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RADIATION FROM SLOT ANTENNAS ON CONES IN THE PRESENCE OF A LAYERED PLASMA SHEATH

FINAL REPORT

by

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ABSTRACT

The radiation from slot antennas on a cone in the presence of an inhomogeneous sheath is treated. The sheath is considered as being made up of one or two conical layers, each of which is homogeneous. The boundary conditions lead to a system of integral equations, which number 4M+4 for a sheath composed of M (= 1 or 2) conical layers. These are reduced to singular integral equations of Cauchy type, which are solved in iterative fashion. For sufficiently fine stratification of the sheath, the first iteration should suffice.

In general, fields of both magnetic and electric types are generated in the presence of a sheath, even though only a field of magnetic type may be generated in free space. For a ring slot, however, in which the excitation is azimuthally symmetrical, only a field of magnetic type is generated even in the presence of a sheath. It is shown that the solution for this case forms the basis of the solution for the general case.

In general, evaluation of the integrals must be accomplished by contour integration, which leads to series expansions that are not convenient for numerical evaluation. For thin layers, however, Taylor's series expansions allow all but one of the coefficients to be evaluated in closed form.

The far field is found by a multi-dimensional saddle point evaluation. This is illustrated in detail for the free-space case, and then the far field patterns in the presence of a sheath are determined. This can be carried out successfully for all components, and to arbitrary orders of iteration.

The calculation of input admittance and mutual coupling between transmitting and receiving slots on the cone is formulated and methods of calculation are discussed.

CONTENTS

Section	Page
I INTRODUCTION	1
II FORMULATION OF INTEGRAL EQUATIONS	4
2.1 GENERAL CONSIDERATIONS	4
2.2 FORMULATION	6
2.3 BOUNDARY CONDITIONS	8
III SINGLE-LAYERED SHEATH	11
3.1 BOUNDARY EQUATIONS	11
3.2 RING SOURCE	12
3.3 SLOT SOURCE	18
IV DOUBLE-LAYERED SHEATH	27
4.1 INTRODUCTION	27
4.2 RING SOURCE	28
V FAR-FIELD PATTERN	35
5.1 INTRODUCTION	35
5.2 RING SOURCE IN FREE SPACE	35
5.3 RING SOURCE IN SHEATH	42
5.4 SLOT SOURCE IN SHEATH	43
5.5 EXTENSIONS OF THE METHOD	48
VI INPUT ADMITTANCE AND MUTUAL COUPLING; EXTENSIONS OF THE ANALYSIS	49
6.1 INPUT ADMITTANCE	49
6.2 MUTUAL COUPLING	51
6.3 EXTENSIONS OF THE ANALYSIS	52
VII CONCLUSIONS	53
APPENDIX A THE K-L TRANSFORM	54
APPENDIX B SOLUTION OF THE BASIC INTEGRAL EQUATION	57

Section	Page
APPENDIX C K-L TRANSFORM OF DERIVATIVE	61
APPENDIX D ZEROS OF THE LEGENDRE FUNCTION	63
REFERENCES	74

SECTION I

INTRODUCTION

The ionized sheath which is formed around a high-speed vehicle or missile in the upper atmosphere has a profound effect on the transmission of electromagnetic waves through it. This is manifested, for example, by the blackout of transmissions from a space vehicle which occurs upon re-entry into the atmosphere. This blackout is accompanied by a large change in impedance of the on-board transmitting antenna due to the reaction of the ambient medium, as shown by in-flight VSWR measurements. The antennas on such vehicles are typically waveguide arrays of resonant longitudinal slots The vehicle is typically conical in shape, and transmitting and receiving antennas are positioned alternately around the cone. At high altitude the sheath is conical in shape, due to the conical form of the shock wave.

The effects of a plasma sheath on antenna radiation and propagation have been studied intensively for slotted cylinder antennas [Ref. 1] and to a certain extent also for spheres [Ref. 2]. In those problems, the formulation is not difficult because the discontinuity in the external medium occurs in the radial coordinate, and this does not affect the separability of the wave equation or orthogonality of the functions involved. For the conical geometry, however, orthogonality does not exist, so that the problem becomes more complicated. In fact, a solution of the electromagnetic boundary value problem of a conducting cone covered by a dielectric or partially conducting conical sheath has not appeared heretofore. This has led to the approximate representation of the sheath as an infinitely thin conical impedance discontinuity. Pridmore-Brown [Ref.3] treated the case of a magnetic dipole antenna (loop), and Baños, et al [Ref. 4] treated the case of an axial electric dipole in the presence of such a sheath. Their analyses, however, were possible only because of the infinitely thin idealization of the sheath, and for special variations of sheath impedance with the radial coordinate R.

A promising new approach to the solution of the sheath problem was developed in a previous report [Ref. 5]. A method of using the Kontorovich-Lebedev (K-L) transform [Ref. 6] was proposed to solve the integral equations which represent the formulation of the boundary value problem. This method is developed successfully here, leading to a technique for solving the basic problem.

In Sec. II, the basic formulation of the problem is developed. Although the work statement of the present contract encompasses the investigation of single- and double-layered sheaths (two- and three-medium environments), the formulation of the basic equations is carried out for a sheath composed of M layers. Consequently, the geometry considered is that of an infinite perfectly conducting cone provided with an infinitesimal radial slot, the cone being overlaid by a sheath consisting of a succession of conical layers, each of which has an arbitrary complex dielectric constant which is constant throughout a given layer. The angular thickness of a given layer is arbitrary, as is the number of layers. The fields in each layer are expressed in terms of magnetic and electric Hertz vectors, and an integral representation is employed for the Hertz vector of each type. Application of the boundary conditions at the cone and at each layer boundary, together with the requirement that the field in the ambient medium be finite along the cone axis, then results in a system of $4(M+1)$ coupled integral equations, where M is the number of layers in the sheath. This system of equations constitutes the formulation of the problem.

In Sec. III, the simplest problem of a single-layered sheath is considered. A ring source, with an excitation which is uniform in azimuth, is considered first, since it turns out that the solution for this case is the basis of the general solution. For this type of source, only a field of magnetic type is generated. Each integral equation is converted, by means of the K-L transform, to a singular integral equation of Cauchy type. The system of these equations then can be reduced to a single equation of Fredholm type, which can be solved in the usual manner by iteration. It is shown that the n^{th} interaction is $O(\delta^n)$, where

$$\delta = 1-\rho^2, \qquad \rho = \gamma_2/\gamma_1,$$

where γ_1 and γ_2 are the complex propagation constants in the sheath and in the surrounding medium, respectively. In general, the evaluation of the integrals in the solution has to be accomplished by contour integration. This leads to a series type of solution in terms of the residues at singularities of the integrand. These include the zeros of the Legendre functions as a function of degree. Accordingly, a computer program was developed for the computation of these zeros. The analysis simplifies considerably when the sheath is thin. For this case, the angular functions are expanded in Taylor's series in the angular thickness, ϑ, of the sheath. It turns out that the first approximation is $O(\vartheta^2)$, and that the integral equation can be solved in closed form to this order.

The case of a slot source is taken up next. This involves fields of both electric and magnetic types. The system of Cauchy type integral equations is reduced to two integral equations, which may be called the excitation and coupling equations, respectively. The coupling equation gives the coupling of the electric-type field to that of magnetic type at the sheath boundary, while the other expresses the excitation of the magnetic-type field by the source, including the effect thereon of the electric-type field. In the case of a uniform medium (i.e., no sheath) the electric-type field is zero, so that only a magnetic-type field is generated. The excitation equation again may be reduced to an equation of Fredholm type, the zero-order term being just the zero-order term for the ring source. From this zero order term, the electric-type coefficient can be determined to first order, which is $O(\delta)$. This first-order electric coefficient then allows the reaction on the excitation of the magnetic coefficient to be found, which is $O(\delta^2)$. For the case of a thin sheath, the magnetic coefficient is determinable in closed form, but the electric coefficient involves an integral which cannot be evaluated in closed form, although an asymptotic evaluation is obtained later in Sec. V, in connection with the determination of the far field.

In Sec. IV, the extension of the analysis to the case of a two-layer sheath is worked out. As in Sec. III, this is carried out first for a ring source. Again, an exact evaluation of the resulting integrals requires contour integration, which leads to a multiple series type of solution. For (angularly) thin layers, however, the integrals again can be evaluated in closed form for the magnetic-type coefficients, but not for the electric-type coefficients.

In Sec. V, the far fields and patterns are calculated. The method employed is the multi-dimensional saddle point method developed by Van der Pol and Bremmer. This is worked out in detail for the free-space case. The case of a single-layered sheath is then taken up. In this case a complication arises because the integrand possesses poles, a situation which does not appear to have been treated adequately before. A method for dealing with this situation is developed, and applied to determine the fields of both a ring source and a slot source in the presence of a thin sheath. The necessary extension for sources of finite extent is also worked out. Although not carried out in detail, it is pointed out that the technique is applicable to sheaths of arbitrary thickness, as well as to the case of a multi-layered sheath.

In Sec. VI, various possible extensions of the method of analysis developed in this report are pointed out. The calculation of input admittance and coupling between adjacent slots, both of which were formulated in a previous report [Ref. 5] involves the near fields, so that asymptotic expansions cannot be used. The evaluation of the integrals by contour integration can be carried out along the lines employed in Sec. III. Alternatively, numerical evaluation of the integrals would be required.

Sec. VII summarizes the work accomplished. Various mathematical developments and the computer program for the Legendre function zeros are presented in Appendices A-D.

SECTION II

FORMULATION OF INTEGRAL EQUATIONS

2.1 GENERAL CONSIDERATIONS

For a conical geometry, a spherical coordinate system (R,θ,φ) is appropriate. The reduced wave equation then can be separated only if the electrical properties of the medium do not depend on the angular variables θ,φ. Hence, for an implied time factor $e^{i\omega t}$,

$$k^2 = -i\omega\mu(\sigma + i\omega\varepsilon)$$

can depend at most on the radial variable R. An infinite cone covered with a conical sheath, in which k depends at most on R, is thus a separable problem in spherical coordinates. Thus, in order to achieve a separable formulation for the practical case where the sheath properties vary wit θ, it is necessary to represent the sheath as a succession of conical layers, in each of which k does not depend on θ. In principle, it is possible to approach a continuous variation of k with θ to any desired degree of approximation by employing a sufficient number of layers.

As the first step in solving the sheath problem, the case of a sheat. whose electrical properties are invariant in the radial direction will be considered. The case of an infinite cone will be considered in this report. The formulation of the equations for a sheath consisting of M uniform conical layers, as in Fig. 1, will be carried out in this Section. The solution of these equations will be carried out in Section III for the case of a single layer ($M=1$), and in Section IV for a double sheath ($M=2$).

A general type of field may be expressed as a superposition of electric (TM) and magnetic (TE) modes, derivable from respective Hertz vectors $\underset{\sim}{R}\Pi^e$, $\underset{\sim}{R}\Pi^m$. If k is independent of R, the electric and magnetic fields are given by

$$\begin{aligned} \underset{\sim}{E}^e &= k^{-1}\,\mathrm{curl\,curl}\,(\underset{\sim}{R}\Pi^e) & \qquad -i\omega\mu\underset{\sim}{H}^e &= k\,\mathrm{curl}\,(\underset{\sim}{R}\Pi^e) \\ \underset{\sim}{E}^m &= \mathrm{curl}\,(\underset{\sim}{R}\Pi^m) & \qquad -i\omega\mu\underset{\sim}{H}^m &= \mathrm{curl\,curl}\,(\underset{\sim}{R}\Pi^m) \end{aligned} \tag{2.1}$$

$\underset{\sim}{R}\Pi^e$, $\underset{\sim}{R}\Pi^m$ each satisfy the differential equation

$$\frac{\partial^2(\underset{\sim}{R}\Pi)}{\partial R^2} + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial(\underset{\sim}{R}\Pi)}{\partial\theta}\right) + \frac{1}{R^2\sin^2\theta}\,\frac{\partial^2(\underset{\sim}{R}\Pi)}{\partial\varphi^2} + k^2(\underset{\sim}{R}\Pi) = \underset{\sim}{S} \tag{2.2}$$

where the inhomogeneous source term $\underset{\sim}{S}$ is different, in general, for the electric and magnetic modes. Solutions of (2.2) then can be expressed in terms of solutions of the corresponding homogeneous equation. Separation of variables by setting

$$\underset{\sim}{R}\Pi = \underset{\sim}{R}\,T(\theta)\,U(R)\,V(\varphi)$$

then leads to the differential equations

$$\frac{d}{dR}\left(R^2\frac{dU}{dR}\right) + \left(k^2R^2 - s^2\right)U = 0 \tag{2.3a}$$

$$\frac{1}{\sin\theta}\,\frac{d}{d\theta}\left(\sin\theta\,\frac{dT}{d\theta}\right) + \left(s^2 - \frac{n^2}{\sin^2\theta}\right)T = 0 \tag{2.3b}$$

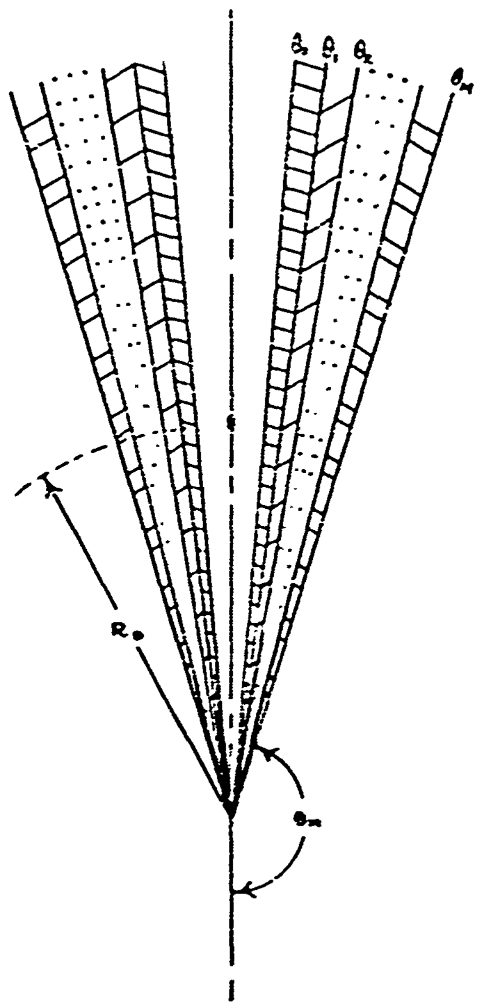

Fig. 1 Geometry for a sheath of M conical layers

$$\frac{d^2V}{d\varphi^2} + n^2V = 0 \tag{2.3c}$$

where s^2 is the separation parameter which couples (2.3a) and (2.3b), and n^2 is the separation parameter which couples (2.3b) and (2.3c). For a geometry unbounded in the φ-coordinate, as in the cases considered here. n is an integer.

In general, a solution of (2.2) has to be built up from a superposition of solutions of (2.3a-c) corresponding to a continuous range of the separation parameter s^2, leading to an integral representation. A method for evaluating the coefficients in such integral representations was proposed in reference 5. The basic machinery for accomplishing this stems from the K-L transform. This method will be extended in this report to attain a solution of the boundary value problem.

2.2 FORMULATION

The antennas of interest in the present problem typically are longitudinal slots or waveguide arrays of resonant longitudinal slots. The radiation problem of such an array can be found quite simply once the pattern due to a δ-function slot is known. If the field of the latter at a typical point is denoted by E_δ, then the field due to the array is

$$E = \int_{\varphi_1}^{\varphi_2} d\varphi_0 \int_{R_1}^{R_2} E_\delta f(R_0,\varphi_0)\, dR_0 \tag{2.4}$$

where $f(R_0,\varphi_0)$ is the distribution of applied field over the array, and R_1, R_2 and φ_1, φ_2 are the bounding coordinates of the array. Consequently, the problem is essentially solved once E_δ has been found, since only the integration of known functions remains. Therefore the analysis to follow will be concerned with finding the field due to an elementary longitudinal slot at (R_0,φ_0), for which the applied field is given by

$$E_\varphi(\theta_0) = E_0\, \delta(R-R_0)\, \delta(\varphi-\varphi_0) \tag{2.5}$$

For reasons which will become apparent below, we now choose

$$s^2 = (\nu-1/2)(\nu+1/2) = \nu^2-1/4 \tag{2.6}$$

For convenience in the application of the K-L transform (see Appendix A), the two independent solutions of the spherical Bessel equation (2.3a) that will be employed here then are

$$i_\nu(\gamma R) \equiv R^{1/2}\, I_\nu(\gamma R) \tag{2.7a}$$

$$k_\nu(\gamma R) \equiv R^{1/2}\, K_\nu(\gamma R) \tag{2.7b}$$

where

$$\gamma = ik \tag{2.7c}$$

I_ν is the Bessel function of the first kind of imaginary argument

$$I_\nu(\gamma R) = e^{-i\nu\pi/2} J_\nu(ikR)$$

and K_ν is the Macdonald function

$$K_\nu(\gamma R) = \frac{\pi}{2} \frac{I_{-\nu}(\gamma R) - I_\nu(\gamma R)}{\sin \nu\pi} \tag{2.8}$$

The product

$$i_\nu(\gamma R_<) k_\nu(\gamma R_>)$$

in which

$$R_< = R,\ R_> = R_0 \text{ for } R < R_0$$
$$R_< = R_0,\ R_> = R \text{ for } R > R_0$$

then is the appropriate solution of (2.3a) to represent a δ-function source at $R = R_0$.

For the two independent solutions of the associated Legendre equation (2.3b), the associated Legendre functions

$$\begin{aligned} p &\equiv P^{-m}_{\nu-1/2}(\cos\theta) \\ q &\equiv P^{-m}_{\nu-1/2}(-\cos\theta) \end{aligned} \tag{2.9}$$

will be employed because they are even functions of ν and the first is finite at $\theta = 0$. Again, this choice is made for ready application of the K-L transform. These functions are represented here by p, q for compactness.

The trigonometric functions $\sin n\varphi$, $\cos n\varphi$ will be employed as the independent solutions of (2.3c).

Consequently, the basic form chosen for the Hertz vectors is

$$\underset{\sim}{R}\pi = \underset{\sim}{i}_R \sum_{n=0}^{\infty} \int_C (\mathcal{A} p + \mathcal{B} q)\, i_\nu(\gamma R_<) k_\nu(\gamma R_>)\, \epsilon_n \,{}^{\sin}_{\cos} n\varphi\, \nu\, d\nu \tag{2.10}$$

where $\mathcal{A}$ and $\mathcal{B}$ are functions of the integration parameter ν, and

$$\epsilon_n = \begin{cases} 1, & n = 0 \\ 2, & n > 0 \end{cases}$$

The contour C and the choice of $\sin n\varphi$ or $\cos n\varphi$ are dictated by the boundary conditions. As will become apparent when the boundary conditions for E_φ and H_φ at the cone and sheath boundaries are expressed, $\cos n\varphi$ will be required for $R\Pi^m$, and $\sin n\varphi$ for $R\Pi^e$. Consequently we adopt the formulation

$$R\Pi_i^m = \sum_{n=0}^{\infty} \int_C (\mathcal{A}_i p + \mathcal{B}_i q)_\nu\, i_\nu(\gamma_i R_<) k (\gamma_i R_>) \epsilon_n \cos n\varphi\, \nu\, d\nu \tag{2.11}$$

$$R\Pi_i^e = \sum_{n=0}^{\infty} \int_C (\mathcal{C}_i p + \mathcal{D}_i q)_\nu\, i_\nu(\gamma_i R_<) k_\nu(\gamma_i R_>)\, \epsilon_n \sin n\varphi\, \nu\, d\nu \tag{2.12}$$

where the subscript i refers to the i^{th} layer of a multilayered sheath, and $(\ )_\nu$ denotes that all quantities within the parentheses are functions of ν.

2.3 BOUNDARY CONDITIONS

At the sheath boundaries $\theta = \theta_1$, the tangential components of the fields must be continuous At the cone, the applied azimuthal electric field, E_φ, is given by (2.5), while the radial electric field, E_R, vanishes. To complete the notation, to the angular functions $\mathfrak{p}$, $\mathfrak{q}$ we affix subscripts denoting the θ-boundary. Thus, for example, at $\theta = \theta_0$ we write, in accordance with (2.9),

$$\left.\begin{aligned} \mathfrak{p}_0 &= P^{-n}_{\nu-1/2}(\cos\theta_0) \\ \mathfrak{q}_0 &= P^{-n}_{\nu-1/2}(-\cos\theta_0) \end{aligned}\right\} \tag{2.9a}$$

Corresponding to the boundary condition (2.5) at the cone, $\theta = \theta_0$, we obtain from (2.1) and (2.11)

$$-\frac{1}{R}\sum_{n=0}^{\infty}\int_C (\mathfrak{a}_1\mathfrak{p}_0' + \mathfrak{b}_1\mathfrak{q}_0')_\nu\, i_\nu(\gamma_1 R_<)\, k_\nu(\gamma_1 R_>)\, \epsilon_n \cos n\varphi\, \nu\, d\nu = E_0\, \delta(R-R_0)\, \delta(\varphi) \tag{2.13}$$

where primes denote derivates with respect to θ; i.e., for example,

$$\mathfrak{p}_0' = \left[\frac{\partial}{\partial\theta} P^{-n}_{\nu-1/2}(\cos\theta)\right]_{\theta=\theta_0}$$

Corresponding to the vanishing of E_R at θ_0, we obtain from (2.1) and (2.12)

$$\sum_{n=0}^{\infty}\int_C (\mathfrak{c}_1\mathfrak{p}_0 + \mathfrak{d}_1\mathfrak{q}_0)_\nu\, i_\nu(\gamma_1 R_<)\, k_\nu(\gamma_1 R_>)\, \epsilon_n \sin n\varphi\, \nu\, d\nu = 0 \tag{2.14}$$

(The ϵ_n is superfluous here, since the $n = 0$ term vanishes; it was used in (2.10) for symmetry.) Multiplying (2.13) by $\cos m\varphi$, (2.14) by $\sin m\varphi$, and integrating over φ between $-\pi$ and π leads to

$$-\frac{1}{R}\int_C (\mathfrak{a}_1\mathfrak{p}_0' + \mathfrak{b}_1\mathfrak{q}_0')_\nu\, i_\nu(\gamma_1 R_<)\, k_\nu(\gamma_1 R_>)\, \nu\, d\nu = (2\pi)^{-1} E_0\, \delta(R-R_0) \tag{2.15}$$

$$\int_C (\mathfrak{c}_1\mathfrak{p}_0 + \mathfrak{d}_1\mathfrak{q}_0)_\nu\, i_\nu(\gamma_1 R_<)\, k_\nu(\gamma_1 R_>)\, \nu\, d\nu = 0 \tag{2.16}$$

for each value of n. (2.16) is of the same form as (2.15), but with $E_0 = 0$.

From (A3a) of Appendix A we have

$$\delta(R-R_0) = \frac{1}{R_0^2\pi i}\int_{-i\infty}^{i\infty} i_\nu(\gamma R)\, k_\nu(\gamma R_0)\, \nu\, d\nu = \frac{1}{R_0^2\pi i}\int_{-i\infty}^{i\infty} i_\nu(\gamma R_0)\, k_\nu(\gamma R)\, \nu\, d\nu \tag{2.17}$$

If (2.17) is inserted on the right side of (2.15), the following deductions may be made:

(a) The contour C may be identified with the imaginary axis, or a contour reconcilable thereto (i.e., without crossing singularities of the integrand).

(b) The integrand of the left-hand integral behaves properly at infinity of the imaginary axis to insure boundedness. In particular,

(c) γ_1 may be considered as real.

(d) From the K-L transform property of the δ-function, the integrand is an even function of ν; this means that $\mathfrak{a}_1$, and $\mathfrak{b}_1$ are even functions of ν, since $\mathfrak{p}$ and $\mathfrak{q}$ were chosen to have that property. Thus, it is not necessary to distinguish between $R_<$ and $R_>$ in the radial functions; either

$$i_\nu(\gamma_i R)\, k_\nu(\gamma_i R_0)$$

or

$$i_\nu(\gamma_i R_0)\, k_\nu(\gamma_i R)$$

may be used interchangeably in (2.11), (2.12) and succeeding integrals. To show this, consider

$$\mathcal{L} = \int_{-i\infty}^{i\infty} E(\nu)\, i_\nu(x)\, k_\nu(x_0)\, \nu\, d\nu$$

where $E(\nu)$ is an even function of ν. By replacing $K_\nu(\gamma R)$ in (2.7b) by its definition (2.8), this becomes

$$\mathcal{L} = \frac{\pi}{2} \int_{-i\infty}^{i\infty} E(\nu)\, i_\nu(x) \left[i_{-\nu}(x_0) - i_\nu(x_0) \right] \frac{\nu\, d\nu}{\sin \nu\pi}$$

Changing the sign of ν in the first term, we obtain

$$\mathcal{L} = \frac{\pi}{2} \int_{-i\infty}^{i\infty} E(\nu) \left[i_\nu(x_0)\, i_{-\nu}(x) - i_\nu(x_0)\, i_\nu(x) \right] \frac{\nu\, d\nu}{\sin \nu\pi} = \int_{-i\infty}^{i\infty} E(\nu)\, i_\nu(x_0)\, k_\nu(x)\, \nu\, d\nu$$

(e) By equating integrands on both sides of (2.15) (or, equivalently, by taking the K-L transform), we obtain

$$(\mathcal{A}_1 p_0' + \mathcal{B}_1 q_0') = \mathcal{E}_0 \tag{2.18}$$

where

$$\mathcal{E}_0 = -(2\pi^2\ R_0 i)^{-1}\, E_0 \tag{2.19}$$

Similarly, from (2.15),

$$(\mathcal{C}_1 p_0 + \mathcal{D}_1 q_0) = 0 \tag{2.20}$$

where $\mathcal{C}_1$ and $\mathcal{D}_1$ are even functions of ν.

At the sheath boundaries, the boundary conditions require that E_R, H_R, E_φ, H_φ be continuous. Writing

$$\left. \begin{aligned} x_i &= \gamma_i R \\ y_i &= \gamma_i R_0 \end{aligned} \right\} \tag{2.21}$$

these lead to the respective equations

$$\int_C (\nu^2 - \tfrac{1}{4}) \left[\frac{1}{\gamma_i} (\mathcal{C}_i p_i + \mathcal{D}_i q_i)_\nu\, i_\nu(x_i)\, k_\nu(y_i) - \frac{1}{\gamma_{i+1}} (\mathcal{C}_{i+1} p_i + \mathcal{D}_{i+1} q_i)_\nu\, i_\nu(x_{i+1})\, k_\nu(y_{i+1}) \right] \nu\, d\nu = 0 \tag{2.22}$$

$$\int_C (\nu^2 - \tfrac{1}{4}) \left[(\mathcal{A}_i p_i + \mathcal{B}_i q_i)_\nu\, i_\nu(x_i)\, k_\nu(y_i) - (\mathcal{A}_{i+1} p_i + \mathcal{B}_{i+1} q_i)_\nu\, i_\nu(x_{i+1})\, k_\nu(y_{i+1}) \right] \nu\, d\nu = 0 \tag{2.23}$$

$$\begin{aligned} \int_C \Big\{ N_i \Big[& \frac{1}{\gamma_i} (\mathcal{C}_i p_i + \mathcal{D}_i q_i)_\nu\, i_\nu'(x_i)\, k_\nu(y_i) - \frac{1}{\gamma_{i+1}} (\mathcal{C}_{i+1} p_i + \mathcal{D}_{i+1} q_i)_\nu\, i_\nu'(x_{i+1})\, k_\nu(y_{i+1}) \Big] \\ & - \Big[(\mathcal{A}_i p_i' + \mathcal{B}_i q_i')_\nu\, i_\nu(x_i)\, k_\nu(y_i) - (\mathcal{A}_{i+1} p_i' + \mathcal{B}_{i+1} q_i')_\nu\, i_\nu(x_{i+1})\, k_\nu(y_{i+1}) \Big] \Big\} \nu\, d\nu = 0 \end{aligned} \tag{2.24}$$

$$\int_C \Big\{ \big[\gamma_i (\mathcal{C}_i p_i' + \mathcal{D}_i q_i')_\nu\, i_\nu(x_i)\, k_\nu(y_i) - \gamma_{i+1} (\mathcal{C}_{i+1} p_i' + \mathcal{D}_{i+1} q_i')_\nu\, i_\nu(x_{i+1})\, k_\nu(y_{i+1}) \big] + N_i \big[(\mathcal{A}_i p_i + \mathcal{B}_i q_i)_\nu\, i_\nu'(x_i)\, k_\nu(y_i) - (\mathcal{A}_{i+1} p_i + \mathcal{B}_{i+1} q_i)_\nu\, i_\nu'(x_{i+1})\, k_\nu(y_{i+1}) \big] \Big\} \nu\, d\nu = 0 \tag{2.25}$$

where

$$N_i = in/\sin\theta_i \tag{2.26}$$

$$i_\nu'(x_i) = \frac{\partial}{\partial R}\left[i_\nu(x_i) \right] \tag{2.27}$$

For the ambient medium in an M-layered sheath, which includes the cone axis $\theta = 0$, finiteness of the field requires that the coefficients of q_{M+1} vanish. Hence

$$\mathcal{B}_{M+1} = 0 \tag{2.28}$$

$$\mathcal{D}_{M+1} = 0 \tag{2.29}$$

The set of equations (2.18), (2.20), (2.22)-(2.25), (2.28), (2.29) represent the formulation of the boundary value problem. The method of solving the integral equations to determine the spectral densities $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ will be carried out in Section III for the case of a single layer, and in Section IV for a two-layered sheath.

SECTION III

SINGLE-LAYERED SHEATH

3.1 BOUNDARY EQUATIONS

For a single-layered sheath, the boundary equations of Section II reduce to the following six equations, where (2.28) and (2.29) already have been applied:

$$(\mathcal{A}_1 p_0' + \mathcal{B}_1 q_0')_\nu = \mathcal{E}_0 \tag{3.1}$$

$$(\mathcal{C}_1 p_0 + \mathcal{D}_1 q_0)_\nu = 0 \tag{3.2}$$

$$\int_C (\nu^2 - \nu_0^2)\left[\frac{1}{\gamma_1}(\mathcal{C}_1 p_1 + \mathcal{D}_1 q_1)_\nu\, i_\nu(x_1)\, k_\nu(y_1) - \frac{1}{\gamma_2}(\mathcal{C}_2 p_2)_\nu\, i_\nu(x_2)\, k_\nu(y_2)\right] \nu\, d\nu = 0 \tag{3.3}$$

$$\int_C (\nu^2 - \nu_0^2)\left[(\mathcal{A}_1 p_1 + \mathcal{B}_1 q_1)_\nu\, i_\nu(x_1)\, k_\nu(y_1) - (\mathcal{A}_2 p_2)_\nu\, i_\nu(x_2)\, k_\nu(y_2)\right] \nu\, d\nu = 0 \tag{3.4}$$

$$\begin{aligned}\int_C \Big\{ N_1\Big[\frac{1}{\gamma_1}(\mathcal{C}_1 p_1 + \mathcal{D}_1 q_1)_\nu\, i_\nu'(x_1)\, k_\nu(y_1) - \frac{1}{\gamma_2}(\mathcal{C}_2 p_2)_\nu\, i_\nu'(x_2)\, k_\nu(y_2)\Big]\\ - \Big[(\mathcal{A}_1 p_1' + \mathcal{B}_1 q_1')_\nu\, i_\nu(x_1)\, k_\nu(y_1) - (\mathcal{A}_2 p_2')_\nu\, i_\nu(x_2)\, k_\nu(y_2)\Big]\Big\} \nu\, d\nu = 0\end{aligned} \tag{3.5}$$

$$\begin{aligned}\int_C \Big\{\Big[\gamma_1(\mathcal{C}_1 p_1' + \mathcal{D}_1 q_1')_\nu\, i_\nu(x_1)\, k_\nu(y_1) - \gamma_2(\mathcal{C}_2 p_2')_\nu\, i_\nu(x_2)\, k_\nu(y_2)\Big]\\ + N_1\Big[(\mathcal{A}_1 p_1 + \mathcal{B}_1 q_1)_\nu\, i_\nu'(x_1)\, k_\nu(y_1) - (\mathcal{A}_2 p_2)_\nu\, i_\nu'(x_2)\, k_\nu(y_2)\Big]\Big\} \nu\, d\nu = 0\end{aligned} \tag{3.6}$$

Since the right side of (3.1) is a constant given by (2.19), it is apparent that $\mathcal{A}_1$ must contain a factor $1/p_0'$ and $\mathcal{B}_1$ a factor $1/q_0'$. Similarly, from (3.2) it follows that $\mathcal{C}_1$ and $\mathcal{D}_1$ must contain the factors $1/p_0$, $1/q_0$, respectively. For later convenience, therefore, the notation is revised slightly at this point by defining

$$\left.\begin{aligned}
A_1 &= (\mathcal{A}_1 p_0')_\nu\, k_\nu(y_1)\\
A_2 &= (\mathcal{A}_2)_\nu\, k_\nu(y_2)\\
B_1 &= (\mathcal{B}_1 q_0')_\nu\, k_\nu(y_1)\\
C_1 &= (\mathcal{C}_1 p_0)_\nu\, k_\nu(y_1)\\
C_2 &= (\mathcal{C}_2)_\nu\, k_\nu(y_2)\\
D_1 &= (\mathcal{D}_1 q_0)_\nu\, k_\nu(y_1)\\
\bar{\mathcal{E}}_0 &= \mathcal{E}_0\, k_\nu(y_1)
\end{aligned}\right\} \tag{3.7}$$

In addition, we introduce the parameter

$$\rho = \gamma_2/\gamma_1 = x_2/x_1 = y_2/y_1 \tag{3.8}$$

and denote x_1 by x, and y_1 by y. <u>It will be assumed provisionally in the solution of the integral equations that ρ is real.</u> The justification of this will come in the eventual integration over R_2 which is to be performed. The boundary equations (3.1)-(3.6) then become

$$(A_1 + B_1) = \bar{E}_0 \tag{3.1a}$$

$$(C_1 + D_1) = 0 \tag{3.2a}$$

$$\int_c (\nu^2 - \tfrac{1}{4}) \left[\frac{1}{\gamma_1} (C_1 p_1/p_0 + D_1 q_1/q_0)_\nu i_\nu(x) - \frac{1}{\gamma_2} (C_2 p_1)_\nu i_\nu(\rho x) \right] \nu \, d\nu = 0 \tag{3.3a}$$

$$\int_c (\nu^2 - \tfrac{1}{4}) \left[(A_1 p_1/p_0' + B_1 q_1/q_0')_\nu i_\nu(x) - (A_2 p_1)_\nu i_\nu(\rho x) \right] \nu \, d\nu = 0 \tag{3.4a}$$

$$\int_c \Big\{ N_1 \Big[\frac{i}{\gamma_1} (C_1 p_1/p_0 + D_1 q_1/q_0)_\nu i_\nu'(x) - \frac{1}{\gamma_2} (C_2 p_1)_\nu i_\nu'(\rho x) \Big] - \Big[(A_1 p_1'/p_0' + B_1 q_1'/q_0')_\nu i_\nu(x) - (A_2 p_1')_\nu i_\nu(\rho x) \Big] \Big\} \nu \, d\nu = 0 \tag{3.5a}$$

$$\int_c \Big\{ \gamma_1 (C_1 p_1'/q_0 + D_1 q_1'/q_0)_\nu i_\nu(x) - \gamma_2 (C_2 p_1')_\nu i_\nu(\rho x) + N_1 \Big[(A_1 p_1/p_0' + B_1 q_1/q_0')_\nu i_\nu'(x) - (A_2 p_1)_\nu i_\nu'(\rho x) \Big] \Big\} \nu \, d\nu = 0 \tag{3.6a}$$

From (3.1a) and (3.2a), it is evident that the source generates only a magnetic-type field at the slot. Coupling between magnetic- and electric-type fields occurs only at the sheath boundary through the E_φ- and H_φ-components, as expressed in equations (3.5a) and (3.6a), respectively.

A special case of interest is that of a ring source, in which the applied field is circularly symmetrical around the cone. This special case will be considered first, as it permits the solution of the integral equations to be developed in its simplest form. It also forms the basis for the general case.

3.2 RING SOURCE

For a circularly symmetrical ring source, (2.5) for the applied field becomes

$$E_\varphi(\theta_0) = E_0 \, \delta(R - R_c) \tag{3.9}$$

The field is then independent of the φ-coordinate, so that only the n = 0 term in the representation (2.10) is required. The terms containing the factor N_1 in (3.5) and (3.6) thus drop out, so that no coupling between electric-type and magnetic-type fields takes places at the sheath boundary. Consequently only a magnetic-type field is transmitted through the sheath, as in the free-space case.

Thus, for the ring source, boundary equation (3.5a) reduces to

$$\int_c \left[(A_1 p_1'/p_0' + B_1 q_1'/q_0')_\nu i_\nu(x) - (A_2 p_1')_\nu i_\nu(\rho x) \right] \nu \, d\nu = 0 \tag{3.10}$$

while (3.2a), (3.3a), and (3.6a) do not apply.

3.2.1 Solution of the Integral Equations

The technique for solving the integral equations (3.4a) and (3.10) is based on the K-L transform, as shown in detail in Appendix B. Taking the K-L transform of (3.10) with respect to x, we obtain

$$(A_1\, \varphi_1'/\varphi_0' + B_1\, q_1'/q_0')_\mu = \frac{1}{\pi i}\int_{C_1} (A_2\varphi_1')_\nu\, \mathcal{H}(\mu,\nu;\rho)\, d\nu \tag{3.11}$$

where μ is a typical point on the imaginary axis, the contour C_1 is parallel to and to the right of the imaginary axis, as shown in Fig. B1 of Appendix B, and $\mathcal{H}(\mu,\nu;\rho)$ is defined by

$$\mathcal{H}(\mu,\nu;\rho) = \sum_{m=0}^{\infty} \frac{1}{2}\rho^m c_m(\nu,\rho)\left(\frac{1}{\nu+2m-\mu}+\frac{1}{\nu+2m+\mu}\right) \tag{3.12}$$

The coefficients $c_m(\nu,\rho)$ are given by (B3) of Appendix B. $\mathcal{H}(\mu,\nu;\rho)$ has singularities at $\nu = \pm\mu-2m$. It is an even function of μ, but not of ν.

Similarly, the transform of (3.4a) yields

$$(\mu^2-\tfrac{1}{4})(A_1\varphi_1/\varphi_0' + B_1 q_1/q_0')_\mu = \frac{1}{\pi i}\int_{C_1}(\nu^2-\tfrac{1}{4})(A_2\varphi_0)_\nu\, \mathcal{H}(\mu,\nu;\rho)\, d\nu \tag{3.13}$$

$(A_1)_\mu$ and $(B_1)_\mu$ may now be eliminated between (3.1a), (3.11) and (3.13). The resulting equation may be written in the form

$$(\bar{\xi}_0)_\mu = \frac{1}{\pi i}\int_{C_1} M(\mu,\nu)\,\varphi(\nu)\,\mathcal{H}(\mu,\nu;\rho)\, d\nu \tag{3.14}$$

where

$$\varphi(\nu) = (A_2\varphi_0')_\nu \tag{3.15}$$

$$M(\mu,\nu) = \left[\alpha_1^\mu(\varphi_1'/\varphi_0')_\nu - \alpha_2^\mu \tau_\mu^\nu (\varphi_1/\varphi_0')_\nu\right] \tag{3.16}$$

$$\alpha_1^\mu = W_{10}/W_1 \tag{3.17}$$

$$\alpha_2^\mu = W_{10}'/W_1 \tag{3.18}$$

$$W_{10} = (\varphi_1 q_0' - q_1\varphi_0')_\mu \tag{3.19}$$

$$W_{10}' = (\varphi_1' q_0' - q_1'\varphi_0')_\mu \tag{3.20}$$

$$W_1 = (\varphi_1 q_1' - q_1\varphi_1')_\mu = \frac{2}{\pi}\,\frac{\sin(\mu-\tfrac{1}{2})\pi}{\sin\theta_1} \tag{3.21}$$

$$\tau_\mu^\nu = (\nu^2-\tfrac{1}{4})/(\mu^2-\tfrac{1}{4}) \tag{3.22}$$

$M(\mu,\nu)$, which is an even function of both μ and ν, has been written in a form such that

$$M(\mu,\mu) = 1 \tag{3.16a}$$

Then, by defining

$$M_1(\mu,\nu) = M(\mu,\nu) - M(\mu,\mu) = M(\mu,\nu) - 1 \tag{3.23}$$

(3.14) may be rearranged into the form

$$(\bar{\xi})_\mu = \frac{1}{\pi i}\int_{C_1}\varphi(\nu)\,\mathcal{H}(\mu,\nu;\rho)\,d\nu + \frac{1}{\pi i}\int_{C_1} M_1(\mu,\nu)\,\varphi(\nu)\,\mathcal{H}(\mu,\nu;\rho)\,d\nu \tag{3.24}$$

In virtue of (3.16a) and (3.23), the integrand of the second integral in

(3.24) is not singular at $\nu = \mu$. Hence we may shift C_1 to the left to C_0, the imaginary axis, in the second integral.

If we denote the second integral in (3.24) by $I(\mu)$,

$$I(\mu) = \frac{1}{\pi i}\int_{C_0} M_1(\mu,\nu)\,\varphi(\nu)\,\mathcal{X}(\mu,\nu;\rho)\,d\nu \tag{3.25}$$

(3.24) may be written as

$$(\bar{\xi}_0)_\mu - I(\mu) = \frac{1}{\pi i}\int_{C_1} \varphi(\nu)\,\mathcal{X}(\mu,\nu;\rho)\,d\nu \tag{3.26}$$

This equation has the form of (B6) of Appendix B. Hence the inversion of (3.26) is given by (B10) of Appendix B:

$$\varphi(\mu) = \frac{1}{\pi i}\int_{C_1}\left[(\bar{\xi}_0)_\nu - I(\nu)\right]\mathcal{X}(\mu,\nu;1/\rho)\,d\nu \tag{3.27}$$

The first term of the integral in (3.27) may be evaluated as follows: By using the definition of $\bar{\xi}_0$ given in the last equation of (3.7) and the form of $\mathcal{X}(\mu,\nu;1/\rho)$ given in (B12), we have

$$\varphi_0(\mu) = \frac{1}{\pi i}\int \bar{\xi}_0\,\mathcal{X}(\mu,\nu;1/\rho)\,d\nu = \frac{\xi_0}{\pi i}\int_{C_1} k_\nu(y)\,\nu\,d\nu\int_0^\infty i_\nu(x)\,k_\mu(\rho x)\,dx/x^2$$

Interchanging the order of integrations, this may be written as

$$\varphi_0(\mu) = \frac{\xi_0}{\pi i}\int_0^\infty k_\mu(\rho x)\frac{dx}{x^2}\int_{C_1} i_\nu(x)\,k_\nu(y)\,\nu\,d\nu$$

But from (A3a) of Appendix A, the ν-integral is a δ-function. Hence we obtain

$$\varphi_0(\mu) = \xi_0\,k_\mu(\rho y) \tag{3.28}$$

so that (3.27) reduces to

$$\varphi(\mu) = \varphi_0(\mu) - \frac{1}{\pi i}\int_{C_1} I(\nu)\,\mathcal{X}(\mu,\nu;1/\rho)\,d\nu \tag{3.29}$$

(3.29) may be solved by an iteration process similar to that employed in the solution of integral equations of Fredholm type. For $I(\nu)$ we have from (3.25)

$$I(\nu) = \frac{1}{\pi i}\int_{C_0} M_1(\nu,\lambda_1)\,\varphi(\lambda_1)\,\mathcal{X}(\nu,\lambda_1;\rho)\,d\lambda_1$$

where for $\varphi(\lambda_1)$ we use (3.29). Then we obtain

$$\begin{aligned}\varphi(\mu) &= \varphi_0(\mu) - \frac{1}{(\pi i)^2}\int_{C_1}\mathcal{X}(\mu,\nu;1/\rho)\,d\nu\int_{C_0} M_1(\nu,\lambda_1)\left[\varphi_0(\lambda_1) - \frac{1}{\pi i}\int_{C_1} I(\lambda_2)\,\mathcal{X}(\lambda_1,\lambda_2;1/\rho)\,d\lambda_2\right]\mathcal{X}(\nu,\lambda_1;\rho)\,d\lambda_1 \\ &= \varphi_0(\mu) + \varphi_1(\mu) + \varphi_2(\mu) + \cdots\end{aligned} \tag{3.30}$$

where

$$\varphi_1(\mu) = \frac{-1}{(\pi i)^2}\int_{C_1}\mathcal{X}(\mu,\nu;1/\rho)\,d\nu\int_{C_0} M_1(\nu,\lambda)\,\varphi_0(\lambda)\,\mathcal{X}(\nu,\lambda;\rho)\,d\lambda \tag{3.31a}$$

$$\varphi_2(\mu) = \frac{-1}{(\pi i)^2}\int_{C_1}\mathcal{X}(\mu,\nu;1/\rho)\,d\nu\int_{C_0} M_1(\nu,\lambda)\,\varphi_1(\lambda)\,\mathcal{X}(\nu,\lambda;\rho)\,d\lambda \tag{3.31b}$$

. . . .

It may be shown that $c_n(\mu)$ is $O(\delta^n)$, where

$$\delta = 1 - \rho^2 \tag{3.32}$$

This follows from the fact that $I(\mu)$ is not singular at $\nu = \mu$; thus the $m = 0$ term of $H(\mu,\nu;\rho)$ in (3.12) gives no contribution as we deform the contour to cross $\nu = \mu$. However, from (B3b) of Appendix B, the coefficients $c_m(\nu,\rho)$ all are $O(\delta)$ for $m > 0$, while $c_0(\nu,\rho) = c_0(\nu,1/\rho) = 1$.

3.2.2 First Iteration

For the first iteration, $c_1(\mu)$, we need to evaluate (3.31a). On using (3.28) for $c_0(\lambda)$, this is

$$c_1(\mu) = \frac{-E_0}{(\pi i)^2} \int_{C_1} H(\mu,\nu;1/\rho)\, d\nu \int_{C_0} M_1(\nu,\lambda)\, h_\lambda(\rho y)\, H(\nu,\lambda;\rho)\, d\lambda \tag{3.33}$$

From (3.23) and (3.16)

$$M_1(\nu,\lambda) = \alpha_1^\nu \{\rho_1' \, \%\}_\lambda - \alpha_2^\nu \tau_\nu^2 (\rho/\%)_\lambda - 1 \tag{3.33a}$$

α_1^ν and α_2^ν, from their definitions (3.17), (3.18), appear to have poles at $W_1 = 0$. However, from (3.21) the zeros of W_1 occur at $\nu - \frac{1}{2} = n$. But then from (3.19)

$$W_{n} = P_n(\cos\theta_1)\, P_n'(-\cos\theta_1) - P_n(-\cos\theta_1)\, P_n'(\cos\theta_1) = 0$$

since $P_n(-\cos\theta) = (-)^n P_n(\cos\theta)$. Similarly, $W_2' = 0$ at $\nu - \frac{1}{2} = n$. Hence α_1^ν and α_2^ν have no singularities, so that $M_1(\nu,\lambda)$ has no singularities in the ν-plane.

The inner integral in (3.3) will be considered first. With $H(\nu,\lambda;\rho)$ in the form (3.12), the integrand converges along the imaginary axis. In order to be able to deform the contour to infinity however, it is necessary to expand $h_\lambda(\rho y)$ in accordance with (2.8), and then deform the term with $i_\lambda(\rho y)$ to the right, the term with $i_{-\lambda}(\rho y)$ to the left. Alternatively the sign of λ can be changed in the second term, whereupon the integral becomes

$$d_1(\nu) \equiv \frac{4}{3} \int_{C_0} M_1(\nu,\lambda)\, i_\lambda(\rho y) \left[H(\nu,\lambda;\rho) + H(\nu,-\lambda;\rho) \right] \frac{d\lambda}{\sin\lambda\pi}$$

The contour may now be closed by an infinite semicircle to the right. Poles of the integrand occur at the zeros of $(\%)_\lambda$ in $M_1(\nu,\lambda)$ (see (3.33a)), at $\lambda = \pm\nu + 2m$ in $H(\nu,-\lambda;\rho)$, and at $\lambda = n$. Residues at $\lambda = n$, however, can be shown to cancel out between $H(\nu,\lambda;\rho)$ and $H(\nu,-\lambda;\rho)$. Thus we obtain

$$\begin{aligned} d_1(\nu) = {} & \pi \sum_{p=0}^{\infty} \bar{M}_1(\nu,\lambda_p)\, i_{\lambda_p}(\rho y) \left[H(\nu,\lambda_p;\rho) + H(\nu,-\lambda_p;\rho) \right] / \sin\lambda_p\pi \\ & + \pi \sum_{m=1}^{\infty} \left[M_1(\nu,\nu+2m)\, i_{\nu+2m}(\rho y)\, h(\nu+2m,\rho) - M_1(\nu,-\nu+2m)\, i_{-\nu+2m}(\rho y)\, h(-\nu+2m,\rho) \right] / \sin\nu\pi \end{aligned}$$

where

$$\begin{aligned} (\%)_{\lambda_p} &= 0 \\ \bar{M}_1(\nu,\lambda_p) &= \left[M_1(\nu,\lambda)\, (\rho_1')_\lambda \Big/ \frac{\partial}{\partial\lambda} (\%)_\lambda \right]_{\lambda=\lambda_p} \\ h(\nu,\rho) &= \rho^\nu c_m(\nu,\rho) \end{aligned}$$

(3.33) then becomes

$$\phi_1(\mu) = \frac{-\xi_0}{\pi i} \int_C \phi_1(\nu)\, \mathcal{H}(\mu, \nu; 1/\rho)\, d\nu$$

A contour integration in the ν-plane encounters the poles of $\mathcal{H}(\mu,\nu;1/\rho)$ and of $\mathcal{H}(\nu,\lambda_p;\rho)$. The result is

$$\begin{aligned}
\phi_1(\mu) = -\frac{\pi}{2} \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} \Bigg\{ & \frac{1}{\sin\mu\pi} \Big[M_1(\mu-2k, \mu-2k-2m)\, A_{-\mu+2k+2m}(\rho\gamma)\, h(\mu-2k-2m,\rho)\, h(\mu-2k, 1/\rho) \\
& - M_1(-\mu-2k, -\mu-2k-2m)\, A_{\mu+2k+2m}(\rho\gamma)\, h(-\mu-2k-2m,\rho)\, h(-\mu-2k, 1/\rho) \Big] \\
& - \sum_{p=1}^{\infty} \frac{A_{\lambda_p}(\rho\gamma)}{\sin\lambda_p \pi} \Big[\bar{M}_1(\mu-2k, \lambda_p)\, h(-\lambda_p,\rho)\, h(\mu-2k, 1/\rho) \left(\frac{1}{-\lambda_p+2m+2k-\mu} + \frac{1}{-\lambda_p+2m+2k+\mu} \right) \\
& + \bar{M}_1(-\mu-2k, \lambda_p)\, h(-\lambda_p,\rho)\, h(-\mu-2k, 1/\rho) \left(\frac{1}{-\lambda_p+2m+2k-\mu} + \frac{1}{-\lambda_p+2m-2k-\mu} \right) \\
& + \bar{M}_1(\mu-2k, \lambda_p)\, h(\lambda_p,\rho)\, h(\mu-2k, 1/\rho) \left(\frac{1}{\lambda_p+2m+2k-\mu} + \frac{1}{\lambda_p+2m-2k+\mu} \right) \\
& + \bar{M}_1(-\mu-2k, \lambda_p)\, h(\lambda_p,\rho)\, h(-\mu-2k, 1/\rho) \left(\frac{1}{\lambda_p+2m+2k-\mu} + \frac{1}{\lambda_p+2m+2k+\mu} \right) \\
& + \bar{M}_1(-\lambda_p-2m, \lambda_p)\, h(\lambda_p,\rho)\, h(-\lambda_p-2m, 1/\rho) \left(\frac{1}{-\lambda_p-2m+2k-\mu} + \frac{1}{-\lambda_p-2m+2k+\mu} \right) \\
& + \bar{M}_1(\lambda_p-2m, \lambda_p)\, h(-\lambda_p,\rho)\, h(\lambda_p-2m, 1/\rho) \left(\frac{1}{\lambda_p-2m-2k-\mu} + \frac{1}{\lambda_p-2m+2k+\mu} \right) \Big] \Bigg\}
\end{aligned} \qquad (3.34)$$

where $\bar{M}_1(\alpha,\beta)$ and $h(\nu,\rho)$ have been defined above.

Since $\phi_2(\mu)$ is $O(\delta^2)$, an iteration stopping at $\phi_1(\mu)$ need retain only $O(\delta)$ terms in $\phi_1(\mu)$. Only the $k = 0$ term is of this order. Then, making use of (B3b) and the expansion

$$\rho^{\nu} = 1 - \frac{\nu}{2}\delta + O(\delta^2)$$

we obtain

$$\begin{aligned}
\phi_1(\mu) = \frac{\pi}{2} \xi_0 \delta \sum_{m=1}^{\infty} \Bigg\{ & \frac{1}{\sin\mu\pi} \Big[(\mu+2m)\, A_{-\mu+2m}(\rho\gamma)\, M_1(\mu, \mu+2m) + (\mu-2m)\, A_{\mu-2m}(\rho\gamma)\, M_1(\mu, \mu-2m) \Big] \\
& - \sum_{p=1}^{\infty} \frac{\lambda_p A_{\lambda_p}(\rho\gamma)}{\sin\lambda_p \pi} \Big[-\bar{M}_1(\mu,\lambda_p) \left(\frac{1}{\lambda_p+2m-\mu} + \frac{1}{\lambda_p+2m+\mu} + \frac{1}{\lambda_p-2m-\mu} + \frac{1}{\lambda_p-2m+\mu} \right) \\
& + \bar{M}_1(\lambda_p-2m, \lambda_p) \left(\frac{1}{\lambda_p-2m-\mu} + \frac{1}{\lambda_p-2m+\mu} \right) + \bar{M}_1(\lambda_p+2m, \lambda_p) \left(\frac{1}{\lambda_p+2m-\mu} + \frac{1}{\lambda_p+2m+\mu} \right) \Big] \Bigg\} + O(\delta^2)
\end{aligned} \qquad (3.35)$$

3.2.3 Second Iteration

The second iteration, $\phi_2(\mu)$, is given by (3.31b), in which $\phi_1(\lambda)$ is given by (3.34). The evaluation of $\phi_2(\mu)$ by contour integration can be carried out in a manner similar to that used in the evaluation of $\phi_1(\mu)$. $\phi_1(\mu)$ already contains a triple summation over the indices k,m,p, so that $\phi_2(\mu)$ involves three additional summations, making six in all. It is evident that the successive iterations pose a severe computational problem which is not attractive. Thus it appears that it may be more economical from the computational standpoint to divide the sheath into a succession of layers of small δ to the point where the first iteration is a sufficient degree of approximation.

3.2.4 Thin Sheath

For the case of a thin sheath, by which is meant one for which the quantity

$$\vartheta_1 = \theta_0 - \theta_1$$

is small, the angular functions can be expanded in Taylor's series around $\theta = \theta_0$. It turns out that the first approximation is $O(\vartheta_1^2)$, and that the integral equation can be evaluated in closed form to this order.

From the Taylor expansions to $O(\vartheta_1^2)$

$$g_1 = g_0 - \vartheta_1 g_0' + \tfrac{1}{2}\vartheta_1^2 g_0'' = [1 - \tfrac{1}{2}\vartheta_1^2(\nu^2-\tfrac14)]g_0 - (\vartheta_1 + \tfrac{1}{2}\vartheta_1^2\cot\theta_0)g_0'$$

$$g_1' = (\vartheta_1 + \tfrac{1}{2}\vartheta_1^2\cot\theta_0)(\nu^2-\tfrac14)g_0 + \{1 - \vartheta_1\cot\theta_0 + \tfrac{1}{2}\vartheta_1^2[\cot^2\theta_0 + \csc^2\theta_0 - (\nu^2-\tfrac14)]\}g_0'$$

we obtain

$$W_{12} = [1 - \tfrac{1}{2}\vartheta_1^2(\mu^2-\tfrac14)]W_0$$

$$W_{12}' = (\mu^2-\tfrac14)(\vartheta_1 + \tfrac{1}{2}\vartheta_1^2\cot\theta_0)W_0$$

$$W_0/W_1 = \sin\theta_1/\sin\theta_0 = 1 - \vartheta_1\cot\theta_0 - \tfrac{1}{2}\vartheta_1^2$$

where

$$W_0 = (g_0 h_0' - h_0 g_0')_\mu = \frac{2}{\pi}\,\frac{\sin(\mu-\tfrac12)\pi}{\sin\theta_0}$$

Then we find

$$M_1(\mu,\nu) = \tfrac{1}{2}\vartheta_1^2(\nu^2-\mu^2) + O(\vartheta_1^3) \tag{3.36}$$

Then (3.31a) for the first iteration $\varphi_1(\mu)$ becomes

$$\varphi_1(\mu) = -\tfrac{1}{2}\vartheta_1^2\frac{\xi_0}{(\pi i)^2}\int_{C_1} K(\mu,s;\rho)\,ds\int_{C_2}(\lambda^2-\nu^2)\,k_\lambda(\rho y)\,K(\nu,\lambda;\rho)\,d\lambda \tag{3.37}$$

Denoting the inner integral by $d_1(\nu)$, as before, we write

$$\lambda^2-\nu^2 = (\lambda^2-\tfrac14) - (\nu^2-\tfrac14)$$

and split $d_1(\nu)$ into two parts, to be denoted by $d_{1s}(\nu)$ and $d_{1d}(\nu)$, respectively. For $d_{1s}(\nu)$ we obtain directly by using (B11) of Appendix B for $K(\nu,\lambda;\rho)$ and (A3c) of Appendix A

$$\begin{aligned} d_{1s}(\nu) &= \frac{1}{\pi i}(\nu^2-\tfrac14)\int_C k_\lambda(\rho y)\lambda\,d\lambda\int_0^\infty i_\lambda(\rho x)\,k_\nu(x)\frac{dx}{x^2} \\ &= (\nu^2-\tfrac14)\,\frac{1}{\pi i}\int_0^\infty k_\nu(x)\frac{dx}{x^2}\int_C i_\lambda(\rho x)\,k_\lambda(\rho y)\lambda\,d\lambda = (\nu^2-\tfrac14)\,k_\nu(y) \end{aligned} \tag{3.38}$$

For $d_{1d}(\nu)$, by using the differential equation for $k_\lambda(\rho y)$

$$y^2\left(\frac{\partial^2}{\partial y^2} - \rho^2\right)k_\lambda(\rho y) = (\lambda^2-\tfrac14)\,k_\lambda(\rho y) \tag{3.39}$$

we have, similarly,

$$
\begin{aligned}
\mathscr{A}_{11}(\nu) &= \frac{1}{\pi i}\int_{C_1}(\lambda^2-1/4)k_\lambda(\rho y)\,\lambda d\lambda \int_0^\infty i_\lambda(\rho x)k_\nu(x)\frac{dx}{x^2} \\
&= y^2\left(\frac{\partial^2}{\partial y^2}-\rho^2\right)k_\nu(y) \\
&= y^2\left(\frac{\partial^2}{\partial y^2}-1+1-\rho^2\right)k_\nu(y) \\
&= \left[(\nu^2-1/4)+\delta y^2\right]k_\nu(y)
\end{aligned} \tag{3.40}
$$

on using the differential equation for $k_\nu(y)$, which is simply (3.39) with $\rho = 1$. Hence, from (3.38) and (3.40), we have

$$
\mathscr{A}_1(\nu) = \mathscr{A}_{11}(\nu) - \mathscr{A}_{12}(\nu) = \delta y^2 k_\nu(y) \tag{3.41}
$$

Inserting this for the inner integral in (3.37), and using (B12) of Appendix B for $\mathscr{H}(\mu,\nu;1/\rho)$, we obtain

$$
\begin{aligned}
\varphi_1(\mu) &= -\frac{1}{2}\vartheta_1^2\delta\,\mathcal{E}_0 y^2 \frac{1}{\pi i}\int_{C_1} k_\nu(y)\nu d\nu\int_0^\infty i_\nu(x)k_\mu(\rho x)\frac{dx}{x^2} \\
&= -\frac{1}{2}\vartheta_1^2\delta\,\mathcal{E}_0 y^2 k_\mu(\rho y)
\end{aligned} \tag{3.42}
$$

The successive iterations now could be written down by inspection. But the next iteration, $\varphi_2(\mu)$, in virtue of (3.31b) and (3.36), is $O(\vartheta_1^4)$, while, from (3.36), $\varphi_1(\mu)$ contains an additional term which is $O(\vartheta_1^4)$, and thus is of higher order than $\varphi_2(\mu)$. Thus, only the first iteration can be retained if Taylor expansions of the angular functions are carried only as far as $O(\vartheta_1^2)$.

3.2.5 Zeros of the Legendre Functions

The poles, λ_p, which arise in the contour integration are the zeros of the Legendre function

$$
(\mathscr{P}_0')_\lambda = P'_{\lambda-1/2}(\cos\theta_0)
$$

More generally, in the case of an unsymmetrical excitation, poles occur at zeros of the associated functions $P^{-m}_{\lambda-1/2}(\cos\theta_0)$ and $P^{-m\prime}_{\lambda-1/2}(\cos\theta_0)$. Accordingly, computer programs were written for the calculation of these zeros. These programs are described in detail in Appendix D. The quantities NU and MU which appear in the printout are defined by

$$
\left[P_\nu^{-m}(\cos\theta)\right]_{\nu=NU} = 0
$$

$$
\left[P_\nu^{-m\prime}(\cos\theta)\right]_{\nu=MU} = 0
$$

Consequently the poles λ_p which are required are given, respectively, by

$$
\lambda_p = NU + 1/2
$$

$$
\lambda_p = MU + 1/2
$$

3.3 SLOT SOURCE

We now return to consideration of an elementary longitudinal slot source, for which the applied field is given by (2.5). The formulation of the integral equations has been given in Section 3.1, where it was pointed out that coupling between magnetic- and electric-type fields takes place at the sheath boundary in this case.

The boundary equations now are (3.1a)-(3.6a). Of these, only (3.5a) and (3.6a) differ from the ring source equations considered in Section 3.2. Since these equations contain derivatives of the radial functions, the K-L transform of the derivative is required in order to solve these equations by the K-L transform technique. This is worked out in Appendix C. The applicability of those results depends on the condition that the spectral densities have no singularities in the strip

$$-(1+\epsilon) < \mathrm{Re}\, \nu < 1+\epsilon$$

The spectral densities turn out to have factors $1/\wp_0$ and $1/\wp_0'$, and the derivative occurs only for angular functions of order $n > 0$. Since the only zero of these functions in this strip occurs for $\wp_0$ with $n = 0$, the condition is met in the present problem.

(3.1a) and (3.2a) can be used to eliminate B_1 and D_1 from (3.3a)-(3.6a). The latter set of equations then becomes

$$\int_{C_0} \frac{1}{\gamma_1} f_1(\nu)\, i_\nu(x)\, \nu\, d\nu = \int_{C_0} \frac{1}{\gamma_2} f_2(\nu)\, i_\nu(\rho x)\, \nu\, d\nu \tag{3.43a}$$

$$\int_{C_0} f_3(\nu)\, i_\nu(x)\, \nu\, d\nu = \int_{C_0} f_4(\nu)\, i_\nu(\rho x)\, \nu\, d\nu \tag{3.43b}$$

$$\int_{C_0} \left[\frac{1}{\gamma_1} f_5(\nu)\, i_\nu'(x) + f_7(\nu)\, i_\nu(x)\right] \nu\, d\nu = \int_{C_0} \left[\frac{1}{\gamma_2} f_6(\nu)\, i_\nu'(\rho x) + f_8(\nu)\, i_\nu(\rho x)\right] \nu\, d\nu \tag{3.43c}$$

$$\int_{C_0} \left[\gamma_1 f_9(\nu)\, i_\nu(x) + f_{11}(\nu)\, i_\nu'(x)\right] \nu\, d\nu = \int_{C_0} \left[\gamma_2 f_{10}(\nu)\, i_\nu(\rho x) + f_{12}(\nu)\, i_\nu'(\rho x)\right] \nu\, d\nu \tag{3.43d}$$

where

$$f_1(\nu) = (\nu^2 - 1/4)\,(C_1 U_{10}/\wp_0 q_0)_\nu \tag{3.44a}$$

$$f_2(\nu) = (\nu^2 - 1/4)\,(\wp_1 C_2)_\nu \tag{3.44b}$$

$$f_3(\nu) = (\nu^2 - 1/4)\left[(A_1 + S_1) W_{10}/\wp_0' q_0'\right]_\nu \tag{3.44c}$$

$$f_4(\nu) = (\nu^2 - 1/4)\,(\wp_1 A_2)_\nu \tag{3.44d}$$

$$f_5(\nu) = N_1\,(C_1 U_{10}/\wp_0 q_0)_\nu \tag{3.44e}$$

$$f_6(\nu) = N_1\,(\wp_1 C_2)_\nu \tag{3.44f}$$

$$f_7(\nu) = -\left[(A_1 + S_2) W_{10}'/\wp_0' q_0'\right]_\nu \tag{3.44g}$$

$$f_8(\nu) = -(\wp_0' A_2)_\nu \tag{3.44h}$$

$$f_9(\nu) = (C_1 U_{10}'/\wp_0 q_0)_\nu \tag{3.44i}$$

$$f_{10}(\nu) = (\wp_1' C_2)_\nu \tag{3.44j}$$

$$f_{11}(\nu) = -N_1\left[(A_1 + S_1) W_{10}/\wp_0' q_0'\right]_\nu \tag{3.44k}$$

$$f_{12}(\nu) = N_1(p_1 A_2)_\nu \tag{3.44l}$$

$$U_{10} = p_1 q_0 - q_1 p_0 \tag{3.44m}$$

$$U_{10}' = p_1' q_0 - q_1' p_0 \tag{3.44n}$$

$$S_1 = \bar{E}_0 q_1 p_0' / W_{10} \tag{3.44o}$$

$$S_2 = \bar{E}_0 q_1' p_0' / W_{10}' \tag{3.44p}$$

Note that

$$f_{11}(\nu)/f_3(\nu) = f_5(\nu)/f_1(\nu) = f_6(\nu)/f_2(\nu) = f_{12}(\nu)/f_4(\nu) = N_1/(\nu^2 - 1/4) \tag{3.45}$$

The transforms of (3.43a,b) are

$$\frac{1}{\gamma_1} f_1(\mu) = \frac{1}{\pi i} \int_{C_1} \frac{1}{\gamma_2} f_2(\nu)\, \mathcal{K}(\mu,\nu;\rho)\, d\nu \tag{3.46a}$$

$$f_3(\mu) = \frac{1}{\pi i} \int_{C_1} f_4(\nu)\, \mathcal{K}(\mu,\nu;\rho)\, d\nu \tag{3.46b}$$

respectively.

From (C6) of Appendix C, (3.43c) and (3.43d) are equivalent to

$$\int_{C_1+1} [F_5(\nu) + f_7(\nu)]\, i_\nu(x)\, \nu\, d\nu = \int_{C_1+1} \{f_6(\nu)[\alpha^+ i_{\nu-1}(\rho x) + \alpha^- i_{\nu+1}(\rho x)] + f_8(\nu) i_\nu(\rho x)\}\, \nu\, d\nu \tag{3.47a}$$

$$\int_{C_1+1} [f_9(\nu) + F_{11}(\nu)]\, i_\nu(x) \cdot \nu\, d\nu = \int_{C_1+1} \rho\{f_{10}(\nu) i_\nu(\rho x) + f_{12}(\nu)[\alpha^+ i_{\nu-1}(\rho x) + \alpha^- i_{\nu+1}(\rho x)]\}\, \nu\, d\nu \tag{3.47b}$$

respectively, where

$$F_{5,11}(\nu) = \frac{1}{2\nu}[(\nu + 3/2) f_{5,11}(\nu+1) + (\nu - 3/2) f_{5,11}(\nu-1)] \tag{3.48a}$$

$$\alpha^{\pm} = (\nu \pm 1/2)/2\nu \tag{3.48b}$$

The transforms of (3.47a,b) are

$$F_5(\mu) + f_7(\mu) = \frac{1}{\pi i} \int_{C_1+1} \{f_6(\nu)\, \mathcal{K}^{\mu,\pm\nu} + f_8(\nu)\, \mathcal{K}(\mu,\nu;\rho)\}\, d\nu \tag{3.49a}$$

$$f_9(\mu) + F_{11}(\nu) = \frac{1}{\pi i} \int_{C_1+1} \rho\{f_{10}(\nu)\, \mathcal{K}(\mu,\nu;\rho) + f_{12}(\nu)\, \mathcal{K}^{\mu,\pm\nu}\}\, d\nu \tag{3.49b}$$

where

$$\mathcal{K}^{\mu,\pm\nu} \equiv \frac{1}{2}\left[\frac{\nu + 1/2}{\nu - 1}\, \mathcal{K}(\mu,\nu-1;\rho) + \frac{\nu - 1/2}{\nu + 1}\, \mathcal{K}(\mu,\nu+1;\rho)\right] \tag{3.50}$$

In virtue of (3.48a), $f_{5,11}(\mu \pm 1)$ are required in (3.49a,b). These quantities can be obtained in the following way: Multiply (3.43a,b) by $N_1 k_{\mu\pm1}(x)/x^2$ and integrate over x from 0 to ∞. There results, in virtue of (3.45),

$$[(\mu \pm 1)^2 - 1/4]\frac{1}{\gamma_1} f_5(\mu \pm 1) = \frac{N_1}{\pi i} \int_{C_1+1} \frac{1}{\gamma_2} f_2(\nu)\, \mathcal{K}(\mu \pm 1,\nu;\rho)\, d\nu \tag{3.51a}$$

$$[(\mu \pm 1)^2 - 1/4]\, f_{11}(\mu \pm 1) = \frac{N_1}{\pi i} \int_{C_1+1} f_4(\nu)\, \mathcal{K}(\mu \pm 1,\nu;\rho)\, d\nu \tag{3.51b}$$

Since

$$(\mu\pm1)^2-1/4 = (\mu\pm 3/2)(\mu\pm 1/2)$$

we obtain

$$(\mu^2-1/4)\frac{\mu\pm 3/2}{2\mu} f_5(\mu\pm1) = \frac{N_1}{\pi i}\int_{C_1+1} \frac{1}{\rho} f_2(\nu)\,\frac{\mu\mp 1/2}{2\mu}\mathcal{H}(\mu\pm1,\nu;\rho)\,d\nu$$

$$(\mu^2-1/4)\frac{\mu\pm 3/2}{2\mu} f_{11}(\mu\pm1) = \frac{N_1}{\pi i}\int_{C_1+1} f_4(\nu)\,\frac{\mu\mp 1/2}{2\mu}\mathcal{H}(\mu\pm1,\nu;\rho)\,d\nu$$

Hence

$$(\mu^2-1/4)\,F_5(\mu) = \frac{N_1}{\pi i}\int_{C_1+1} \frac{1}{\rho} f_2(\nu)\,\mathcal{H}^{\pm\mu,\nu}\,d\nu \qquad (3.52a)$$

$$(\mu^2-1/4)\,F_{11}(\mu) = \frac{N_1}{\pi i}\int_{C_1+1} f_4(\nu)\,\mathcal{H}^{\pm\mu,\nu}\,d\nu \qquad (3.52b)$$

where

$$\mathcal{H}^{\pm\mu,\nu} \equiv \frac{1}{2\mu}\left[(\mu-1/2)\mathcal{H}(\mu+1,\nu;\rho)+(\mu+1/2)\mathcal{H}(\mu-1,\nu;\rho)\right] \qquad (3.53)$$

The set of equations (3.46a,b), (3.49a,b), and (3.52a,b) can be used to eliminate $(A_1)_\mu$ and $(C_1)_\mu$. This yields the integral equations

$$\frac{1}{\pi i}\int_{C_1} M^m(\mu,\nu)\,\varphi^m(\nu)\,\mathcal{H}(\mu,\nu;\rho)\,d\nu - \frac{1}{\pi i}\int_{C_1+1} M_2^m(\mu,\nu)\,\varphi^e(\nu)\,\mathcal{H}^{\pm\mu,\pm\nu}\,d\nu = (\bar{E})_\mu \qquad (3.54a)$$

$$\frac{1}{\pi i}\int_{C_1} M^e(\mu,\nu)\,\varphi^e(\nu)\,\mathcal{H}(\mu,\nu;\rho)\,d\nu = \frac{1}{\pi i}\int_{C_1+1} M_2^e(\mu,\nu)\,\varphi^m(\nu)\,\mathcal{H}^{\pm\mu,\pm\nu}\,d\nu \qquad (3.54b)$$

where

$$\varphi^m(\nu) = (\rho_0' A_2)_\nu \qquad (3.55a)$$

$$\varphi^e(\nu) = (\rho_0 C_2)_\nu \qquad (3.55b)$$

$$M^m(\mu,\nu) = \left(\frac{W_0}{W_1}\right)_\mu\left(\frac{\rho_1'}{\rho_0'}\right)_\nu - \tau_\rho^\nu\left(\frac{W_{10}'}{W_1}\right)_\mu\left(\frac{\rho_1}{\rho_0}\right)_\nu \qquad (3.55c)$$

$$M^e(\mu,\nu) = \left[\rho^2\left(\frac{U_{10}}{W_1}\right)_\mu\left(\frac{\rho_1'}{\rho_0}\right)_\nu - \tau_\mu^\nu\left(\frac{U_{10}'}{W_1}\right)_\mu\left(\frac{\rho_1}{\rho_0}\right)_\nu\right]\Big/\left(1-\delta\,\frac{U_{10}}{W_1}\,\frac{\rho_1'}{\rho_0}\right)_\mu \qquad (3.55d)$$

$$M_2^m(\mu,\nu) = N_1\left(\frac{W_{10}}{W_1}\right)_\mu\left(\frac{\rho_1}{\rho_0}\right)_\nu \qquad (3.55e)$$

$$M_2^e(\mu,\nu) = -\rho^2 N_1\left(\frac{U_{10}}{W_1}\right)_\mu\left(\frac{\rho_1}{\rho_0'}\right)_\nu\Big/\left(1-\delta\,\frac{U_{10}}{W_1}\,\frac{\rho_1'}{\rho_0}\right)_\mu \qquad (3.55f)$$

$$\begin{aligned}\mathcal{H}^{\pm\mu,\pm\nu} \equiv \mathcal{H}^{\mu,\pm\nu} - \frac{\tau_\mu^\nu}{\rho}\mathcal{H}^{\pm\mu,\nu} = \frac{1}{2}\Big\{&\frac{\nu+1/2}{\nu-1}\mathcal{H}(\mu,\nu-1;\rho)+\frac{\nu-1/2}{\nu+1}\mathcal{H}(\mu,\nu+1;\rho)\\ &-\frac{\tau_\mu^\nu}{\rho}\left[\frac{\mu-1/2}{\mu}\mathcal{H}(\mu+1,\nu;\rho)+\frac{\mu+1/2}{\mu}\mathcal{H}(\mu-1,\nu;\rho)\right]\Big\}\end{aligned} \qquad (3.55g)$$

The structure of (3.54a,b) is worthy of note. (3.54b) gives $\varphi^e(\nu)$ in terms of $\varphi^m(\nu)$. In the case $\gamma_1 = \gamma_2 = \gamma$ of a homogeneous medium (no sheath), $\rho = 1$ and the $\mathcal{H}(\mu\pm1,\nu\pm1;1)$, in accordance with (B12) of Appendix B and (A4a) of Appendix A, become δ-functions:

$$\mathcal{H}(\mu\pm1,\nu;1) = \nu\int_0^\infty i_\nu(x)\,k_{\mu\pm1}(x)\,dx/x^2 = \delta(\mu\pm1-\nu)$$

$$\mathcal{H}(\mu,\nu\pm1;1) = \delta(\mu-\nu\mp1)$$

so that

$$\mathcal{H}^{\pm\mu,\nu} = \frac{1}{2\mu}\left[(\mu-\tfrac{1}{2})\,\delta(\mu-\nu+1) + (\mu+\tfrac{1}{2})\,\delta(\mu-\nu-1)\right]$$

$$\mathcal{H}^{\mu,\pm\nu} = \frac{1}{2}\left[\frac{\nu+\frac{1}{2}}{\nu-1}\,\delta(\mu-\nu+1) + \frac{\nu-\frac{1}{2}}{\nu+1}\,\delta(\mu-\nu-1)\right]$$

and a typical integral

$$\int_{C_{+1}} F(\nu)\,\mathcal{H}^{\pm\mu,\pm\nu}\,d\nu = \int_{C_{+1}} F(\nu)\left[\frac{1}{2}\left(\frac{\nu+\frac{1}{2}}{\nu-1} - \frac{\nu^2-\frac{1}{4}}{\mu^2-\frac{1}{4}}\,\frac{\mu-\frac{1}{2}}{\mu}\right)\delta(\mu-\nu+1) + \frac{1}{2}\left(\frac{\nu-\frac{1}{2}}{\nu+1} - \frac{\nu^2-\frac{1}{4}}{\mu^2-\frac{1}{4}}\,\frac{\mu+\frac{1}{2}}{\mu}\right)\delta(\mu-\nu-1)\right]d\nu = 0$$

Thus the right-hand side of (3.54b) vanishes when $\gamma_1 = \gamma_2$. But the left-hand side, since

$$\mathcal{H}(\mu,\nu;1) = \delta(\mu-\nu)$$

becomes

$$M^e(\mu,\mu)\,\varphi^e(\mu) = \varphi^e(\mu)$$

Thus for $\gamma_1 = \gamma_2$, $\varphi^e(\mu) = 0$. For $\gamma_1 \neq \gamma_2$, the right-hand side of (3.54b) does not vanish, so that then $\varphi^e(\mu)$ does not vanish. Thus (3.54b) expresses the excitation of the electric-type field by the magnetic-type field at the sheath boundary.

In (3.54a), the second integral similarly vanishes for $\gamma_1 = \gamma_2$, so that the equation then expresses the excitation of the magnetic-type field. For $\gamma_1 \neq \gamma_2$, the second integral represents the alteration in the excitation of the magnetic-type field due to the creation of the electric-type field at the sheath boundary.

3.3.1 Solution of the Integral Equations

The solution of the integral equations (3.54a,b) can be effected in a manner similar to that employed in Sec. 3.2.1 for the case of a ring source.

As can be seen from (3.55c), $M^m(\mu,\nu)$ in (3.54a) is the same as the function $M(\mu,\nu)$ which was obtained in the ring source case, and given by (3.16), so that $M^m(\mu,\mu) = 1$. Since $\varphi^e(\mu)$ is zero for $\delta = 0$ ($\rho = 1$), it follows that $\varphi^e(\nu)$ is at most $O(\delta)$; i.e., $\mathcal{H}^{\pm\mu,\pm\nu}$ must lead to terms that are $O(\delta)$ at most. In fact, it is shown in Appendix B that $\mathcal{H}^{\pm\mu,\pm\nu}$ leads to terms which are $O(\delta)$. Then it follows that the second integral in (3.54a) must be $O(\delta^2)$. Thus, to $O(\delta)$, the generation of the electric-type field at the sheath boundary does not affect the excitation of the magnetic-type field.

In view of this property, we can define

$$M_1^m(\mu,\nu) = M^m(\mu,\nu) - M^m(\mu,\mu) \qquad M_1^m(\mu,\mu) = 0 \tag{3.56}$$

and write (3.54a) in the form

$$\frac{1}{\pi i}\int_{C_1} \varphi^m(\nu)\,\mathcal{H}(\mu,\nu;\rho)\,d\nu = (\bar{\xi}_0)_\mu - I^m(\mu) + J^e(\mu) \tag{3.57}$$

where

$$I^{m}(\mu) = \frac{1}{\pi i}\int_{C_0} M_1^{m}(\mu,\nu)\,\varphi^{m}(\nu)\,\mathcal{X}(\mu,\nu;\rho)\,d\nu \tag{3.58a}$$

$$J^{e}(\mu) = \frac{1}{\pi i}\int_{C_{+1}} M_2^{m}(\mu,\nu)\,\varphi^{e}(\nu)\,\mathcal{X}^{\pm\mu,\pm\nu}\,d\nu \tag{3.58b}$$

$I^{m}(\mu)$, as in the ring source case, is $O(\delta)$. As pointed out above, $J^{e}(\mu)$ is $O(\delta^2)$.

As in Sec. 3.2.1, the inversion of (3.57) is

$$\varphi^{m}(\mu) = \frac{1}{\pi i}\int_{C_1}\left[(\bar{E})_{\nu} - I^{m}(\nu) + J^{e}(\nu)\right]\mathcal{X}(\mu,\nu;1/\rho)\,d\nu \tag{3.59}$$

(3.59) can be evaluated iteratively, just as in Sec. 3.2.1. In this procedure, the results differ from those obtained there in only one essential respect, namely the contribution to $\sigma^{m}(\mu)$ from $J^{e}(\nu)$.

In (3.54b), $M^{e}(\mu,\nu)$ has been defined in such a way that $M^{e}(\mu,\mu) = 1$. Hence we can define

$$M_1^{e}(\mu,\nu) = M^{e}(\mu,\nu) - M^{e}(\mu,\mu) \qquad M_1^{e}(\mu,\mu) = 0 \tag{3.60}$$

and write (3.54b) in the form

$$\frac{1}{\pi i}\int_{C_1}\varphi^{e}(\nu)\,\mathcal{X}(\mu,\nu;\rho)\,d\nu = -I^{e}(\mu) + J^{m}(\mu) \tag{3.61}$$

where

$$I^{e}(\mu) = \frac{1}{\pi i}\int_{C_0} M_1^{e}(\mu,\nu)\,\varphi^{e}(\nu)\,\mathcal{X}(\mu,\nu;\rho)\,d\nu \tag{3.62a}$$

$$J^{m}(\mu) = \frac{1}{\pi i}\int_{C_{+1}} M_2^{e}(\mu,\nu)\,\varphi^{m}(\nu)\,\mathcal{X}^{\pm\mu,\pm\nu}\,d\nu \tag{3.62b}$$

(3.61) differs in form from (3.57) only in the absence of a source term on the right-hand side. The inversion of (3.61) thus is

$$\varphi^{e}(\mu) = \frac{1}{\pi i}\int_{C_1}\left[-I^{e}(\mu) + J^{m}(\nu)\right]\mathcal{X}(\mu,\nu;1/\rho)\,d\nu \tag{3.63}$$

The "source" of $\sigma^{e}(\mu)$ is the second term, $J^{m}(\nu)$, on the right, which stems from $\varphi^{m}(\nu)$ in virtue of (3.62b). Due to the property of $\mathcal{X}^{\pm\mu,\pm\nu}$, this integral is $O(\delta)$. Then the integral of the first term, $I^{e}(\nu)$, is $O(\delta^2)$ in virtue of (3.62a).

If we express $\sigma^{m}(\mu)$ and $\sigma^{e}(\mu)$ as a sum of iterations of successively higher orders of δ, i.e.,

$$\varphi^{m}(\mu) = \sum_{j=0}^{\infty}\varphi_j^{m}(\mu) \tag{3.64a}$$

$$\varphi^{e}(\mu) = \sum_{j=0}^{\infty}\varphi_j^{e}(\mu) \tag{3.64b}$$

where

$$\varphi_j^{e,m}(\mu) = O(\delta^{j})$$

then (3.58a,b) and (3.62a,b) become

$$I^m(\mu) = \sum_{j=1}^{\infty} I_j^m(\mu) = \sum_{j=1}^{\infty} \frac{1}{\pi i} \int_{C_0} M_1^m(\mu,\nu)\varphi_{j-1}^m(\mu)\,\mathcal{H}(\mu,\nu;\rho)\,d\nu \tag{3.65a}$$

$$J^e(\mu) = \sum_{j=1}^{\infty} J_j^e(\mu) = \sum_{j=1}^{\infty} \frac{1}{\pi i} \int_{C_{+1}} M_2^m(\mu,\nu)\varphi_{j-1}^m(\mu)\,\mathcal{H}^{\pm\mu,\pm\nu}\,d\nu \tag{3.65b}$$

$$I^e(\mu) = \sum_{j=1}^{\infty} I_j^e(\mu) = \sum_{j=1}^{\infty} \frac{1}{\pi i} \int_{C_0} M_1^e(\mu,\nu)\varphi_{j-1}^e(\mu)\,\mathcal{H}(\mu,\nu;\rho)\,d\nu \tag{3.65c}$$

$$J^m(\mu) = \sum_{j=1}^{\infty} J_j^m(\mu) = \sum_{j=1}^{\infty} \frac{1}{\pi i} \int_{C_{+1}} M_2^e(\mu,\nu)\varphi_{j-1}^m(\mu)\,\mathcal{H}^{\pm\mu,\pm\nu}\,d\nu \tag{3.65d}$$

respectively. As pointed out earlier, the $I^m(\mu)$ and $I^e(\mu)$ integrals are $0(\delta)$ relative to the φ-function in the integrand because $M_1^m(\mu,\mu) = 0 = M_1^e(\mu,\mu)$. Thus $I_j^m(\mu)$ and $I_j^e(\mu)$ are $0(\delta^j)$. Similarly, because of the property of $\mathcal{H}^{\pm\mu,\pm\nu}$, $J_j^e(\mu)$ and $J_j^m(\mu)$ are $0(\delta^j)$. Thus it is evident from (3.59) that $\varphi_j^m(\mu)$, $j>0$, arises from I_j^m, i.e., φ_{j-1}^m, and J_j^e. But from (3.65b) J_j^e stems from φ_{j-1}^e: Hence $\varphi_j^m(\mu)$ comes from φ_{j-1}^m and φ_{j-1}^e.

Similarly, from (3.63) it follows that $\varphi_j^e(\mu)$ arises from I_j^e and J_j^m, i.e., from φ_{j-1}^e and φ_{j-1}^m.

Since $\varphi^e(\mu)$ has no source term,

$$\varphi_0^e(\mu) = 0$$

Then it follows that $\varphi_0^m(\mu)$ and $\varphi_1^m(\mu)$ are given by exactly the same expressions as for the ring source case considered in Sec. 3.2. The only difference in these terms between the ring source and the slot source cases is that for the ring source the azimuthal parameter n of (2.10) is 0, while for the slot source all values of n are involved. Thus from (3.28) we obtain

$$\varphi_0^m(\mu) = \mathcal{E}_0\,\hat{k}_\mu(\rho y) \tag{3.66}$$

while $\varphi_1^m(\mu)$ is given by (3.34), (3.35), or (3.42), or in integral form, by (3.32).

Correspondingly, $\varphi_1^e(\mu)$ is given in integral form by

$$\begin{aligned}\varphi_1^e(\mu) &= \frac{1}{\pi i} \int_{C_1} J^m(\nu)\,\mathcal{H}(\mu,\nu;1/\rho)\,d\nu \\ &= \frac{\mathcal{E}_0}{(\pi i)^2} \int_{C_1} \mathcal{H}(\mu,\nu;1/\rho)\,d\nu \int_{C_{+1}} M_2^e(\nu,\lambda)\,\hat{k}_\lambda(\rho y)\,\mathcal{H}^{\pm\mu,\pm\nu}\,d\nu\end{aligned} \tag{3.67}$$

The evaluation of (3.67) by contour integration can be carried out as in the evaluation of (3.32). In (3.67), $\mathcal{H}^{\pm\nu,\pm\lambda}$ takes the place of $\mathcal{H}(\nu,\lambda;\rho)$ in (3.32), and $M_2^e(\nu,\lambda)$ appears instead of $M_1(\nu,\lambda)$. The poles of $\mathcal{H}(\nu,\lambda;\rho)$ which gave rise to residues at $\lambda = \pm\nu\mp 2m$ now are supplanted by the poles of $\mathcal{H}^{\pm\nu,\pm\lambda}$ at $\lambda = \pm\nu+2m+1$. While there was no pole in (3.32) at $m = 0$ because $M_1(\nu,\nu) = 0$, there is such a pole in (3.67) because $M_2^e(\nu,\nu)$ is not zero for $\rho \neq 1$. The poles, λ_p, of $M_1(\nu,\lambda)$ occurred at the zeros of $\mathcal{P}_0^{\lambda\,\prime}$, while the poles of $M_2^e(\nu,\lambda)$ occur at the zeros of $\mathcal{P}_0^{\lambda}$, as can be seen from (3.55d). Except for these differences, the evaluation of $\varphi_1^e(\mu)$, and higher iterations if desired, can be carried as in Sec. 3.2.

3.3.2 Thin Sheath

In Sec. 3.2.4, the thin sheath approximation was introduced by defining

$$\vartheta_1 = \theta_1 - \theta_0$$

and developing the Legendre functions in Taylor's series around θ_0. The first iteration, which is $0(\delta)$, then turned out to be $0(\vartheta_1^2)$ and could be evaluated in closed form. The second iteration then would be $0(\vartheta_1^4)$, so that, if the Taylor's series development is carried no higher than ϑ_1^2, one is necessarily restricted to the first iteration. A similar procedure will be followed here. In particular, terms will be limited to the lowest order in ϑ_1.

From (3.36)

$$M_1^m(\mu,\nu) = \frac{1}{2}\vartheta_1^2(\nu^2-\mu^2) + O(\vartheta_1^3)$$

Similarly, we find

$$M^e(\mu,\nu) = 1 + \frac{1}{2}\vartheta_1^2(\nu^2-\mu^2) + \delta\left\{\left[\left(\frac{p_1'}{p_0}\right)_\nu - \left(\frac{p_1'}{p_0}\right)_\mu\right]\vartheta_1 + \left[\left(\frac{p_2'}{p_0}\right)_\mu - \left(\frac{p_2'}{p_0}\right)_\nu - (\nu^2-\mu^2)\right]\vartheta_1^2\right\} + O(\delta^2) + O(\vartheta_1^3)$$

It has already been pointed out that $\varphi^e(\nu)$ is $0(\delta)$. Consequently, if a solution is limited to $0(\delta)$ terms, it is evident from (3.54b) that $M^e(\mu,\nu)$ must be limited to $0(1)$ terms. Hence we take

$$M^e(\mu,\nu) = 1 + \frac{1}{2}\vartheta_1^2(\nu^2-\mu^2)$$

so that, in accordance with (3.60)

$$M_1^e(\mu,\nu) = \frac{1}{2}\vartheta_1^2(\nu^2-\mu^2) = M_1^m(\mu,\nu) \tag{3.68a}$$

Also

$$M_2^m(\mu,\nu) = N_1 + O(\vartheta_1) \tag{3.68b}$$

$$M_2^e(\mu,\nu) = -N_1\vartheta_1\left(\frac{P_1}{P_0'}\right)_\nu + O(\vartheta_1^2) \tag{3.68c}$$

Of the iterative terms in (3.59) for $\varphi^m(\mu)$, we then find

$I^m(\nu) = 0(\vartheta_1^2\delta)$ from (3.58a) and (3.68a)
$J^e(\nu) = 0(\delta^2)$ from (3.58b) and (3.68b)

while in (3.63) for $\varphi^e(\mu)$,

$I^e(\nu) = 0(\vartheta_1^2\delta^2)$ from (3.62a) and (3.68a)
$J^m(\nu) = 0(\vartheta_1\delta)$ from (3.62b) and (3.68c)

As in Section 3.2.4, we then find

$$\varphi_1^m(\mu) = -\frac{1}{2}\delta\vartheta_1^2\xi_0 y^2 k_\nu(\rho y) \tag{3.69}$$

$\varphi_1^e(\mu)$ stems from $J^m(\nu)$ in (3.62b). This integral, however, cannot be evaluated in closed form, so that we have

$$J^{m}(\mu) = -\frac{N_1 \vartheta_1}{\pi i}\, \varepsilon_0 \int_{C_{\nu 1}} \left(\frac{\rho_1}{\rho_0}\right)_{\nu} k_{\nu}(\rho y)\, \mathcal{H}^{2\mu,2\nu}\, d\nu \tag{3.70}$$

$$\varphi_1^{e}(\mu) = \frac{N_1 \vartheta_1}{(\pi i)^2}\, \varepsilon_0 \int_{C_1} \mathcal{H}(\mu,\nu;1/\rho)\, d\nu \int_{C_{\nu 1}} \left(\frac{\rho_1}{\rho_0}\right)_{\lambda} k_{\lambda}(\rho y)\, \mathcal{H}^{2\nu,2\lambda}\, d\lambda \tag{3.71}$$

Although (3.71) cannot be evaluated in closed form, nevertheless it is possible to obtain saddle point developments of the far field of electric type (i.e., stemming from φ_1^{e}). This will be carried out in Sec. 5.4.

SECTION IV

DOUBLE-LAYERED SHEATH

4.1 INTRODUCTION

The formulation of the boundary equations for a sheath composed of M uniform conical layers was given in Section II, and in Section III the case of a single-layered sheath was treated. In this Section, a double-layered sheath will be treated.

For an M-layered sheath, the two source equations (2.18) and (2.20) at the cone surface, the four equations (2.22)-(2.25) expressing the continuity of the tangential components of field at each sheath boundary, plus the two equations (2.28), (2.29 expressing the finiteness of the field along the cone axis in the ambient medium, make up a totality of 4(M+1) boundary equations. After introducing the notation

$$
\begin{aligned}
A_i &= \mathcal{A}_i\, k_\nu(y_i) \\
B_i &= \mathcal{B}_i\, k_\nu(y_i) \\
C_i &= \mathcal{C}_i\, k_\nu(y_i) \\
D_i &= \mathcal{D}_i\, k_\nu(y_i)
\end{aligned}
\tag{4.1}
$$

these equations become

$$(A_1 p_0' + B_1 q_0')_\nu = \bar{E}_0 \tag{4.2a}$$

$$(C_1 p_0 + D_1 q_0)_\nu = 0 \tag{4.2b}$$

$$\int_C (\nu^2 - 1/4)\left[\frac{1}{\gamma_i}(C_i p_i + D_i q_i)_\nu\, i_\nu(x_i) - \frac{1}{\gamma_{i+1}}(C_{i+1} p_i + D_{i+1} q_i)_\nu\, i_\nu(x_{i+1})\right]\nu\, d\nu = 0 \tag{4.2c}$$

$$\int_C (\nu^2 - 1/4)\left[(A_i p_i + B_i q_i)_\nu\, i_\nu(x_i) - (A_{i+1} p_i + B_{i+1} q_i)_\nu\, i_\nu(x_{i+1})\right]\nu\, d\nu = 0 \tag{4.2d}$$

$$\begin{aligned}\int_C \Big\{ N_i\Big[\frac{1}{\gamma_i}(C_i p_i + D_i q_i)_\nu\, i_\nu'(x_i) - \frac{1}{\gamma_{i+1}}(C_{i+1} p_i + D_{i+1} q_i)_\nu\, i_\nu'(x_{i+1})\Big] \\ - \Big[(A_i p_i' + B_i q_i')_\nu\, i_\nu(x_i) - (A_{i+1} p_i' + B_{i+1} q_i')_\nu\, i_\nu(x_{i+1})\Big]\Big\}\nu\, d\nu = 0\end{aligned} \tag{4.2e}$$

$$\begin{aligned}\int_C \Big\{ \gamma_i (C_i p_i' + D_i q_i')_\nu\, i_\nu(x_i) - \gamma_{i+1}(C_{i+1} p_i' + D_{i+1} q_i')_\nu\, i_\nu(x_{i+1}) \\ + N_i\Big[(A_i p_i + B_i q_i)_\nu\, i_\nu'(x_i) - (A_{i+1} p_i + B_{i+1} q_i)_\nu\, i_\nu'(x_{i+1})\Big]\Big\}\nu\, d\nu = 0\end{aligned} \tag{4.2f}$$

The solution of the problem of a single-layered sheath was developed in Section III, first for a circularly symmetrical ring source, and then for an elementary slot source. It turned out that the solution for the ring source forms the basis for the more general case; that is, the ring source solution is the "zero order" solution for the general case. For the multiple-layered sheath, the case of a general source distribution also is an extension of the ring source solution. Consequently the case of a ring source will be considered first, in particular for a double sheath (M=2).

4.2 RING SOURCE

For a circularly symmetrical source, there is no coupling at the sheath boundaries between magnetic-type and electric-type fields. Hence for a ring slot excited by an azimuthally-directed electric field, only a magnetic-type field is created. Thus, the terms in (4.2e,f) which contain derivatives of the radial function drop out, and the boundary equations (4.2b,c,f) become superfluous. (4.2e) then reduces to

$$\int_C (A_i p_i' + B_i q_i')_\nu \, i_\nu(x_i)\, \nu \, d\nu = \int_C (A_{i+1} p_i' + B_{i+1} q_i')_\nu \, i_\nu(x_{i+1})\, \nu \, d\nu \tag{4.3}$$

Following the procedure used in Sec. 3.2.1, $i_\nu(x_{i+1})$ on the right side of (4.2d) and (4.3) is expanded into a series in $i_{\nu+2s}(x_i)$ and the K-L transform of the equation taken with respect to x_i. From the transform property discussed in Appendix A, each x_i is considered as real. This gives

$$(\mu^2 - 1/4)(A_i p_i + B_i q_i)_\mu = \frac{1}{\pi i}\int_{C_1} (\nu^2 - 1/4)(A_{i+1} p_i + B_{i+1} q_i)_\nu \, \mathcal{H}_i^{\mu,\nu} \, d\nu \tag{4.4a}$$

$$(A_i p_i' + B_i q_i')_\mu = \frac{1}{\pi i}\int_{C_1} (A_{i+1} p_i' + B_{i+1} q_i')_\nu \, \mathcal{H}_i^{\mu,\nu} \, d\nu \tag{4.4b}$$

where

$$\mathcal{H}_i^{\mu,\nu} = \mathcal{H}(\mu,\nu;\rho_i) \tag{4.5}$$

$$\rho_i = r_{i+1}/r_i \tag{4.6}$$

Again, each ρ_i is considered as real, since ultimately an integration over the source coordinate (i.e., the y_i) can be taken along the contour for which ρ_i is real.

By means of (4.2a), B_1 may be eliminated from the i = 1 equation of (4.4a,b), and then A_1 may be eliminated from the resulting two equations. This leads to

$$(\bar{E}_0)_\mu = \frac{1}{\pi i}\int_{C_1} \left[M_1^a(\mu,\nu)(A_2 p_0')_\nu + K_1^b(\mu,\nu)(B_2 q_0')_\nu \right] \mathcal{H}_1^{\mu,\nu} \, d\nu \tag{4.7}$$

where $(\bar{E}_0)_\mu$ is defined by (3.7), and

$$M_1^a(\mu,\nu) = \left(\frac{W_{10}}{W_1}\right)_\mu \left(\frac{p_1'}{p_0'}\right)_\nu - \tau_\mu^a \left(\frac{W_{10}'}{W_1}\right)_\mu \left(\frac{p_1}{p_0'}\right)_\nu \qquad M_1^a(\mu,\mu) = 1 \tag{4.8a}$$

$$K_1^b(\mu,\nu) = \left(\frac{W_{10}}{W_1}\right)_\nu \left(\frac{q_1'}{q_0'}\right)_\nu - \tau_\mu^b \left(\frac{W_{10}'}{W_1}\right)_\mu \left(\frac{q_1}{q_0'}\right)_\nu \qquad K_1^b(\mu,\mu) = 1 \tag{4.8b}$$

The remaining equations of (4.4a,b) (that is, for i = 2) are solved simultaneously to eliminate $(B_2)_\mu$. The result can be expressed in the form

$$\varphi_2^a(\mu) \equiv (A_2 p_0')_\mu = \frac{1}{\pi i}\int_{C_1} \left[M_2^a(\mu,\nu)\,\varphi_2^a(\nu) + M_2^b(\mu,\nu)\,\varphi_2^b(\nu) \right] \mathcal{H}_2^{\mu,\nu} \, d\nu \tag{4.9}$$

where

$$M_2^a(\mu,\nu) = \tau_\mu^a \left(\frac{q_2' p_0}{W_2}\right)_\mu \left(\frac{p_2}{p_0'}\right)_\nu - \left(\frac{q_2 p_0'}{W_2'}\right)_\mu \left(\frac{p_2'}{p_0'}\right)_\nu \tag{4.10a}$$

$$M_2^b(\mu,\nu) = \tau_\mu^a \left(\frac{q_2' p_0}{W_2}\right)_\mu \left(\frac{q_2}{q_0'}\right)_\nu - \left(\frac{q_2 p_0'}{W_2'}\right)_\mu \left(\frac{q_2'}{q_0'}\right)_\nu \tag{4.10b}$$

$$\varphi_{i,3}^{a}(\nu) = (A_{2,3}\xi_0')_\nu \tag{4.10c}$$

$$\varphi_{2,3}^{b}(\nu) = (B_{2,3}\xi_0')_\nu \tag{4.10d}$$

(4.9) was obtained by eliminating $(B_2)_\mu$ from (4.4a,b). Alternatively, we can eliminate $(A_2)_\mu$ instead and obtain

$$\varphi_2^b(\mu) \equiv (B_2\xi_0')_\mu = \frac{1}{2\pi i}\int_{C_1} [K_2^a(\mu,\nu)\varphi_3^a(\nu) + K_2^b(\mu,\nu)\varphi_3^b(\nu)]\, \mathcal{H}_2^{\mu,\nu} d\nu \tag{4.11}$$

where

$$K_2^a(\mu,\nu) = \left(\frac{f_2\xi_0'}{W_2}\right)_\mu \left(\frac{f_2'}{f_0'}\right)_\nu - \tau_\mu^\nu \left(\frac{f_2'\xi_0'}{W_2}\right)_\mu \left(\frac{f_2}{f_0'}\right)_\nu \qquad K_2^a(\mu,\mu) = 0 \tag{4.12a}$$

$$K_2^b(\mu,\nu) = \left(\frac{f_2\xi_0'}{W_2}\right)_\mu \left(\frac{\xi_2'}{\xi_0'}\right)_\nu - \tau_\mu^\nu \left(\frac{f_2'\xi_0'}{W_2}\right)_\mu \left(\frac{\xi_2}{\xi_0'}\right)_\nu \qquad K_2^b(\mu,\mu) = 1 \tag{4.12b}$$

For $i = 1$, (4.7) is obtained once more.

Adding (4.9) and (4.11), and defining

$$q_1(\mu) = (\xi_0)_\mu = \xi_0 k_\mu(y) \tag{4.13a}$$

$$q_i(\mu) = \varphi_i^a(\mu) + \varphi_i^b(\mu) \qquad i = 2,3 \tag{4.13b}$$

$$\mathfrak{M}_i^{a,b}(\mu,\nu) = M_i^{a,b}(\mu,\nu) + K_i^{a,b}(\mu,\nu) \qquad \mathfrak{M}_i^{a,b}(\mu,\mu) = 1 \tag{4.13c}$$

we obtain

$$q_i(\mu) = \frac{1}{2\pi i}\int_{C_1} [\mathfrak{M}_i^a(\mu,\nu)\varphi_{i+1}^a(\nu) + \mathfrak{M}_i^b(\mu,\nu)\varphi_i^b(\nu)]\, \mathcal{H}_i^{\mu,\nu} d\nu \tag{4.14}$$

Now by introducing

$$\bar{\mathfrak{M}}_i^{a,b}(\mu,\nu) = \mathfrak{M}_i^{a,b}(\mu,\nu) - \mathfrak{M}_i^{a,b}(\mu,\mu) \qquad \bar{\mathfrak{M}}_i^{a,b}(\mu,\mu) = 0 \tag{4.15}$$

(4.14) may be written in the form

$$\frac{1}{2\pi i}\int_{C_1} q_{i+1}(\nu)\, \mathcal{H}_i^{\mu,\nu} d\nu = q_i(\mu) - I_i^a(\mu) - I_i^b(\mu) \tag{4.16}$$

where

$$I_i^{a,b}(\mu) = \frac{1}{2\pi i}\int_{C_1} \bar{\mathfrak{M}}_i^{a,b}(\mu,\nu)\, \varphi_{i+1}^{a,b}(\nu)\, \mathcal{H}_i^{\mu,\nu} d\nu \tag{4.17}$$

Since $\bar{\mathfrak{M}}_i^{a,b}(\mu,\mu)_\nu = 0$, $I_i^{a,b}(\mu)$ is not singular at $\nu = \mu$. Thus, from the property of $\mathcal{H}_i^{\mu,\nu}$, these integrals are $O(\delta_i)$ relative to $\varphi_{i+1}^{a,b}(\nu)$, where

$$\delta_i = 1 - \beta_i^2 \tag{4.18}$$

Hence (4.16) may be inverted by iteration in a manner quite analogous to the way in which the single-layered sheath was handled. Thus we obtain

$$q_{i+1}(\mu) = \frac{1}{2\pi i}\int_{C_1} [q_i(\nu) - I_i^a(\nu) - I_i^b(\nu)]\, \bar{\mathcal{H}}_i^{\mu,\nu} d\nu \tag{4.19}$$

where

$$\bar{\mathcal{H}}_i^{(\mu,\nu)} = \mathcal{H}(\mu,\nu;1/\rho_i) \tag{4.20}$$

Since $I_i^{a,b}(\mu)$ are $O(\delta_i)$, if we write

$$\varphi_i^{a,b}(\mu) = \sum_{j=0}^{\infty} \varphi_{i,j}^{a,b}(\mu) \tag{4.21a}$$

$$I_i^{a,b}(\mu) = \sum_{j=1}^{\infty} I_{i,j}^{a,b}(\mu) \tag{4.21b}$$

where

$$\varphi_{i,j}^{a,b}(\mu) = O(\delta_i^j), \qquad I_{i,j}^{a,b}(\mu) = O(\delta_i^j) \tag{4.21c}$$

then $I_{i,0}^{a,b}(\mu) = 0$. so that from (4.19)

$$\varphi_{i+1,0}(\mu) = \frac{1}{\pi i}\int_{C_1} \varphi_{i,0}(\mu)\bar{\mathcal{H}}_i^{\mu,\nu}\,d\nu \tag{4.22a}$$

In particular, in virtue of (4.13b),

$$\varphi_{2,0}(\mu) = \frac{1}{\pi i}\int_{C_1} \varphi_1(\nu)\bar{\mathcal{H}}_1^{\mu,\nu}\,d\nu = \frac{\xi_0}{\pi i}\int_{C_1} k_\nu(y)\mathcal{H}(\mu,\nu;1/\rho_1)\,d\nu = \xi_0 k_\mu(\rho_1 y) \tag{4.22b}$$

This result may be inserted in (4.22a) with $i = 2$ to yield

$$\varphi_{3,0}(\mu) = \frac{\xi_0}{\pi i}\int_{C_1} k_\nu(\rho_1 y)\bar{\mathcal{H}}_2^{\mu,\nu}\,d\nu = \xi_0 k_\mu(\rho_2 y)$$

Hence

$$\varphi_{i+1,0}(\mu) = \xi_0 k_\mu(\rho_i y) \qquad i = 1,2 \tag{4.23}$$

For $i = 2$, it follows from (2.28) that $B_3(\nu) = 0$, so that $\varphi_3^b(\nu) = 0$. Hence from (4.16) with $i = 2$ and (4.13a)

$$\frac{1}{\pi i}\int_{C_1} \varphi_{3,0}^a(\nu)\mathcal{H}_2^{\mu,\nu}\,d\nu = \varphi_{2,0}(\mu) = \frac{\xi_0}{\pi i}\int_{C_1} k_\nu(\rho_2 y)\mathcal{H}_2^{\mu,\nu}\,d\nu = \xi_0 k_\mu(\rho_1 y) = \varphi_{2,0}^a(\mu) \tag{4.24a}$$

from (4.22b). Obviously, then, $\varphi_{2,0}^b(\mu) = 0$. Then from (4.16) with $i = 1$ we obtain

$$\frac{1}{\pi i}\int_{C_1} \varphi_{2,0}^a(\nu)\mathcal{H}_1^{\mu,\nu}\,d\nu = \varphi_{1,0}(\mu) = \frac{\xi_0}{\pi i}\int_{C_1} k_\nu(\rho_1 y)\mathcal{H}_1^{\mu,\nu}\,d\nu = \xi_0 k_\mu(y) = \varphi_{1,0}^a(\mu) \tag{4.24b}$$

so that $\varphi_{1,0}^b(\mu) = 0$. Hence

$$\varphi_{i,0}^b(\mu) = 0 \qquad i = 1,2,3 \tag{4.25a}$$

Thus $\varphi_i^b(\mu)$ is $O(\delta_i)$. (4.22a) then becomes

$$\varphi_{i+1,0}^a(\mu) = \frac{1}{\pi i}\int_{C_1} \varphi_{i,0}^a(\mu)\bar{\mathcal{H}}_i^{\mu,\nu}\,d\nu \qquad i = 1,2 \tag{4.25b}$$

(4.25a,b) are a result of the fact that $I_i^{a,b}(\mu) = O(\delta_i)$ relative to $\varphi_{i+1}^{a,b}(\nu)$.

The generation of $\varphi_i^b(\mu)$ takes place via (4.13b), which can be written as

$$\varphi_1(\mu) = \varphi_1^a(\mu) + \varphi_1^b(\mu) = \sum_{j=0}^{\infty}[\varphi_{1,j}^a(\mu) + \varphi_{1,j}^b(\mu)] = \xi_0 k_\mu(y)$$

But from (4.24b), $\xi_0 k_\mu(y)$ is just the value of $\varphi_{1,0}^a(\mu)$. Hence it follows with

(4.25a) that

$$\varphi_{1,j}(\mu) = 0 \qquad j = 1, 2, \cdots \tag{4.26a}$$

so that

$$\varphi^{b}_{1,j}(\mu) = -\varphi^{a}_{1,j}(\mu) \qquad j = 1, 2, \cdots \tag{4.26b}$$

Thus, from (4.19) for i = 1

$$\varphi_2(\mu) = \frac{1}{\pi i}\int_{C_1}[\varphi^{a}_{1,2}(\nu) - I^{a}_1(\nu) - I^{b}_1(\nu)]\bar{\mathcal{H}}_1^{\mu,\nu}\,d\nu \tag{4.27a}$$

where $I_1^{a,b}(\nu)$ are given by (4.17). Subtracting out the j = 0 terms from this equation by means of (4.25a,b) and using (4.17), we obtain

$$\begin{aligned}\sum_{j=1}^{\infty}\varphi_{2,j}(\mu) &= -\frac{1}{\pi i}\int_{C_1}[I^{a}_1(\nu) + I^{b}_1(\nu)]\bar{\mathcal{H}}_1^{\mu,\nu}\,d\nu \\ &= -\frac{\varepsilon_2}{(\pi i)^2}\int_{C_1}\bar{\mathcal{H}}_1^{\mu,\nu}\,d\nu\int_{C_0}[\bar{\mathfrak{M}}^{a}_1(\nu,\lambda)\,\varphi^{a}_2(\lambda) + \bar{\mathfrak{M}}^{b}_1(\nu,\lambda)\varphi^{b}_2(\lambda)]\mathcal{H}_1^{\nu,\lambda}\,d\lambda\end{aligned} \tag{4.27b}$$

The j = 1 term is obtained by using $\varphi^{a,b}_{2,0}(\lambda)$ in the integral. But, since $\varphi^{b}_2(\lambda)$, by (4.25a), is $O(\delta_2)$, the $I_1^{b}(\lambda)$ integral is $O(\delta_2^2)$, so it does not contribute to the j = 1 term. Then (4.27b) for j = 1 becomes

$$\varphi_{2,1}(\mu) = -\frac{1}{\pi i}\int_{C_1}I^{a}_{1,1}(\nu)\bar{\mathcal{H}}_{1,1}^{\mu,\nu}\,d\nu = -\frac{\varepsilon_2}{(\pi i)^2}\int_{C_1}\bar{\mathcal{H}}_{1,1}^{\mu,\nu}\,d\nu\int_{C_0}\bar{\mathfrak{M}}^{a}_1(\nu,\lambda)\,\varphi^{a}_{2,0}(\lambda)\mathcal{H}_{1,1}^{\nu,\lambda}\,d\lambda \tag{4.27c}$$

where $\mathcal{H}_{1,1}^{\nu,\lambda}$ is the portion of $\mathcal{H}_1^{\nu,\lambda}$ which is $O(\delta_1)$; similarly for $\bar{\mathcal{H}}_{1,1}^{\mu,\nu}$. From (B7) and (B3a,b) of Appendix B, for a general value of i,

$$\mathcal{H}_{i,1}^{\mu,\nu} = \sum_{m=1}^{\infty} -\tfrac{1}{2}\delta_i\,\nu\left(\frac{1}{\nu+2m-\mu} + \frac{1}{\nu+2m+\mu}\right) \tag{4.28a}$$

$$\bar{\mathcal{H}}_{i,1}^{\mu,\nu} = -\mathcal{H}_{i,1}^{\mu,\nu} \tag{4.28b}$$

in virtue of the relations

$$\rho_i^{-\nu} = 1 + O(\delta_i)$$

$$1 - 1/\rho_i^2 = -\delta_i + O(\delta_i^2)$$

Next, from (4.19) for i = 2, we get in a similar way

$$\varphi_{3,1}(\mu) = \frac{1}{\pi i}\int_{C_1}[\varphi_{2,1}(\nu) - I^{a}_{2,1}(\nu)]\bar{\mathcal{H}}_{2,1}^{\mu,\nu}\,d\nu = \varphi^{a}_{3,1}(\mu) \tag{4.29}$$

since $\varphi^{b}_3(\mu) = 0$ from the boundary condition (2.28).

Thus all the $\varphi_{i,1}(\mu)$ have been determined. However, except for i = 1 and i = 3, the components $\varphi^{a}_{i,1}$ and $\varphi^{b}_{i,1}$ are yet to be found; i.e., we need to find $\varphi^{a,b}_{2,1}$. For these, it is necessary to solve (4.9) and (4.11) separately. This can be carried out in exactly the same way that was used in the solution of (4.14). Thus we define

$$\bar{M}^{a}_i(\mu,\nu) = M^{a}_i(\mu,\nu) - M^{a}_i(\mu,\mu) = M^{a}_i(\mu,\nu) - 1, \qquad \bar{M}^{a}_i(\mu,\mu) = 0 \tag{4.30a}$$

$$\bar{K}^{b}_i(\mu,\nu) = K^{b}_i(\mu,\nu) - K^{b}_i(\mu,\mu) = K^{b}_i(\mu,\nu) - 1, \qquad \bar{K}^{b}_i(\mu,\mu) = 0 \tag{4.30b}$$

Note that we already have from (4.10b) and (4.12a)

$$M_i^b(\mu,\mu) = K_i^a(\mu,\mu) = 0$$

For notational convenience, we can write subsequently

$$M_i^b(\mu,\nu) = \bar{M}_i^b(\mu,\nu) \tag{4.31a}$$

$$K_i^a(\mu,\nu) = \bar{K}_i^a(\mu,\nu) \tag{4.31b}$$

so that

$$\bar{M}_i^{a,b}(\mu,\mu) = \bar{K}_i^{a,b}(\mu,\mu) = 0 \tag{4.32}$$

Then (4.9) and (4.11) may be written as

$$\frac{1}{\pi i}\int_{C_i} \varphi_{i+1}^a(\nu)\mathcal{X}_i^{\mu,\nu}d\nu = \varphi_i^a(\mu) - J_i^a(\mu) - J_i^b(\mu) \tag{4.33a}$$

$$\frac{1}{\pi i}\int_{C_i} \varphi_{i+1}^b(\nu)\mathcal{X}_i^{\mu,\nu}d\nu = \varphi_i^b(\mu) - \mathcal{G}_i^a(\mu) - \mathcal{G}_i^b(\mu) \tag{4.33b}$$

respectively, where

$$J_i^{a,b}(\mu) = \frac{1}{\pi i}\int_{C_i} \bar{M}_i^{a,b}(\mu,\nu)\varphi_{i+1}^{a,b}(\nu)\mathcal{X}_i^{\mu,\nu}d\nu \tag{4.34a}$$

$$\mathcal{G}_i^{a,b}(\mu) = \frac{1}{\pi i}\int_{C_i} \bar{K}_i^{a,b}(\mu,\nu)\varphi_{i+1}^{a,b}(\nu)\mathcal{X}_i^{\mu,\nu}d\nu \tag{4.34b}$$

In virtue of (4.32), the J_i and $\mathcal{G}_i$ integrals are non-singular, so that they are $O(\delta_i)$. Hence (4.33a,b) may be inverted to yield

$$\varphi_{i+1}^a(\mu) = \frac{1}{\pi i}\int_{C_i}[\varphi_i^a(\nu) - J_i^a(\nu) - J_i^b(\nu)]\bar{\mathcal{X}}_i^{\mu,\nu}d\nu \tag{4.35a}$$

$$\varphi_{i+1}^b(\mu) = \frac{1}{\pi i}\int_{C_i}[\varphi_i^b(\nu) - \mathcal{G}_i^a(\nu) - \mathcal{G}_i^b(\nu)]\bar{\mathcal{X}}_i^{\mu,\nu}d\nu \tag{4.35b}$$

The zero-order approximations, $j = 0$, of $\varphi_{i,j}^{a,b}$ have already been found in (4.25a,b). Hence the $j = 1$ approximations can be evaluated directly by subtracting out the $j = 0$ terms of (4.35a,b) to yield

$$\varphi_{i+1,1}^a(\mu) = \frac{1}{\pi i}\int_{C_i}[\varphi_{i,1}^a(\nu) - J_{i,1}^a(\nu) - J_{i,1}^b(\nu)]\bar{\mathcal{X}}_{i,1}^{\mu,\nu}d\nu \tag{4.36a}$$

$$\varphi_{i+1,1}^b(\mu) = \frac{1}{\pi i}\int_{C_i}[\varphi_{i,1}^b(\nu) - \mathcal{G}_{i,1}^a(\nu) - \mathcal{G}_{i,1}^b(\nu)]\bar{\mathcal{X}}_{i,1}^{\mu,\nu}d\nu \tag{4.36b}$$

where $J_{i,1}^{a,b}$ and $\mathcal{G}_{i,1}^{a,b}$ are given by (4.34a,b) with $\varphi_{i+1,0}^{a,b}$. In particular, for $i = 2$

$$\varphi_3^a(\mu) = \frac{1}{\pi i}\int_{C_i}[\varphi_{2,1}^a(\nu) - J_{2,1}^a(\nu) - J_{2,1}^b(\nu)]\bar{\mathcal{X}}_{2,1}^{\mu,\nu}d\nu \tag{4.37a}$$

$$\varphi_3^b(\mu) = \frac{1}{\pi i}\int_{C_i}[\varphi_{2,1}^b(\nu) - \mathcal{G}_{2,1}^a(\nu) - \mathcal{G}_{2,1}^b(\nu)]\bar{\mathcal{X}}_{2,1}^{\mu,\nu}d\nu \tag{4.37b}$$

Since $\varphi_3^b = 0$ from the boundary condition, $\mathcal{G}_2^b(\nu) = 0$ from (4.34b), so from (4.37b) and (4.34b) it follows that

$$\varphi_{2,1}^b(\mu) = \mathcal{G}_{2,1}^a(\mu) = \frac{\varepsilon_a}{\pi i}\int_{C_i}\bar{K}_{2,1}^a(\mu,\nu)\chi_1(\rho_2\nu)\mathcal{X}_{2,1}^{\mu,\nu}d\nu \tag{4.38}$$

Hence

$$\varphi^{a}_{2,1}(\mu) = \varphi_{2,1}(\mu) - \varphi^{b}_{2,1}(\mu) \tag{4.39}$$

But $\varphi_{2,1}(\mu)$ already has been evaluated in (4.29a) for i+1 = 2. Hence $\varphi^{a}_{2,1}(\mu)$ is determined by (4.39).

Next, the same type of procedure can be followed to evaluate the j = 2 terms. In this $I^{b}_{1,2}(\mu)$ and $J^{b}_{1,2}(\mu)$ will first appear, since $\varphi^{b}_{i,1}(\mu)$ is now non-zero. In this way, all terms of the representation of $\varphi^{a,b}_{i}(\mu)$ in (4.21a) can be found. In principle, all the integrals can be evaluated by contour integration as in Section 3.2.2. However, due to the multiple series involved in this procedure, this form of solution is not attractive. Consequently, the thin sheath approximation, which allows most of the integrals to be evaluated in closed form, will now be developed.

4.2.1 Thin Sheath

In the thin sheath approximation, the angular functions are developed in Taylor's series in the angular increment

$$\vartheta_i = \theta_0 - \theta_i \tag{4.40}$$

Furthermore, this approximation is limited to the first iteration, or $O(\delta_i)$ terms. Since $\varphi^{b}_{i}(\nu)$ is $O(\delta_i)$, the $I^{b}_{i}(\mu)$, $J^{b}_{i}(\mu)$, and $g^{b}_{i}(\mu)$ terms defined by (4.17) and (4.34a,b), respectively, are $O(\delta_i^2)$, so that these terms can be neglected.

For the kernel functions $\mathfrak{M}^{a,b}_{i}(\mu,\nu)$ defined by (4.15) and (4.13c), we find from (4.10a,b), (4.12a,b), and (4.30a,b)

$$\overline{\mathfrak{M}}^{a}_{i}(\mu,\nu) = \left(\frac{W_{io}}{W_i}\right)_{\mu}\left(\frac{P_i'}{P_o'}\right)_{\nu} - \tau^{\nu}_{\mu}\left(\frac{W_{io}'}{W_i}\right)_{\mu}\left(\frac{P_i}{P_o'}\right)_{\nu} - 1 \tag{4.41a}$$

$$\overline{\mathfrak{M}}^{b}_{i}(\mu,\nu) = \left(\frac{W_{io}}{W_i}\right)_{\mu}\left(\frac{Q_i'}{Q_o}\right)_{\nu} - \tau^{\nu}_{\mu}\left(\frac{W_{io}'}{W_i}\right)_{\mu}\left(\frac{Q_i}{Q_o'}\right)_{\nu} - 1 \tag{4.41b}$$

where W_{io}, W_{io}', and W_i are obvious generalizations of (3.19)-(3.21). Then, analogous to (3.39), we find

$$\overline{\mathfrak{M}}^{a}_{i}(\mu,\nu) = \overline{\mathfrak{M}}^{b}_{i}(\mu,\nu) = \overline{M}^{a}_{i}(\mu,\nu) = \overline{R}^{b}_{i}(\mu,\nu) = \frac{1}{2}\vartheta_i^2(\nu^2-\mu^2) + O(\vartheta_i^3) \tag{4.42}$$

(4.42) does not involve the angular functions, so that all the integrals can be evaluated in closed form exactly as in Section 3.2.4. From (4.17) and (4.25b) and from (4.34a) for example,

$$I^{a}_{i,1}(\mu) = J^{a}_{i,1}(\mu) = \frac{\vartheta_i^2}{2\pi i}\int_{C_0}(\nu^2-\mu^2)k_\nu(\rho_i y)\mathcal{H}^{\mu,\nu}_{i,1}\,d\nu = \frac{1}{2}\vartheta_i^2\delta_i E_0 y^2 k_\mu(\rho_{i-1}y) \qquad i=1,2 \tag{4.43}$$

which is the counterpart of (3.45).

The only integrals yet to be determined are the $g^{a}_{i,1}(\mu)$. From (4.34b), (4.23), and (4.31b), they are given by

$$g^{a}_{i,1}(\mu) = \frac{1}{\pi i}\int_{C_0} K^{a}_{i}(\mu,\nu)\,k_\nu(\rho_i y)\,\mathcal{H}^{\mu,\nu}_{i}\,d\nu \tag{4.44}$$

where $K^{a}_{i}(\mu,\nu)$ is defined by (4.12a). Due to the form of $K^{a}_{i}(\mu,\nu)$, this integral cannot be evaluated in closed form. However, it is possible to obtain a saddle point development of the far field involving this integral, in the manner developed in Sec. 5.

SECTION V

FAR-FIELD PATTERN

5.1 INTRODUCTION

In this Section, the far-field pattern of a sheath-covered slot antenna will be determined. The method employed is the saddle point method, due to Debye, which is frequently used for the asymptotic evaluation of integrals. Van der Pol and Bremmer [Ref. 8] extended this method to the evaluation of a multi-dimensional (i.e., multiple) integral encountered in the diffraction of radio waves around a sphere. The technique requires that all functions which occur in the field representation be expressed as exponential integrals. A Taylor series expansion of the exponent to second order is made about its stationary point, or saddle point,* whereupon the integral becomes a multiple Fresnel integral. In the present problem, a complication is encountered because the integrand possesses poles, the poles being those which occur in the function $N(\mu,\nu;\rho)$, defined in (B7) of Appendix B. This situation does not appear to have been treated in the literature heretofore. A method for dealing with this situation is developed here.

As an illustration of the method, the derivation of the far field of a ring source on a cone in free space by the multi-dimensional saddle point method, which does not seem to have been determined by this method previously, will be developed in detail first. This method will then be extended to the case of a sheath-covered slot. This will be carried out first for a ring source, in which only a field of magnetic type is set up. Following this, the case of a slot source, where fields of both magnetic and electric type are generated, will be worked out. The far-field patterns, for an infinitesimal slot, as well as a half-wave slot, will then be determined. The extension to an array of slots will also be given.

5.2 RING SOURCE IN FREE SPACE

For a ring slot source in free space, excited by a circularly symmetrical azimuthal electric field E_0, the only electric field component is E_φ. Then from (2.11), (2.21), and (3.1) with $n = 0$,

$$E_\varphi = -\frac{1}{R}\frac{\partial(R\mathcal{J}^m)}{\partial\theta} = -\frac{1}{R}\int_C (a_{\varphi 1}')_\nu\, i_\nu(x)\, k_\nu(y)\,\nu\, d\nu$$

$$= -\frac{E_0}{R}\int_{-i\infty}^{i\infty}\left(\frac{\varphi'}{\varphi_0'}\right)_\nu i_\nu(x)\, k_\nu(y)\,\nu\, d\nu \qquad (5.1)$$

Introducing the defining relations (2.7a,b) for i_ν, k_ν, this may be written as

$$-RE_\varphi = (RR_0)^{1/2} E_0 \mathcal{A} \qquad (5.2a)$$

where

$$\mathcal{A} = \int_{-i\infty}^{i\infty}\left(\frac{\varphi'}{\varphi_0'}\right)_\nu I_\nu(x)\, K_\nu(y)\,\nu\, d\nu \qquad (5.2b)$$

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* The terms "saddle point" and "stationary point" will be used interchangably hereafter.

For the application of the saddle point method, integral representations are required for the functions in the integrand. For the I_ν and K_ν we have the Sommerfeld integral representations

$$I_\nu(x) = \frac{1}{2\pi} \int_{i\infty}^{2\pi+i\infty} e^{-x\cos w + i\nu(w-\pi)}\, dw \qquad (5.3a)$$

$$K_\nu(y) = \frac{1}{2i} \int_{-\infty}^{i\infty} e^{-y\cos w + i\nu w}\, dw \qquad (5.3b)$$

For the angular functions, it is desirable at this point to replace the Legendre function $\varphi = P_{\nu-\frac{1}{2}}(\cos\theta)$ by the traveling wave angular functions used by Felsen [Ref. 9], which are defined by

$$\varphi^{(1),(2)} \equiv P_{\nu-\frac{1}{2}}(\cos\theta) \pm \frac{2}{\pi i} Q_{\nu-\frac{1}{2}}(\cos\theta)$$

where $P_{\nu-\frac{1}{2}}$, $Q_{\nu-\frac{1}{2}}$ are the usual Legendre functions of zero order. Hence

$$\varphi = \frac{1}{2}(\varphi^{(1)} + \varphi^{(2)}) \qquad (5.4)$$

$\varphi^{(1),(2)}$ have the Laplace integral representations

$$\varphi^{(1),(2)} = \mp \frac{1}{\pi} \int_{-i\infty}^{i\infty} (\cos\theta \mp i \sin\theta \cos\varphi)^{-(\nu+\frac{1}{2})}\, d\varphi \qquad (5.5a)$$

Then

$$\varphi^{(1),(2)\prime} = \frac{1}{\pi i} \int_{-i\infty}^{i\infty} (\nu+\tfrac{1}{2})\, \alpha(\varphi) \exp\{-(\nu+\tfrac{3}{2}) \ln(\cos\theta \mp i \sin\theta\cos\varphi)\}\, d\varphi \qquad (5.5b)$$

where

$$\alpha(\varphi) = \frac{\cos\theta\cos\varphi \mp i \sin\theta}{\cos\theta \mp i \sin\theta\cos\varphi} \qquad (5.5c)$$

The traveling wave nature of the functions $\varphi^{(1),(2)}$ can be seen from their asymptotic forms

$$\varphi^{(1),(2)} \sim \kappa\, e^{\pm i(\nu\theta - \pi/4)} \qquad (5.6a)$$

where

$$\kappa = \left(\frac{2}{\pi \sin\theta}\right)^{1/2} \frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu+1)} \sim \left[\frac{2}{\pi(\nu+\frac{1}{2})\sin\theta}\right]^{1/2} \qquad (5.6b)$$

(5.6a) results from a saddle point evaluation of (5.5a). The ratio

$$\sigma^{+} \equiv \frac{\varphi_0^{(1)\prime}}{\varphi_0^{(2)\prime}} \sim e^{i2(\nu\theta_0 - \pi/4)} = -i\, e^{i2\nu\theta_0} \qquad (5.7a)$$

thus converges along the upper branch of the imaginary axis, $\mathcal{Im}\, \nu > 0$, while the ratio

$$\sigma^{-} \equiv \frac{1}{\sigma^{+}} \sim i\, e^{-i2\nu\theta_0} \qquad . \quad (5.7b)$$

converges along the lower branch of the imaginary axis, $\mathcal{Im}\, \nu < 0$. The product $\varphi_0^{(1)\prime}\varphi_0^{(2)\prime}$, however, is asymptotically not of exponential character, since

$$\varphi_0^{(1)\prime}\varphi_0^{(2)\prime} \sim \nu^2\kappa^2 \approx \frac{2\nu}{\pi \sin\theta_0} \qquad (5.7c)$$

For the ratio φ'/φ_0' in (5.2), we can then obtain the series representation

$$\frac{\varphi'}{\varphi_0'} = \frac{\varphi^{(1)'} + \varphi^{(2)'}}{\varphi_0^{(1)'} + \varphi_0^{(2)'}} = \frac{\varphi^{(1)'} + \varphi^{(2)'}}{\varphi_0^{(1),(2)'}} \sum_{k=0}^{\infty} (-\sigma^{\pm})^k \qquad \mathcal{I}m\,\nu \gtrless 0$$

Thus different expansions are required in the upper and lower half-planes. Multiplying numerator and denominator by $\varphi_0^{(1)(2)'}$, we obtain

$$\frac{\varphi'}{\varphi_0'} = \frac{\varphi_0^{(1)(2)'}(\varphi^{(1)'} + \varphi^{(2)'})}{\varphi_0^{(1)'} \varphi_0^{(2)'}} \sum_{k=0}^{\infty} (-\sigma^{\pm})^k \qquad \mathcal{I}m\,\nu \gtrless 0 \tag{5.8}$$

It will be shown later that a stationary point of the integral $\mathcal{J}$ in (5.2) occurs only for k = 0. Thus, retaining only the k = 0 term of (5.8), and introducing the integral representations (5.3a,b) and (5.5b) into (5.2b), we obtain for the integral $\mathcal{J}$

$$\mathcal{J} = \int_{-i\infty}^{i\infty} A(\nu)\,d\nu \int_{-\infty}^{2\pi+i\infty} dw_1 \int_{-i\infty}^{i\infty} dw_2 \int_{-i\infty}^{i\infty} a(\varphi_1)\,d\varphi_1 \int_{-i\infty}^{i\infty} a(\varphi_2)\,d\varphi_2 \left(e^{S_1} + e^{S_2}\right) \equiv \mathcal{G}_1 + \mathcal{G}_2 \tag{5.9}$$

where $\mathcal{G}_{1,2}$ represent the integrals with the exponentials $S_{1,2}$ respectively, and

$$\left.\begin{aligned}
A(\nu) &= \frac{i\nu(\nu+1/2)^2}{4\pi^2 \varphi_0^{(1)'} \varphi_0^{(2)'}} \\
S_{1,2} &= \Phi_0 + \Phi_{1,2} - x\cos w_1 - y\cos w_2 + i\nu(w_1 + w_2 - \pi) \\
\Phi_0 &= -(\nu+1/2)\ln(\cos\theta_0 \mp i\sin\theta_0\cos\varphi_1) \\
\Phi_1 &= -(\nu+1/2)\ln(\cos\theta - i\sin\theta\cos\varphi_2) \\
\Phi_2 &= -(\nu+1/2)\ln(\cos\theta + i\sin\theta\cos\varphi_2)
\end{aligned}\right\} \tag{5.10}$$

The saddle point of each integral $\mathcal{G}_{1,2}$ is determined by simultaneously equating to zero the partial derivatives of the exponent $S_{1,2}$ with respect to each of the integration variables,

$$\frac{\partial S_{1,2}}{\partial \nu} = \frac{\partial S_{1,2}}{\partial w_1} = \frac{\partial S_{1,2}}{\partial w_2} = \frac{\partial S_{1,2}}{\partial \varphi_1} = \frac{\partial S_{1,2}}{\partial \varphi_2} = 0 \tag{5.11}$$

to determine the stationary value of each variable. As shown by Van der Pol and Bremmer [Ref. 8], the stationary value of the integral is given by

$$\mathcal{G}_{1,2} = A(\nu_{1,2}) \frac{(i\sqrt{2\pi})^n}{\Delta_{1,2}^{1/2}} e^{S_{1,2}^s} \tag{5.12}$$

where n is the dimensionality of the integral, $S_{1,2}^s$ is the value of the exponent and $A(\nu_{1,2})$ the amplitude coefficient at the stationary point $\nu_{1,2}$, and $\Delta_{1,2}$ is the Hessian determinant [Ref. 10] of order n, evaluated at the corresponding stationary point

$$\Delta_{1,2} = \left|\frac{\partial^2 S_{1,2}}{\partial z_i\,\partial z_j}\right|_s \tag{5.13}$$

From (5.11) we obtain for the stationary point of S_1, with $\mathcal{I}m\,\nu > 0$,

$$\left.\begin{aligned}
\varphi_1 &= 0 & \Phi_0^s &= i(\nu_1 + 1/2)\theta_0 \\
\varphi_2 &= 0 & \Phi_1^s &= i(\nu_1 + 1/2)\theta_0 \\
\sin w_{1s} &= \frac{-i\nu_1}{x} & \sin w_{2s} &= \frac{-i\nu_1}{y}
\end{aligned}\right\} \tag{5.14}$$

and, introducing the stationary values $\varphi_1 = \varphi_2 = 0$,

$$w_{11} + w_{21} + \theta_0 + \theta = \pi \tag{5.15a}$$

where w_1 and w_2 are represented as w_{11}, w_{21}, respectively, to denote association with the stationary point of S_1. From (5.15a), w_{11}, w_{21}, and $(\theta_0+\theta)$ form a triangle. This leads to the geometrical relation shown in Fig. 2(a). Similarly, for the stationary point of S_2, with $\mathcal{I}m\ \nu > 0$, we obtain the same relations as for S_1, except that the sign of θ is changed, so that, denoting the corresponding values of w_1 and w_2 by w_{12} and w_{22}, respectively, we have

$$w_{12} + w_{22} + \theta_0 - \theta = \pi \tag{5.15b}$$

w_{12}, w_{22}, and $(\theta_0-\theta)$ now form a triangle, as shown in Fig. 2(b).

We still have to consider the portion of the ν-integration for which $\mathcal{I}m\ \nu < 0$. For this situation, we need merely change the sign of θ_0, corresponding to the interchange of $\varphi_0^{(1)}$ and $\varphi_0^{(2)}$ in (5.8). Then (5.15a) becomes

$$w_{11}' + w_{21}' - (\theta_0-\theta) = \pi$$

or

$$(\pi - w_{11}') + (\pi - w_{21}') + (\theta_0-\theta) = \pi \tag{5.15c}$$

Similarly, (5.15b) becomes

$$(\pi - w_{12}') + (\pi - w_{22}') + (\theta_0+\theta) = \pi \tag{5.15d}$$

From (5.15c) and (5.15b) it is evident that w_{11}' is the supplement of w_{12}, and w_{21}' is the supplement of w_{22}. Similarly, from (5.15d) and (5.15a), it is evident that w_{12}' and w_{22}' are the supplements of w_{11} and w_{21}, respectively. Thus the geometrical interpretation is the same as for $\mathcal{I}m\ \nu > 0$, the stationary points merely being interchanged. Consequently the result is just twice the contribution due to the stationary points in the range $\mathcal{I}m\ \nu > 0$.

The relation (5.15a) for the stationary point of S_1 can be fulfilled only if $\theta_0+\theta \le \pi$. For $\theta_0+\theta > \pi$, S_1 does not have a stationary point.

The values of S_1 and S_2 and ν at the stationary points are then

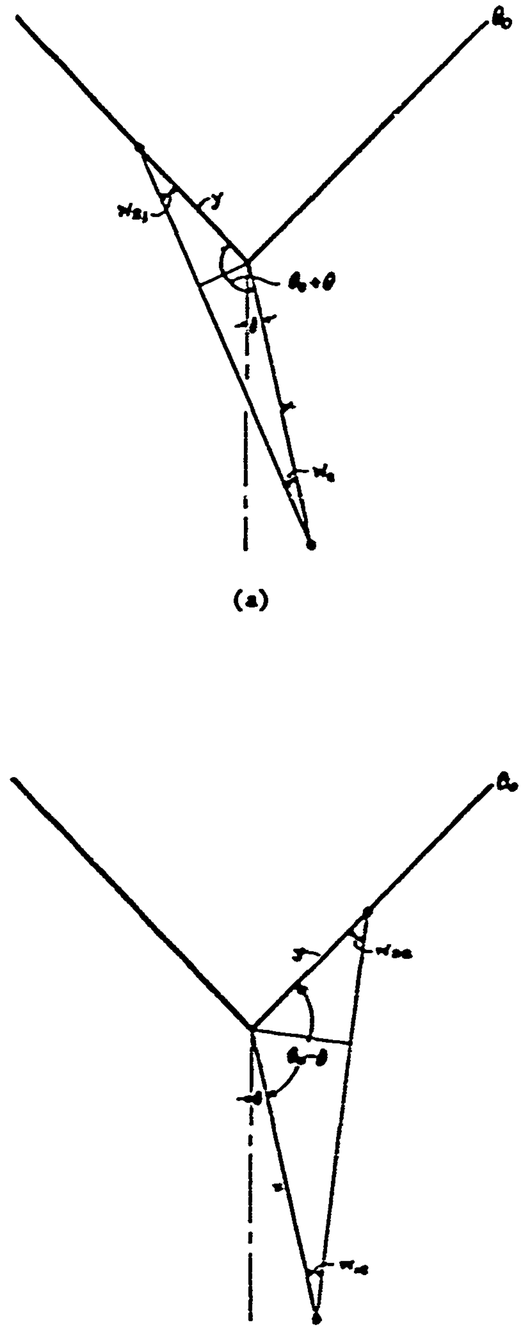
$$\left.\begin{aligned} S_1^s &= D_1 + i\tfrac{1}{2}(\theta_0+\theta) \\ S_2^s &= D_2 + i\tfrac{1}{2}(\theta_0-\theta) \end{aligned}\right\} \tag{5.16}$$

$$\left.\begin{aligned} \nu_1 &= ipy \sin w_{21} = ipy \sin[\pi - (\theta_0+\theta) - w_{11}] \\ \nu_2 &= ipy \sin w_{22} = ipy \sin[\pi - (\theta_0-\theta) - w_{12}] \end{aligned}\right\} \tag{5.17}$$

where

$$D_1 = -x \cos w_{11} - y \cos w_{21}$$

$$D_2 = -x \cos w_{12} - y \cos w_{22}$$



(a)

(b)

Fig. 2 Geometry of stationary paths

Substituting the values of x and y given by (2.21) and using (2.7c), we obtain

$$\left.\begin{aligned} D_1 &= -ik(R\cos w_{11} + R_0\cos w_{21}) \equiv -ikr_1 \\ D_2 &= -ik(R\cos w_{12} + R_0\cos w_{22}) \equiv -ikr_2 \end{aligned}\right\} \qquad (5.18)$$

For the far field we allow $x \to \infty$, so that, as can be seen from Fig. 2(b),

$$r_1 \to R + R_0\cos(\pi - \theta_0 - \theta) = R - R_0\cos(\theta_0 + \theta)$$

$$r_2 \to R + R_0\cos(\pi - \theta_0 + \theta) = R - R_0\cos(\theta_0 - \theta)$$

$$w_{21} \to \pi - (\theta_0 + \theta) \qquad\qquad w_{22} \to \pi - (\theta_0 - \theta)$$

so that

$$\left.\begin{aligned} \nu_1 &\to -kR_0\sin(\theta_0 + \theta) \\ \nu_2 &\to -kR_0\sin(\theta_0 - \theta) \end{aligned}\right\} \qquad (5.17a)$$

Then (5.18) becomes

$$\left.\begin{aligned} D_1 &= -ik[R - R_0\cos(\theta_0 + \theta)] \\ D_2 &= -ik[R - R_0\cos(\theta_0 - \theta)] \end{aligned}\right\} \qquad (5.19)$$

The only non-zero second derivatives of S required to evaluate $\Delta_{1,2}$ are the following:

$$\frac{\partial^2 S_1}{\partial w_1^2} = x\cos w_{11} \qquad \frac{\partial^2 S_2}{\partial w_1^2} = x\cos w_{12} \qquad \frac{\partial^2 S_1}{\partial w_2^2} = y\cos w_{21} \qquad \frac{\partial^2 S_2}{\partial w_2^2} = y\cos w_{22}$$

$$\frac{\partial^2 S_{12}}{\partial w_{12}\partial\nu} = i \qquad \frac{\partial^2 S_{12}}{\partial \varphi_1^2} = -i(\nu_1 + \tfrac{1}{2})\sin\theta_0\, e^{i\theta} \qquad \frac{\partial^2 S_{12}}{\partial \varphi_2^2} = \mp i(\nu_2 + \tfrac{1}{2})\sin\theta\, e^{\mp i\theta}$$

Then, denoting the determinant Δ by $|a_{ij}|$, where the order of the elements i and j is w_1, w_2, φ_1, φ_2, ν, we obtain

$$\Delta = -a_{11}a_{44}(a_{22}a_{45}a_{53} + a_{43}a_{22}a_{52})$$

$$\left.\begin{aligned} \Delta_1 &= -(\nu_1 + \tfrac{1}{2})^2 \sin\theta_0 \sin\theta\, e^{i(\theta_0 + \theta)} \\ \Delta_2 &= (\nu_2 + \tfrac{1}{2})^2 \sin\theta_0 \sin\theta\, e^{i(\theta_0 - \theta)} \end{aligned}\right\} \qquad (5.20)$$

Inserting (5.16), (5.17a), (5.20), and (5.18a) into (5.12), using the asymptotic value (5.7c) for $\varphi_1^{\nu\ell}\varphi_2^{\nu\ell'}$ in $A(\nu_s)$, including the factor 2 to account for the contribution of $\mathcal{I}m\,\nu < 0$, as discussed earlier, and introducing the appropriate value of E_0 for a ring source (2π times the value given in (2.19)), we obtain for E_φ in (5.1)

$$E_\varphi = \frac{e^{-i(kR - \pi/4)}}{kR}\left(\frac{2kR_0}{\pi}\frac{\sin\theta_0}{\sin\theta}\right)^{1/2}\Big[\sin(\theta_0 - \theta)\, e^{-ikR_0\cos(\theta_0 - \theta)}$$

$$-\,i\sin(\theta_0 + \theta)\, e^{-ikR_0\cos(\theta_0 + \theta)}\mathcal{H}(\pi - \theta_0 - \theta)\Big] E_0 \qquad (5.21)$$

where $H(x)$ is the Heaviside step function

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

The (unnormalized) far-field pattern (i.e., the dependence of E_σ on θ), thus is given by

$$F(\theta) = (\sin\theta)^{-1/2}\left[\sin(\theta_0-\theta)\, e^{-ikR_0\cos(\theta_0-\theta)} - i\sin(\theta_0+\theta)\, e^{-ikR_0\cos(\theta_0+\theta)} H(\pi-\theta_0-\theta)\right] \tag{5.22}$$

(5.21) and (5.22) do not hold in the vicinity of $\theta = 0$, since then $\Delta_{1,2} = 0$, corresponding to the vanishing of the second derivatives $\partial^2 S_{1,2}/\partial\sigma_2^2$. In this region a higher order expansion of $S_{1,2}$ is required. It suffices to evaluate the pattern function directly for $\theta = 0$. Then

$$F(\theta = 0) = P_{\nu-1/2}(1) = 1$$

$$F'(\theta = 0) = 0$$

Thus there is a null along the cone axis for the ring source in free space.

The far-field pattern given in (5.22) is for a slot of infinitesimal radial extent. For a slot of finite radial extent, it is necessary to integrate the field across the slot in accordance with the applied field distribution. This is an elementary integration. For a half-wave slot with a sinusoidal distribution, the result is

$$\begin{aligned} F(\theta) = (\sin\theta)^{-1/2}\Big\{&\cot(\theta_0-\theta)\cos\left[\tfrac{\pi}{2}\cos(\theta_0-\theta)\right] e^{-ikR_0\cos(\theta_0-\theta)} \\ &- i\cot(\theta_0+\theta)\cos\left[\tfrac{\pi}{2}\cos(\theta_0+\theta)\right] e^{-ikR_0\cos(\theta_0+\theta)} H(\pi-\theta_0-\theta)\Big\} \end{aligned} \tag{5.23}$$

The first term becomes indeterminate when $\theta_0 - \theta = 0$, the second when $\theta_0 + \theta = \pi$. In such a case, the indeterminate coefficient of the exponential is zero. In the neighborhood of $x = 0$ (π), where x denotes $\theta_0 - \theta$ ($\theta_0 + \theta - \pi$), the critical coefficient is approximately equal to $\pi x/4$.

It has yet to be shown that the terms in the expansion (5.8) for $k > 0$ do not yield stationary points of the integrals $g_{1,2}$. Obviously it is sufficient to show this for $\mathcal{I}m\, \nu > 0$. By writing

$$(\sigma^*)^k = \left(\frac{\xi_0^{\sigma\nu'}}{\xi_0^{\sigma\nu}}\right)^k = \left(\frac{\xi_1^{\sigma\nu'}\xi_0^{\sigma\nu'}}{\xi_0^{\sigma\nu}\xi_0^{\sigma\nu'}}\right)^k$$

using (5.5b) for each of the $\xi_i^{\sigma\nu'}$ terms in the numerator, and (5.7c) for the denominator, we obtain for the exponentials $S_{1,2}$ in (5.10)

$$S_{1,2} = \Phi_0 + \Phi_{1,2} + \sum_{j=1}^{m} \Phi_j$$

where

$$\Phi_j = -(\nu + 1/2)\ln(\cos\theta_0 - i\sin\theta_0\cos\phi_j)$$

Then $\partial S_{1,2}/\partial\phi_j = 0$ leads to $\phi_j = 0$, so that

$$(\Phi_j)_s = i(\nu + \tfrac{1}{2})\theta_0$$

Then $\partial S_{-2}/\partial\nu = 0$ results in

$$w_1 + w_2 + (2k+1)\theta_0 \pm \theta = \pi \tag{5.24}$$

Obviously, since $\theta_0 > \frac{\pi}{2}$, (5.24) cannot be satisfied for $k > 0$.

The evaluation of the determinant Δ, given by (5.13), can be simplified when one (or more) of the variables is uncoupled to the others, so that, for that variable, only one entry appears in its row and column. In the free-space example considered above, this happens for the variables φ_1 and φ_2. The result then is equivalent to inserting the asymptotic expansions for the angular functions p', p_0' at the outset.

5.3 RING SOURCE IN SHEATH

We consider next a ring slot on a cone covered by a sheath. For simplicity, we consider first a single homogeneous layer, and the thin sheath approximation.

For the far field, we are concerned with the field in the ambient medium. Then we obtain from (2.11), (3.7), and (3.15)

$$\begin{aligned} -RE_\varphi &= \int_{C_0} a_2 p' \lambda_\nu(\rho x)\, k_\nu(\rho y)\, \nu\, d\nu \pm \int_{C_0} A_2 p' i_\nu(\rho x)\, \nu\, d\nu \\ &= \int_{C_0} \varphi(\nu) \left(\frac{p'}{p_0'}\right)_1 i_\nu(\rho x)\, \nu\, d\nu \end{aligned} \tag{5.25}$$

$\varphi(\nu)$ is given in general form by the representation (3.30). The first term of (3.30), $\varphi_0(\nu)$, is simply the free-space term (5.1) (with arguments ρx and ρy replacing x and y, respectively), for which the far field was evaluated in Sec. 5.2. The first iteration, $\varphi_1(\nu)$, is given by (3.45). On inserting this into (5.25), we have

$$-RE_\varphi^1 = (RR_0)^{1/2} \mathcal{E}_0 d_1 \tag{5.26a}$$

where

$$d_1 = -\frac{1}{2} \vartheta_1^2 y^2 \delta \cdot \int_{-i\infty}^{i\infty} \left(\frac{p'}{p_0'}\right)_1 I_\nu(\rho x) K_\nu(\rho y)\, \nu\, d\nu \tag{5.26b}$$

The integral in (5.26b) is the same as (5.2b), except for the replacement of x and y by ρx and ρy, respectively. Consequently, by comparison of (5.26a) with (5.2a), we can immediately write for the field E_φ^1 due to the first iteration

$$E_\varphi^1 = -\frac{1}{2} \vartheta_1^2 y^2 \delta E_\varphi^0 \tag{5.27}$$

where E_φ^0 is the free-space field. Since $R_0 \vartheta_1$ is the thickness, t, of the sheath at the source radius,

$$t = R_0 \vartheta_1 \tag{5.28}$$

(5.27) may be written as

$$E_\varphi^1 = -\frac{1}{2}(\gamma_1 t)^2 \delta E_\varphi^0 \tag{5.29}$$

Thus we obtain

$$E_\varphi = E_\varphi^0 + E_\varphi^1 + O(\delta^2) = \left[1 - \frac{1}{2}(\gamma_1 t)^2 \delta\right] E_\varphi^0 + O(\delta^2) \tag{5.30}$$

where E_φ^0 is given by (5.21). Thus, in first approximation, the sheath decreases the far field, by an amount which is proportional to δ and to the square of the electrical thickness of the sheath at the source position on the cone.

5.4 SLOT SOURCE IN SHEATH

A slot source covered by a thin sheath was treated in Sec. 3.3.2, where (3.72) and (3.74) for $\sigma_1^m(\mu)$ and $\sigma_1^e(\mu)$, respectively, were obtained. Whereas for the ring source, the formulation (2.11) for $R\Pi^m$ involves only the term $n = 0$ because of the symmetrical excitation, for the slot source all values of n are involved. Then, as pointed out following (2.15) and (2.16), the coefficients σ_1^m and σ_1^e are the same for each value of n. The relative excitations of the various orders, n, will then be determined by the applied excitation as a function of the azimuthal coordinate φ.

Since (3.72) for $\sigma_1^m(\mu)$ is identical with (3.45) for the ring source, the azimuthal electric far field of magnetic type, E_φ^m, for $n = 0$ is the same as for the ring source case, which is given by (5.30). For $n > 0$, it can be seen from the n^{th} order asymptotic expansion of $(\mathscr{P}'/\mathscr{P}_0')_\nu$

$$\frac{P_{\nu-1/2}^{-n\,\prime}(\cos\theta)}{P_{\nu-1/2}^{-n\,\prime}(\cos\theta_0)} \sim \left(\frac{\sin\theta_0}{\sin\theta}\right)^{1/2} \frac{\sin(\nu\theta - \pi/4 - n\pi/2)}{\sin(\nu\theta_0 - \pi/4 - n\pi/2)}$$
$$\sim \left(\frac{\sin\theta_0}{\sin\theta}\right)^{1/2} \left[e^{i\nu(\theta_0-\theta)} + (-)^n i\, e^{i\nu(\theta_0+\theta)}\right] \sum_{l=0}^{\infty} e^{il(2\nu\theta_0 - \pi/2 - n\pi)} \tag{5.31a}$$

that it is only necessary to multiply the coefficient of the second term in brackets in (5.21) by $(-)^n$ and the entire expression by $\cos ln\pi$.

For the slot source, there is also a meridional component of magnetic type, E_θ^m, as well as components of electric type, E_φ^e and E_θ^e. Far-field expressions for these components will now be obtained.

For E_θ^m, we have

$$E_\theta^m = \frac{1}{R\sin\theta}\frac{\partial R\Pi^m}{\partial\theta} = -\sum_{n=1}^{\infty} \frac{n\sin n\varphi}{R\sin\theta} \int_{C_0} \sigma^m(\nu) \left(\frac{\mathscr{P}}{\mathscr{P}_0'}\right)_\nu i_\nu(\rho\lambda)\,\nu\, d\nu \tag{5.32}$$

The integral differs from that in (5.25) only by the replacement of $\mathscr{P}'$ by $\mathscr{P}$. The asymptotic expansion of $(\mathscr{P}/\mathscr{P}')_\nu$ is

$$\frac{P_{\nu-1/2}^{-n}(\cos\theta)}{P_{\nu-1/2}^{-n\,\prime}(\cos\theta_0)} \sim -\left(\frac{\sin\theta_0}{\sin\theta}\right)^{1/2} \frac{\cos(\nu\theta - \pi/4 - n\pi/2)}{\nu\sin(\nu\theta_0 - \pi/4 - n\pi/2)}$$
$$\sim \left(\frac{\sin\theta_0}{\sin\theta}\right)^{1/2} \frac{i}{\nu}\left[e^{i\nu(\theta_0-\theta)} - (-)^n i\, e^{i\nu(\theta_0+\theta)}\right] \sum_{l=0}^{\infty} e^{il(2\nu\theta_0 - \pi/2 - n\pi)} \tag{5.31b}$$

which differs from (5.31a) principally by the factor ν^{-1}. Consequently the saddle-point evaluation of (5.32) yields for the n^{th} term (with $k_2 = k$)

$$E_\phi^m = n \sin n\varphi \, \frac{e^{-i(kR-\pi/4)}}{kR \sin\theta} \left(\frac{2}{\pi k R_0} \frac{\sin\theta_0}{\sin\theta}\right)^{1/2} \Big[-i \, e^{-ikR_0 \cos(\theta_0-\theta)}$$

$$+ (-)^n e^{-ikR_0 \cos(\theta_0+\theta)} \mathcal{H}(\pi-\theta_0-\theta) \Big] E_0 \qquad (5.33)$$

The field components of electric type involve $\sigma^e(\nu)$. $\sigma_0^e(\nu)$ is zero, so that $\sigma_1^e(\nu)$, which is $O(\delta)$, is the term of lowest order. For this we have the integral expression (3.74). Since the thin sheath approximation is limited to terms which are $O(\delta)$, we need be concerned with the evaluation of (3.74) only to terms which are $O(\delta)$. Then, as shown in Appendix B, (3.74) reduces to

$$\sigma^e(\mu) = \frac{N_1 \varphi_0 \mathcal{E}_0}{(\pi i)^2} \int_{C_0} \left(\frac{\rho_0}{\rho_0'}\right)_\lambda k_\lambda(\rho y) \, \mathcal{H}^{\pm\nu,\pm\lambda} d\lambda$$

For E_ϕ^e we then have from (2.16), (3.7), and (3.58b)

$$E_\phi^e = \frac{1}{kR \sin\theta} \frac{\partial^2 R \Pi^e}{\partial R \partial \phi} = \sum_{n=1}^{\infty} \frac{2n N_1 \varphi_0 \mathcal{E}_0 R_0^{1/2} \sin n\varphi}{\pi^2 k R \sin\theta} \frac{\partial}{\partial R}\left(R^{1/2} d^e\right) \qquad (5.34a)$$

where

$$d^e = \sum_{m=0}^{\infty} I_m \qquad (5.34b)$$

$$I_m = \int_0^{\infty} \left(\frac{J}{\rho_0'}\right)_\nu I_\nu(\rho x) \nu \, d\nu \int_{-i\infty}^{+i\infty} \left(\frac{\rho_0}{\rho_0'}\right)_\lambda K_\lambda(\rho y) \mathcal{H}_m^{\pm\nu,\pm\lambda} d\lambda \qquad (5.34c)$$

in which

$$\left.\begin{aligned} \mathcal{H}_0^{\pm\nu,\pm\lambda} &= -\frac{\delta}{2} \sum_{k=1}^{4} \frac{h_k(\lambda)}{\lambda + a_k} \qquad h_3 = h_4 = 0 \\ \mathcal{H}_m^{\pm\nu,\pm\lambda} &= -\frac{\delta}{4} \sum_{k=1}^{4} \frac{h_k(\lambda)}{\lambda + a_k} \qquad m > 0 \end{aligned}\right\} \qquad (5.34d)$$

$$a_1 = 2m - 1 - \nu$$

$$a_2 = 2m + 1 - \nu$$

$$a_3 = -2m + 1 - \nu$$

$$a_4 = -2m - 1 - \nu$$

$$h_1 = \frac{\lambda + 1/2}{\lambda} - \frac{\lambda^2 - 1/4}{\nu(\nu + 1/2)}$$

$$h_2 = \frac{\lambda - 1/2}{\lambda} - \frac{\lambda^2 - 1/4}{\nu(\nu - 1/2)}$$

$$h_3 = h_4 = 0, \qquad m = 0$$

$$\left.\begin{aligned} h_3 &= h_2 \\ h_4 &= h_1 \end{aligned}\right\rangle \qquad m > 0$$

(5.34c) has been couched in such a form that the poles of $\mathcal{H}^{\pm\nu,\pm\lambda}$ occur in the upper half-plane, $\mathcal{I}m\, \lambda > 0$. Furthermore, since it turns out that the saddle points of (5.26) in the λ-plane occur at the poles, the lower half of contour C_0 can be neglected.

The poles of $\mathcal{H}^{\pm\nu,\pm\lambda}$ in (5.34c) pose a complication. This can be overcome in the following way:

Denoting the λ-integral by $I(\theta_o)$, we have

$$I(\theta_o) \equiv \sum_{k=1}^{4} I_m^{(k)}(\theta_o) = \sum_{k=1}^{4} \int_0^{i\infty} \left(\frac{f_1}{f_o'}\right)_\lambda K_\lambda(\rho y) \frac{h_k(\lambda)}{\lambda + a_k} \, d\lambda$$

where the a_k represent the poles at $\lambda = \pm 2m \pm 1 - \nu$. Multiplying by $e^{ia_k\theta_o}$ and then differentiating with respect to θ_o, we obtain the differential equation

$$e^{-ia_k\theta_o} \frac{\partial}{\partial\theta_o}\left[e^{ia_k\theta_o} I_m^{(k)}(\theta_o)\right] = \frac{\partial I_m^{(k)}(\theta_o)}{\partial\theta_o} + ia_k I_m^{(k)}(\theta_o) = i\int_0^{i\infty} \left(\frac{f_1}{f_o'}\right)_\lambda K_\lambda(\rho y) h_k(\lambda)\, d\lambda$$

The solution of this differential equation is

$$I_m^{(k)}(\theta_o) = e^{-ia_k\theta_o} \int_{L'}^{\theta_o} e^{ia_k\chi} I_o(\chi)\, d\chi = \int_L^{0} e^{ia_k\chi} I_o(\theta_o + \chi)\, d\chi \tag{5.35}$$

where the lower limit L is chosen so that the constant of integration is zero. Since $\mathcal{I}m\ a_k = -\nu < 0$, ia_k has a positive real part. Hence χ must be negative, and $L = -\infty$ insures that the integral in (5.30) vanishes at the lower limit.

Thus (5.34c) may be written as

$$I_m = \sum_{k=1}^{4} I_m^{(k)}$$

where

$$I_m^{(k)} = i\int_0^{i\infty} e^{i\nu\theta_o}\left[-i(-)^m e^{i\nu\theta} + e^{-i\nu\theta}\right] I_\nu(\rho x) \int_{-\infty}^{0} e^{ia_k\chi} d\chi \int_0^{i\infty} e^{i\lambda(\theta_o+\chi)}\left[(-)^m e^{i\lambda\theta_1} + i e^{-i\lambda\theta_1}\right] K_\lambda(\rho y) h_k(\lambda)\, d\lambda \tag{5.36}$$

In this form, the poles have been eliminated at the expense of an additional exponential integral, which increases by one the dimensionality of the multi-dimensional integral to be evaluated.

There are four varieties of multi-dimensional integrals in (5.36), corresponding to the four combinations of the exponents $\pm i\nu\theta$ and $\pm i\lambda\theta_1$. Thus we have to evaluate the four integrals

$$J_k^{\pm\theta,\pm\theta_1} = \int_{-\infty}^{0} e^{im_k\chi} d\chi \int_0^{i\infty} e^{i\nu(\theta_o \pm\theta - \chi)} I_\nu(\rho x)\, \nu\, d\nu \int_0^{i\infty} e^{i\lambda(\theta_o \pm\theta_1 + \chi)} K_\lambda(\rho y) h_k(\lambda)\, d\lambda \tag{5.37a}$$

where

$$m_k = a_k + \nu \qquad m_{1,2} = 2m \mp 1 \qquad m_{3,4} = -2m \pm 1 \tag{5.37b}$$

Replacing $I_\nu(\rho x)$ and $K_\lambda(\rho y)$ by their integral representations (5.3a,b) as in Sec. 5.2, we obtain a five-dimensional exponential integral. By equating to zero the first partial derivatives of the exponent S with respect to the integration variables, we obtain the following equations determining the stationary point:

$$\sin w_1 = \nu / i\rho x \tag{5.38a}$$

$$\sin w_2 = \lambda / i\rho y \tag{5.38b}$$

$$w_1 + \theta_o \pm \theta - \chi = \pi \tag{5.38c}$$

$$w_2 + \theta_o \pm \theta_1 + \chi = 0 \tag{5.38d}$$

$$\nu = \lambda + m_k \tag{5.38e}$$

For the far field, $x \to \infty$, so that we obtain

$$\sin w_1 = 0 \qquad w_1 = 0, \pi, 2\pi$$

From (5.38c) it can be seen that, in order that χ be negative, we must choose $w_1 = 0$, so that

$$-\chi_s = \pi - (\theta_0 \pm \theta) \tag{5.39a}$$

For the upper sign, the right-hand side is positive if $\theta_0 + \theta < \pi$. For $\theta_0 + \theta > \pi$, a stationary point does not exist.

From (5.39a) and (5.38d),

$$w_2 = \pi - (\theta_0 \pm \theta_1 + \theta_0 \pm \theta) \tag{5.39b}$$

Thus

$$\sin w_2 = \sin(2\theta_0 \pm \theta_1 \pm \theta) \tag{5.39c}$$

so that, from (5.38b)

$$\lambda_s = i\rho y \sin(2\theta_0 \pm \theta_1 \pm \theta) \tag{5.39d}$$

whence, from (5.38e),

$$\nu_s = i\rho y \sin(2\theta_0 \pm \theta_1 \pm \theta) + m_k \tag{5.39e}$$

The value of the exponent S at the stationary point then is

$$S_s = -\rho x - \rho y \cos w_2 + i m_k \chi_s = -\rho x + \rho y \cos(2\theta_0 \pm \theta_1 \pm \theta) + i m_k(\theta_0 \pm \theta - \pi) \tag{5.40a}$$

and the value of the determinant Δ_s is

$$\Delta_s = \rho x + \rho y \cos w_2 \to \rho x \quad \text{as } x \to \infty \tag{5.40b}$$

The geometrical interpretation of (5.39b) is shown in Figs. 3(a) and 3(b). Fig. 3(a) corresponds to $-\theta$ and $-\theta_1$. The angle

$$2\theta_0 - \theta_1 - \theta = \theta_0 - \theta_1 + \theta_0 - \theta$$

and the distance

$$x - y\cos(2\theta_0 - \theta_1 - \theta)$$

which occurs in S_s show that the field of electric type appears to originate at the image of the sheath in the cone. The situation depicted in Fig. 3(b) corresponds to $+\theta$ and $-\theta_1$.

Since

$$2\theta_0 + \theta_1 \pm \theta \geq 2\theta_0 > \pi$$

the combination $\theta_1 \pm \theta$ in (5.39b) cannot lead to a realizable stationary point.

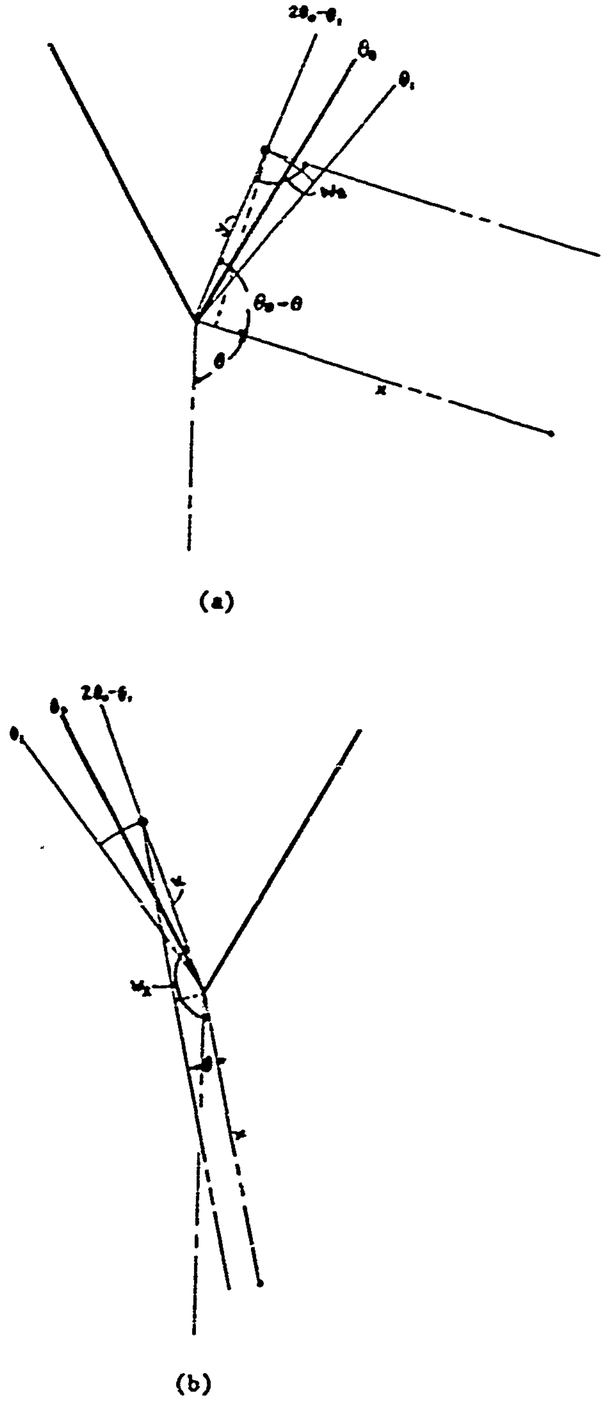


Fig. 3 Stationary paths for electric-type field

On inserting (5.40a) and (5.40b) into (5.12) to evaluate the integrals $J^{\pm\theta\theta_1}$, we obtain for E^e_φ in (5.34a)

$$E^e_\varphi = E_0 \frac{e^{-i(k_2R-\pi/4)}}{k_2R\sin\theta}\,\frac{i\theta_1\delta}{2\pi^2\sin\theta_1}\left(\frac{\cdot}{2\pi k_2R_0}\,\frac{\sin\theta_0}{\sin\theta}\right)^{1/2}\sum_{n=1}^{\infty} n^2\sin n\varphi \sum_{m=0}^{\infty}\sum_{k=1}^{\infty}\cdot$$

$$\cdot h'_k\left[-i\,e^{i[k_2R_0\cos(2\theta_0-\theta_1-\theta)+m_k(\theta_0-\theta)]} - (-1)^m e^{i[k_2R_0\cos(2\theta_0-\theta_1+\theta)+m_k(\theta_0+\theta)]}\,H(\pi-\theta_0-\theta)\right] \tag{5.41}$$

$$h'_{1,2} = (4m\mp 1)\,\frac{(\lambda_3\mp\ldots)(\lambda_3+m\mp\ldots)}{\lambda_3(\lambda_3+2m\mp\ldots)} \qquad h'_3 = h'_4 = 0,\ m=0;\qquad \begin{matrix} h'_3 = h'_2 \\ h'_4 = h'_1 \end{matrix}\Big\}\ m>0$$

In a similar way, we obtain for E^e_θ

$$E^e_\theta = E_0 \frac{e^{-i(k_2R-\pi/4)}}{k_2R}\,\frac{i\theta_1\delta}{2\pi^2\sin\theta_1}\left(\frac{k_2R_0}{2\pi}\,\frac{\sin\theta_0}{\sin\theta}\right)^{1/2}\sum_{n=1}^{\infty} n\cos n\varphi \sum_{m=0}^{\infty}\sum_{k=1}^{4}\cdot$$

$$\cdot h'_k\Big\{-\left[\sin(2\theta_0-\theta_1-\theta) - \frac{i m_k}{k_2R_0}\right] e^{i[k_2R_0\cos(2\theta_0-\theta_1-\theta)+m_k(\theta_0-\theta)]} \tag{5.42}$$

$$+ i\left[\sin(2\theta_0-\theta_1+\theta) - \frac{i m_k}{k_2R_0}\right] e^{i[k_2R_0\cos(2\theta_0-\theta_1+\theta)+m_k(\theta_0+\theta)]}\,H(\pi-\theta_0-\theta)\Big\}$$

The far-field expressions (5.33), (5.41), and (5.42), as well as the expression corresponding to (5.21) for the slot source, hold for a slot of infinitesimal width, corresponding to the $\delta(\varphi)$ factor on the right-hand side of (2.13). For a slot of finite angular width $2\phi < \lambda/2$, the excitation can be assumed to be uniform in φ across the slot. Then $\delta(\varphi)$ in (2.13) is first replaced by $\delta(\varphi-\varphi_0)$, where φ_0 represents the azimuthal location of the infinitesimal slot, followed by an integration with respect to φ_0 over the azimuthal extent, 2ϕ, of the slot. Thus E_0 in (2.19) can be replaced by

$$E_0\int_{-\pi}^{\pi}\cos n\varphi_0\left[H(\varphi_0+\phi) - H(\varphi_0-\phi)\right]d\varphi_0 = 2E_0\,\frac{\sin n\phi}{n}$$

The factor

$$\frac{2\sin n\phi}{n} \tag{5.43}$$

thus should be affixed to all of the far-field expressions to account for the φ-distribution of a slot of finite width.

5.5 EXTENSIONS OF THE METHOD

The examples worked out in Sec. 5.3 and Sec. 5.4 were for a single-layered sheath, and in Sec. 5.4 use was made of the thin sheath approximation to $O(\delta)$. There is no inherent difficulty, however, in extending the treatment. This can be done to any order in δ without invoking the thin sheath approximation, since asymptotic expansions can be used for the angular functions in the kernel functions $M_1(\nu,\lambda)$ and $M_2(\nu,\lambda)$ of the single-layered sheath analysis, or $\overline{m}_i^{\pm,\lambda}(\nu,\lambda)$ and $\overline{K}_i^{\pm,\lambda}(\nu,\lambda)$ of the double-layered sheath analysis. Thus the far fields and patterns can be determined for the general case to any desired degree of precision $O(\delta^k)$.

SECTION VI

INPUT ADMITTANCE AND MUTUAL COUPLING; EXTENSIONS OF THE ANALYSIS

6.1 INPUT ADMITTANCE

The reaction of the sheath on the input admittance of the antenna is of great importance, since experimentally it is found that the change in admittance can be severe, and can profoundly affect the excitation of the antenna. The method of calculation of input admittance was presented in Reference 5, where, for a radial slot energized by a voltage V across its center, the input admittance, Y, was given as

$$Y = -V^{-2}\int_0^{2\pi} d\varphi \int_a^{\infty} E_\varphi(\theta_o) H_R(\theta_o) R_o \sin\theta_o \, dR_o \tag{6.1}$$

Since $H_R(\theta_o)$ is zero for the electric-type field, this reduces to

$$Y = -V^{-2}\int_0^{2\pi} d\varphi \int_a^{\infty} E_\varphi^m(\theta_o) H_R^m(\theta_o) R_o \sin\theta_o \, dR_o \tag{6.2}$$

Since

$$H_R^m(\theta_i) = \frac{1}{-i\omega\mu}\left(\frac{\partial^2}{\partial R^2} - \gamma_i^2\right) R\Pi^m(\theta_i) = \frac{1}{-i\omega\mu R^2}(\nu^2 - \tfrac{1}{4}) R\Pi^m(\theta_i) \tag{6.3}$$

(6.2) becomes

$$Y = (i\omega\mu V^2)^{-1}\int_0^{2\pi} d\varphi \int_a^{\infty} E_\varphi(\theta_o)\left(\frac{\partial^2}{\partial R^2} - \gamma_i^2\right) R\Pi^m(\theta_o) R_o \sin\theta_o \, dR_o \tag{6.4}$$

$R\Pi^m$ is given in general form by (2.11), and for the input admittance $i = 1$.

In the case of a single-layered sheath, for example, the notation of Section III is applicable. Then, just as was done for the coefficient A_2, the boundary equations may be solved for A_1. The result for the ring source, for example, can be expressed in the form

$$A_1(\nu) = \psi_0(\nu) + \psi_1(\nu) + \psi_2(\nu) + \cdots \tag{6.5}$$

where

$$\psi_0(\nu) = \xi_0 \, k_\nu(y) \tag{6.6a}$$

$$\psi_1(\nu) = \frac{1}{(2\pi i)^2}\int_{C_1} K^{\nu,\lambda_1} d\lambda_1 \int_{C_2} L(\lambda_1,\lambda_2)\psi_0(\lambda_2) K^{\lambda_1,\lambda_2} d\lambda_2 \tag{6.6b}$$

$$\psi_2(\nu) = \frac{1}{(2\pi i)^2}\int_{C_1} K^{\nu,\lambda_1} d\lambda_1 \int_{C_2} L(\lambda_1,\lambda_2)\psi_1(\lambda_2) K^{\lambda_1,\lambda_2} d\lambda_2 \tag{6.6c}$$

. . . .

$$L(\lambda_1,\lambda_2) = \left(\frac{\xi_1 \xi_2'}{W_1}\right)_{\lambda_1}\left(\frac{\xi_1'}{\xi_2'}\right)_{\lambda_2} - \zeta_{\lambda_2}^{\lambda_1}\left(\frac{\xi_1' \xi_2}{W_1}\right)_{\lambda_1}\left(\frac{\xi_1}{\xi_2'}\right)_{\lambda_2} \tag{6.6d}$$

$$L(\lambda_1,\lambda_1) = 0 \tag{6.6e}$$

In virtue of (6.6e), $\psi_1(\nu)$ is $0(\delta)$ and $\psi_2(\nu)$ is $0(\delta^n)$.

From (3.1a), (6.5) and (6.6a), we have

$$B_1(\nu) = -\left[\psi_1(\nu) + \psi_2(\nu) + \cdots\right]$$

so that

$$H_R = (-i\omega\mu)^{-1} \sum_{n=0}^{\infty} \epsilon_n \cos n\varphi \int_{C_0} (\nu^2 - \tfrac{1}{4}) \left\{ \psi_0(\nu) \left(\frac{p_0}{p_0'}\right)_\nu + \left[\psi_1(\nu) + \psi_2(\nu) + \cdots\right] \left(\frac{W_0}{p_0' q_0'}\right)_\nu \right\} i_\nu(x)\, \nu\, d\nu \qquad (6.7)$$

Then (6.2) becomes

$$\begin{aligned} Y = {} & (i\omega\mu V^2)^{-1} \sum_{n=0}^{\infty} \epsilon_n \cos n\varphi \int_0^{2\pi} d\varphi \int_0^{\infty} E_\varphi^{m}(\theta_0) R_0 \sin\theta_0 \, dR_0 \cdot \\ & \cdot \int_{C_0} (\nu^2 - \tfrac{1}{4}) \left\{ \psi_0(\nu) \left(\frac{p_0}{p_0'}\right)_\nu + \left[\psi_1(\nu) + \psi_2(\nu) + \cdots\right] \left(\frac{W_0}{p_0' q_0'}\right) \right\} i_\nu(x)\, \nu\, d\nu \\ = {} & Y_0 + Y_1 + Y_2 + \cdots \end{aligned} \qquad (6.8)$$

where

$$\begin{aligned} Y_0 = {} & (i\omega\mu V^2)^{-1} \xi_0 \sum_{n=0}^{\infty} \epsilon_n \cos n\varphi \, d\varphi \int_0^{\infty} E_\varphi^{m}(\theta_0) R_0 \sin\theta_0 \, dR_0 \cdot \\ & \cdot \int_{C_0} (\nu^2 - \tfrac{1}{4}) \left(\frac{p_0}{p_0'}\right)_\nu i_\nu(x)\, k_\nu(y)\, \nu\, d\nu \end{aligned} \qquad (6.9a)$$

$$\begin{aligned} Y_n = {} & (i\omega\mu V^2)^{-1} \xi_0 \sum_{n=0}^{\infty} \epsilon_n \cos n\varphi \, d\varphi \int_0^{\infty} E_\varphi^{m}(\theta_0) R_0 \sin\theta_0 \, dR_0 \cdot \\ & \cdot \int_{C_0} (\nu^2 - \tfrac{1}{4}) \left(\frac{p_0}{p_0'}\right)_\nu \psi_n(\nu) \left(\frac{W_0}{p_0' q_0'}\right)_\nu i_\nu(x)\, k_\nu(y)\, \nu\, d\nu \end{aligned} \qquad (6.9b)$$

Y_0 is the free-space admittance, so that the remaining terms represent the effect of the sheath. In (6.9a,b) $(p_0/p_0')_\nu$ denotes

$$\left(\frac{p_0}{p_0'}\right)_\nu = \frac{P_{\nu-1/2}^{-m}(\cos\theta_0)}{P_{\nu-1/2}^{-m\,\prime}(\cos\theta_0)}$$

and similarly for the other angular functions in $L(\lambda_1, \lambda_2)$ in $\psi_2(\nu)$, so that the $\sum_n$ also involves the Legendre functions.

There is no difficulty in extending the analysis to the case of a slot source, and to the general case of an M-layered sheath, so that expressions can be obtained for the input admittance for these cases as well.

$E_\varphi(\theta_0)$ is the applied field at the slot source. For a single half-wave slot of width $2w$, for example, we have with $\phi = w/R_C$, where R_C is the location of the center of the slot

$$E_\varphi(\theta_0) = -\frac{V}{2w} \cos k_1(R_0 - R_C)\left[\mathcal{H}(R_0 - R_1) - \mathcal{H}(R_0 - R_2)\right]\left[\mathcal{H}(\varphi - \varphi_1) - \mathcal{H}(\varphi - \varphi_2)\right] \qquad (6.10)$$

where

$$R_{1,2} = R_C \mp \lambda_1/4$$

$$\varphi_{1,2} = \mp\phi = \mp w/R_C$$

and in (2.19), $E_0 = -V/2w$, so that

$$\xi_0 = -V/(4\pi^2 k_1 R_0 w) \qquad (6.11)$$

Since $\psi_0(\nu)$ is known, all of the integrals contain only known functions. They can be evaluated exactly by contour integration as in Section III. This procedure is not very appealing, in view of the lengthy series representations. Since the admittance is strongly affected by the near field, asymptotic expansions of the angular functions are not suitable. The only alternative appears to be direct numerical evaluation of the integrals. In this connection, it appears that even the impedance of a cone slot antenna in free space has not been evaluated analytically.

If the thin sheath approximation is introduced into (6.6d) for $L(\lambda_1, \lambda_2)$, then it turns out that there are terms which are $O(\vartheta_1{}^{0})$, and these cannot be evaluated in closed form either.

6.2 MUTUAL COUPLING

The coupling between a transmitting and a receiving slot on the cone can be expressed in terms of the mutual admittance between the two antennas. This was discussed in Reference 5. The mutual admittance, Y_{12}, is the complex mutual power per unit voltages across the two slots

$$Y_{12} = Y_{21} = \iint_A (V_1^{-1}\underset{\sim}{E}_2) \times (V_2^{-1}\underset{\sim}{H}_2)\, d\underset{\sim}{A} \tag{6.12}$$

Where $\underset{\sim}{H}_2$ is the magnetic field strength at slot 1 produced by the excitation of slot 2. For example, for two infinitesimal slots at the same distance R_0 from the cone tip, but spaced an angle Φ around the cone

$$Y_{12} = (V_1 V_2)^{-1} \int_0^{2\pi} E_{\varphi_1}(\theta_0) H_{R_2}(\theta_0) R_0 \sin\theta_0\, d\theta_0 \tag{6.13}$$

where

$$E_{\varphi_1}(\theta_0) = E_1\, \delta(\varphi) + E_2\, \delta(\varphi - \Phi) \tag{6.14}$$

Thus in (2.13), $E_0\,\delta(\varphi)$ is to be replaced by

$$E_1\, \delta(\varphi) + E_2\, \delta(\varphi - \Phi)$$

By putting $E_1 = 0$, it can be seen that the field E_2 produces a field at slot 1 whose φ-variation is shifted by Φ relative to that produced by a field applied to slot 1. Then for $H_{R_2}(\theta_0)$, we use the value (6.6) with φ replaced by $\varphi - \Phi$.

Similarly, for two infinitesimal slots on the same radial, but at distances R_0, R_1, respectively, from the cone tip, $E_0(\varphi)\delta(R-R_0)$ in (2.13) is to be replaced by

$$E_1\, \delta(R - R_0) + E_2\, \delta(R - R_1)$$

Consequently, with $E_1 = 0$, it can be seen that for $H_{R_2}(\theta_0)$ we use the value (6.7), in which the $\psi_0(\nu)$ have $k_\nu(y) = k_\nu(\gamma, R_0)$ replaced by $(R_1^2/R^2)\, k_\nu(\gamma, R_1)$.

Thus the evaluation of the mutual admittance between antennas can be handled in a manner very analogous to the calculation of the self admittance.

6.3 EXTENSIONS OF THE ANALYSIS

In an actual plasma sheath which forms about a re-entry vehicle, the plasma parameters vary in all three coordinate directions, in general. In this report, the plasma has been idealized by assuming that it is uniform in azimuth and in the radial direction. The continuous meridional variation has been replaced by a two-step variation. It is interesting to see in what directions this idealization can be liberalized.

The two-step layering procedure can be extended to an arbitrary number of steps, thus taking into account the meridional variation. The separability of the wave equation in spherical coordinates allows an arbitrary variation in the radial direction. The radial differential equation then is affected by the variation of κ with R. Thus if κ is other than constant, the Bessel differential equation has to be replaced by one which depends on the radial variation of κ. Then the K-L transform also can no longer be employed, since it stems from the differential equation of the cylinder functions. Instead, a transform applicable to the new radial differential equation is required. The technique for developing such a transform exits. Although the radial electric and magnetic Hertz vectors, $\underset{\sim}{R\Pi}^{e,m}$ satisfy somewhat different differential equations, this difference can be neglected if κ does not vary appreciably in a (local) wavelength.

SECTION VII

CONCLUSIONS

The problem of radiation from slot antennas on a cone in the presence of an inhomogeneous sheath has been solved. In this report, the sheath is considered as being made up of one or two conical layers, each of which is homogeneous. In the method of analysis used, the field is expressed as an integral representation. The boundary conditions then lead to a system of integral equations, which number 4M+4 for a sheath composed of M(= 1 or 2) conical layers. By an extension of the K-L transform technique, these equations are reduced to singular integral equations of Cauchy type. An inversion technique is developed which reduces this system to Fredholm equations, which can be solved in iterative fashion. By introducing the parameter

$$\delta_i = 1 - \rho_i^2$$

where

$$\rho_i = \gamma_{i+1}/\gamma_i$$

is the ratio of propagation constants of adjacent layers of the sheath, it is shown that the successive iterations proceed in powers of δ_i. For a sufficiently fine stratification of the sheath, the first iteration should suffice.

In general, fields of both magnetic and electric types are generated in the presence of a sheath, even though, in the case of a radial slot, only a field of magnetic type is generated in free space. The field of electric type then is created at the sheath boundary. For a ring slot, however in which the excitation is azimuthally symmetrical, only a field of magnetic type is generated even in the presence of a sheath. It is shown that the solution for this case forms the basis of the solution for the general case.

In general, the evaluation of the integrals must be accomplished by contour integration, which leads to lengthy series expansions. These are not convenient for numerical evaluation. For the case of thin layers, however, Taylor's series expansions of the angular functions allow all but one of the coefficients to be evaluated in closed form.

The far field is determined by a multi-dimensional saddle point evaluation of the integral representations. This is illustrated in detail for the free-space case, and is then applied to determine the far field patterns in the presence of a sheath. This can be carried out successfully for all components, and to arbitrary orders of iteration.

The calculation of input admittance and mutual coupling between transmitting and receiving slots on the cone is formulated and methods of calculation are discussed.

Extensions of the technique to more general situations are discussed.

APPENDIX A

THE K-L TRANSFORM

The K-L transform can be written as

$$f(x) = \frac{1}{\pi i}\int_{-i\infty}^{i\infty} \nu I_\nu(x)\,d\nu \int_0^\infty f(\xi) K_\nu(\xi)\,d\xi/\xi \tag{A1}$$

This is equivalent to the transform pair

$$f(x) = \frac{1}{\pi i}\int_{-i\infty}^{i\infty} F(\nu) I_\nu(x)\,\nu\,d\nu \tag{A2a}$$

$$F(\nu) = \int_0^\infty f(x) K_\nu(x)\,dx/x \tag{A2b}$$

Since $K_\nu(x)$ is an even function of ν, it follows from (A2b) that $F(\nu)$ likewise is an even function. $F(\nu)$ must be an analytic function of ν in a strip of finite width $-\delta \le \mathrm{Re}\,\nu \le \delta$, $\delta > 0$.

By setting $f(x) = \delta(x-x_0)$ in (A2b) and then inserting the value of $F(\nu)$ in (A2a), we obtain

$$x_0\,\delta(x-x_0) = \frac{1}{\pi i}\int_{-i\infty}^{i\infty} I_\nu(x) K_\nu(x_0)\,\nu\,d\nu \tag{A3}$$

A second δ-function relation obtainable from (A2a,b) is

$$\nu^{-1}\delta(\nu-\mu) = \frac{1}{\pi i}\int_0^\infty I_\nu(x) K_\mu(x)\,dx/x = \frac{1}{\pi i}\int_0^\infty I_\mu(x) K_\nu(x)\,dx/x \tag{A4}$$

By replacing x by γR, the transform pair (A2) is frequently written in the form

$$\begin{aligned} f(R) &= \frac{1}{\pi i}\int_{-i\infty}^{i\infty} F(\nu)\, I_\nu(\gamma R)\,\nu\,d\nu \\ F(\nu) &= \int_0^\infty f(R) K_\nu(\gamma R)\,dR/R \end{aligned} \tag{A5}$$

In terms of the spherical functions

$$i_\nu(\gamma R) = R^{1/2}\, I_\nu(\gamma R)$$

$$k_\nu(\gamma R) = R^{1/2}\, K_\nu(\gamma R)$$

(A5) becomes

$$\begin{aligned} \tilde{f}(R) &= \frac{1}{\pi i}\int_{-i\infty}^{i\infty} F(\nu)\, i_\nu(\gamma R)\,\nu\,d\nu \\ F(\nu) &= \int_0^\infty \tilde{f}(R)\, k_\nu(\gamma R)\,dR/R^2 \end{aligned} \tag{A6}$$

and (A3) and (A4) become

$$R_0^2\,\delta(R-R_0) = \frac{1}{\pi i}\int_{-i\infty}^{i\infty} i_\nu(\gamma R) k_\nu(\gamma R_0)\,\nu\,d\nu = \frac{1}{\pi i}\int_{-i\infty}^{i\infty} i_\nu(\gamma R_0) k_\nu(\gamma R)\,\nu\,d\nu \tag{A3a}$$

$$\nu^{-1}\delta(\nu-\mu) = \int_0^\infty i_\mu(\gamma R) k_\nu(\gamma R)\,dR/R^2 = \int_0^\infty i_\nu(\gamma R) k_\mu(\gamma R)\,dR/R^2 \tag{A4a}$$

respectively.

(A5) is equivalent to the form originally given by Kantorovich and Lebedev [Ref. 6]. A fact which is not generally realized (again stemming from the treatment in Ref. 6) is that in (A5) γ is to be considered as real. Actually, the proper way to write the second equation of (A5) is

$$F(\nu) = \int_0^\infty f(R)\, K_\nu(\gamma R) \frac{d(\gamma R)}{\gamma R} = \int_0^{\infty \exp(-i \arg \gamma)} f(R)\, K_\nu(\gamma R) \frac{dR}{R} \qquad \text{(A5a)}$$

That γ is to be considered as real in (A5) is forcibly brought out by analyzing in detail the manner in which the δ-function properties

$$\delta(R-R_0) = \begin{cases} 0 & R \neq R_0 \\ \infty & R = R_0 \end{cases}$$
$$\int_{R_1}^{R_2} \delta(R-R_0)\, dR = 1 \qquad R_1 < R_0 < R_2 \qquad \text{(A7)}$$

are displayed by the integral representation

$$R_0\, \delta(R-R_0) = \frac{1}{\pi i} \int_{-i\infty}^{i\infty} I_\nu(\gamma R)\, K_\nu(\gamma R_0)\, \nu\, d\nu \qquad \text{(A8)}$$

In (A8), replace $K_\nu(\gamma R_0)$ by

$$K_\nu(\gamma R_0) = \frac{\pi}{2} \; \frac{I_{-\nu}(\gamma R_0) - I_\nu(\gamma R_0)}{\sin \nu\pi}$$

Then the integrand has the asymptotic behavior

$$\nu I_\nu(\gamma R) K_\nu(\gamma R_0) \sim \frac{1}{2}\left(\frac{R}{R_0}\right)^\nu - \frac{1}{4 \sin \nu\pi} \exp\{\pm[\ln(\gamma^2 R R_0/4) + 2 - 2\ln \nu]\} \qquad |\arg \nu| > 0 \qquad \text{(A9)}$$

The first term of (A9) is bounded as $|\nu| \to \infty$ along the imaginary axis. If we write $\gamma = \rho e^{i\varphi}$, then the magnitude of the second term of (A8) along the imaginary axis is

$$\frac{1}{2} \exp(-i 2\nu\varphi)$$

The exponent thus is positive real along either the upper or lower half of the imaginary axis, depending on the sign of φ. Thus, <u>the second term of (A9) is not bounded along the imaginary axis unless $\varphi = 0$; i.e., unless γ is real.</u> If $\varphi = 0$, the magnitude of the second term $\to \frac{1}{2}$. Then the integrand of (A8) is bounded along the imaginary axis. The second term vanishes on an infinite semicircle in the right half-plane; the first term vanishes along an infinite semicircle in the right half-plane if $R < R_0$, in the left half-plane if $R > R_0$. Since the integrand has no poles, the integral vanishes in either case.

But if $R = R_0$, the first term of the integrand does not vanish on an infinite semicircle in either half-plane. Consequently, the integral is unbounded for $R = R_0$. But if an integration over R which straddles R_0 is first performed, a factor $(\nu+1)^{-1}$ is acquired in the first term of (A9) which introduces a convergence factor as well as a pole at $\nu = -1$. Then the integrand vanishes on an infinite semicircle; for $R_2 (>R_0)$ the integral in the left half-plane encloses the pole, and it is easily shown that the residue is R_0. The same result is obtained if an integration is performed over R_0 over a range which straddles R. Thus the δ-function properties (A7) are all properly exhibited if γ is real.

The integration over R over a range straddling R_0, as sketched above, also provides the means whereby the assumption that γ is real is justified.

It is merely necessary to integrate R along a contour rotated by the angle $-\phi$, as in (A5a), which makes γR real. This is merely another aspect of the δ-function property of (A3); i.e., it does not make "sense" mathematically until an integration over R (or R_0) is performed. In the case of the K-L transform of a general $f(R)$, the justification for considering γ as real is performed by the inverse transform (A5a).

APPENDIX B

SOLUTION OF THE BASIC INTEGRAL EQUATION

The basic integral equation is of the form

$$\int_{C_0} A(\nu)\, i_\nu(x)\, \nu\, d\nu = \int_{C_0} B(\nu)\, i_\nu(\rho x)\, \nu\, d\nu \tag{B1}$$

where $A(\nu)$ and $B(\nu)$ are even functions of ν, and contour C_0 is the imaginary axis of the ν-plane. It will be assumed that ρ is real, in order to avoid convergence difficulties in later inversions of the integral equation derived from (B1). This can be justified in the same way as in Appendix A, by noting that eventually integration over the source coordinate will be required. Hence in the ultimate integration over the source, the path need merely be rotated so that ρx is real.

$i_\nu(\rho x)$ in the right-hand integral of (B1) is now expanded [Ref. 7] as the series in $i_{\nu+2m}(x)$:

$$i_\nu(\rho x) = x^{1/2}\, I_\nu(\rho x) = x^{1/2} \rho^\nu \sum_{m=0}^{\infty} \frac{\nu+2m}{\nu}\, c_m(\nu,\rho)\, I_{\nu+2m}(x) = \rho^\nu \sum_{m=0}^{\infty} \frac{\nu+2m}{\nu}\, c_m(\nu,\rho)\, i_{\nu+2m}(x) \tag{B2}$$

where the coefficients $c_m(\nu,\rho)$ are polynomials in ρ^2

$$c_m(\nu,\rho) = (-)^m \frac{\nu\Gamma(\nu+m)}{m!\,\Gamma(\nu+1)}\, {}_2F_1(\nu+m, -m; \nu+1; \rho^2) = \sum_{j=0}^{m} \frac{\nu\Gamma(\nu+j+m)}{\Gamma(\nu+1+j)} \frac{(-)^{m-j}}{(m-j)!} \frac{\rho^{2j}}{j!} \tag{B3}$$

The following properties of these coefficients are important for later developments:

By writing

$$\delta = 1 - \rho^2 \tag{B4}$$

(B3) may be expressed in the form

$$c_m(\nu,\rho) = \sum_{r=0}^{\infty} c_m^r(\nu)\, \delta^r \tag{B3a}$$

where

$$\left.\begin{aligned} c_m^r(\nu) &= \frac{(-)^r (m-1)!}{r!\,(r-1)!\,(m-r)!} \frac{\nu\Gamma(\nu+m+r)}{\Gamma(\nu+1+m)} \\ c_0(\nu) &= c_0^0(\nu) = 1 \\ c_m^0(\nu) &= 0 \\ c_m^1(\nu) &= -\nu \end{aligned}\right\} m>0 \tag{B3b}$$

Thus the first term in the expansion (B2) is $O(1)$, while the remaining terms are $O(\delta)$.

With (B2), (B1) becomes

$$\int_{C_0} A(\nu)\, i_\nu(x)\, \nu\, d\nu = \int_{C_0} \rho^\nu B(\nu) \sum_{m=0}^{\infty} c_m(\nu,\rho)\, (\nu+2m)\, i_{\nu+2m}(x)\, d\nu \tag{B5}$$

Taking the K-L transform of (B5), by multiplying by $k_\mu(x)\,dx/x^2$ and integrating over 0 to ∞, we get

$$A(\mu) = \frac{1}{\pi i} \int_{C_1} B(\nu)\, \mathcal{K}(\mu,\nu;\rho)\, d\nu \tag{B6}$$

where μ is a typical point on the imaginary axis, C_1 is a contour parallel to the imaginary axis to the right of μ, as shown in Fig. B1,

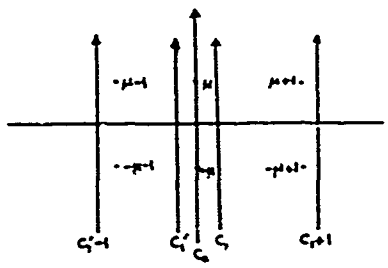


Fig. B1

and $\mathcal{K}(\nu,\mu;\rho)$ is given by

$$\mathcal{K}(\mu,\nu;\rho) = \sum_{m=0}^{\infty} \rho^{\nu} \frac{(\nu+2m)\, c_m(\nu,\rho)}{(\nu+2m)^2 - \mu^2} = \sum_{m=0}^{\infty} \frac{1}{2} \rho^{\nu} c_m(\nu,\rho) \left(\frac{1}{\nu+2m-\mu} + \frac{1}{\nu+2m+\mu} \right) \tag{B7}$$

Alternatively, instead of expanding $i_\nu(\rho x)$ in (B1) in terms of $i_{\nu+2m}(x)$, we can expand $i_\nu(x)$ in terms of a series in $i_{\nu+2m}(\rho x)$. This can be done by replacing x in (B1) by y/ρ and using (B2):

$$i_\nu(x) = i_\nu(y/\rho) = \rho^{-\nu} \sum_{m=0}^{\infty} c_m(\nu, 1/\rho)\, \frac{\nu+2m}{\nu}\, i_{\nu+2m}(y) \tag{B8}$$

Then (B1) becomes

$$\int_{C_0} B(\nu)\, i_\nu(y)\, \nu\, d\nu = \int_{C_0} \rho^{-\nu} A(\nu) \sum_{m=0}^{\infty} c_m(\nu, 1/\rho)\, (\nu+2m)\, i_{\nu+2m}(y)\, d\nu \tag{B9}$$

The transform of (B9) then is

$$B(\mu) = \frac{1}{\pi i} \int_{C_1} A(\nu)\, \mathcal{K}(\mu,\nu;1/\rho)\, d\nu \tag{B10}$$

Thus <u>(B10) is the inversion of (B6), and vice versa</u>. $\mathcal{K}(\mu,\nu;1/\rho)$ is given by (B7) with the replacement of ρ by $1/\rho$.

It should be noted that $\mathcal{K}(\mu,\nu;\rho)$ is equivalent to the integral

$$\mathcal{K}(\mu,\nu;\rho) = \nu \int_0^{\infty} i_\nu(\rho x)\, k_\mu(x)\, dx/x^2 \tag{B11}$$

Similarly,

$$\mathcal{K}(\mu,\nu;1/\rho) = \nu \int_0^{\infty} i_\nu(x)\, k_\mu(\rho x)\, dx/x^2 \tag{B12}$$

In Sec. 3.3, the related function $\mathcal{K}^{\pm\mu,\pm\nu}$ is encountered. This is defined by

$$\mathcal{K}^{\pm\mu,\pm\nu} = \frac{1}{2} \Big\{ \frac{\nu+1/2}{\nu-1} \mathcal{K}(\mu,\nu-1;\rho) + \frac{\nu-1/2}{\nu+1} \mathcal{K}(\mu,\nu+1;\rho)$$
$$- \frac{\nu}{\rho} \Big[\frac{\mu-1/2}{\mu} \mathcal{K}(\mu+1,\nu;\rho) + \frac{\mu+1/2}{\mu} \mathcal{K}(\mu-1,\nu;\rho) \Big] \Big\}$$

Using (B7) for the $\mathcal{H}$-functions, it is obvious from (B3a) and (B3b) that only the $m = 0$ terms are $O(1)$, the remaining terms being $O(\delta)$. Separating off the $m = 0$ terms, we have, since $c_0(\nu,\rho) = i$,

$$[\mathcal{H}^{\pm\mu,\pm\nu}]_{m=0} = \frac{1}{4}\left\{ \frac{\nu+\frac{1}{2}}{\nu-1}\rho^{\nu-1}\left(\frac{1}{\nu-1-\mu}+\frac{1}{\nu-1+\mu}\right) + \frac{\nu-\frac{1}{2}}{\nu+1}\rho^{\nu-1}\left(\frac{1}{\nu+1-\mu}+\frac{1}{\nu+1+\mu}\right) - \frac{\tau_\nu^\nu}{\rho}\left[\frac{\mu-\frac{1}{2}}{\mu}\rho^\nu\left(\frac{1}{\nu-1-\mu}+\frac{1}{\nu+1+\mu}\right) + \frac{\mu+\frac{1}{2}}{\mu}\left(\frac{1}{\nu+1-\mu}+\frac{1}{\nu-1+\mu}\right)\right]\right\} \tag{B13}$$

In a typical integral

$$\int_{C_1+1} F(\nu)\mathcal{H}^{\pm\mu,\pm\nu}\,d\nu$$

where $F(\nu)$ is an even function, we can change the sign of ν in the second term of each pair of terms in large parentheses in (B13), for which the contour becomes $C_1'-1$ (see Fig. B1). A shift of contour from $C_1'-1$ to C_1+1 allows these terms to be combined with the first term of each pair in (B13), and also collects residues at the poles $\nu = 1\pm\mu$. The residues cancel, leaving

$$\int_{C_1+1} F(\nu)[\mathcal{H}^{\pm\mu,\pm\nu}]_{m=0}\,d\nu = \frac{1}{4}\int_{C_1}\left\{\frac{1}{\nu-1-\mu}\left[\frac{\nu+\frac{1}{2}}{\nu-1}\left(\rho^{\nu-1}-\rho^{-(\nu-1)}\right) - \frac{\tau_\nu^\nu}{\rho}\frac{\mu-\frac{1}{2}}{\mu}\left(\rho^\nu-\rho^{-\nu}\right)\right] + \frac{1}{\nu+1-\mu}\left[\frac{\nu-\frac{1}{2}}{\nu+1}\left(\rho^{\nu+1}-\rho^{-(\nu+1)}\right) - \frac{\tau_\nu^\nu}{\rho}\frac{\mu+\frac{1}{2}}{\mu}\left(\rho^\nu-\rho^{-\nu}\right)\right]\right\}F(\nu)\,d\nu \tag{B14}$$

Since

$$\rho^\nu - \rho^{-\nu} = (1-\delta)^{\nu/2} - (1-\delta)^{-\nu/2} = -\nu\delta + O(\delta^2)$$

each of the terms of the integrand of (B14) is $O(\delta)$. Hence integrals containing $\mathcal{H}^{\pm\mu,\pm\nu}$ are $O(\delta)$.

In subsequent developments, we encounter double integrals of the type

$$d(\mu) = \frac{1}{\pi i}\int_{C} \mathcal{H}(\mu,\nu;1/\rho)\,d\nu \int_{C_1+1} M(\nu,\lambda)\,\mathcal{H}^{\pm\nu,\pm\lambda}\,d\lambda \tag{B15}$$

where $M(\nu,\lambda)$ is an even function of both ν and λ. Since it has just been shown that the inner integration yields a result which is $O(\delta)$, this result may be represented as

$$\delta E(\nu)$$

where $E(\nu)$ is an even function of ν. Hence, if we wish to evaluate $d(\mu)$ to $O(\delta)$, we need be concerned only with the first term of the series representing $\mathcal{H}(\mu,\nu;1/\rho)$, since the coefficients of all higher terms of the series are all $O(\delta)$ (see (B3b)). Thus we have

$$\begin{aligned} d(\mu) &= \frac{\delta}{2\pi i}\int_{C_1} \rho^{-\nu}E(\nu)\left(\frac{1}{\nu-\mu}+\frac{1}{\nu+\mu}\right)d\nu + O(\delta^2) \\ &= \frac{\delta}{2\pi i}\left(\int_{C_1}\frac{\rho^{-\nu}E(\nu)}{\nu-\mu}\,d\nu - \int_{C_1'}\frac{\rho^{\nu}E(\nu)}{\nu+\mu}\,d\nu\right) + O(\delta^2) \end{aligned}$$

Since

$$\rho^\nu = 1 - \frac{\nu}{2}\delta + O(\delta^2)$$

we find

$$d(\mu) = \frac{\delta}{2\pi i}\left(\int_{C_1} - \int_{C_1'}\right)\frac{E(\nu)}{\nu-\mu}\,d\nu + O(\delta^2) = \delta E(\mu) + O(\delta^2)$$

Consequently, to $O(\delta)$ (B15) may be written as

$$\partial(\mu) = \int_{C_0} M(\mu,\lambda)\mathcal{H}^{(\mu,\lambda)}\,d\lambda + O(\delta^2) \tag{B16}$$

Similarly, for integrals of the type

$$\partial(\mu) = \frac{1}{\pi i}\int_{C_0}\mathcal{H}(\mu,\nu;\tfrac{1}{2}\rho)\,d\nu\int_{C_1} M_1(\nu,\lambda)\mathcal{H}(\nu,\lambda;\rho)\,d\lambda \tag{B17}$$

where $M_1(\nu,\nu) = 0$, so that the inner integral is $O(\delta)$, we obtain

$$\partial(\mu) = \int_{C_0} M_1(\mu,\lambda)\mathcal{H}(\mu,\lambda;\rho)\,d\lambda + O(\delta^2) \tag{B18}$$

APPENDIX C

K-L TRANSFORM OF DERIVATIVE

Given

$$f(r) = \frac{1}{\pi i}\int_{-i\infty}^{i\infty} F(\nu)\, I_\nu(\gamma r)\,\nu\, d\nu \qquad \text{(C1)}$$

we seek the transform representation of $f'(r) = \frac{d}{dr}f(r)$. That is, if $f'(r)$ is represented as

$$f'(r) = \frac{1}{\pi i}\int_{-i\infty}^{i\infty} F_1(\nu)\, I_\nu(\gamma r)\,\nu\, d\nu$$

what is the form of $F_1(\nu)$?

Differentiating (C1), we obtain

$$f'(r) = \frac{1}{\pi i}\int_{-i\infty}^{i\infty} F(\nu)\frac{d}{dr} I_\nu(\gamma r)\,\nu\, d\nu = \frac{1}{\pi i}\int_{-i\infty}^{i\infty} \frac{\gamma}{2} F(\nu)\left[I_{\nu+1}(\gamma r) + I_{\nu-1}(\gamma r)\right]\nu\, d\nu$$

In the first term, replace $\nu+1$ by ν, in the second, $\nu-1$ by ν. Then

$$f'(r) = \frac{\gamma}{2\pi i}\left[\int_{1-i\infty}^{1+i\infty} F(\nu-1)\, I_\nu(\gamma r)(\nu-1)\, d\nu + \int_{-1-i\infty}^{-1+i\infty} F(\nu+1)\, I_\nu(\gamma r)(\nu+1)\, d\nu\right]$$

Now shift the contour in the first integral to the right, in the second integral to the left, to the imaginary axis. Then, <u>providing F(ν) has no singularities in the strip</u>

$$-(1+\epsilon) < \operatorname{Re}\nu < 1+\epsilon \qquad \epsilon > 0$$

we obtain

$$f'(r) = \frac{1}{\pi i}\int_{-i\infty}^{i\infty} \frac{1}{2\nu}\left[(\nu-1)F(\nu-1) + (\nu+1)F(\nu+1)\right]\nu\, d\nu$$

Hence

$$F_1(\nu) = \frac{\gamma}{2\nu}\left[(\nu-1)F(\nu-1) + (\nu+1)F(\nu+1)\right] \qquad \text{(C2)}$$

Similarly, for the transform pair

$$f(r) = \frac{1}{\pi i}\int_{-i\infty}^{i\infty} G(\nu)\, i_\nu(\gamma r)\,\nu\, d\nu$$

$$G(\nu) = \int_0^\infty f(r)\, k_\nu(\gamma r)\, dr/r^2 \qquad \text{(C3)}$$

in virtue of the relation

$$\frac{d}{dr} i_\nu(\gamma r) = \frac{\gamma}{2\nu}\left[(\nu+1/2)\, i_{\nu-1}(\gamma r) + (\nu-1/2)\, i_{\nu+1}(\gamma r)\right] \qquad \text{(C4)}$$

we obtain

$$f'(r) = \frac{1}{\pi i}\int_{-i\infty}^{i\infty} G_1(\nu)\, i_\nu(\gamma r)\,\nu\, d\nu \qquad \text{(C5)}$$

where

$$G_1(\nu) = \frac{\gamma}{2\nu}\left[(\nu-3/2)\, G(\nu-1) + (\nu+3/2)\, G(\nu+1)\right] \qquad \text{(C6)}$$

providing $G_1(\nu)$ has no singularities in the strip.

$$-(1+\epsilon) < \mathrm{Re}\,\nu < 1+\epsilon \qquad \epsilon > 0$$

APPENDIX D

ZEROS OF THE LEGENDRE FUNCTION

The purpose of this program is to compute the zeros, ν_0, of the associated Legendre function of the first kind, $P_\nu^{-m}(\cos\theta)$, as a function of degree ν, and the derivative $\frac{\partial}{\partial\nu}P_\nu^{-m}(\cos\theta)$ at ν_0. A method of computing these zeros was described by Wilcox* of the University of Michigan. However, a different method is employed here, which is more accurate and faster when θ is close to 180 degrees, which is the region of interest here. Double precision arithmetic is used in order to achieve the desired accuracy.

The following equation was used to evaluate the associated Legendre function:

$$P_\nu^{-m}(\cos\theta) = -\frac{\sin\nu\pi}{\pi}\tan^m\frac{\theta}{2}\left\{\sum_{n=0}^{m-1}(-1)^n\frac{\Gamma(n-\nu)\Gamma(n+1+\nu)}{\Gamma(m-\nu)\Gamma(m+1+\nu)}\frac{(m-n-1)!}{n!}\cos^{2n}\frac{\theta}{2}\right.$$
$$\left. + (-1)^m\sum_{n=0}^{\infty}\frac{\Gamma(n+m-\nu)\Gamma(n+m+1+\nu)}{\Gamma(m-\nu)\Gamma(m+1+\nu)\,n!\,(m+n)!}\cos^{2(m+n)}\frac{\theta}{2}\left[h_n-\log\cos^2\frac{\theta}{2}\right]\right\} \tag{D1}$$

Then $\frac{\partial}{\partial\nu}P_\nu^{-m}(\cos\theta)$ is given by:

$$\frac{\partial}{\partial\nu}P_\nu^{-m}(\cos\theta) = \pi\cot\nu\pi\, P_\nu^{-m}(\cos\theta) - \frac{1}{\pi}\sin\nu\pi\tan^m\frac{\theta}{2}\left\{\sum_{n=0}^{m-1}(-1)^n\frac{\Gamma(n-\nu)\Gamma(n+1+\nu)}{\Gamma(m-\nu)\Gamma(m+1+\nu)}\,g_n\frac{(m-n-1)!}{n!}\cos^{2n}\frac{\theta}{2}\right.$$
$$\left. + (-1)^m\sum_{n=0}^{\infty}\frac{\Gamma(n+m-\nu)\Gamma(n+m+1+\nu)}{\Gamma(m-\nu)\Gamma(m+1+\nu)\,n!\,(m+n)!}\cos^{2(m+n)}\frac{\theta}{2}\left[\left(h_n-\log\cos^2\frac{\theta}{2}\right)k_n+k_n'\right]\right\} \tag{D2}$$

In these equations,

$$g_n = \psi(n+1+\nu) - \psi(n-\nu) + \psi(m-\nu) - \psi(m+1+\nu) \tag{D3}$$

$$h_n = \psi(n+m+1+\nu) - \psi(n+m-\nu) + \psi(m-\nu) - \psi(m+1+\nu) \tag{D4}$$

$$k_n = \psi(n+1) + \psi(n+m+1) - \psi(n+m-\nu) - \psi(n+m+1+\nu) \tag{D5}$$

$$k_n' = \psi'(n+m-\nu) - \psi'(n+m+1+\nu) \tag{D6}$$

The first term in the derivative vanishes at ν_0, of course. However, the complete expression is needed in the iteration process to refine successive trial values of the zero.

In order to take advantage of the expressions which are common to $P_\nu^{-m}(\cos\theta)$ and $\frac{\partial}{\partial\nu}P_\nu^{-m}(\cos\theta)$, the arrays ZRA, ZRB, ZKL are set up. These are defined as follows:

$$ZRA_n = (-1)^n\frac{\Gamma(n-\nu)\Gamma(n+1+\nu)}{\Gamma(m-\nu)\Gamma(m+1+\nu)}\frac{(m-n-1)!}{n!}\cos^{2n}\frac{\theta}{2} \tag{D7}$$

$$ZRB_n = \frac{\Gamma(n+m-\nu)\Gamma(n+m+1+\nu)}{\Gamma(m-\nu)\Gamma(m+1+\nu)\,n!\,(m+n)!}\cos^{2(m+n)}\frac{\theta}{2} \tag{D8}$$

$$ZKL_n = h_n - \log\cos^2\frac{\theta}{2} \tag{D9}$$

In terms of these quantities,

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* PETER H. WILCOX, "The Zeros of $P_\nu^m(\cos\theta)$ and $(\partial/\partial\theta)P_\nu^m(\cos\theta)$," Mathematics of Computation, Vol. 22, No. 101, Jan 1968.

$$P_\nu^{-m}(\cos\theta) = -\frac{1}{\pi}\sin\nu\pi\,\tan^m\frac{\theta}{2}\left[\sum_{n=0}^{m-1} ZRA_n + (-)^m \sum_{n=0}^{\infty} ZRB_n\,ZKL_n\right] \qquad \text{(D10)}$$

and

$$\frac{\partial}{\partial\nu}P_\nu^{-m}(\cos\theta) = -\frac{1}{\pi}\sin\nu\pi\,\tan^m\frac{\theta}{2}\left[\sum_{n=0}^{m-1} ZRA_n\,g_n + (-)^m \sum_{n=0}^{\infty} ZRB_n\,(ZKL_n\,h_n + k_n')\right] + \pi\cot\nu\pi\;P_\nu^{-m}(\cos\theta) \qquad \text{(D11)}$$

The ratios of the gamma functions in ZRA and ZRB are computed using the recurrence relationship, so that

$$\frac{\Gamma(n-\nu)\,\Gamma(n+1+\nu)}{\Gamma(m-\nu)\,\Gamma(m+1+\nu)} = 1\Big/\prod_{\ell=1}^{m-n}(m-\ell-\nu)(m-\ell+1+\nu) \qquad \text{(D12)}$$

and

$$\frac{\Gamma(n+m-\nu)\,\Gamma(n+m+1+\nu)}{\Gamma(m-\nu)\,\Gamma(m+1+\nu)} = \prod_{\ell=1}^{n}(m+\ell+\nu)(m+\ell-1-\nu) \qquad \text{(D13)}$$

Subroutine XP computes $P_\nu^{-m}(\cos\theta)$, and in the process computes the arrays ZRA, ZRB, and ZKL by calling other subroutines. XP calls three other subroutines, RA to compute the ZRA array, RB to compute the ZRB array, and EK to compute k_n, which is needed for the ZKL array.

Subroutine XPPR computes $\frac{\partial}{\partial\nu}P_\nu^{-m}(\cos\theta)$ using the arrays ZRA, ZRB, and ZKL. It calls three other subroutines, EKP to compute k_n', G to compute g_n, and H to compute h_n.

The EK, G, and H subroutines, which compute k_n, g_n, and h_n, respectively, all call the PSA subroutine, which computes the ψ function. The EKP function, which computes k_n, calls the PSB subroutine to obtain the ψ' function.

The ψ function is computed by using the recursion formula to step the argument upward until it is greater than 10, and then using the asymptotic formula. Thus the formula used is

$$\psi(z) = \log(z+n) - \frac{1}{z} - \frac{1}{z+1} - \cdots - \frac{1}{z+n-1} - \frac{1}{2(z+n)} - \sum_{m=1}^{5} B_m'(z+n)^{-2m}$$
$$z+n > 10, \quad B_m' = B_{2m}/2m \qquad \text{(D14)}$$

where the B_{2m} are the Bernoulli numbers.

Similarly, the ψ' function is computed using the formula

$$\psi'(z) = \frac{1}{z+n} + \frac{1}{z^2} + \frac{1}{(z+1)^2} + \cdots + \frac{1}{(z+n+1)^2} + \frac{1}{2(z+n)^2} + \sum_{m=1}^{5} B_m''(z+n)^{-2m-1}$$
$$z+n > 10, \quad B_m'' = 2m\,B_m' = B_{2m} \qquad \text{(D15)}$$

To compute the derivatives with respect to θ, the recurrence relations are used.

$$\frac{\partial}{\partial\theta}P_\nu^{-m}(\cos\theta) = \frac{1}{\sin\theta}\left[\nu\cos\theta\,P_\nu^{-m}(\cos\theta) - (\nu-m)P_{\nu-1}^{-m}(\cos\theta)\right] \qquad \text{(D16)}$$

and

$$\frac{\partial^2}{\partial\theta\,\partial\mu} P_\mu^{-m}(\cos\theta) = \frac{1}{\sin\theta}\left[\mu\cos\theta\,\frac{\partial}{\partial\mu}P_\mu^{-m}(\cos\theta) - (\mu-m)\frac{\partial}{\partial\mu}P_{\mu-1}^{-m}(\cos\theta) + \cos\theta\,P_\mu^{-m}(\cos\theta) - P_{\mu-1}^{-m}(\cos\theta)\right] \quad \text{(D17)}$$

In the actual computations, the first values computed are

$$\frac{\partial}{\partial\theta} P_\nu^{-m}(\cos\theta)/(-AB/\sin\theta) \qquad \frac{\partial^2}{\partial\theta\,\partial\mu} P_\nu^{-m}(\cos\theta)/(-AB/\sin\theta)$$

where

$$-AB = -(\tan^m \tfrac{\theta}{2}\,\sin\nu\pi)/\pi$$

Then in (D16) the following is used

$$-\frac{1}{\sin\nu\pi}(\nu-m)\,P_{\nu-1}^{-m}(\cos\theta) = \frac{1}{\sin(\nu-1)\pi}(\nu-m)\,P_{\nu-1}^{-m}(\cos\theta)$$

and similarly for the corresponding term in (D17).

The program consists of a main program and ten subroutines. All routines are written in Fortran IV.

The main program first reads a single control card which contains the parameters for the first set of zeros. There will be one additional control card for each additional set of zeros. The program terminates by reading an end of file while attempting to read another control card. A description of this control card is given on page 67.

The zeros of $P_\nu^{-m}(\cos\theta)$ are denoted in the program by Nu, and the zeros of $\frac{\partial}{\partial\theta}P_\nu^{-m}(\cos\theta)$ by Mu. The program uses an initial (estimated) value of Nu from the control card to compute the first values of $P_\nu^{-m}(\cos\theta)$ and $\frac{\partial}{\partial\nu}P_\nu^{-m}(\cos\theta)$. The derivative is then used in the following way to obtain a second value of Nu which gives a $P_\nu^{-m}(\cos\theta)$ with a smaller absolute value:

$$\nu_k = \nu_{k-1} - P_{\nu_{k-1}}^{-m}\Big/\left(\frac{\partial}{\partial\nu}P_\nu^{-m}\right)_{\nu=\nu_{k-1}} \quad \text{(D18)}$$

This process is continued until $P_\nu^{-m}(\cos\theta)$ is sufficiently small. The criterion of smallness is selectable by means of a tolerance, which has been set at 10^{-10}. An additional tolerance of 10^{-8} for the second term in (D18) is also included. This insures that 8-decimal accuracy is obtained for the roots. These tolerances are arbitrary and could be changed if a different accuracy is desired. Usually only a few iterations are required to obtain ten significant figures of accuracy.

In a similar way, using an initial value of Mu supplied by the control card, a value of Mu is found for which $\frac{\partial}{\partial\theta}P_\mu^{-m}(\cos\theta)$ is sufficiently small.

The four values for the first zero, ν_0, $\frac{\partial}{\partial\nu}P_{\nu_0}^{-m}$, μ_0, and $\frac{\partial^2}{\partial\nu\partial\theta}P_{\mu_0}^{-m}$ are stored, and the next zero is then processed. The initial values of Nu and Mu for the next zero are obtained from the final values of Nu and Mu for the zero which has just been processed. π/θ is used as an approximation for the difference between consecutive zeros for both Nu and Mu.

When all the zeros have been processed that were called for in the control card, a page of output is created containing a tabular listing of the four values corresponding to each order of zero. The program then attempts to read a new control card. This is the usual procedure. Additional printouts at various stages of the computation are selectable by means of appropriate entries in the control card.

A flow chart of the program is on page 68, and the complete program listing on pages 69 - 73.

CONTROL CARD FORMAT

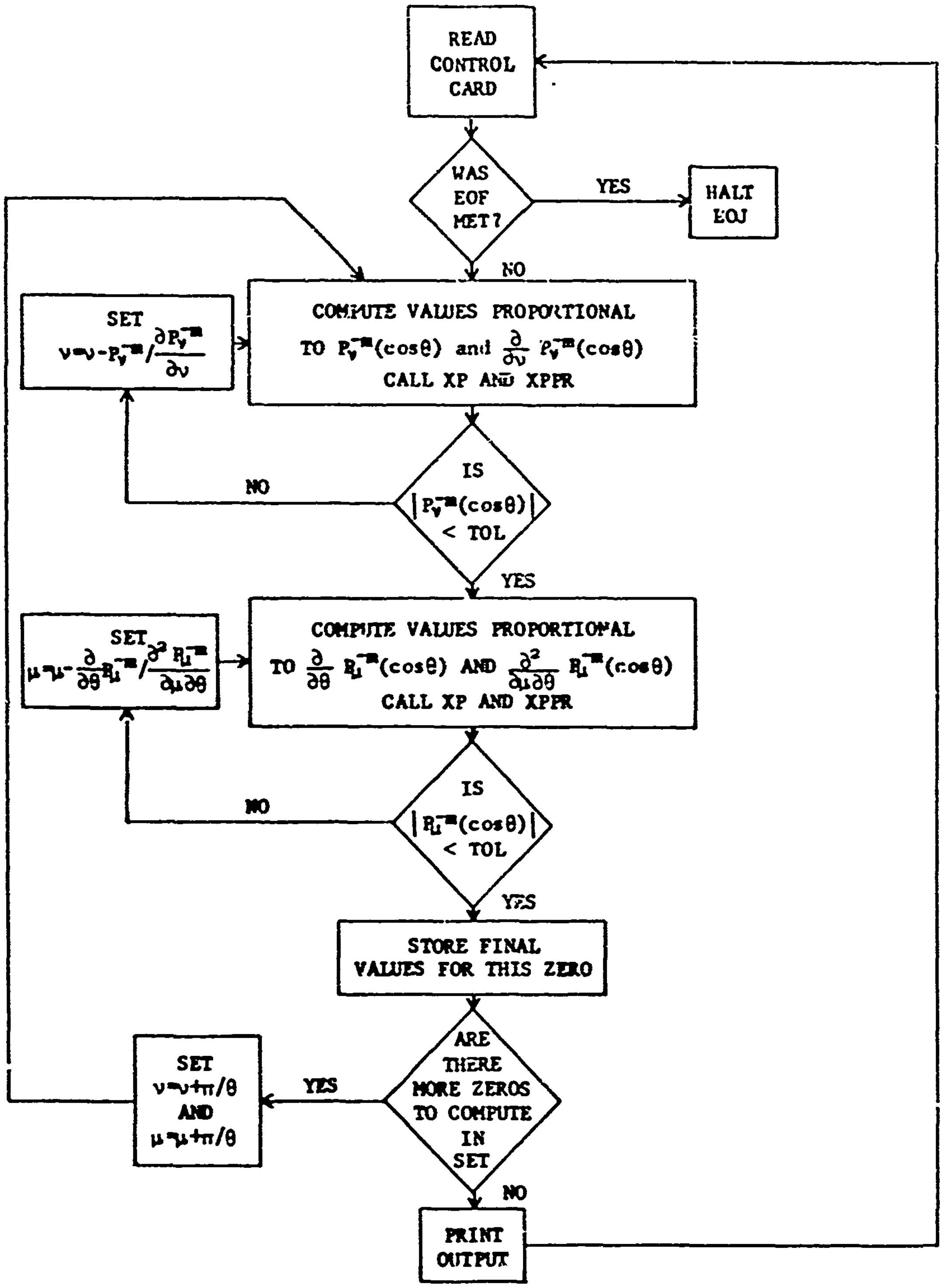
The control card is read by the following Fortran statement:

```
      READ (5,55)    KW,NZ,M,AA,BB,CC
   55 FORMAT (5I1,2I5,3F10.5)
```

KW is a dimensioned variable (DIMENSION KW(5))

The parameters in the control card are described below.

Col. 1	Non-zero means print values of ν, $P_\nu^{-m}(\cos\theta)/(-AB)$, and $\frac{\partial}{\partial\nu} P_\nu^{-m}(\cos\theta)/(-AB)$ for each trial value of ν for each zero.
Col. 2	Non-zero means print final values of ν, $P_\nu^{-m}(\cos\theta)/(-AB)$, and $\frac{\partial}{\partial\nu} P_\nu^{-m}(\cos\theta)$ for each zero.
Col. 3	Non-zero means print values of μ, $\frac{\partial}{\partial\theta} P_\mu^{-m}(\cos\theta)/(-AB/\sin\theta)$, and $\frac{\partial^2}{\partial\mu\partial\theta} P_\mu^{-m}(\cos\theta)/(-AB/\sin\theta)$ for each trial value of μ for each zero.
Col 4	Non-zero means print final values of μ, $\frac{\partial}{\partial\theta} P_\mu^{-m}(\cos\theta)/(-AB/\sin\theta)$, and $\frac{\partial^2}{\partial\mu\partial\theta} P_\mu^{-m}(\cos\theta)$ for each zero.
Col. 5	Non-zero means print a page containing final values of ν, $\frac{\partial}{\partial\nu} P_\nu^{-m}(\cos\theta)$, μ, $\frac{\partial^2}{\partial\mu\partial\theta} P_\mu^{-m}(\cos\theta)$ for each zero in the set. This is the normal production output.
Col. 6-10	Number of zeros in the set to be computed.
Col. 11-15	Order of Legendre function (m in $P_\nu^{-m}(\cos\theta)$).
Col. 16-25	Argument in degrees (θ in $P_\nu^{-m}(\cos\theta)$). This should be greater than 160 and less than 180.
Col. 26-35	First trial value for ν in first zero of the set.
Col. 36-45	First trial value for μ in first zero of the set.


READ CONTROL CARD
WAS EOF MET?
YES
HALT EOJ
NO
SET $\nu = \nu - P_\nu^{-m} / \frac{\partial P_\nu^{-m}}{\partial \nu}$
COMPUTE VALUES PROPORTIONAL TO $P_\nu^{-m}(\cos\theta)$ and $\frac{\partial}{\partial \nu} P_\nu^{-m}(\cos\theta)$ CALL XP AND XPPR
IS $|P_\nu^{-m}(\cos\theta)| <$ TOL
NO
YES
SET $\mu = \mu - \frac{\partial}{\partial\theta} P_\mu^{-m} / \frac{\partial^2 P_\mu^{-m}}{\partial\mu\partial\theta}$
COMPUTE VALUES PROPORTIONAL TO $\frac{\partial}{\partial\theta} P_\mu^{-m}(\cos\theta)$ AND $\frac{\partial^2}{\partial\mu\partial\theta} P_\mu^{-m}(\cos\theta)$ CALL XP AND XPPR
IS $|P_\mu^{-m}(\cos\theta)| <$ TOL
NO
YES
STORE FINAL VALUES FOR THIS ZERO
ARE THERE MORE ZEROS TO COMPUTE IN SET
YES
SET $\nu = \nu + \pi/\theta$ AND $\mu = \mu + \pi/\theta$
NO
PRINT OUTPUT

```
$JOB            178101-A,25,1000,JOHNS
$EXECUTE        IBJOB
$IBJOB
$IBFTC TEST
      DOUBLE PRECISION PI,ANG,TH,ENU,EMU,A,B,S,ST
      DOUBLE PRECISION Q,QPR,ZRA,ZRB,ZKL,XPNU,XPMU
      DOUBLE PRECISION C,CT,EM,CSQ,TOL,P,PA,PPR,PPRA,XM
      DIMENSION ZRA(10),ZRB(100),ZKL(100)
      DIMENSION KW(5)
      DIMENSION XNU(200),XMU(200),XPNU(200),XPMU(200)
  777 CONTINUE
      READ (5,55) KW,NZ,M,AA,BB,CC
   55 FORMAT (5I1,2I5,3F10.5)
      EM=M
      ANG=AA
      ENU=BB
      EMU=CC
      TOL=1.0E-10
      TTOL=1.0E-8
      PI=3.141592653589793
      TH=ANG*PI/180.
      CT=DCOS(TH)
      ST=DSIN(TH)
      C=DCOS(TH/2.)
      S=DSIN(TH/2.)
      CSQ=C*C
      DO 300 J=1,NZ
      N=1
    1 CONTINUE
      A=DSIN(ENU*PI)
      CALL XP(M,ENU,CSQ,P,ZRA,ZRB,ZKL,NN)
      CALL XPPR(M,ENU,CSQ,PPR,ZRA,ZRB,ZKL,NN)
      PPR=PI*P*DCOS(ENU*PI)/A+PPR
      IF (KW(1).EQ.0) GO TO 11
      WRITE (6,66) N,ENU,P,PPR
   11 CONTINUE
      IF (ABS(P).LT.TOL) GO TO 888
      IF (ABS(P/PPR).LT.TTOL) GO TO 888
      N=N+1
      IF (N.GT.10) GO TO 888
      ENU=ENU-P/PPR
      GO TO 1
  888 CONTINUE
      B=((S/C)**M)/PI
      PPR=-B*PPR
      PPR=PPR*A
      XNU(J)=ENU
      XPNU(J)=PPR
      IF (KW(2).EQ.0) GO TO 12
      WRITE (6,66) N,ENU,P,PPR
   66 FORMAT (I10,3D30.16)
   12 CONTINUE
      N=1
    2 CONTINUE
      A=DSIN(EMU*PI)
      CALL XP(M,EMU,CSQ,P,ZRA,ZRB,ZKL,NN)
      CALL XPPR(M,EMU,CSQ,PPR,ZRA,ZRB,ZKL,NN)
      CALL XP(M,EMU-1.,CSQ,PA,ZRA,ZRB,ZKL,NN)
      CALL XPPR(M,EMU-1.,CSQ,PPRA,ZRA,ZRB,ZKL,NN)
      Q=EMU*CT*P+(EMU-EM)*PA
```

```
      QPR=EMU*CT*PPR+(EMU-EM)*PPRA+CT*P*PA
      QPR=Q*PI*DCOS(EMU*PI)/A+QPR
      IF (KW(3).EQ.0) GO TO 13
      WRITE (6,66) N,EMU,Q,QPR
   13 CONTINUE
      IF (ABS(Q).LT.TOL) GO TO 999
      IF (ABS(Q/QPR).LT.TTOL) GO TO 999
      N=N+1
      IF (N.GT.10) GO TO 999
      EMU=EMU-Q/QPR
      GO TO 2
  999 CONTINUE
      QPR=-B*QPR/ST
      QPR=QPR*A
      XNU(J)=EMU
      XPNU(J)=QPR
      IF (KW(4).EQ.0) GO TO 14
      WRITE (6,66) N,EMU,Q,QPR
   14 CONTINUE
      EMU=EMU+PI/TH
      EMU=EMU+PI/TH
  300 CONTINUE
      IF (KW(5).EQ.0) GO TO 15
      WRITE (6,161) AA,M
  161 FORMAT (1H1,20X,20HANGLE IN DEGREES IS ,F9.5,20X,9HORDER IS ,I3)
      WRITE (6,162)
  162 FORMAT (/5X,3HZNO,20X,2HNU,20X,4HP'NU,20X,2HMU,20X,4HP'MU//)
      WRITE (6,163) (I,XNU(I),XPNU(I),XMU(I),XPMU(I),I=1,NZ)
  163 FORMAT (5X,I3,15X,F11.7,11X,D14.7,10X,F11.7,11X,D14.7)
      WRITE (6,164)
  164 FORMAT (1H1)
      WRITE(12,178) NZ,M,AA
  178 FORMAT (2I5,F10.5)
      WRITE(12,177) (I,XNU(I),XPNU(I),XMU(I),XPMU(I),I=1,NZ)
  177 FORMAT (I3,4D17.10)
   15 CONTINUE
      GO TO 777
      END
$IBFTC XP.
      SUBROUTINE XP(M,EMU,CSQ,P,ZRA,ZRB,ZKL,NM)
      DOUBLE PRECISION P,CSQ,EMU,RA,ZRA,ZB,RB,ZRB,EK,ZKL,A,B,BB
      DOUBLE PRECISION TOL
      DIMENSION ZRA(10),ZRB(100),ZKL(100)
      TOL=1.0E-10
      A=0.
      IF (M.LT.1) GO TO 101
      DO 100 NN=1,M
      N=NN-1
      ZRA(NN)=RA(N,M,EMU,CSQ)
      A=A+ZRA(NN)
  100 CONTINUE
  101 CONTINUE
      B=0.
      NN=1
  300 CONTINUE
      N=NN-1
      ZRB(NN)=RB(N,M,EMU,CSQ)
      ZKL(NN)=EK(N,M,EMU)-DLOG(CSQ)
      ZB=ZRB(NN)*ZKL(NN)
      B=B+ZB
```

```
      IF (NN.GT.99) GO TO 201
      IF (ABS(ZB).LT.TOL) GO TO 201
      NN=NN+1
      GO TO 300
  201 CONTINUE
      WRITE (6,66) NN
   66 FORMAT (20H NUMBER OF TERMS IS ,I4)
      BB=(-1)**M
      P=A+BB*B
      RETURN
      END
$IBFTC XPPR.
      SUBROUTINE XPPR(M,ENU,CSQ,PPR,ZRA,ZRB,ZKL,LL)
      DOUBLE PRECISION CSQ,ENU,PPR,ZRA,ZRB,ZKL,ZH,ZEKP,EKP,G,H,B,CA,CB
      DIMENSION ZRA(10),ZRB(100),ZKL(100)
      B=0.
      IF (M.LT.1) GO TO 101
      DO 100 NN=1,M
      N=NN-1
      B=B+ZRA(NN)*G(N,M,ENU)
  100 CONTINUE
  101 CONTINUE
      CA=(-1)**M
      CB=0.
      DO 200 NN=1,LL
      N=NN-1
      ZEKP=EKP(N,M,ENU)
      ZH=H(N,M,ENU)
      CB=CB+ZRB(NN)*(ZKL(NN)*ZH+ZEKP)
  200 CONTINUE
      PPR=B+CA*CB
      RETURN
      END
$IBFTC RA.
      DOUBLE PRECISION FUNCTION RA(N,M,ENU,CSQ)
      DOUBLE PRECISION RA,ENU,CSQ,PROD,EJ,EI,EM,EL
      DOUBLE PRECISION FACT(11)
      DATA (FACT(I),I=1,11)/1.D0,1.D0,2.D0,6.D0,24.D0,
     1   12.D1,72.D1,504.D1,4032.D1,36288.D1,36288.D2/
      EM=M
      K=M-N
      PROD=1.
      IF (K.LT.1) GO TO 101
      DO 100 L=1,K
      EL=L
      PROD=PROD/((EM-EL-ENU)*(EM-EL+1.+ENU))
  100 CONTINUE
  101 CONTINUE
      PROD=PROD*FACT(K)/FACT(N+I)
      PROD=PROD*((-CSQ)**N)
      RA=PROD
      RETURN
      END
$IBFTC RB.
      DOUBLE PRECISION FUNCTION RB(N,M,ENU,CSQ)
      DOUBLE PRECISION RB,ENU,CSQ,A,EK,EJ,EL,EN
      DOUBLE PRECISION FACT(11)
      DATA (FACT(I),I=1,11)/1.D0,1.D0,2.D0,6.D0,24.D0,
     1   12.D1,72.D1,504.D1,4032.D1,36288.D1,36288.D2/
      IF (N.GT.0) GO TO 1
```

```
      RB=CSQ**M/FACT(M+1)
      GO TO 999
    1 EN=N
      EJ=K+N
      EL=F+N-1
      A=EN*EJ
      RB=RB*CSQ*(EJ+ENU)*(EL-ENU)/A
  999 RETURN
      END
$IBFTC PSA.
      DOUBLE PRECISION FUNCTION PSA(Z)
      DOUBLE PRECISION Z,X,B,SUM,PSA
      DIMENSION B(5)
      DATA (B(I),I=1,5)/
     1  0.8333333333333333 D-01,
     2 -0.8333333333333333 D-02,
     3  0.3968253968253968 D-02,
     4 -0.4166666666666667 D-02,
     5  0.7575757575757576 D-02/
      X=Z
      SUM=0.
    1 IF (X.GT.10.) GO TO 10
      SUM=SUM-1./X
      X=X+1.
      GO TO 1
   10 SUM=SUM+DLOG(X)-1./(X+X)
      DO 100 I=1,5
      II=I+I
      SUM=SUM-B(I)/(X**II)
  100 CONTINUE
      PSA=SUM
      RETURN
      END
$IBFTC PSB.
      DOUBLE PRECISION FUNCTION PSB(Z)
      DOUBLE PRECISION Z,X,B,SUM,PSB
      DIMENSION B(5)
      DATA (B(I),I=1,5)/
     1  1.6666666666666667 D-01,
     2 -0.3333333333333333 D-01,
     3  0.2380952380952381 D-01,
     4 -0.3333333333333333 D-01,
     5  0.7575757575757576 D-01/
      X=Z
      SUM=0.
    1 IF (X.GT.10.) GO TO 10
      SUM=SUM+1./(X*X)
      X=X+1.
      GO TO 1
   10 SUM=SUM+1./X+.5/(X*X)
      DO 100 I=1,5
      II=I+I+1
      SUM=SUM+B(I)/(X**II)
  100 CONTINUE
      PSB=SUM
      RETURN
      END
$IBFTC EK.
      DOUBLE PRECISION FUNCTION EK(N,M,ENU)
      DOUBLE PRECISION PSA
```

```
      DOUBLE PRECISION ENU,EK,EN,EM,ZA,ZB,ZC,ZD
      EN=N
      EM=M
      ZA=EN+1.
      ZB=EN+EM+1.
      ZC=EN+EM-ENU
      ZD=EN+EM+1.+ENU
      EK=PSA(ZA)+PSA(ZB)-PSA(ZC)-PSA(ZD)
      RETURN
      END
$IBFTC EKP.
      DOUBLE PRECISION FUNCTION EKP(N,M,ENU)
      DOUBLE PRECISION PSB
      DOUBLE PRECISION EKP,EN,EM,ENU,ZA,ZB,ZC,ZD
      EN=N
      EM=M
      ZC=EN+EM-ENU
      ZD=EN+EM+1.+ENU
      EKP=PSB(ZC)-PSB(ZD)
      RETURN
      END
$IBFTC G.
      DOUBLE PRECISION FUNCTION G(N,M,ENU)
      DOUBLE PRECISION PSA
      DOUBLE PRECISION ENU,G,EN,EM,ZA,ZB,ZC,ZD
      EN=N
      EM=M
      ZA=EN-ENU
      ZB=EN+1.+ENU
      ZC=EM-ENU
      ZD=EM+1.+ENU
      G  =-PSA(ZA)+PSA(ZB)+PSA(ZC)-PSA(ZD)
      RETURN
      END
$IBFTC H.
      DOUBLE PRECISION FUNCTION H(N,M,ENU)
      DOUBLE PRECISION PSA
      DOUBLE PRECISION H,EN,EM,ZA,ZB,ZC,ZD,ENU
      EN=N
      EM=M
      ZA=EN+EM-ENU
      ZB=EN+EM+1.+ENU
      ZC=EM-ENU
      ZD=EM+1.+ENU
      H  =-PSA(ZA)+PSA(ZB)+PSA(ZC)-PSA(ZD)
      RETURN
      END
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13. ABSTRACT

The radiation from slot antennas on a cone in the presence of an inhomogeneous sheath is treated. The sheath is considered as being made up of one or two conical layers, each of which is homogeneous. The boundary conditions lead to a system of integral equations, which number 4M+4 for a sheath composed of M(=1 or 2) conical layers. These are reduced to singular integral equations of Cauchy type, which are solved in iterative fashion. For sufficiently fine stratification of the sheath, the first iteration should suffice.

In general, fields of both magnetic and electric types are generated in the presence of a sheath, even though only a field of magnetic type may be generated in free space. For a ring slot, however, in which the excitation is azimuthally symmetrical, only a field of magnetic type is generated even in the presence of a sheath. It is shown that the solution for this case forms the basis of the solution for the general case.

In general, evaluation of the integrals must be accomplished by contour integration, which leads to series expansions that are not convenient for numerical evaluation. For thin layers, however, Taylor's series expansions allow all but one of the coefficients to be evaluated in closed form.

(continued)

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Missile antenna radiation	8	3				
Plasma sheath	3	3				

ABSTRACT (Continued)

The far field is found by a multidimensional saddle point evaluation. This is illustrated in detail for the free-space case, and then the far field patterns in the presence of a sheath are determined. This can be carried out successfully for all components, and to arbitrary orders of iteration.

The calculation of input admittance and mutual coupling between transmitting and receiving slots on the cone is formulated and methods of calculation are discussed.