2)}$$

simultaneously with its associated sensitivity equation

$$\frac{d}{dt}\left(\frac{\partial K}{\partial Q}\right) = - A^T \frac{\partial K}{\partial Q} - \frac{\partial K}{\partial Q} A + \frac{\partial K}{\partial Q} \underline{bb}^T K + K\underline{bb}^T \frac{\partial K}{\partial Q} - \frac{\partial Q}{\partial Q} \qquad \text{(III-3)}$$

with the boundary condition

$$\frac{\partial K(t_f)}{\partial Q} = 0. \qquad \text{(III-4)}$$

Then the state equation

$$\dot{\underline{x}} = A\underline{x} + \underline{b}u \qquad \underline{x}(0) = \underline{x}_o \qquad \text{(III-5)}$$

must be solved simultaneously with its associated sensitivity function equation

$$\frac{d}{dt}\left(\frac{\partial \underline{x}(t)}{\partial Q}\right) = (A - \underline{b}\underline{b}^T K)\frac{\partial \underline{x}}{\partial Q} + bb^T \frac{\partial K}{\partial Q}\underline{x} \qquad \text{(III-6)}$$

with the boundary condition

$$\frac{\partial(\underline{x}(0))}{\partial Q} = 0. \qquad \text{(III-7)}$$

For an $n^{th}$-order system, Equation III-5 contains n first-order differential equations. Equation III-1 contains $n^2$ first-order differential equations of which $\frac{n\cdot(n+1)}{2}$ are unique since Q, K, and $\dot{K}$ are symmetric matrices. Equations III-3 and III-6 contain $\left(\frac{n\cdot(n+1)}{2}\right)^2$ and $n\left(\frac{n\cdot(n+1)}{2}\right)$ unique first-order differential equations, respectively. Table I demonstrates the rapidity with which the computational complexity increases.

Table I. The number of differential equations, N, that must be solved for a system of order n to evaluate the state-trajectory sensitivity, $\partial \underline{x}/\partial Q$

| n | N |
|---|---|
| 1 | 4 |
| 2 | 20 |
| 3 | 63 |
| 4 | 154 |
| 5 | 320 |
| 10 | 11025 |

The state-trajectory sensitivity matrix, $\frac{\partial \underline{x}}{\partial \underline{Q}}$, must be observed over the entire trajectory to obtain a relationship between it, the state trajectory and a new weighting matrix. A subjective decision in the selection of a new Q is required since there does not exist a direct analytical relationship between the design objectives for a finite final-time problem and the weighting matrix Q.

## C. THE DESIGN MEASURE $\underline{m}$

### 1. Definition

The approach that seemed most promising required the formulation of a vector design measure,

$$\underline{m} = \underline{f}(Q; x^*, t). \qquad \text{(III-8)}$$

Typical examples of the elements of $\underline{m}$ are listed below

a. $\underset{t}{\text{Max}} |u(t)| \qquad 0 \leq t \leq t_f$

b. Percent overshoot

c. Time to maximum overshoot

d. Time to first zero crossing

e. Time to reach and remain within 5 percent of the final value

f. $|x_i(t_f)|$

g. $\int_0^{t_f} u(t)^2 dt$

h. $\int_0^{t_f} \frac{\partial \underline{x}(t)^T}{\partial \underline{a}} s \frac{\partial \underline{x}(t) dt}{\partial \underline{a}}$, where $\frac{\partial \underline{x}(t)}{\partial \underline{a}}$ is the state-trajectory sensitivity to variation in system parameters, $\underline{a}$, and s is a symmetric positive definite matrix.

The elements of $\underline{m}$ are further restricted to be defined so that they will be positive. (It should be noted that an analytical expression for the partial differential of the elements of $\underline{m}$ with respect to the elements of Q does not generally exist.)

The desired vector design measure was defined to be $\underline{m}^*$; the vector, $\underline{f}$, has the elements defined by

$$f_j \triangleq |m_j - m_j^*| \qquad j = 1,2,\ldots,L \qquad \text{(III-9)}$$

where L is a positive integer. The elements of $\underline{f}$ are to be made acceptable by adjusting the weighting matrix Q.

2. Difference Approximation to the Sensitivity Matrix $\partial\underline{m}/\partial Q$

The sensitivity matrix, $\frac{\partial \underline{m}}{\partial Q}$ can be approximated by perturbing the weighting matrix elements $q_{ii}$ to form a finite-difference approximation to the partials, $\partial\underline{m}_i/\partial q_{ii}$. The vector differential $\underline{dm}$ is given by

$$\underline{dm} = \frac{\partial \underline{m}}{\partial Q}^T d\underline{Q}, \qquad \text{(III-10)}$$

where $\frac{\partial \underline{m}}{\partial Q}$ has the form

$$\frac{\partial \underline{m}}{\partial Q} = \begin{bmatrix} \frac{\partial m_1}{\partial q_{11}} & \frac{\partial m_2}{\partial q_{11}} & \cdots & \frac{\partial m_L}{\partial q_{11}} \\ \frac{\partial m_1}{\partial q_{22}} & \frac{\partial m_2}{\partial q_{22}} & & \vdots \\ \vdots & \vdots & & \vdots \\ \frac{\partial m_1}{\partial q_{nn}} & \frac{\partial m_2}{\partial q_{nn}} & & \frac{\partial m_L}{\partial q_{nn}} \end{bmatrix} \qquad \text{(III-11)}$$

and $d\underline{m}$ and $d\underline{Q}$ are both column vectors, with dimensions (Lx1) and (mx1) respectively. Equation (III-10) is used to select a $d\underline{Q}$ that will reduce as many of the elements of the design-measure difference vector, $\underline{f}$, as possible.

3. Computational Algorithm for Selecting Q

The steps of the basic algorithm for finding the "best" weighting matrix are outlined in Flow-Chart 1, with details given by the following list.

a. Transformation of the system into the phase-variable form, in order to use a diagonal weighting matrix Q.

b. Selection of an initial diagonal weighting matrix.

c. Formation of the design measure $\underline{m}$ by:

(1) Integrating the Riccati equation

(2) Finding the worst-case initial conditions $\underline{x}(0)$ from $K(0)$.

(3) Integrating the state equations with $K(t)$ and $\underline{x}(0)$.

(4) Setting $\underline{m} = \underline{f}(Q;\underline{x}^*,t,t_f)$.

d. If $\underline{m}$ is not defined or is unreasonable, a restart is made at step b; m is not defined if some element does not exist. For example, if the optimal system is overdamped, percent overshoot has no meaning.

e. Formation of the design-measure sensitivity matrix $\frac{\partial \underline{m}}{\partial Q}$ by:

(1) Perturbing the Q elements, $q_{ii}$, i=1,2,...,n.

(2) Integrating the n sets of $\left(\frac{n\cdot(n+1)}{2}\right)$ Riccati equations, $\dot{K}^i$, with $Q^i$ = diagonal $(q_{11},q_{22},\ldots,q_{ii}+dq_{ii}\ldots,q_{nn})$ for $i = 1,2,\ldots,n$.

(3) Integrating the n state equations using $K^i(t)$.

(4) Forming the perturbed vector performance measures, $\underline{m}^i$, $i = 1,\ldots,n$.

(5) $\frac{\partial m_j}{\partial q_{ii}} \triangleq \frac{m_j^i - m_j}{q_{ii}}$ $i = 1,2,\ldots,n$; $j = 1,2,\ldots,L$.

f. Selection of a normalized $d\underline{Q}$, using a $\frac{\partial \underline{m}}{\partial \underline{Q}}$, that will improve as many of the elements $m_j$ as possible. This procedure is discussed in more detail in section III C4.

g. Selection of the magnitude, $\alpha$, for dQ and evaluation of $\underline{m}'$ by:

(1) Initializing $\alpha$

(2) Setting $Q' = Q + \alpha I d\underline{Q}$ where I is the nxn identity matrix.

(3) Forming the vector performance measure $\underline{m}$ for this value of $Q'$ by:

(a) Integrating the Riccati equation's using $Q'$.

(b) Finding the worst-case initial conditions $\underline{x}'(0)$ for $K'(0)$.

(c) Integrating the state equation's with $\underline{x}'(0)$ and $K'(t)$.

(d) Setting $\underline{m}' = \underline{f}(Q';\underline{x}',t,t_f)$

h. Formulation of subjective decisions based on

$$\underline{f}_i = \left|\underline{m}_i - \underline{m}_i^*\right| \quad i = 1,2,\ldots,L \qquad \text{(III-12)}$$

and

$$\underline{f}_i' = \left|\underline{m}' - \underline{m}_i^*\right| \quad i = 1,2,\ldots,L \qquad \text{(III-13)}$$

(1) $\underline{f}'$ is considered to be an improvement on $\underline{f}$ if any of the following conditions exists:

(a) $f_i' \leq f_i \quad i = 1,2,\ldots,L$

(b) The number of values of i for which $f_i' \leq f_i$ is greater than $L/2$

(2) $\underline{f}'$ is considered to be acceptable at this step if any of the following conditions exist:

(a) $f_i' \leq \epsilon_i'$ for a majority of the elements $f_i'$ where $\epsilon_i'$ is a positive small number; $i = 1,2,\ldots,L$.

(b) $\alpha \leq \epsilon p$; where $\epsilon p$ is a small positive number.

(3) $\underline{f}'$ is considered to be acceptable for the design if any of the following conditions exist:

(a) $f_i' \leq \epsilon_i''$ for a majority of the elements of $\underline{f}'$ where $\epsilon_i''$ is a small positive number ($\epsilon_i'' < \epsilon_i'$) and it is clear that no further significant improvement can be made in the other elements by moving locally.

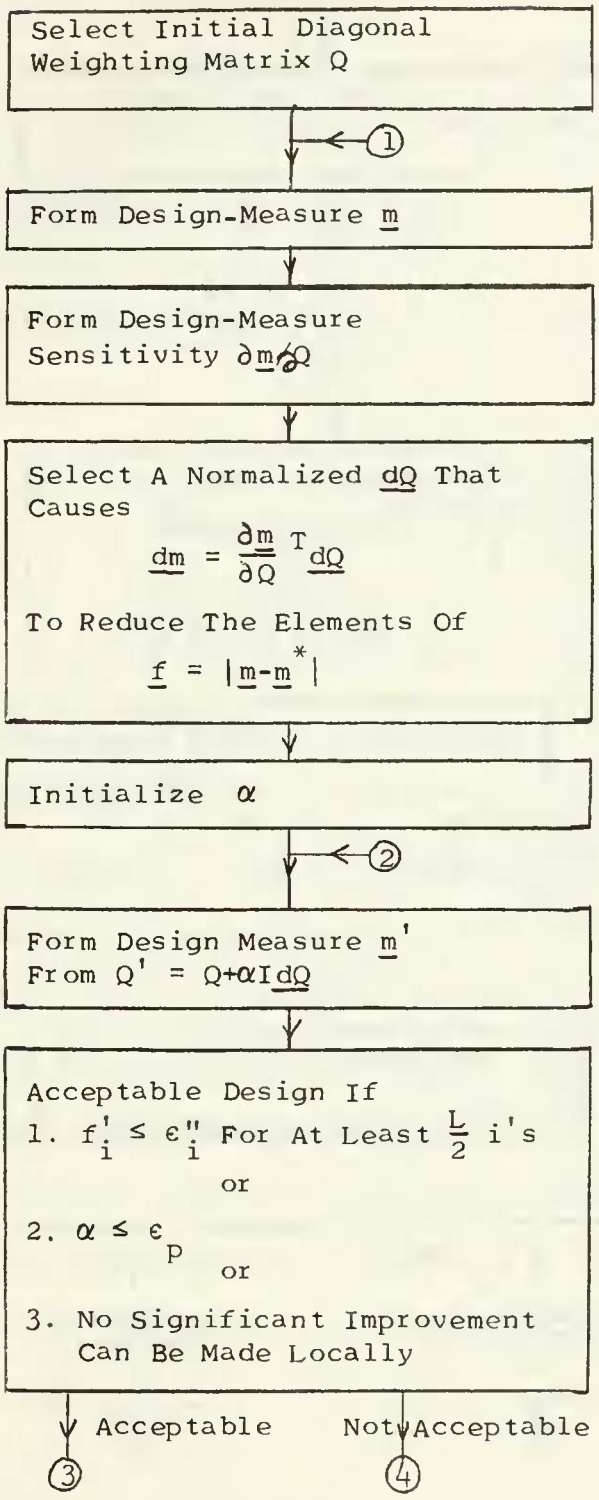
(b) $\alpha \leq \epsilon_p'$ where $\epsilon_p'$ is a small positive number and the inequality $\epsilon_p' << \epsilon_p$ holds.

(4) (a) If h. (1) is true, increase $\alpha$ and go to step g.(2).

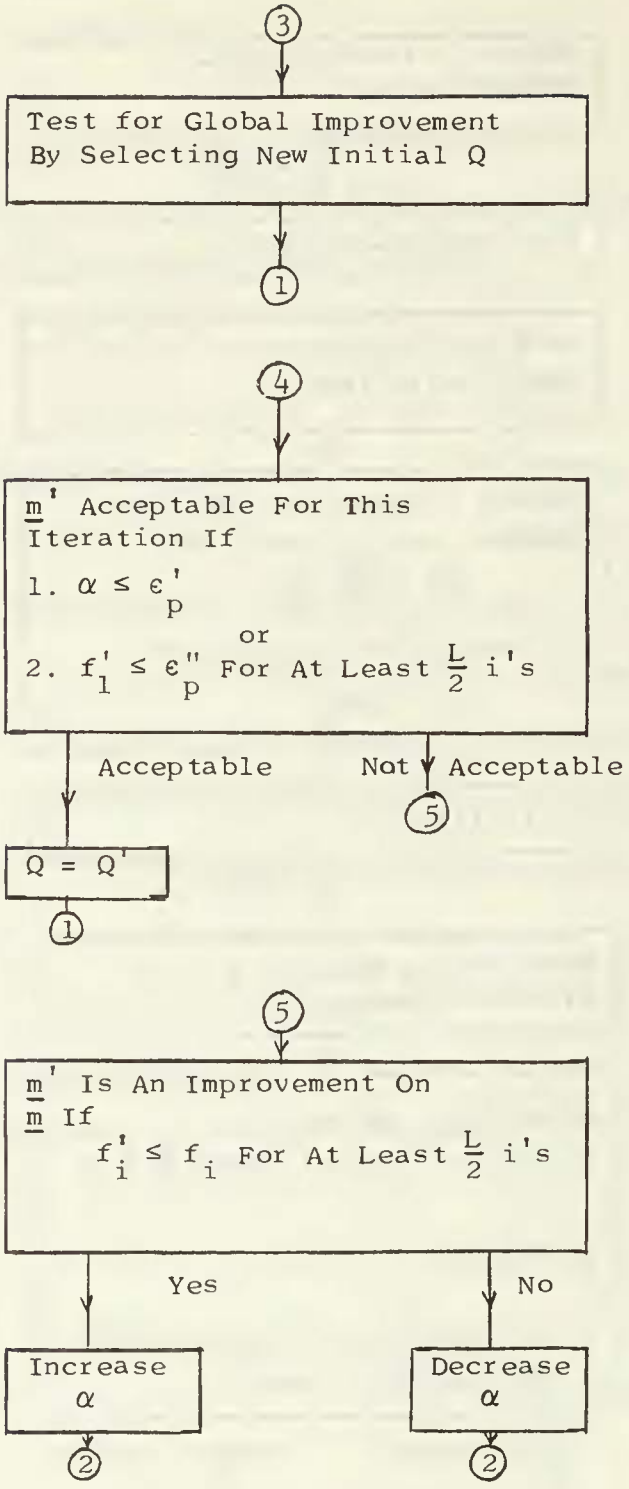
(b) If h.(1) is not true, decrease $\alpha$ and go to step g.(2).

(c) If h.(2) is true, go to step e, with $Q = Q'$.

Flow-Chart 1. Basic Steps of One Iteration of Design Algorithm



Flow Chart 1. continued



i. Testing the weighting matrix found by perturbing $Q'$ to determine if a significant reduction can be made on the elements of $\underline{f}'$. If a significant improvement can be made go to step e, with $Q = Q'$.

4. Considerations for Selecting $\underline{dQ}$

a. Problem Formulation

The vectors, $\frac{\partial \underline{m}_j}{\partial Q}$, $j = 1,2,\ldots,L$, are normalized to the vectors $\underline{w}_j$ by

$$\underline{w}_j = \frac{\frac{\partial \underline{m}_j}{\partial Q}}{\sqrt{\sum_{i=1}^{n} \left(\frac{\partial \underline{m}_j}{\partial Q}\right)^2}} \quad j = 1,2,\ldots L. \tag{III-14}$$

There exist L hyperplanes $H_j$ defined by

$$H_j: \left(\left\langle \underline{w}_j, \underline{dQ} \right\rangle\right) = 0 \qquad j = 1,2,\ldots,L \tag{III-15}$$

where $\langle . , . \rangle$ denotes the inner product in n-dimensional Euclidian space, $E^n$. $\underline{w}$ is in the direction of the largest rate of increase of $\underline{dm}_j$. Therefore the best $\partial \underline{Q}$ to select for a particular $\partial \underline{m}_j$ is colinear with $\pm \underline{w}_j$. The sign selected must be that which causes $\underline{dm}_j$ to minimize $f_j = |m_j - m_j^*|$. For $L > 1$, a systematic procedure must be established to select a $\underline{dQ}$ that will minimize as many $f_j$'s as possible.

The unit normals, $\underline{n}_j$, $j = 1,2,\ldots,L$, that result in a reduction of $f_j$ are given by

$$\underline{n}_j = \begin{cases} \underline{w}_j & m_j \leq m_j^* \\ -\underline{w}_j & m_j > m_j^* \end{cases} \quad j = 1,2,\ldots,L. \tag{III-16}$$

Defining $\underline{y} = \underline{\partial\Omega}$, there exist L half-spaces $S_j$, where $S_j$ is the set of all $\underline{y}$ that satisfy the relation

$$\left\langle \underline{n}_j, \underline{y} \right\rangle \geq 0 \qquad j = 1,2,\ldots,L. \tag{III-17}$$

Thus the hyperplane $H_j$ and its normal $\underline{n}_j$ define a half-space $S_j$ which is the set of all points that reduce $f_j$.

The intersection of all the $S_j$'s, given by

$$S = \bigcap_{j=1}^{L} S_j, \tag{III-18}$$

defines the set of all points that reduce all $f_j$ simultaneously, provided that S is not the set, $\varphi$, which contains only the origin. If $S = \varphi$, some of $f_j$'s must be allowed to increase so that a partial improvement can be made at this step. This relaxation should be made only for one iteration.

b. General Problem Considerations

(1) First-Order System. All normals $\underline{n}_j$, for a first-order system, must be of equal sign to improve all components locally. If this is not the case, the selection of $\underline{d\Omega}$ becomes subjective for a particular trial point.

(2) Second-Order System. A second-order system is considered next with $L < 2$. Since all the normals, $\underline{n}_j$, have been normalized to be unit vectors, the inner products

$$R_{ij} = \left\langle \underline{n}_i, \underline{n}_j \right\rangle \; ; \; i, j = 1,2,\ldots L; \; i \neq j \tag{III-19}$$

indicate the nature of the sets, $S_{ij}$, given by

$$S_{ij} = S_i \cap S_j. \tag{III-20}$$

$R_{ij} = 1.0$ implies that $\underline{n}_i$ and $\underline{n}_j$ are parallel, therefore

$$S_{ij} = S_i = S_j. \tag{III-21}$$

as indicated in Figure 1.

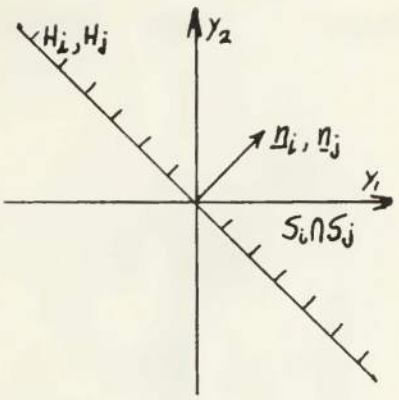



Figure 1. Intersection of 2 half-spaces in 2-space. $R_{ij} = 1$.

$R_{ij} = -1.0$ implies that $\underline{n}_i$ and $\underline{n}_j$ are anti-parallel; therefore

$$S_{ij} = H_i = H_j \qquad \text{(III-22)}$$

as indicated in Figure 2.

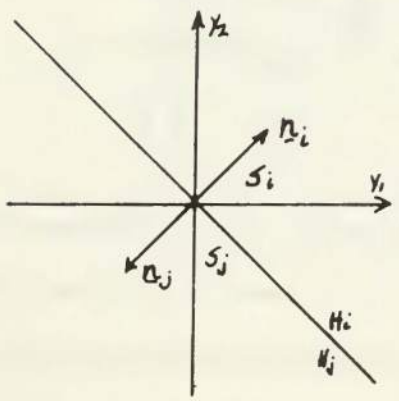



Figure 2. Intersection of 2 half-spaces in 2-space. $R_{ij} = -1$.

If $R_{ij} \neq \pm 1.0$ then there exists a region, $R_s$, where both inner products $\langle n_i, \underline{y} \rangle$ and $\langle \underline{n}_j, \underline{y} \rangle$ are greater than zero, as required to reduce the design-measure difference elements $f_i$ and $f_j$ simultaneously. This is depicted in Figure 3.

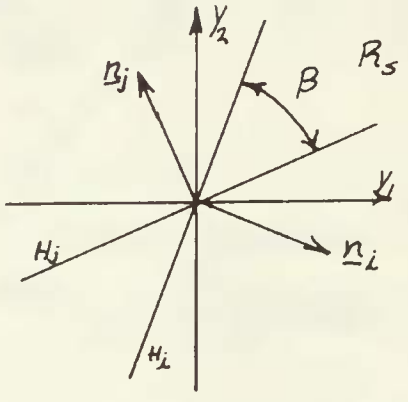



Figure 3. Intersection of 2 half-spaces. $R_{ij} \neq \pm 1$.

Suppose that L = 3 for a second-order system. If $R_{12} \neq \pm 1$ then $\underline{n}_1$ and $\underline{n}_2$ are linearly independent and $\underline{n}_3$ can be expressed as a linear combination of $\underline{n}_1$ and $\underline{n}_2$ by the relationship

$$\underline{n}_3 = N\underline{a} \qquad \text{(III-23)}$$

where N is the partitioned matrix given by

$$N = \left[ \underline{n}_1 \vdots \underline{n}_2 \right] \qquad \text{(III-24)}$$

and $\underline{a}$ is a column vector with components $a_1$, $a_2$.

For example, Figures 4 thru 11 depict the relationships that can exist between half-spaces $S_1$,$S_2$ and $S_3$

for various combinations of $a_1$ and $a_2$. In Figure 4, $S_3$ contains the intersection of $S_1$ and $S_2$; therefore $S_3$ may be dropped from consideration.

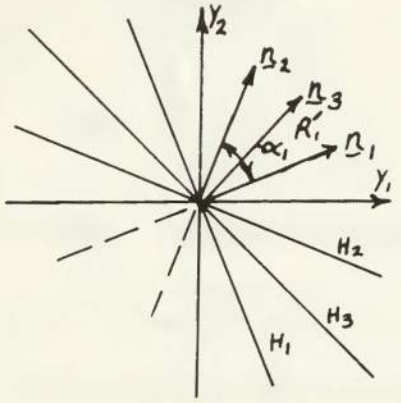



Figure 4. Intersection relationships for $a_1>0$, $a_2>0$.

The same situation holds in Figures 5 and 6, where $S_1$ and $S_2$, respectively, may be dropped from consideration.

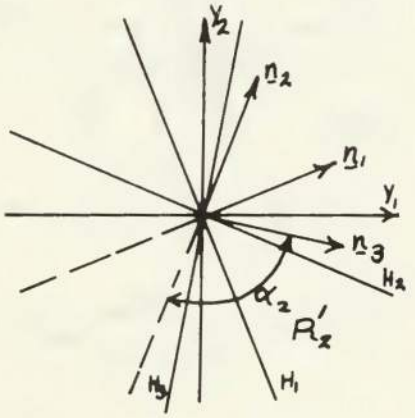



Figure 5. Intersection relationships for $a_1>0$, $a_2<0$.

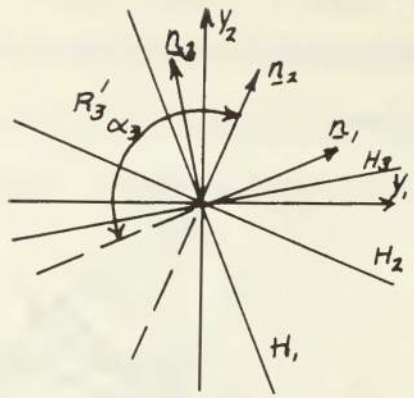



Figure 6. Intersection relationships for $a_1<0$, $a_2>0$.

In Figure 7, $S_3 = S_1$ and in Figure 8, $S_3 = S_2$; therefore $S_3$ may be dropped from consideration in both cases.

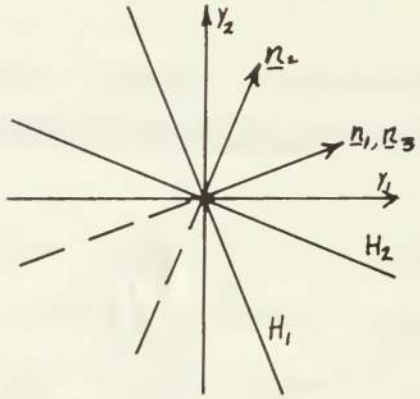



Figure 7. Intersection relationships for $a_1=1$, $a_2=0$.

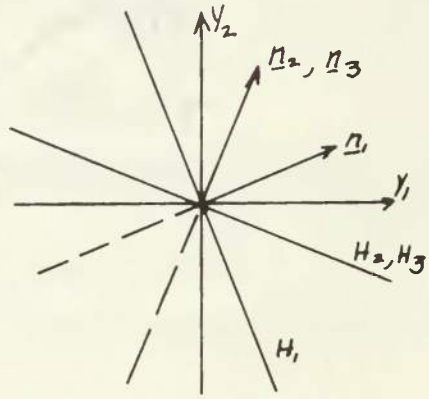



Figure 8. Intersection relationships for $a_1=0$, $a_2=1$.

In Figures 9, 10 and 11, $S_1 \cap S_2 \cap S_3 = 0$; therefore at least one $S_i$ must be dropped from consideration to have a non-zero $\underline{dQ}$.

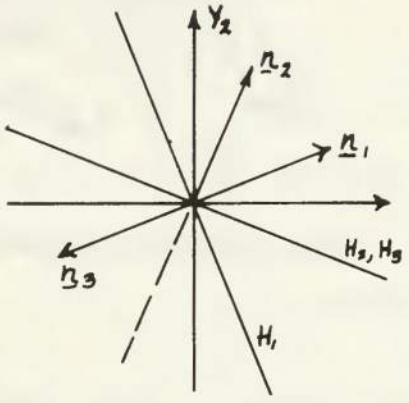



Figure 9. Intersection relationships for $a_1=1.$, $a_2=0.$

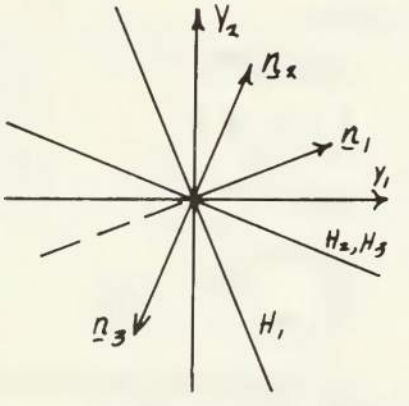



Figure 10. Intersection relationships for $a_1=0.$, $a_2=-1.$

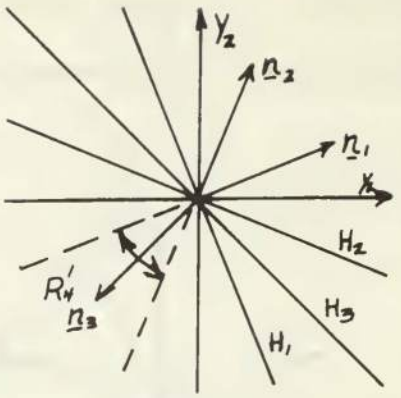



Figure 11. Intersection relationships for $a_1<0$, $a_2<0$.

$$S_1 \cap S_2 \cap S_3 = 0.$$

The general results related to these figures are tabulated in Table II.

| Figure | $a_1$ | $a_2$ | Relationship | Drop From Consideration |
|---|---|---|---|---|
| 4 | >0 | >0 | $S_3 \supseteq S_1 \cap S_2$ | $S_3$ |
| 5 | >0 | <0 | $S_1 \supseteq S_2 \cap S_3$ | $S_1$ |
| 6 | <0 | >0 | $S_2 \supseteq S_1 \cap S_3$ | $S_2$ |
| 7 | >0 | =0 | $S_3 = S_1$ | $S_3$ |
| 8 | =0 | >0 | $S_2 = S_3$ | $S_3$ |
| 9 | <0 | =0 | $S_1 \cap S_3 = H_1 = H_3$; $S_1 \cap S_2 \cap S_3 = \varphi$ | $S_1$ or $S_3$ |
| 10 | =0 | <0 | $S_1 \cap S_2 \cap S_3 = \varphi$; $S_2 \cap S_3 = H_2 = H_3$ | $S_2$ or $S_3$ |
| 11 | <0 | <0 | $S_1 \cap S_2 \cap S_3 = \varphi$ | $S_1$ or $S_2$ or $S_3$ |

Table II. General relationships for sets $S_j$, $j = 1,2,3$ for a second-order system when $\langle \underline{n}_1, \underline{n}_2 \rangle \neq \pm 1$.

As an example, if Table II has indicated that $S_3$ should be dropped from consideration, the normals $\underline{n}_1, \underline{n}_2$ may be used to form a vector, $\underline{dQ}'$, that lies in the region $R_1$ given by the angle $\alpha_1$ in Figures 12 and 13. $\underline{dQ}'$ is defined by

$$\underline{dQ}' = \left[ \underline{n}_1 \mid \underline{n}_2 \right] \underline{c} \qquad \text{(III-25)}$$

where $\underline{c}$ is a column vector with components $c_1, c_2$. If $c_1$ and $c_2$ are restricted to be positive, then $\underline{dQ}'$ lies in the region $R_1'$ which is contained by $R_1$ if $\langle n_1, n_2 \rangle \geq 0$, as in Figure 12. $R_1'$ contains $R_1$ if $\langle n_1, n_2 \rangle < 0$, as in Figure 13. For $\underline{dQ}'$ to lie within the region $R_1$, the angles $\theta_1$, and $\theta_2$ must both be less than or equal to 90 degrees. Let

$$\underline{dQ} = \frac{\underline{dQ}'}{\sqrt{(dQ_1')^2 + (dQ_2')^2}} . \qquad \text{(III-26)}$$

Then the restrictions on $\theta_1$ and $\theta_2$ become

$$\left\{ \begin{array}{c} \langle \underline{n}_1, \underline{dQ} \rangle \geq 0 \\ \text{and} \\ \langle \underline{n}_2, \underline{dQ} \rangle \geq 0 \end{array} \right\} . \qquad \text{(III-27)}$$

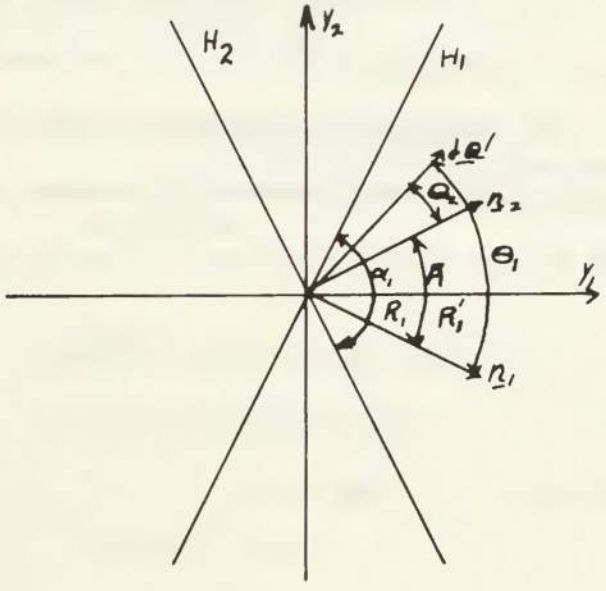



Figure 12. Region relationships for $\langle \underline{n}_1, \underline{n}_2 \rangle \geq 0$.

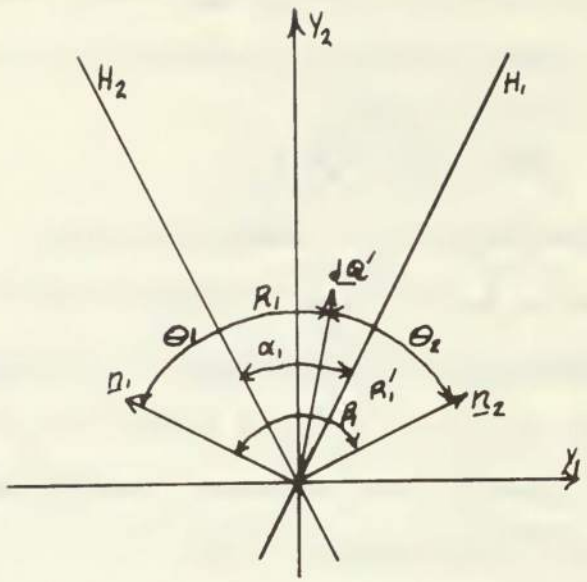



Figure 13. Region relationships for $\langle \underline{n}_1, \underline{n}_2 \rangle < 0$.

The procedure discussed above considers only three normals, $\underline{n}_j$, $j = 1,2,3$, at a time, but can be extended to the case where $L > 3$ in an iterative fashion. At each iteration, one $\underline{n}_j$ is dropped from consideration using Table (II). This process is continued until all $\underline{n}_j$'s have been considered.

(3) Third-Order Systems. For a third-order system, similar reasoning is used to develop a basis of three linearly independent $\underline{n}_j$'s. A fourth normal $\underline{n}_4$ can be represented by

$$\underline{n}_4 = a_1\underline{n}_1 + a_2\underline{n}_2 + a_3\underline{n}_3 \qquad \text{(III-28)}$$

$$= \left[\underline{n}_1 \,\vdots\, \underline{n}_2 \,\vdots\, \underline{n}_3\right] \underline{a} \qquad \text{(III-29)}$$

$$= N\underline{a} \qquad \text{(III-30)}$$

If all $a_j \geq 0$, then $S \underset{4}{\supseteq} \bigcap_{j=1}^{3} S_j$ and $S_4$ can be dropped from consideration. If all $a_j < 0$, $\bigcap_{j=1}^{4} S_j = \varphi$ and one of the four $S_j$'s must be dropped from consideration. Table III gives the appropriate relationships for n = 3, L = 4, where any 3 of the normals $\underline{n}_j$ are linearly independent.

| $a_1$ | $a_2$ | $a_3$ | Relationship | Drop From Consideration |
|---|---|---|---|---|
| $\geq 0$ | $\geq 0$ | $\geq 0$ | $S \underset{4}{\supseteq} S_1 \cap S_2 \cap S_3$ | $S_4$ |
| $<0$ | $<0$ | $<0$ | $\bigcap_{j=1}^{4} S_j = \varphi$ | $S_1$ or $S_2$ or $S_3$ or $S_4$ |
| $=0$ | $>0$ | $<0$ | $S_2 \underline{\supseteq} S_1 \cap S_3 \cap S_4$ | $S_2$ |
| $=0$ | $<0$ | $<0$ | $\bigcap_{j=1}^{4} S_j = \varphi$ | $S_2$ or $S_3$ or $S_4$ |
| $>0$ | $<0$ | $<0$ | $S_1 \underline{\supseteq} S_2 \cap S_3 \cap S_4$ | $S_1$ |
| $>0$ | $>0$ | $<0$ | $S_1 \cap S_2 \cap S_3$ is intersected by $S_4$ in such a way as to form a fourth bounding hyperplane. | |

Table III. General relationships for sets $S_j$, $j = 1,..,4$ for a third-order system when $\underline{n}_1, \underline{n}_2$ and $\underline{n}_3$ are linearly independent.

The relationships that exist for $a_1 > 0$, $a_2 > 0$ and $a_3 < 0$ in Table III imply that many boundary hyperplanes could exist for systems of order higher than two when the number of hyperplanes, L, is greater than the

order of the system under consideration. Results from linear algebra [10] can be used to show that there may be L-(n+1) bounding hyperplanes and therefore L-(n+1) normals to consider. For this reason, the algebraic development was not pursued further for cases where $L \geq n+1$.

The set of hyperplanes not excluded from consideration form a convex polyhedral cone in n-dimensional Euclidean space. Unit vectors, $\underline{u}_i$, defined as being colinear with the extreme edges of this cone, when summed with positive weighting, form an interior vector of the cone. These unit vectors, $\underline{u}_i$, can be found using linear programming techniques [11].

The problem may be reformulated into the Simplex format as follows. The linear function

$$Z = \sum_{i=1}^{n} y_i \qquad \text{(III-31)}$$

is to be maximized subject to the constraints

$$\langle \underline{n}_j, \underline{y} \rangle \geq 0 \qquad j = 1,2,\ldots,L \qquad \text{(III-32)}$$

$$|y_i| < \infty \qquad i = 1,2,\ldots,n \qquad \text{(III-33)}$$

$$\sum_{i=1}^{n} y_i \leq B \qquad \text{(III-34)}$$

where B is a positive number. If Z can be maximized by the Simplex routine, Z will be equal to B, and there may be more than one solution, $\underline{y}_j$. If there is no solution, one of the constraints in equation (III-32) must be relaxed. If there are multiple solutions, each solution, $\underline{y}_j$, forms a vector which is colinear

with an extreme edge of the polyhedral cone formed by the set

$$S = \bigcap_{j=1}^{L} S_j \qquad \text{(III-35)}$$

(See Figure 14.)

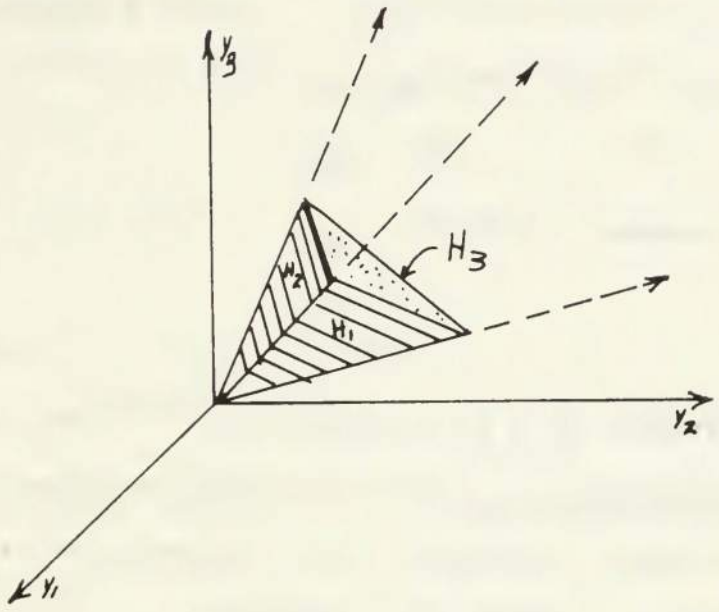



Figure 14. Intersection of 3 hyperplanes in 3-space.

The Simplex approach requires from one to L - n applications of the Simplex routine. It was felt that the computation required was too complex and time-consuming to allow the overall algorithm to be a feasible design tool.

If L is restricted to be less than or equal to n + 1, the algebraic procedures considered previously may be applied to find the normals of the boundary hyperplanes. For ease of computation and to make the algorithm feasible, L was further restricted to be less than four.

c. Problem Considerations for $L \leq n + 1$

If the partitioned matrix

$$N = \left[\underline{n}_1, \underline{n}_2, \ldots, \underline{n}_L\right] \tag{III-36}$$

is considered, the rank of N, r(N), gives the number of independent normals. If r(N) = L then all normals are independent. The vector $\underline{dQ}'$ given by

$$\underline{dQ}' = N\underline{c} \tag{III-37}$$

will be interior to the set

$$S = \bigcap_{j=1}^{L} S_j \tag{III-38}$$

if the elements of $\underline{c}$ are non-negative. If r(N) < L then the L-r(N) half-spaces $S_j$, associated with the linearly dependent normals removed from N (for this step only) to give the reduced matrix N'. Then $\underline{dQ}'$ is given by

$$\underline{dQ}' = N'\underline{c} \tag{III-39}$$

where $\underline{c}$ is now an (r(N)x1) row vector with positive equal elements.

Table II gives the general relationships that exist for an $n^{th}$-order system when L = 3 and r(N) = 2. Similar relationships are given in Table IV for an $n^{th}$-order system with L = 3 and r(N) = 1. Table V gives the relationships that exist for an $n^{th}$-order system with L = 2 and r(N) = 1.

Table IV. General relationships for sets $S_i$, $S_j$ and $S_k$ where n = 2, L = 3 and r(N) = 1.

$\underline{n}_k = a_1\underline{n}_i \cdot \underline{n}_j = a_2\underline{n}_i$

| $a_1$ | $a_2$ | Relationship | Drop From Consideration |
|---|---|---|---|
| >0 | >0 | $S_i = S_j = S_k$ | none |
| >0 | <0 | $S_i \cap S_j \cap S_k = H_{i,j,k}$ | $S_j$, or $S_i$ and $S_k$ |
| <0 | >0 | $S_i \cap S_j \cap S_k = H_{i,j,k}$ | $S_k$, or $S_i$ and $S_j$ |
| <0 | <0 | $S_i \cap S_j \cap S_k = H_{i,j,k}$ | $S_i$, or $S_j$ and $S_k$ |

Table V. General relationships for sets $S_i$ and $S_j$ where $n \geq 2$, L = 2, r(N) = 1. $\underline{n}_j = a\underline{n}_i$

| a | Relationship | Drop From Consideration |
|---|---|---|
| >0 | $S_i = S_j$ | none |
| <0 | $S_i \cap S_j = H_i$ or $H_j$ | $S_i$ or $S_j$ |

The vector $\underline{dQ}'$ found by Equations III-37 or III-39, when normalized to unit length gives the $\underline{dQ}$ required in step f. of the algorithm for selecting a new weighting matrix Q.

# IV. EXAMPLES

## A. A SECOND-ORDER EXAMPLE

The second-order system selected for consideration is given in phase-variable form by

$$\dot{\underline{x}}(t) = \begin{bmatrix} 0.0 & 1.0 \\ -20. & -5.0 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t). \qquad \text{(IV-1)}$$

The final time, $t_f$, was arbitrarily set at 2.5 seconds. The design measure, $\underline{m}$, selected for examination, had the following elements:

$m_1$ - percent overshoot for $x_1(t)$

$m_2$ - time to maximum overshoot for $x_1(t)$

$m_3$ - max $|u(t)|$, $0 \leq t \leq t_f$

which had the desired values

$$\underline{m}^* = \begin{bmatrix} 5.0 & \% \\ 0.925 & \text{seconds} \\ 0.0 & \end{bmatrix} . \qquad \text{(IV-2)}$$

Percent overshoot is defined by

$$m_1 = \left| \frac{\text{overshoot of } x_1(t)}{x_1(0)} \right| \times 100. \qquad \text{(IV-3)}$$

### 1. Estimation Procedure

The first method used to select a suitable weighting matrix Q was to guess an initial Q, solve for the design measure $\underline{m}$ and then make a new estimate for Q based on $\underline{m}$. This, of necessity, was accomplished on-line with the IBM 360-67 computer in a time-sharing mode.

The estimation procedure used is given in the following steps.

a. Guessing $Q^i$

b. Evaluating $\underline{m}$

c. Repeating steps a. and b. until one element of $\underline{m}$, $m_j$, was close to $m_j^*$.

d. Selecting $Q^{i+1}$ so that $m_j$ remained close to $m_j^*$ and improvement was made on one of the other elements of $\underline{m}$. This was accomplished by

(1) Perturbing $Q^i$

(2) Evaluating $\underline{m}^i$

(3) Repeating steps (1) and (2) until a direction of change was established that improved one of the other elements of $\underline{m}$ while keeping $\underline{m}_i$ close to $m_i^*$.

(4) Changing $Q^i$ as far as possible in the direction determined above.

e. Step d. was repeated until all of the elements of $\underline{m}$ were as close as possible to their associated elements in $\underline{m}^*$. This procedure changes Q along a contour of $m_i$ in the $q_{11}$-$q_{22}$ plane.

Using this procedure led to the weighting matrix $Q = \text{diag}\,[85.6,\ 13.6]$, with a computation time estimated at 25 minutes.

2. Exhaustive Search

The exhaustive search for the second-order example was arbitrarily restricted to the region R defined by

$$R: \begin{cases} 0 \le q_{11} \le 100. \\ 0 \le q_{22} \le 20. \end{cases} \qquad \text{(IV-4)}$$

R was divided into grid points with intervals of $dq_{11}=2.0$ and $dq_{22}=0.25$; $dq_{22}$ was required to be this small by the low sensitivity of $m_1$ and $m_2$ with respect to $q_{11}$. This low sensitivity can be seen in Figures 15 and 16. Approximately 100 minutes of IBM 360-70 computation time was required to complete the evaluation of $\underline{m}$ at all grid points.

Figures 15, 16 and 17, obtained from this search, give contours for selected values of $m_1$, $m_2$ and $m_3$. Figure 18 is a composite of these figures. Graphical construction indicates that the desired weighting matrices is in a neighborhood of Q = diag [81.0, 12.8].

3. Application of Design Algorithm to Select Q

The algorithm given in Chapter III was applied with all three elements of $\underline{m}$ being considered in the selection procedure for $\underline{dQ}$, for three initial Q's. Two initial Q's were selected to test the algorithm with only $m_1$ and $m_3$ being considered in the selection procedure for $\underline{dQ}$. The results of these trials are given in Table VI.

Attempts were made to test the algorithm with only $m_1$ and $m_2$ or $m_2$ and $m_3$ being considered in the selection procedure for $\underline{dQ}$. The moves made by the algorithm did not tend to improve the element not being considered, or approach a best Q for all three elements of $\underline{m}$. This is partially due to the insensitivity of $m_2$, with respect to Q. This insensitivity was exaggerated by the non-continuous nature of time in the digital computer calculations.

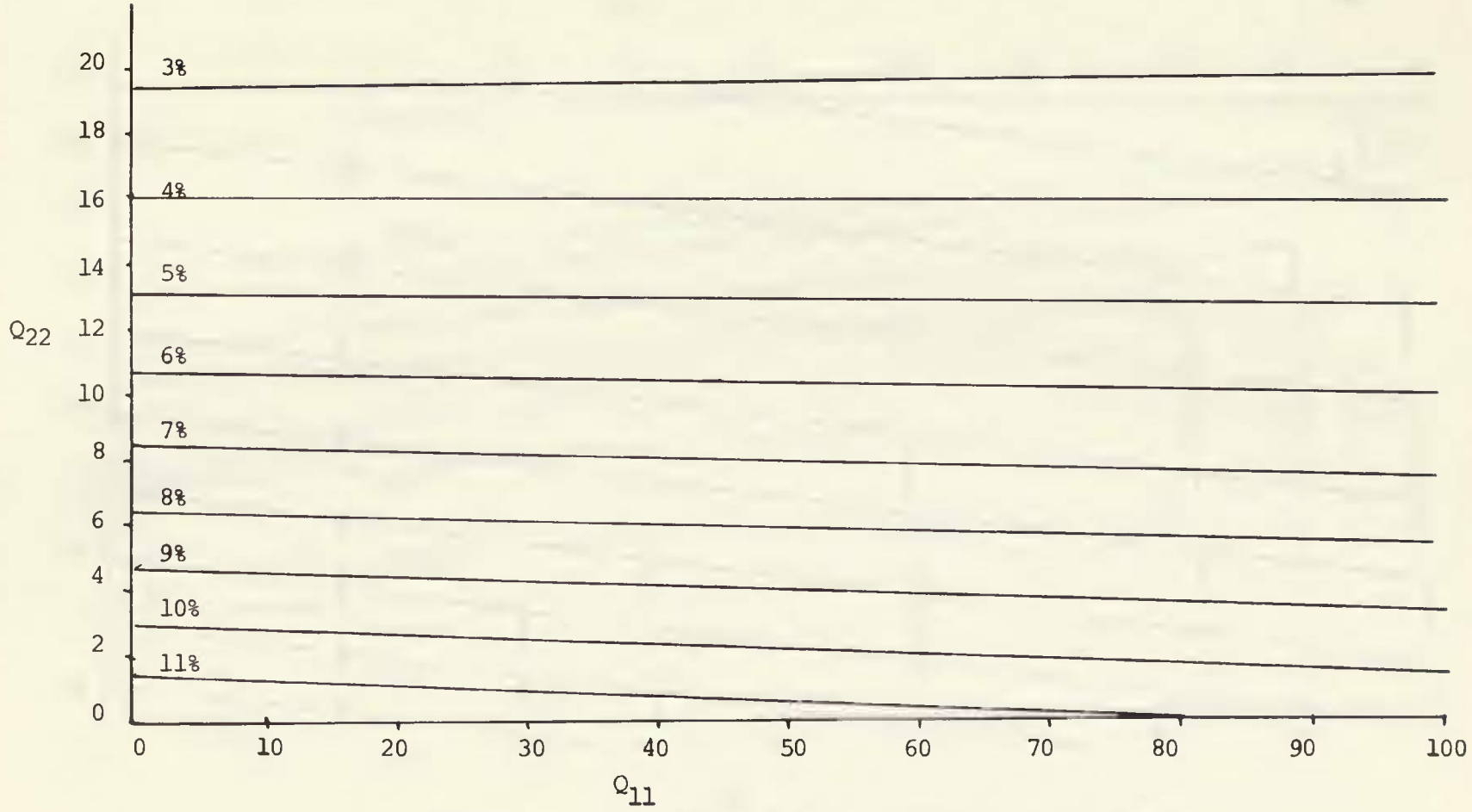



Figure 15. $Q_{11}$-$Q_{22}$ Plane showing contours for selected values of percent overshoot of $x_1$.

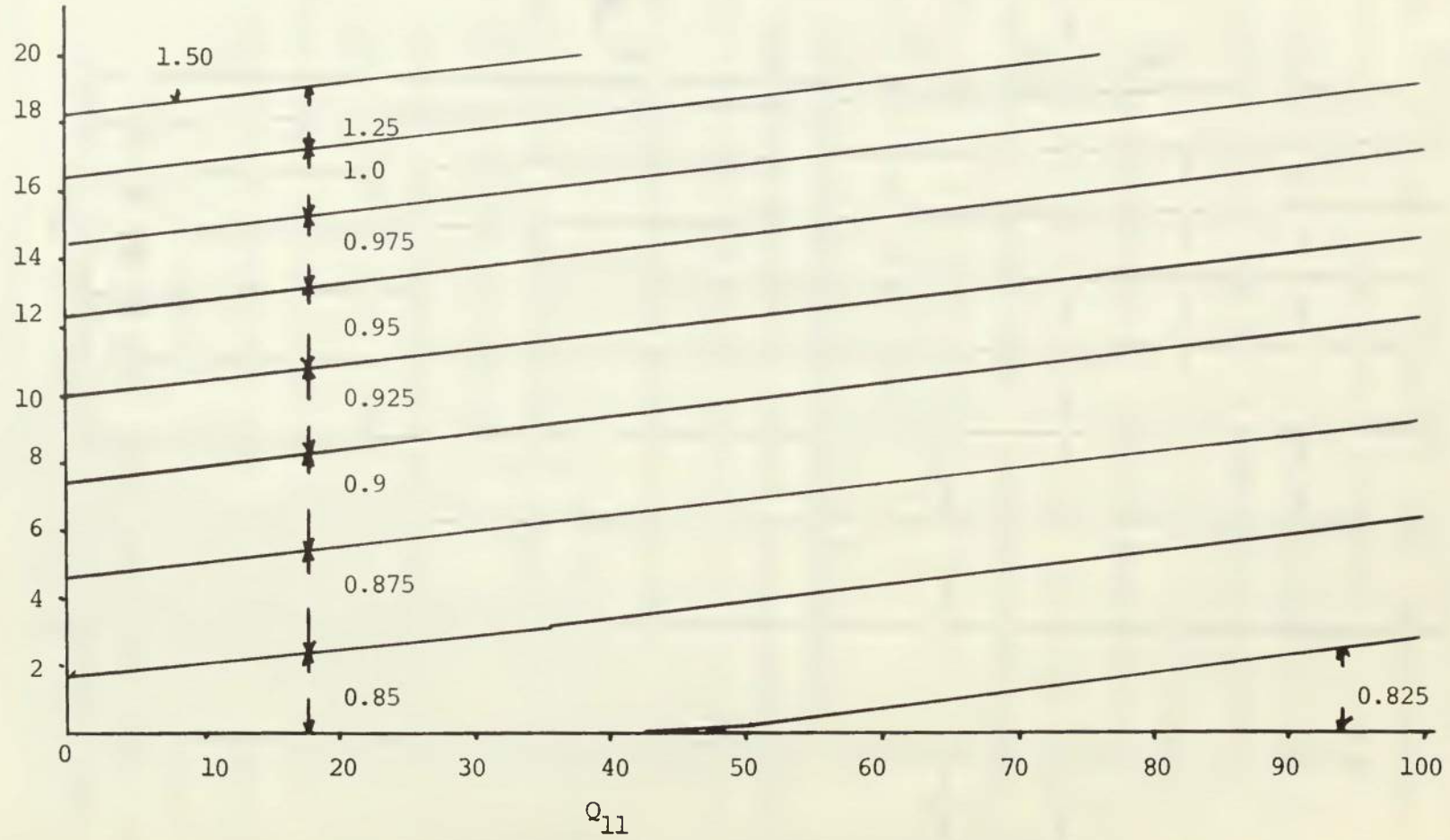



Figure 16. $Q_{11}$-$Q_{22}$ Plane showing regions for time to maximum overshoot of $x_1$.

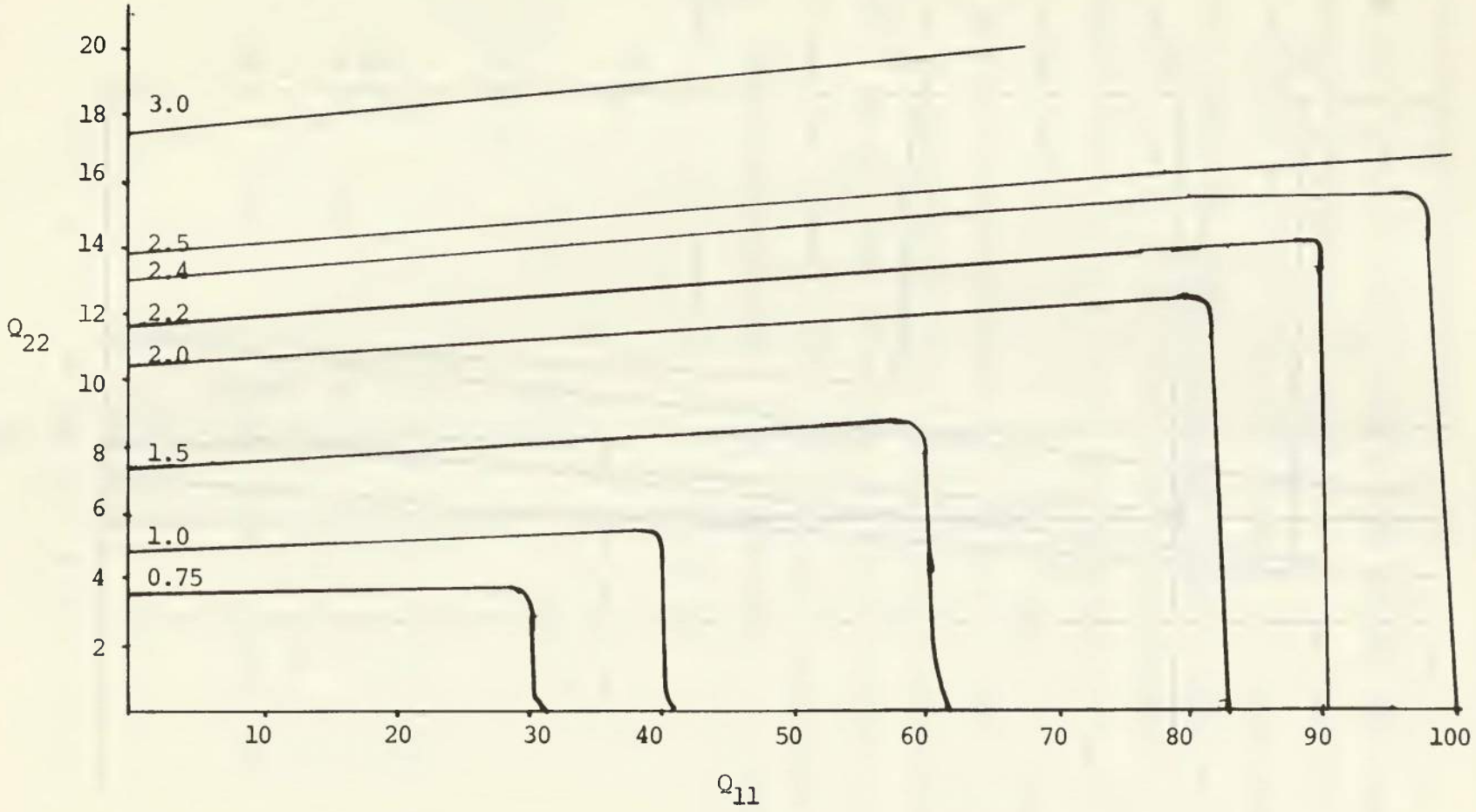



Figure 17. $Q_{11}$-$Q_{22}$ Plane showing contours for selected values of $\max_t |U(t)|$.

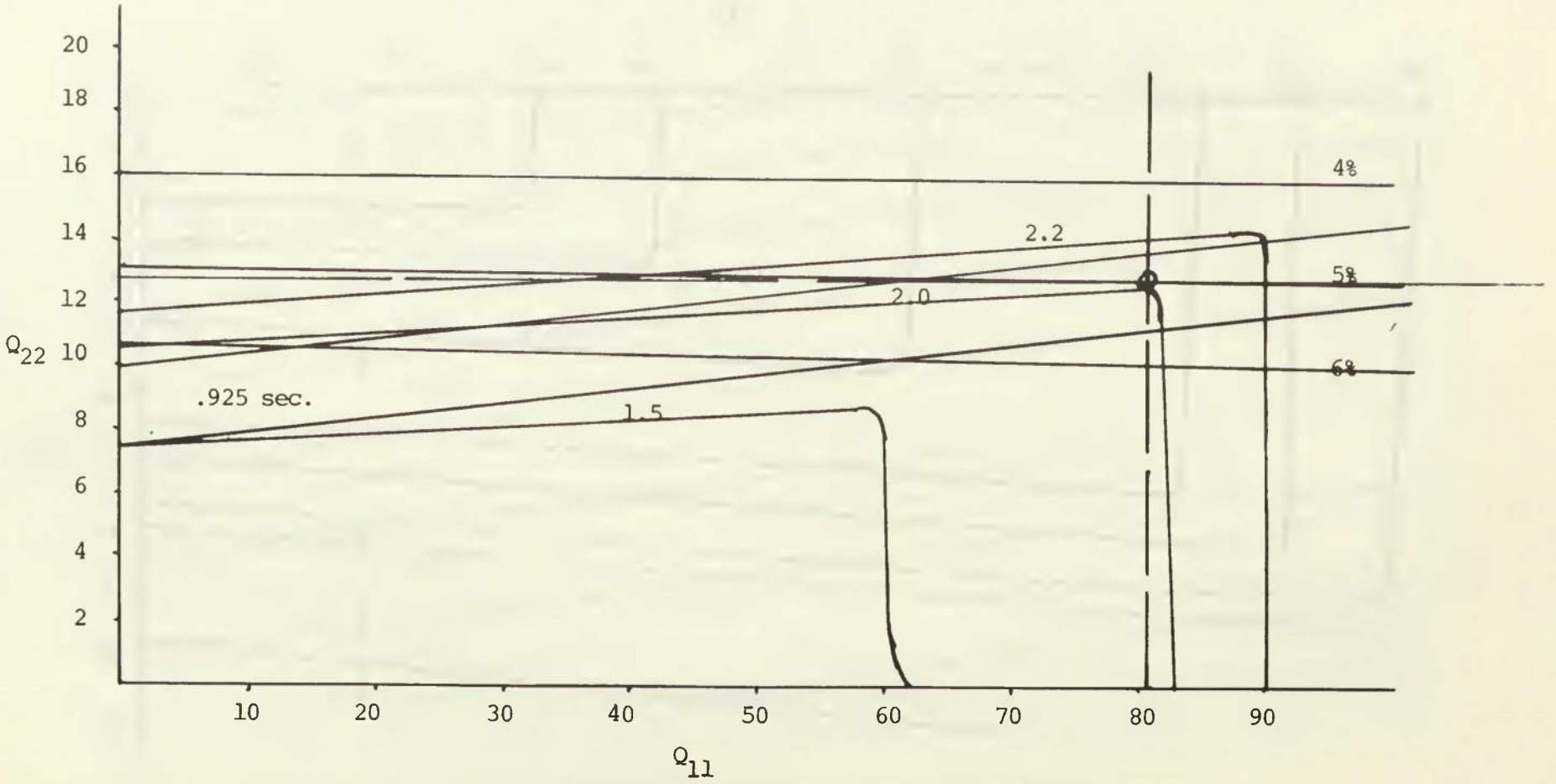



Figure 18. Composite of Figures 15,16,17.

Figures 19, 20 and 21 show the initial iteration moves made by the algorithm for trials 1, 2 and 4 in the $q_{11}$-$q_{22}$ plane. For the initial points tested, the largest number of iterations required to move $Q^i$ (the superscript i refers to the latest iteration number) to close proximity to the best Q was eight. (No claim as to the uniqueness of a best Q is made for this algorithm as this depends on the system, initial $Q^0$, and design measure being considered.) From the point, $Q^i$, which is in close proximity to the best Q, at least 20 and not more than 30 iterations were necessary to arrive at a locally unimprovable weighting matrix, denoted by $Q^f$. The $Q^f$'s thus found were not equal, although all design measure elements were considered to be satisfactory. The largest variation being in $q_{11}$, to which all elements of $\underline{m}$ are relatively insensitive in this region.

Table VI. Results for Second-Order Example

| L | $m_j$'s Considered to select $\underline{dQ}$ | Initial Q | | Final Q | | Final $\underline{m}$ | | |
|---|---|---|---|---|---|---|---|---|
| | | 1,1 | 2,2 | 1,1 | 2,2 | 1 | 2 | 3 |
| 3 | $m_1, m_2, m_3$ | 10.0 | 1.0 | 81.37 | 12.897 | 5.0057 | 0.925 | 2.075 |
| 3 | $m_1, m_2, m_3$ | 1.0 | 10.0 | 83.46 | 12.864 | 5.016 | 0.925 | 2.062 |
| 3 | $m_1, m_2, m_3$ | 100.0 | 1.0 | 80.44 | 12.683 | 5.08 | 0.925 | 2.044 |
| 2 | $m_1, m_3$ | 10.0 | 1.0 | 79.81 | 12.916 | 5.0001 | 0.925 | 2.076 |
| 2 | $m_1, m_3$ | 100.0 | 1.0 | 81.41 | 12.914 | 4.9999 | 0.925 | 2.0226 |

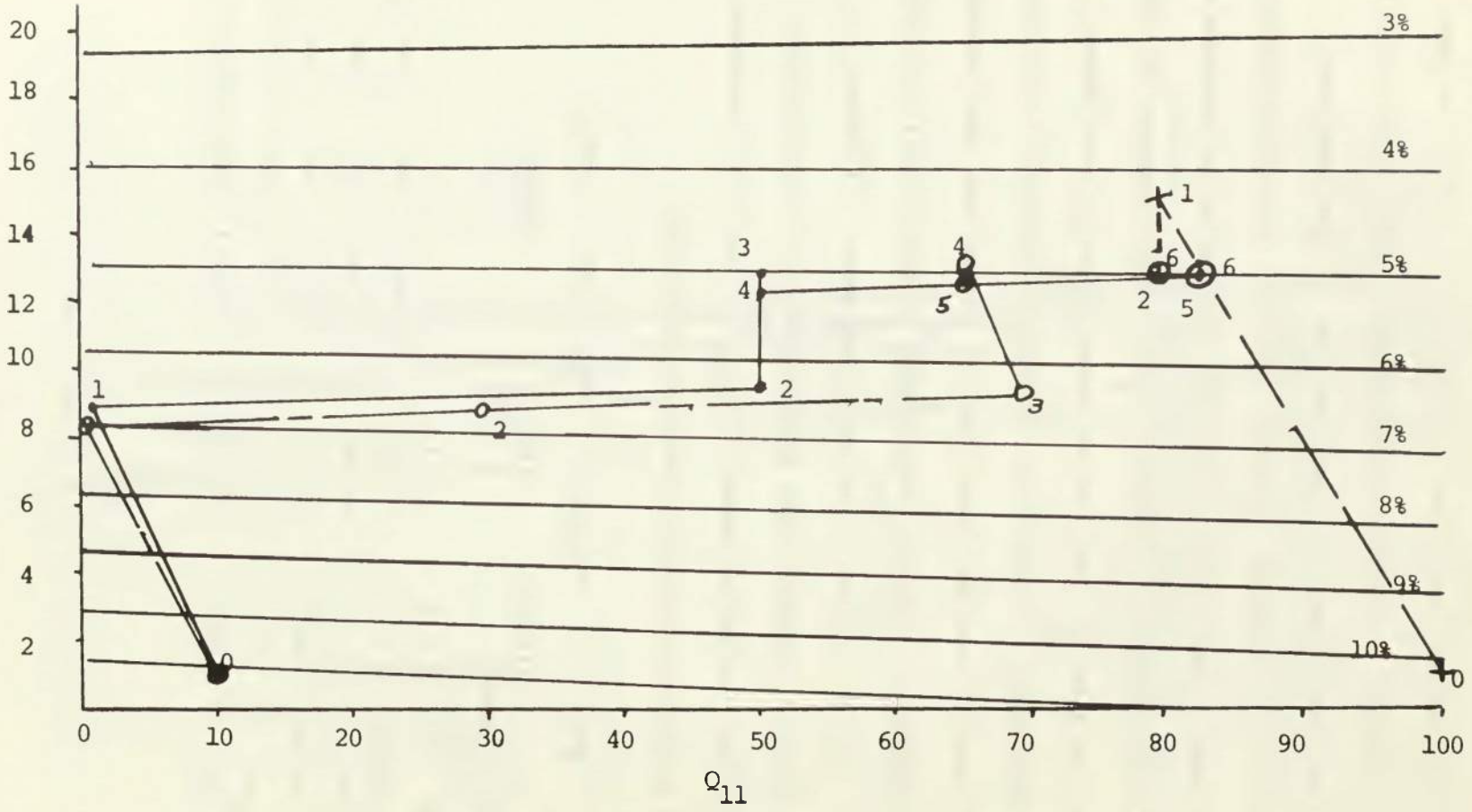



Figure 19. Algorithm moves for trials 1(·), 2(+), and 4(0) on overshoot contour plot (initial moves only).

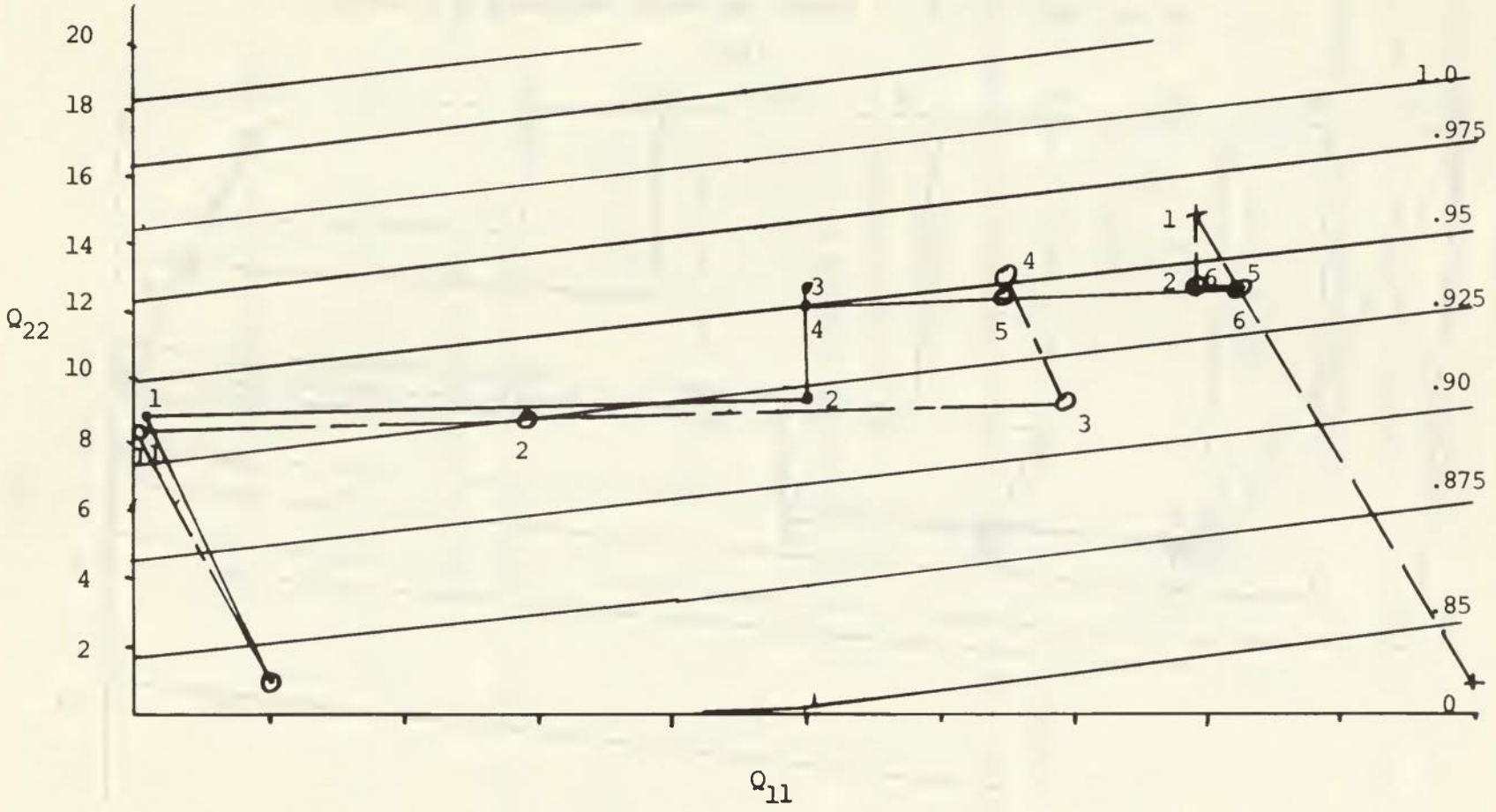



Figure 20. Algorithm moves for trials 1(·), 2(+), and 4(0) on time-to-overshoot regions (initial moves only).

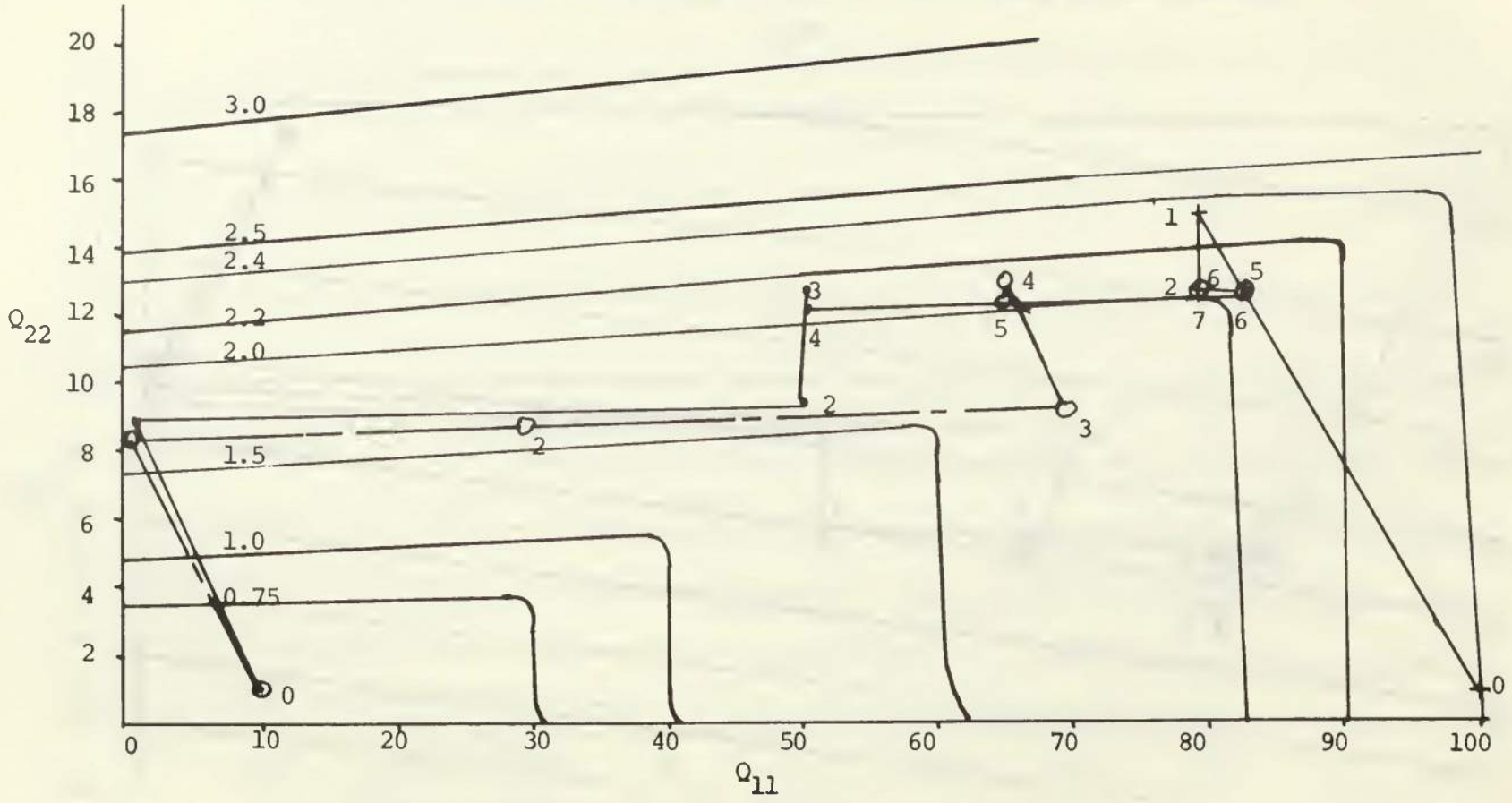



Figure 21. Algorithm moves for trials 1(·), 2(+), and 4(0) on $\max_t |U(t)|$ contours (initial moves only).

B. A THIRD-ORDER EXAMPLE

The third-order system considered was taken from [4]. It is for the linearized incremental longitudiual motion of an aircraft. The system is given by

$$\dot{\underline{x}}(t) = \begin{bmatrix} -0.0016 & 1.0 & -.0002 \\ -0.1569 & -0.0015 & -.1131 \\ 0.0 & 0.0 & -.666 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0.0 \\ 0.0 \\ 0.666 \end{bmatrix} u(t) \quad . \qquad \text{(IV-5)}$$

The final-time, $t_f$, was arbitrarily set at 2.5 seconds. The design measure, $\underline{m}$, selected for examination, had the following elements:

$m_1$ - percent overshoot for $x_1(t)$

$m_2$ - time to maximum overshoot for $x_1(t)$

$m_3$ - $\max_t |u(t)|$, $0 \leq t \leq t_f$

which were assigned the desired value

$$m^* = \begin{bmatrix} 4.5 & \% \\ 0.91 & \text{sec} \\ 0.0 & \end{bmatrix} \quad . \qquad \text{(IV-6)}$$

The system was transformed to the phase-variable form in the manner of [5]. The transformation matrix

$$T = \begin{bmatrix} -41.329 & -0.08026 & 0.0 \\ -3.41561 & -41.329 & 1.536 \times 10^{-7} \\ 52.809 & 1.406 & 6.71141 \end{bmatrix} \qquad \text{(IV-7)}$$

results in a system matrix given by

$$A_0 = \begin{bmatrix} 0. & 1. & 0. \\ 0. & 0. & 1. \\ -53. & -8.88 & -6.8 \end{bmatrix} \quad . \qquad \text{(IV-8)}$$

Therefore the system is given by

$$\dot{\underline{z}}(t) = A_0\underline{z}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \quad . \qquad \text{(IV-9)}$$

The design measure, $\underline{m}$, is defined with respect to the original states. Since it was desired to observe the state trajectories for both the phase-variable form and for the original states, the computer program was written to transform the phase state $\underline{z}(t)$ to $\underline{x}(t)$ at each integration step, in order to evaluate $\underline{m}$. A more direct method is to transform the diagonal weighting matrix to the general form, $Q_g$ and perform all integration in the original state form. $Q_g$ is given by

$$Q_g = (T^{-1})^T Q (T^{-1}) . \qquad \text{(IV-10)}$$

Kriendler's [4] selection for the original system weighting matrix is $Q_g$ = diag [10.0, 0.5, 0.0]. This $Q_g$ led to the design measure

$$\underline{m} = \begin{bmatrix} 4.497 \\ 1.1 \\ 97.2 \end{bmatrix} \quad . \qquad \text{(IV-11)}$$

No attempt was made to guess the best Q, or to use the exhaustive search method, as it was felt the expected results would not have been worth the time or effort involved.

The algorithm given in Chapter III was applied to the third-order example, with one, two and three design-measure elements, $m_j$, being considered in the selection procedure for $\underline{dQ}$. The results of this application, with test points, are tabulated for all trials in Table VII.

Favorable results on any one trial depended on the initial weighting matrix, $Q^0$, selected and on the elements $m_j$, being considered. The first trials (1-8 in Table VII) were made considering either all three elements, $m_j$, or elements $m_1$ and $m_2$. Most initial points resulted in an $m_1^f$ close to $m_1^*$, but $m_2^f$ could not be reduced to $m_2^*$; $m_3^f$ was at an apparent local minimum in Q space. Those trials that considered all design measure elements in the selection of $\underline{dQ}$ tended to reduce $m_3^f$ more rapidly in the local region.

Since $m_2^f$ seemed to be least changeable for these first trials, a starting $Q^0$ was selected that had resulted in $m_1^f$ close to $m_1^*$. A trial (9) was then made considering only $m_2$ for selecting $\underline{dQ}$. The result was a significant reduction in $\left|m_2-m_2^*\right|$ while $\left|m_1-m_1^*\right|$ did not increase unreasonably. Subsequent trials alternated between considering $m_2$ and considering both $m_1$ and $m_2$ in the selection procedure for $\underline{dQ}$, with a resulting $Q^f$ that reduced both $\left|m_1-m_1^*\right|$ and $\left|m_2-m_2^*\right|$ to an acceptable level. The $Q^f$ thus found was used as the initial $Q^0$ in a trial (14) where all three elements $m_j$, were considered in the selection procedure for $\underline{dQ}$. This trial (14) significantly reduced both $\left|m_1-m_1^*\right|$ and $\left|m_2-m_2^*\right|$ in two iterations. Tests in the neighborhood of the $Q^f$ found in trial (14) showed that the element, $q_{33}$, was zero for this system.

TABLE VII . RESULTS OF TESTING ALGORITHM ON A THIRD-ORDER EXAMPLE

| TRIAL NO. | ACTIVE CONSTRAINTS ELEMENTS | O:INITIAL F:FINAL T:TEST | I | Q(I,I) | M(I) |
|---|---|---|---|---|---|
| 1 | 1,2 | O | 1 | 17000.000 | 4.6 |
| | | | 2 | 850.00 | 1.1 |
| | | | 3 | 1.0E-9 | 95.8 |
| | | F | 1 | 17200.00 | 4.56 |
| | | | 2 | 855.80 | 1.1 |
| | | | 3 | 1.0E-9 | 96.17 |
| 2 | 1,2 | O | 1 | 1282.0 | 5.55 |
| | | | 2 | 649,0 | 1.48 |
| | | | 3 | 1.0E-9 | 32.46 |
| | | F | 1 | 11526.0 | 4.50 |
| | | | 2 | 812.5 | 1.175 |
| | | | 3 | 1.0E-9 | 73.89 |
| 3 | 1,2,3 | O | 1 | 11530.0 | 4.5 |
| | | | 2 | 812.5 | 1.175 |
| | | | 3 | 1.0E-9 | 73.89 |
| | | F | 1 | 11524.2 | 4.511 |
| | | | 2 | 811.2 | 1.175 |
| | | | 3 | 1.0E-9 | 73.884 |
| 4 | 1,2 | O | 1 | 17090.0 | 4.5584 |
| | | | 2 | 855.0 | 1.1 |
| | | | 3 | 1.0E-9 | 95.776 |
| | | F | 1 | 17489.83 | 4.4969 |
| | | | 2 | 866.6843 | 1.1 |
| | | | 3 | 1.0E-9 | 97.231 |
| | | T | 1 | 17490.0 | 4.4968 |
| | | | 2 | 866.7 | 1.1 |
| | | | 3 | 1.0E-9 | 97.231 |
| 5 | 1,2,3 | O | 1 | 17489.3 | 4.497 |
| | | | 2 | 866.68 | 1.1 |
| | | | 3 | 1.0E-9 | 97.231 |
| | | F | 1 | 17490.21 | 4.5021 |
| | | | 2 | 865.9717 | 1.1 |
| | | | 3 | 1.0E-9 | 97.231 |
| 6 | 1,2 | O | 1 | 10000.0 | 16.717 |
| | | | 2 | 1.0 | 1.05 |
| | | | 3 | 0.0001 | 65.6 |
| | | F | 1 | 10034.00 | 9.53 |
| | | | 2 | 586.23 | 1.0125 |
| | | | 3 | - 28.93 | 65.45 |
| | | O | 1 | 10031.27 | 6.78 |
| | | | 2 | 585.9265 | 1.15 |
| | | | 3 | 0.01 | 66.95 |
| | | F | 1 | 10049.01 | 4.5028 |
| | | | 2 | 786.7891 | 1.225 |
| | | | 3 | 4.6321 | 67.561 |
| | | O | 1 | 10049.01 | 6.78 |
| | | | 2 | 786.7891 | 1.225 |
| | | | 3 | 4.6321 | 67.561 |
| | | F | 1 | 10049.07 | 4.491 |
| | | | 2 | 786.2485 | 1.225 |
| | | | 3 | 5.4707 | 67.596 |
| | | F | 1 | 10049.07 | 4.6164 |
| | | | 2 | 736.2485 | 1.2 |
| | | | 3 | 0.01 | 67.353 |
| | | F | 1 | 10057.14 | 4.5021 |
| | | | 2 | 802.4175 | 1.2 |
| | | | 3 | -1.65517 | 67.338 |
| 7 | 1,2,3 | O | 1 | 10057.14 | 4.5021 |
| | | | 2 | 802.4175 | 1.2 |
| | | | 3 | -1.65517 | 67.338 |
| | | F | 1 | 10057.13 | 4.4764 |
| | | | 2 | 802.783 | 1.2 |
| | | | 3 | -0.72456 | 67.382 |
| | | O | 1 | 10500.00 | 4.5053 |
| | | | 2 | 802.00 | 1.1875 |
| | | | 3 | 1.0E-8 | 69.40 |

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TABLE VII . CONTINUED

| TRIAL NO. | ACTIVE CONSTRAINT ELEMENTS | O:INITIAL F:FINAL T:TEST | I | Q(I,I) | M(I) |
|---|---|---|---|---|---|
| | | F | 1 | 10523.85 | 4.1638 |
| | | | 2 | 83 .5637 | 1.2 |
| | | | 3 | 1.0E-8 | 69.56 |
| 8 | 1,2 | O | 1 | 20000.00 | 3.7668 |
| | | | 2 | 1000.00 | 1.0875 |
| | | | 3 | 1.0E-7 | 106.12 |
| | | F | 1 | 20007.16 | 4.5082 |
| | | | 2 | 886.54 | 1.075 |
| | | | 3 | 1.0E-7 | 106.00 |
| 9 | 2 | O | 1 | 20000.00 | 4.5115 |
| | | | 2 | 886.0 | 0.9 |
| | | | 3 | 1.0E-7 | 105.98 |
| | | F | 1 | 46490.00 | 5.4 |
| | | | 2 | 886.0 | 0.9 |
| | | | 3 | 1.0E-7 | 178.14 |
| 10 | 1,2 | O | 1 | 46490.0 | 5.4 |
| | | | 2 | 886.0 | 0.9 |
| | | | 3 | 1.0E-8 | 178.14 |
| | | F | 1 | 46493.05 | 4.4895 |
| | | | 2 | 1089.55 | 0.925 |
| | | | 3 | 1.0E-8 | 178.33 |
| 11 | 2 | O | 1 | 46493.05 | 4.4895 |
| | | | 2 | 1089.55 | 0.925 |
| | | | 3 | 1.0E-7 | 178.33 |
| | | F | 1 | 47493.05 | 5.2906 |
| | | | 2 | 914.547 | 0.9 |
| | | | 3 | 1.0E-7 | 180.45 |
| 12 | 1,2 | O | 1 | 47493.05 | 5.2906 |
| | | | 2 | 914.547 | 0.9 |
| | | | 3 | 1.0E-7 | 180.45 |
| | | F,O | 1 | 47242.65 | 4.5338 |
| | | | 2 | 1083.722 | 0.9125 |
| | | | 3 | 1.0E-7 | 180.03 |
| | | F | 1 | 46246.18 | 4.5065 |
| | | | 2 | 1083.933 | 0.925 |
| | | | 3 | 1.0E-7 | 177.76 |
| 13 | 2 | O | 1 | 46246.18 | 4.5065 |
| | | | 2 | 1083.933 | 0.925 |
| | | | 3 | 1.0E-7 | 177.76 |
| | | F | 1 | 50000.00 | 4.6105 |
| | | | 2 | 1083.933 | 0.9125 |
| | | | 3 | 1.0E-7 | 186.19 |
| | | O | 1 | 50000.00 | 4.6103 |
| | | | 2 | 1083.933 | 0.9125 |
| | | | 3 | 0.01 | 186.19 |
| | | F | 1 | 51000.00 | 4.6371 |
| | | | 2 | 1083.933 | 0.9 |
| | | | 3 | 0.01 | 188.38 |
| 14 | 1,2,3 | O | 1 | 51000.00 | 4.6371 |
| | | | 2 | 1083.933 | 0.9 |
| | | | 3 | 0.01 | 188.38 |
| | | F | 1 | 51001.00 | 4.5005 |
| | | | 2 | 1116.933 | 0.9125 |
| | | | 3 | 0.01 | 188.42 |
| | | T,O | 1 | 51001.00 | 4.5007 |
| | | | 2 | 1116.933 | 0.9125 |
| | | | 3 | 1.0E-7 | 188.42 |
| | | F | 1 | 51000.85 | 4.5015 |
| | | | 2 | 1116.74 | 0.9125 |
| | | | 3 | 1.0E-7 | 188.41 |
| | | T,O | 1 | 51000.85 | 4.5015 |
| | | | 2 | 1116.74 | 0.9125 |
| | | | 3 | 0.001 | 188.41 |
| | | F | 1 | 51000.85 | 4.5033 |
| | | | 2 | 1116.75 | 0.9125 |
| | | | 3 | -0.0997 | 188.41 |

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TABLE VII . CONTINUED

| TRIAL NO. | ACTIVE CONSTRAINT ELEMENTS | O:INITIAL F:FINAL T:TEST | I | Q(I,1) | M(I) |
|---|---|---|---|---|---|
| 15 | 1,2,3 | T,O | 1 | 51000.85 | 4.5056 |
| | | | 2 | 1116.754 | 0.9 |
| | | | 3 | -0.198 | 188.40 |
| | | F | 1 | 510000.84 | 4.5014 |
| | | | 2 | 1116.83 | 0.9125 |
| | | | 3 | -0.0131 | 188.41 |
| | | T | 1 | 51000.84 | 4.5011 |
| | | | 2 | 1116.83 | 0.9125 |
| | | | 3 | 1.0E-7 | 188.41 |
| | | T | 1 | 51000.84 | 4.5011 |
| | | | 2 | 1116.83 | 0.9125 |
| | | | 3 | 0.0 | 188.41 |
| | | T | 1 | 51000.0 | 4.5011 |
| | | | 2 | 1116.83 | 0.9125 |
| | | | 3 | 0.0 | 188.41 |
| | | T | 1 | 51000.0 | 4.5044 |
| | | | 2 | 1116.0 | 0.9125 |
| | | | 3 | 0.0 | 188.41 |
| | | T | 1 | 51000.0 | 4.5008 |
| | | | 2 | 1116.9 | 0.9125 |
| | | | 3 | 0.0 | 188.41 |
| | | T | 1 | 51000.0 | 4.5004 |
| | | | 2 | 1117.0 | 0.9125 |
| | | | 3 | 0.0 | 188.41 |
| | | T | 1 | 51000.0 | 4.4984 |
| | | | 2 | 1117.5 | 0.9125 |
| | | | 3 | 0.0 | 188.41 |
| | | T | 1 | 51000.0 | 4.4994 |
| | | | 2 | 1117.25 | 0.9125 |
| | | | 3 | 0.0 | 188.41 |
| | | T,O | 1 | 51000.0 | 4.5000 |
| | | | 2 | 1117.1 | 0.9125 |
| | | | 3 | 0.0 | 188.41 |
| | | F | 1 | 51000.0 | 4.5 |
| | | | 2 | 1117.1 | 0.9125 |
| | | | 3 | 0.0 | 188.41 |

The best Q found, using the algorithm, transforms to the original system form, $Q_g$, given by

$$Q_g = \begin{bmatrix} 29.87207 & -0.112076 & -1.39 \times 10^{-6} \\ -0.112076 & 6.54338 & 2.705 \times 10^{-9} \\ -1.393 \times 10^{-6} & 2.705 \times 10^{-9} & 6.497 \times 10^{-14} \end{bmatrix} \quad \text{(IV-12)}$$

which is positive definite.

C. DEMONSTRATION EXAMPLE

A second-order example, that had not been considered previously, was selected for a comparison of the trial-and-error estimation procedure and the design algorithm. This system is given by

$$\dot{\underline{x}}(t) = \begin{bmatrix} 0.0 & 1.0 \\ -1.0 & -1.0 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0.0 \\ 1.0 \end{bmatrix} u(t). \quad \text{(IV-13)}$$

The final-time was arbitrarily set at 2.5 seconds. The design measure, $\underline{m}$, considered had the following elements:

$m_1$ - percent overshoot for $x_1(t)$

$m_2$ - time to maximum overshoot for $x_1(t)$

$m_3$ - $\max_t |u(t)| \quad 0 \le t \le t_f$ ,

which had the desired value

$$\underline{m}^* = \begin{bmatrix} 5.0 & \% \\ 0.925 & \text{sec.} \\ 0.0 & \end{bmatrix} . \quad \text{(IV-14)}$$

The estimation procedure was applied for several starting points, with no success in satisfying both $m_1^*$ and $m_2^*$. Q = diag [500.0, 0.0] was the most satisfactory point found; the design measure for this Q was $\underline{m}^T = [4.629, .925, 21.97]$. CPU time was seven minutes.

The design algorithm was applied for an initial $Q = \text{diag}\,[100.0,\ 0.1]$, with only $m_1$ and $m_2$ being considered in the selection of $\underline{dQ}$. After five iterations, an unimprovable weighting matrix, $Q = \text{diag}\,[100.56,\ 0.11539]$, was obtained, which had the design measure, $\underline{m}^T = [5.026,\ 1.4,\ 9.59]$. With only $m_1$ and $m_2$ being considered in the selection of $\underline{dQ}$, an initial $Q = \text{diag}\,[500.,\ 0.1]$ resulted in $Q = \text{diag}\,[499.9998,\ -1.125]$ and $\underline{m}^T = [5.0022,\ 0.925,\ 21.966]$ in one iteration. Considering all three elements of $\underline{m}$ in the selection of $\underline{dQ}$, an initial $Q = \text{diag}\,[500.,\ -1.125]$ resulted in $Q = \text{diag}\,[499.5,\ -1.11969]$ and $\underline{m}^T = [5.0008,\ 0.925,\ 21.955]$ in six iterations. Five minutes of CPU time were required to perform these calculations.

Application of the algorithm, considering one, two or three elements of $\underline{m}$ in the selection of $\underline{dQ}$, for the initial point $Q = \text{diag}\,[1000.,\ 0.1]$, failed to result in a satisfactory design measure. All moves stayed in close proximity to the initial point. This failure was apparently due to the relative insensitivity of the time to overshoot with respect to Q, which was exaggerated by the discrete time intervals used in the calculations. Fourteen iterations were performed in 4.6 minutes CPU time.

# V. SUMMARY AND CONCLUSION

Since the optimal state trajectory for a linear regulator system depends upon the weighting matrices chosen, it is desirable to establish a relationship between the weighting matrix, Q, and the design objectives.

Many authors have considered methods for selecting a weighting matrix for the infinite-final-time ($t_f = \infty$) problem that result in a system that meets their requirements. The only methods surveyed that can be applied to the finite-final-time problem ($t_f < \infty$) are exhaustive search in Q space and trial-and-error adjustment of the Q matrix.

The exhaustive-search method appears to be a feasible method for selecing Q under very restrictive conditions. The region that is investigated must be small or computation time borders on the ridiculous. This method is only feasible for systems where only two elements of the weighting matrix, Q, are allowed to vary.

The estimation procedure does not require as many calculations as the exhaustive method does for the same system, provided the subjective decisions are made properly and with efficient use of the computer. The procedure could be applied in situations where more than two elements of the weighting matrix are allowed to vary, but the information that must be processed by the designer becomes excessive.

It has been shown that the sensitivity of the optimal state trajectory with respect to the weighting matrix can be evaluated using the influence function approach. Because this evaluation is computationally complex, and it is necessary to consider the time-varying trajectory sensitivity to select a change for the time-invariant weighting matrix, it was decided that this approach would not be fruitful.

The procedure selected for development required the formulation of a vector design measure, $\underline{m}$. This design measure was composed of generally used time-domain system characteristics such as percent overshoot, time to maximum overshoot, etc., and desired values were selected for each element of $\underline{m}$. The purpose in selecting Q, therefore became the minimization of the absolute difference between each element of $\underline{m}$ and its desired value $m_j^*$. This difference is denoted by

$$f_j = |m_j - m_j^*|; \quad j = 1,2,\ldots,L. \tag{V-1}$$

The sensitivity of the design measure, denoted by $\frac{\partial \underline{m}}{\partial Q}$, must be approximated by finite-differences. $\frac{\partial \underline{m}}{\partial Q}$ is related to the finite differential $\underline{dQ}$ by

$$\underline{dm} = \frac{\partial \underline{m}^T}{\partial Q} \underline{dQ}. \tag{V-2}$$

This relationship was utilized to develop a procedure for selecting a $\underline{dQ}$ that results in the reduction of as many of the elements, $f_j$, as possible.

The procedure for selecting $\underline{dQ}$ was incorporated in an iterative algorithm that attempts to select a weighting matrix Q that minimizes each design-measure difference, $f_j$.

The algorithm was tested on three examples, two second-order systems and one third-order system. These tests were carried out on the IBM 360-67 computer system operating in a time-sharing mode. Operation in this mode was mandatory due to the subjective decisions that are required by the procedure for selecting $\underline{dQ}$.

Computer CPU time for the third-order example was approximately 91.3 minutes. It should be noted that this time includes many calculations that were made solely to check out the algorithm. A designer who was reasonably familiar with this algorithm should require no more than one-third that time to develop an acceptable design for a third-order system.

The tests indicate that the procedure for selecting $\underline{dQ}$ is a valid extension of Equation (V-2), and does lead to a reduction in the elements $f_j$ in a neighborhood of the Q in question. The test further indicates that the algorithm does lead to a minimization of the elements $f_j$, for a third-order system, if the number of elements, $m_j$, used in the selection procedure for $\underline{dQ}$ is alternated in a judicious manner between one, two and three. Consideration must be given to the relative magnitudes of the various sensitivities, $\frac{\partial m_j}{\partial Q_i}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, L$, and to the current values of $\underline{m}$ so that needless and useless computation is avoided.

The algorithm developed in Chapter III, modified by the consideration discussed above, appears to be a feasible procedure for use in designing optimal controls for first-, second-, and third-order linear regulations. An extension of the algorithm to linear tracking systems can be readily made.

The extension of the algorithm to higher-order systems should be limited in feasibility only by the computer time required to integrate large-order systems of differential-equations. The selection procedure for $\underline{dQ}$ developed in Chapter III applies for $n^{th}$-order systems, provided that the number of design measure elements is less than or equal to n+1. This presupposes that the L hypersurfaces in n-dimensional Q-space, corresponding to the elements of $\underline{m}^*$, have a common intersection.

It is felt that further testing of the algorithm should be undertaken before incorporation in a general design procedure. This testing should include higher-order systems and different combinations of design measures.

A possible direction for future research concerns a time-varying weighting matrix, Q(t). A less time-consuming integration scheme, such as on a hybrid analog-digital computer, could make the evaluation of the state-trajectory sensitivity, $\frac{\partial \underline{x}}{\partial Q}$, feasible for higher-order systems. It seems reasonable to assume $\frac{\partial \underline{x}}{\partial Q}$ could be used to adjust the weighting matrix, Q(t), to reduce the absolute difference between the elements of the state vector and those of a desired state vector.

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13. ABSTRACT

The relationship between the weighting matrices and the design objectives for finite-final-time linear regulator systems is considered. An iterative algorithm is presented for selecting a weighting matrix that reduces the absolute difference between the actual and desired values of a vector design measure. The algorithm utilizes the sensitivities of the vector design measure to determine changes for the weighting matrix. These sensitivities are approximated by finite-difference perturbations of the weighting matrix elements. Examples are presented that illustrate the design procedure.

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