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E. M. Hofstetter

Simple Estimates of Wake Velocity Parameters

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ABSTRACT

A technique for using a sequence of two-pulse bursts for obtaining estimates of mean wake velocity and wake velocity width has been proposed by W. D. Rummler of Bell Telephone Laboratories. The derivation of this technique was based on heuristic reasoning. The present note derives the same estimates from the theory of maximum likelihood estimation.

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Chief, Lincoln Laboratory Office

In a recent series of memoranda, W. D. Rummler of the Bell Telephone Laboratories has proposed simple estimates of mean wake velocity and wake velocity width. These estimates were obtained on the basis of purely heuristic arguments. The purpose of the present note is to show that these same estimates can be derived from fundamentals using the method of maximum likelihood.

The transmitted signal s(t), to be used for obtaining these estimates is a sequence of two-pulse bursts such as shown in Figure 1. The pulse width Δ is chosen to yield the desired range resolution and the interpulse spacing is chosen so that the wake returns from the two pulses of the pair do not overlap. The time between bursts is made large enough so that the returns from a range cell located a fixed distance behind the re-entry vehicle are statistically independent. This condition of statistical independence can also be achieved by suitably jumping the carrier frequency between successive two-pulse bursts.

The signal[+] r(t), received when a single burst is transmitted will be taken to be

$$r(t) = c(t) + n(t) \tag{1}$$

where n(t) is white, Gaussian receiver noise and

$$c(t) = \iint a(\tau, f) s(t-\tau) e^{j2\pi ft} d\tau \, df \tag{2}$$

is the uncorrupted wake return. The function $a(\tau, f)$ is the complex amplitude density of that portion of the wake return having delay τ relative to the re-entry vehicle and Doppler shift f. This function is random with the properties.

$$E[a(\tau, f)] = 0$$

$$E[a(\tau, f) a(\tau', f')] = 0$$

$$E[a(\tau, f) a^*(\tau', f')] = \sigma(\tau, f) \delta(\tau - \tau') \delta(f - f') \tag{3}$$

[+]All time functions will be understood to be the complex envelopes of their corresponding real, r.f. signals.

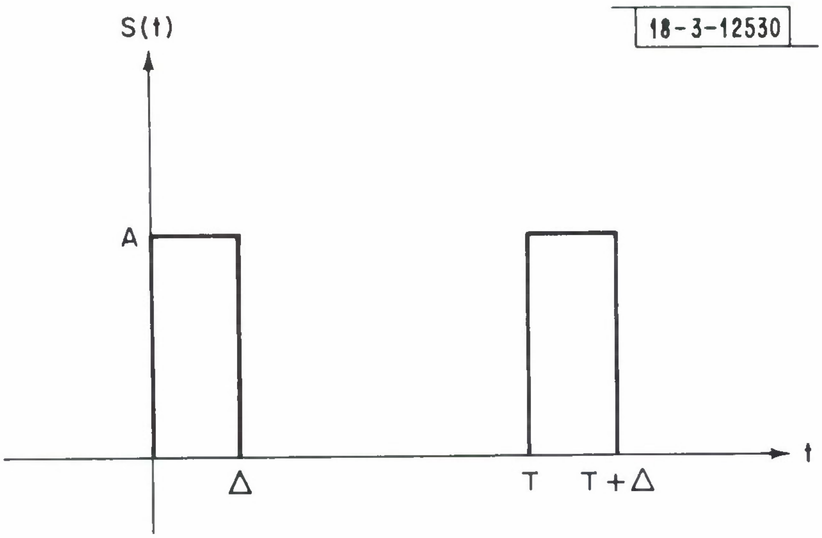

Fig. 1. The two-pulse bursts.

where $\sigma(\tau, f)$ is the average scattering cross-section density of that part of the wake located at delay τ and Doppler shift f. The basic problem at hand is to estimate the "f-centroid" and "f-width" of $\sigma(\tau, f)$ for various values of τ.

The conditions (3) correspond to assuming that the returns from different wake scatterers are statistically independent and that the phases of the returns are uniformly distributed. It will further be assumed that the total number of scatterers is large enough so that some form of the Central Limit Theorem holds and c(t) can be modelled as a Gaussian process. This being the case, c(t) can be characterized completely by the three functions.

$$E[c(t)] = 0$$

$$E[c(t)c(t')] = 0$$

$$E[c(t)c^*(t')] = \iint \sigma(\tau, f) s(t-\tau) s^*(t'-\tau) e^{j2\pi f(t-t')} d\tau df \tag{4}$$

In order to facilitate the forthcoming analysis, it will be assumed that the radar receiver operates in a range-sampled mode. This means that the received signal is passed through an i. f. filter whose bandwidth is roughly equal to the signal bandwidth and then sampled once per range resolution cell over the range interval of interest; in the present case this is the range interval over which the wake return is to be studied. The result of this process is that each transmitted pulse pair produces a pair of complex samples for each range gate, $r_1(i) \equiv r(\tau_o + i\Delta)$, $r_2(i) \equiv r(\tau_o + T + i\Delta)$ where τ_o denotes the range delay at which the first wake return of interest occurs and i denotes the number of the particular range gate under study. These complex samples are zero-mean Gaussian random variables whose covariances are given by

$$E[r_1(i) r_1(i)] = E[r_2(i) r_2(i)] = E[r_1(i) r_2(i)] = 0$$

$$E[r_1(i) r_2^*(i)] = A^2 \Delta \int \sigma(\tau_o + i\Delta, f) e^{-j2\pi f T} df \tag{5}$$

$$\begin{aligned} E[r_1(i) r_1^*(i)] &= E[r_2(i) r_2^*(i)] \\ &= A^2 \Delta \int \sigma(\tau_o + i\Delta, f) df + \alpha^2 \end{aligned}$$

where α^2 denotes the variance of the range-sampled receiver front-end noise. Equations (5) follow directly from equations (4) when it is assumed that the function $\sigma(\tau, f)$ can be regarded as constant over a delay interval of length Δ so that the τ-integration can be performed explicitly and it is assumed that $\sigma(\tau, f)$ is only non-zero in a delay interval of length less than T. This last assumption is equivalent to the earlier assumption that the interpulse spacing of the two-pulse burst is large enough to straddle the wake.

The pair of random variables $r_1(i)$, $r_2(i)$ is Gaussianly distributed with a density function given by

$$\ln p[r_1(i), r_2(i)] \equiv \ln p[\underline{r}(i)] = -\underline{r}^*(i)\Lambda_i^{-1}\underline{r}(i) - \ln[\pi^2|\Lambda_i|] \tag{6}$$

where

$$\underline{r}(i) = \begin{bmatrix} r_1(i) \\ r_2(i) \end{bmatrix} \tag{7}$$

$$\Lambda_i = \begin{bmatrix} \beta_i^2+\alpha^2 & \beta_i^2\rho_i \\ \beta_i^2\rho_i^* & \beta_i^2+\alpha^2 \end{bmatrix} \tag{8}$$

and

$$\beta_i^2 = A^2\Delta\int\sigma(\tau_o+i\Delta,\ f)df$$

$$\beta_i^2\rho_i = A^2\Delta\int\sigma(\tau_o+i\Delta,\ f)e^{-j2\pi fT}df \tag{9}$$

The assumption that the returns from successive two-pulse bursts are statistically independent now can be used to write down the joint probability density function for the returns from N successive bursts as follows,

$$\ln p[\underline{r}_1(i), \ldots \underline{r}_N(i)] = -\sum_{k=1}^{N}[\underline{r}_k^*(i)\Lambda_i^{-1}\underline{r}_k(i) + \ln[\pi^2|\Lambda_i|]] \tag{10}$$

The maximum likelihood strategy for forming estimates of the mean wake velocity and wake velocity width in each of the range cells of interest is to maximize the function given by Equation (10) with respect to these parameters and with respect to any other unknown parameters. To accomplish this end, it is first necessary to give precise definitions of these quantities. There will be done by assuming that the function $\sigma(\tau_o+i\Delta, f)$ has a precisely known shape but that its scale, center and width are unknown. Stated in symbols, it is assumed that

$$\sigma(\tau_o+i\Delta, \quad f) = \frac{\sigma_i}{w_i}\sigma_o(i, \quad \frac{f-\overline{f}_i}{w_i}) \tag{11}$$

where the function $\sigma_o(i, x)$ is known for each i but σ_i, $\overline{f}_i$ and w_i are unknown parameters. It is convenient, but not necessary for most of the following analysis to further assume that $\sigma_o(i, f)$ has been normalized so that

$$\int\sigma_o(i, f)df = 1$$

$$\int\sigma_o(i, f)fdf = 0$$

$$\int\sigma_o(i, f)f^2df = 1 \tag{12}$$

The normalizations defined by Equation (12) are equivalent to defining σ_i, w_i, and $\overline{f}_i$ in the manner usually used in radar estimation problems; namely,

$$\sigma_i = \int\sigma(\tau_o+i\Delta, \quad f)df$$

$$\sigma_i\overline{f}_i = \int\sigma(\tau_o+i\Delta, \quad f)fdf$$

$$\sigma_i w_i^2 = \int\sigma(\tau_o+i\Delta, \quad f)(f-\overline{f}_i)^2df \tag{13}$$

The wake parameter estimation problem now has been reduced to a purely mathematical problem, that of maximizing the function defined by Equation (10) with respect to the parameters σ_i, $\overline{f}_i$, w_i. Since $\beta_i^2 = A^2\Delta\sigma_i$, maximizing this function with respect to σ_i is the same as maximizing it with respect to β_i^2. This will be done in the following analysis

Denoting $\frac{1}{N}$ times the left-hand side of Equation (1) by L and referring to Equation (8) allows Equation (10) to be written in the form,

$$L = -2y_i \cdot \frac{(\beta_i^2+\alpha^2) - \mathrm{Re}\,\beta_i^2 \rho_i x_i^*}{(\beta_i^2+\alpha^2)^2 - \beta_i^4 |\rho_i|^2} - \ln \pi^2 [(\beta_i^2+\alpha^2)^2 - \beta_i^4 |\rho_i|^2] \tag{14}$$

where

$$y_i \equiv \frac{1}{2N} \sum_{k=1}^{N} [|r_{1k}(i)|^2 + |r_{2k}(i)|^2] \tag{15}$$

$$x_i y_i \equiv \frac{1}{N} \sum_{k=1}^{N} r_{1k}(i) r_{2k}^*(i) \tag{16}$$

The parameters $\overline{f}_i$ and w_i enter Equation (14) only through the complex quantity ρ_i. It is easy to show, using Equations (9) and (11), that any value of ρ_i inside the unit circle in the complex plane can be achieved by a suitable choice of $\overline{f}_i$ and w_i and conversely, that any choice of $\overline{f}_i$ and w_i results in a value of ρ_i inside the unit circle. Thus, maximizing L with respect to $\overline{f}_i$ and w_i is equivalent to maximizing it with respect to ρ_i with the constraint that ρ_i lie inside the unit circle.

The maximum of L with respect to $\arg\rho_i$ obviously occurs when $\arg\rho_i = \arg\hat{\rho}_i$ where

$$\arg\hat{\rho}_i = \arg x_i \tag{17}$$

and is given by

$$\max_{\arg\rho_i} L = -2y_i \cdot \frac{(\beta_i^2+\alpha^2) - |\beta_i^2 \rho_i x_i|}{(\beta_i^2+\alpha^2)^2 - \beta_i^4 |\rho_i|^2} - \ln \pi^2 [(\beta_i^2+\alpha^2)^2 - \beta_i^4 |\rho_i|^2] \tag{18}$$

The maximization of Equation (18) with respect to β_i^2 and $|\rho_i|$ is facilitated by the change of variables,

$$f_i = \beta_i^2 (1 + |\rho_i|) + \alpha^2 \tag{19}$$
$$g_i = \beta_i^2 (1 - |\rho_i|) + \alpha^2$$

in terms of which Equation (18) assumes the form

$$\underset{\arg \rho_i}{\max L} = - y_i \frac{1 - |x_i|}{g_i} - \ln \pi g_i - y_i \frac{1 + |x_i|}{f_i} - \ln \pi f_i \qquad (20)$$

Differentiating Equation (20) with respect to f_i and g_i and setting the results equal to zero yields the following equations for $\hat{f}_i$ and $\hat{g}_i$ the maximizing values of f_i and g_i

$$y_i \frac{1 - |x_i|}{\hat{g}_i^2} - \hat{g}_i = 0$$

$$y_i \frac{1 + |x_i|}{\hat{f}_i^2} - \hat{f}_i = 0 \qquad (21)$$

therefore,

$$\hat{g}_i = y_i(1 - |x_i|)$$

$$\hat{f}_i = y_i(1 + |x_i|) \qquad (22)$$

and the corresponding estimates of β_i and $|\rho_i|$ are given by

$$\hat{\beta}_i^2 = y_i - \alpha^2$$

$$|\hat{\rho}_i| = \frac{y_i |x_i|}{y_i - \alpha^2} \qquad (23)$$

The estimates given by Equation (23) are correct (i. e. the values of β_i^2 and $|\rho_i|$ that maximize L subject to constraints $\beta_i^2 > 0$, $|\rho_i| < 1$), if, and only if, $\hat{\beta}_i^2 > 0$ and $|\hat{\rho}_i| < 1$. If either of these conditions is violated, then the required maximum of L occurs for degenerate values of the parameters such as $\hat{\beta}_i^2 = 0$ and/or $|\hat{\rho}_i| = 1$. It can be shown that as long as the integrated wake-to-noise ratio $N\beta_i^2/\alpha^2$ is sufficiently greater than unity, this situation will only occur with very small probability. In the sequel it will be assumed that the wake-to-noise ratio is large enough so that the occurrence of the above mentioned degeneracies can be ignored.

The results of the above analysis now can be summarized by combining Equations (17) and (23) with Equations (9) and (11) yielding the results that the estimates of β_i^2 and $\bar{f}_i$ and w_i are given by the equations,

$$\hat{\beta}_i^2 = y_i - \alpha^2 \tag{24}$$

$$\arg\left[\frac{1}{\hat{w}_i}\int \sigma_o\left(i, \frac{f-\hat{\bar{f}}_i}{\hat{w}_i}\right) e^{-j2\pi f T} df\right] = \arg x_i \tag{25}$$

$$\left|\frac{1}{\hat{w}_i}\int \sigma_o\left(i, \frac{f-\hat{\bar{f}}_i}{\hat{w}_i}\right) e^{-j2\pi f T} df\right| = \frac{y_i |x_i|}{y_i - \alpha^2} \tag{26}$$

Equation (24) is an explicit expression for $\hat{\beta}_i^2$, whereas, Equations (25) and (26) only define $\hat{\bar{f}}_i$ and $\hat{w}_i$ implicitly. These last two equations can be further simplified by making the change of variables

$$x = \frac{f-\hat{\bar{f}}_i}{\hat{w}_i}$$

resulting in the equations.

$$\arg\left[e^{-j2\pi\hat{\bar{f}}_i T}\int \sigma_o(i, x) e^{-j2\pi \hat{w}_i T x} dx\right] = \arg x_i \tag{25$'$}$$

$$\left|\int \sigma_o(i, x) e^{-j2\pi \hat{w}_i T x} dx\right| = \frac{y_i |x_i|}{y_i - \alpha^2} \tag{26$'$}$$

These equations still define the exact maximum likelihood estimates $\hat{\bar{f}}_i$ and $\hat{w}_i$ in an implicit manner; however, they suggest an approximation that will yield explicit expressions for these estimates. If the estimate $\hat{w}_i$ is sufficiently small compared to $\frac{1}{T}$ ($\hat{w}_i T << 1$), then the exponential appearing on the left-hand side of Equations (25$'$) and (26$'$) can be approximated by the first three terms of its Taylor series.

$$e^{-j2\pi\hat{w}_i Tx} \cong 1 - j2\pi\hat{w}_i Tx - \frac{1}{2}(2\pi\hat{w}_i T)^2 x^2 \qquad (27)$$

Substituting Equation (27) into Equations (25') and (26') and making use of the normalizations defined by Equation (12) yields the result

$$\hat{\overline{f}}_i \cong \frac{1}{2\pi T} \arg x_i \qquad (25'')$$

$$(\hat{w}_i)^2 \cong \frac{1}{2(\pi T)^2} \left[1 - \frac{y_i |x_i|}{y_i - \alpha^2} \right] \qquad (26'')$$

Note that these approximate expressions for the maximum likelihood estimates $\hat{\overline{f}}_i$ and $\hat{w}_i$ do not require explicit knowledge of the shape of the wake distribution $\sigma_o(i, f)$ as do the exact Equations (25') and (26'). Equations (25'') and (26'') define exactly the same estimates that Rummler has proposed on an intuitive basis. They are now seen to be approximations to the maximum likelihood estimates of $\overline{f}_i$ and w_i valid as long as the estimated wake velocity width $\hat{w}_i$ is sufficiently small compared to $\frac{1}{T}$, the unambiguous frequency interval defined by the two-pulse burst.

This concludes the derivation of Rummler's wake velocity estimates from basic statistical principles. For a discussion of the bias and accuracy of these estimates, the reader is referred to the references cited below.

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13. ABSTRACT

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