DISCRETIZING DYNAMIC PROGRAMS

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I. INTRODUCTION

Except in rare cases, it is necessary to discretize uncountable-state dynamic programs to obtain even an approximate solution. ~~We~~ assume in the first two sections that the state space is *compact*, and so ~~we can construct~~ a finite grid so that any point in the space is in the neighborhood of a grid point. The problem is to find conditions such that the approximations converge to the solution of the original problem as the mesh becomes finer.

This paper may be regarded as a companion to Fox [3], although the papers can be read independently. Except for the difference in the cardinality of the state space, the setups are roughly the same. In the last section, the approaches are combined. The result is a recipe for finite-state approximations to uncountable-state dynamic programs, where the state space need not be compact. This complements the analogous result in Fox [3] for denumerable-state programs.

Let H_δ have domain and range V, indexed by the states, and

$$Av = \sup \{H_\delta v : \delta \in \Delta\}.$$

The policy space Δ is the Cartesian product of decision sets, one set for each state. We assume that H_δ is affine, monotone, and a uniform contraction in policies. That is, with state space S, V the L_∞ space of real-valued (payoff) functions on S, and sup metric d,

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$$H_\delta v(x) = r_\delta(x) + \int_S q_\delta(x,y)v(y)dy$$

$$q_\delta(x,y) \geq 0$$

$$d(H_\delta u, H_\delta v) \leq cd(u,v), \quad c < 1$$

for all δ. Then A is a contraction (e.g., see Denardo [1]). (Our results can be extended to the case where H_δ is a monotonic, N-stage contraction, as defined in [1], but the notation becomes more involved.) This formulation encompasses discounted, discrete-time Markov programs; we do not require that the time intervals between points where decisions are made to be identical.

For a given (finite) grid G_n (corresponding to the n-th approximation) we partition S into sections such that each section contains exactly one grid point. Suppose that S_{ni} is the section containing the grid point g_{ni}. For $x \in S_{ni}$, $y \in S_{nj}$, define

$$q_{n\delta}(x,y) = \int_{S_{nj}} q_\delta(g_{ni},t)dt$$

$$H_{n\delta}v(x) = r_\delta(g_{ni}) + \sum_j q_{n\delta}(g_{ni},g_{nj})v(g_{nj}).$$

$$A_n v = \sup \{H_{n\delta} v : \delta \in \Delta\}.$$

This setup is equivalent to a finite-state problem on G_n. One easily verifies that

$$d(H_{n\delta}u, H_{n\delta}v) \leq cd(u,v)$$

and so $H_{n\delta}$ and A_n are contractions. Let the unique bounded fixed points corresponding respectively to H_δ, $H_{n\delta}$, A, and A_n be v^δ, $v^{n\delta}$, v^*, and v^{n*}. By results in [1],

$$v^{n*} = \sup \{v^{n\delta} : \delta \in \Delta\}$$

$$v^{*} = \sup \{v^{\delta} : \delta \in \Delta\}.$$

Without discretizing, it may not be possible to specify a finite algorithm for the original problem that finds even an approximately optimal policy. However, we can find v^{n*} to within any positive tolerance in a finite number of steps by schemes given in [1]. If each decision set is finite, v^{n*} can be found exactly in a finite number of steps.

In the next section, we show that $v^{n*} \to v^{*}$ uniformly over S, under weak conditions.

Uncountable-state stochastic games can be discretized in an analogous manner and under similar conditions the values of the discretized games converge uniformly to the value of the original game. The proof is similar. One first shows uniform convergence when both players' policies are fixed. The remainder of the argument goes through with minor modifications.

II. THE RESULT

By assumption,

(i) $\sup||r_\delta|| < \infty$.

(ii) r_δ and q_δ are continuous over S for each δ.

(iii) S is a metric space and, for any ball B in S with arbitrary positive radius, S_{ni} is in B for all sufficiently large n, for some i depending on n.

(iv) S_{ni} is convex for all n and i.

(v) for any fixed $v \in V$ and $x \in S$, $H_\delta v(x)$ is continuous in δ in a topology for which the decision set for x is compact.

Observe that (i) implies that

$$\sup_{\delta,n} (||v^\delta||,||v^{n\delta}||) \le \sup_\delta ||r_\delta||/(1-c) < \infty.$$

On the boundary of S_{ni}, $v^{n\delta}$ can be discontinuous. So we define $w^{n\delta}$ to agree with $v^{n\delta}$ on G_n and otherwise to be given by a continuous interpolation formula such that the sequence $\{w^{n\delta}\}$ is equicontinuous on S. In view of assumptions (i)-(iv), this is possible. For example, if S is a subset of the real line with usual metric, linear interpolation works.

LEMMA 1. $\{w^{n\delta}\}$ *has a uniformly convergent subsequence with a continuous limit* u^δ.

PROOF. This follows from the equicontinuity of $\{w^{n\delta}\}$ and a standard selection theorem (Feller [2], Theorem 3 on p. 263).||

LEMMA 2. $d(v^{n\delta},w^{n\delta}) \to 0$ *as* $n \to \infty$.

PROOF. Follows from (i)-(iv).||

LEMMA 3. $w^{n\delta} \to u^{\delta}$ *uniformly implies* $u^{\delta} = v^{\delta}$.

PROOF. Since contraction mappings have unique fixed points, it suffices to show that $H_{\delta}u^{\delta} = u^{\delta}$. Since H_{δ} is a contraction, $w^{n\delta} \to u^{\delta}$ uniformly implies that $H_{\delta}w^{n\delta} \to H_{\delta}u^{\delta}$ uniformly. It remains to show that $d(w^{n\delta}, H_{\delta}w^{n\delta}) \to 0$. From the definitions, $x \in G_n$ implies that $v^{n\delta}(x) = H_{\delta}v^{n\delta}(x)$. Hence, by Lemma 2, $w^{n\delta}(x) - H_{\delta}w^{n\delta}(x) \to 0$ for all x in a set dense in S. (By (iii), $\{G_n\}$ converges to a grid dense in S and, by Lemma 2, $d(H_{\delta}v^{n\delta}, H_{\delta}w^{n\delta}) \to 0$ since H_{δ} is a contraction.) Noting that the sequence $\{w^{n\delta}-H_{\delta}w^{n\delta}\}$ is equicontinuous, it follows that $d(w^{n\delta}, H_{\delta}w^{n\delta}) \to 0$ since S is compact.||

LEMMA 4. $w^{n\delta} \to v^{\delta}$ *uniformly.*

PROOF. Changing the notation slightly in Lemma 3, it is easily seen that every convergent subsequence of $\{w^{n\delta}\}$ converges to v^{δ}. By Lemma 1 a convergent subsequence exists, and so it follows easily from a contradiction argument that the entire sequence converges to v^{δ}.||

Defining w^{n*} analogously to $w^{n\delta}$, we have the

THEOREM. $w^{n*} \to v^{*}$ *uniformly and* $v^{n*} \to v^{*}$ *uniformly.*

PROOF. By assumption (v) and Lemma 4,

$$\sup_{\delta} d(w^{n\delta}, v^{\delta}) \to 0.$$

This implies that

$$d(\sup_{\delta} w^{n\delta}, \sup_{\delta} v^{\delta}) \to 0$$

by a simple contraction argument (not depending on the definitions), showing that $d(w^{n*}, v^{*}) \to 0$. Using Lemma 2, the proof of the second assertion is similar.||

COROLLARY. v^{*} *is continuous.*

III. WHEN S IS NOT COMPACT

So far we have assumed that S was compact. If S is not compact we propose a two-step approximation procedure. First, define $B_\delta(v(x); S_n) = H_\delta v(x)$ if $x \in S_n$ and 0 otherwise, where S_n is compact. This effectively truncates the problem to one with a compact state space S_n, which we then discretize as before. Let

$$S_n \to S, \quad T_n v = \sup_\delta B_\delta(v;S_n), \quad \text{and} \quad z^{n*}$$

be the unique fixed point of T_n. Using an argument similar to that in Fox [3], $z^{n*} \to v^*$ *pointwise*.

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