Spheroidal Geodesics, Reference Systems, & Local Geometry
ABSTRACT

A discussion of the geodesic on the oblate spheroid (reference ellipsoid) is given with formulae of geodetic accuracy (second order in the flattening, distance and azimuths) for the noniterative direct and inverse solutions over the hemisphere, requiring no root extraction and no tabular data except 8-place tables of the natural trigonometric functions.

Forms are presented for use with any ellipsoid of reference and the formulae are adaptable to high speed electronic computers. Instructions for use of the forms in desk computations are given with the parameters for ten known ellipsoids of reference and the radii of spherical approximations.

A discussion is included of the computation of a long reference line in stations and of reference systems in the vicinity of a station as may be useful in oceanography, seismology, or other geophysical disciplines.

While the formulae introduced are satisfactory for short as well as long lines, the emphasis is on long lines out to maximum spheroidal geodesic length under the shortest distance property of the geodesic. The use of certain types of map projections for such base line work is also discussed.

The direct and inverse solutions as presented here have been adapted to high speed computers by the Earth Sciences Division of Teledyne, Inc., Alexandria, Virginia under the direction of Dr. E. F. Chiburis. The Fortran statements for the inverse solution are given in Appendix 4.

January 1970

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FOREWORD

This report fills a void in the theory and computation of long geodetic distances on the reference ellipsoid. The results will be particularly useful to long range navigation systems such as the Omega, and to several geophysical disciplines such as oceanography, seismology, and geodesy.

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PREFACE

The exposition of the computation of geodesics on the reference ellipsoid (oblate spheroid with small eccentricity) is based on the mathematical investigation I have conducted and included as Appendix I to this report. The many papers which have appeared on the subject since the early work of Legendre and Bessel are evidence of the dissatisfaction with the classic methods. This paper is no exception. It is a "fresh" investigation, but shows the influence of literature search. Where results were identifiable in other treatises, I have made reference to them. All the published works consulted are listed in the bibliography. Many of the results presented here are new. The emphasis is on long lines, based upon somewhat arbitrary criteria, i.e., an accuracy of at least 1 meter in position-geodetic length within 1 meter; latitude, longitude, and azimuth within .035 second—over the longest possible hemispheroidal geodesics employing no tables except 8-place natural trigonometric for desk computations—in any case meeting the 1/100,000 distance and 1 second azimuth requirement as specified by ACIC in their special studies (bibliographical reference [22] of this report), easy adaptation to any reference ellipsoid by merely changing the defining parameters; no root extraction or iteration with formulae limited to first and second powers of the flattening and which are compatible with both desk and large electronic computers.

Since the investigation included the longest possible geodesics, the following questions had to be resolved in the evaluation: If we take an arbitrary point on a given nonplanar spheroidal geodesic, can we find a second limiting point on the geodesic beyond which the unique shortest distance property fails? While Euler's differential equation is a necessary condition, is it sufficient? For example, in a limiting case, the equator as well as a meridian on the spheroid are geodesics (both satisfy Euler's condition) and both contain a common equatorial diameter—is there an arc of the equator which satisfies the shortest distance criteria? Are there more than two consecutive geodesic vertices or more than two nodes (equatorial crossings) in a hemispheroid? Are there any antipodal points on nonplanar spheroidal geodesics? What happens antipodally in a family of geodesics each having a vertex in a common meridian? How can we independently check approximation equations for very long geodesics?

In the 1957 report of the study group No. 2 on long lines, International Association of Geodesy, we find the statement: "Consequently, if two points are situated near the equator and are separated by nearly 180° of longitude there is a certain ambiguity as to what is meant by the geodesic between them." In his paper "The distance between two widely separated points on the surface of the earth" (Bibliographical reference [17] below), Dr. W. D. Lambert stated concerning the ambiguity: "There appears to be no comprehensive treatment readily available in English. The author hopes to publish one shorty." This was never done.
From my investigation (Appendix 1) it was concluded that the maximum lengths of all oblate spheroidal geodesics, under the shortest distance property, each having a vertex in a common semimeridian (pole to pole) are contained in a hemispheroid (on the same side of the meridian orthogonal to that containing the vertices). This permitted determination of maximum distances over which approximation formulae to geodesics need hold under assumed accuracy criteria.

The antipodal zones were investigated for such a family of geodesics (each with a vertex in a common meridian) and formulae developed for determining the axes of the geodesic evolutes (envelopes). A formula for the latitude of the conjugate of an arbitrary point on the spheroidal geodesic (the point beyond which the unique shortest distance property fails) was found.

Formulae were developed in terms of vertex latitude of the geodesic for longitude difference and length to serve as control checks on approximation formulae, and to check already published lines to be used for comparative purposes. A new direct solution was developed, and the inverse solution (previously published in NAVOCEANO TR-182, 1966) improved in form layout, azimuths to second order in the flattening were added and the quadrant search for azimuths eliminated. Where possible or feasible the formulae presented were developed through at least two different analyses, the details of which are presented in Appendix 1.

No apology is made for including the computations of a large number of numerical results throughout the discourse of Appendix 1, or for those included as a group in Appendix 3. One of the disappointing aspects of the literature review (Bibliography to this report), was the frequency of a single or at most two numerical examples presented in verification of formulae, such formulae being subsequently unacceptable when applied to lines differing considerably from those presented. The numerical results of Appendix 3 are also useful as checks, should individual programming of the equations be attempted, and all the ACIC test lines, already published in reference [22], have been included in Appendix 3 for check purposes. Appendix 2 contains the parameters for ten reference ellipsoids, the radii of spherical approximations, antipodal zone axes and areas, coordinate systems and other useful formulae.

The formulae presented here for the direct and inverse (reverse) solutions of geodesics in terms of parametric latitude have been programmed (Fortran) by the Earth Sciences Division, Teledyne, Inc., Alexandria, Virginia, under the supervision of Dr. E. F. Chiburis. The Fortran statements for the inverse solution are given in Appendix 4, and the card deck is available.

Finally, it seemed desirable to devote a section to a discussion of the use of forms presented for desk computations, and in applications such as the computation of reference lines and local associated geometry in the neighborhood of stations on the base line as may be needed in geophysical surveys and studies.

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HISTORICAL NOTE

The French Mathematician, Legendre, published papers in 1806 and 1811 on the theory of spheroidal geodesics, consolidating and extending his work in the Traité des Fonctions Elliptiques, 1825.

The German astronomer, Bessel, published an approximation solution to the spheroidal geodesic in 1825, [1],* and since that time an almost endless stream of publications on the subject has appeared. Other famous 19th century scientists who studied the problem include Bennet (1850, '51), Christoffel (1868), Hansen (1868), Cayley (1870), Jacobi (posthumous publication), Halphen (1888), Darboux (1894), A.R. Forsyth (1895). Cayley was the first to use the term "parametric latitude" for the eccentric angle of the meridian ellipse, [25], preferring it to Legendre's "reduced latitude." Two outstanding 19th century treatises, in each of which the geodesic problem is presented with approximation solutions (iterative), are those of the British geodesist Clarke and of Helmert, the German contemporary, both volumes appearing in 1880, [2].

The Bessel-Helmert method, which is an iterative type computation of the development of the projection on the sphere of the spheroidal geodesic, has been modified by some investigators to eliminate the iterative process and the use of tables other than natural trigonometric, but usually involving root extraction, [3], [4]. Others have followed Clarke's method which in general involves tables for a particular reference ellipsoid, and may involve root extraction, [5].

Since the difference in length between the elliptic normal sections or the great elliptic section and the geodesic is of the 4th order in the eccentricity of the meridian ellipse, formulae have appeared computing these lengths rather than the geodesic, some using also azimuths of these sections rather than geodesic azimuths and with the option, in some cases, of applying difference or differential correction formulae for finally converting to geodesic length and geodesic azimuths, [6], [7], [8], [9]. Particularly with respect to long geodesic lines, the literature is quite extensive, [10], [11], [12], [13], [14]. Many of these formulae as published were developed to give distance up to a fixed predetermined maximum length with a given accuracy and fail almost immediately on lines in excess of that maximum. Many involve coefficients of many terms in powers of the eccentricity or other associated parameter. None of these examined appeared capable of supplying the versatility required under the criteria adopted for the present study.

THE GEODESIC ON THE OBLATE SPHEROID

The longest plane closed curve on the oblate ellipsoid of revolution is the circular equator, and the shortest closed curve, which is also a geodesic, is the meridian. The equator and the meridians are the only

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*Bracketed numbers refer to the bibliography attached to this report.
plane geodesics and the only closed geodesics. All other geodesics are three dimensional space curves, that is they have at each point two principal radii of curvature (a radius of curvature and a radius of torsion). The nonplanar geodesic oscillates symmetrically between tangencies to its two associated symmetric parallels with respect to the equator, and because of the flattening retrogresses through each revolution about the spheroid and thus cannot close on itself, as shown in Figure 1.

The geodesic is fundamentally defined as the curve of shortest distance between two points on a surface. From the integral for arc length we may, by the calculus of variations, determine the conditions on the integrand for the arc length to be a minimum. Actually maximum or minimum (extrema). That is the distance by way of the geodesic around the back side of the oblate spheroid, between two points within a hemispheroid, would be the longest geodesic distance. In Figure 1, note the geodesic arc $P_1P_2$ where $P_2$ is a first point of crossing after one revolution about the spheroid. Around the backside, the geodesic distance from $P_1$ to $P_2$ is the long geodesic arc $P_1CDEFP_2$.

If the geodesic is traced from a point A, a node, on the equator in the direction of the tangents as shown it will pass again through A after a complete revolution but will cross the equator at a point E as shown. The course is $AP_2P_1CDEFP_2G\ldots; P_2$ is the first point where the geodesic crosses itself.

Figure 1. Pictorial representation of the nonplanar geodesic on the oblate ellipsoid of revolution.
From the results of the extremal conditions may be deduced the property that the osculating plane at each point of a geodesic contains the normal to the surface, or equivalently that at each point of a geodesic the principal normal to the curve must coincide with the normal to the surface, [16]. But from simple mechanics, considering a string stretched under tension between two points on a smooth spheroid, we can show that the curve assumed by the string is a geodesic, [15].

Analogy with the subsatellite trace.

The normal projection of the orbit of an earth artificial satellite upon an ellipsoid of reference simulates the geodesic. The normal projection of an equatorial orbit is very near the equator and that of a polar orbit is close to a meridian. For other orbits, the satellite responds in greater degree to the flattening (the equatorial bulge) of the geoid (sea level surface) which is approximated by the reference ellipsoid. This effect on the satellite (sustained by its velocity-falling very slowly back to earth) with the rotation of the earth under the orbit, causes the trace of the trajectory (orbit) as projected normally upon the reference ellipsoid to oscillate between two parallels symmetric with respect to the equator as shown in Figure 2. The symmetric parallels are in latitude ± 48° corresponding to the satellite inclination (the angle between the orbit and the equator). Note also in Figure 2 that the longitude difference between successive equatorial traces is 30°. Hence for each half revolution of the satellite the earth turns 15° to the east under the orbit which is itself in an easterly direction. Hence the longitude difference, node to node (N1 to N2) of the continuous trace is 165° as shown. The orbit also retrogresses but only about 3° per day as shown at the injection point of the orbit. Now a geodesic on the Clarke 1866 ellipsoid with vertex parametric latitude 48° has a longitude difference node to node, of about 179° 36' (see TABLE 8), and no geodesic on it can have a longitude difference, node to node, of less than about 179° 24' and this is along the equator itself. Hence the subsatellite trace is not a geodesic on the reference ellipsoid but it behaves like one, oscillating between two symmetric parallels in latitude equal to the inclination of the orbit, and with no more than two nodes or two vertices (of the trace) within a hemispheroid (on the same side of a meridian). But this digression is useful to remind us that the nonplanar geodesic tries to climb to the nearest pole.

Geodesic antipodal zones.

The behavior of the geodesic, when the geodesic arc end points are nearly antipodal has been discussed in several sources [17], [24], [25]. Clearly if the two points are 180° apart on the equator, then the shortest distance between them on the surface is the meridional semilength. In fact the shortest distance on the surface between the end points of any diameter of the spheroid is either of the two equal arcs of the meridian subtended by the diameter—that is the meridians are the only antipodal geodesics. This is clearly so because of all the plane elliptic sections through any diameter of the oblate spheroid, the one with the largest eccentricity and therefore shortest length is the meridian.

Only the circular length π along the equator belongs to the hemispheroidal family of geodesics (a vertex of each geodesic in a common meridian) and it is the shortest member. There are no antipodal points on nonplanar spheroidal geodesics. See Appendix 1 to this report for the proofs.

If the difference in longitude of two points on the equator is not π radians but π(1-k) radians, where k is a small quantity, k < f (f is the flattening of the spheroid) then there are two geodesics, symmetric with
respect to the equator, which take advantage of the flattening and climb toward the poles. Note the geodesics (1) and (2) in Figure 3. If \( k = f \), the geodesic consists of the equatorial arc \( DD' = CC' = n\alpha(1 - e^2)^{1/2} = n\alpha(1 - f) \).

Continuing the discussion, with the help of Figure 3, we suppose that \( TT' \) is an equatorial diameter of the spheroid orthogonal to a fixed meridian as shown. An arbitrary point \( P \) on the meridian has the symmetric \( R' \) with respect to the equator, the symmetric \( R \) with respect to the polar axis, and the symmetric \( P' \) with respect to the spheroidal center. There are thus four equal geodesics, two each with vertex latitude \( \pm \theta_0 \), determined by every point \( P \) and all are orthogonal to the fixed meridian. In the limit as \( k \to f \), geodesics (1) and (2) coincide with the arc \( DD' \) of the equator and analogously geodesics (3) and (4) coincide with the arc \( CC' \). When \( k \to 0, \theta_0 \to n/2, -\theta_0 \to -n/2 \) and then geodesics (1) and (3), (2) and (4) respectively coincide with the upper and lower halves of the meridian \( ABA'B' \) (plane of the paper in Figure 3).

![Figure 3](image_url)

Figure 3. Pictorial representation of the two geodesic antipodal zones with respect to a given meridian of the oblate spheroid.
It has been shown that this family of geodesics, as depicted in Figure 3, has two evolutes. Cayley, [25], called these geodesic evolutes with respect to a given meridian. These are shown pictorially in Figure 3 as the figures ABCD, A'B'C'D' and they resemble the evolute of the meridian ellipse or a hypocycloid of four cusps since the eccentricity of the meridian is small with respect to earth reference ellipsoids. See also Figure 12 of Appendix 1. (The evolute of a given plane curve is the curve tangent to all normals or perpendiculars to the given curve—also called the envelope of the normals. On the spheroid, arcs of geodesics correspond to straight line segments in the plane relative to the shortest distance property.) Determination of the meridional arc axes of the geodesic evolutes (AB = A'B' of Figure 3) requires the solution of a transcendental equation and is discussed in Appendix 1.

The spheroidal areas enclosed by the geodesic evolutes are called the geodesic antipodal zones with respect to a given meridian. Note from Figure 3 that only two consecutive nodes (equator crossings) occur in a hemispheroid and that they always lie in the geodesic antipodal zones with respect to the meridian containing the geodesic vertex. Because of the symmetry about the equator, the distance between consecutive nodes is the same as between consecutive vertices. Hence we may within a hemispheroid (on the same side of a meridian) have a maximum of two consecutive nodes and the vertex between them; or a maximum of two vertices and the node between them. For proof see Appendix 1 to this report.

Other properties of the geodesic.

The differential equation of the spheroidal geodesics may be found using the property of coincidence of principal normal to the curve and the normal to the surface at an arbitrary common point and it can be shown that the integral length depends on the evaluation of an elliptic integral. Since the eccentricity and the flattening are small quantities for earth reference ellipsoids, the series expansion of the integral in terms of eccentricity, flattening, or other associated parameter converges rapidly and evaluation is usually made in this way rather than by interpolation in elliptic integral tables.

An easily demonstrated but very important well known property of the geodesic on the oblate spheroid (or of the geodesic on any revolute) is that at each point of the geodesic, the product of the radius of the parallel and the sine of the angle which the geodesic makes with the meridian is constant. The mathematical demonstration is found in Appendix 1.

The problem of determining azimuths or geographic position of an end point of a geodesic arc involves solution of a polar spheroidal triangle and is usually approximated by solution of a corresponding spherical triangle or a sequence of them (iteration).

ACCURACY CRITERIA FOR COMPUTATIONS

While sophisticated computer systems are becoming more available universally, there is a need additionally or alternatively to have some computing forms which will give a reasonable geodetic accuracy over hemispheroidal geodesics for both direct or inverse (reverse) solutions with minimum requirements of a desk computer, only 8-place tables of natural trigonometric functions—no iteration or root extraction.
Accordingly the following criteria were adopted relative to the mathematical study included in Appendix I to this report:

1. An accuracy of 1 meter in position-geodetic length within 1 meter; latitude, longitude, and azimuth within .035 second-over the longest possible hemispheroidal geodesics, but in any case equalling the 1/100,000 distance and 1 second azimuth requirement adopted by ACIC, [22].

2. No tabular data required except 8-place natural trigonometric for desk computations.

3. No iteration or root extraction with formulae also adaptable to large electronic computing systems.

4. Easy adaptation to any reference ellipsoid by merely changing the scale parameters a, f, etc.

**DIRECT SOLUTION**

All direct solutions of the spheroidal triangle involve approximations by one or more spherical triangles. They differ with respect to the variables, parameters, required tabular data, arithmetic operations and subsequent accuracy. The formulae to be presented here involve corrections to a single spheroidal triangle. The variables are longitude, λ, parametric latitude, θ. Parameters are a, f, ε, where a, f are the semimajor axis and flattening of the reference ellipsoid and ε is the parametric latitude of the geodesic vertex. The only tabular data required is a table, such as Peters, of the natural trigonometric functions. No root extraction or iteration is required in arithmetic operations.

We are given the point P₁ (φ₁, λ₁) on the spheroid, where φ₁, λ₁ are geodetic latitude and longitude (geographic coordinates); the forward azimuth α₁, and distance S to a second point P₂ (φ₂, λ₂); and from these we are to find the geographic coordinates φ₂, λ₂ and the back azimuth α₂, S. The given quantities are φ₁, λ₁, α₁, S.

Formulas. (The derivations are given in Appendix 1)

**Second Order in f.**

\[ \tan \theta_1 = (1 - f) \tan \phi_1, M = \cos \theta_0 = \cos \theta_1 \sin \alpha_{1,2}, \]
\[ N = \cos \theta_1 \cos \alpha_{1,2}, C_1 = \sqrt{M}, C_2 = (1/4)(1 - M^2), D = (1 - C_2)(1 - C_1 - C_2 M), \]
\[ P = C_2 (1 + (1/2)C_1 M) D, \cos \alpha_1 = \sin \theta_1 \sin \theta_0, d = S/a D, u = 2(ε₁ - d), \]
\[ W = 1 - 2P \cos u, V = \cos (u + d) = \cos u \cos d - \sin u \sin d, X = C_1 \sin d \cos d (2V^2 - 1), \]
\[ Y = 2PVW \sin d, \Delta \omega = d + X - Y, 2a = 2a_1 - \Delta \omega, \]
\[ \tan \alpha_{1,2} = M / (N \cos \Delta \omega - \sin \theta_1 \sin \Delta \omega), \]
\[ \tan \phi_0 = - (\sin \theta_1 \cos \Delta \omega + N \sin \Delta \omega) \sin \alpha_{1,2} / (M(1 - f)M), \]
\[ \tan \Delta \omega = \sin \Delta \omega \sin \alpha_{1,2} (\cos \theta_1 \cos \Delta \omega - \sin \theta_1 \sin \Delta \omega \cos \alpha_{1,2}), \]
\[ H = C_1 (1 - C_2) \Delta \omega - C_2 C_2 \sin \Delta \omega \cos \Delta \lambda, \Delta \lambda = \Delta \eta - H, \lambda_2 = \lambda_1 + \Delta \lambda. \]
First Order in \( f(f^2 = 0) \)

We place terms in \( f^2 \) equal to zero in the above equations which will remain the same except for the following:

\[
P = 1 - 2c_2 - c_1 M, P = c_2/D, X = 0, \Delta a = 0 - Y, H = c_1 \Delta a.
\]

**Spherical \((f - 0)\)**

If we place \( f - 0 \) in the above equations we have

\[
\tan \theta_1, \cos \theta_1, \sin \theta_1,
\]

And

\[
\Delta \lambda = \sin d \sin a_{12} / (\cos \theta_1 \cos d - \sin \theta_1 \sin d \cos a_{12}),
\]

\[
\lambda_0 = \lambda_1 + \Delta \lambda; \ a \ may \ be \ the \ radius \ of \ a \ spherical \ approximation \ such \ as \ given \ in \ Appendix \ 2.
\]

**Sign Conventions for Azimuth and Longitude**

We take the initial point to be west of the terminus in the direct solution and then always

\[180^\circ < \phi_2 < 360^\circ.\] We also have \( O < \Delta \eta < \Delta \lambda < \pi. \) If two arbitrary points are both in the southern hemispheroid (both in negative latitude), we solve as though both were in the northern hemispheroid and write symmetric elements with respect to the equator. While not necessary, these conventions simplify somewhat the determination of azimuth and longitude difference in desk computing.

From the quantities above in the formulae we find the first quadrant angles \( u \) and \( v \) given by

\[
\tan u = \tan a_{12} \tan \Delta \eta, \quad \tan v = \tan \Delta \eta.
\]

If \( \tan a_{12} > 0, \) then \( a_{12} = 180^\circ + u; \) if \( \tan a_{12} < 0, \) then \( a_{12} = 360^\circ - u. \) If \( \tan \Delta \eta > 0, \) then \( \Delta \eta = v; \) if \( \tan \Delta \eta < 0, \) then \( \Delta \eta = 180^\circ - v. \)

The conventions are sufficient, under the assumptions, as demonstrated by the following:

Always \( 0 < a_{12} < 180^\circ. \) When \( \tan a_{12} > 0, \) then \( a_{12} \) is in the third quadrant and is of the form \( 180^\circ + u, \) since \( \tan (180^\circ + u) = \tan u. \) When \( \tan a_{12} < 0, \) then \( a_{12} \) is in the fourth quadrant and is of the form \( 360^\circ - u, \) since \( \tan (360^\circ - u) = -\tan u. \)

Since always \( 0 < \Delta \eta < \Delta \lambda < \pi, \) when \( \tan \Delta \eta > 0, \) \( \Delta \eta \) is in the first quadrant and \( \Delta \eta = v. \) When \( \tan \Delta \eta < 0, \) \( \Delta \eta \) is in the second quadrant and is of the form \( 180^\circ - v, \) since \( \tan (180^\circ - v) = -\tan v. \)

The arrangement of the direct formulae into a computing form is shown in Figure 18, Appendix 1.

**Inverse (Reverse) Solution**

The published inverse solutions have been more varied than the direct. The series expansion for the geodesic length in the flattening \( f, \) spherical length \( d \) (with reference to the geodesic latitude of the vertex of the great elliptic arc) in the form

\[
S = a [d - F_1(d)f + F_2(d)f^2 + \ldots]
\]

was published by A.R. Forsyth in 1895, [20]. Errors in \( F_2(d) \), making unstable the use of the second order term, remained undetected until 1965, [21]. The more recent examinations also revealed that the
Andoyer-Lambert expansions to first order in the flattening are merely those of Forsyth to first order in the flattening, [18]. An independent verification of the corrections to Forsyth’s equations was found in the work of Gougenheim, [23]. Gougenheim’s work has apparently gone unnoticed although he has had a correct expansion in terms of geodetic latitude to second order in the flattening since 1950.

Forsyth had the expansion in parametric latitude to first order in the flattening. This was extended to second order as reported in [18]. The formulae for distance to be used here are basically those from [18]. The azimuth formulae are adaptations of those presented by Gougenheim in [23]. See Appendix 1, Equations (143).

We are given the points \( P_1 (\phi_1, \lambda_1), P_2 (\phi_2, \lambda_2) \) on the spheroid and are to find the distance \( S \) between the points and the forward and back azimuths, \( a_{1-2} \) and \( a_{2-1} \). Given quantities are \( \phi_1, \lambda_1, \phi_2, \lambda_2 \). It is assumed that east longitudes are positive and that \( P_1 \) is west of \( P_2 \).

Formulae

Second Order in \( f \)

\[
\tan \theta_1 = (1 - f) \tan \phi_1, \quad i = 1, 2
\]
\[
\theta_m = (1/2)(\theta_1 + \theta_2), \quad \Delta \theta_m = (1/2)(\theta_2 - \theta_1), \quad \Delta \lambda = \lambda_2 - \lambda_1
\]
\[
\Delta \lambda_m = (1/2) \Delta \lambda, \quad H = \cos^2 \Delta \theta_m - \sin^2 \theta_m = \cos^2 \theta_m - \sin^2 \Delta \theta_m.
\]
\[
L = \sin^2 \Delta \theta_m + H \sin^3 \Delta \lambda_m = \sin^2 (1/2)d, \quad 1 - L = \cos^2 (1/2)d, \quad \cos d = 1 - 2L.
\]
\[
U = 2 \sin^2 \theta_m \cos^2 \Delta \theta_m (1 - L), \quad V = 2 \sin^2 \Delta \theta_m \cos^2 \theta_m / L, \quad \lambda = U + V.
\]
\[
Y = U - V, \quad T = d / \sin d, \quad D = 4T^2, \quad E = 2 \cos d, \quad A = DE, \quad B = 2D.
\]
\[
C = T - (1/2)(A - E); \quad \text{check: } C = \Delta E + AD / B = T.
\]
\[
n_1 = X(A + CX), \quad n_2 = Y(B + EY), \quad n_3 = DXY, \quad \delta_1 d = (1/4)(TX - Y),
\]
\[
\delta_2 d = (t^2 / 64) (n_1 - n_2 + n_3), \quad S_1 = \sin d (T - \delta_1 d), \quad S_2 = \sin d (T - \delta_1 d + \delta_2 d),
\]
\[
F = 2Y - E(4 - X), \quad M = 32T - (20T - A)X - (B + 4) Y,
\]
\[
G = (1/2)YT + (t^2 / 64) M, \quad Q = -(FG \tan \Delta \lambda) / 4, \quad \Delta \lambda_m = (1/2) (\Delta \lambda + Q),
\]
\[
c_1 = - \Delta \theta_m / \sin \theta_m \tan \Delta \lambda_m, \quad u = \arctan c_1, \quad a_1 = v - u,
\]
\[
c_2 = \cos \Delta \theta_m / \sin \theta_m \tan \Delta \lambda_m, \quad v = \arctan c_2, \quad a_2 = v + u,
\]

<table>
<thead>
<tr>
<th>( c_1 )</th>
<th>( c_2 )</th>
<th>( a_{1-2} )</th>
<th>( a_{2-1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-</td>
<td>+</td>
<td>( a_1 )</td>
<td>360 - ( a_1 )</td>
</tr>
<tr>
<td>+</td>
<td>+</td>
<td>( a_2 )</td>
<td>360 - ( a_2 )</td>
</tr>
<tr>
<td>-</td>
<td>-</td>
<td>180 - ( a_2 )</td>
<td>180 + ( a_1 )</td>
</tr>
<tr>
<td>+</td>
<td>-</td>
<td>180 - ( a_1 )</td>
<td>180 + ( a_2 )</td>
</tr>
</tbody>
</table>

First Order in \( f (f^2 = 0) \)

\[
\tan \theta_1 = (1 - f) \tan \phi_1, \quad i = 1, 2; \quad \theta_m = (1/2)(\theta_1 + \theta_2), \quad \Delta \theta_m = (1/2)(\theta_2 - \theta_1),
\]
\[
\Delta \lambda = \lambda_2 - \lambda_1, \quad \Delta \lambda_m = (1/2) \Delta \lambda, \quad H = \cos^2 \Delta \theta_m - \sin^2 \theta_m = \cos^2 \theta_m - \sin^2 \Delta \theta_m.
\]
\[
L = \sin^2 \Delta \theta_m + H \sin^3 \Delta \lambda_m = \sin^3 \frac{1}{3} d, \quad 1 - L = \cos^3 \frac{1}{3} d, \quad \cos d = 1 - 2L.
\]
\[ U = 2 \sin^2 \theta_m \cos^2 \Delta \theta_m / (1 - L), \]
\[ V = 2 \sin^2 \Delta \theta_m \cos \theta_m / L, \]
\[ X = U + V, \]
\[ Y = U - V, \]
\[ T = d / \sin d, \]
\[ \delta_1 = (1/4)(TX - Y), \]
\[ S = a \sin d (T - \delta_1 d), \]
\[ F = 2 [Y - (1 - 2L)(4 - X)], \]
\[ G = (1/2)T, \]
\[ Q = -(FG \tan \Delta \lambda) / 4, \]
\[ \Delta \lambda' = (1/2)(\Delta \lambda + Q); \]
\[ the \ rest \ of \ the \ azimuth \ solution \ is \ the \ same \ as \ for \ the \ original \ formulae \ above. \]

**Spherical \ (f = 0)**

With \( f = 0 \) in the above formulae we have:

\[ \tan \phi_1 = \tan \theta_1, \phi_2 = \theta_1, \theta_m = (1/2)(\theta_1 + \theta_2), \]
\[ \Delta \lambda = \lambda_2 - \lambda_1, \Delta \theta_m = (1/2)\Delta \lambda, \]
\[ H = \cos^2 \Delta \theta_m - \sin^2 \theta_m = \cos^2 \theta_m - \sin^2 \Delta \theta_m, \]
\[ L = \sin^2 \Delta \theta_m + H \sin^2 \Delta \lambda_m, \]
\[ \cos d = 1 - 2L, S = ad, Q = 0, \Delta \lambda' = \Delta \lambda_m, \]
\[ c_1 = -\sin \Delta \theta_m / \cos \theta_m \tan \Delta \lambda_m, \]
\[ c_2 = \cos \Delta \theta_m / \sin \theta_m \tan \Delta \lambda_m; \]
\[ the \ rest \ of \ the \ azimuth \ solution \ is \ the \ same \ as \ above. \]

**Azimuth Determination—Elimination of Quadrant Search**

In the above formulae we find two first quadrant angles given by \( u = \arctan c_1, v = \arctan c_2 \). We then form \( a_1 = v - u, a_2 = v + u \) and determine the azimuths according to the signs of \( c_1 \) and \( c_2 \) from the array:

<table>
<thead>
<tr>
<th>( c_1 )</th>
<th>( c_2 )</th>
<th>( a_1 )</th>
<th>( a_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-</td>
<td>+</td>
<td>( a_1 )</td>
<td>360 - ( a_2 )</td>
</tr>
<tr>
<td>+</td>
<td>+</td>
<td>( a_2 )</td>
<td>360 - ( a_1 )</td>
</tr>
<tr>
<td>-</td>
<td>-</td>
<td>180 - ( a_2 )</td>
<td>180 + ( a_1 )</td>
</tr>
<tr>
<td>+</td>
<td>-</td>
<td>180 - ( a_1 )</td>
<td>180 + ( a_2 )</td>
</tr>
</tbody>
</table>

This, in effect, eliminates the quadrant search since it has been done in advance. For the development of these expressions see Appendix 1.

The arrangement of the inverse formulae into a computing form is shown in figure 25, Appendix 1.

### Desk Computations of Direct and Inverse Solutions

For a demonstration of the direct and inverse forms, Figures 18, 25—Appendix 1, the long line published in reference [4] will be used. Its elements as given there are:

**ORIGIN** \( \phi_1 = 20^\circ, \lambda_1 = 0; S = 964941.2505 \) meters

**TERMINUS** \( \phi_2 = 45^\circ, \lambda_2 = 106^\circ; a_1 = 42^\circ 56' 30.035. \)

\[ f = .003367003367, a = 6378388 \text{ meters, } a_{2,1} = 295^\circ 17' 18.860. \]

To provide a check for this line we use equations (49), (50) of Appendix 1 to make an independent computation as follows:

\[ f = .003367003367, a = 6378388 \text{ m, } \cos \theta_o = .64042078, \sin \theta_o = .76802423, \]
\[ c_1 = f \cos \theta_o = .215629982 \times 10^2, A = c_1 (1 - c_2 c_0) = .215522628 \times 10^2. \]
\[
c_2 = \frac{1}{4} f \sin^3 \theta_0 = 0.49651618 \times 10^{-3}, \quad B = \frac{1}{2} c_1 c_2 c_3 = 0.53606 \times 10^{-4}
\]

\[
c_4 = c_1 + c_3 \cos \theta_0 = 1.0013809386, \quad D = 2 + c_2(c_3 + c_4) - (1 + c_2) c_4 - c_2 = 0.9976269631
\]

\[
c_5 = c_2 + c_3 = 0.1008774548, \quad E = \frac{1}{2} c_3 [2 + c_3(c_3 - 1) - c_4] = 0.496859424 \times 10^{-3}
\]

\[
\eta_1 = 72^\circ 23' 36.933, \quad F = \frac{4}{5} c_2 (2c_4 - 1) = 0.6186 \times 10^{-7}
\]

\[
\eta_2 = 33^\circ 47' 36.595, \quad \Sigma \eta = \eta_1 + \eta_2 = 106^\circ 11' 13.628 = 1.8533148482 \text{ rad.}
\]

\[
o_1 = 63^\circ 38' 26.269, \quad \Sigma \alpha = \alpha_1 + \alpha_2 = 86^\circ 50' 29.583 = 1.5156709899 \text{ rad.}
\]

\[
o_2 = 23^\circ 12' 03.314, \quad \sin \Sigma \alpha = 0.99848098, \quad \Delta \alpha = \alpha_1 - \alpha_2 = 40^\circ 26' 22.955,
\]

\[
\cos \Delta \alpha = 0.76108893, \sin 2\Sigma \alpha = 1.1002746, \quad \rho = 2 \sin \Sigma \cos \Delta \alpha = 1.51986564,
\]

\[
\cos 2\Delta \alpha = 0.15851273, \quad q = 2 \sin 2\Sigma \cos 2\Delta \alpha = 0.34881506.
\]

<table>
<thead>
<tr>
<th>(\Sigma \eta)</th>
<th>1.8533148482</th>
<th>(D \Sigma \alpha)</th>
<th>1.5120742467</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-A \Sigma \alpha)</td>
<td>-0.0032666139</td>
<td>(+E \alpha)</td>
<td>+0.0007551596</td>
</tr>
<tr>
<td>1.8500482343</td>
<td>1.5128294063</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(+B \rho)</td>
<td>+0.0000008147</td>
<td>(-F \alpha)</td>
<td>-</td>
</tr>
<tr>
<td>(\Delta \lambda(\text{rad}))</td>
<td>1.8500490440</td>
<td>(S/a)</td>
<td>1.5128294041</td>
</tr>
<tr>
<td>(\Delta \lambda)</td>
<td>106.00' 00.0009</td>
<td></td>
<td>9649412.917 m</td>
</tr>
</tbody>
</table>

Figures 4 to 9 are respectively the direct and inverse solutions of this line—second order in \(f\), first order in \(f^2 = 0\), and spherical \((f = 0)\). The crosses in spaces of the first order and spherical examples indicate values to be omitted in the computation.

**Computation of the direct solution.**

**Second order in \(f\).** We first identify the reference ellipsoid to be used at the top of the form and enter the indicated spheroidal constants from Appendix 2. The given quantities \(\phi_1, \alpha_1, \Sigma, \lambda_1\) are then entered in the spaces provided with heavy underline as shown in figure 4; \(\sin \alpha_1, \cos \alpha_1, \tan \phi_1\) are found from the Peters (or other) 8-place tables of natural trigonometric functions; \(\tan \phi_1\) is multiplied by \(1 - f\) to get \(\tan \phi_1\) as shown, and then \(\sin \phi_1, \cos \phi_1\) are found from the tables. In using linear interpolation in the Peters Tables, always take the tabular difference at the particular second in the table unless the difference is constant for the particular column as marked top and bottom. For instance, at 41° 52' the tabular difference is a constant 361 for the sine column as indicated top and bottom, but this is seldom so. Convenient checks are provided by the identities \(\sin \theta / \cos \theta = \tan \theta, \sin^2 \theta + \cos^2 \theta = 1\).

After \(M, N, \theta_0, \sin \theta_0\) have been found we compute the constants \(c_1, c_2, D, P\). We may compute \(c_2\) in two ways since \(1 - B_2 = \sin^3 \theta_0\). We next find \(\alpha_1\) than \(d = S/aD\) which is in radians. At the top of the form find 1 radian = 206264.8062 seconds. This factor is multiplied by \(d\) (radians) and then divided by 3600 seconds (1 degree) which will give an integral number of degrees plus a decimal part of a degree. This decimal part is multiplied by 60 to get an integral number of minutes plus a decimal part of a minute. The decimal part of a minute is multiplied by 60 to get seconds retaining three decimals. If the total number of seconds is less than 3600, but more than 60, we divide by 60 to get minutes and then continue as above. Always check by reversing the process to get the radians \(d\).

With \(\alpha_1\) and \(d\) in degrees we form \(u = 2(\alpha_1 - d)\) and find \(s, d, \cos d, \sin u, \cos u\). These are checked by \(\sin^2 x + \cos^2 x = 1\). \(V\) and \(W\) may now be computed and then \(X\) and \(Y\). Note that \(X\) may be ignored if

11
it is less than \(3 \times 10^{-4}\). Next \(\Delta \delta\) is computed from the radian values of \(d, X, Y\) and converted to degrees. \(\Delta \delta\) and \(d\) always differ by only a few minutes. \(\Sigma \delta\) is formed in degrees and then sin \(\Delta \delta\), cos \(\Delta \delta\), cos \(\Sigma \delta\) found from the tables. We are then able to compute tan \(\alpha_{2-1}\). The first quadrant solution for tan \(u = 2.11661579\) is \(u = 64^\circ 42' 41.399'\). Since the sign of tan \(\alpha_{2-1}\) as computed is negative, we have \(\alpha_{2-1} = 360^\circ - u = 295^\circ 17' 18.601'\). sin \(\alpha_{2-1} = \sin (360^\circ - u) = -\sin u = -\sin 64^\circ 42' 41.399' = -0.90416831\). We may now compute tan \(\phi_1\), which is found to be 1.00000003 \(= 1 + (3 \times 10^{-4})\) and from the table \(\phi_2 = 45^\circ 00' 00.003'\). Next find tan \(\Delta \eta = 3.4449133\). From the Peters tables we find for tan \(v = 3.4449133\), that \(v = 73^\circ 48' 46.375'\), but since the sign of tan \(\Delta \eta\) is negative, \(\Delta \eta = 180^\circ - v = 106^\circ 11' 13.625'\). Now the computation for \(H\) is in radians and converting to angular value, \(H = 11' 13.620.\) We subtract \(H\) from \(\Delta \eta\) and add the difference, \(\Delta \lambda\), to \(\lambda_1\) to get \(\lambda_2 = 106^\circ 00' 00.005'\) as shown.

**First order in \(f^2 = 0\).** The input quantities are the same as shown in figure 4. We "cross out" the quantities to be omitted as shown in Figure 6, and the computational procedure is then the same.

**Spherical (\(f = 0\)).** We must adopt a spherical radius. For figure 8 we have adopted the great normal radius for \(\phi = 20^\circ\), see Appendix 2, equations (11) and (22). The quantities to be omitted are then "crossed out" and the simplified computations made as shown.

**Computation of the inverse solution.**

**Second order in \(f\).** We enter the name of the reference ellipsoid to be used and the corresponding spheroidal constants from Appendix 2. The given quantities \(\phi_1, \phi_2, \lambda_1, \lambda_2\) are entered in the spaces with heavy underline as shown in figure 5. We find tan \(\phi_1\), and tan \(\phi_2\) from the tables and compute tan \(\theta_1\), tan \(\theta_2\) as shown; then back to the tables to find \(\delta_1, \delta_2\). We then form \(\theta_m\) and \(\Delta \theta_m\) and check by adding, since \(\theta_m + \Delta \theta_m = \theta_2\). Next find \(\Delta \lambda, \Delta \lambda_m\) and then from the tables sin \(\theta_m\), cos \(\theta_m\); sin \(\Delta \theta_m\), cos \(\Delta \theta_m\); sin \(\Delta \lambda_m\), tan \(\Delta \lambda\). We next compute two values of \(H\) as shown which should agree within 5 in the 9th place of decimals. Take the mean and retain 8 decimals. \(L\) is then computed retaining 8 decimals.

With the value of \(L\), we form \(1 - L, \cos d = 1 - 2L\) as shown; then find \(d, \sin d\) from the tables. Now compute \(U, V, X, Y, T, E, D, B, A, C\). Note that \(B = 2D, A = DE, D = 4T^4, C = T - (1/2)(A - E)\), so that these are relatively easy to compute. The check is given by \(T = C - HE + ADVB = d/\sin d\).

Compute \(n_1, n_2, n_3\), and then \(\delta_1, \delta_2, \delta_3\). We can now compute \(S_1\) (first order for comparison) or go directly to \(S_2\) for the second order distance as shown in figure 5.

For the azimuths, we compute in order \(F, M, G, Q, \Delta \lambda_m, \tan \Delta \lambda_m\). Then \(c_1, c_2, u, v\) and in that order. We add and subtract the quantities \(u, v\) to get \(a_1 = v - u, a_2 = v + u\). Now the signs of \(c_1, c_2\) are \(-, +\) as shown in figure 5. Hence the azimuths are \(a_{1-2} = a_1, a_{2-1} = 360 - a_2\) as shown.

**First order in \(f\).** The heading information to first order in \(f\) and input quantities are the same as in figure 5. The quantities to be omitted are "crossed out" as shown in figure 7 and then the computations are done as before computing \(S_1\) after finding the first order correction \(\delta_1 d\).

**Spherical (\(f = 0\)).** We need a radius approximation to the ellipsoid and use that determined for the spherical direct solution which is the great normal length for \(\phi = 20^\circ, r = 6380097.5\) meters (International ellipsoid). The omitted quantities are then "crossed out" as shown in figure 9, and the simplified computation made analogously as shown.
DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given \( \phi_1, \lambda_1, \alpha_1, \beta_1, S \) to find \( \phi_2, \lambda_2, \alpha_2, \beta_2 \). East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

\[
\text{INTERNATIONAL SPHEROID a } 6378.389 \text{ m } \quad f = 00.3367 \quad 00.3367
\]

1 - f = 9966.37996

1 radian = 206264.8062 seconds

**Line** from Initial to Terminus

\[
\begin{align*}
\phi_1 & = 40.0000 \\
\alpha_1 & = 50.36.10.35 \sin \theta_1 \\
\cos \alpha_1 & = 0.7320.4755 \quad N \quad \cos \theta_1 \sin \alpha_1 \quad 1.688.17033 \quad \sin \theta_0 \quad 0.7680 \quad 24.23 \\
c_1 & = \tan \phi_1 \times 2156.29910 \times \theta_1 \\
c_2 & = \frac{1}{3} (1 - M^2) \times 49.65.16.2 \times 10^{-3} \\
D & = (1 - c_2) (1 - c_2 - c_1 M)^{-1} \times 99.76.266617 \\
\cos \alpha_1 & = \sin \theta_1 \times 0.44399.989 \quad \phi_1 \quad 6.3 \quad 38 \quad 26.77 \\
d/S & = d/ \sin \theta_1 \times 6.5164.27946 \quad \text{(rad)} \\
\sin \theta_1 & = 2 \cos \theta_1 \sin \theta_0 \times 0.337.50 \times 0.6479 \quad \sin u \times 0.7287.316 \\
\cos d & = 0.5473.9160 \quad W \quad 1 - 2 \cos u \times 0.9993.14199 \quad \cos u \times 0.6874.914 \\
D & = 2 \cos \theta_1 \sin \theta_1 \times 0.7615.29694 \\
\gamma & = 2 PVW \sin d \times 0.0018.56.9553 \\
X & = c_2 \sin d \cos d (2V^2 - 1) \times 6.214.10^{-6} \times (19.46) \\
\Delta \omega & = d + X - Y \times 1.5164.709.911 \quad \text{(rad)} \\
\sin \Delta \omega & = 0.9984.8098 \cos \Delta \omega \times 0.9550.9772 \quad \Delta \omega \times 0.80.50 \quad 23.583 \\
\cos \Lambda & = 0.7610.8282 \quad \Delta \omega \times 2 \Delta \omega \times 4.0.64.72.959 \\
\tan \alpha_2 & = \frac{M (N \cos \Delta \omega - \sin \theta_1 \sin \Delta \omega)}{1 - \sin \theta_1} \times 1.146.1572 \quad \alpha_2 \quad 2.95 \quad 17.14.80.1 \\
\tan \phi_2 & = \frac{\tan \theta_1 \cos \Delta \omega + N \sin \theta_1 \sin \Delta \omega 1 + (3 \times 10^{-8})}{1 - \sin \alpha_2} \times 0.9041.6831 \\
\phi_2 & = 45.00 \quad 00.0003 \\
\tan \Delta \eta & = \frac{\sin \Delta \omega \sin \alpha_1}{\cos \theta_1 \cos \Delta \omega - \sin \theta_1 \sin \Delta \omega \cos \alpha_1} \times -2.4449.133 \quad \Delta \eta \times 104.11 \quad 10.625 \\
H & = c_1 (1 - c_2) \Delta \omega - c_1 c_2 \sin \Delta \omega \cos \Delta \omega \times 0.03 \times 0.583 \quad \text{(rad)} \\
\Delta \lambda & = \Delta \eta - H \times 104.00 \quad 00.0005 \\
\lambda_1 & = 0 \quad 00 \quad 00.0005 \\
\end{align*}
\]

**CHECK**

\[
\begin{align*}
M & = \cos \theta_1 \cos \theta_1 \sin \alpha_1 \cdot \cos \theta_2 \sin (180 + \alpha_2) \\
\lambda_2 & = \lambda_1 + \Delta \lambda \times 104.00 \quad 00.0005 \\
\end{align*}
\]

Figure 4. Direct computation—second order in f.
| \( \phi_1 \) | 30 0 0 | 1. Initial | \( \lambda_1 \) | 0 0 0 |
| \( \phi_2 \) | 45 0 0 | 2. TERMINUS | \( \lambda_2 \) | 106 0 0 |
| tan \( \phi_1 \) | 3639.2023 | \( \tan \theta = (1 - \tan \phi \Delta \lambda_m = \frac{1}{2}\Delta \lambda \) | 53 |
| \( \theta_2 \) | 0.54 \( \lambda \) & 0.168 | tan \( \theta_2 \) | 0.99663300 |
| \( \psi_1 \) | 75 56 16.736 | tan \( \psi_1 \) | 0.266714475 |
| \( \theta_m \) | 0.32 \( \lambda \) & 0.25 \( \lambda \) & 0.1437 | sin \( \theta_m \) | 0.53613146 |
| \( \Delta \theta_m \) | 0.5 \( \theta_2 - \theta_1 \) | 0.216 \( \lambda \) & 0.4907 | cos \( \theta_m \) | 0.84913472 |
| H = cos^2 \( \Delta \theta_m \) - sin^2 \( \theta_m \) - sin^2 \( \Delta \theta_m \) & 0.68514445 & 1 - L & 0.52859337 |
| L = \sin^2 \( \theta_m \) & + H \sin^2 \( \Delta \theta_m \) | 0.47140663 | cos d = 1 - 2L | + 0.05718274 |
| U = 2 \sin^2 \( \theta_m \) & cos^2 \( \theta_m \) | 0.4036745079 | d & 0.43 | 1.7950 |
| V = 2 \sin \( \Delta \theta_m \) \cos \( \theta_m \) & L & 0.41534672 | sin d & 0.99633500 | d (rad) & 1.51878374 |
| X = U + \lambda & 0.1779811251 | T = d/sin d & 1.5160594053 | E = 2 \cos d & 0.14373498 |
| Y = U - \lambda & 0.8935084070 | D = 4\( \lambda \) & 0.1937444427 | B = 2D & 18.32748876 |
| A = DE & 1.051520551 | C = T - \( \frac{1}{2} \) (A - E) & 0.04748586 | CHECK C - \( \frac{1}{2} \) E + AD/B = T |
| n_1 = X (A + CX) & 0.692339385 | n_2 = Y (B + EY) & 0.6557781707 | n_3 = DXB & 0.698407923 |
| \( \delta_d \) & 0.0647 & 0.19 | \( \delta_d \) & 0.47 | 0.238 \times 10^{-6} |
| S_1 = a \sin d (T - \delta_d) & 0.9469417344 | S_2 = a \sin d (T - \delta_d + \delta_d) & 0.9649412792 |
| F = 2Y - E & 0.4X & 1.9482527 | M = 32T - (20T - A) X & (B + 4) Y & 6.01349299 |
| G = \frac{1}{4} \tan \phi & (f/64) M & 0.0285501777377 | Q = - (FG \tan \Delta \lambda)/4 + 11 & 13.635 |
| \( \Delta \lambda_m \) & \frac{1}{2} \Delta \lambda + Q & \tan \Delta \lambda_m & 1.3315 \times 1.317 |
| \( \psi_1 \) & arctan \( \lambda_1 \) & \psi_2 & \psi_2 & \psi_2 & \psi_2 |
| \( \lambda_1 \) & \psi_2 & \psi_2 & \psi_2 & \psi_2 |
| C = \psi_1 \psi_2 \psi_2 \psi_2 \psi_2 | 4.24 \psi_2 \psi_2 \psi_2 \psi_2 \psi_2 | U = \psi_1 \psi_2 \psi_2 \psi_2 \psi_2 | 4.14 | 0.04 |

Figure 5. Inverse computation—second order in \( f \).
DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1$, $\lambda_1$, $\alpha_{1-2}$, S to find $\phi_2$, $\lambda_2$, $\alpha_{2-1}$. East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

**INTERNATIONAL SPHEROID** $a = 6378388$ m $f = 0.003367$ $00^{22}76$

$1 - f = 9966329966$

1 radian = 206264.8062 seconds

<table>
<thead>
<tr>
<th>LINE</th>
<th>INITIAL</th>
<th>TO TERMINUS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_1$</td>
<td>42 0 0</td>
<td>3639 7023</td>
</tr>
<tr>
<td>$\alpha_{1-2}$</td>
<td>42 56 30.035</td>
<td>$54^0102^67$</td>
</tr>
<tr>
<td>$\sin \alpha_{1-2}$</td>
<td>$6877 5353$</td>
<td>$M = \cos \theta = \cos \phi_1 \sin \alpha_{1-2}$</td>
</tr>
<tr>
<td>$\cos \alpha_{1-2}$</td>
<td>$7320 4755$</td>
<td>$N = \cos \phi_1 \cos \alpha_{1-2}$</td>
</tr>
<tr>
<td>$c_1 = FM$</td>
<td>$2156299.910^2$</td>
<td>$D = (1 - c)^2/(2N + c M)$</td>
</tr>
<tr>
<td>$c_2$</td>
<td>$\sqrt{(1 - f)^2}$</td>
<td>$P = c_2 (1 + f P M^2)^{-1}1.497698 \times 10^{-3}$</td>
</tr>
<tr>
<td>$\cos \alpha_1 = \sin \phi_1 / \sin \alpha_0$</td>
<td>$4439.9989$</td>
<td>$\alpha_0 = 63 39 26.271$</td>
</tr>
</tbody>
</table>

$d = D/4D$ | $1.164729361$ (rad) |

$d = 1.56$ | $53 06.008$ |

$\sin d = 0.9985 2248^1$ | $u = \chi(\alpha_1 - d) = 96 59 19.774$ | $\sin u = -72.50 9.111$ |

$\cos d = 0.543 4019$ | $W = 1 - 2 \sin u = 0.99314673$ | $\cos u = 1.684 9708$ |

$V = \cos u \cos d - \sin u \sin d = 16 58 06.17$ | $Y = 2 \sin \phi V \sin d = 0.000156 435$ |

$X = c_2^2 \sin d \cos (2V^2 - 1)$ | $\Delta \phi = d \times Y = d \times 1.54072926$ (rad) |

$\sin \Delta \phi = 0.9828 8109$ | $\cos \Delta \phi = 0.0550 9649$ | $\Delta \phi = 86 50 49.982$ |

$\cos \Sigma_0 = \chi$ | $\Sigma_0 = 2 \alpha_1 - \Delta \phi$ | $\chi$ |

$\tan \alpha_{2-1} = M/(N \cos \Delta \phi - \sin \phi_1 \sin \Delta \phi) = -2.116 0624^4$ | $\alpha_{2-1} = 295 17 18.960$ |

$\tan \phi_2 = -((\sin \phi_1 \cos \Delta \phi + N \sin \phi_1 \sin \Delta \phi) \sin \alpha_{2-1}) \times (1 - f)M$ | $\phi_2 = 44 59 59.330$ |

$\sin \Delta \phi = \cos \phi_1 \cos \Delta \phi - \sin \phi_1 \sin \Delta \phi \cos \alpha_{1-2}$ | $\Delta \phi = -3.44 48816$ |

$H = c_1(1 - X) \Delta \phi - c_2 \sin \Sigma_0 \cos \alpha_{2-1}$ | $0.003367 4999$ (rad) |

$\Delta \lambda = \Delta \phi - H$ | $\Delta \lambda = 14.124$ |

$\lambda_1 = 14.124$ |

CHECK

$M = \cos \theta = \cos \phi_1 \sin \alpha_{1-2}$ | $\cos \Delta \phi = \cos \phi_2$ |

$\lambda_1 = 14.124$ | $\lambda_1 + \Delta \lambda = 106 00 00.011$ |

**Figure 6. Direct computation—first order in f.**
INVERSE POSITION COMPUTATION FORM FOR LONG LINES. Given \( \phi_1, \lambda_1, \phi_2, \lambda_2 \) to find \( S, a_1, a_2, a_3 \).

Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

**INTERNATIONAL SPHEROID** a 6378.289 m b 6356.911.445 m

\( 1 - f = b/a = 0.9966329966 \) m 0016835017 m 0000891175085

\( f^2/64 = 0 \) m 1 radian = 206264.8062 seconds

\[
\begin{array}{lll}
\phi_1 & 20^\circ 0' 0'' & 1. \text{ INITIAL} \\
\phi_2 & 45^\circ 0' 0'' & 2. \text{ TERMINUS} \\
\tan \phi_1 & 36.397073 & \Delta \lambda = \lambda_2 - \lambda_1 106 \\
\tan \phi_2 & 1 & \Delta \lambda_m = \frac{1}{2} \Delta \lambda 52 \\
\theta_1 & 94^\circ 54' 12.168 & \tan \theta_2 = \frac{94663300}{\sin \Delta \lambda_m} = 1.92863551 \\
\theta_1 & 19^\circ 56' 16.706 & \tan \Delta \lambda = -3.48731941 \\
\theta_m = \frac{1}{2}(\theta_1 + \theta_2) & \sin \theta_m = \frac{53613146}{\cos \theta_m} = 0.84413450 \\
\Delta \theta_m = \frac{1}{2}(\theta_2 - \theta_1) & \sin \Delta \theta_m = \frac{31614487}{\cos \Delta \theta_m} = 0.97636130 \\
H = \cos^2 \Delta \theta_m - \sin^2 \theta_m = \cos^2 \theta_m - \sin^2 \Delta \theta_m = 6658.44451 - L 5.2859337 \\
L = \sin^2 \Delta \theta_m + H \sin^2 \Delta \lambda_m = 4714.0663 \cos d = 1 - 2L 4.05718674 \\
U = 2 \sin^2 \theta_m \cos^2 \Delta \theta_m (1 - L) = 1.036745079 \ d = \frac{E_0}{E} = 17.950 \\
V = 2 \sin^2 \Delta \theta_m \cos \theta_m / L = 0.14136672 \sin d = 0.998363500 \ d (\text{rad}) = 1.5135783741 \\
X = U + V = 1.177981175 \ T = d / \sin d = 41.620592405 \ E = 2 \cos d = 1.1432484 \\
Y = U - V = 1.855084077 \ D = 4T^2 \ X = B = 2D \ X \\
A = DE \ X \ C = T - \frac{1}{2}(A - E) \ X \ \text{CHECK} C = \frac{1}{2} E + AD / B = T \\
n_1 = X (A + CX) \ X \ n_2 = Y (B + EY) \ X \ n_3 = DXY \ X \\
\delta_1 = \frac{4\pi}{f^2} (TX - Y) = 0.004749479 \ \delta_2 = \frac{4\pi}{f^2} (n_1 - n_2 + n_3) \ X \\
S_1 = a \sin d (T - \delta_1 d) = 94449417.4944 \ m \ S_2 = a \sin d (T - \delta_1 d + \delta_2 d) \ X \ X \ m \\
F = 2Y - E (4X - X) = 1.44682527 \ M = 32T - (20T - A) X - (B + 4Y) \ X \\
G = \frac{4\pi T}{f^2} \ M = 0.023522886 \ Q = - (FG \tan \Delta \lambda) / 4 \ X \\
\Delta \lambda_m = \frac{1}{2} (\Delta \lambda + Q) 5.05 28.5673 \ tan \Delta \lambda_m = 1.33156506 \\
v = \arctan \frac{c_1}{c_2} 53.49 35.828 \ c_2 = \cos \Delta \theta_m / (\sin \theta_m \tan \Delta \lambda_m) + 1.36765600 \\
u = \arctan \frac{c_1}{c_1} 10.53 25.627 \ c_1 = - \sin \Delta \theta_m / (\cos \theta_m \tan \Delta \lambda_m) = -1.19229628 \\
\alpha_1 = v - u 42.56 29.951 \\
\alpha_2 \cos^{-1} 1.2 \ 2.3 \ 1.2 \ 360 - \alpha_2 = 0.45 17.875 \\
\alpha_3 \cos^{-1} 1.2 \ 2.3 \ 1.2 \ 360 - \alpha_3 = 0.45 17.875 \\
}\]

Figure 7. Inverse solution—first order in \( f \).
DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1, \alpha_{1,2}, S$ to find $\phi_2, \lambda_2, \alpha_{2,1}$. East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

<table>
<thead>
<tr>
<th>SPHERE</th>
<th>SPHEROID a=6370998 m f=0</th>
</tr>
</thead>
</table>

$1 - f = 0$  

1 radian = 206264.8062 seconds

**LINE INITIAL TO TERMINUS**

<table>
<thead>
<tr>
<th>$\phi_1$</th>
<th>$\alpha_{1,2}$</th>
<th>$\sin \alpha_{1,2}$</th>
<th>$\cos \alpha_{1,2}$</th>
<th>$c_1$</th>
<th>$c_2$</th>
<th>$M$</th>
<th>$N$</th>
<th>$D$</th>
<th>$P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$20^\circ 00' 00&quot;$</td>
<td>0.52.54.30.06</td>
<td>$0.6814.535.8$</td>
<td>$0.7320.475.5$</td>
<td>$0.6878.996.8$</td>
<td>$0.520.0.0.0$</td>
<td>$0.3420.201.4$</td>
<td>$0.9996.926.2$</td>
<td>$0.2100.489.0$</td>
<td>$0.9678.996.8$</td>
</tr>
</tbody>
</table>

**CHECK**

$M = \cos \theta_0 = \cos \theta_1 \sin \alpha_{1,2} = \cos \theta_2 \sin (180^\circ + \alpha_{2,1})$  

$\lambda_2 = \lambda_1 + \Delta \lambda = 105^\circ 59' 41.961'$
INVERSE POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1; \phi_2, \lambda_2$ to find $S_1, \alpha_1-2, \alpha_2-1$.

Azimuths clockwise from north: east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

\[
S_{\text{SPHERE}} = \text{SPHEROID:} \quad a = 6378137 \quad b = \frac{a}{\sqrt{2}} \quad m \quad \text{b} = \frac{X}{b/a} \quad \chi \quad \frac{b}{a} \quad X \\
f^2/64 = X \quad 1 \text{ radian} = 206264.8062 \text{ seconds}
\]

1. **INITIAL**
   \[
   \phi_1 \quad 0 \quad 0 \quad 0 \\
   \phi_2 \quad 0 \quad 45 \quad 0 \quad 0 \\
   \tan \phi_1 \quad \chi \quad \chi \quad \chi \\
   \tan \phi_2 \quad \chi \quad \chi \quad \chi \\
   \tan \theta_1 \quad \chi \\
   \tan \theta_2 \quad \chi \\
   \theta_1 = \frac{\phi_1 + \phi_2}{2} \quad 32 \quad 30 \quad \sin \theta_m = 0.53729461 \quad \cos \theta_m = 0.84339145 \\
   \theta_2 = \frac{\phi_1 + \phi_2}{2} \quad 12 \quad 30 \quad \sin \theta_m = 0.21642961 \quad \cos \theta_m = 0.97629601 \\
   \Delta \theta = \frac{\theta_2 - \theta_1}{2} \quad \chi \quad \chi \\
   \Delta \theta_m = \sin^2 \theta_m - \sin^2 \theta_m = \cos^2 \theta_m - \sin^2 \theta_m = 0.04446321 \quad 1 - \chi \\
   L = \sin^2 \Delta \theta_m + H \sin^2 \Delta \theta_m = 0.0475664 \quad \cos d = 1 - 2L \quad 0.05862943 \\
   U = 2 \sin^2 \theta_m \cos^2 \theta_m (1 - L) \quad \chi \\
   V = 2 \sin^2 \Delta \theta_m \cos^2 \theta_m / L \quad \chi \\
   X = U + V \quad \chi \quad \chi \quad \chi \\
   T = d / \sin d \quad \chi \\
   E = 2 \cos d \quad \chi \\
   Y = U - V \quad \chi \quad \chi \quad \chi \\
   A = DE \quad \chi \quad \chi \quad \chi \\
   B = 2D \quad \chi \\
   C = T - \frac{1}{2} (A - E) \quad \chi \\
   n_1 = X (A + CX) \quad \chi \\
   n_2 = Y (B + EY) \quad \chi \\
   \delta_1 d = W (TX - Y) \quad \chi \\
   S_1 = \tan \left( \frac{\delta_1 d}{2} \right) \quad 94.82355 \quad m \\
   F = 2Y - E (4 - X) \quad \chi \\
   G = \frac{1}{2} + \frac{1}{2} \tan \Delta \theta_m / 4 \quad \chi \\
   \Delta \lambda_m = \frac{1}{2} (\Delta \lambda + Q) \quad \chi \\
   v = \arctan \left( \frac{\delta_1 d}{2} \right) \quad 53 \quad 51 \quad 0.2273 \\
   u = \arctan \left( \frac{\delta_1 d}{2} \right) \quad 10 \quad 56 \quad 42.667 \\
   \alpha_1 = v + u \quad 23 \quad 54 \quad 24.006 \\
   \alpha_2 = v + u \quad 29 \quad 25 \quad 11 \quad 6259
\]

Figure 9. Inverse computation—spherical.
Table I gives a comparison of the given line elements, the control computation, second order, first orders, and spherical computations. See Appendix 3 for more examples of direct and inverse solutions for several line lengths and in several azimuths. Also see the evaluation and comparison in Appendix 1.

NOTE: Appendix 4 gives the Fortran statements for the inverse solution as presented here. The card deck including the arctangent library function (ATAN2) is available.

DISCUSSION OF PROBLEMS INVOLVING LONG GEODETIC LINES, LOCAL COORDINATE SYSTEMS, AND ASSOCIATED GEOMETRY

General Remarks.

If we wish to compute reference lines connecting islands, continents, shoals in ocean areas, there are several alternatives available depending on the purpose for which the reference is needed and the accuracy required. Direct scaling from a large accurate globe may be used. If a mean spherical representation of the reference ellipsoid can be tolerated, then a plot of computed great circle intervals on a authalic (equal area), autogonal (true angles about points), or sphyllactic (neither authalic nor autogonal) projection may suffice. Within a radius of 10 n.m. of a station, simple plane coordinates, appropriately scaled, will be adequate for most geodetic work, and small relative errors will be incurred as far as 100 n.m. See Table I4, Appendix 2, for errors in distance from the origin associated with plane coordinates involving several types of geometric projection. Also included there is a discussion of plane coordinates. See also reference [9].

For track line reference, the azimuthal equidistant or doubly equidistant projection may be useful, although both are sphyllactic. Appendix 2 has a discussion of the doubly equidistant projection with its equations. The Department of Scientific and Industrial Research, New Zealand, has found the azimuthal equidistant projections useful in their South Pacific studies, see reference [34].

<table>
<thead>
<tr>
<th>φ₁</th>
<th>λ₁</th>
<th>S(meters)</th>
<th>φ₁-1</th>
<th>λ₁-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>45</td>
<td>106</td>
<td>9649412.505</td>
<td>42 56 30.033</td>
<td>295 17 18.600</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>φ₂</td>
<td>λ₂</td>
<td>S(meters)</td>
<td>φ₂-1</td>
<td>λ₂-1</td>
</tr>
<tr>
<td>56</td>
<td>30.033</td>
<td>9649412.505</td>
<td>42 56 30.033</td>
<td>295 17 18.600</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>φ₃</td>
<td>λ₃</td>
<td>S(meters)</td>
<td>φ₃-1</td>
<td>λ₃-1</td>
</tr>
<tr>
<td>42</td>
<td>56 30.033</td>
<td>9649412.505</td>
<td>42 56 30.033</td>
<td>295 17 18.600</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>φ₄</td>
<td>λ₄</td>
<td>S(meters)</td>
<td>φ₄-1</td>
<td>λ₄-1</td>
</tr>
<tr>
<td>295 17 18.600</td>
<td>9649412.505</td>
<td>42 56 30.033</td>
<td>42 56 30.033</td>
<td>295 17 18.600</td>
</tr>
</tbody>
</table>

Table I. Comparison of direct and inverse direct computations.
The diagonal (skew, oblique) Mercator (cylindrical) projection, is often useful, since the base line (the track) is one of the reference axes for rectangular coordinates, the scale may be held true along the base line or along two parallels symmetric to the base line, and the projection is autogonal. A mathematical development is given in reference [16]. General tables exist to provide coverage for route charting, see reference [33].

For detail and greater accuracy in local area surveys connected with a base line, rectangular spherical coordinates may be more convenient, particularly for point to point computation away from the base line. The formulas for this kind of computation are included in Appendix 2. Appendix 2 also includes transformations from local rectangular spherical coordinates to space rectangular at a point of the ellipsoid referred to the normal, great normal section tangent, and meridional tangent, and this system in turn referred to the rectangular system at ellipsoid center, with the axis of rotation a coordinate axis. These may be useful relative to the adoption of the World Geodetic Reference System, 1967, see Appendix 2.

For oceanographic surveys, the positioning problem may not be essentially different from the navigation track plot. The gnomonic linear plot, with projection center on the track, gives the geographical coordinates of the great circle which can then be transferred to any suitable projection, the resulting curve being the great circle track. Distances may then be scaled from the map or chart, azimuths or bearings measured directly, if the map is autogonal, etc. Where accuracy requirements are not high, the possibility of using existing maps and charts should be considered since U. S. Government agencies such as AMS, GMIRADA, ACIS, C&GS, USGS, NAVOCEANO; the National Geographic Society; the State governments; mapping and charting agencies of other countries, collectively publish large numbers of maps, charts and grids on various projections and at several scales. Direct scaling from a large globe may suffice.

For world reference, positions may be expressed in terms of the Universal Transverse Mercator coordinate system, reference [35]. See also an extensive study of world plane coordinate reference systems and recommendations as given in reference [36]. Positions may also be referenced in rectangular coordinates at ellipsoid center, see Appendix 2.

Long spheroidal geodesics—partitioning.

If the end points are in triangulation nets on different spheroids, one station can be transferred to the ellipsoid of the other or both can be transferred to a third. The equations used in the NASA tracking system will be found in reference [37]. See also references [9] and [38].

With the end point coordinates on the same ellipsoid, an inverse computation will give the distance and azimuth. This may be done by use of a form such as Figure 5. The distance is partitioned according to a preplot of the base line on a globe, into stations to fit islands, shoal areas, etc. Beginning with the first distance and forward azimuth of the base line, the coordinates of the first station and back azimuth are computed by a direct solution using the form of Figure 4. For best accuracy use the order \( f^3 \) computation, keep the initial azimuth and position but increase the distance incrementally as partitioned until the terminal point is reached. The desk computing could be formidable if the line is very long and the stations numerous. Use of a large scale computer is then indicated if available.
Alternatively one may compute from station to station along the base line, but this requires additional computation, even if first order in $f$ suffices, since all input elements change for each succeeding computation.

**Spherical case.**

A method of computing stations along a great circle and parallels to the great circle simultaneously is given in reference [18]. Alternatively the forms as given in Figures 8 and 9, can be used. See also reference [39]. The best spherical radius to use is probably the ellipsoidal mean radius computed for the mean latitude of the base line terminals, see Appendix 2, equation (12).

**Problems in local geometry.**

**Problem.** To compute the geographic coordinates of a point at distance $S$ from a base line station and at angle $a$ with the base line. The geographic coordinates $\phi_1, \lambda_1$, and azimuth $a_1$ at the particular station are known, which with given $S$ and $a$, provide the input $\phi_1, \lambda_1, a_1 + a, S$ for a direct solution from the form as shown in Figures 4, 6 or 8, depending on the magnitude of $S$ and accuracy required. For a point at distance $S$ on the perpendicular to the base line, $a = 90^\circ$. If $S$ is constant and $a = 90^\circ$ at each base station, the direct computation at each station provides points on a parallel at a given distance $S$ from the geodetic base line (this geodesic parallel is not itself a geodesic). If the base line is a great circle, a circle parallel to it is generated. If geographic coordinates along a partitioned spherical base line with corresponding coordinates along two symmetric parallels are required, the method as given in reference [18] may be used.

**Problem.** Given the geographic coordinates $Q_1(\phi_1, \lambda_1), Q_2(\phi_2, \lambda_2)$ of two stations of a spherical base line, to find the perpendicular distance $s$ from an arbitrary third point $p(\phi_0, \lambda_0)$ to the base line.

From equations (3) and (4), page (23), reference [18] solve for $\phi_0, \lambda_0$:

\[
\tan \lambda_0 = (\tan \phi_2 \cos \lambda_1 - \tan \phi_1 \cos \lambda_2) / (\tan \phi_1 \sin \lambda_2 - \tan \phi_2 \sin \lambda_1)
\]

\[
\cot \phi_0 = \cot \phi_1 \cos (\lambda_0 - \lambda_1) = \cot \phi_2 \cos (\lambda_0 - \lambda_2).
\]

From the two figures, page (27) of reference [18], using the spherical formula $\cos a = \cos b \cos c + \sin b \sin c \cos A$, with $a = s$, find

\[
\sin s = \pm [\sin \phi_1 \cos \phi_0 - \cos \phi_1 \sin \phi_0 \cos (\lambda_0 - \lambda_2)],
\]

where the $+$ sign corresponds to $k = p$, the $-$ sign to $k = p'$, relative to the points $p(\phi_0, \lambda_0), p'(\phi_0', \lambda_0')$ respectively as shown in Figure 3, page 26, reference [18].

Note also the solution in Appendix 2 following equations (47), with reference to the distance $s$ of Figure 34. Additionally $s = y$-coordinate of the doubly-equidistant projection, see the discussion following equations (56), Appendix 2.

**Problem.** An observer at the known station $Q(\phi_0, \lambda_0)$, $h_o$ meters above the spherical surface (assumed sea level), Figure 31, measures a linear distance $D$ to $S_o$ (target on a hill, island mountain peak, etc.) at a measured angle of elevation $\delta$, and in measured or known azimuth $a$. If the spheroid at $Q$ is approximated with a sphere of radius $N_0$ (the great normal length for $\phi_0$, equation 11, Appendix 2) find the rectangular space coordinates of $S_o$ referred to the normal and tangents to the parallel and meridian at $Q$, the geographic
coordinates of the normal projection P of S₀ upon the sphere, the spherical distance d = PQ and the height h of S₀ above the sphere (sea level). We have a, D, N₀, δ, φ₀, λ₀, h₀, to find X, Y, Z, h, d, φ, λ. From Figure 31, and some trigonometric identities we have $D_1 = D \cos \delta, X = D_1 \cos \alpha, Y = D_1 \sin \alpha, Z = h_0 + D \sin \delta, \tan \tau = D_2/(N_0 + Z), d = N_0 \tau$ (radians), $h = (N_0 + Z) \sec \tau - N_0, h = D_2 \sec \tau - N_0, \sin \phi = \cos d \sin \phi_0 + \sin d \cos \phi_0 \cos \alpha, \cot \Delta \lambda = (\cos \phi_0 \cos d - \sin \phi_0 \sin d \cos \alpha)/\sin d \sin \alpha, \lambda = \lambda_0 - \Delta \lambda$.

These problems illustrate the use of the geodetic line computing forms, and the formulae of Appendix 2, for solving local problems of computation for a station configuration. For very long base lines, it may be desirable to compute the positions of the stations along them very accurately, but in the vicinity of a particular station, a spherical approximation or plane coordinate configuration may suffice. Additional formulae such as for dip; maximum separation, chord-arc; geographic coordinates of point of maximum separation, etc. will be found in reference [18]. Other coordinate problems are discussed in Appendix 2. For uniform high accuracy over a considerable extent of the spheroid, a plane rectangular coordinate system based on one of the autogonal projections as used for geodesy may be more appropriate, see references [9], [16], [36].

**BIBLIOGRAPHY**


[6] Tobe, W.M. Geodesy, Geodetic Survey of Canada Publication No. 11, 1928. Note on page 24, that if we replace D, N, I, m with their functions of latitude and azimuth, the expression for the difference in length of the normal section and the geodesic may be written:

$$S_n - S = (b^2S^3 \sin^2 \alpha \cos^2 \alpha \cos^4 \phi/90)^{1/2} + F(b, S, \alpha, \phi)e^a + \ldots$$

which is the 4th order in the eccentricity, e.

[7] Ward, L.E. Geodesics and Plane Arcs on an Oblate Spheroid, American Mathematical Monthly, Aug.-Sept. 1943. Note that the separate expansions for the geodesic and the great elliptic arc in the same parameters, as found on pages 426 and 428 respectively, are identical to terms in e^a.

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[39] Carver, H.C. Distance and Azimuth Computations (Spherical) with Tables, Engineering Research Institute, University of Michigan, for Air Research and Development Command. 1954.
APPENDIX I. MATHEMATICAL DISCUSSION OF THE SPHEROIDAL GEODESIC
MATHEMATICAL DISCUSSION OF THE SPHEROIDAL GEODESIC

In Figure 10, \( \alpha \) is the angle which the differential arc length, \( ds \), makes with the meridian at \( P_1 \). The radius of the parallel in parametric latitude \( \theta \) is \( a \cos \theta \). Then \( a \cos \theta \, d\lambda \) is the differential arc length along the parallel in latitude \( \theta \). Now the element of arc length along the meridian is defined as \( Rd\phi \) where \( R \) is the radius of curvature in the meridian given by \( R = a(1-e^2)/(1-e^2 \sin^2 \phi)^{3/2} \), see reference [16], page 59. The transformation between geodetic and parametric latitude is \( \tan \psi = \tan \theta/(1-e^2)^{1/2} \).

From the differential right triangle \( PP_1P_2 \), we have \( ds^2 = \sin^2 \theta \, d\lambda^2 + r^2 \, d\phi^2 \) where \( r^2 = a^2(1-e^2 \cos^2 \theta) \).

Figure 10. Differential arc length on the oblate spheroid as obtained from a differential right triangle.
whence
\[
\frac{1}{(1 - e^2 \sin^2 \phi)^{3/2}} = \frac{(1 - e^2 \cos^2 \phi)^{3/2}}{(1 - e^2)^{3/2}}, \quad \sin \phi = (1 - e^2)^{1/2} \cos \theta/(1 - e^2 \cos^2 \theta),
\]
and
\[
Rd\phi = a(1 - e^2)^{3/2} \frac{(1 - e^2)(1 - e^2 \cos^2 \phi)^{3/2}}{(1 - e^2)^{3/2}} = \frac{a(1 - e^2)(1 - e^2 \cos^2 \theta)^{3/2}}{(1 - e^2)^{3/2}},
\]
or
\[
Rd\phi = r d\theta, \quad \text{where} \quad r = a(1 - e^2 \cos^2 \theta)^{1/2} = R(1 - e^2)^{1/2}/(1 - e^2 \cos^2 \theta).
\]

**NOTE** that \( r \) is not the radius of curvature in the spheroidal meridian, but \( r \) is the differential arc length along the meridian in terms of parametric latitude and applying the pythagorean theorem to the right differential triangle \( P_1P_2P_3 \) we have at once the formula for the general differential arc length on the spheroid in terms of parametric latitude:

\[
ds^2 = a^2 [(1 - e^2 \cos^2 \theta) d\theta^2 + \cos^2 \theta d\lambda^2].
\]

**Differential equation from Euler’s Condition**

We may write (1) as

\[
s = f H d\theta
\]

where

\[
H = a[1 - e^2 \cos^2 \theta + \cos^2 \theta \lambda'^2]^{1/2}, \quad \lambda' = d\lambda/d\theta.
\]

Now along geodesics, the Euler equation \( d(\partial H/\partial \lambda')/d\theta - \partial H/\partial \lambda = 0 \) must be satisfied.

Since \( \partial H/\partial \lambda = 0 \), the equation is \( d(\partial H/\partial \lambda')/d\theta = 0 \), a first integral being then

\[
\partial H/\partial \lambda' = c (\text{constant}).
\]

From (2) \( \partial H/\partial \lambda' = (a\lambda' \cos^2 \theta)/H \) and this value placed in (3) gives

\[
a\lambda' \cos^2 \theta = cH = ac[1 - e^2 \cos^2 \theta + \cos^2 \theta \lambda'^2]^{1/2}
\]

Solving (4) for \( \lambda' \) and then placing \( \lambda' = d\lambda/d\theta \) gives

\[
d\lambda = \frac{c}{\cos \theta} \cdot (1 - e^2 \cos^2 \theta)^{1/2} \cos \theta d\theta.
\]

From (2), \( H = ds/d\theta \) and this value placed in (4) gives

\[
a \cos^2 \theta d\lambda/ds = c, \quad \text{or} \quad a \cos^2 \theta d\lambda/ds = ac.
\]

From the differential right triangle \( P_1P_2P_3 \) of Figure 10

\[
\cos (90^\circ - \alpha) = a \cos \theta d\lambda/ds = \sin \alpha.
\]

The value from (7) placed in (6) gives

\[
\cos \theta \sin \alpha = c, \quad \text{or} \quad \cos \theta \sin \alpha = ac.
\]

Since \( \cos \theta \) is the radius of the parallel in latitude \( \theta \) and \( \alpha \) is the angle which the geodesic makes with the meridian as shown in Figure 10, equation (8) states that the product of the radius of the parallel and the sine of the azimuth, \( \alpha \), is a constant along the geodesic.

Now the geodesic will be orthogonal to a meridian when \( \alpha = 90^\circ \), and using this value in (8) we have

\[
c = \cos \theta_0, \quad \text{where} \quad \theta_0 \text{ is the parametric latitude of the vertex of the geodesic.}
\]

With this value of \( c \), equation (5) becomes

\[
d\lambda = \cos \theta_0 \frac{(1 - e^2 \cos^2 \theta)^{1/2}}{\cos \theta} \frac{d\theta}{(\cos^2 \theta - \cos^2 \theta_0)^{1/2}} d\theta,
\]

where always

\[
\cos \theta \sin \alpha = \cos \theta_0.
\]
With the differential equation to the geodesics in this form, equation (9), we can make several observations concerning the behavior of the geodesic. The substitution of \( \pm \theta \) does not alter the coefficient of \( d\theta \), since \( \cos \pm \theta = \cos \theta \) and therefore the curve is symmetric about the equatorial plane. When \( \theta = \pm \theta_0 \), \( d\theta = 0 \), which means that the geodesic is tangent to the parallels \( \theta = \pm \theta_0 \) and hence undulates alternately between tangencies to them.

From (10), with \( \theta = 0 \), we have \( \sin a_0 = \cos \theta_0 \). That is at a node (a point where the geodesic crosses the equator) the sine of the angle which the geodesic makes with the meridian is equal to the cosine of the parametric latitude of the vertex, or \( a_0 = 90 - \theta_0 \).

For reference in the developments to follow we include here a short resume of elliptic integrals and functions to be used, [26].

**Elliptic Integrals (Legendre Forms)**

\[
S = F(k, \phi) = \int_0^\phi \frac{da}{\sqrt{1 - k^2 \sin^2 a}}, k < 1
\]

\[
E(k, \phi) = \int_0^\phi \sqrt{1 - k^2 \sin^2 a} da = \int_0^\phi \Delta a da
\]

\[
\Pi(n, k, \phi) = \int_0^\phi \frac{da}{(1 + n \sin^2 a)(1 - k^2 \sin^2 a)^{1/2}} = \int_0^\phi \frac{da}{\delta a \Delta a}
\]

**Complete Elliptic Integrals**

\[
K = F(k, \pi/2) = \int_0^{\pi/2} \frac{da}{\sqrt{1 - k^2 \sin^2 a}} = \int_0^{\pi/2} \frac{da}{\Delta a}
\]

\[
E = E(k, \pi/2) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 a} da = \int_0^{\pi/2} \Delta a da
\]

**Elliptic Functions**

In the elliptic integral of the first class, \( \phi \) is called the amplitude of \( S = F(k, \phi) \) and \( k < 1 \) is the modulus. \( \sin \phi, \cos \phi, \Delta \phi \) are called the sine, cosine, delta of the amplitude of \( S \) and we have the following:

**Definitions (Jacobi)**

\[ a = \sin S, \sin a = \sin S, \cos a = \cos S, \tan a = \tan S, \]

\[ \Delta a = (1 - k^2 \sin^2 a)^{1/2} = \mathrm{d}n S. \]

**Identities**

\[ \sin^2 S + \cos^2 S = \sin^2 S + k^2 \sin^2 S = 1, \tan S = \sin S/\cos S, \]

\[ \Delta a = (1 - k^2 \sin^2 a)^{1/2} = \mathrm{d}n S. \]
\[ E(k, \sigma_1) \pm E(k, \sigma_2) = E(k, \sigma_1 \pm \sigma_2) \pm k^2 \sin \sigma_1 \sin \sigma_2 \]
\[ \cos(S + S') = (\cos S \cos S' \pm \sin S \sin S' \cos(S + S))/(1 - \cos^2 S \sin^2 S) \]
\[ E(x_1 + 2K, k) = E(x_1, k) + 2E, E - cK = 2cc'dK/dc \]
\[ E(x_1, k) - c'x_1 = c \int_{0}^{x_1} \cos^2 x \, dx \]

**Special Values:**

- \( S = 0 \); \( \sin(0) = 0 \), \( \cos(0) = \tan(0) = 0 \);
- \( S = K \); \( \sin K = 1, \cos K = 0, \tan K = (1 - k^2)^{1/2} \);
- \( S = 2K \); \( \sin(2K) = 0, \cos(2K) = -\sin(K) = -1, \tan(2K) = \tan(K) = 1; \)
- \( \cos(S + 2K) = -\sin S \).

**Differentials:**

\[ d\sin S = \sigma \cos \sigma d \sigma = \cos S \sin S dS \]
\[ d\cos S = -\sin \sigma d \sigma = -\sin S \cos S dS \]
\[ d\tan S = \sec^2 \sigma d \sigma = \sec^2 S \cos^2 S dS \]
\[ d\cot S = \csc^2 \sigma d \sigma = \csc^2 S \sin^2 S \cos S dS \]

Note that the elliptic functions as determined by (13) have an analogy with trigonometric functions but \( S \) is not an angle as is clear from its integral definition, (12). Like trigonometric functions they have a real period and like exponential functions have a pure imaginary period and are thus doubly periodic.

If we define \( K' = F(k', \pi/2) \) where \( k' = (1 - k^2)^{1/2} \), that is \( K' \) is the complete integral \( K \) of (12) with the modulus \( k \) replaced by \( k' \) then the periods of the elliptic functions \( \sin S, \cos S, \tan S \) are:

<table>
<thead>
<tr>
<th>Function</th>
<th>Periods</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin S )</td>
<td>( 4K, 2iK' )</td>
</tr>
<tr>
<td>( \cos S )</td>
<td>( 4K, 2K + 2iK' )</td>
</tr>
<tr>
<td>( \tan S )</td>
<td>( 2K, 4iK' )</td>
</tr>
</tbody>
</table>

where \( i = \sqrt{-1} \), \( K = F(k, \pi/2) \), \( K' = F(k', \pi/2) \), \( k' = (1 - k^2)^{1/2} \), \( k < 1 \).

**Expression of longitude and arc length in elliptic integrals.**

If we let \( \cos \sigma = \cos S \sin \theta \), \( \sin \theta \), we have then:

\[ 1 - e^2 \cos^2 \theta = (1 - e^2 \cos^2 \theta_0)(1 - k^2 \sin^2 \sigma) = (1 - e^2 \cos^2 \theta_0) \Delta \sigma^2 \]
\[ \cos^2 \theta - \cos^2 \theta_0 = \sin^2 \theta_0 \sin^2 \sigma \]
\[ \sin^2 \theta = 1 - \sin^2 \theta = 1 - \sin^2 \theta_0 \cos^2 \sigma = \cos^2 \theta_0 (1 + n \sin^2 \sigma) \delta \cos^2 \theta_0 \]
\[ \cos^2 \sigma = \csc^2 \theta_0 \sin^2 \theta = \csc^2 \theta_0 (1 - \cos^2 \theta_0) \]
\[ = \csc^2 \theta_0 - \cot^2 \theta_0 (1 + n \sin^2 \sigma) = \csc^2 \theta_0 - \delta \cot^2 \theta_0 \]
\[ \sin^2 \sigma = 1 - \cos^2 \sigma = -\cot^2 \theta_0 + \cot^2 \theta_0 (1 + n \sin^2 \sigma) = -\cot^2 \theta_0 (1 - \delta \sigma) \]
\[ k^2 = e^2 \sin^2 \theta_0/(1 - e^2 \cos^2 \theta_0), n = \tan^2 \theta_0. \]
Eliminating $d\lambda$ between equations (1) and (9) we have

$$ds = \frac{e(1 - e^2 \cos^2 \theta)^{1/2} \cos \theta}{(\cos^2 \theta - \cos^2 \theta_0)^{1/2}} d\theta$$

(16)

Applying the transformation equations (15) to (16) and (9) we get

$$S = \frac{e \sin \theta_0}{k} \int_0^\alpha (1 - k^2 \sin^2 \alpha)^{1/2} d\alpha = \frac{e \sin \theta_0}{k} \int_0^\alpha \Delta \alpha d\alpha,$$

$$\Delta \lambda = \frac{e \tan \theta_0}{k} \int_0^\alpha (1 - k^2 \sin^2 \alpha)^{1/2} \frac{d\alpha}{1 + n \sin^2 \alpha} = \frac{e \tan \theta_0}{k} \int_0^\alpha \Delta \alpha d\alpha.$$  

(17)

In the second of (17), multiply numerator and denominator of the integrand by $(1 - k^2 \sin^2 \alpha)^{1/2}$ and in the resulting numerator replace $\sin^3 \alpha$ with its value from (15) which then allows the integral to be written in the form

$$\Delta \lambda = \frac{e \tan \theta_0}{k} \left[ \int_0^\alpha (1 + k\cot^2 \theta_0) \frac{d\alpha}{\Delta \alpha \sin \alpha} \right].$$  

(18)

Now comparing the first integral of (17) and the integrals of (18) with the elliptic integrals (12) we can then write

$$S = \frac{e \sin \theta_0}{k} E(k, \alpha),$$

$$\Delta \lambda = \frac{e \tan \theta_0}{k} \left[ (1 + k^2 \cot^2 \theta_0) \Pi(n, k, \alpha) - k^2 \cot^2 \theta_0 F(k, \alpha) \right],$$

(19)

Where the modulus is $k = e \sin \theta_0/(1 - e^2 \cos^2 \theta_0)^{1/2}; n = \tan^2 \theta_0$; and the amplitude is $\alpha = \arccos(\sin \theta_0/\sin \theta_0)$ or the spherical length from the vertex of the geodesic in parametric latitude $\theta_0$ to a point in parametric latitude $\theta$ on the geodesic as shown in Figure 11, $|\theta| < |\theta_0|$. That is, the formulae (19) give longitude and distance along the geodesic measured from the geodesic vertex in terms of the spherical distance.

The elliptic functions in terms of the amplitude $\alpha$ and modulus $k; \alpha = \arccos(\sin \theta_0/\sin \theta_0), k = e \sin \theta_0/(1 - e^2 \cos^2 \theta_0)^{1/2}.$

From the definitions (13) we have:

$$cS_n = \cos \alpha = \sin \theta_0/\sin \theta_0; \sin S_n = \sin \alpha = (\sin^2 \theta_0 - \sin^2 \theta)^{1/2}/\sin \theta_0;$$

$$tS_n = \tan \alpha = \sin S_n/\cos S_n = (\sin^2 \theta_0 - \sin^2 \theta)^{1/2}/\sin \theta;$$

$$\Delta \alpha = \Delta S_n = (1 - k^2 \sin^2 \alpha)^{1/2} = (1 - e^2 \cos^2 \theta)^{1/2}/(1 - e^2 \cos^2 \theta_0)^{1/2};$$

$$\delta \alpha = 1 + n \sin^2 \alpha = 1 + n \sin^2 S_n = sec^2 \theta_0 \cos^2 \theta, n = \tan^2 \theta_0.$$  

(19a)

Since $\alpha = \arccos(\sin \theta_0/\sin \theta_0)$, we have the correspondences $\theta = 0, \alpha = \pi/2; \theta = \theta_0, \alpha = 0$. From (13c), (17) and (19a), we may write for the geodesic, vertex to vertex or node to node:

$$S_0 = 2\alpha(1 - e^2 \cos^2 \theta_0)^{1/2} \int_0^{\pi/2} \sin^2 S_n dS$$

$$\Delta \lambda_0 = \frac{2(1 - e^2 \cos^2 \theta_0)^{1/2}}{\cos \theta_0} \int_0^{\pi/2} \sin^2 S_n dS.$$  

(19b)
When \( \theta = \theta_0 = 0 \), we have from (19)a that \( 1 + n \sin^2 S = \text{dn} S = 1 \), and from (19)b,

\[ \Delta \lambda_0 = 2(1 - e^2)^{1/2} \int_0^{\pi a} \text{ds} = \pi(1 - e^2)^{1/2} = \pi b/a. \]

\( S_0 = \sin(1 - e^2)^{1/2} = \pi b; \) where \( a, b \) are the semimajor, semiminor axes of the spheroid. This shows that an arc of the equator of length \( \pi b \) is a limiting position of spheroidal geodesics and that there are no antipodal points on nonplanar spheroidal geodesics.

Since the vertex, \( \theta_0 \), may be negative and internal or external to a segment \( S_{1,0} \) of the geodesic, all alternatives are included from the first of (19) by writing

\[ S_{1,0} = \frac{e a}{k} \sin \theta_0 \{ E(k, \sigma_1) \pm E(k, \sigma_2) \}, \]

and by use of the addition formula for elliptic integrals of the second class with the same modulus, from (13)a, we may write (20) as

\[ S_{1,0} = \frac{e a}{k} \sin \theta_0 \{ E(k, \sigma_1 \pm \sigma_2) \pm k^2 \sin \sigma_1 \sin \sigma_2 \sin (\sigma_1 \pm \sigma_2) \}, \]

where \( \sigma_1 = \arccos (\sin \theta_1 / \sin \theta_0), \sigma_2 = \arccos (\sin \theta_2 / \sin \theta_0), \) \( k = e \sin \theta_0 / (1 - e^2 \cos^2 \theta_0)^{1/2}. \)

Similar expressions may be written for the longitude difference from the second of (19).

Integration of differential equations

Since many tables of the elliptic integrals exist it would appear that evaluation of expressions like (21) would be simple. But (21) is in terms of \( \theta_0 \), the parametric latitude of the vertex of the geodesic, and
not obtainable directly from the geographic coordinates of two given points on the nonplanar geodesic. Interpolation in the tables is not easy. Since the eccentricity and flattening of oblate spheroids, as used for the earth representation, are small, series expansions in them converge rapidly and numerical evaluation is then relatively simple. Now the elliptic integrals themselves can be expanded in series of $e$ or $f$ since the modulus $k$ is a function of $e$—see equations (19)—but we will first expand the differential equations (9) and (16) in powers of $e$ and of $f$ and integrate term by term. The eccentricity, $e$, and flattening, $f$, are connected by the relation $1 - f = (1 - e^2)^{1/2}$, or $e^2 = 2f - f^2$.

From (9) and (16) we write again for reference

$$d\lambda = \frac{\cos \theta (1 - e^2 \cos^2 \theta)^{1/2}}{\cos \theta (\cos^2 \theta - \cos^2 \theta_0)^{1/2}} \, d\theta$$

$$dz = \frac{s \cos^2 \theta}{\cos \theta} \, d\lambda = \frac{s \cos \theta (1 - e^2 \cos^2 \theta)^{1/2}}{(\cos^2 \theta - \cos^2 \theta_0)^{1/2}} \, d\theta$$

(22)

The expansion by the binomial formula of $(1 - e^2 \cos^2 \theta)^{1/2}$ to sixth order in $e$ is

$$(1 - e^2 \cos^2 \theta)^{1/2} = 1 - (1/2)e^2 \cos^2 \theta - (1/8)e^4 \cos^4 \theta - (1/16)e^6 \cos^6 \theta - \ldots$$

(23)

If we place $e^2 = 2f - f^2$, $e^4 = 4f^2 - 4f^3$, $e^6 = 8f^3$, then

$$(1 - e^2 \cos^2 \theta)^{1/2} = 1 - f \cos^2 \theta + (f^2/2) \cos^3 \theta - \cos^4 \theta + (f^3/2) \cos^5 \theta - \cos^6 \theta + \ldots$$

(24)

Substituting from (23) and (24), in (22) we find

$$\Delta \lambda = l_1 - (e^2/2) \cos \theta_0 l_2 + (e^4/8) \cos \theta_0 l_3 - (e^6/16) \cos \theta_0 l_4 - \ldots$$

$$S = s[l_1 - (e^2/2) l_2 + (e^4/8) l_3 - (e^6/16) l_4 - \ldots]$$

(25)

Where

$$l_1 = \int \cos \theta \, d\theta = \int \frac{\cos \theta}{\cos \theta_0 (\cos^2 \theta - \cos^2 \theta_0)^{1/2}} = \int \frac{\sec \theta}{\tan \theta_0 (\tan \theta)^{1/2}} = \int \sec \theta = \gamma,$$

$$L_1 = \int \frac{\cos \theta \, d\theta}{\cos \theta_0 (\cos^2 \theta - \cos^2 \theta_0)^{1/2}} = \int \frac{\cos \theta}{\sin \theta_0 (\tan \theta)^{1/2}} = \int \frac{\cos \theta}{\sin \theta_0} = \beta,$$

$$l_3 = \int \frac{(1 - \sin^2 \theta) \cos \theta \, d\theta}{\sin \theta_0 (\sin^2 \theta - \sin^2 \theta_0)^{1/2}} = \int \frac{(1 - \sin^2 \theta) \cos \theta}{\sin \theta_0 (\sin^2 \theta - \sin^2 \theta_0)^{1/2}} = \int \frac{(1 - x^2) \, dx}{(c^2 - x^2)^{1/2}}$$

$$L_4 = \int \frac{(1 - \sin^2 \theta)^2 \cos \theta \, d\theta}{\sin \theta_0 (\sin^2 \theta - \sin^2 \theta_0)^{1/2}} = \int \frac{(1 - \sin^2 \theta)^2 \cos \theta}{\sin \theta_0 (\sin^2 \theta - \sin^2 \theta_0)^{1/2}} = \int \frac{(1 - x^2)^2 \, dx}{(c^2 - x^2)^{1/2}}$$

(27)

and where $x = \sin \theta$, $c = \sin \theta_0$. 33
We let $x = c \sin \beta$ in the three last integrals of (27), whence $dx = c \cos \beta \, d\beta$, and the integrals may be written

$$I_1 = \int (1 - c^2 \sin^2 \beta) d\beta, \quad I_4 = \int (1 - c^2 \sin^2 \beta)^2 d\beta, \quad I_5 = \int (1 - c^2 \sin^2 \beta)^3 d\beta$$

where $c = \sin \theta_0$ and $\beta$ is the integral $I_3$ of (26).

Now

$$\sin^2 \beta = \frac{1}{2} (1 - \cos 2\beta)$$

$$\sin^4 \beta = \frac{1}{8} (3 - 4 \cos 2\beta + \cos 4\beta)$$

By expanding the integrands in equations (28) and using the identities (29) we are able to integrate term by term and find that the integrals (28) are

$$I_2 = \frac{1}{4} \left[ 2(1 + \cos^2 \theta_0) \beta + \sin^2 \theta_0 \sin 2\beta \right]$$

$$I_4 = \frac{1}{32} \left[ 4(8 \cos^2 \theta_0 + 3 \sin^6 \theta_0) \beta + 8 \sin^2 \theta_0 (1 + \cos^2 \theta_0) \sin 2\beta + \sin^4 \theta_0 \sin 4\beta \right]$$

$$I_5 = \frac{1}{192} \left[ 12(1 + \cos^2 \theta_0)(8 \cos^2 \theta_0 + 5 \sin^6 \theta_0) \beta + 9(16 \cos^2 \theta_0 + 5 \sin^6 \theta_0) \sin^2 \theta_0 \sin 2\beta + 9(1 + \cos^2 \theta_0) \sin^4 \theta_0 \sin 4\beta + \sin^6 \theta_0 \sin 6\beta \right]$$

where

$$\beta = I_3 = \text{arc sin} \left( \frac{\sin \theta}{\sin \theta_0} \right)$$

Formulas referred to a node

If we place the values of the integrals $I_1, I_2, I_3, I_4, I_5$ from (26) and (30) in (25) we may write in terms of $e$

$$\Delta \lambda = \gamma - e^3 \cos \theta_o \beta - \frac{e^3}{32} \cos \theta_o [2(1 + \cos^2 \theta_0) \beta + \sin^2 \theta_0 \sin 2\beta] - \frac{e^6}{512} \cos \theta_o \left[ 4(8 \cos^2 \theta_0 + 3 \sin^6 \theta_0) \beta + 8 \sin^2 \theta_0 (1 + \cos^2 \theta_0) \sin 2\beta \right] + \sin^4 \theta_0 \sin 4\beta$$

$$S/a = \beta - \frac{e^3}{8} \left[ 2(1 + \cos^2 \theta_0) \beta + \sin^2 \theta_0 \sin 2\beta \right] - \frac{e^3}{256} \left[ 4(8 \cos^2 \theta_0 + 3 \sin^6 \theta_0) \beta + 8 \sin^2 \theta_0 (1 + \cos^2 \theta_0) \sin 2\beta \right] + \sin^4 \theta_0 \sin 4\beta$$

$$- \frac{e^6}{3072} \left[ 12(1 + \cos^2 \theta_0)(8 \cos^2 \theta_0 + 5 \sin^6 \theta_0) \beta + 9(16 \cos^2 \theta_0 + 5 \sin^6 \theta_0) \sin^2 \theta_0 \sin 2\beta + 9(1 + \cos^2 \theta_0) \sin^4 \theta_0 \sin 4\beta + \sin^6 \theta_0 \sin 6\beta \right]$$

and in terms of $f$

$$\Delta \lambda = \gamma - f \cos \theta_0 \beta + \frac{f^3}{8} \cos \theta_0 \sin^2 \theta_0 (2\beta - \sin 2\beta)$$

$$+ \frac{f^3}{64} \cos \theta_0 \sin^2 \theta_0 [4(1 + 3 \cos^2 \theta_0) \beta - 8 \cos^2 \theta_0 \sin 2\beta - \sin^2 \theta_0 \sin 4\beta]$$

$$+ \frac{f^6}{512} \left[ 12(1 + \cos^2 \theta_0)(8 \cos^2 \theta_0 + 5 \sin^6 \theta_0) \beta + 9(16 \cos^2 \theta_0 + 5 \sin^6 \theta_0) \sin^2 \theta_0 \sin 2\beta + 9(1 + \cos^2 \theta_0) \sin^4 \theta_0 \sin 4\beta + \sin^6 \theta_0 \sin 6\beta \right]$$

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\[ S/a = \beta - \left( \frac{1}{16} \right) [2(1 + \cos^2 \theta) \delta + \sin^2 \theta \sin 2\delta] \]
\[
\frac{f^2}{64} \sin^3 \theta \left[ 4(1 + 3 \cos^2 \theta) \delta - 8 \cos^2 \theta \sin 2\delta - \sin^2 \theta \sin 4\delta \right]
\]
\[
+ \frac{f^2}{384} \sin^3 \theta \left[ 12(1 + 2 \cos^2 \theta) \delta + 5 \cos^2 \theta \right]
\]
\[
- 3(1 + 3 \cos^2 \theta) \sin^2 \theta \sin 4\delta - \sin^2 \theta \sin 6\delta \]

Where \( \gamma = \text{arc sin} \left( \frac{\tan \theta}{\tan \theta} \right), \beta = \text{arc sin} \left( \frac{\sin \theta}{\sin \theta} \right) \).

Limiting cases of integral equations

We first make some preliminary evaluations of equations (32). First we find the values of \( \Delta \lambda, S \) between \( \theta = 0, \theta = \theta_0 \) or from a node to the first vertex. For \( \theta = 0, \beta = \gamma = \text{arc sin} \, 0 = 0 \). For \( \theta = \theta_0, \beta = \gamma = \text{arc sin} \, 1 = \pi/2 \), and from (32) we have (doubling the result)
\[
\Delta \lambda_0 = \pi \left[ 1 - f \cos \theta_0 + \left( \frac{f^2}{4} \right) \cos \theta_0 \sin^2 \theta_0 + \left( \frac{f^3}{16} \right) \cos \theta_0 \sin^2 \theta_0 (1 + 3 \cos^2 \theta_0) \right],
\]
\[
S_0 = \pi \left[ 1 - \left( \frac{f}{2} \right) \left( 1 + \cos^2 \theta_0 \right) + \left( \frac{f^3}{16} \right) \sin^2 \theta_0 (1 + 3 \cos^2 \theta_0) \right]
\]
\[
+ \left( \frac{f^3}{32} \right) \sin^2 \theta_0 (1 + 2 \cos^2 \theta_0 + 5 \cos^4 \theta_0),
\]

which will subsequently be shown to give all hemispheroidal geodetic, vertex to vertex or node to node, compare (19)b.

The expressions (33) are even functions of \( \theta_0, f(\theta_0) = f(\theta_0) \), which would be expected from the discussion of symmetry following equation (10). Therefore, the expressions (33) give longitude and distance between successive vertices and also between successive nodes.

From the first of equations (33) we have \( \pi - \Delta \lambda_0 = \pi \cos \theta_0 \cdot \left[ 1 - \left( \frac{f^2}{4} \right) \sin^2 \theta_0 \right] \)
\[
-(\frac{f^3}{16}) \sin^2 \theta_0 (1 + 3 \cos^2 \theta_0),
\]
which shows again that except for the meridian \( \theta_0 = \pi/2 \), two consecutive vertices of the geodesic on the oblate spheroid cannot be antipodal (end points of a diameter).

From equations (32) and (33) we have with
\[
\theta_0 = 0: \quad \Delta \lambda_0 = \pi (1 - f) = \pi (1 - e^2)^{1/2} = \pi b/a
\]
\[
S_0 = \pi (1 - f) = \pi b = \pi \Delta \lambda_0;
\]
\[
\theta_0 = \pi/2: \quad \Delta \lambda_{\pi/2} = \pi,
\]
\[
S_{\pi/2} = \pi (1 - f/2 + f^2/16 + f^4/32 + \ldots).
\]

If we take the derivative of \( S_0 \) with respect to \( \theta_0 \) and place equal to zero we obtain
\[
\sin \theta_0 \cos \theta_0 \left[ 15 f^2 \cos^6 \theta_0 + 6f(2 - f) \cos^3 \theta_0 + 16 - 4f - f^2 \right] = 0.
\]

The discriminant of the quadratic factor in \( \cos^4 \theta_0 \) is \( 48f^4 [2f(1 + f) - 17] < 0 \), since \( f < 1 \), hence the only real values are given by \( \sin \theta_0 = 0, \cos \theta_0 = 0 \), or by \( \theta_0 = 0, \theta_0 = \pi/2 \); equations (34) are actually the upper and lower limits to hemispheroidal geodetic length (vertex to next vertex or node to next node.)

Along the equator, only the arc \( \pi b \) satisfies the fundamental definition of the geodesic, i.e. the longest hemispheroidal geodetic is the semimercator, the shortest is the spherical arc \( \pi b \). The values of \( S_0, \Delta \lambda_0 \) from (33) satisfy the inequalities
\[
\arcsin (1 - f/2 + f^2/16 + f^4/32) > S_0 > \pi b, \pi \Delta \lambda_0 > \pi b/a.
\]

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If derivatives of the second and third order terms in equations (33) are placed equal to zero, we find that for the Clarke 1866 ellipsoid:

\[ \Delta \delta_0: \quad \Delta \delta_0(f^2)(\text{max.}) \text{ occurs at } \theta_0 = 54^\circ 44' 08'' 197 \]
\[ \Delta \delta_0(f^3)(\text{max.}) \text{ occurs at } \theta_0 = 43^\circ 28' 31'' \]
\[ S_0: \quad S_0(f^2)(\text{max.}) \text{ occurs at } \theta_0 = 54^\circ 44' 06'' 197 \]
\[ S_0(f^3)(\text{max.}) \text{ occurs at } \theta_0 = 43^\circ 29' 31'' \] (36)

With the values of \( \theta_0 \) from (36) placed in (33) we find the maximum contribution of second and third order terms over the Clarke 1866 ephemeris:

\[ \Delta \delta_0(f^2)(\text{max.}) = 3474.2 \times 10^{-3} \text{ radians} \approx 3.5 \text{ seconds} \]
\[ \Delta \delta_0(f^3)(\text{max.}) = 0.8 \times 10^{-5} \text{ radians} \approx 0.0012 \text{ seconds} \] (37)

Formulas referred to an ephemeris

Now equations (31) and (32) are referred to a node, (equator crossing) of the geodesic.

If we subtract, respectively, the equations for longitude and distance in (32) from those of (33), than place \( \gamma = (\pi/2) - \eta \), \( \sigma = (\pi/2) - \beta \) we have:

\[ \Delta \varsigma = \eta - f \cos \theta_0 \sigma + (f^2/8) \cos \theta_0 \sin^2 \theta_0 (2\sigma + \sin 2\sigma) \]
\[ + (f^2/64) \cos \theta_0 \sin^2 \theta_0 \left[ 8 \cos^2 \theta_0 \sin 2\sigma - 8 \sin^2 \theta_0 \sin \frac{\pi}{4} \right] \]
\[ S = \sigma - (f/4) \left[ 2 (1 + \cos^2 \theta_0) \sigma - \sin^2 \theta_0 \sin 2\sigma \right] \]
\[ + (f^2/64) \sin^2 \theta_0 \left[ (1 + 3 \cos^2 \theta_0) \sigma + 8 \cos^2 \theta_0 \sin 2\sigma - \sin^2 \theta_0 \sin 4\sigma \right] \]
\[ + (f^3/384) \sin^3 \theta_0 \left[ 12 (1 + 2 \cos^2 \theta_0 + 5 \cos^4 \theta_0) \sigma \right] \]
\[ + 5 (1 + 3 \cos^2 \theta_0) (1 + 3 \cos^2 \theta_0 - 1) \sin^3 \sigma \]
\[ - 3 (1 + 3 \cos^2 \theta_0) \sin^3 \theta_0 \sin 4\sigma + \sin^2 \theta_0 \sin 6\sigma \] (38)

where now \( \sigma = \sec \left( \sin \theta_0 \right) \), \( \eta = \sec \cos \left( \tan \theta_0 \right) \), and the formulas (38) give longitude and distance from the vertex of the geodesic to a point on the geodesic in parametric latitude \( \theta \), where \( 0 < \theta < \pi \).

Note that \( \eta \) and \( \sigma \) are spherical longitude and spherical distance from the geodesic vertex, see figure 11. To show that \( \Delta \varsigma \) and \( S \) of equations (38) are in fact the expansions of equations (19), we write from the first of (17) using the binomial formula,

\[ S = \sum \sin \theta_0 \int_0^0 (1 - k^2 \sin^2 \phi)^{k^2} d\phi \]
\[ = \sum \frac{\sin \theta_0}{k} \int_0^0 \left[ 1 - (1/2)k^2 \sin^2 \phi - (1/8)k^4 \sin^4 \phi - (1/16)k^6 \sin^6 \phi \ldots \right] d\phi \] (39)
\[ = a \sum \frac{\sin \theta_0}{k} \left[ (1/2)k \sin \theta_0 \sin^2 \phi - (1/8)k^3 \sin \theta_0 \sin^4 \phi - (1/16)k^5 \sin \theta_0 \sin^6 \phi \right] d\phi \]

From (15), \( k = \sin \theta_0/(1 - \cos^2 \theta_0)^{1/2} \), and
\[ (a/k) \sin \theta_0 = (1 - \cos^2 \theta_0)^{1/2} = 1 - (1/2)k^2 \cos^2 \theta_0 - (1/8)k^4 \cos^4 \theta_0 - (1/16)k^6 \cos^6 \theta_0 \]
\[ ek \sin \theta_0 = e^2 \sin^2 \theta_0 [1 + (1/2)k^2 \cos^2 \theta_0 + (3/8)k^4 \cos^4 \theta_0] \] (40)

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ek^3 \sin \theta_o = e^4 \sin^4 \theta_o [1 + (3/2)e^2 \cos^2 \theta_o] \\

ek^3 \sin \theta_o = e^6 \sin^6 \theta_o.

Now from (29), with \beta replaced by \sigma, we have

f \sin^3 \sigma \, d\sigma = f(1/2)[1 - \cos 2\sigma] \, d\sigma = (1/4)[2\sigma - \sin 2\sigma] \\

f \sin^4 \sigma \, d\sigma = f(1/8)[4 - \cos 4\sigma + \cos 4\tau \tan \tau] \, d\sigma = (1/8)[3\sigma - 2\sin 2\sigma + (1/4)\sin 4\sigma] \\

f \sin^5 \sigma \, d\sigma = f(1/32)[10 - 15 \cos 2\sigma + 6 \cos 4\sigma - \cos 6\sigma] \, d\sigma \\

= (1/32)[10\sigma - (15/2) \sin 2\sigma + (3/2) \sin 4\sigma - (1/6) \sin 6\sigma] \\

With the values from (40) and (41) we may evaluate (39) and we have then

S/a = [1 - (1/2)e^2 \cos^2 \theta_o - (1/8)\sin^4 \theta_o - (1/16)\sin^6 \theta_o] \, d\sigma \\

= (1/8) \sin^2 \theta_o [1 + (1/2)\sin^2 \theta_o + (3/8)\sin^4 \theta_o] [2\sigma - \sin 2\sigma] \\

= (1/16) \sin^4 \theta_o [1 + (3/2)\sin^2 \theta_o - 2\sin 2\sigma + (1/4)\sin 4\sigma] \\

= (1/512) \sin^6 \theta_o [10\sigma - (15/2) \sin 2\sigma + (3/2) \sin 4\sigma - (1/6) \sin 6\sigma] \\

Collecting the coefficients of the terms in like powers of \epsilon, letting \epsilon^2 = 2f - \rho, \epsilon^4 = 4f^2 - 4f^3, e^6 = 8f^3, and using some elementary trigonometric identities in the coefficients of the powers of \rho, we find that equation (42) becomes exactly the second of equations (38).

Similarly from the second of equations (17) we have:

\[ \Delta \lambda = \frac{e \tan \theta_o}{k} \int_0^\sigma (1 - k^3 \sin^3 \sigma)^{1/2} \, d\sigma = e \tan \theta_o \tan \theta_1 \tan \theta_2 \\

= (1/8)ek^3 \tan \theta_1 \tan \theta_2 - (1/16)ek^3 \tan \theta_3 \tan \theta_4 \] \\

Where

\[ \tan \theta_1 \tan \theta_2 = \cos \theta_o \tan \theta_o \tan \theta_1 \tan \theta_2 = \cos \theta_o \eta, \text{(see Figure 11)} \]

\[ \tan \theta_1 \tan \theta_2 = \cos^2 \theta_o (\sigma - 1) \] \\

Now \( k = e \sin \theta_o (1 - e^2 \cos^2 \theta_o)^{1/2} \), \( \tan \theta_o / k = \sec \theta_o (1 - e^2 \cos^2 \theta_o)^{1/2} \) and expanding by the binomial formula to sixth order terms in \epsilon we have

\[ e \tan \theta_o / k = \sec \theta_o - (1/2)e^2 \cos \theta_o - (1/8)e^4 \cos^4 \theta_o - (1/16)e^6 \cos^6 \theta_o \] \\

\[ - (1/2)ek \tan \theta_o - (e^2/2) \sin \theta_o \tan \theta_o [1 + (1/2)e^2 \cos^2 \theta_o + (3/8)e^4 \cos^4 \theta_o] \] \\

\[ - (1/8)ek^3 \tan \theta_o - (1/8)ek^3 \sin^2 \theta_o \tan \theta_o [1 + (3/2)e^2 \cos^2 \theta_o] \] \\

\[ - (1/16)ek^3 \tan \theta_o - (1/16)ek^3 \sin^3 \theta_o \tan \theta_o \]
Placing the values from (44) and (45) in (43), collecting like terms and employing some trigonometric identities we have

\[
\frac{e^2 e^4}{AX} = \frac{i^2}{\cos^2 \theta_0 - \cos^2 \theta_0} - \cos^2 \theta_0 \left(2 + \cos^2 \theta_0\right) \sin 2 \theta_0 - \cos^2 \theta_0 \sin 2 \theta_0 \sin 4 \theta_0 \left(1 + \cos^2 \theta_0\right) \sin 2 \theta_0 - \sin^2 \theta_0 \sin 4 \theta_0 \right]
\]

(46)

Placing \(e^2 = 2f - f^2, e^4 = 4f^2 - 4f^4, e^6 = 8f^3\) in (46) find

\[
\frac{\Delta \lambda}{\cos \theta_0 \sin^2 \theta_0} = \frac{f^2}{8} \left[\frac{4(1 + 3 \cos^2 \theta_0)}{\cos \theta_0 \sin 2 \theta_0} + 8 \cos^2 \theta_0 \sin 2 \theta_0 - \sin^2 \theta_0 \sin 4 \theta_0 \right]
\]

which is exactly the first of equations (38).

Collecting like terms in \(\alpha\) and \(\alpha\), equations (32) and (38) may be written with longitude and arc length measured from the geodesic node;

\[
\Delta \lambda = \eta - A \alpha - B \sin \theta_0 - C \sin \theta_0, \eta = \arccos \left(\tan \theta_0 / \tan \theta_0\right),
\]

(47)

longitude and arc length measured from the geodesic vertex;

\[
\Delta \lambda = \eta - A \alpha - B \sin \theta_0 - C \sin \theta_0, \eta = \arccos \left(\tan \theta_0 / \tan \theta_0\right),
\]

\[
\frac{\Delta \lambda}{\cos \theta_0 \sin \theta_0} = \frac{f^2}{8} \left[\frac{4(1 + 3 \cos^2 \theta_0)}{\cos \theta_0 \sin 2 \theta_0} + 8 \cos^2 \theta_0 \sin 2 \theta_0 - \sin^2 \theta_0 \sin 4 \theta_0 \right]
\]

(48)

and where in both cases with \(c_1 = f \cos \theta_0, c_2 = (1/4)f \sin^2 \theta_0, c_3 = 1 + c_1 \cos \theta_0, c_4 = c_2 + c_3,\) we have

\[
A = c_1 (1 - c_2 c_4), B = (1/2)c_1 c_2 c_3, C = (1/4)c_1 c_2^2 - c_3, \]

\[
D = 2 + c_2 (c_2 - c_3) - (1 + c_2) c_4 - c_2, E = (1/2)c_2 [2 + c_2 (c_3 - 1) - c_2 ^2],
\]

(49)

\[
F = (1/4)c_2^2 (2c_4 - 1), \quad G = (1/6)c_3^3,
\]

and \(c_1, c_2, c_3\) satisfy \(c_1^2 - 4c_2 (c_2 - 1) + c_3 (c_2 - c_3) = 1.\)

Formulae for longitude and arc length between two arbitrary points on the hemispheroidal geodesic

From (47), for a geodesic arc containing a vertex

\[
\Delta \lambda = \Sigma \eta - A \Sigma \alpha + B \eta - C \eta, \quad \Sigma \eta = \eta_1 + \eta_2, \Sigma \alpha = \alpha_1 + \alpha_2, \Delta \alpha = \alpha_2 - \alpha_1,
\]

(50)

\[
\frac{\Delta \lambda}{\Sigma \alpha} = \frac{p}{2} = 2 \sin \Sigma \alpha \cos \Delta \alpha, q = 2 \sin \Sigma \alpha \cos \Delta \alpha,
\]

Also from (48) for a geodesic arc containing neither node nor vertex

\[
\Delta \lambda = \Delta \eta - A \Delta \alpha + B \eta - C \eta, \quad \Delta \eta = \eta_1 - \eta_1, \Delta \alpha = \alpha_2 - \alpha_1, \Sigma \alpha = \alpha_1 + \alpha_2,
\]

(51)

\[
\frac{\Delta \lambda}{\Sigma \alpha} = \frac{p}{2} = 2 \cos \Sigma \alpha \sin \Delta \alpha, q = 2 \sin \Delta \alpha \cos \Sigma \alpha.
\]

From (47) for a geodesic arc containing a node

\[
\Delta \lambda = \Delta \gamma - A \Delta \beta - B \eta - C \eta, \quad \Delta \gamma = \gamma_1 + \gamma_2, \Delta \beta = \beta_1 + \beta_2, \Delta \beta = \beta_2 - \beta_1,
\]

(52)

\[
\frac{\Delta \lambda}{\Sigma \beta} = \frac{p}{2} = 2 \sin \Sigma \beta \cos \Delta \beta, q = 2 \sin \Sigma \beta \cos \Delta \beta.
\]

\[
\gamma_1 = \arccos \left(\tan \theta_1 / \tan \theta_0\right), \beta_1 = \arccos \left(\sin \theta_1 / \sin \theta_0\right)
\]

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Also from (47) for a geodesic arc containing neither node nor vertex

\[\Delta \lambda = \Delta \gamma - A \Delta \delta - B \rho - C q\]
\[S/\alpha = D \Delta \beta - E \psi - F q - G r\]
\[r = 2 \cos 3 \Delta \beta \sin 3 \Delta \delta\]
\[\eta_1 = \arcsin \left( \tan \theta_1 / \tan \theta_0 \right), \beta_1 = \arcsin \left( \sin \theta_1 / \sin \theta_0 \right)\]

The constants \(A, B, C, D, E, F, G\), of formulae (50), (51), (52), (53) are given by (49). Since (51) and (53) should give the same results one should transform into the other if we make the substitutions respectively from \(\alpha + \beta = \pi/2, \eta + \gamma = \pi/2\). For instance in (53)

\[\Delta \gamma = \gamma_1 - \gamma_2 = (\pi/2) - \eta_1 - (\pi/2) + \eta_2 - \eta_1 = \Delta \eta,\]
\[\Delta \delta = \beta_1 - \beta_2 = (\pi/2) - \alpha_1 - (\pi/2) + \alpha_2 - \alpha_1 = \Delta \alpha, \Sigma \beta = \beta_1 + \beta_2 = \pi - \Sigma \alpha.\]

These substitutions in \(\Delta \lambda\) and \(S/\alpha\) of (53) give

\[\Delta \lambda = \Delta \eta - A \Delta \sigma + B \rho - C q\]
\[S/\alpha = D \Delta \alpha + E \psi - F q - G r\]
\[r = 2 \cos 3 \Delta \alpha \sin 3 \Delta \delta\]

Which are formulae (51). Now in (50) with \(\theta_1 = \theta_2 = 0\), we have \(\Delta \alpha = \Sigma \eta = \pi, p = q = r = 0\). Analogously for (52) with \(\theta_1 = \theta_2 = \theta_0\) we have \(\Delta \gamma = \Sigma \psi = \pi, p = q = r = 0\) and both therefore give for length and longitude of hemispheroidal geodesics, node to node or vertex to vertex,

\[\Delta \alpha_0 = \pi(1 - A), S_0 = \pi a D. \quad (54)\]

Equations (54) are thus a shorter version of equations (33). Referring to equations (49), (54), when \(\theta_0 = \pi/2, c_1 = 0, c_2 = (1/4)f, c_3 = 1, c_4 = 1 + (1/4)f, A = 0, D = 1 - f/2 + (1/16)f^2 + (1/32)f^3\), and again for the semi-meridian \(\Delta \lambda = \pi, S = \sin \left[1 - f/2 + (1/16)f^2 + (1/32)f^3\right]\). When \(\theta_0 = 0, c_1 = f, c_2 = 0, c_3 = c_4 = 1 + f, A = f, D = 2 - (1 + f) = 1 - f\) and we have again the equatorial limiting arc \(\Delta \lambda = \pi(1 - f), S = \sin(1 - f)\).

Throughout this discussion \(\Delta \alpha\) has been used to represent two quantities. When dealing with elliptic integrals and functions, \(\Delta \alpha = \left(1 - k^2 \sin^2 \theta \right)^{1/2}\), see equations (12). When dealing with computational formulae for distance and longitude, \(\Delta \alpha = \sigma_2 - \sigma_1\), see equations (50), (51). The usage is clearly indicated in each case, and no ambiguity occurs.

We now have equations to third order in the flattening which may be used to check approximation formulae to the geodesic and to check known or published geodetic lines. After a discussion of the spheroidal triangle, some of these formulae will be used in the derivation of the direct solution for the long geodetic line. But we next examine the antipodal zones and conjugate points with respect to the nonplanar geodesic.

**Antipodal zones**

The hemispheroidal geodesic is that part included between two consecutive vertices or two consecutive nodes since no more than two consecutive of either nodes or vertices can be contained in the same hemispheroid (on the same side of a meridian).

The antipodal zones are the two equal areas bounded by the two symmetric geodesic evolutes (envelopes) of all oblate spheroidal geodesics which have a vertex in a common fixed meridian. Cayley, reference [25].

**NOTE:** The evolute of a given curve is the curve tangent to all normals (perpendiculars) of the given curve, or the envelope of the normals. The normal to the meridian ellipse in terms of parametric latitude.
\( \theta \) (eccentric angle of the ellipse) is \( F(\theta) = ax / \cos \theta - by / \sin \theta - (a^2 - b^2) = 0 \), where \( a, b \) are semimajor, semiminor axes of the spheroid, \( x \) and \( y \) are rectangular coordinates in the plane of the meridian, the \( y \)-axis coinciding with the ellipsoidal polar axis. The evolute (envelope) is obtained by eliminating \( \theta \) between \( F(\theta) = F'(\theta) = 0 \) where the prime denotes differentiation with respect to \( \theta \). The result is the equation \( a^2 \sqrt[3]{x^2} + b^2 \sqrt[3]{y^3} = (a^2 - b^2)^{1/3} \), its graph resembling the geodetic evolutes as displayed in Figure 12.

The geodesic evolutes are the figures \( ADBC, A'D'B'C' \) which resemble the meridional evolute \( EFE'F' \). Geodesic arcs \( PN, OM \) are equal. Location of the nodes \( N, N' \) within the antipodal zones is known from equations (33). When \( \Theta = 0 \), \( \Theta_0 \) then \( \Theta_0 \rightarrow 0 \), and \( \Theta \rightarrow N \rightarrow D, \Theta' \rightarrow N' \rightarrow D' \), when \( \Theta \rightarrow N \rightarrow T, N' \rightarrow T' \), and \( \Theta \rightarrow A, \Theta' \rightarrow A' \).

Figure 12. Geodesic evolutes and antipodal zones on the oblate spheroid (pictorial).

Two consecutive nodes are in the geodesic antipodal zones with respect to the meridian containing the included vertex of the geodesic. From the first of the inequalities (35) we have, when \( \theta_0 = 0 \),
\[ S_0 = \pi b = \pi(1 - f). \] Hence the equatorial axis of the geodesic evolute is then \( \pi - \pi(1 - f) = \pi f \) as shown in Figure 12.

The distance from node to node \((N \to N' \text{ in the diametrically opposite antipodal zone})\) is given by equations (33) and by symmetry this is the same distance as that between two consecutive vertices. Is the geodesic distance thus obtained the maximum under the shortest distance property of the geodesic? Apparently this is so from the limits given by inequalities (35). But for any point \( P \) on a given geodesic, is there a point \( P' \) on the geodesic beyond which the unique shortest distance property does not hold? Before we attempt to answer this question we find the length of the meridional axis of the geodetic evolute (antipodal zone), the segments \( AB = A'B' \) of Figure 12.

In Figure 12, note that for the geodesic with vertex \( P(0, \pi/2) \) we have

\[ \Delta = (\pi/2) - (1/2)\Delta_0. \] (55)

As \( \theta \to \pi/2, N \to T, Q \to A, \) and there exists the value \( +\theta \) which is the parametric latitude of \( A \) as given by (55). From equations (32), (33) and (55) we have to terms in \( f^2 \):

\[
\gamma = -f \cos \theta \beta + (f^2/8) \cos \theta \sin^2 \theta (2\beta - \sin 2\beta) = \pi/2 - \pi/2 + (\pi/2) \cos \theta - (\pi/2) \cos \theta \sin^2 \theta
\]

or

\[ F(\theta, \theta_0) = (\gamma/\cos \theta_0) - f(\beta + \pi/2) + (f^2/8) \sin^2 \theta_0 (2\beta - \sin 2\beta) = 0. \] (56)

Where \( \gamma = \arcsin(\tan \theta \cot \theta_0), \beta = \arcsin(\sin \theta \csc \theta_0). \)

We must therefore solve for \( \theta \) in the equation remaining by taking

\[ \lim \theta_0 \to \pi/2 F(\theta, \theta_0) = 0. \]

Only the first term of (56) is bothersome in determining the required limit:

\[ \lim \theta_0 \to \pi/2 \frac{\gamma}{\cos \theta_0} = \lim \frac{\arcsin(\tan \theta \cot \theta_0)}{\cos \theta_0} = \lim \frac{(d/d\theta_0) \arcsin(\tan \theta \cot \theta_0)}{(d/d\theta_0) \cos \theta_0}. \] (57)

We have then from (56) and (57)

\[ \lim \theta_0 \to \pi/2 F(\theta, \theta_0) = \tan \theta - (f/2)(\pi + 2\beta) + (f^2/8)(\pi + 2\beta - \sin 2\beta) = 0, \]

or

\[ \tan \theta - B \sin 2\beta = A(\pi + 2\beta) \]

\[ B = f^2/8, A = (f/2) - B \]

In (58) let \( \tan \theta = \theta, B = 0, \) to get the approximation

\[ 2\beta = \pi f/(1 - f). \] (59)

With the value \( f = 0.03399075283 \) (Clarke 1866 ellipsoid—Appendix 2), \( \pi = 3.1415926536 \), (59) gives

\[ \theta = 18' 22.121 \] which fails to satisfy (58) by \( 0.00000457, \) i.e.

\[ \tan \theta = 2\alpha - B \sin 2\beta > A \pi \text{ by } 0.00000457. \]

Now the tangent different for 1 second at \( 18' 22' \) is \( 0.00000485 \) (Peters Tables). For 1 second change, \( 2\alpha \) changes by \( 2 \times 10^{-8} \) but there is no change in \( B \sin 2\beta. \) Hence we take 459/485 = .946 second and reduce the first estimate by that amount since \( \tan \theta > \theta > \sin \theta, \) i.e. \( \theta = 18' 22.121 - .946 = 18' 21.175. \)

This last value checks (58) to 1 in the 8th decimal.
Since the flattening does not vary much among the 10 reference ellipsoids of Appendix 2, we may alter the approximation (59) to give a solution for any reference ellipsoid. This was accomplished by changing .946 second to radians, factoring irf, writing \( \frac{1}{1 - f} = 1 + f + \ldots \) and then adjusting for the variation in \( f \) among the values as given in Appendix 2. The resulting solution to terms in \( f^2 \) is
\[
2\theta = \pi f(1 + .7495f).
\] (60)

Seven of the values of \( \theta \) computed from (60) checked (58) exactly to 8 decimals and 3 were within 1 in the 8th decimal. The computations are included in Appendix 2, where the axes and approximate areas of the antipodal zones for the 10 spheroids are also given.

**Conjugate points on spheroidal geodesics**

For an arbitrary point \( P_1 \) on a spheroidal geodesic there exists a second point \( P_2 \) on that geodesic beyond which the unique shortest distance property fails. Forsyth, citing Jacobi, called such pairs conjugate points, reference [28].

Because of symmetry, the distance, node to node is the same as vertex to vertex, or point \( P_1 \) to \( P_2 \) in numerically equal but opposite signed latitudes when the longitude difference is the same as node to node or vertex to vertex. But consecutive nodes are not conjugate since there exist two equal geodesics symmetric with respect to the equator with these common nodes, see Figure 12. Again, but in the meridian, any diameter bisects the meridianal arc length, hence the diametral end points are both antipodal and conjugate. Hence by inference two consecutive vertices should be conjugate.

That this is so may be demonstrated in Figure 13. The equal symmetric nodal hemispheroidal geodesics are \( N_1 Q N_2, N_1 R N_2 \). Arc lengths \( N_1 P_1, N_2 P_2, P_1 T \) are equal, hence the hemispheroidal geodesics \( P_1 Q P_2, P_1 S P_2 \) are both equal to \( N_1 Q N_2 \) or \( N_1 R N_2 \). By symmetry the longitude difference, \( \Delta \lambda_0 \), node to node, is equal to that from \( P_1 \) to \( P_2 \). Again, the arc lengths \( V_1 P_1', V_2 P_2', P_1' U \) are equal and therefore the geodesics \( P_1' O P_2', P_1' M P_2', V_1 O V_2, N_1 Q N_2 \) are all equal and the longitude difference, \( P_1' \) to \( P_2' \), is \( \Delta \lambda_0 \), the same as from \( N_1 \) to \( N_2 \).

For the mathematical demonstration we will maximize the equation for longitude difference between two points on the geodesic. As a preliminary, note from the inequalities (35), that for hemispheroidal geodesics we have the longitude difference and length satisfying
\[
\pi > \Delta \lambda_0 > \pi b/a
\]

\[
\pi[1 - (f/2) + (f^2/16) + f^3/32] > S_0 > \pi b,
\]
i.e. along the equator from a given point one can extend the length to \( \pi b \) before two equal and symmetric geodesics of length shorter than the subtended equatorial arc exist. In Figure 13 we can extend the distance along the equator from \( N_1 \) to \( T_1, N_1 T_1 = \pi b \), before the two symmetric geodesics \( N_1 Q N_2, N_1 R N_2 \) exist. That is \( N_1 N_2 > N_1 Q N_2 = N_1 R N_2 > N_1 T_1 = \pi b \) and the points \( N_1, T_1 \) are conjugate.

From equations (17) we may write
\[
\Delta \lambda = 1F, F = \sec \theta (1 - e^2 \cos^2 \theta) \theta/	heta' = \varepsilon \tan \theta_0/k,
\]

\[
L = \int_{\theta_2}^{\theta_1} \frac{\Delta \theta}{\theta^2} d\theta, \Delta \theta = (1 - c \sin^2 \sigma) \theta/	heta', \theta_2 = 1 + n \sin^3 \sigma,
\] (61)
Figure 13. Conjugate points on the oblate spheroid.

\[ n = \tan^2 \theta_0, \quad c = e^2 \sin^2 \theta_0/(1 - e^2 \cos^2 \theta_0) = k^2, \]
\[ a_1 = \arccos \left( \frac{\sin \theta_1}{\sin \theta_0} \right), \quad |\theta_1| \ll |\theta_0| \]

Now
\[ d\Delta \lambda = F \frac{dl}{d\theta} + F d\theta = 0, \]
\[ \frac{d\Delta \lambda}{d\theta} = \frac{dF}{d\theta} \]

or equivalently
\[ F \frac{d\Delta \lambda}{d\theta} = \frac{dF}{d\theta} \]
\[ \frac{d\Delta \lambda}{d\theta} = 0 \]
\[ (62) \]

Since \(a_1, a_2, \Delta \sigma, \delta \sigma\) are all functions of \(\theta_0\), we have
\[ \frac{dl}{d\theta_0} = \int_{a_1}^{a_2} \frac{\partial}{\partial \theta_0} \left( \frac{\Delta \sigma}{\delta \sigma} \right) da + \frac{\Delta a_1}{\delta a_1} \frac{d\sigma}{d\theta_0} \frac{\Delta a_2}{\delta a_2} \frac{d\sigma}{d\theta_0} \]
\[ \frac{\partial}{\partial \delta \sigma} \left( \frac{\Delta \sigma}{\delta \sigma} \right) = \frac{1}{\delta \sigma^2} \left( \frac{\delta \sigma}{d\theta_0} - \Delta \sigma \frac{d\delta \sigma}{d\theta_0} \right) \]
\[ (63) \]
\[ (64) \]

From (61) we have
\[ \frac{d\Delta \sigma}{d\theta_0} = \frac{\sin^2 \sigma \, dc \, dc}{2 \Delta \sigma \, d\theta_0} \frac{2c^2(1-e^2)}{e^4} \cot \theta_0 \csc^2 \theta_0, \]
\[ \frac{43}{4} \]
\[
\frac{d\alpha}{d\theta} = \sin^2 \alpha \frac{dn}{d\theta_0}, \quad \frac{dn}{d\theta_0} = 2 \tan \theta_0 \sec^2 \theta_0,
\]

(65)

\[
\frac{d\alpha}{d\theta_0} = \cot \alpha \cot \theta_0, \quad \frac{dF}{d\theta_0} = (c/e^2) \csc \theta_0 \sec \theta_0
\]

With the values of \(d\alpha/d\theta_0\) from (65) the last two terms of (63) may be written

\[
\frac{\Delta_1 \Delta_3}{\sin_1} - \frac{\Delta_2 \Delta_3}{\sin_2} = \cot \theta_0 \left( \frac{\Delta_1}{\sin_1} \cot \alpha_1 - \frac{\Delta_2}{\sin_2} \cot \alpha_2 \right)
\]

(66)

With the values of \(\Delta_1, \Delta_2\) from (61) we find

\[
\frac{d}{d\alpha} (\csc \theta_0) = \frac{\Delta \alpha}{\delta \alpha} = \frac{c \cos^2 \alpha}{\delta \alpha - \Delta \delta \delta} = \frac{2 \tan^2 \theta_0 \cos^2 \alpha \Delta \alpha}{\delta \alpha^2}
\]

(67)

With the values of \(d\Delta \alpha/d\theta_0, d\delta \alpha/d\theta_0\) from (65), we may write (64) as

\[
\frac{\delta}{d\delta} (\frac{\Delta \alpha}{\delta \alpha}) = \frac{c^2 (1 - e^2)}{e^2 \Delta \alpha \delta \alpha} \cot \theta_0 \csc^2 \theta_0 \Delta \alpha \sin^2 \alpha - 2 \tan \theta_0 \sec^2 \theta_0 \frac{\Delta \alpha}{\delta \alpha} \sin^2 \alpha
\]

(68)

With the value of \((1/F) dF/d\theta_0\) from (65) and the value of \(l\) from (61) we have

\[
\frac{I}{F \delta \theta_0} = \int_{\alpha_1}^{\alpha_2} (c/e^2) \sec \theta_0 \sec \theta_0 \Delta \alpha \delta \alpha d\sigma.
\]

(69)

Now with the value of (67) placed in (66) and the result returned to (63), together with the value of \(\partial_{\theta_0} (\Delta \alpha/\delta \alpha)\) from (68) for the first term of (63), we may, with the resulting value of \(dl/d\theta_0\) and the value of \((1/F) dF/d\theta_0\) from (69) write the condition (62) as

\[
\cot \theta_0 \left\{ \begin{array}{c}
\int_{\alpha_1}^{\alpha_2} \frac{d\alpha}{\cos^2 \alpha \Delta \alpha \delta \alpha \sin^2 \alpha} \\
\int_{\alpha_1}^{\alpha_2} c^2 (1 - e^2) \csc \theta_0 \Delta \alpha \sin^2 \alpha + 2e^2 n \csc \theta_0 \Delta \alpha \sin^2 \alpha \\
+ e^2 \delta \alpha \Delta \alpha \sin^2 \alpha + c e^2 \delta \alpha \sin^2 \alpha \cos^2 \alpha \\
\int_{\alpha_1}^{\alpha_2} + 2e^2 n \Delta \alpha \sin^2 \alpha \cos^2 \alpha \cos^2 \theta_0 = 0 \\
\end{array} \right\}
\]

(70)

where \(n = \tan^2 \theta_0, \Delta \alpha = 1 - c \sin^2 \alpha, \delta \alpha = 1 + n \sin^2 \alpha, c = e^2 \sin^2 \theta_0/(1 - e^2 \cos^2 \theta_0).

In (70), within the braces, we first combine the terms (2) and (5) to get

(2) + (5) = 2e^2 n \Delta \alpha \sin^2 \alpha.

(71)

We next combine terms (1), (4), and (6) to get analogously

(1) + (4) + (6) = -ne^2 \delta \alpha \sin^2 \alpha + cne^2 \delta \alpha \sin^2 \alpha.

(72)

With the values from (71) and (72) returned to (70), we now have for the quantity within the braces

\(\int \delta \alpha (e^2 n \Delta \alpha \sin^2 \alpha + e^2 \Delta \alpha \delta \alpha - e^2 n \Delta \alpha \sin^2 \alpha) = \left\{ \begin{array}{c}
e^2 \Delta \alpha \delta \alpha \sin^2 \alpha \\
\end{array} \right\}

(73)

where in the reductions we have used the identities given with equation (70). The value from (73) placed in (70) gives
\[(1/F) \frac{d\Delta\lambda}{d\vartheta_o} = (d\lambda/d\vartheta_o) + (1/F)(dF/d\vartheta_o) = -\cot \vartheta_o \int_0^{\alpha_1} \frac{e^2 \Delta\alpha^2 \delta\alpha^2 \, d\alpha}{e^2 \Delta\alpha \delta\alpha^2 \sin^2 \alpha} \]
\[= -\cot \vartheta_o \int_0^{\alpha_1} \frac{\Delta\alpha \, d\alpha}{\sin^2 \alpha} = 0. \tag{74}\]

Since the equation to the geodesic evolute will be given by the elimination of \(\vartheta_o\) between \(d\Delta\lambda/d\vartheta_o = 0\), and \(\Delta\lambda = FI\), equations (61) and (74), we should be able to get an equation for determining the parametric latitude of the meridional vertex of the geodesic evolute and thus provide a check for equation (58). In fact the equation should be given by (74), that is from

\[\lim_{\theta_o \to \pi/2} \int_{-\pi/2}^{\pi/2} \Delta\alpha \, d\alpha/\sin^2 \alpha = 0.\]

With \(\theta_o = \pi/2\), \(c = \epsilon^2\), \(\Delta\alpha = (1 - \epsilon^2 \sin^2 \alpha)\), we have

\[\Delta\alpha/\sin^2 \alpha = \csc^2 \alpha [1 - (1/2)\epsilon^2 \sin^2 \alpha - (1/8)\epsilon^4 \sin^4 \alpha - ...] = \csc^2 \alpha - \epsilon(1 - \epsilon/4 + (1/4)\epsilon^2 \cos 2\alpha).\]

Integrating this last expression with respect to \(\alpha\) and evaluating for the limits \(\pi/2 + \theta\), \(-\pi/2 + \theta\) we obtain equation (58).

In equation (74), the factor \(\cot \theta_o = 0\) implies the meridian, \(\theta_o = (1/2)\varpi\). Now from (13) and (13)c, \(\Delta\alpha = \Delta S, \sin^2 \alpha = \sin^2 S, \alpha = \alpha S, \alpha S = d \alpha S = dS, dS = dS dS S\) and the integral (74) may be written

\[\int_{S_1}^{S_2} \sin^2 S \, dS/\sin^2 S = 0. \tag{75}\]

By manipulation of the identities (13)a and differentials (13)c, we can write the integral (75), indefinite, as

\[\int \sin^2 S \, dS = 0, \tag{76}\]

\[\int \sin^2 S \, dS = \c' S - \epsilon \sin S \, dS/\epsilon = 0. \tag{77}\]

From (12), (13), (13)c we have

\[E(k, \sigma) = \int_0^\sigma \Delta\alpha \, d\alpha = \int_0^S \sin^2 S \, dS = E(S, k). \tag{78}\]

From (77) and (78) the definite integral (75) may be written

\[\int_{S_1}^{S_2} \sin^2 S \, dS/\sin^2 S = [c'(S - S_2) - E(S, k)] - [c(S_2, k) - E(S, k)] = 0. \tag{79}\]

We expand (79) and write the result in the form

\[c'(S_1 - S_2) = E(S_1, k) - E(S_2, k) - [(cS_1 \sin S_1/\sin S_2) - (cS_1 \sin S_1/\sin S_1)]. \tag{80}\]

Using the difference formula for two elliptic integrals of the second class with the same modulus, and the difference formula for the sine amplitude from (13)a, we can write the right members of (80) respectively as

\[E(S_1, k) - E(S_2, k) = c S_1 - S_2, k - c \Delta S_1, \Delta S_2, \Delta S_1 \sin(S_1 - S_2) \]
\[[-(cS_1 \sin S_1/\sin S_2) + (cS_2 \sin S_2/\sin S_2)] = \sin(S_1 - S_2)(1 - \Delta S_1 \sin S_2/\sin S_1, \sin S_2). \tag{81}\]

Placing the values from (81) in (80) and solving for \(\sin(S_1 - S_2)\) we find

\[\sin(S_1 - S_2) = \sin S_1, \sin S_2 \sin(S_1 - S_2, k) - c(S_1 - S_2), \tag{82}\]

where \(c' = 1 - \epsilon^2 = (1 - \epsilon^2)/(1 - \epsilon^2 \cos^2 \theta_0), c = \epsilon^2 \sin^2 \theta_0/(1 - \epsilon^2 \cos^2 \theta_0).\)
If we place $S_1 = S_2$ in equation (82), the equation is satisfied since from (12) and (13b), $sn(0) = E(0, k) = 0$. If we place, in equation (82), $S_1 = 2K, S_2 = 0$, the equation is satisfied since $sn2K = sn(0) = 0$. Hence the value of $S_1$ required is the root of (82) next greater than $S_2$ where $0 < S_2 < 2K$. Note that $K$ is a complete elliptic integral, see (12)a.

For an approximation we write $x$ for $S_1$ in (82) and consider the intersection of the functions (curves)
\[ y = \frac{sn(x - S_2)/sn(x)snS_2}{sn(x)snS_2} = E(x - S_2, k) - c'(x - S_2), c' = 1 - k^2 \]
\[ = (1 - e^2)/(1 - e^2 \cos^2 \theta_0) < 1. \tag{83} \]

As shown in Figure 14, the next value of $x$ for which equations (83) are satisfied is $S_1$ and we have
\[ 0 < x_1 < 2K - S_2, \text{ where } x_1 = S_1 - (S_2 + 2K). \tag{84} \]
If we solve (84) for $S_1$ and place this value, $x = S_1 = x_1 + S_2 + 2K$, in (83), we may write, using the values from (13b)
\[ y_1 = \frac{sn(x_1 + 2K)/sn(x_1 + S_2 + 2K)snS_2}{sn(x_1 + S_2)snS_2} = E(x_1 + 2K, k) - c'(x_1 + 2K). \tag{85} \]

Using the appropriate identities from (13a), we transform the right member of (85) as follows:
\[ E(x_1 + 2K, k) - c'(x_1 + 2K) = E(x_1, k) - c'x_1 + 2(E - c'k) = E(x_1, k) - c'x_1 + 4cc'dk/dc \]
\[ = c \int_0^{x_1} cn^2 x \, dx + 4cc'dk/dc. \tag{86} \]

Using the value of $K$ from (12a), we have
\[ dK/dc = \int_0^{\pi/2} \frac{3}{dc} (1 - c \sin^2 x)^{3/2} \, dx = (1/2) \int_0^{\pi/2} (1 - c \sin^2 x)^{3/2} \sin^2 x \, dx. \tag{87} \]

From (85), (86), and (87) we have, for the determination of $x_1$, the equation
\[ y_1 = \frac{snx_1/snx_1 + S_2)snS_2}{snx_1 + S_2)snS_2} = c \int_0^{x_1} cn^2 x \, dx \]
\[ + 2c(1 - c) \int_0^{\pi/2} (1 - c \sin^2 x)^{3/2} \sin^2 x \, dx. \tag{88} \]

Since $c = e^2 \sin^2 \theta_0/(1 - e^2 \cos^2 \theta_0) = 2f \sin^2 \theta_0 + \ldots$, and $2f = e^2 + f^2$, then always $2f > e^2 > c$. For earth reference ellipsoids $2f - e^2 = f^2 \approx 1 \times 10^{-3}$. We consider here that $2f, e^2, c$ are of the same order and reject terms of second and higher order in $c$ or $f$. Since $x_1, snx_1$ are of the same order as $c$, we place $snx_1 = x_1, sn(x_1 + S_2) = snS_2$ and write (88) as
\[ y_1 = x_1 /sn^2 S_2 = c \int_0^{x_1} (1 - \ldots) \, dx + 2c \int_0^{\pi/2} (1/2 - \ldots) \, dx = 0 + (1/2)cw, \]
and we find
\[ x_1 = (1/2)cw \, sn^2 S_2. \tag{89} \]

Placing $c = 2f \sin^2 \theta_0$ in (89) we may write
\[ x_1 = \pi f \sin^2 \theta_0 \, sn^2 S_2 = y_1 \sin^2 S_2, y_1 = \pi f \sin^2 \theta_0. \tag{90} \]

From (84) and (90) we obtain
\[ S_1 = 2K + S_2 + x_1 = 2K + S_2 + \pi f \sin^2 \theta_0 \, sn^2 S_2. \tag{91} \]
Note that the value given by (91) is analogous to that obtained by Forsyth, see reference [28]. By direct integration of the integral for $K$, equations (12)a, we find to first order in $f$ that $2K = \pi \left(1 + \frac{1}{2} f \sin^2 \theta_0\right)$

$= \pi + \frac{1}{2} y_1$. Substitution in (91) gives $S_1 - S_2 = \pi + x_1 + \frac{1}{2} y_1$, as shown in Figure 14.

From (61), (13), (13)a, (13)b we have

\[ \text{cn} S_1 = \cos \theta_1 = \sin \theta_1 \sin \theta_0, \quad \text{cn}(2K + S) = - \text{cn} S, \quad m^2 S = 1 - \text{cn}^2 S. \]

(92)
We write from (91), with the help of (92)
\[
\begin{align*}
\text{cn}S_1 &= \text{cn}[2K + (S_2 + x_1)] = -\text{cn}(S_2 + x_1), \\
\sin \theta_1 &= -\sin \theta_0 \text{cn}(S_2 + x_1), x_1 = \pi f \sin^2 \theta_0 \sin^2 S_1 = \pi f (\sin^2 \theta_0 - \sin^2 \theta_2),
\end{align*}
\]
(93)
which to first order in \( f \) relates the parametric latitudes \( \theta_1, \theta_2 \) of conjugate points on spheroidal geodesics.

When \( \theta_2 = \theta_0 \), a geodesic vertex, then \( x_1 = 0 \), \( \text{cn}S_2 = 1 \), and we have \( \sin \theta_1 = -\sin \theta_0 \), or \( \theta_1 = -\theta_0 \).

That is the conjugate of a vertex of the geodesic is the next vertex, a result obtained by another argument. See the discussion following equations (33) and the geometric demonstration, Figure 13. A special case of this last is given by \( \theta_0 = (1/2) \pi \), whence \( \sin \theta_1 = -1, \theta_1 = -\theta_0 \), i.e., the poles are conjugate as well as antipodal for the meridian, a known result. When \( \theta_0 = \theta_2 = 0 \), then \( \theta_1 = 0 \) and we have the end points of the equatorial limiting geodesic arc \( \alpha \), the segment \( N_1 T_1 \) of Figure 13.

Since along the geodesic, \( |x| < |\theta| \), the range of \( x_1 \) for a particular geodesic is \( 0 < x_1 < \sin^2 \theta_0 = y_1 \), and over the spheroid is \( 0 < x_1 < \pi f \).

Note that when \( x_1 = 0 \), then \( \theta_1 = \theta_0 \) and we have the end points of the spheroidal geodesic arc \( \alpha \), the segment \( N_1 T_1 \) of Figure 13.

The values from (96), and the value of \( x_1 \) from (93), placed in (95), retaining terms of first order in \( f \), give
\[
\sin \theta_1 = -\sin \theta_0 \text{cn}S_2 - x_1 \text{sn}S_2 \text{dn}S_2.
\]
(95)
Now from (19a), with \( e^2 = 2f \) and retaining terms in \( f \), we have
\[
\text{sn}S_2 = (\sin^2 \theta_0 - \sin^2 \theta_2)\sqrt{\sin \theta_0}, \dnS_2 = [1 - 2f (\sin^2 \theta_0 - \sin^2 \theta_2)]^{1/2},
\]
\[
\text{cn}S_2 = \sin \theta_2/\sin \theta_0.
\]
(96)
The values from (96), and the value of \( x_1 \) from (93), placed in (95), retaining terms of first order in \( f \), give
\[
\sin \theta_1 = -\sin \theta_0 + \pi f (\sin^2 \theta_0 - \sin^2 \theta_2)^{1/2} \sin \theta_0 | \theta_2 | < |\theta_0|, \]
(97)
which to first order in \( f \) is the equation relating the parametric latitudes \( \theta_1, \theta_2 \) of conjugate points on spheroidal geodesics but free of elliptic functions. Note that (97) also gives the special cases discussed following equations (93), as it should.

Discussion. We have demonstrated mathematically and geometrically (pictorially in Figure 13) that along the equator, the end points of the segment \( \alpha \) are conjugate. We have proved that consecutive vertices are conjugate from both (93) and (97). Now if we ignore the term in \( f \) in equation (97), we have \( \theta_1 = -\theta_2 \), with the longitude difference that of the hemispherical geodesic in vertex parametric latitude \( \theta_0 \), x-hence we get two equal geodesics as demonstrated in Figure 13. Hence to test approximation formulæ to the geodesic we need not exceed the length of the hemispherical geodesic (node to node or vertex to vertex) since it is maximum under the unique shortest distance criterion.

Note that equation (74) provides through the subsequent discussion, the sufficient condition for maximum geodesic length under the shortest distance property, the Euler equation, equation (3) above, being the necessary condition.
Also note that the parallels $\theta = \pm \theta_0$ are envelopes of all the geodesics whose vertex latitudes are $\pm \theta_0$, and the points $V_1, V_2$ (vertices), Figure 13, are points of tangency to the envelopes. But any conjugate point, according to the analytic definition, is a contact point of an envelope, reference [29] page 34. Finally note that two types of envelopes are involved. The envelope of all the geodesics having the same vertex parameter latitude $|\theta_0|$ are the parallels $\theta = \pm \theta_0$; the envelopes of all geodesics with a vertex in a common meridian are the two symmetric geodesic evolutes as shown in Figure 12.

Hemispheroidal geodesics under the shortest distance property

With the help of equations (33), (34), (35), (97) we establish that the nonplanar geodesic distance between two given spheroidal points, under the shortest distance property of the spheroidal geodesic, lies on an arc of one of the four equivalent spheroidal geodesics, as shown in Figure 15, where $\theta_0$ is the vertex parameter latitude of the geodesic through the two given points.

Graphically, if a wire frame were constructed connecting the semiequator, the semimeridian $m$, the meridian NTST' and the four hemispheroidal geodesics with vertex latitude $\theta_0$ as shown in Figure 15, then

$$\cos \theta_0 = \cos \theta \sin \alpha. \text{ The arc } DD' = \sin |1 - \theta| = h.$$

Figure 15. The four equal nonplanar hemispheroidal geodesics determined by a given vertex parameter latitude.
the rotation of the ellipsoid about the polar axis under this frame would bring the two given points into coincidence with one of these four equal nonplanar spheroidal geodesics. Note that when $\theta_0 \to 0$, all four geodesics coincide with the equatorial limiting distance $DD'$ and when $|\theta_0| \to 0/2\pi$ both spheroidal geodesics (1) and (2) coincide with the semimeridian $m$ while the spheroidal geodesics (3) and (4) coincide with the meridian $NT$.

We may then construct a table of possible cases of hemispheroidal geodesics to be considered in the testing of approximation formulae to the geodesic. With the help of equations (32), (33), (35), (38), (50)-(54), (97) the possible cases are listed in Table 2.

Note in Figure 15 that hemispheroidal geodesics (2) and (4) are the reflections in the equator of (1) and (3). Also the meridian $NQS$ bisects all four hemispheroidal geodesics with the same vertex latitude $\theta_0$. We can treat a geodesic in the southern hemispheroid as though it were in the northern and translate computed elements symmetrically with respect to the equator. Thus all possible cases required to test approximation formulae to a geodesic with vertex latitude $\theta_0$ are as shown in Figure 15. Note that arc $AB$ contains a node, arc $CE$ contains a vertex, and arc $CF$ contains neither vertex nor node.

### Table 2. Hemispheroidal geodesics.

$\Delta \lambda_0$ is the longitude difference, node to node or vertex to vertex, of the spheroidal geodesic whose vertex parametric latitude is $\theta_0$, see equations (33) and Table 8.

#### CASE

I. $|\theta_2| \neq |\theta_1|$, $\Delta \lambda_1 \pm \Delta \lambda_2 < \Delta \lambda_0$

a. $|\theta_0| > \theta_2 > \theta_1 > 0$

b. $|\theta_0| > \theta_1 > \theta_2 > 0$

c. $\theta_2 < 0, \theta_1 > 0$

   1. $|\theta_0| > |\theta_2| > |\theta_1| > 0$

   2. $|\theta_0| > |\theta_1| > |\theta_2| > 0$

d. $\theta_1 < 0, \theta_2 > 0$

   1. $|\theta_0| > |\theta_1| > |\theta_2| > 0$

   2. $|\theta_0| > |\theta_1| > |\theta_2| > 0$

II. $|\theta_2| = |\theta_1|$, $\Delta \lambda_1 = \pm \Delta \lambda_2 < (1/2)\Delta \lambda_0$

a. $|\theta_0| > |\theta_1| > 0, \theta_2 = -\theta_1$

b. $\theta_1 < 0, \theta_2 = |\theta_1| < |\theta_0|$

c. $\theta_2 = \theta_1, |\theta_0| > |\theta_1|$

III. $\Delta \lambda_1 = \pm \Delta \lambda_2 < (1/2)\Delta \lambda_0$

a. $|\theta_2| = |\theta_1| = |\theta_0| = 0$

b. $\theta_1 = \theta_2 = 0, \theta_0 \neq 0$

   c. $\theta_1 = -\theta_2, \Delta \lambda = \Delta \lambda_0$

*General* — the geodesic arc may not include either a vertex or a node; may include one vertex or one node.

*Symmetric* — with respect to a node or vertex but not maximum; contains one node or one vertex.

*Special case of I.*

*Maximum* — between two consecutive vertices or two consecutive nodes or between two points as in IIIc. Special case of II.
Some numerical considerations

Since there are 206264.8062 seconds in one radian, a 6 in the ninth decimal place of one radian represents .001 second. Maximum hemispheroidal radian geodesic length is the semimeridian which is slightly under π radians. Table 3 shows the effect of radian decimal places and significant figures in computing geodetic distances over the hemispheroid. Note that with 10 decimal places of radians there will be some uncertainty in the third decimal of meters at maximum hemispheroidal geodetic length.

The spheroidal triangle

We first indicate some analogies between spherical and spheroidal right triangles. From the definitions, equations (13), we have \( \sin u = \sin S, \cos a = \cos S, \tan a = \tan S; \) where \( a = \text{am} S, \) amplitude of the elliptic integral of the first kind, \( S = F(k, \phi), \) and where the modulus is \( k = e \sin \theta_0/(1 - e^2 \cos^2 \theta)^{1/2} \)—see equations

<table>
<thead>
<tr>
<th>radians</th>
<th>decimals</th>
<th>significant figures</th>
<th>meters (aradians)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi/32 = .0981748 )</td>
<td>7</td>
<td>6</td>
<td>626179</td>
</tr>
<tr>
<td>.77</td>
<td>8</td>
<td>7</td>
<td>8.9</td>
</tr>
<tr>
<td>... .04</td>
<td>10</td>
<td>9</td>
<td>.949</td>
</tr>
<tr>
<td>...... .25</td>
<td>12</td>
<td>11</td>
<td>.94904</td>
</tr>
<tr>
<td>( \pi/4 = .7853982 )</td>
<td>7</td>
<td>7</td>
<td>5009432</td>
</tr>
<tr>
<td>.16</td>
<td>8</td>
<td>8</td>
<td>1.6</td>
</tr>
<tr>
<td>... .3</td>
<td>9</td>
<td>9</td>
<td>.59</td>
</tr>
<tr>
<td>...... .4</td>
<td>10</td>
<td>10</td>
<td>.592</td>
</tr>
<tr>
<td>( \pi/2 = 1.5707963 )</td>
<td>7</td>
<td>8</td>
<td>10018863</td>
</tr>
<tr>
<td>.3</td>
<td>8</td>
<td>9</td>
<td>3.2</td>
</tr>
<tr>
<td>.27</td>
<td>9</td>
<td>10</td>
<td>.19</td>
</tr>
<tr>
<td>...... .68</td>
<td>10</td>
<td>11</td>
<td>.185</td>
</tr>
<tr>
<td>( \pi = 3.1415927 )</td>
<td>7</td>
<td>8</td>
<td>20037727</td>
</tr>
<tr>
<td>.65</td>
<td>8</td>
<td>9</td>
<td>6.3</td>
</tr>
<tr>
<td>... .4</td>
<td>9</td>
<td>10</td>
<td>.37</td>
</tr>
<tr>
<td>...... .36</td>
<td>10</td>
<td>11</td>
<td>.369</td>
</tr>
</tbody>
</table>
(15) and (19)a. Hence for the spherical and spheroidal triangles $NP_a F$, $NP_b P$ as shown in Figure 16, we have the following analogies:

<table>
<thead>
<tr>
<th>Spherical</th>
<th>Spheroidal</th>
</tr>
</thead>
<tbody>
<tr>
<td>A.1 $\cos \theta_a = \cos \theta \sin \alpha$</td>
<td>B.1 $\cos \theta_a = \cos \theta \sin \alpha$</td>
</tr>
<tr>
<td>A.2 $\cot \theta_a = \tan \alpha \sin \sigma$</td>
<td>B.2 $\cot \theta_a = \tan \alpha \sin S$</td>
</tr>
<tr>
<td>A.3 $\sin \theta = \sin \theta_a \cos \alpha$</td>
<td>B.3 $\sin \theta = \sin \theta_a \cos S$</td>
</tr>
<tr>
<td>A.4 $\tan \sigma = \cos \theta_a \tan \eta$</td>
<td>B.4 $\tan S = \cos \theta_a \tan \Delta \lambda$</td>
</tr>
</tbody>
</table>

From Figure 10 and the identity (98) (A.1 = B.1) we have respectively:

$$\cos \alpha = \sin \left( \frac{\pi}{2} - \alpha \right) = \alpha(1 - \varepsilon \cos \theta)^{1/2} \left( -\frac{d\theta}{ds} \right)$$

$$\sin \alpha = \cos \theta \sec \theta,$$

whence

$$\tan \alpha = -\frac{\cos \theta \sec \theta}{\alpha(1 - \varepsilon \cos ^2 \theta)^{1/2}} \cdot \frac{ds}{-\cos \theta d\theta}$$

\(\phi\) is geocentric latitude for the sphere, parametric for the ellipsoid; \(\phi\) is geodetic latitude

**Figure 16.** Illustrating analogies between spherical and spheroidal right triangles.

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From (98), B.3 and (13)c
\[ -\cos \theta \, d\theta = \sin \theta_0 \, dn \, dS \, dS \]  
(100)

From (13) and (15)
\[ (1 - e^2 \cos^2 \theta)^{1/2} = (1 - e^2 \cos^2 \theta_0)^{1/2} \, dn \]  
(101)

From (13)c and (17)
\[ ds = \frac{e \, \sin \theta_0 \, dn^2 \, dS}{k} \]  
(102)

From (100) and (102), with \( k = e \, \sin \theta_0 (1 - e^2 \cos^2 \theta_0)^{1/2} \),
\[ ds = a \frac{(1 - e^2 \cos^2 \theta_0)^{1/2} \, dn \, dS}{\sin \theta_0} \]  
(103)

Substituting from (101) and (103) in (99) find
\[ \cos \theta_0 \, d\theta = \frac{1}{\sin \theta_0} \, (1 - e^2 \cos^2 \theta_0)^{1/2} \, \frac{ds}{\sin \theta_0} = \frac{\cos \theta_0}{\sin \theta_0} \]  
(104)

which is (98) B.2 and becomes (98) A.2 when \( e \to 0, k \to 0, S \to \int_0^\theta \, d\sigma = \sigma \), see equations (12), and
\( sn \to \sin \sigma, cn \to \cos \sigma \), where \( \sigma \) is then the spherical distance from the vertex of the geodesic (parametric latitude \( \theta_0 \)) to a point on the geodesic in parametric latitude, \( \theta \), i.e. \( \sigma = \arccos(\sin \theta/\sin \theta_0) \) or (98) A.3; see also Figure 11.

To find an expression for \( \Delta \lambda \) in (98) B.4, we have from equation (17)
\[ (k/e) \cot \theta_0 \, d\lambda = (1 - k^2 \sin^2 \sigma)^{1/2} \, d\sigma/(1 + n \sin^2 \sigma), n = \tan^2 \theta_0 \]  
(105)

and from (13)c and (19)a find
\[ (1 - k^2 \sin^2 \sigma)^{1/2} = dn \, dS, \sin^2 \sigma = sn^2 S, \, d\sigma = dn \, dS, \]  
whence (105) becomes
\[ (k/e) \cot \theta_0 \, d\lambda = dn^2 S \, dS/(1 + n \sin^2 S) \]  
(106)

If we let \( \tan U = \sec \theta_0 \, \tan S \), then
\[ \sec^2 U \, dU = \sec \theta_0 \, d(\tan S) = \sec \theta_0 \, dn \, dS/cn^2 S \]  
(107)

where we have used \( d(\tan S) = dn \, dS/cn^2 S \) from (13)c.

Now
\[ \sec^2 U = 1 + \tan^2 U = 1 + \sec^2 \theta_0 \, \tan^2 S \]  
\[ = 1 + (1 + \tan^2 \theta_0) \tan^2 S \]  
\[ = (cn^2 S + sn^2 S + \tan^2 \theta_0 \, sn^2 S)/cn^2 S \]  
\[ \sec^2 U = (1 + n \sin^2 S)/cn^2 S. \]  
(108)

(from the identities (13)a, \( sn^2 S + cn^2 S = 1 \), \( \tan S = sn S/cn S \))

From (107) and (108) we have
\[ \cos \theta_0 \, dU = dn \, dS/(1 + n \sin^2 S), n = \tan^2 \theta_0. \]  
(109)

Subtracting respective members of (106) from (109), find
\[ d\lambda = (e/k) \tan \theta_0 \, [\cos \theta_0 \, dU - \frac{dn S - dn^2 S}{1 + n \sin^2 S} \, d\lambda], \]  
or
\[ \Delta \lambda = (e \sin \theta_0/k) \int_0^\theta \frac{dn S - dn^2 S}{1 + n \sin^2 S} \, dS \]  
(110)

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where \( U = \arctan \left( \sec \theta_0 \tan S \right) \), \( k = e \sin \theta_0 / (1 - e^2 \cos^2 \theta_0) \), \( n = \tan^2 \theta_0 \). Solving for \( U \), (110) may be written

\[
\tan S = \cos \theta_0 \tan \left[ \left( k/e \sin \theta_0 \right) \Delta \lambda + \sec \theta_0 \int_0^S \frac{dnS}{1 + n \sin^2 S} \right].
\]  

(111)

When \( e \to 0 \), \( k/e \sin \theta_0 \to 1 \to dsS, \tan S \to \tan \sigma \) and (111) becomes the spherical formula \( \tan \sigma = \cos \theta_0 \tan \Delta \lambda \) where \( \Delta \lambda = \eta \), the spherical longitude, i.e. (111) becomes (98) A.4 when \( e \to 0 \). Thus the analogies (98) are implicit in the spherical approximation to the spheroidal triangle as demonstrated in Figure 11.

The approximate solution for geodesy

Direct Solution

For the direct solution we are given the geodesic length \( S \) from a given point \( P_1(\phi_1, \lambda_1) \) in given azimuth \( \alpha_{1,2} \) to find the geographic coordinates \( \phi_2, \lambda_2 \) of a point \( P_2(\phi_2, \lambda_2) \) and the azimuth \( \alpha_{2,1} \). A solution, reliable over the hemispheroid, will be sought consistent with the following criteria:

1. An accuracy of 1 meter in position—geodetic distance within 1 meter: latitude, longitude, azimuth within .035 second over the longest possible hemispheroidal geodesics; at least, in the limiting case, equalling the 1/100,000 distance and 1 second azimuth requirement adopted by ACIC, reference [22].

2. No tables required in the computations except natural trigonometric as Peters 8-place for desk computing.

3. Easy adaptation to any reference ellipsoid by merely changing the ellipsoid defining parameters.

4. No root calculation or iteration and formulae adaptable to both desk computation and large electronic computers with terms no higher than second order in the flattening.

Now the parametric latitude \( \theta_1 \) of \( P_1 \) may be computed from \( \cos \theta_0 = \sin \alpha_{1,2} \cos \theta_1 = - \sin \alpha_{2,1} \cos \theta_2 \).

We place \( d = S/aD \),

and from equations (48) write

\[
\xi_1 = S_1/aD = \sigma_1 + P \sin 2\sigma_1 - Q \sin 4\sigma_1 + R \sin 6\sigma_1
\]

(114)

\[
\xi_2 = d - \xi_1 = S_2/aD = \sigma_2 + P \sin 2\sigma_2 - Q \sin 4\sigma_2 + R \sin 6\sigma_2
\]

(115)

where \( \sigma_1 = \arccos \left( \sin \theta_1 / \sin \theta_0 \right) \), \( \sigma_2 = \arccos \left( \sin \theta_2 / \sin \theta_0 \right) \)

and \( P = E/D, Q = F/D, R = G/D \),

with \( D, E, F, G \) from equations (49).

Since we have \( \cos \theta_0 \) from (112), the constants \( c_1, c_2, c_3, c_4 \) and \( A, B, C, D, E, F, G \) may be computed from (49). Since we have \( \xi_1 \) and \( \theta_0 \) we can compute \( \alpha_1 \) and then \( \xi_2 \) from (114), \( \delta_2 \) from (115), i.e. from \( \xi_2 = d - \xi_1 \). But we need \( \delta_3 \) and therefore the series (115) must be reversed. Figure 17 shows the spherical triangle being used.

Now the ranges of \( c_1, c_2, c_3, c_4 \) are for \( 0 < |\theta_0| < \pi/2; f > c_1 > 0, 0 < c_2 < f/4, 1 + f > c_3 > 1, 1 + f > c_4 > 1 + f/4 \). Since the maximum hemispheroidal geodesic length under the shortest distance criterion is the semimeridian, given when \( \theta_0 = (1/2)\pi \), we have for this value of \( \theta_0 \):
A = B = C = 0, D = .9983056819, E = .8475185 \times 10^3, F = .1799 \times 10^3.
G = 1 \times 10^{-10}, P = E/D = .84889569 \times 10^3, Q = F/D = .1802 \times 10^4,
R = G/D = 1.001 \times 10^5, \text{where } f = .3399075283 \times 10^3 \text{ (Clarke 1866).}

The maximum contributions of the terms E \sin 2\theta, F \sin 4\theta, G \sin 6\theta are: 
2aE = 10.811.296m, 2aF = 2.295m, 2aG = .0013m. An examination of Table 3 shows that there will be a maximum angular error of .001 second in holding 8 decimals of radians. We arbitrarily reject all decimal radian terms of the order .3 \times 10^{-6} or less in the analysis to follow. We have at once from (116) that G = R = 0.

We write (115) as
\begin{align*}
\xi_2 &= \sigma_2 + \Sigma = P \sin 2\sigma_2 - Q \sin 4\sigma_2 \\
\text{whence } \sin \xi_2 &= \sin 2\sigma_2 \cos 2\Sigma + \cos 2\sigma_2 \sin 2\Sigma \\
\sin 4\xi_2 &= \sin 4\sigma_2 \cos 4\Sigma + \cos 4\sigma_2 \sin 4\Sigma
\end{align*}

We use the series approximations, \sin X = X - X^3/6, 
\cos X = 1 - X^2/2, and find, rejecting terms of the order of .3 \times 10^{-6}

\begin{align*}
\sum \sigma &= \sigma_1 + \sigma_2 \\
2\sigma_{2-1} &= 2\theta - \infty
\end{align*}

Figure 17. Spherical triangle used in approximating the spheroidal triangle. Azimuths are from north with east longitudes positive.
or less (using the values of \( P \) and \( Q \) from (116)): 
\[
\sin 2\zeta = 2(P \sin \frac{\pi}{2} - Q \sin 4\theta) \\
\cos 2\zeta = 1 - 2P^2 \sin^2 2\theta \\
\sin 4\zeta = 4(P \sin 2\theta - Q \sin 4\theta) \\
\cos 4\zeta = 1 - 8P^2 \sin^2 2\theta
\]  
(119)

The values from (119) returned to (118), with the help of some trigonometric identities, give 
\[
\sin 2\zeta = [1 - (3/2)P^2 - Q] \sin 2\theta + P \sin 4\theta + [(1/2)P^2 - Q] \sin 6\theta \\
\sin 4\zeta = -2P \sin 2\theta + (1 - 4P^2) \sin 4\theta + 2P \sin 6\theta + 2P^2 - Q \sin 8\theta \\
\]  
(120)

If we multiply (120) by \( P \), rejecting terms of the order \( .3 \times 10^8 \) or less, with the values of \( P \) and \( Q \) from (116), we get 
\[
P \sin 2\zeta = P \sin 2\theta + P^2 \sin 4\theta \\
\]  
(122)

Now subtract respective members of (122) from (117) to get 
\[
\xi - P \sin 2\xi = \sigma_2 - (Q + P^2) \sin 4\theta \\
\]  
(123)

Next multiply (121) by \( Q + P^2 \), rejecting terms of order \( .3 \times 10^8 \) or less, to get 
\[
(Q + P^2) \sin 4\zeta = (Q + P^2) \sin 4\theta \\
\]  
(124)

From (123) and (124) we have then 
\[
\sigma_2 = \xi - P \sin 2\xi + M \sin 4\theta \\
\]  
(125)

where \( M = P^2 + Q \).

Now from (116), \( Q = F = .3 \times 10^8 \), hence we may place \( Q = F \) and we write from (114), (115), and (125) with \( G = R = 0, Q = F, P = E/D, M = P^2 + F, d = S/aD, \)
\[
\sigma_2 = U - P \sin 2\theta + M \sin 4\theta \\
U = d - \xi = d - \xi - P \sin 2\theta + F \sin 4\theta \\
\]  
(126)

We next examine our fundamental coefficients for exclusion of terms of order \( .3 \times 10^8 \) or smaller.

From (49),
\[
C = (1/4) c_1 c_2, c_1 = f \cos \theta, c_2 = (1/4) f \sin^2 \theta, \\
\frac{dC}{d\theta} = (1/4) c_1 (c_2 + 2c_1 c_2 + c_2 c_2) = 0, \\
f = \frac{dC}{d\theta} = (1/2)f \sin \theta \cos \theta, \\
\frac{dC}{d\theta} = (1/4)f \sin \theta \cos \theta \\
\]  
and we find 
\[
\frac{dC}{d\theta} = (1/4) c_2 f \sin \theta (-c_2 + c_1 \cos \theta) = 0, \\
\]  
whence the minimum is given by \( \theta = 0 \) (the equatorial limiting arc), and the maximum by \( c_2 = c_1 \cos \theta \) or \( (1/4)f \sin^2 \theta = f \cos^2 \theta \), whence \( \tan \theta = 2, \theta = 63^\circ 26' 05.816 \) and maximum value of \( C \) is 
\[
C = (1/4) c_1 c_2 = .174 \times 10^9. \\
\]  
Hence we place \( C = 0 \) with respect to our rejection criterion \( .3 \times 10^8 \).

Since \( c_2^2 = (1/64) f^2 \sin^2 \theta \), the maximum value is at \( \theta = \pi/2, \) when \( c_2^2 = .6 \times 10^9 \) (Clarke 1866).

Hence we neglect terms of the order \( c_2^2 \) in the coefficients \( D \) and \( E \) and write:
\[
A = c_1(1 - c_2 c_2), B = (1/2)c_1 c_2 c_3, D = 2 + c_2 c_4(c_4 - 1) - (c_3 + c_4), \\
E = (1/2)c_2 (2 + c_2(c_2 - 1)), F = (1/4)c_2^2 (2c_4 - 1), \\
c_1 = f \cos \theta, c_2 = (1/4)f \sin^2 \theta, c_3 = 1 + c_1 \cos \theta, c_4 = c_3 + c_3. \\
\]  
(127)
We next consider the effect of omitting the terms in $f^4$ in the coefficients. If this is done they become

\[ A = c_1(1 - c_2), \quad B = (1/2)c_1c_2, \quad D = (1 - c_2)(2 - c_2 - c_3), \quad E = (1/2)c_2(1 + c_3), \]
\[ F = (1/4)c_2^2, \quad c_1 = fn, \quad c_3 = (1/4)(1 - N^2), \quad c_3 = 1 + fn^2, \quad N = \cos \theta_0, \]

identity: \( c_1^2 - 4c_2(c_3 - 1) + c_3(2 - c_2) = 1. \)

Now we form the differences of coefficient values from (127) and (128) and examine for maximum values:

\[ |\Delta A| = c_1 c_2 |(c_3 - 1)| = (1/16) f^2 N(1 - N^2)(1 + 3N^2) = \frac{4N}{1 + 3N^2} |\Delta A| \]
\[ |\Delta B| = (1/2)c_1 c_2 |1 - c_3| = (1/8) f^2 N^3(1 - N^2) = \frac{2N^2}{1 + 3N^2} \frac{4N}{1 + 3N^2} |\Delta B| \]
\[ |\Delta D| = c_2 |[c_4(2 - c_4) - 1]| = (1/64) f^2 (1 + 3N^2)^2 (1 - N^2) \]
\[ |\Delta E| = (1/2)c_3 |(2c_3 - 1 - c_3^2)| = (1/8) f^2 N^4(1 - N^2) = N |\Delta B| = N \frac{2N^2}{1 + 3N^2} \frac{4N}{1 + 3N^2} |\Delta D| \]
\[ |\Delta F| = (1/2)c_2^2 |(1 - c_4)| = (1/128) f^2 (1 + 3N^2) (1 - N^2) = \frac{1 - N^2}{2(1 + 3N^2)} |\Delta D| \]

Since \[ |\theta_0| < \pi/2, \quad 0 < N = \cos \theta_0 < 1, \quad 0 < \frac{1 - N^2}{2(1 + 3N^2)} < .5, \]
\[ 0 < \frac{2N^2}{1 + 3N^2} < .5, \quad 0 < \frac{4N}{1 + 3N^2} < 1.2, \]
we have from (129) that

\[ |\Delta A| < (1.2) |\Delta D| \text{ max}, \quad |\Delta B| < (0.6) |\Delta D| \text{ max}, \quad |\Delta E| < (0.6) |\Delta D| \text{ max}, \quad |\Delta F| < (0.5) |\Delta D| \text{ max}. \]

Thus we have only to find \( |\Delta D| \text{ max} \) and show that both it and \( |\Delta A| \) are less than the rejection criterion, \( .3 \times 10^{-8} \). We find

\[ \frac{d |\Delta D|}{d \theta_0} = (f^2/64) \cdot 2(1 + 3N^2) NN'(5 - 9N^2) = 0, \]
whence \( N = 0, \theta_0 = \pi/2, \quad N' = 0, \quad |\theta_0| > 0, \quad N' \text{ max} = 5/9. \) With this last value \( |\Delta D| \text{ max} = .2 \times 10^{-8}, \]
\( |\Delta A| < (1.2) |\Delta D| \text{ max} = .24 \times 10^{-8} \) which is sufficient to justify the values (128).

Now \( \sin \theta = \sin \theta_0 \cos \phi, \quad \tan \phi_1 = \tan \theta_0/(1 - \ell), \)
\( \sin \phi_1 = \cos \theta_0 \cos \phi_1 = \cos \theta_0 \sin a_1/cos \theta_0 \]

From (50)

\[ \Sigma = \eta_1 + \eta_2, \quad \Sigma = \sigma_1 + \sigma_2, \quad \Delta = \sigma_1 - \sigma_2, \]
\[ P = 2 \sin \Sigma \cos \Delta, \quad \Delta = \Sigma - A \Sigma + B \Pi, \quad \lambda_1 = \lambda_1 + \Delta, \]
\( \eta_1 = \arccos \left( \tan \theta_0/\tan \theta_0 \right) = \arccos \left( \cos \theta_0 \cos \phi_1/\cos \theta_2 \right) \]
\( a_1 = \arcsin \left( \sin \theta_0/\sin \theta_0 \right). \)

**Summary of first direct solution, given \( \phi_1, \lambda_1, S, a_1, \lambda. \)**

1. Convert \( \phi_1 \) to parametric latitude from \( \tan \theta_1 = (1 - f) \tan \phi_1 \).
2. Compute \( \cos \theta_0 = \cos \theta_1 \sin a_{1.2} \) (geodesic vertex)
3. Compute \( a_1 = \arccos \left( \frac{\sin \theta_1}{\sin \theta_0} \right) \sin 2a_1, \sin 4a_1 \)
4. Compute A, B, D, E, F from (128) and
   \[ P = E/D, M = F + D^3, d = S/aD \]
5. From (126), \( U = d - a_1 - P \sin 2a_1 + F \sin 4a_1, \sin 2U, \sin 4U \)
6. \( a_2 = U - P \sin 2U + M \sin 4U, \cos a_2 \)
7. \( \theta_2 = \arcsin \left( \frac{\cos \theta_0 \cos a_2}{\cos \theta_1} \right) \) \( a_{1.2} = \frac{2\pi - \arcsin \left( \frac{\cos \theta_0 / \cos \theta_1}{\cos \theta_1} \right)}{\cos \theta_1}, \tan \theta_2, \tan \theta_2/(1 - \ell) \)
8. \( \eta_1 = \arccos \left( \frac{\tan \ell}{\tan \theta_0} \right), \eta_2 = \arccos \left( \frac{\cos \theta_0 \cos a_2 / \cos \theta_2}{} \right) \)
9. From (131), \( \Sigma \eta = \eta_1 + \eta_2, \Sigma a = a_1 + a_2, \Delta \sigma = a_1 - a_2, \Delta \lambda = \Sigma \eta - \Sigma \sigma + Bp, \lambda_2 = \lambda_1 + \Delta \lambda \)

**Alternative trigonometric formulae**, reference [19].

When \( \Sigma a = a_1 + a_2 \) has been found
\[
\tan \alpha_{1.3} = \frac{\cos \theta_2 / (\Sigma \sigma \sin \theta_1 - N \cos \Sigma \sigma), N = \sin \theta_0 \sin a_1 = \cos \theta_0 \cos a_{1.3}}{\sin \alpha_{2.3}}
\]
\[
\tan \phi_2 = \left( \cos \Sigma \sigma \sin \theta_1 + N \sin \Sigma \sigma \right) \sin \alpha_{2.3} / (1 - \ell) \cos \theta_0
\]
\[
\Sigma \eta = \arctan \left( \sin \Sigma \sigma \cos \theta_0 / (\cos \Sigma \sigma - \sin \theta_1 \sin \theta_2) \right)
\]
\[
= \arctan \left( \sin \Sigma \sigma \sin \alpha_{1.3} / (\cos \theta_1 \cos \Sigma \sigma - \sin \theta_1 \sin \Sigma \sigma \cos \alpha_{1.3}) \right)
\]

We make the following changes for a geodesic arc that will contain no vertex, but will contain a node:
\[
a_{1.2} = \pi + \alpha = \pi + \arcsin \left( \frac{\cos \theta_0 / \cos \theta_1}{\cos \theta_1} \right), \quad U = a_1 - d + P \sin 2a_1 - F \sin 4a_1 \]
\[
\tan \Delta \eta = \tan (\eta_1 - \eta_2) = \sin \Delta \sigma \sin a_{1.2} / (\cos \theta_1 \cos \Delta \sigma - \sin \theta_1 \sin \Delta \sigma \cos a_{1.3})
\]
\[
\Delta \lambda = \Delta \eta - \Delta \sigma + 2B \sin \Delta \sigma \cos \Sigma \sigma, \text{ from (51).}
\]

**General hemispheroidal direct solution.** (First form).

Now the formulae (132), (134) respectively, suggest the following general direct solution over the hemispheroid:

from
\[
\begin{align*}
U &= a_1 - d + P \sin 2a_1 - F \sin 4a_1 \\
a_1 &= U + d - P \sin 2a_1 + F \sin 4a_1 \\
a_2 &= U - P \sin 2U + M \sin 4U
\end{align*}
\]
we have \( \Delta \sigma = a_1 - a_2 = d + P \sin (2U - \sin 2a_1) + F \sin 4a_1 - M \sin 4U \)
\[
\sin 2U = \sin 2(a_1 - d) \cos 2\Sigma + \cos 2(a_1 - d) \sin 2\Sigma
\]
\[
\sin 4U = \sin 4(a_1 - d) \cos 4\Sigma + \cos 4(a_1 - d) \sin 4\Sigma
\]
where
\[
\Sigma = P \sin 2a_1 - F \sin 4a_1
\]

With the approximations \( \sin x = x - x^3/6, \cos x = 1 - x^2/2 \), where \( x = 2\Sigma, 4\Sigma \) and rejecting terms whose coefficients are \( 3 \times 10^{-6} \) or less in using the values of \( P, F, M = P^2 + Q \) from (116), we find
\[
P \sin 2U = P \sin 2(a_1 - d) + 2P^2 \sin 2a_1 \cos 2(a_1 - d)
\]
\[
M \sin 4U = (F + P^2) \sin 4(a_1 - d)
\]

The values from (137) placed in (135) and use of some trigonometric identities enable us to write

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\[ \Delta = a_1 - c_1 = d - 2\pi \sin d \cos (2\alpha_1 - d) \{1 - 2\pi \cos 2(\alpha_1 - d)\} + 2F \sin 2d \cos 2(\alpha_1 - d) \]
\[ \Sigma = a_3 + c_3 = 2\alpha_1 - \Delta \sigma. \]

From equations (128), (133), and (138) we assemble the formulae for the general direct hemispheroidal solution:

\[ \tan \theta_1 = (1 - \Gamma) \tan \phi_1, M = \cos \theta_0 = \cos \theta_1 \sin a_{1,2}, N = \cos \theta_1 \cos a_{1,2}, \]
\[ c_1 = fM, c_2 = (1/4)(1 - M^2), A = c_1 - 2B, B = (1/2)c_1 c_2, D = (1 - c_2)^2 - AM, \]
\[ E = c_2 + BM, F = (1/4)c_2^2, P = E/D, \text{Check: } AM = 2BM + D + 2E - 4F = 1. \]
\[ a_1 = \arccos(\sin \theta_1 / \sin \theta_0), \Delta \sigma = d - 2\pi \sin d \cos (2\alpha_1 - d) \{1 - 2\pi \cos 2(\alpha_1 - d)\} + 2F \sin 2d \cos 2(2\alpha_1 - d) \]
\[ \cos (2\alpha_1 - d) = \cos 2(\alpha_1 - d) \cos d - \sin 2(\alpha_1 - d) \sin d, \]
\[ \cos 2(2\alpha_1 - d) = 2 \cos^2 (2\alpha_1 - d) - 1, \]
\[ \Sigma = 2\alpha_1 - \Delta, \tan \phi_2 = M/(N \cos \Delta \sigma - \sin \theta_1 \sin \Delta \sigma) \]
\[ \tan \phi_2 = (\sin \theta_1 \cos \Delta \sigma + N \sin \Delta \sigma) \sin a_{1,2}/(1 - \Gamma)M, \]
\[ \tan \Delta \eta = \sin \Delta \sigma \sin a_{1,2}/(\cos \theta_1 \cos \Delta \sigma - \sin \theta_1 \sin \Delta \sigma \cos a_{1,2}), \]
\[ \Lambda = \Delta \eta - A \Delta \sigma + 2B \sin \Delta \sigma \cos \Delta \sigma, \text{Check: } M = \cos \theta_1, \sin a_{1,2} = \cos \theta_1 \sin (\pi + a_{1,2}). \]

We arrange equations (139) as follows for construction of a computing form:

\[ \tan \theta_1 = (1 - \Gamma) \tan \phi_1, M = \cos \theta_0 = \cos \theta_1 \sin a_{1,2}, N = \cos \theta_1 \cos a_{1,2}, \]
\[ c_1 = fM, c_2 = (1/4)(1 - M^2), D = (1 - c_2)(1 - c_2 - c_1 M), P = c_2 [1 + (1/2)c_1 M]/D. \]
\[ \cos \alpha_1 = \sin \theta_1 / \tan \theta_0, d = S/aD, \tan \phi_2 = (\sin \theta_1 \cos \Delta \sigma + N \sin \Delta \sigma) \sin a_{1,2}/(1 - \Gamma)M, \]
\[ \tan \Delta \eta = \sin \Delta \sigma \sin a_{1,2}/(\cos \theta_1 \cos \Delta \sigma - \sin \theta_1 \sin \Delta \sigma \cos a_{1,2}), \]
\[ H = c_1(1 - c_2) \Delta \sigma - c_1 c_2 \sin \Delta \sigma \cos \Sigma, \Lambda = \Delta \eta - H, \lambda_1 = \lambda_1 + \Delta \lambda. \]

Check:
\[ M = \cos \theta_0 = \cos \theta_1 \sin a_{1,2} = \cos \theta_1 \sin (\pi + a_{1,2}). \]

Figure 18 shows equations (140) arranged in a computing form.

**General hemispheroidal direct solution. (Second form)**

With the hope of reducing the number of trigonometric functions involved, a second solution was developed which involves successive solutions on two spheres. The formulae are identical in some instances to those of the first solution. The quantities are the same in some cases but appear in different form with respect to formulae. The principal difference is in obtaining \( \Delta \sigma \). The solution from there on is identical.

The formulae are:

\[ \tan \theta_1 = (1 - \Gamma) \tan \phi_1, M = \cos \theta_0 = \cos \theta_1 \sin a_{1,2}, N = \cos \theta_1 \cos a_{1,2}, c_1 = fM, c_2 = Q/\sqrt{1 - M^2}, \]
\[ \cos \alpha_1 = \sin \theta_1 / \sin \theta_0, d = S/b, T = d / \sin d, V = 1 + h \sin^2 \theta_1, \gamma = 1/2(1 - \theta^2 - 1), A = V(1 - M^2), \]
\[ \sin \theta_3 = \sin \theta_1 \cos d + N \sin d, B = V \sin \theta_1, C = T - \cos d, L = AC + 2B, D = (B + L - A) \cos d, \]
\[ E = 8B(B + L) \cos d, P = 2AD \sin^2 d, Q = 3CA^2 + E, \Delta \sigma = \sin \theta [\gamma - \gamma cos \theta_1 + h^2/16] + Q). \]
\[ (141) \]
DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1, \alpha_{1-2}, S$ to find $\phi_2, \lambda_2, \alpha_{2-1}$. East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

SPHEROID $a \quad m \quad f$

$1 - f \quad \frac{1}{1} \quad \text{radian} = 206264.8062 \text{ seconds}$

<table>
<thead>
<tr>
<th>LINE</th>
<th>$\phi_1$</th>
<th>$\cos \theta_1$</th>
<th>$\tan \phi_1$</th>
<th>$\tan \theta_1 = (1 - f) \tan \phi_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_{1-2}$</td>
<td>$\sin \theta_1$</td>
<td>$\cos \theta_1$</td>
<td>$\cos \alpha_{1-2}$</td>
<td>$\sin \alpha_{1-2}$</td>
</tr>
<tr>
<td>$\sin \alpha_{1-2}$</td>
<td>$M = \cos \theta_0 = \cos \theta_1 \sin \alpha_{1-2}$</td>
<td>$\theta_0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\cos \alpha_{1-2}$</td>
<td>$N = \cos \theta_1 \cos \alpha_{1-2}$</td>
<td>$\sin \theta_0$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$c_1 = fM$ 
$c_2 = \frac{1}{2}(1 - M^2)f$
$c_2 = \frac{1}{2}(1 + \frac{1}{2}c_1 M)$

$D = (1 - c_2)(1 - c_2 - c_1 M)$

$P = c_2 (1 + \frac{1}{2}c_1 M)/D$

$\cos \sigma_1 = \sin \theta_1 \sin \theta_0$

$d = S/aD$ 

$u = 2(\sigma_1 - d)$

$W = 1 - 2P \cos u$

$V = \cos u \cos d - \sin u \sin d$

$Y = 2P VW \sin d$

$X = c_1 \sin d \cos d (2V^2 - 1)$

$\Delta \sigma = d + X - Y$

$\sin \sigma_1 \cos \sigma_1 = \cos \sigma_2$

$\Sigma \sigma = 2\alpha_{1-2} - \Delta \sigma$

$\tan \alpha_{2-1} = M/(N \cos \Delta \sigma - \sin \theta_1 \sin \Delta \sigma)$

$\tan \phi_2 = -(\sin \theta_1 \cos \Delta \sigma + N \sin \Delta \sigma) \sin \alpha_{2-1}$

$(1 - f)M$

$\phi_2$

$\tan \Delta \eta = \frac{\sin \Delta \sigma \sin \alpha_{1-2}}{\cos \theta_1 \cos \Delta \sigma - \sin \theta_1 \sin \Delta \sigma \cos \alpha_{1-2}}$

$H = c_1 (1 - c_2) \Delta \sigma - c_1 c_2 \sin \Delta \sigma \cos \Sigma \sigma$

$\Delta \lambda = \Delta \eta - H$

$\lambda_1$

$M = \cos \theta_0 = \cos \theta_1 \sin \alpha_{1-2} = \cos \theta_2 \sin (180 + \alpha_{2-1})$

$\lambda_2 = \lambda_1 + \Delta \lambda$

Figure 18. First direct solution computing form.
\[
\Sigma \alpha = 2\alpha_1 - \Delta \alpha, \tan \alpha_{2-1} = M/(N \cos \Delta \alpha - \sin \theta_1 \sin \Delta \alpha), \tag{141}
\]
\[
\tan \theta_2 = -(\sin \theta_1 \cos \Delta \alpha + N \sin \Delta \alpha) \sin \alpha_{2-1}/(1 - f)M, \tag{141}
\]
\[
\tan \Delta \eta = \sin \Delta \alpha \sin \alpha_{1-2}/(\cos \theta_1 \cos \Delta \alpha - \sin \theta_1 \sin \Delta \alpha \cos \alpha_{1-2}) \tag{141}
\]
\[
H = c_1(1 - c_2)\Delta \alpha - c_1 c_2 \sin \Delta \alpha \cos \Sigma \alpha, \Delta \lambda = \Delta \eta - H, \lambda_2 = \lambda_1 + \Delta \lambda. \tag{141}
\]
Check: \[
M = \cos \theta_\phi \sin \alpha_{1-2} = \cos \theta_2 \sin (\pi + \alpha_{2-1}). \tag{141}
\]

In essence, one solves for \(\Delta \alpha\) through two spherical triangles. With \(\alpha_{1-2}, \theta_1,\) and \(d = S/b\) one solves for \(\theta'_j\) in the triangle of Figure 19, by the formula \(\sin \theta'_j = \sin \theta_1 \cos \Delta \alpha + N \sin \Delta \alpha \). With this value of \(\theta'_2\), one computes the several quantities including \(\Delta \alpha\) and then one solves for \(\alpha_{2-1}, \theta_2, \Delta \eta\) in the triangle of Figure 20 as was done in the first general direct solution, equations (140).

The second method appears to be slightly less accurate than the first, and little if anything is saved in computation. Figure 21 shows equations (141) arranged in a computing form.

Conventions for azimuth and longitude.

We assume the initial is west of the terminus in the direct solution and then always \(0 \leq \alpha_{1-2} < 180^\circ\), \(0 \leq \Delta \eta < \Delta \lambda < \pi\). We find the first quadrant angles \(\nu\) and \(\nu'\) given by \(\tan \nu = |\tan \alpha_{2-1}|, \tan \nu' = |\tan \Delta \eta|\).

If \(\tan \alpha_{2-1} > 0\), then \(\alpha_{2-1} = 180^\circ + \nu\); if \(\tan \alpha_{2-1} < 0\), then \(\alpha_{2-1} = 360^\circ - \nu\). If \(\tan \Delta \eta > 0\), then \(\Delta \eta = \nu\); if \(\tan \Delta \eta < 0\), then \(\Delta \eta = 180^\circ - \nu\).

**Figure 19.** First spherical solution-second direct solution method.

**Figure 20.** Second spherical solution-second direct solution method.
DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given \( \phi_1, \lambda_1, \alpha_{1-2}, S \) to find \( \phi_2, \lambda_2, \alpha_{2-1} \). East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

\[
\begin{align*}
\text{SPHEROID} & \quad a \quad m \quad b \quad m \\
1 - f &= \frac{b}{a} \quad f \quad \text{m} \quad h &= \frac{1}{2} \left[ \frac{1}{(b/a)^2} - 1 \right] \text{m} \\
1 \text{ radian} &= 206264.8062 \text{ seconds}
\end{align*}
\]

\[
\begin{align*}
\text{LINE} & \quad \phi_1, \quad \tan \phi_1, \quad \sin \phi_1, \quad \cos \phi_1 \\
\theta_1, \sin \theta_1, \cos \theta_1 & \\
\alpha_{1-2} = \sin \alpha_{1-2}, \cos \alpha_{1-2} & \\
d(\text{rad}) &= \frac{S}{b} + d \equiv \sin d, \cos d & \\
M &= \cos \theta_1 \sin \alpha_{1-2} \quad T = d / \sin d, \cos d & \\
N &= \cos \theta_1 \cos \alpha_{1-2} \quad V = 1 + h \sin^2 \theta_1, + & \\
A &= V (1 - M^2), + B = V \sin \theta_1 \left( N \sin d + \sin \theta_1 \cos d \right) & \\
C &= T - \cos d \quad L = AC + 2B & \\
D &= 4(L + B) - A \cos d \quad E = 8B(2L + B) \cos d & \\
P &= 2AD \sin^2 d \quad Q = 3A^2C + E \quad P + Q & \\
\Delta \sigma &= \sin d \left[ T - \left( h/2 \right) L + \left( h^2/16 \right) \left( P + Q \right) \right] + & \\
\sin \Delta \sigma &= \sin \alpha_{1-2} \quad \cos \Delta \sigma \equiv \Delta \sigma & \\
\cos \Sigma \sigma &= \Sigma \sigma = \Delta \sigma, - \Delta \sigma & \\
\tan \alpha_{2-1} &= M / \left( N \cos \Delta \sigma - \sin \theta_1 \sin \Delta \sigma \right) \quad \alpha_{2-1} & \\
\tan \phi_2 &= \left( \sin \theta_1, \cos \Delta \sigma + N \sin \Delta \sigma \right) \sin \alpha_{2-1} \equiv \sin \alpha_{2-1} \quad \left( 1 - f \right) M & \\
\phi_2 & \quad \phi_2 & \\
\tan \Delta \eta &= \frac{\sin \Delta \sigma \sin \alpha_{1-2}}{\cos \theta_1 \cos \Delta \sigma - \sin \theta_1 \sin \Delta \sigma \cos \alpha_{1-2}} \quad \Delta \eta & \\
H &= c_1 \left( 1 - c_2 \right) \Delta \sigma - c_1 c_2 \sin \Delta \sigma \cos \Delta \sigma \equiv (\text{rad}) H & \\
c_1 &= fM \quad \Delta \lambda = \Delta \lambda = \Delta \eta - H & \\
c_2 &= \pi \left( 1 - M^2 \right) \quad \lambda_1 & \\
\text{CHECK} & \\
M &= \cos \theta_2 = \cos \theta_1 \sin \alpha_{1-2} \cos \theta_2 \sin \left( 180 + \alpha_{2-1} \right) \equiv \lambda_2 = \lambda_1 + \Delta \lambda & \\
\lambda_2 & \equiv \lambda_1 + \Delta \lambda & \\
\end{align*}
\]

Figure 21. Second direct solution computing form.
General hemiobroidal inverse (reverse) solution

The following geodetic length approximation for the inverse (reverse) solution between two points \( P_1(\theta_1, \lambda_1), P_2(\theta_2, \lambda_2) \) of the reference ellipsoid, was developed by the author, following the method of Forsyth [20], and published in [18]:

\[
S = a \left[ \frac{d}{4} \right] (Xd - Y \sin d) + \left( \frac{d^2}{64} \right) (AX - BY + CX^2 + DXY - EY^2),
\]

\[
\cos d = \sin \theta_1 \sin \theta_2 + \cos \theta_1 \cos \theta_2 \cos \Delta \lambda, \Delta \lambda = \lambda_2 - \lambda_1,
\]

\[
B = \frac{8d^2}{\sin d}, A = B \cos d, D = \frac{B}{2}, E = 2 \sin d \cos d, C = d + \left( \frac{1}{2} \right)(E - A),
\]

\[
X = (\sin \theta_1 + \sin \theta_2)^2 (1 + \cos d) + (\sin \theta_1 - \sin \theta_2)^2 (1 - \cos d) = 2 \sin^2 \theta_0,
\]

\[
Y = (\sin \theta_1 + \sin \theta_2)^2 (1 - \cos d) - (\sin \theta_1 - \sin \theta_2)^2 (1 - \cos d) = X \cos(d_1 + d_2),
\]

where \( \theta_0 \) is the parameter latitude of the great elliptic section through \( P_1, P_2 \) (contains the center of the ellipsoid) and \( d_1, d_2 \) are the spherical distances from this vertex to the points \( P_1, P_2; (d = d_1 - d_2) \). Other trigonometric formulae may be used to obtain the most accurate value of \( d \). Figure 22 shows the spherical elements involved.

\[\theta_0\] is the parametric latitude of the vertex of the great elliptic section. In the spherical triangle \( N P_1 P_2 \) we have \( \cos d = \sin \theta_1 \sin \theta_2 + \cos \theta_1 \cos \theta_2 \cos \Delta \lambda \). In the right spherical triangles \( N P_0 P_1, N P_0 P_2 \) we have respectively \( \cos d_1 = \sin \theta_1 \sin \theta_0, \cos d_2 = \sin \theta_2 \sin \theta_0 \). Thus \( d_1 \) and \( d_2 \) are analogous to \( d_1 \) and \( d_2 \), equations (114) and Figure 11, where \( \theta_0 \) is the parametric latitude of the geodetic vertex.

Figure 22. The spherical triangles used in the inverse approximation.
To assure the best trigonometric solution for \( d \), we adapt mid-latitude formulae, reference [18], page 87. We factor \( \sin d \) out of each term and write equations (142) in the following form for computing (east longitudes considered positive):

**Inverse (Reverse) Solution Formula**

\[
\tan \theta_1 = (1 - \ell) \tan \phi_1, \quad i = 1, 2, \quad \theta_m = (1/2)(\theta_1 + \theta_2), \quad \Delta \theta_m = (1/2)(\theta_2 - \theta_1),
\]

\[
\Delta \lambda = \lambda_2 - \lambda_1, \quad \Delta \lambda_m = (1/2)\Delta \lambda, \quad H = \cos^2 \Delta \theta_m - \sin^2 \theta_m = \cos^2 \theta_m - \sin^2 \Delta \theta_m,
\]

\[
L = \sin^2 \Delta \theta_m + H \sin^2 \Delta \lambda_m = \sin^2 \left(\frac{1}{2}\right)d, \quad 1 - L = \cos^2 \left(\frac{1}{2}\right)d, \quad \cos d = 1 - 2L,
\]

\[
U = 2 \sin^2 \theta_m \cos^2 \Delta \theta_m /(1 - L), \quad V = 2 \sin^2 \Delta \theta_m \cos^2 \theta_m /L, \quad X = U + V,
\]

\[
Y = U - V, \quad T = d / \sin d, \quad D = 2 \cos d, \quad A = DE, \quad B = 2D,
\]

\[
C = T - \left(\frac{1}{2}\right)(A - E); \quad Check: \quad C - \left(\frac{1}{2}\right)E + AD/B = T.
\]

\[
n_1 = X(A + CX), \quad n_2 = Y(B + EY), \quad \delta_1 d = (1/4)(TX - Y),
\]

\[
\delta_2 d = (f^2/64)(n_1 - n_2 + n_3), \quad S_1 = \sin d (T - \delta_1 d), \quad S_2 = \sin d (T - \delta_1 d + \delta_2 d),
\]

\[
F = 2Y - E(4 - X), \quad M = 32T - (20T - A)(X - (B + 4) Y),
\]

\[
G = (1/2)(T + f^2/64)M, \quad Q = - (GF \tan \Delta \lambda) / A, \quad \Delta \lambda_m = (1/2)(\Delta \lambda + Q),
\]

\[
c_1 = - \sin \Delta \theta_m / (\cos \theta_m \tan \Delta \lambda_m), \quad u = \sec \tan |c_1|, \quad a_1 = v - u,
\]

\[
c_2 = \cos \Delta \theta_m / (\sin \theta_m \tan \Delta \lambda_m'), \quad v = \sec \tan |c_2|, \quad a_2 = v + u,
\]

\[
\begin{array}{cccc}
c_1 & c_2 & a_{11} & a_{21} \\
- & + & a_1 & 360 - a_2 \\
+ & + & a_2 & 360 - a_1 \\
- & - & 180 - a_2 & 180 + a_1 \\
+ & - & 180 - a_1 & 180 + a_2
\end{array}
\]

The principal difference in equations (143) and those of reference [18] page 87, is the arrangement for \( P_1 \) to be always west of \( P_2 \), east longitudes positive, and the addition of azimuth equations to second order in \( f \). The azimuths are an adaptation of Guggenheim's equations, reference [23], where conversion has been made to parametric latitude and terms transformed into the parameters used in the length computations. The arrangement for identifying the azimuths without the quadrant search, as displayed in the last of (143), will be generated in a discussion of azimuths to follow.

**Azimuth determination in the inverse solution**

With the point \( P_1 \) always west of \( P_2 \), east longitudes positive, we must establish some conventions in order to determine the azimuths from north. In a spherical triangle \( P_1P_2P \), as shown in Figure 23, we have the corresponding parts as indicated: \( B = a_{11}, A = 360 - a_{21}, a = 90 - \theta_1, b = 90 - \theta_2, C = \Delta \lambda' \) and

\[
(1/2)(A + B) = 180^\circ + (1/2)(a_{11} - a_{21}), \quad (1/2)(A - B) = 180^\circ - (1/2)(a_{11} + a_{21}),
\]

\[
(1/2)(a - b) = (1/2)(\theta_2 - \theta_1) = \Delta \theta_m, \quad (1/2)(a + b) = 90^\circ - (1/2)(\theta_1 + \theta_2) = 90 - \theta_m.
\]

\[
C/2 = (1/2)\Delta \lambda' = \Delta \lambda_m'.
\]

From Gauss's equations, reference [19] page 162:

\[
\tan \left(1/2\right)(A + B) = \cos \left(1/2\right)(a - b) / \cos \left(1/2\right)(a + b) \tan \left(1/2\right)C,
\]

\[
\tan \left(1/2\right)(A - B) = \sin \left(1/2\right)(a - b) / \sin \left(1/2\right)(a + b) \tan \left(1/2\right)C.
\]

(145)
The values from (144) placed in (145) give
\[
\tan \left( \frac{1}{2}(a_1 + a_2,1) \right) = -\sin \Delta \theta_m / \cos \theta_m \tan \Delta \lambda'_m = c_1,
\]
\[
\tan \left( \frac{1}{2}(a_1 - a_2,1) \right) = \cos \Delta \theta_m / \sin \theta_m \tan \Delta \lambda'_m = c_2. \tag{146}
\]

The formulae (146) were given with equations (143) where \( \Delta \lambda'_m \) is the mean longitude difference as corrected to account for the ellipsoid.

Since \( |\theta_m| = |(1/2)(\theta_1 + \theta_2)| \leq 90^\circ \) and \( |\Delta \theta_m| = |(1/2)(\theta_2 - \theta_1)| \leq 90^\circ \), then always \( \cos(\pm \theta_m) > 0 \), \( \cos(\pm \Delta \theta_m) > 0 \). Always, since east longitudes are positive, with \( P_1 \) west of \( P_2 \), \( \Delta \lambda > 0 \), \( \Delta \lambda'_m > 0 \). Hence the signs of \( c_1 \) and \( c_2 \), in equations (146) depend only on the signs of \( \sin \Delta \theta_m \) and of \( \sin \theta_m \) respectively. Now Figure 24 shows all the possible azimuth situations, \( \theta_1 \neq \theta_2 \), from which the corresponding signs of \( \sin \Delta \theta_m = \sin (\Delta \theta_m) \), \( \sin \theta_m = \sin (\Delta \theta_m) \) can be determined.

A summary of sign conventions as obtained from Figure 24 and equations (146) is given in Table 4.

If we find the first quadrant angles \( u \) and \( v \) corresponding to \( \tan u = |c_1| \), \( \tan v = |c_2| \) and then form \( a_1 = v - u \), \( a_2 = v + u \), we may determine all azimuths from Table 5.

![Figure 23. Azimuths in the equivalent spherical triangle.](image-url)
Figure 34. Aircraft directions over the hemispheroid.
Table 4. Summary of azimuth sign conventions.

| Figure | \( \sin \Delta \theta_m \) | \( \sin \theta_m \) | \( \theta_2 > 0, \theta_1 > |\theta_1| \) | \( \theta_2 > 0, \theta_1 > |\theta_2| \) | \( \theta_1 < 0, |\theta_1| > |\theta_2| \) | \( \theta_2 < 0, |\theta_2| > |\theta_1| \) |
|--------|------------------|------------------|-----------------|-----------------|-----------------|-----------------|
| IA, IIA | +                | +                | -               | +               | -               | +               |
| IB, IIB | -                | -                | +               | +               | +               | -               |
| IIIA, IVA | +                | -                | -               | -               | -               | -               |
| IIIB, IVB | -                | -                | -               | -               | -               | -               |

Table 5. Azimuth determination in the inverse solution.

<table>
<thead>
<tr>
<th>Figure</th>
<th>( \theta_1 )</th>
<th>( \theta_2 )</th>
<th>( \lambda_2 - \lambda_1 )</th>
<th>( \alpha_{1-2} )</th>
<th>( \alpha_{2-1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>IA, IIA</td>
<td>( \theta_1 &gt; 0, \theta_2 &gt;</td>
<td>\theta_1</td>
<td>)</td>
<td>+</td>
<td>( \alpha_1 )</td>
</tr>
<tr>
<td>IB, IIB</td>
<td>( \theta_1 &gt; 0, \theta_1 &gt;</td>
<td>\theta_2</td>
<td>)</td>
<td>+</td>
<td>( \alpha_2 )</td>
</tr>
<tr>
<td>IIIA, IVA</td>
<td>( \theta_1 &lt; 0,</td>
<td>\theta_1</td>
<td>&gt;</td>
<td>\theta_2</td>
<td>)</td>
</tr>
<tr>
<td>IIIB, IVB</td>
<td>( \theta_2 &lt; 0,</td>
<td>\theta_2</td>
<td>&gt;</td>
<td>\theta_1</td>
<td>)</td>
</tr>
</tbody>
</table>

The last four columns of Table 5 are given with equations (143) and in effect eliminate the quadrant search since it has been done in advance. Figure 25 shows equations (143) arranged in a computing form.

Direct and inverse solutions of maximum spheroidal geodesics, node to node, vertex to vertex

Vertex to vertex. The direct and inverse are identical since the end points of the arc are the vertices and the longitude difference and length are given by equations (33) or (54). Azimuths are 90° and 270°.

Node to node. For the direct, \( \theta = 90° - \alpha_{1-2} \). Longitude and length are then given by equations (35) or (54). The back azimuth is given by \( \alpha_{2-1} = 270° + \theta \). For the inverse, we are given \( \Delta \lambda = \lambda_3 - \lambda_1 \), i.e. the end points are \( P(0, \lambda_1), P(0, \lambda_3) \) on the equator, and we have two cases:
1. \( \Delta \lambda < \pi(1 - \delta) \). The distance \( P_1, P_2 \) is \( S = \Delta \lambda \) and azimuths are 90° and 270°.

2. \( \pi(1 - \delta) < \Delta \lambda < \pi \). The nodes are in the respective antipodal zones. In the first of equations (33) we place \( \sin^2 \theta = 1 - \cos^2 \theta \) and write

\[
(1 - \Delta \lambda / \pi) = (1 - 1/4 - \eta/16) \cos \theta + (1/4)^2 (1 - \eta/2) \cos^3 \theta + 3(1/4)^3 \cos^5 \theta /16. \tag{147}
\]

Using \( 1/(1 - x) = 1 + x + x^2 + \ldots \), we may write

\[
D = (1/\phi)(1 - 1/4 - \eta/16) = (1/\phi)[1 + 1/4 + 2(1/4)^2 + 3(1/4)^3 + 4(1/4)^4 + \ldots] \]

We then write (147) as

\[
\cos \theta + u \cos^3 \theta = v = D(1 - \Delta \lambda / \pi), u = D(1/4)^2 (1 - \eta/2) = 1/4 - (1/4)^2, \tag{148}
\]

where unnecessary terms have been omitted.

Finally the formula for \( v \) is reversed in (148) and with the equation for \( S_0 \) from (33) we write for the inverse solution

\[
\cos \theta = v - u \cos^2 \theta, v = D(1 - \Delta \lambda / \pi), u = 1/4 - (1/4)^2, D = (1/\phi)[1 + 1/4 + 2(1/4)^2], \]

\[
\alpha_{1-2} = 90° - \theta, \alpha_{2-1} = 270° + \theta, S_0 = u (1 - 2(1/4)A + (1/4)^2 B + 2(1/4)^3 C), \tag{149}
\]

\[
A = 1 + \cos^2 \theta, B = (1 + 3 \cos^2 \theta)(1 - \cos^2 \theta), C = (1 + 2 \cos^2 \theta + 5 \cos^4 \theta) \cdot (1 - \cos^2 \theta). \]
INVERSE POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1; \phi_2, \lambda_2$ to find $S, \alpha_1, \alpha_2$. Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters). no root extraction.

<table>
<thead>
<tr>
<th>SPHEROID</th>
<th>a</th>
<th>m</th>
</tr>
</thead>
<tbody>
<tr>
<td>b</td>
<td>m</td>
<td></td>
</tr>
</tbody>
</table>

\[ 1 - f = \frac{b}{a} \]

\[ \frac{f^2}{64} \]

1 radian = 206264.8062 seconds

<table>
<thead>
<tr>
<th>$\phi_1$</th>
<th>$\phi_2$</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
</tr>
</thead>
</table>

\[ \tan \phi_1 \]

1. always west of 2.

\[ \tan \phi_2 \]

\[ \tan \theta_1 \]

\[ \tan \theta_2 \]

\[ \delta \theta_m = \frac{1}{2} (\theta_1 + \theta_2) \]

\[ \sin \delta \theta_m \]

\[ \cos \delta \theta_m \]

\[ H = \cos^2 \delta \theta_m - \sin^2 \delta \theta_m - \cos^2 \theta_m - \sin^2 \Delta \theta_m \]

\[ L = \sin^2 \delta \theta_m + H \sin^2 \Delta \theta_m \]

\[ \cos d = 1 - 2L \]

\[ U = 2 \sin^2 \delta \theta_m \cos^2 \delta \theta_m / (1 - L) \]

\[ V = 2 \sin^2 \delta \theta_m \cos^2 \delta \theta_m / L \]

\[ X = U + V \]

\[ T = \frac{d}{\sin d} \]

\[ Y = U - V \]

\[ D = 4T^2 \]

\[ A = DE \]

\[ C = T - \frac{1}{2} (A - E) \]

CHECK $C - \frac{1}{2} E + AD/B = T$

\[ n_1 = X (A + CX) \]

\[ n_2 = Y (B + EY) \]

\[ n_3 = DXY \]

\[ \delta d = \frac{f}{4} (T - Y) \]

\[ S_1 = 2 \sin d (T - \delta d) \]

\[ F = 2Y - E (4 - X) \]

\[ G = \frac{4}{4} (T + \frac{f^2}{64}) M \]

\[ M = 32T - (20T - A) X - (B + 4) Y \]

\[ \Delta \lambda_m' = \frac{1}{2} (\Delta \lambda + Q) \]

\[ \tan \Delta \lambda_m' \]

\[ v = \arctan l \]

\[ u = \arctan l \]

\[ \alpha_1 = v - u \]

\[ \alpha_2 = v + u \]

\[ \alpha_{3, 4} \]

\[ \alpha_{2, 3} \]

\[ 360 - \alpha_1 \]

\[ 360 - \alpha_2 \]

\[ 180 + \alpha_1 \]

\[ 180 + \alpha_2 \]

Figure 35. Inverse position computation form.
Now when $\theta_0 = 0$ in (149) we get $B = C = 0$, $A = 2$, $S_0 = \sin(1 - \ell)$, $v = Df = 1 + f/4 - 2f(1/4)^2$, $uv^3 = f/4 + 2f(1/4)^2$, $\cos \theta_0 = v - uv^3 = 1$, $a_{1-3} = 90^\circ$, $a_{2-4} = 270^\circ$, which is case 1 with the equality sign and $\Delta \lambda_0 = \pi(1 - \ell)$. When $\theta_0 = \pi/2$, $\Delta \lambda_{1/2} = \pi$, $v = 0$, $uv^3 = 0$, $\cos \theta_0 = 0$, $A = B = C = 1$, $S_{1/2} = \sin(1 - f/2 + f^2/16 + f^3/32)$, the meridian semilength, see equations (34).

Direct and inverse computation of the ACIC 6000 mile lines

To begin the evaluation of equations (140) and (143), arranged in the forms of Figures 18 and 25, the nine ACIC 6000 mile lines were computed. The results are compared in Table 6 and the actual computations displayed in Appendix 3. Note, for the meridional limiting case of the direct solution, that when $\theta_0 = 90^\circ$, then $\sin a_1 = \cos (90^\circ - \theta_1)$ or $a_1 = 90^\circ - \theta_1$, $a_2 = 90^\circ - \theta_2$, $N = \cos \theta_1$, $\Delta \alpha = a_1 - a_2 = \theta_2 - \theta_1$. Using the identity $\cos \phi_1 \sin a_1 = -\cos \theta_2 \sin a_{2-1}$, we have

$$\tan \phi_1 = \frac{(\sin \theta_1 \cos \Delta \alpha + \cos \theta_1 \sin \Delta \alpha) \sin a_{1-1}}{(1 - f) \cos \theta_1 \sin a_{1-1}} = \frac{\sin (\theta_1 + \Delta \alpha)}{(1 - f) \cos \theta_1} = \frac{\sin \theta_2}{(1 - f) \cos \theta_2} = \tan \theta_2$$

and $a_{1-1} = 0$, $a_{2-1} = 360^\circ$, $\theta_2 = 90^\circ + \alpha_1 - \Delta \alpha$. Hence in the limit $\tan \phi_1 = \tan (90^\circ + \alpha_1 - \Delta \alpha)/(1 - f)$.

Table 6 shows that good results were obtained using only 8-place tables, (Petert). The maximum difference in length for the control value of 9653977.366 meters is +189 meter, the minimum difference is +0.044 meter, and the mean difference for the nine line positions is -0.044 meter. All the angular values are flat checks or at most .003 second from the control value. These results are better, at 6000 miles, than the adopted criteria (1 meter., .035 sec.) by a factor of 10 for both distance and angular quantities.

Complete check of direct and reverse solutions over a hemisphere geodesic

In order to test for all the cases as delineated in Table 2, we construct a geodesic model as given in Figure 26 containing the given initial and terminal points of the ACIC 6000 mile check line having the largest vertex parametric latitude (excluding the meridian), i.e.

Init. $\phi_1 = 70^\circ$, $\theta_1 = 69^\circ 56' 14.590'$, $\lambda_1 = -18^\circ$

Ter. $\phi_2 = 17^\circ 06' 38.317'$, $\theta_2 = 17^\circ 05' 217.296'$, $\lambda_2 = 114^\circ 18' 43.900$

$e_0 = 76^\circ 00' 26.541'$, $\theta_0 = 75^\circ 55' 42.053'$, $S = 9653977.366$ meters (150)

$a_{1-2} = 45^\circ$, $a_{2-3} = 345^\circ 17' 56.277$

From our geodesic model, Figure 26, we choose the arcs:

$$V_1P_1 \quad \Delta \lambda_1 \quad S_1 \quad \text{A vertex and point}$$
$$V_1V_2 \quad S_2 \quad \text{Contains two vertices (and points)}$$
$$P_1N_1 \quad \Delta \lambda_3 \quad S_3 \quad \text{A node and point}$$
$$N_1P_2 \quad S_4 \quad \text{A node and point}$$
$$P_2P_2 \quad \Delta \lambda_3 + \Delta \lambda_3 \quad S_2 + S_3 \quad \text{Contains a node}$$
$$P_1I \quad \Delta \lambda_4 \quad S_4 \quad \text{Contains neither node nor vertex}$$
$$P_1P_3 \quad S_5 \quad \text{Contains a node and a vertex}$$
$$V_2T \quad \Delta \lambda_4 + \Delta \lambda_4 \quad S_1 + S_4 \quad \text{A vertex and point}$$
$$I T \quad \lambda_1 - \lambda_1 = 2\Delta \lambda_4 + \Delta \lambda_4 \quad S_2 + S_4 \quad \text{Given ACIC line—contains a vertex}$$
$$TN_1 \quad \Delta \lambda_5 \quad S_5 \quad \text{A node and point}$$
$$N_1N_2 = V_1N_1 + N_1V_2 \quad \Delta \lambda_6 \quad S_6 \quad \text{Contains two nodes (and points)}$$
Table 6. Computed values for 6000 mile lines.

<table>
<thead>
<tr>
<th>ORIGIN</th>
<th>TERMINUS</th>
<th>8(meters)</th>
<th>ΔS(m)</th>
<th>0° 1-2</th>
<th>90° 2-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>W W</td>
<td>8° 11'</td>
<td>12° 30'</td>
<td>0° 2'</td>
<td>8° 11'</td>
<td>12° 30'</td>
</tr>
<tr>
<td>1° 18'</td>
<td>D</td>
<td>-89° 270'</td>
<td>12° 30'</td>
<td>0° 2'</td>
<td>8° 11'</td>
</tr>
<tr>
<td>(1)</td>
<td>I</td>
<td>-89° 270'</td>
<td>0° 2'</td>
<td>0° 2'</td>
<td>8° 11'</td>
</tr>
<tr>
<td>70 18</td>
<td>23 18</td>
<td>40,100</td>
<td>162</td>
<td>500</td>
<td>500</td>
</tr>
<tr>
<td>D</td>
<td>-40,100</td>
<td>162</td>
<td>500</td>
<td>500</td>
<td>500</td>
</tr>
<tr>
<td>(5)</td>
<td>I</td>
<td>-40,100</td>
<td>500</td>
<td>500</td>
<td>500</td>
</tr>
<tr>
<td>10 18</td>
<td>45 54</td>
<td>05,051</td>
<td>77 25</td>
<td>25,649</td>
<td>45</td>
</tr>
<tr>
<td>D</td>
<td>-05,051</td>
<td>77 25</td>
<td>25,649</td>
<td>45</td>
<td></td>
</tr>
<tr>
<td>(4)</td>
<td>I</td>
<td>-05,051</td>
<td>77 25</td>
<td>25,649</td>
<td>45</td>
</tr>
<tr>
<td>40 18</td>
<td>35 18</td>
<td>47,064</td>
<td>102 02</td>
<td>29,821</td>
<td>45</td>
</tr>
<tr>
<td>D</td>
<td>-47,064</td>
<td>102 02</td>
<td>29,821</td>
<td>45</td>
<td></td>
</tr>
<tr>
<td>(9)</td>
<td>I</td>
<td>-47,064</td>
<td>102 02</td>
<td>29,821</td>
<td>45</td>
</tr>
<tr>
<td>70 18</td>
<td>17 08</td>
<td>38,317</td>
<td>114 18</td>
<td>45,800</td>
<td>45</td>
</tr>
<tr>
<td>D</td>
<td>-38,317</td>
<td>114 18</td>
<td>45,800</td>
<td>45</td>
<td></td>
</tr>
<tr>
<td>(6)</td>
<td>I</td>
<td>-38,317</td>
<td>114 18</td>
<td>45,800</td>
<td>45</td>
</tr>
<tr>
<td>10 18</td>
<td>0 50</td>
<td>55,829</td>
<td>68 47</td>
<td>05,259</td>
<td>90</td>
</tr>
<tr>
<td>D</td>
<td>-55,829</td>
<td>68 47</td>
<td>05,259</td>
<td>90</td>
<td></td>
</tr>
<tr>
<td>(7)</td>
<td>I</td>
<td>-55,829</td>
<td>68 47</td>
<td>05,259</td>
<td>90</td>
</tr>
<tr>
<td>40 18</td>
<td>1 56</td>
<td>36,356</td>
<td>69 27</td>
<td>01,113</td>
<td>90</td>
</tr>
<tr>
<td>D</td>
<td>-36,356</td>
<td>69 27</td>
<td>01,113</td>
<td>90</td>
<td></td>
</tr>
<tr>
<td>(8)</td>
<td>I</td>
<td>-36,356</td>
<td>69 27</td>
<td>01,113</td>
<td>90</td>
</tr>
<tr>
<td>70 18</td>
<td>2 55</td>
<td>17,426</td>
<td>70 50</td>
<td>04,891</td>
<td>90</td>
</tr>
<tr>
<td>D</td>
<td>-17,426</td>
<td>70 50</td>
<td>04,891</td>
<td>90</td>
<td></td>
</tr>
<tr>
<td>(9)</td>
<td>I</td>
<td>-17,426</td>
<td>70 50</td>
<td>04,891</td>
<td>90</td>
</tr>
</tbody>
</table>

First listed values are those given for the AGED 6000 mile lines, Appendix 3. D and I indicate direct and inverse values from the computations, Appendix 3.
For control we compute $\Delta\lambda_0$, $\Delta\lambda_1$, $\Delta\lambda_2$, $\Delta\lambda_3$, $\Delta\lambda_4$ and $S_0$, $S_1$, $S_2$, $S_3$ from equations (47)-(54). This provides incidentally a check for the ACIC line (150). The computations are included in Appendix 3.

The values obtained are:

<table>
<thead>
<tr>
<th></th>
<th>$\Delta\lambda_0$</th>
<th>$\Delta\lambda_1$</th>
<th>$\Delta\lambda_2$</th>
<th>$\Delta\lambda_3$</th>
<th>$\Delta\lambda_4$</th>
<th>$S_0$</th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$S_3$</th>
<th>$S_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>o</td>
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<td>46</td>
<td>4</td>
<td>4</td>
<td>38</td>
<td>20001779.136</td>
<td>1611471.024</td>
<td>8389418.545</td>
<td>1956383.534</td>
<td>6433035.010</td>
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<tr>
<td>r</td>
<td>51</td>
<td>46</td>
<td>6</td>
<td>2</td>
<td>45</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>$\lambda$</td>
<td>0.553</td>
<td>49.167</td>
<td>44.619</td>
<td>9.1%</td>
<td>0.464</td>
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</tr>
</tbody>
</table>

From the longitude values of (152) and the given line (150) we have the coordinates of the points:

<table>
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<tr>
<th>Point</th>
<th>$\theta$</th>
<th>$\lambda$</th>
<th>$\theta$</th>
<th>$\lambda$</th>
<th>$\theta$</th>
<th>$\lambda$</th>
</tr>
</thead>
<tbody>
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<td>-151</td>
<td>57</td>
<td>04</td>
<td>21.296</td>
<td>18.387</td>
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<td>-104</td>
<td>56</td>
<td>17</td>
<td>45</td>
<td>29.220</td>
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<tr>
<td>$N_1$</td>
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<td>-01</td>
<td>56</td>
<td>08</td>
<td>14.590</td>
<td>44.610</td>
</tr>
<tr>
<td>$P_2$</td>
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<td>-56</td>
<td>05</td>
<td>45</td>
<td>0.464</td>
<td></td>
</tr>
<tr>
<td>$I$</td>
<td>+69</td>
<td>-18</td>
<td>56</td>
<td>0</td>
<td>14.590</td>
<td></td>
</tr>
<tr>
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<tr>
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<td>+75</td>
<td>56</td>
<td>33</td>
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<td></td>
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<tr>
<td>$T$</td>
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<td>+114</td>
<td>05</td>
<td>18</td>
<td>21.296</td>
<td>43.798</td>
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<tr>
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<td>+118</td>
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<td>42</td>
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</tr>
</tbody>
</table>

From (150), (152) we may write the values for (151) including azimuths:

<table>
<thead>
<tr>
<th>Line</th>
<th>$\Delta\lambda$</th>
<th>$S$ (meters)</th>
<th>$\alpha_{1,2}$</th>
<th>$\alpha_{3,4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_1P_1$</td>
<td>46</td>
<td>49.167</td>
<td>1611471.024</td>
<td>90</td>
</tr>
<tr>
<td>$V_1V_2$</td>
<td>179</td>
<td>51</td>
<td>20001779.136</td>
<td>90</td>
</tr>
<tr>
<td>$P_1N_1$</td>
<td>43</td>
<td>08</td>
<td>8389418.545</td>
<td>45</td>
</tr>
<tr>
<td>$N_1P_2$</td>
<td>4</td>
<td>23</td>
<td>1956383.534</td>
<td>14, 02, 17.947</td>
</tr>
<tr>
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<td>10345802.079</td>
<td>45</td>
</tr>
<tr>
<td>$P_3T$</td>
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<td>45</td>
<td>5433035.010</td>
<td>14, 42, 03.723</td>
</tr>
<tr>
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<td>179</td>
<td>51</td>
<td>20001779.136</td>
<td>45</td>
</tr>
<tr>
<td>$V_3T$</td>
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<td>31</td>
<td>8044506.034</td>
<td>90</td>
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<td>$I$</td>
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<td>18</td>
<td>9659977.058</td>
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<td>$T$</td>
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<td>179</td>
<td>51</td>
<td>20001779.136</td>
<td>14, 02, 17.947</td>
</tr>
</tbody>
</table>

By comparing the properties of the lines, as delineated in (151), with Table 2, it is seen that the computation of hemispheroidal geodesics and arcs of (154) is sufficient. Note that the lines $V_1V_2$, $P_1P_3$, $N_1N_2$ are maximum hemispheroidal geodesics under the unique shortest distance property, i.e. node to
node; vertex to vertex; points in equal but opposite signed latitudes separated by maximum longitude (that between successive nodes or successive vertices).

We first dispose of the computation of the equal maximum hemispheroidal geodesics $V_1V_2, N_1N_2, P_1P_2$.

$V_1V_2$. Since the direct and inverse are identical, the end points of the arc are the vertices $\pm \theta_0$; one may compute $A$ and $D$ from (49) and then $\Delta \lambda_0$ and $S_0$ from (54). This has already been done and the computations are given in Appendix 3. Azimuths are always $\alpha_{1-3} = 90^\circ$, $\alpha_{2-3} = 270^\circ$ (second vertex always east of the first).

Note in equations (143) that the term $T = \frac{d}{\sin d}$ grows very large when $d \to \pi$. Now from equations (142) with $\theta_2 = -\theta_1 = -\theta_0$, $\Delta \lambda_0 = -2\lambda_0$ we have:

$$Y = X = 2 \sin^2 \theta_0 \cos \lambda_0 \cos \lambda_0/(1 + \cos d) = 2 \sin^2 \theta_0/\sin^2(d/2).$$

$$\tan \theta_0 = \sin \theta_0 / \cos \theta_0 \cos \lambda_0, \cos(d/2) = \cos \theta_0 \sin \lambda_0, \cos d = 2 \cos^2(d/2) - 1,$$

$$S = s [d = (4(1 + \sin d) + (4(1 + \cos d) \{X(\pi - \sin d \cos d) + 4d^2(2 - X \cos d) \} \cot(d/2))]$$

(155)

where $\theta_0$ is the vertex parametric latitude of the great elliptic section, see Figure 27.

The great elliptic section containing the geodesic vertices $P_1, P_2$ has the antipodal vertices $P_1', P_2'$ and passes through the point $Q$ as shown. Its plane has the equation $\tan \theta = \tan \theta_0 \sec \lambda_0 \cos \lambda_0 \Delta \lambda_0$, as given by equations (33), is related to $\lambda_0$ by $\Delta \lambda_0 = \pi - 2\lambda_0$, and $\alpha_0 = \pi/2 - \theta_0$. The arc lengths $P_1Q, Q_2R, Q'P'$ are all equal maximum hemispheroidal geodesics under the shortest distance property.

Figure 27. The great elliptic section containing two consecutive vertices of the geodesic.
From the control computations, Appendix 3, $\Delta \lambda_0 = 179^\circ 51' 07.554$, and hence \( \lambda_0 = (1/2)(\pi - \Delta \lambda_0) \)
\[ = 4' 26.7223; \]
\[
\sin \lambda_0 = .00129069, \sin \theta_0 = .97013371, \cos \theta_0 = .24257076, \cos (d/2) = \cos \theta_0 \sin \lambda_0
\]
\[ = .00031308, \]
\[
\sin (d/2) = .99999995, \sin d = 2 \sin (d/2) \cos (d/2) = .00062616, \cos d = 2 \cos (d/2) - 1
\]
\[ = -.99999980, \]
\[
\cot (d/2) = \cos (d/2)/\sin (d/2) = .00031308, d = 179^\circ 57' 50'.846
\]
\[ = 3.140966498 \text{ radians}, \]
\[
d = \sin d \cos d = \sin d + \sin d = 3.141592658 = \pi, a = 6378206.4 \text{ meters}
\]
\[
S = a(3.140966498 - .005011785 + .000001999) = (6378206.4)(3.135956712)
\]
\[ = 20001779.171 \text{ m}, \]
which is within .035 meter of the control, Appendix 3.

IIpN. For the direct solution, $\theta_0 = 90^\circ - a_{1,-2}$. $A, D$ are computed from (49) and $\Delta \lambda_0, S_0$ from (54). For the inverse we are given $\Delta \lambda_0$, whence we have two possible cases as described in (147). For our case the second solution is appropriate and we solve for $\theta_0$ and then $S_0$ from equations (149). The calculations are given in Appendix 3. Note that there are two solutions symmetric with respect to the equator for this reverse problem.

$P_1 P_3$. For the direct solution we are given $\theta_1, a_{1,-2}$ and we have $\theta_0$ from equation (10), $\cos \theta_0 = \cos \theta_1 \sin a_{1,-2}$. $\Delta \lambda_0, S_0$ are then given by (54) after computing $A$ and $D$ from (49). $\theta_2 = -\theta_1$, $a_{2,-1} = 360^\circ - a_{1,-2}$. For the reverse solution we are given $\theta_1, \theta_2, \lambda_1, \lambda_2$ where $\theta_2 = -\theta_1, \Delta \lambda = \lambda_2 - \lambda_1 = \Delta \lambda_0$. From $\Delta \lambda_0$ we may solve for $\cos \theta_0$ and then $S_0$ from equations (149). Then $\sin a_{1,-2} = \cos \theta_0/\cos \theta_1, a_{2,-1} = 360^\circ - a_{1,-2}$. Since there are two solutions (see Figure 13) the alternative azimuths are $a_{1,-2} = 180^\circ - a_{1,-2}, a_{2,-1} = 180^\circ + a_{1,-2}$.

Comparison of direct and inverse computations of the geodesic line segments of (154) are given in Table 7, and the computations are included in Appendix 3. Over lengths of 1.5, 2, 6, 8, 9.5, 10, 20 megameters, maximum length error was .26 m, and maximum angular error was .018 second. All values were a factor of 2 to 10 better than the assumed criteria.

A geometric limitation in the inverse solution

Since $T = d/\sin d$ grows large when $d \rightarrow \pi$, some increase in accuracy is made for long almost antipodal geodesic arcs by returning $\sin d$ to the formulae, that is using them in the form of equations (142). However, when two spheroidal points are in nearly the same small latitude, and separated by maximum hemispheroidal geodetic longitude difference, as shown in Figure 28, a limitation is imposed which is purely geometric. An examination of Figure 28 shows that the separation of geodesic and great elliptic vertices may be large where $P_1, P_2$ are in the same latitude and near the equator (in the antipodal zones), because the great elliptic section through $P_1, P_2$ always contains the diameter $AA'$, while the geodesic does only in the limiting case of the meridian. In fact for the complete hemispheroidal geodesic, node to node, the great elliptic section coincides with the equator, and $S = a\Delta \lambda$, for all such hemispheroidal geodetics as given by equations (142) but which is true for only the limiting case of the geodesic equatorial limiting arc when $\Delta \lambda = \pi(1 - \Delta)$, see equations (34).
<table>
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<th>Arc</th>
<th>(d_1)</th>
<th>(\lambda_1)</th>
<th>(d_2)</th>
<th>(\lambda_2)</th>
<th>S(meters)</th>
<th>(\Delta S)</th>
<th>(\Delta \lambda_{1-2})</th>
<th>(\Delta \lambda_{2-1})</th>
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<td>-131 04 18,387</td>
<td>-70 00 00,000</td>
<td>-104 17 29,220</td>
<td>1611471,024</td>
<td>90 00 00,000</td>
<td>225 00 00,012</td>
<td>34 00 00,012</td>
</tr>
<tr>
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<td></td>
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<td></td>
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<tr>
<td>Inverse</td>
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<td>270</td>
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<td>34 00 00,012</td>
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<td>Inverse</td>
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<tr>
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<tr>
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</tbody>
</table>
The plane of the great elliptic now has for equation \( \tan \theta = \tan \theta_1 \sin \lambda / \sin \lambda_1 \). Hence the vertex of the great elliptic is given by \( \tan \theta_0 = \tan \theta_1 / \sin \lambda_1 \).

Figure 26. The geodesic and great elliptic section through two points in the same latitude.

This geometric limitation applies also, unfortunately, to the inverse solution as given in reference [4]. This geometric singularity is also inherent in any solution based on the normal section for when two points on the geodesic are near the equator (same latitude) separated by maximum hemispheroidal geodesic longitude difference, the plane common to the normals at the geodesic arc end points, containing the common plane section vertex, lies near the equator, while the geodesic vertex is near the pole.

To obtain some estimates of this limitation, hemispheroidal geodesics, vertex to vertex, were computed from equations (33) and (155) simultaneously for several geodesic vertex parametric latitudes as shown in the summary, Table 8. From Figure 28 we have the vertex parametric latitude of the great
<table>
<thead>
<tr>
<th>( \theta_0 )</th>
<th>( \Delta \lambda_0 )</th>
<th>( \theta )</th>
<th>( \Delta \lambda )</th>
<th>( \Sigma ) (meters)</th>
<th>( S = S_0 )</th>
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<td>20003776,086</td>
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<td>+3,665</td>
<td>19972075,865</td>
</tr>
<tr>
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<td>2275,260</td>
<td>+0,120</td>
<td>19972075,863</td>
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<tr>
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<td>-67671,401</td>
<td>+1,435</td>
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<tr>
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<td>-254,907</td>
<td>+0,063</td>
<td>19970035,241</td>
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</tr>
<tr>
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<td>20037726,369</td>
<td>-6929,401</td>
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</tr>
<tr>
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<td>-6929,401</td>
<td>19969796,968</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Top line: \( S_0 \) from equations (33)
Lower line: \( S \) from equations (155)

\[ \text{Clarke 1866 Ellipsoid} \]
elliptic section by tan \( \theta' = \tan \theta \sin \lambda \) and with \( \theta_2 = \theta \), we have from (47), (128), and (142) the following formulas:

\[
\theta' = \arcsin \left( \frac{\sin \phi}{\cos \lambda} \right) = \arctan \left( \frac{\tan \phi}{\cos \lambda} \right), \quad X = Y = 2 \sin^2 \theta' ,
\]

\[
\sin d = 1 - 2 \sin^2 \frac{d}{2} , \quad \lambda_1 = \frac{\pi}{2} - \Delta \lambda_0 / 2 , \quad \Delta \lambda_1 = \pi - 2 \lambda_1 , \quad \Delta \lambda_n = \theta_n - \theta_0 ,
\]

\[
S_1 = \pi [d - (\pi/4)(d - \sin d) + (\pi/64)X(d - \sin d \cos d) - 4d^3 (1 - \cos d)(2 - X)/\sin d] .
\]

\[
\delta \lambda = \gamma - A \lambda - B \sin 2 \delta , \quad \gamma = \arcsin \left( \tan \frac{\theta_n}{\tan \theta_0} \right) , \quad S = \arcsin \left( \sin \theta / \sin \theta_0 \right) ,
\]

\[
S_2 = D \delta - E \sin 2 \delta - F \sin 4 \delta , \quad M = \cos \theta_0 , \quad c_1 = \sin \phi - c_2 = (1/4)(1 - M^2) , \quad A = c_1 - 2B ,
\]

\[
\delta \lambda = \frac{c_1 c_2 / 2}{1 - c_2^2} - \lambda M , \quad E = c_2 + BM , \quad F = c_1 / 4 , \quad a_{1-3} = \arcsin \left( \cos \theta_0 / \cos \theta_1 \right) .
\]

From Table 8 we have \( S_0 \) and \( \Delta \lambda_0 \) for the hemispherical geodesics with vertex parametric latitudes \( \theta_0 = 5, 15, 30, 45, 60, 75, 85 \) degrees. The values of \( \Delta \theta = \theta_0 - \theta \) given there are for the hemispherical geodesic, north to node, as the same and longitude difference is the same, distances and longitude differences were computed between \( P_1(\theta, \lambda') \) and \( P_2(\theta, \lambda_1) \), Figure 28, as follows:

With the values of \( \theta_1 = 30', 1', 5', 10' \) for each value of \( \theta_0 \), the values of \( \delta \lambda_1, S_1 \) were computed from their formulas as given in (156). Thus \( \lambda_1 = \delta \lambda_1 + \pi/2 - \Delta \lambda_0 / 2 , \quad \Delta \lambda_1 = \pi - 2 \lambda_1 , \quad S = S_0 - 2S_1 \) were correspondingly determined which define the control for each geodesic line \( P_1P_2 \). Then \( \theta_0' \) and \( S_1 \) were computed from (156) and the corresponding values of \( S = S_1 - S, \Delta \lambda_n = \theta_n - \theta_0' \), \( S' = S_1 / S \) obtained.

Only geodesic arcs with end points in the same latitude and separated by maximum geodesic longitude were thus obtained.

Table 9 gives the results of the computations. Figure 29 shows the graphs of \( \theta_0 \) versus \( \Delta \theta_n = \theta_n - \theta_0' \) for \( \theta_1 = 30', 1', 5', 10' \) and corresponding distance errors over maximum geodesic lengths, 10.6 to 19.9 megameters. Some conclusions may be drawn from these results. Under the distance criterion of one meter, when two points are in about the same latitude, \( \theta_1 > 10' \), separated by maximum hemispherical longitude difference for that common latitude and particular geodesic, the inverse solution holds for geodesic vertex latitude range \( 10' < \theta_0 < 90' \). Under the ACIC criterion 1/100000 for distance, the inverse is satisfactory for two points in the same latitude \( \theta_1 > 1' \), for values of geodesic vertex latitude \( 1' < \theta_0 < 90' \), and with longitude separation maximum for a given geodesic. The formulas will also hold under the ACIC distance criterion for \( \theta_1 = 30' \) at maximum longitude separation for \( 0 < \theta_0 < 30' \), \( 87' < \theta_0 < 90' \). All these values are approximations as deduced from Table 9. Obviously if the longitude separation between two points in the same latitude is less than the maximum possible for hemispherical geodesics, the formulas will give better results since the separation between geodesic and great elliptic vertices will be less, see Figure 28.
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<th>90°</th>
<th>50°</th>
<th>9.0</th>
<th>43</th>
<th>60</th>
<th>75</th>
<th>85</th>
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<tr>
<td>$S_1$ (a)</td>
<td>18499003.98</td>
<td>19943579.19</td>
<td>19756609.79</td>
<td>19650123.51</td>
<td>19466279.23</td>
<td>19289806.13</td>
</tr>
<tr>
<td>$\Delta \lambda$ (a)</td>
<td>187.42'</td>
<td>175.41</td>
<td>177.94</td>
<td>178.34</td>
<td>178.32</td>
<td>179.63</td>
</tr>
<tr>
<td>$\Delta A$ (a)</td>
<td>+1.15</td>
<td>+31.44</td>
<td>+205.06</td>
<td>+227.99</td>
<td>-587.51</td>
<td>-939.32</td>
</tr>
<tr>
<td>$\Delta \theta$ (a)</td>
<td>+14' 05&quot; 576</td>
<td>1.95 97 936</td>
<td>6.04 45 237</td>
<td>9.39 46 819</td>
<td>11.21 32 366</td>
<td>8.12 07 999</td>
</tr>
<tr>
<td>1/8&quot;</td>
<td>1/635622</td>
<td>1/635622</td>
<td>1/635622</td>
<td>1/635622</td>
<td>1/635622</td>
<td>1/635622</td>
</tr>
<tr>
<td>$S_2$</td>
<td>174206991.12</td>
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<td>19534621.60</td>
<td>19673027.09</td>
<td>19738922.14</td>
<td>19777610.42</td>
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<tr>
<td>$\Delta \lambda$</td>
<td>186.27 11 722</td>
<td>171.57 56 354</td>
<td>176.01 04 300</td>
<td>177.34 30 622</td>
<td>178.32 58 988</td>
<td>179.18 28 784</td>
</tr>
<tr>
<td>$\Delta \theta$</td>
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<td>1.00 51 723</td>
<td>3.12 18 992</td>
<td>5.28 45 147</td>
<td>6.02 50 161</td>
<td>4.92 11 371</td>
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<tr>
<td>1/8&quot;</td>
<td>1/1532085</td>
<td>1/1532085</td>
<td>1/1532085</td>
<td>1/1532085</td>
<td>1/1532085</td>
<td>1/1532085</td>
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<tr>
<td>$S_3$</td>
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<td>15602415.24</td>
<td>1776855.99</td>
<td>18652905.61</td>
<td>18711665.31</td>
<td>18832821.00</td>
</tr>
<tr>
<td>$\Delta \lambda$</td>
<td>152.30 04 082</td>
<td>161.23 35 234</td>
<td>162.09 55 802</td>
<td>169.55 51 644</td>
<td>171.55 26 646</td>
<td>177.09 52 966</td>
</tr>
<tr>
<td>$\Delta A$</td>
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<td>+5.24</td>
<td>+8.85</td>
<td>-5.71</td>
<td>-6.02</td>
</tr>
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<td>9.53 035</td>
<td>20.58 075</td>
<td>1.06 51 943</td>
<td>1.12 22 498</td>
<td>4.73 35 955</td>
</tr>
<tr>
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<td>1/2252645</td>
<td>1/2252645</td>
<td>1/2252645</td>
<td>1/2252645</td>
<td>1/2252645</td>
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<tr>
<td>$S_4$</td>
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<td>10881780.30</td>
<td>9869234.87</td>
<td>10832340.58</td>
<td>17228653.96</td>
<td>17705456.52</td>
</tr>
<tr>
<td>$\Delta \lambda$</td>
<td>106.02 26 922</td>
<td>97.22 59 984</td>
<td>144.01 52 299</td>
<td>159.19 50 240</td>
<td>166.02 54 066</td>
<td>174.26 32 326</td>
</tr>
<tr>
<td>$\Delta A$</td>
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<td>-42</td>
<td>+6.44</td>
<td>-38</td>
<td>+1.25</td>
<td>+7.0</td>
</tr>
<tr>
<td>$\Delta \theta$</td>
<td>1.14 19 152</td>
<td>1.40 719</td>
<td>16.25 198</td>
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<td>53.31 980</td>
<td>22.05 464</td>
</tr>
<tr>
<td>1/8&quot;</td>
<td>1/2956361</td>
<td>1/2956361</td>
<td>1/2956361</td>
<td>1/2956361</td>
<td>1/2956361</td>
<td>1/2956361</td>
</tr>
</tbody>
</table>
\[ \Delta \theta (\text{deg}) = \theta - \theta_0 \]

\( \theta_0 \) - parametric latitude of geodesic vertex

\( \theta \) - parametric latitude of great elliptic vertex

Numbers under the column \( \Delta \theta \) are geodetic distances in megameters and corresponding errors in meters.

Figure 29. Graph of \( \theta \) versus \( \Delta \theta \) for \( \theta_0 = 30^\circ, 1^\circ, 3^\circ, 10^\circ \) and corresponding distance errors (maximum geodetic longitude separation for a given \( \theta_0 \)).
DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1, \alpha_{1,1}, S$ to find $\phi_2, \lambda_2, \alpha_{2,1}$. East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

Spheroid

$$a = \frac{1 - f}{1 - f^2}$$

1 radian = 206264.8062 seconds

Line to

$$\phi_1 \quad \tan \phi_1 \quad \tan \theta_1 = (1 - f) \tan \phi_1$$

$$\alpha_{1,1} \quad \sin \theta_1 \quad \cos \theta_1$$

$$\sin \alpha_{1,1} \quad M = \cos \theta_0 = \cos \theta_1 \sin \alpha_{1,1}$$

$$\cos \alpha_{1,1} \quad N = \cos \theta_1 \cos \alpha_{1,1}$$

$$c_1 = M \quad D = (1 - c_1)(1 - c_1 - c_1)$$

$$c_2 = \frac{1}{2}(1 - M^2)$$

$$P = c_2 (1 + Hc_1) / D$$

$$\cos \alpha_1 = \sin \theta_1 / \sin \theta_0$$

$$d = \frac{S}{aD} \quad \text{(rad)}$$

$$\sin d \quad u = 2(\alpha_1 - d) \quad \sin u$$

$$\cos d \quad W = 1 - 2P \cos u \quad \cos u$$

$$V = \cos u \cos d - \sin u \sin d \quad Y = \frac{P}{V} \sin d$$

$$X = c_1 \sin d \cos d (2V_1 - 1) \quad \Delta \alpha = d + X - Y \quad \text{(rad)}$$

$$\sin \Delta \alpha \quad \cos \Delta \alpha$$

$$\cos \Sigma_0 \quad \Sigma_0 = \Delta \alpha - \Delta \phi$$

$$\tan \alpha_{2,1} = M(N \cos \Delta \alpha - \sin \theta_1 \sin \Delta \alpha)$$

$$\tan \phi_2 = \frac{(\sin \theta_1 \cos \Delta \alpha + N \sin \Delta \alpha) \sin \alpha_{2,1}}{(1 - f) \Delta \alpha}$$

$$\tan \Delta \eta = \frac{\sin \Delta \alpha \sin \alpha_{2,1}}{\cos \theta_1 \cos \Delta \alpha - \sin \theta_1 \sin \Delta \alpha \cos \alpha_{2,1}}$$

$$H = c_1(1 - c_1) \Delta \alpha - c_1 \Delta \phi \cos \Sigma_0$$

CHECK

$$M = \cos \theta_0 = \cos \theta_1 \sin \alpha_{1,1} = \cos \theta_1 \sin (180 + \alpha_{1,1})$$

$$\lambda_2 = \lambda_1 + \Delta \lambda$$
INVERSE POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1; \phi_2, \lambda_2$ to find $S, a_1, a_2.$ Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

SPHEROID $a$ $m$ $b$ $m$

$1 - f = b/a$ $\frac{1}{4f}$ $\frac{1}{4f}$

$t^2/64$ $m^2$ 1 radian $= 206264.8062$ seconds

$\phi_1$ $\lambda_1$

$\phi_2$ $\lambda_2$

$\tan \phi_1$ 1. always west of 2. $\Delta \lambda = \lambda_2 - \lambda_1$

$\tan \phi_2$ $\tan \theta = (1 - f) \tan \phi$ $\Delta \lambda_m = \frac{1}{2} \Delta \lambda$

$\phi_m = \frac{1}{2} (\phi_1 + \phi_2)$ $\sin \Delta \lambda_m$

$\theta_m = \frac{1}{2} (\theta_1 - \theta_2)$ $\tan \Delta A$

$\Delta \theta_m = \frac{1}{2} (\theta_1 - \theta_2)$ $\sin \Delta \theta_m$

$H = \cos^2 \Delta \theta_m - \sin^2 \theta_m = \cos^2 \theta_m - \sin^2 \Delta \theta_m$

$L = \sin^2 \Delta \theta_m + H \sin^2 \Delta \lambda_m$ $\cos d = 1 - 2L$

$U = 2 \sin^2 \theta_m \cos^2 \Delta \theta_m / (1 - L)$ $d (\text{rad})$

$V = 2 \sin^2 \Delta \theta_m \cos \theta_m / L$ $\sin d$

$X = U + V$ $T = d / \sin d$ $E = 2 \cos d$

$Y = U - V$ $D = 4T^2$ $B = 2D$

$A = DE$ $C = T - \frac{1}{2} (A - E)$ CHECK $C - \frac{1}{2} E + AD/B = T$

$n_1 = X (A + CX)$ $n_2 = Y (B + EY)$ $n_3 = DXY$

$\delta d = \frac{4f}{64} (TX - Y)$ $\delta_2 d = (t^2/64)(n_1 - n_2 + n_3)$

$S_1 = \sin d (T - \delta, d)$ $S_2 = \sin d (T - \delta, d + \delta_2 d)$

$F = 2Y - E (4 - X)$ $M = 32 T - (20 T - A) X -(B + 4) Y$

$G = \sqrt{f + t^2 / 64}$ $Q = - (FG \tan \Delta \lambda)/4$

$\Delta \lambda_m = \frac{1}{4} (\Delta \lambda + Q)$ $\tan \Delta \lambda_m$

$v = \arctan k_1 l$ $c_1 = \cos \Delta \theta_m / (\sin \theta_m \tan \Delta \lambda_m)$

$u = \arctan k_1 l$ $c_1 = - \tan \Delta \theta_m / (\cos \theta_m \tan \Delta \lambda_m)$

$a_1 = v - u$ $a_2 = v + u$

$\frac{a_1 + a_2}{2}$ $360 - a_2$

$+ a_1$ $360 - a_1$

$- 180 - a_2$ $180 + a_1$

$+ 180 - a_1$ $180 + a_2$
Appendix 2.
SPHEROID PARAMETERS; SPHERICAL APPROXIMATIONS; SPACE COORDINATES
AT A POINT OF THE SPHEROID; OTHER USEFUL FORMULAE
<table>
<thead>
<tr>
<th>Ellipsoid</th>
<th>a meters</th>
<th>1/f</th>
<th>b meters</th>
<th>1-f</th>
<th>append x10^-3</th>
<th>f</th>
<th>f/2</th>
<th>f/4</th>
<th>f^2/60</th>
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<tr>
<td>FISCHER (MERCURY)</td>
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<td>6556784.28361</td>
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<td>CLARKE 1880</td>
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<td>EVEREST</td>
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</tbody>
</table>

**NOTE:** The parameters a and 1/f were held fixed in the above derivations. The AIRY, CLARKE 1880, and EVEREST are revised ellipsoids. The star values are for the MERCURY ellipsoid which was changed in 1968, see reference[41].
APPROXIMATING SPHERES FOR THE OBLATE SPHEROID

Equivalent area or volume

The area and volume of the oblate spheroid are given by

\[ A = 4\pi(a/b^2) \int_0^b [b^2 + (a^2 - b^2)y^2] \, dy \]

\[ = 2\pi \left[ a^2 + (b^2/e)(1/2)\ln \left( \frac{1 + e}{1 - e} \right) \right] = 2\pi [a^2 + (b^2/e) \text{arc tanh}(e)] \]  

(1)

\[ V = 2\pi(b/a) \int_0^a (a^2 - x^2)^{1/2} \, dx = (4/3)\pi a^2 b, \]

where the meridian ellipse (y-axis polar), is

\[ b^2x^2 + a^2y^2 = a^2b^2, \quad b^2 = a^2(1 - e^2) \]

The area and volume of the sphere are

\[ A_s = 4\pi R_s^2, \quad V_s = \frac{4}{3}\pi R_s^3. \]  

(2)

From (1) and (2), the equalities \( A_s = A, \quad V_s = V \) lead to

\[ 2R_s^2 = a^2 + (b^2/e) \text{arc tanh}(e), \]

\[ R_s^2 = a^2b^2, \quad b = a(1 - e^2)^{1/2}. \]  

(3)

(4)

Now

\[ \frac{1}{e} \text{arc tanh}(e) = (1/2e)\ln \left( \frac{1 + e}{1 - e} \right) = 1 + e^2/3 + e^4/5 + e^6/7 + \ldots \]

and this substitution in (3) gives

\[ 2R_s^2 = a^2 + a^2(1 - e^2)(1 + e^2/3 + e^4/5 + e^6/7 + \ldots) \]

which may be written, after expanding and combining like terms as

\[ R_s = a[1 - e^2(1/3 + e^4/15 + e^6/35 + \ldots)]^{1/2}. \]  

(5)

Expanding the radical in (5) to 6th order terms in \( e \) leads to

\[ R_s = a[1 - e^2/6 - 17e^4/360 - 67e^6/3024 - \ldots]. \]  

(6)

From (4) we have

\[ R_s^2 = a^2(1 - e^2)^{1/2}, \quad R_s = a(1 - e^2)^{1/2} \]

and expanding the radical to 6th order terms in \( e \) we find

\[ R_s = a[1 - e^2/6 - 5e^4/72 - 55e^6/1296 - \ldots]. \]  

(7)

From (6) and (7) we have

\[ \Delta R = R_s - R_s = a(e^6/45 + 23e^8/1134 + \ldots). \]  

(8)

With \( e^2 = 2f - f^2, \quad e^4 = 4f^2 - 4f^3, \quad e^6 = 8f^3 \), we may write from (7) and (8)

\[ R_s = a[1 - f/3 - f^2/9 - 5f^3/81 + \ldots]. \]

\[ \Delta R = (4/45)a^2f^2(1 + 52f/63) \]

\[ R_A = R_V + \Delta R \]

85
Mean spherical approximations

\[ r_A = (1/2)(a + b), \quad r_G = (ab)^{1/2}, \quad r_H = 2ab/(a + b) = ab/r_A, \quad (r_G^2 = r_A \cdot r_H) \]  

are respectively the radii equal to the arithmetic, geometric, and harmonic means of the ellipsoid semiaxes and \( r_A > r_G > r_H \). Since \( a \) and \( b \) differ very little, \( r_G - (1/2)(r_A + r_H) \) is a satisfactory formula for reference ellipsoids.

Principal radii of curvature

The radii of curvature of the meridian and the normal section perpendicular to the meridian at a given point of the reference ellipsoid are the principal radii of curvature, i.e.

Meridian:

\[ R = a(1 - e^2)(1 - e^2 \sin^2 \phi)^{1/2} = a(1 - e^2 \cos^2 \theta)^{1/2}/(1 - e^2)^{1/2} \]

Great Normal:

\[ N = a(1 - e^2 \sin^2 \phi)^{1/2} = a(1 - e^2 \cos^2 \theta)^{1/2}/(1 - e^2)^{1/2} \]

\[ = a(1 + e^2 \sin^2 \phi/2 + 3e^4 \sin^2 \theta/8 + 5e^6 \sin^4 \theta/16) \]

\[ = a[1 + f \sin \phi - (1/2)f^2 \sin^2 \phi - (1 - 3 \sin^2 \phi - (1/2)f^2 \sin^4 \phi - (3 - 5 \sin^2 \phi) \ldots ] \]

\[ = a[1 + (1/2)e^2 \sin^2 \theta + (1/8)e^4 [4 + (1 + \cos^2 \theta)^2] + (1/16)e^6 \sin^2 \theta [4 + (1 + \cos^2 \theta)^2] \]

\[ + \ldots ] \]

\[ \sin \phi = \sin \theta(1 - e^2 \cos^2 \theta)^{1/2}, \quad \cos \phi = (1 - e^2)^{1/2} \cos \theta, \quad \cos \phi = (1 - e^2)^{1/2} \cos \theta/(1 - e^2 \cos^2 \theta)^{1/2}, \]

\[ \tan \phi = \tan \theta/(1 - e^2)^{1/2} = \tan \theta/(1 - f), \quad t^2 = 2f - f^2. \]

Mean radius of the spheroid at a given point of the surface

The mean radius of the spheroid at a given point of its surface is the geometric mean of the principal radii of curvature. From (11) we have

\[ R_m = (RN)^{1/2} = a(1 - e^2)(1 - e^2 \sin^2 \phi)^{1/2} = b/(1 - e^2 \sin^2 \phi), \quad e^2 = 2f - f^2, \]

where \( \phi \) is geodetic latitude, or in terms of parametric latitude

\[ R_m = (a(1 - e^2)^{1/2}) (1 - e^2 \cos^2 \theta) = (a^2/b)(1 - e^2 \cos^2 \theta), \]

see references [6], [9], or [16].

Table 11 gives the corresponding radii \( R_A, R_G, r_A, r_G, r_H \) for each of the 10 reference ellipsoids included here. Equations (9) and (10) above were used for the computations.

Meridional and equatorial arc axes and area of antipodal zones

From equations (58), (60)--Appendix I—with the constants for the 10 given ellipsoids, the parametric latitudes of the endpoints of the meridional arc axes of the antipodal zones were computed as shown in Table 12.

From the second of equations (32)—Appendix I—with \( \theta_0 = \pi/2 \), and from Figure 12, we have for the arc length of the antipodal zone axes:

\[ S_M = a[2\theta - (1/2)(2\theta + \sin 2\theta)] \] (meridional),

\[ S_E = a\theta \] (equatorial), \( 2\theta = \pi(1 + .7495f). \)

An approximation to the area of the antipodal zone is that of the hypocycloid of four cusps, i.e.

\[ A = (3/8)m^2, \quad t = (1/4)(S_M + S_E). \]

With the values of \( \theta \) from Table 12, \( S_M, S_E \) were computed from (13) and then \( A \) from (14) for each of the 10 given spheroids. The computations are displayed in Table 13.
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<th>SPHEROID</th>
<th>( R_a ) (AREA)</th>
<th>( R_v ) (VOLUME)</th>
<th>( \frac{R_a}{\sqrt{2(a+b)}} )</th>
<th>( \frac{1}{2}(R_a + R_v) )</th>
<th>( \frac{a + b}{2} )</th>
<th>( a/b )</th>
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A space coordinate system referred to the normal and tangents to the meridian and parallel through a given point of the reference ellipsoid.

In Figure 30, note that a change of coordinates from the center, O, of the ellipsoid with axes \( x_1, y_1, z_1 \) to the point \( Q \) on the surface with axes \( X, Y, Z \) involves a translation from \( O \) to \( Q \) in the \( x_1, z_1 \)-plane, and then a rotation about \( Q \) in that plane through the angle \( \phi_0 \). If we are interested in the slant range, \( D \), from a point \( S_0 \) at a height \( h \) above or below the ellipsoidal surface, to a point \( Q_0 \) at a height \( h_0 \) above or below \( Q \) then the following derivation will give \( D \).

From Figure 30, the parametric representation of the point \( P(x_1, y_1, z_1) \) on the ellipsoidal surface relative to the rectangular system with origin, \( O \), the ellipsoid center, is

\[
x_1 = N \cos \phi \cos \Delta \lambda, \quad y_1 = N \cos \phi \sin \Delta \lambda, \quad z_1 = N(1 - e^2) \sin \phi
\]

where \( \phi \) is geodetic latitude, \( \Delta \lambda \) is the longitude computed from the meridian through \( Q \), \( N = a/(1 - e^2 \sin^2 \phi)^{1/2} \) is the great normal, see equation (11) above.

![Figure 30. Space coordinate system referred to the normal and tangents to the meridian and parallel at an arbitrary point of the ellipsoid.](image)
The coordinates of any point $S_0$ at a height $h$ above or below $P(x_1,y_1,z_1)$ on the normal to the surface at $P$ are

$$\begin{align*}
x_2 &= (N \pm h) \cos \phi \cos \Delta \lambda, \quad y_2 = (N \pm h) \cos \phi \sin \Delta \lambda, \quad z_2 = [(1 - e^2)N \pm h] \sin \phi.
\end{align*}$$  \hspace{1cm} (16)

Now the transformation equations which give the coordinates of $S_0$ referred to the normal and tangents to the meridian and parallel through a point $Q$ in latitude $\phi_0$ (translation from 0 to Q and rotation about Q through $\phi_0$ in the $x_1,z_1$-plane) are

$$\begin{align*}
X &= (x_2 - N_0 \cos \phi_0) \sin \phi_0 - [z_2 - N_0(1 - e^2) \sin \phi_0] \cos \phi_0, \\
Y &= y_2, \\
Z &= (x_2 - N_0 \cos \phi_0) \cos \phi_0 + [z_2 - N_0(1 - e^2) \sin \phi_0] \sin \phi_0.
\end{align*}$$  \hspace{1cm} (17)

Placing the values of $x_2,y_2,z_2$ from (16) in (17) we have

$$\begin{align*}
X &= u_1 \cos \phi \cos \Delta \lambda - u_2 \sin \phi - c_1, \\
Y &= (N \pm h) \cos \phi \sin \Delta \lambda, \\
Z &= v_1 \cos \phi \cos \Delta \lambda + v_2 \sin \phi - c_1, \\
u_1 &= (N \pm h) \sin \phi_0, \quad u_2 = [N(1 - e^2) \pm h] \cos \phi_0, \\
v_1 &= (N \pm h) \cos \phi_0, \quad v_2 = [N(1 - e^2) \pm h] \sin \phi_0, \\
c_1 &= N_0(1 - e^2 \sin^2 \phi_0).
\end{align*}$$  \hspace{1cm} (18)

With the coordinates from (18) we have then, as seen from Figure 30,

$$D_1^2 = X^2 + Y^2, \quad E_1 = Z \mp h_0, \quad D = (D_1^2 + E_1^2)^{1/2} = [X^2 + Y^2 + (Z \mp h_0)^2]^{1/2}. \hspace{1cm} (19)$$

In the computation of the coordinates (18), the values of $N,N_0$ may be taken from tables, if available, or computed from the series given above in equations (11).

Now the coordinates (16), with $h = 0$, represent a point on the ellipsoid. Hence if we solve (17) for $x_2,y_2,z_2$ we obtain

$$\begin{align*}
x_3 &= Z \cos \phi_0 + X \sin \phi_0 + N_0 \cos \phi_0, \\
y_3 &= Y, \\
z_3 &= Z \sin \phi_0 - X \cos \phi_0 + N_0(1 - e^2) \sin \phi_0
\end{align*}$$  \hspace{1cm} (20)

and $x_3,y_3,z_3$, with $h = 0$, must satisfy the ellipsoidal equation

$$\begin{align*}
(x_3^2 + y_3^2)/a^2 + z_3^2/b^2 &= 1, \quad \text{or since } b^2 = a^2(1 - e^2), \\
(1 - e^2)(x_3^2 + y_3^2) + z_3^2 &= a^2 (1 - e^2). \hspace{1cm} (21)
\end{align*}$$

Now (21) may be written as $x_3^2 + y_3^2 + z_3^2 - e^2(x_3^2 + y_3^2 - z_3^2) = a^2$ which, when $e = 0$, represents the sphere of radius $a$. Analogously, if we place $x_1,y_1,z_1$ from (20) in (21), we obtain the equation of the ellipsoid referred to the point $Q$ as origin (See Figure 30),

$$\begin{align*}
X^2 + Y^2 + (Z + N_0)^2 &= N_0^2 - \frac{e^2}{1 - e^2} (X \cos \phi - Z \sin \phi)^2, \hspace{1cm} (22)
\end{align*}$$

Now when $e = 0$, equation (22) becomes the equation to a sphere tangent to the ellipsoid at $Q$ with radius $N_0$, the great normal length at $Q$. Hence the justification for using the great normal radius at the initial point when the spherical forms of the direct and inverse geodetic line solutions are used. See Figures 8 and 9.
The spherical case

If we place \( e = 0 \) in equations (18), we get
\[
X = (N_0 \pm h)(\cos \phi \sin \theta \cos \Delta \lambda - \sin \theta \cos \phi), \\
Y = (N_0 \pm h)(\cos \phi \sin \theta \cos \Delta \lambda + \sin \theta \cos \phi), \\
Z = (N_0 \pm h)(\cos \phi \cos \theta), \\
D_2 = X^2 + Y^2, \\
E_2 = Z + h_0, \\
D = (D_2 + E_2)^{1/2} = [X^2 + Y^2 + (Z + h_0)^2]^{1/2}. 
\]

see Figure 31. Also note in Figure 31, the quantities, \( u, v, c, a, b, \alpha, \) \( \tau \) is the azimuth of \( P \) from \( Q \). \( \delta \) is the angle of the horizontal at \( Q_0 \). \( \tau \) is the central angle subtended by the arc distance
\[
t = PQ, \tau(\text{rad}) = s/N_0. \]

We may write the following formulas involving the spherical triangle \( P'TQ \) and other quantities as indicated in Figure 31.

\[
\cot \alpha = (\cos \phi \sin \theta \cos \Delta \lambda - \sin \theta \cos \phi), \\
\cot \theta = (\tan \phi \sin \theta \cos \phi), \\
\tan \phi = (\cos \phi \sin \theta \cos \phi \cos \Delta \lambda = N_0/(N_0 + c), \\
\sin \phi = \sin \theta \cos \phi \cos \Delta \lambda = N_0/(N_0 + c), \\
\cot \Delta \lambda = (\cos \phi \sin \theta \cos \phi \cos \Delta \lambda - \sin \theta \cos \phi), \\
\cot \theta = (\tan \phi \sin \theta \cos \phi \cos \Delta \lambda = N_0/(N_0 + c). \\
\]

(24)

Now \( h, h_0 \) in equations (24) can have opposite or like signs, negative signs indicating below the surface of the sphere, see Figure 31. Note that further simplification of this type local reference system is possible for \( d = PQ \leq 8 \) minutes = 8 nautical miles, for then \( \tau = \sin \tau = \tan \tau, \cos \tau = 1 \).

Rectangular spherical coordinates

In Figure 32, we have the space rectangular coordinate system \( X, Y, Z \) with axes the normal and the tangents to the parallel and meridian through a point \( Q \) of the surface. Now the tangent at \( Q \) to the parallel is also tangent to the great circle containing the poles \( C, C' \) of the meridian through \( Q \). A rectangular spherical system on the surface may be used where \( x \)-coordinates are measured along the circular meridian from \( Q \) and \( y \)-coordinates are measured along the great circles through the poles \( C, C' \) of the meridian through \( Q \). The points \( P \) and \( T \) as shown have the spherical coordinates \( x, y \) and \( x', y' \) respectively. The angles \( \beta, \beta' \) at \( P, T \) respectively are measured from the line \( PT = s \) to parallels through \( P, T \) having the same poles \( C, C' \) as the meridian through \( Q \).

Now in the spherical triangle \( P'TQ \) we have
\[
P = 90^\circ - \beta, \quad T = 90^\circ + \beta', \quad C = (x' - x)/N_0, \\
b = 90^\circ - \gamma, \quad a = (a' + b)/N_0, \\
an + b = (y' - y)N_0. 
\]

(25)

To solve for \( x', y', \beta \) we need the following three spherical formulas (Napier's fourth analogy, sine, and cosine laws)
\[
\tan \frac{1}{2}(P + T) = \cos \frac{1}{2}(a - b) \sec \frac{1}{2}(a + b) \cot \frac{1}{2}c, \\
\cos a = \cos b \cos c + \sin b \sin c \cos P, \quad \sin C = \sin c \sin P/\sin a, \\
\]

(26)

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Figure 31: Local space coordinates system at a point of the sphere.
The values from (25) placed in (26) give

\[
\tan \frac{1}{2} (\beta - \beta') = \tan \frac{1}{2N_0} (x' - x) \sec \frac{1}{2N_0} (y' - y) \sin \frac{1}{2N_0} (y' + y)
\]

\[
\sin (y'/N_0) = \cos (u/N_0) \sin (y/N_0) \sin (u/N_0) \cos (y/N_0) \sin \beta
\]

\[
\frac{1}{N_0} (x' - x) \sin (u/N_0) \cos \beta \sec (y'/N_0)
\]

For local coordinate systems the angles \( u/N_0, y'/N_0, (x' - x)/N_0, (y' - y)/2N_0, (x' - x)/2N_0, (y' + y)/2N_0 \) are small and we may use the first two terms of their series expansion, i.e.
\[
\sin x = x - x^3/6, \cos x = 1 - x^2/2, \tan x = x + x^3/3. \quad \text{Hence we have}
\]
\[
\sin (y'/N_0) = y'/N_0 - y'^3/6 N_0^3, \quad \sin (s/N_0) = s/N_0 - s^3/6 N_0^3, \quad \cos (s/N_0) = 1 - s^2/2 N_0^2, \quad \cos (y/N_0) = 1 - y^2/2 N_0^2.
\]

The values from (28) placed in the second of (27) give
\[
y' - y'^3/6 N_0^3 = (1 - s^2/2 N_0^2 y - y'^3/6 N_0^3) + (s - s^3/6 N_0^3) (1 - y^2/2 N_0^2) \sin \beta
\]
\[
y' - y'^3/6 N_0^3 = y + s \sin \beta - \frac{1}{6 N_0^2} (y^2 + 3y^2 s \sin \beta + 3y^2 + 3 \sin \beta)
\]

where the terms in \( s^2 y^2/12 N_0^6 \) have been ignored.

Now in (29), if we ignore the terms in \( 1/6 N_0^6 \) we have the first term of the series which is \( y' = y + s \sin \beta \).

We now place this value of \( y' \) in the term in \( y'^3 \) and write (29) as
\[
y' = y + s \sin \beta + \frac{1}{6 N_0^2} [(y + s \sin \beta)^3 - y^3 - 3y^2 s \sin \beta - 3ys^2 - s^3 \sin \beta]
\]
\[
y' = y + s \sin \beta + \frac{1}{6 N_0^2} (3y^2 \sin \beta - 3ys^2 + s^3 \sin \beta - s^3 \sin \beta)
\]
\[
y' = y + s \sin \beta - s^3 \cos \beta (3y + s \sin \beta)/6 N_0^6.
\]

Similarly from the last of equations (27) we have
\[
(x' - x) = \left(\frac{x' - x}{6 N_0^2} + \left(\frac{y + s \sin \beta}{2 N_0^2} \cdot \frac{s}{6 N_0^2}\right) \cos \beta \right)
\]

and if we ignore the terms in \( 1/N_0^6 \) we get as first approximation \( x' - x = s \cos \beta \). This value returned to the term in \( (x' - x)^3 \) in (31) allows us to write
\[
x' - x = s \cos \beta + \frac{s \cos \beta}{6 N_0^2} (s^2 \cos \beta + 3y^2 - s^2)
\]
\[
x' - x = s \cos \beta + \frac{s \cos \beta}{6 N_0^2} (3y^3 - s^3 \sin \beta)
\]

From the first of (27), since
\[
\sin \frac{1}{2N_0} (y' + y) = \sin \frac{1}{2N_0} (y' - y) + \frac{y}{N_0},
\]
and
\[
\sec \frac{1}{2N_0} (y' - y) \sin \frac{1}{2N_0} (y' + y) = \tan \frac{1}{2N_0} (y' - y) \cos \frac{y}{N_0} + \sin \frac{y}{N_0},
\]
we may write
\[
\tan \beta (\beta - \beta') = \tan \frac{1}{2N_0} (x' - x) \left[ \tan \frac{1}{2N_0} (y' - y) \cos \frac{y}{N_0} + \sin \frac{y}{N_0} \right]
\]

(33)

Using \( \tan x = x + x^3/3 \) and the values of \( \sin y/N_0, \cos y/N_0 \) from (28) we can write the right member of (33) as
\[
\left[ \frac{x' - x}{2N_0} \right] \left[ \left( \frac{y' - y}{2N_0} + \frac{y'^3 - y^3}{24N_0^4} \right) \left( 1 - \frac{y^3}{2N_0^2} + \frac{y}{6N_0^2} \right) \right]
\]

(34)

Retaining terms in \( 1/N_0^6 \), we, from (34), write (33) as
\[
\tan \beta (\beta - \beta') = \frac{1}{4N_0^6} \left[ (x' - x)(y' - y) + 2(x' - x)y \right]
\]
(35)

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With $x' = x \cos \beta$, $y' = y \sin \beta$ from (32) and (30), equation (35) becomes
\begin{equation}
\tan \frac{\beta - \beta'}{2} = s \cos \beta (2y + s \sin \beta) / 4N_0^2
\end{equation}
\label{eq:36}

or
\begin{equation}
\beta' = \beta - 2 \arctan\left[\frac{s \cos \beta (2y + s \sin \beta)}{4N_0^2}\right]
\end{equation}
\label{eq:37}

With $\arctan u = u - u^3/3$, we may write (36) as
\begin{equation}
\beta' = -s \cos \beta (2y + s \sin \beta) / 4N_0^2.
\end{equation}
\label{eq:38}

Finally, equations (30), (32), and (37) may be written as
\begin{align}
x' &= x + u [1 + (3y^2 - v^2)/6N_0^2],
\beta' &= -s \cos \beta (2y + s \sin \beta) / 4N_0^2,
\end{align}
\label{eq:39}

since
\begin{equation}
\frac{u}{2N_0^2} (y + y') = \frac{u}{2N_0^2} \left[\frac{y + y + v - u^2 (3y + v)}{6N_0^2}\right] = \frac{u}{2N_0^2} (2y + v)
\end{equation}
\label{eq:40}

If we place $x = y = 0$, $\beta = \alpha$, $s = d$, $p = Q$ and equations (38) become
\begin{align}
x &= d \cos a (1 + d^2 \sin^2 a / 3N_0^2),
y &= d \sin a (1 - d^2 \cos^2 a / 6N_0^2),
\beta' &= \alpha - d^4 \cos a \sin a / 2N_0^2 \sin 1''.
\end{align}
\label{eq:41}

The terms in $1/N_0^2$ are corrections to plane coordinates. If the ellipsoid is to be taken into account one uses instead of $1/N_0^2$, the value $1/R_{0}N_0$ which is the square of the mean radius in latitude $\phi_0$, see equations (12). Equations (38), in equivalent form, are found in references [15], [32]. Note that $\beta$, $\beta'$ are not azimuths as usually defined, that is the lines $PA'$, $TB'$ in Figure 32 are parallels to the meridian.

Transformations between rectangular space coordinates $X, Y, Z$ and local spherical space coordinates $x, y, h$.

In Figure 32, if $S_0(X, Y, Z)$ is a point at altitude $h$ above or below the point $P(x, y)$, where $x, y$ are the spherical coordinates as shown, we may from (24) and some formulae for right spherical triangles establish some transformations between the $x, y, h$ system and the $X, Y, Z$ system.

We have
\begin{align}
\cos \tau &= \cos \left(\frac{x}{N_0}\right) \cos \left(\frac{y}{N_0}\right) \approx \left(1 - \frac{x^2}{2N_0^2}\right) \left(1 - \frac{y^2}{2N_0^2}\right) \approx 1 - (x^2 + y^2) / 2N_0^2, \\
D_1 &= (h + N_0) \sin \tau, \quad \sin a = \sin \left(\frac{y}{N_0}\right) / \sin \tau, \quad \cos a = \tan \left(\frac{x}{N_0}\right) / \tan \tau, \\
X &= D_1 \cos a = (h + N_0) \cos \left(\frac{x}{N_0}\right) \cos \tau \approx (h + N_0) \left(\frac{x}{N_0} + \frac{x^3}{3N_0^2}\right) \left[1 - (x^2 + y^2) / 2N_0^2\right], \\
X &= x \left[1 + h/N_0 - (x^2 + 3y^2) / 6N_0^2\right] \approx x + 2h \frac{x^3}{N_0^2}, \\
Y &= D_1 \sin a = (h + N_0) \sin \left(\frac{y}{N_0}\right) \approx (h + N_0) \left(\frac{y}{N_0} + \frac{y^3}{6N_0^2}\right), \\
Y &= y \left[1 + h/N_0 - y^2 / 6N_0^2\right] = y - y^3 / 6N_0^2, \\
Z &= h \cos \tau - N_0 (1 - \cos \tau) \approx h - (N_0 + h)(x^2 + y^2) / 2N_0^2, \\
Z &= h - (x^2 + y^2) / 2N_0.
\end{align}
\label{eq:42}
Now

\[
\sin \left( \frac{x}{N_0} \right) = \frac{X}{h + N_0} \sec \left( \frac{y}{N_0} \right)
\]

\[
x = \frac{x^3}{h + N_0} = \frac{X}{h + N_0} \left( 1 + \frac{y^2}{2N_0^2} \right)
\]

\[
x \approx \frac{x^3}{6N_0^2} + \frac{X}{h + N_0} \left( 1 + \frac{X^3}{2N_0^3} \right) \approx X + (X/2N_0)^3
\]

\[
\sin \left( \frac{y}{N_0} \right) = \frac{1}{h + N_0} \frac{Y}{N_0} \]

\[
y = \frac{y^3}{N_0} \approx \frac{1}{N_0} Y, y \approx Y + (Y^3)/6N_0^2
\]

\[
h = Z + (x^2 + y^2)/2N_0 \approx Z + (X^2 + Y^2)/2N_0
\]

This last implies \( x \approx X \approx D_2 \cos \alpha, y \approx Y = D_2 \sin \alpha \)

Figure 33. Tangent-arc-chord.
Plane coordinates and map projections

Figure 33 shows the familiar tangent-arc-chord relationship as inherent in the spherical approximation as given in Figure 31. We have the following formulae relating \( T, v, d, L, H, r, N_0 \) as shown in Figure 33:

\[
\begin{align*}
    d &= \tau N_0 = L + L^2/24N_0^2 = v - v^3/3N_0^3 \\
    L &= 2N_0 \sin \left(\frac{r}{2}\right) = N_0 \left(\tau - \tau^3/24\right) = d - d^3/24N_0^2 = v - 3v^3/8N_0^3 \\
    v &= N_0 \tan \tau = N_0 \left(\tau + \tau^3/3\right) = d + d^3/3N_0^2 = L + 3L^2/8N_0^2 \\
    \tau &= d/N_0 = L/N_0 + L^3/24N_0^2 = v/N_0 - v^3/3N_0^3 \\
    H &= N_0 \tan \frac{\tau}{2} = \frac{1}{2}N_0 \left(\tau + \tau^3/12\right) = \frac{1}{2}(d + d^3/12N_0^2) \\
    T &= N_0 \sin \tau = N_0 \left(\tau - \tau^3/6\right) = d - d^3/6N_0^2 \\
    d - L &= d^3/24N_0^2, \quad 2H - d = d^3/12N_0^2 = 2(d - L), \quad d - T = d^3/6N_0^2 = 4(d - L), \\
    v - 2H &= d^3/4N_0^2 = 6(d - L), v - d = d^3/3N_0^2 = 8(d - L), v - L = (3/8)d^3/N_0^2, \\
    v - T &= d^3/2N_0^2 = 12(d - L). 
\end{align*}
\]

Equations (41) thus give the plane coordinates \( X, Y \) as functions of the geographic coordinates of the points \( P, Q \), that is of \( \phi, \phi_0, \Delta \lambda = \lambda_0 - \lambda \). The last of equations (42) show the solution for the geographic coordinates \( \phi, \lambda \) of \( P \) when the plane coordinates \( X, Y \) of \( S \), referenced to the tangents to the meridian and parallel at \( Q \), are given, assuming \( S \) is gnomonically projected.

Now if we let \( v = D_2 \) = \( d = N_0 r \), the resulting plane coordinates map a Lambert azimuthal equidistant projection on the tangent plane; if \( v = D_2 = L = 2N_0 \sin \tau/2 \) the resulting projection is the Lambert azimuthal authalic (equal area); if \( v = D_2 = T = N_0 \sin \tau \) the projection is orthographic, points \( P \) are projected on the tangent plane at \( Q \) by lines parallel to \( OQ \), see Figure 33; if \( D_2 = 2H = 2N_0 \tan \tau/2 \) the projection is stereographic, angles are preserved about each point of the projection (conformal or autogonal).

The last four columns of Table 14 show the error in the radius \( v = D_2 \) about \( Q \) when we allow \( v = D_2 \) to be \( 2H, d, L, \) or \( T \), i.e. the point \( P \) to be projected upon the tangent plane at \( Q \) stereographically, equidistantly, equal-areally, or orthographically.
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<th>d - T (m)</th>
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Table 14. Differences for d, L, 2H, T, v from equations (41).
Now from equations (24), (26), (41), (42) we have
\[ U = \cos \lambda = \sin \phi \sin \phi_0 + \cos \phi \cos \phi_0 \cos \Delta \lambda \]
\[ V = \sin \phi \cos \phi_0 - \cos \phi \sin \phi_0 \cos \Delta \lambda, \quad W = \cos \phi \sin \Delta \lambda, \]
\[ \sin \alpha = \frac{W}{\sin \tau}, \quad \cos \alpha = \frac{V}{\sin \tau} \tag{43} \]

With the help of (43) we can express the rectangular plane coordinates of the several projections as functions of U, V, W and hence of the geographical coordinates of P and Q, i.e. as functions of \( \phi, \phi_0, \Delta \lambda = \lambda_0 - \lambda \).

**Gnomonic.** \( D_2 = N_0 \tan \tau \)
\[ X = D_2 \cos \alpha = N_0 \tan \tau \cos \alpha = \frac{N_0 \sin \tau}{\cos \tau} \cdot \frac{V}{\sin \tau} = N_0 \frac{V}{U} \]
\[ Y = D_2 \sin \alpha = N_0 \tan \tau \sin \alpha = \frac{N_0 \sin \tau}{\cos \tau} \cdot \frac{W}{\sin \tau} = N_0 \frac{W}{U} \]

**Azimuthal equidistant.** \( D_2 = d = N_0 \tan \tau \)
\[ X = D_2 \cos \alpha = s \cos \alpha = N_0 \tau \cos \alpha = N_0 \frac{V}{\sin \tau} \]
\[ = N_0 \frac{V}{\sin \tau} \cos \alpha \cos U \sin (\alpha \cos U) \]
\[ Y = D_2 \sin \alpha = s \sin \alpha = N_0 \sin \tau = N_0 \frac{W}{\sin \tau} \cos \alpha \cos U \sin (\alpha \cos U) \]

**Azimuthal equal area (authalic).** \( D_2 = D = 2N_0 \sin \tau/2 = N_0 \sin \tau/[\sqrt{1 + \cos \tau}]^{1/2} \)
\[ X = D_2 \cos \alpha = \frac{N_0 \sin \tau}{[\sqrt{1 + \cos \tau}]^{1/2}} \cdot \frac{V}{\sin \tau} = N_0 \frac{V}{[\sqrt{1 + \cos \tau}]^{1/2}} \]
\[ Y = D_2 \sin \alpha = \frac{N_0 \sin \tau}{[\sqrt{1 + \cos \tau}]^{1/2}} \cdot \frac{W}{\sin \tau} = N_0 \frac{W}{[\sqrt{1 + \cos \tau}]^{1/2}} \]

**Orthographic.** \( D_2 = T = N_0 \sin \tau \)
\[ X = N_0 \sin \tau \cos \alpha = N_0 \sin \tau \cdot \frac{V}{\sin \tau} = N_0 \frac{V}{U} \tag{44} \]
\[ Y = N_0 \sin \tau \sin \alpha = N_0 \sin \tau \cdot \frac{W}{\sin \tau} = N_0 \frac{W}{U} \]

**Stereographic.** \( D_2 = 2H = 2N_0 \tan \frac{\tau}{2} = 2N_0 \sin \tau/(1 + \cos \tau) \)
\[ X = \frac{2N_0 \sin \tau \cos \alpha}{1 + \cos \tau} = \frac{2N_0 \sin \tau \cdot \frac{V}{\sin \tau}}{1 + \cos \tau} = 2N_0 \frac{V}{\sin \tau}(1 + \cos \tau) \]
\[ Y = \frac{2N_0 \sin \tau \sin \alpha}{1 + \cos \tau} = \frac{2N_0 \sin \tau \cdot \frac{W}{\sin \tau}}{1 + \cos \tau} = 2N_0 \frac{W}{\sin \tau}(1 + \cos \tau) \]

Spherical coordinates relative to a great circle arc determined by two given points of the sphere.

In Figure 34, Q is the midpoint (but may be any point) of the great circle arc determined by two given points \( P_1(\phi_1, \lambda_1), P_2(\phi_2, \lambda_2) \) of length 2S. The azimuth at Q is \( \alpha \) and Q is taken as origin of the
spherical coordinate system as shown. At an arbitrary point P of $P_1P_2$, at distance $d$ from Q, a perpendicular $PT = s$ is constructed. Note that $\beta = \pi/2 - \beta_1$, hence the spherical rectangular coordinates $x', y'$ of $T$ may be computed using the value of $x$ and $y$ from $(38)$ and the value $\beta = \pi/2 - \beta', \text{where } \beta'$ is given by the expression in $(38)$, i.e.

$$y' = y + v - u(3y + v)/6N^2, \quad x' = x + u[1 + (3y^2 - v^2)]/6N^2, \quad (45)$$

where $u = s \cos \beta, v = s \sin \beta, \beta = \pi/2 - \alpha + d^2 \cos \alpha \sin \alpha/2N^2 \sin 1^{\circ}, x = d \cos \alpha(1 + d^2 \sin^2 \alpha/3N^2), y = d \sin \alpha(1 - d^2 \cos^2 \alpha/6N^2).$
Note that in the mapping of spherical coordinates, the y-coordinates are laid off perpendicular to the central meridian, which causes an increase in the latitude scale as the distance from the central meridian increases. The magnification is given by \( K = 1 + \left( y + y' \right)^2 - y^2 \) and for short lines we may let \( y = y' \), giving \( K = 1 + y^2 \cos^2 \beta/2N^2 \). When \( \beta = \pi/2 \), \( K = 1 \), and the map gives then true longitude differences. When \( K = 0 \), \( K \) is maximum with the value \( K = 1 + y^2/2N^2 \).

Formulas relating spherical coordinates to geographic coordinates

The given reference line \( P_1, P_2 \) of Figure 34, having been already established, we may wish to compute the distance \( s \) to \( P_1, P_2 \) from an arbitrary point \( T(O', X') \), or given \( s \), find geographical coordinates of \( T(O', X') \) and \( P(\phi, \lambda) \) at a given distance \( d \) from \( Q \) along \( P_1, P_2 \). From the right spherical triangles \( T'P_2, Q'T, P'Q' \), \( T'T'P \), we have

\[
\begin{align*}
\cos s' &= \cos x' \cos y' = \cos s \cos d = \cos s \cos x \cos y \\
\tan x &= \tan d \cos a, \sin y' = \sin d \sin a, \tan s &= \tan (\alpha - \alpha') \sin d, \\
\sin \phi' &= \cos y' \sin (x' + \phi_0), \sin y' &= \cos \phi' \sin (\lambda' - \lambda_0), \\
\sin s &= \sin (\alpha - \alpha') \sin s', \\
\cos (\phi_0 + x') &= \tan y' \cos (\lambda' - \lambda_0), \tan x' &= \tan s' \cos \alpha', \\
\tan d &= \cos (\alpha - \alpha') \tan s', \\
\sin y' &= \sin s' \sin \alpha', \cos r &= \cos s \cos (S - d), \cos r' &= \cos s \cos (S + d) \\
\end{align*}
\]

Since \( P_1, Q, P_2 \) are fixed, the constants \( 2S = P_1, P_2, \alpha_1, \alpha_2, a, \phi_1, \phi_2, \phi_0, \lambda_1, \lambda_0, \lambda_2, \) are known. Some of the oblique spherical triangles involving these known parameters and the coordinates of \( T \) and \( P \) are \( P_1, P_2, Q'T, P'Q', T, P'T, P'T', T', P'T \). From these we obtain the following spherical formulae from the sine and cosine laws for spherical triangles:

\[
\begin{align*}
P'TP': & \cos r = \sin \phi' \sin \phi_2 + \cos \phi' \cos \phi_2 \cos (\lambda_2 - \lambda') \\
P'QT: & \cos s' = \sin \phi_0 \sin \phi' + \cos \phi_0 \cos \phi' \cos (\lambda' - \lambda_0) \\
& \sin s' \sin \alpha' = \cos \phi' \sin (\lambda' - \lambda_0) \\
& \sin \phi' = \cos s' \sin \phi_0 + \sin s' \cos \phi_0 \cos \alpha' \\
& \sin \phi = \sin \phi_0, \cos d + \sin d \cos \phi_0 \cos \alpha \\
P'QP: & \cos d = \sin \phi_0 \sin \phi + \cos \phi_0 \cos \phi \cos (\lambda - \lambda_0) \\
& \sin d \sin \alpha = \cos \phi \sin (\lambda - \lambda_0) \\
QTP': & \cos r = \cos s' \cos S + \sin s' \sin S \cos (\alpha - \alpha') \\
P_1, P_2: & \cos 2S = \cos r \cos r' + \sin r \sin r' \cos (\alpha_2 - \alpha_1) \\
P'P: & \cos (S + d) = \sin \phi_1 \sin \phi + \cos \phi_1 \cos \phi \cos (\lambda - \lambda_1) \\
& \sin \alpha_1 = \sin (S + d) \cos \phi \sin (\lambda - \lambda_1) \\
P'P': & \cos (S - d) = \sin \phi_1 \sin \phi + \cos \phi_1 \cos \phi \cos (\lambda - \lambda_1) \\
& \cos \phi \sin (\lambda - \lambda_1) = \sin \alpha_2 = \sin (S - d) \\
P_1, P'T: & \cos r' = \sin \phi_1 \sin \phi' + \cos \phi_1 \cos \phi' \cos (\lambda - \lambda_1) \\
TP': & \cos s = \cos \phi \sin \phi' + \cos \phi \cos \phi' \cos (\lambda - \lambda_1) \\
\end{align*}
\]

From (38), (38)a, (45), (46), (47) we have the formulae to handle most of the geometric problems which may occur in the local geometry of a given base line. For instance if \( s \) is constant but \( d \)
varies, then \(\theta\) varies, and the rectangular coordinates \(x', y'\), as given by (45), give points \(T\) on the parallel at distance \(s\) from the base line \(P_1P_2\).

Suppose that we are given the geographic coordinates \(\phi', \lambda'\) of an arbitrary point \(T\) to find the perpendicular distance \(s\) to the base line, the geographic coordinates \(\phi, \lambda\) of the foot, \(P\), of the perpendicular, and the distance \(d\) from the origin \(Q\) to \(P\). Now the known constants are \(S = \sqrt{\frac{1}{2}P_1P_2}a_P, a_{P+1}, a, \phi', \phi_1, \phi_2, \phi_0, \lambda_1, \lambda_2, \lambda', \lambda\) and we are to find \(s, d, \phi, \lambda\). From (46) we have

\[
\cos (S + d) = \cos S \cos d - \sin S \sin d = \cos \frac{r'}{\cos s}
\]

\[
\cos (S - d) = \cos S \cos d + \sin S \sin d = \cos r / \cos s
\]

Adding and subtracting respective members of (48) get

\[
\cos s \sin d = (\cos r - \cos r')/2 \sin S
\]

\[
\cos s \cos d = (\cos r + \cos r')/2 \cos S
\]

Dividing respective members of (49) we find

\[
\tan d = \cot S (\cos r - \cos r') / (\cos r + \cos r')
\]

From (46) and triangle \(P'QT\) of (47) we have

\[
\cos s = \cos s' / \cos d,
\]

\[
\cos s' = \sin \phi' \sin \phi_1 + \cos \phi' \cos \phi_1 \cos (\lambda - \lambda')
\]

From (50) and triangle \(P'QP\) of (47) we have

\[
\tan d = \cot S \cos \phi \cos d + \sin \phi_1 \cos \phi_0 \cos (\lambda' - \lambda_0).
\]

\[
\sin (\lambda - \lambda_0) = \sin d \sin a_1 / \cos \phi \text{ or } \lambda = \arcsin \left[ \sin d \sin a_1 / \cos \phi \right] + \lambda_0.
\]

Note also a type of spherical rectangular coordinate system referenced to the base line and a great circle orthogonal to the base line at its midpoint as presented in Reference [18].

The doubly equidistant projection

This is a useful projection for investigators in fields such as seismology (earthquakes or microseisms), meteorology (long range location of cyclone trajectories), electronic distance measuring systems as Hiran or Shiran, location of aurorae or of meteors, studies of water waves, tsunamis or swells, oceanography. It is obtained by constructing the spheroidal (spherical) triangle \(P_1P_2T\), of Figure 34, in the plane as shown in Figure 35. The true length (to scale) of the base line \(P_1P_2 = 2S\) is drawn as a straight line in the plane. Points \(T\) are located with respect to the base line from the intersection of circular arcs about \(P_1, P_2\) with radii the true lengths (scaled) of \(r', r\). Either spheroidal or spherical distances for \(S, r', r\) may be used. The projection is not conformal, that is angles are not preserved about every point of the projection.

The equations relating the several parameters as shown in Figure 35 are:

\[
x = p \cos \theta = \frac{1}{2S} (S^2 + p^2 - r^2) = \frac{1}{4S} (r^2 - r^2) = r' \cos a_1 \sin \phi - S = r \cos a_2.
\]

\[
y = p \sin \theta = \pm (p^2 - x^2)^{1/2} = r' \sin a_1 = r \sin a_2 = \pm \frac{1}{4S} [16S^2 r^2 - (r^2 - r^2 + 4S^2)^{1/2}]
\]

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Given distances on the spheroid (sphere) are $S, i', r$.  

Figure 35. The doubly equidistant projection.

\[ p^2 = \frac{1}{4}(r'^2 + r^2) - S^2, \cos \theta = \frac{x}{p} = \frac{1}{2pS}(r'^2 - r^2) = \frac{1}{2pS}(S^3 + p^2 - r^3), \]

\[ \cos a_1 = \frac{(4S^3 + r'^2 - r^2)}{4pS}, \cos a_2 = \frac{(4S^3 - r'^2 + r^2)}{4rS} \]  

(53)

and where for the spherical case, we have from (47)

\[ r = \text{arc cos} \left[ \sin \phi' \sin \phi + \cos \phi' \cos \phi \cos (\lambda_2 - \lambda_1) \right] \]

\[ r' = \text{arc cos} \left[ \sin \phi' \sin \phi + \cos \phi' \cos \phi \cos (\lambda_1 - \lambda_1) \right] \]

\[ 2S = \text{arc cos} \left[ \sin \phi_1 \sin \phi_2 + \cos \phi_1 \cos \phi_2 \cos (\lambda_1 - \lambda_1) \right] \]

Discussion. The doubly equidistant projection may also be called bi-polar, since we have two radii $r', r$ from two foci $P_1, P_2$ and the angles $a_1, a_2$. We have given, or compute, the values $S, i', r$ and then from (53)

\[ \cos a_1 = \frac{(4S^3 + r'^2 - r^2)}{4S}, x = i' \cos a_1 - S, y = i' \sin a_1, \]

\[ \cos a_2 = \frac{(4S^3 - r'^2 + r^2)}{4S}, x = S - r \cos a_2, y = r \sin a_2 \]  

(54)

which provide a check for rectangular coordinate computation.

Given the rectangular coordinates $x, y$ of the point $T$ on the doubly equidistant projection and the constants $S, a, \theta, \lambda_0$ of the base line as shown in Figure 34; find the geographic coordinates $\phi', \lambda'$, of the point $T$ on the sphere.

From $x, y, S$ we have

\[ r' = \sqrt{(x + S)^2 + y^2}^{1/2}, r = \sqrt{(S - x)^2 + y^2}^{1/2}. \]
From (50) we have
\[ \tan d = \cot S (\cos r - \cos r')/(\cos r + \cos r'), \quad (55) \]

From (46) find
\[ \cos s' = \cos r' \cos d/(\cos S + d) = \cos r \cos d/(\cos S - d), \]

From (46) (QTP) we have
\[ \alpha' = \alpha - \arccos(\tan d/\tan s'), \]

Finally from P'QT of (47) find
\[ a'n = x + \arcsin(\sin s' \sin \alpha'/\cos \phi'). \]

From equation (50) we have
\[ \cos r = \cos r' = 1, \quad \sin r' = r', \quad \sin r = r, \quad \text{then (56) becomes} \]
\[ d = \sqrt{(r' - r^2)A^2} = x \text{-coordinate of the doubly equidistant projection}, \quad \text{equations (53).} \]

Consider \( d/N_0, r/N_0, r'/N_0, S/N_0 \) to be small enough to place \( \tan d/N_0 = d/N_0, \tan S/N_0 = S/N_0, \)

The word geodetic reference system 1967

The International Union of Geodesy and Geophysics has tentatively adopted a new geodetic reference system, see reference [30]. It is defined by the three constants: equatorial radius \( a = 6378160 \text{ meters}; \)

earth geocentric gravitational constant including the atmosphere \( GM = 398603 \times 10^6 \text{m}^3/\text{s}^2; \)

earth dynamical form factor \( J_2 = 10827 \times 10^{-4}. \)

Now the earth's rotational velocity is given by
\[ \omega = \sqrt{2m(1 + s_1/86400)/(s_1 + p \cos e/1500)}, \quad \text{s}^{-1}, \]

where \( s_1 = 31556925.9747 \text{ (ephemeris seconds in one tropical year, 1900)}; \)

\[ p = 5025.64 \text{ (seconds general precession in longitude per tropical century, 1900)}; \]

\[ \epsilon = 23^° 27' 08.3126 \text{ (obliquity of the ecliptic, 1900)}; \]

\[ \omega = 7.292115144 \times 10^2 \text{s}^{-1}. \]

Since \( J_2 = e^2(1 - 2me'[1500]/(1 + e^2)), \)

and \( a, GM, J_2, \omega \) are known we may solve for \( e'. \)
\[ e' = 0.08209582892 \text{ (the second eccentricity)}. \]

We can then solve for \( b, 1/f, \) and \( e^2: \)
\[ b = a(1 + e^2)^{-1/2} = 6356745.9166 \text{ m}; \]

\[ 1/f = a/(a - b) = 298.2471675. \]

\[ e^2 = (a^2 - b^2)/a^2 = 0.00669437999. \]

The formulae for gravity at the pole, equator, general (normal) are:
\[ g_p = GM(1 + m'e'/6dq)/a^2; \]

\[ g_e = GM(1 - m'e'/6dq)/b^2; \]

\[ g = (aq' \cos^2 \phi + bg' \sin^2 \phi)/(a^2 \cos^2 \phi + b^2 \sin^2 \phi)^{1/2}. \]
where \( q' = 3 \left(1 + \frac{1}{e^2} \right) \left(1 - \frac{1}{e^2} \arctan e' \right) - 1. \)

With the above values of constants, these become:

\[
\begin{align*}
\varepsilon_p &= 983.21777279 \text{ gal}, \quad \varepsilon_e = 978.0318456 \text{ gal} \\
g &= 978.0318 \left(1 + 0.0053024 \sin^2 \phi - 0.0000058 \sin^2 2\phi \right) \text{ gal}
\end{align*}
\]

(this last the series expansion of the above general formula).

The numerical values are preliminary. The official values will be published by the International Association of Geodesy.

**Summary Values**

**Given**

- \( a = 6378160 \text{ m}, GM = 398603 \times 10^8 \text{ m}^3 \text{ s}^{-2}, \quad J_2 = 10827 \times 10^{-7} \)

**Computed**

- \( \omega = 7.292115144 \times 10^{-5} \text{ s}^{-1}, \quad e^2 = 0.006694605326 \),
- \( e' = 0.006739725126, \quad g_p = 983.21777279 \text{ gal} \),
- \( g_e = 978.0318456 \text{ gal}, \quad \frac{1}{f} = 298.2471675, \quad b = 6356774.516 \text{ m} \),
- \( g = 978.0318 \left(1 + 0.0053024 \sin^2 \phi - 0.0000058 \sin^2 2\phi \right) \text{ gal} \)

**Comment.** A comparison with the AUSTRALIAN ellipsoidal constants, Table 10, shows that, for practical geodetic purposes, using 8-place tables, we may use the AUSTRALIAN ellipsoid. Tables of mid-latitude arc length, principal radii of curvature, and latitude functions have been published for several countries by the Department of the Army as technical manuals. For instance, CLARKE 1866 (TMS-241-19), INTERNATIONAL (AMS TM-67); AUSTRALIAN (TMS-241-33); FISCHER (MERCURY) (TMS-241-35).

If we take \( a = 6378160 \text{ m}, \omega = 7.292115 \times 10^{-5} \text{ s}^{-1}, \frac{1}{f} = 298.25 \),

- \( f' = 0.3352892 \times 10^{-3}, \quad g_p = 978.0318 \text{ cm} \text{ s}^{-2}, \) we find \( m = \omega^2 a / \varepsilon_a = 346.77 \times 10^{-3} \),
- \( f/m = 0.1162703 \times 10^4 \). From reference [31], page 366, we have

\[
g = g_p \left[1 + \beta \sin^2 \phi - \beta_1 \sin^2 2\phi \right], \quad \text{where}
\]

\[
\beta = \left(5/2\right)g - f - \left(17/14\right)nf, \quad \beta_1 = \left(5/8\right)nf - (1/8)f^2
\]

With the stated values of \( f, m, \) and \( nf \) we have

- \( \beta = 0.0053024, \quad \beta_1 = 0.00000586, \) and
- \( g = 978.0318 \left[1 + 0.0053024 \sin^2 \phi - 0.00000586 \sin^2 2\phi \right] \text{ gal}. \)

This gravity formula gives only \(0.06 \sin^2 2\phi \text{ mgal} \) less than the recommended formula above, the maximum difference being \(0.06 \text{ mgal at } \phi = 45^\circ. \)
Appendix 3.
THE ACIC CHECK LINES, 50-6000 MILES,
AND GEODETIC LINE COMPUTATIONS
ACIC CHECK LINES

Taken directly from ACIC Technical Reports 59, 80. See bibliographical reference [22]. The 500 mile lines are repeated since they were given in both publications.
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<table>
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Latitudes north; longitudes west, except those prefixed by a minus sign.
| Distance (Miles) | Latitude 10°  | 0°  | 40° | 70° | 0°  | 40° | 70° | 0°  | 40° | 70° | 0°  | 40° | 70° | 0°  | 40° | 70° | 0°  | 40° | 70° |
|-----------------|---------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 300 Miles       | 80,466,478    | 80,466,478 | 80,466,478 | 80,466,478 | 80,466,478 | 80,466,478 | 80,466,478 | 80,466,478 | 80,466,478 | 80,466,478 | 80,466,478 | 80,466,478 | 80,466,478 | 80,466,478 | 80,466,478 | 80,466,478 | 80,466,478 | 80,466,478 | 80,466,478 |
| 100 Miles       | 80,466,478    | 80,466,478 | 80,466,478 | 80,466,478 | 80,466,478 | 80,466,478 | 80,466,478 | 80,466,478 | 80,466,478 | 80,466,478 | 80,466,478 | 80,466,478 | 80,466,478 | 80,466,478 | 80,466,478 | 80,466,478 | 80,466,478 | 80,466,478 | 80,466,478 |
| Azimuth         | 0°            | 0°  | 0°  | 0°  | 45° | 45° | 45° | 70° | 70° | 70° | 10° | 10° | 10° | 40° | 40° | 40° | 70° | 70° | 70° |

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### ACTUAL FORWARD AZIMUTH OF LINES

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112
### ORIGIN AND TERMINAL POSITIONS OF ALL TEST LINES

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**CLARKE 1866 ELLIPSOID**
DIRECT AND REVERSE COMPUTATIONS OF ALL ACIC 6000 MILE CHECK LINES –
Clarke 1866 Ellipsoid
DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1, \alpha_{1,2}, S$ to find $\phi_2, \lambda_2, \alpha_{2,1}$. Fast longitudes positive; azimuths clockwise from north; no root extraction: only 8 place trigonometric natural tables (as Peters) required for desk work.

**Clarke 1866 Spheroid** $a = 6378206.4$ m $f = 2.90075283 \times 10^{-3}$

$1 - f = 0.99970997$ 1 radian $= 206264.8062$ seconds

**Line 1**

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<td>$\phi_1, \lambda_1, 0, 0, 0$</td>
<td>$\tan \phi_1, 0.17632698$, $\tan \theta_1 = (1-f) \tan \phi_1 - 1.97239427$</td>
</tr>
<tr>
<td>$\alpha_{1,2} = 0$</td>
<td>$\tan \theta_1, 1.7307716$, $\cos \theta_1, 0.9840827$, $\theta_1, 9.5800408$</td>
</tr>
<tr>
<td>$\sin \alpha_{1,2} = 0$</td>
<td>$M = \cos \theta_0 = \cos \theta_1 \sin \alpha_{1,2}$, $0_0, 90, 0, 0$</td>
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<tr>
<td>$\cos \alpha_{1,2} = 1$</td>
<td>$N = \cos \theta_1 \cos \alpha_{1,2} = \cos \theta_1 \sin \theta_0$, $1$</td>
</tr>
<tr>
<td>$c_1 = \frac{f}{\cos \theta_1}$</td>
<td>$D = (1 - c_2) (1 - c_2 - c_1)$, $0.9868056806$</td>
</tr>
<tr>
<td>$c_2 = \frac{f}{(1 - f^2)}$, $0.847513872 \times 10^{-3}$</td>
<td>$P = c_2 (1 + 2c_1 M)/(1 - c_2)$, $0.008848957225$</td>
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<td>$\cos \alpha_1 = \sin \theta_1 / \sin \theta_0 = \cos (\theta_0 - \alpha_1)$, $\phi_1$, $0^\circ \quad 0^\circ \quad 0^\circ$</td>
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<td>$\sin \alpha_1 = \sin \theta_1$</td>
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<tr>
<td>$d = S/4D$, $1.514710635$ (rad)</td>
<td>$d = 86.5314.610$ S, 9655977.366 m</td>
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<tr>
<td>$\sin \beta = 0.99952475$</td>
<td>$u = 2(\alpha_1 - d) - 13.42.30.036$, $\sin u = -2369.7967$</td>
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<tr>
<td>$\cos d = 0.05473855$</td>
<td>$W = 1 - 2P \cos u + 0.94235045 \cos u + 0.9715.1467$</td>
</tr>
<tr>
<td>$V = \cos u \cos d - \sin u \sin d$, $0.289381451$</td>
<td>$Y = 2PV \sin d$, $+0.1498.1147 \times 10^{-3}$</td>
</tr>
<tr>
<td>$X = c_2^2 \sin \alpha_1$, $\cos (\alpha_2 - 1)$, $-3.32422 \times 10^{-7}$</td>
<td>$\Delta \phi = J + X - Y$, $1.5159.12.196$ (rad)</td>
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<td>$\sin \Delta \alpha$, $\cos \Delta \alpha$</td>
<td>$\Delta \alpha = 86.51.32.572$</td>
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<td>$\cos \Sigma$, $\cos \Delta \alpha$</td>
<td>$\Sigma = 2\alpha_1 - \Delta \phi$</td>
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<tr>
<td>$\tan \alpha_{2,1} = M/(N \cos \Delta \alpha - \sin \theta_1 \sin \Delta \alpha)$, $0$, $\alpha_{2,1}$</td>
<td>$36.0$</td>
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<td>$\tan \phi_2 = (\sin \theta_1 \cos \Delta \phi + N \sin \Delta \alpha) \sin \alpha_{2,1}$, $0.38222927$, $\phi_2$, $82.11$, $48.546$</td>
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<td>$\tan \Delta \eta = \cos \theta_1 \cos \Delta \eta - \sin \theta_1 \sin \Delta \alpha \cos \alpha_{1,2}$</td>
<td>$\Delta \eta = 180$</td>
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<tr>
<td>$H = c_1 (1 - c_1) \Delta \alpha - c_1 c_2 \sin \Delta \alpha \cos \Delta \alpha$, $0$</td>
<td>$\Delta \lambda = \Delta \eta - H$, $180$</td>
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<tr>
<td>$\lambda_1$</td>
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</table>

CHECK

$M = \cos \theta_0 = \cos \theta_1 \sin \alpha_{1,2} = \cos \theta_2 \sin (180 + \alpha_{1,2})$, $\lambda_2 = \lambda_1 + \Delta \lambda$, $117$
Inverse Position Computation Form for Long Lines. Given $\phi_1, \lambda_1; \phi_2, \lambda_2$ to find $S, a_1, a_2$. Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

**Clarke 1866 Spheroid**

$$a = 6378206.4 \text{ m}$$  
$$b = 6356583.8 \text{ m}$$

$$1 - \ell = \frac{29960924777}{1690237645.6 \times 10^{-3}} = 0.584751892075 \times 10^{-3}$$

$$f^2/64 = \frac{1745.720379}{10^{-6}}$$

1 radian = 206264.8062 seconds

<table>
<thead>
<tr>
<th>$\phi_1$</th>
<th>10°</th>
<th>0'</th>
<th>0&quot;</th>
<th>1. ORIGIN (ALIC)</th>
<th>$\lambda_1$</th>
<th>-18°</th>
<th>0'</th>
<th>0&quot;</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_2$</td>
<td>03°</td>
<td>11'</td>
<td>48.545&quot;</td>
<td>2. TERMINUS</td>
<td>$\lambda_2$</td>
<td>162°</td>
<td>0'</td>
<td>0&quot;</td>
</tr>
</tbody>
</table>

**Tables**

- $\Delta \lambda = \lambda_2 - \lambda_1$
- $\cos \theta_m = \cos \phi_1 + \cos \phi_2$
- $\sin \Delta \lambda = \sin \phi_1 \cos \phi_2 - \cos \phi_1 \sin \phi_2$
- $\sin \Delta \phi = \sin \phi_2 - \sin \phi_1$

**Formulas**

- $\Delta \lambda = \lambda_2 - \lambda_1$
- $\cos \theta_m = \cos \phi_1 + \cos \phi_2$
- $\sin \Delta \lambda = \sin \phi_1 \cos \phi_2 - \cos \phi_1 \sin \phi_2$
- $\sin \Delta \phi = \sin \phi_2 - \sin \phi_1$

**Equations**

- $\Delta \lambda = \lambda_2 - \lambda_1$
- $\cos \theta_m = \cos \phi_1 + \cos \phi_2$
- $\sin \Delta \lambda = \sin \phi_1 \cos \phi_2 - \cos \phi_1 \sin \phi_2$
- $\sin \Delta \phi = \sin \phi_2 - \sin \phi_1$

**Constants**

- $a = 6378206.4 \text{ m}$$
- $b = 6356583.8 \text{ m}$$
- $f^2/64 = \frac{1745.720379}{10^{-6}}$

**Chains**

- $1 - \ell = \frac{29960924777}{1690237645.6 \times 10^{-3}} = 0.584751892075 \times 10^{-3}$
- 1 radian = 206264.8062 seconds
DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1, \alpha_1, \lambda_2, \alpha_2, \alpha_3$. To find $\phi_2, \lambda_2, \alpha_2, \alpha_3$. East longitudes positive; azimuths clockwise from north. No root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

**Clarke 1866 Spheroid a = 6378,100 km \[1.5920718 \times 10^{-3}\]**

1 radian = 206,264.8062 seconds

### Line Origin To Terminator (Clarkic)

<table>
<thead>
<tr>
<th>$\phi_1$</th>
<th>$\lambda_1$</th>
<th>$\phi_2$</th>
<th>$\lambda_2$</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$\alpha_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0° 0' 0&quot;</td>
<td>0° 0' 0&quot;</td>
<td>tan $\phi_1$ 0.83904463</td>
<td>tan $\theta_1 = (1 - f) tan \phi_1$ 0.83625507</td>
<td>$\alpha_1$</td>
<td>0</td>
<td>$\alpha_1$</td>
</tr>
<tr>
<td>sin $\alpha_1$</td>
<td>0</td>
<td>M = cos $\theta_0$ = cos $\phi_1$ sin $\alpha_1$</td>
<td>$\theta_0$ 90° 0' 0&quot;</td>
<td>$\alpha_1$</td>
<td>0</td>
<td>$\alpha_1$</td>
</tr>
<tr>
<td>cos $\alpha_1$</td>
<td>1</td>
<td>N = cos $\phi_1$ cos $\alpha_1$</td>
<td>$\psi_0$</td>
<td>$\theta_0$ 90° 0' 0&quot;</td>
<td>$\alpha_1$</td>
<td>0</td>
</tr>
<tr>
<td>$c_1 = \pi M$</td>
<td>0</td>
<td>D = (1 - $c_1$)(1 - $c_1$ - $c_1$M) 0.9984056806</td>
<td>$\alpha_1$</td>
<td>0</td>
<td>$\alpha_1$</td>
<td>0</td>
</tr>
<tr>
<td>$c_2 = \frac{1}{2}(1 - M^2) f$ 0.3475/882075x10^-3</td>
<td></td>
<td>P = $c_1$ (1 + $c_1$M)/D 0.00849547254</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\cos \alpha_1 = \sin \theta_1 / \sin \theta_2 = \cos (90° - \phi_1)$</td>
<td>0</td>
<td>$\delta_0 0' 0&quot; 44.797$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$\omega$ = $S/A$ $\pm$ $2\pi$ 0.5147/10635 (rad) d 96.53 14.612 0.9655 977.366 m

$\sin d = 9485, 2475$ $u = 2(c_1 - d) - 73.34/9.926 \sin u - 95.597 212.9$

$\cos d + 0.052 9.555 W = 1 - 2P \cos u + 0.5995 \tan d 1.7 = 2PVW \tan d 4.8$ 4.28 26.2 22.4

$V = \cos u \cos d - \sin u \sin d - 4.37/16.19 Y = 2PVW \sin d 0.0164 91.67$

$X = c_1 \sin d \cos d (2V^2 - 1) + 4.3487 \times 10^{-7}$ $\Delta_0 = d + X - Y 1.5482/19.16$ (rad)

$\sin \Delta_0 / \cos \Delta_0$ $\Delta_0 86.47 48.416$

$\cos \Delta_0$ $\Delta_0 0 - \Delta_0$

$\tan \Delta_0 0 - \Delta_0$

$\tan \phi_1 = (\tan \theta_1 \cos \Delta_0 + N \sin \Delta_0) / \sin \alpha_1$ 1.3463 0731 $\sin \alpha_1$ $\sin \alpha_1$ $\sin \alpha_1$

$\cos \theta_1 = \tan (90° + \phi_1 - \alpha_1') / (1 - f) \phi_1$ 53.23 35.786

$\tan \eta_0 = \sin \Delta_0 \sin \alpha_1$ 0 $\Delta_0$ $\Delta_0$

$H = c_1(1 - \alpha_2') \Delta_0 - c_1 c_3 \sin \Delta_0 \cos \Sigma_0$ 0 (rad) $\Delta_0 = \Delta_0 - \Delta_0 180°$

$\lambda_1 = -18$

CHECK

$M = \cos \theta_0 = \cos \phi_1 \sin \alpha_1 = \cos \phi_2 \sin (180 + \alpha_2, \alpha_3)$ $\lambda_2 = \lambda_1 + \Delta_0 162°$
INVERSE POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1$, $\lambda_1$, $\phi_2$, $\lambda_2$ to find $S$, $a$, $u$, $v$.

Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters), no root extraction.

\[ \text{CLARKE } 1866 \quad \text{Crs.} \quad \text{ORD \ } b = 6256.583 m \quad b = 6256.583 \times 10^{-3} \]

\[ 1 - f = b/a = 3986.092 \times 2.717 \quad \text{if } 1.6971276 \times 10^{10} \quad 0.0 \quad 1.075 \times 10^{-2} \]

\[ \text{f} = 0.01295 \times 10^{-2} \quad 1.8 \times 10^{-2} \]

1 radian = 206264.8062 seconds

\[ \theta_1 = \tan^{-1} \left( \frac{\tan \phi_1}{\cos \lambda_1} \right) \]

\[ \theta_2 = \tan^{-1} \left( \frac{\tan \phi_2}{\cos \lambda_2} \right) \]

\[ \theta = \frac{\theta_1 - \theta_2}{2} \]

\[ \Delta \theta = \frac{\theta_1 + \theta_2}{2} \]

\[ \Delta \varphi = \frac{\phi_1 - \phi_2}{2} \]

\[ \Delta \lambda = \frac{\lambda_1 - \lambda_2}{2} \]

\[ \Delta = \frac{\lambda_1 + \lambda_2}{2} \]

\[ \Delta \theta = \frac{\theta_1 + \theta_2}{2} \]

\[ H = \cos^2 \Delta + \Delta \cos \Delta \cos \Delta \]

\[ \Delta \lambda = \frac{\lambda_1 - \lambda_2}{2} \]

\[ \Delta \varphi = \frac{\phi_1 - \phi_2}{2} \]

\[ \Delta = \frac{\lambda_1 + \lambda_2}{2} \]

\[ \Delta \theta = \frac{\theta_1 + \theta_2}{2} \]

\[ \Delta \varphi = \frac{\phi_1 - \phi_2}{2} \]

\[ \Delta = \frac{\lambda_1 + \lambda_2}{2} \]

\[ \Delta \theta = \frac{\theta_1 + \theta_2}{2} \]

\[ \Delta \varphi = \frac{\phi_1 - \phi_2}{2} \]

\[ \Delta = \frac{\lambda_1 + \lambda_2}{2} \]

\[ \Delta \theta = \frac{\theta_1 + \theta_2}{2} \]

\[ \Delta \varphi = \frac{\phi_1 - \phi_2}{2} \]

\[ \Delta = \frac{\lambda_1 + \lambda_2}{2} \]

\[ \Delta \theta = \frac{\theta_1 + \theta_2}{2} \]

\[ \Delta \varphi = \frac{\phi_1 - \phi_2}{2} \]

\[ \Delta = \frac{\lambda_1 + \lambda_2}{2} \]

\[ \Delta \theta = \frac{\theta_1 + \theta_2}{2} \]

\[ \Delta \varphi = \frac{\phi_1 - \phi_2}{2} \]

\[ \Delta = \frac{\lambda_1 + \lambda_2}{2} \]

\[ \Delta \theta = \frac{\theta_1 + \theta_2}{2} \]

\[ \Delta \varphi = \frac{\phi_1 - \phi_2}{2} \]

\[ \Delta = \frac{\lambda_1 + \lambda_2}{2} \]

\[ \Delta \theta = \frac{\theta_1 + \theta_2}{2} \]

\[ \Delta \varphi = \frac{\phi_1 - \phi_2}{2} \]

\[ \Delta = \frac{\lambda_1 + \lambda_2}{2} \]

\[ \Delta \theta = \frac{\theta_1 + \theta_2}{2} \]

\[ \Delta \varphi = \frac{\phi_1 - \phi_2}{2} \]

\[ \Delta = \frac{\lambda_1 + \lambda_2}{2} \]

\[ \Delta \theta = \frac{\theta_1 + \theta_2}{2} \]

\[ \Delta \varphi = \frac{\phi_1 - \phi_2}{2} \]

\[ \Delta = \frac{\lambda_1 + \lambda_2}{2} \]

\[ \Delta \theta = \frac{\theta_1 + \theta_2}{2} \]

\[ \Delta \varphi = \frac{\phi_1 - \phi_2}{2} \]

\[ \Delta = \frac{\lambda_1 + \lambda_2}{2} \]

\[ \Delta \theta = \frac{\theta_1 + \theta_2}{2} \]

\[ \Delta \varphi = \frac{\phi_1 - \phi_2}{2} \]

\[ \Delta = \frac{\lambda_1 + \lambda_2}{2} \]

\[ \Delta \theta = \frac{\theta_1 + \theta_2}{2} \]

\[ \Delta \varphi = \frac{\phi_1 - \phi_2}{2} \]

\[ \Delta = \frac{\lambda_1 + \lambda_2}{2} \]

\[ \Delta \theta = \frac{\theta_1 + \theta_2}{2} \]

\[ \Delta \varphi = \frac{\phi_1 - \phi_2}{2} \]

\[ \Delta = \frac{\lambda_1 + \lambda_2}{2} \]

\[ \Delta \theta = \frac{\theta_1 + \theta_2}{2} \]

\[ \Delta \varphi = \frac{\phi_1 - \phi_2}{2} \]

\[ \Delta = \frac{\lambda_1 + \lambda_2}{2} \]

\[ \Delta \theta = \frac{\theta_1 + \theta_2}{2} \]

\[ \Delta \varphi = \frac{\phi_1 - \phi_2}{2} \]

\[ \Delta = \frac{\lambda_1 + \lambda_2}{2} \]

\[ \Delta \theta = \frac{\theta_1 + \theta_2}{2} \]

\[ \Delta \varphi = \frac{\phi_1 - \phi_2}{2} \]

\[ \Delta = \frac{\lambda_1 + \lambda_2}{2} \]

\[ \Delta \theta = \frac{\theta_1 + \theta_2}{2} \]

\[ \Delta \varphi = \frac{\phi_1 - \phi_2}{2} \]

\[ \Delta = \frac{\lambda_1 + \lambda_2}{2} \]

\[ \Delta \theta = \frac{\theta_1 + \theta_2}{2} \]

\[ \Delta \varphi = \frac{\phi_1 - \phi_2}{2} \]

\[ \Delta = \frac{\lambda_1 + \lambda_2}{2} \]
DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1$, $\lambda_1$, $\alpha_{1,2}$, $S$ to find $\phi_2$, $\lambda_2$, $\alpha_{2,3}$, East longitudes positive, azimuths clockwise from north, no root extraction, only 8-place trigonometric natural tables (as Peters) required for desk work.

**Clarke 1866 Spheroid** $a = 6378206.4 \text{ m}$, $f = 3.390075283 \times 10^{-3}$

1 radian = 206264.8062 seconds

<table>
<thead>
<tr>
<th>LINE</th>
<th>ORIGIN</th>
<th>TO</th>
<th>TERMINUS (A11C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_1$</td>
<td>32° 0' 0&quot;</td>
<td>$\tan \phi_1 = 2.74747794$</td>
<td>$\tan \phi_1 = (1 - f) \tan \phi_1 = 2.72816326$</td>
</tr>
<tr>
<td>$\alpha_{1,2}$</td>
<td>0</td>
<td>$\sin \theta_1$</td>
<td>$\cos \theta_1 = 0.99960992717$</td>
</tr>
<tr>
<td>$\sin \alpha_{1,2}$</td>
<td>0</td>
<td>$M = \cos \theta_0 = \cos \theta_1 \sin \alpha_{1,2}$</td>
<td>$0.0$</td>
</tr>
<tr>
<td>$\cos \alpha_{1,2}$</td>
<td>1</td>
<td>$N = \cos \theta_1 \cos \alpha_{1,2}$</td>
<td>$\cos \theta_0 = 0.900$</td>
</tr>
<tr>
<td>$c_1 = \tan \alpha_1$</td>
<td>0</td>
<td>$D = (1 - c_2)(1 - c_2 - c_1 M) = 9983056206$</td>
<td></td>
</tr>
<tr>
<td>$c_2 = (1 - M^2) f = 847518821.1 \times 10^{-3}$</td>
<td>$P = c_1 (1 + \tan \alpha_1 M)/D$</td>
<td>$0.9983056206$</td>
<td></td>
</tr>
<tr>
<td>$\cos \sigma_1 = \sin \theta_1 / \sin \theta_0 = \cos (90 - \theta_1)$</td>
<td>$\sigma_1 = 3060^{\circ}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$d = \frac{S}{aD}$</td>
<td>1.5164700635 (rad)</td>
<td>$d = 86.5314.612$</td>
<td></td>
</tr>
<tr>
<td>$\sin d = +0.99852475$</td>
<td>$u = 2(\sigma_1 - d) = 133.8858.600$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\cos d = 1.05427455$</td>
<td>$W = 1 - 2P \cos u + 1001171978 \cos u = -6922.456$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$V = \cos \theta \cos d - \sin \theta \sin d \cos (0.6850283295)$</td>
<td>$Y = 2PVW \cos d$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$X = c_1 \sin d \cos d (2V^2 - 1) - 2.39410^{-8}$ (right)</td>
<td>$\Delta \theta = d + X - Y$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sin \Delta \alpha$</td>
<td>$\cos \Delta \alpha$</td>
<td>$\Delta \alpha = 384.9914.773$</td>
<td></td>
</tr>
<tr>
<td>$\cos \Sigma \alpha$</td>
<td>$\Sigma \alpha = 2\alpha_1 - \Delta \alpha$</td>
<td>$360^{\circ}$</td>
<td></td>
</tr>
<tr>
<td>$\tan \alpha_{2,1} = M/(N \cos \Delta \alpha - \sin \theta_1 \sin \Delta \alpha)$</td>
<td>$0$</td>
<td>$\alpha_{2,1} = 23.18^{\circ}$</td>
<td></td>
</tr>
<tr>
<td>$\tan \phi_2 = -((\sin \theta_1 \cos \Delta \alpha + N \sin \Delta \alpha) \sin \alpha_{2,1})$</td>
<td>$0$</td>
<td>$\phi_2 = 180^{\circ}$</td>
<td></td>
</tr>
<tr>
<td>$\sin \alpha_{2,1}$</td>
<td>$48092616$</td>
<td>$\sin \sigma_{2,1}$</td>
<td>$180^{\circ}$</td>
</tr>
<tr>
<td>$\Delta \eta = \cos \theta_1 \cos \Delta \alpha - \sin \theta_1 \sin \Delta \alpha \cos \alpha_{1,2}$</td>
<td>$0$</td>
<td>$\Delta \theta = \Delta \eta - H$</td>
<td></td>
</tr>
<tr>
<td>$H = c_1 (1 - c_2) \Delta \alpha - c_1 c_2 \sin \Delta \alpha \cos \Sigma \theta$</td>
<td>$0$</td>
<td>$H = 0$</td>
<td></td>
</tr>
<tr>
<td>$\Delta \lambda = \Delta \eta - H = 180^{\circ}$</td>
<td>$\lambda_1$</td>
<td>$-18^{\circ}$</td>
<td></td>
</tr>
</tbody>
</table>

**CHECK**

$M = \cos \theta_0 = \cos \theta_1 \sin \alpha_{1,2} = \cos \theta_2 \sin (180 + \alpha_{2,1})$

$\lambda_2 = \lambda_1 + \Delta \lambda = 182^{\circ}$
INVERSE POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1; \phi_2, \lambda_2$ to find $S, \alpha, \beta$. Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

| $\phi_1$ | 20 o 0' 0" | 1. ORIGIN (NCIC) | $\lambda_1$ | 18 o 0' 0" |
| $\phi_2$ | 23 19 44.908 | 2. TERMINUS | $\lambda_2$ | 162 o 0' 0" |
| $\tan \phi_1$ | 0.42747 774' | 1. always west of 2. | $\Delta \lambda = \lambda_2 - \lambda_1$ | 180 |
| $\tan \phi_2$ | 0.4309 261' | $\tan \theta = (1 - f) \tan \phi$ | $\Delta \lambda_m = \frac{\Delta \lambda}{2}$ | 90 |
| $\theta_1$ | 69.56 14.549 | $\tan \theta_2 = 0.42456529$ | $\sin \Delta \lambda_m$ | 1 |
| $\theta_2$ | 72 14 30.638 | $\sin \Delta \lambda_m$ | $\Delta \lambda_m$ | 0 |
| $\theta_m = \frac{\pi}{2}(\theta_1 + \theta_2)$ | 75.35 22.614 | $\sin \theta_m + 0.6174 \cos \theta_m$ | $\cos \theta_m$ | 142 |
| $\Delta \theta_m = \frac{\pi}{2}(\theta_2 - \theta_1) = 23.20 \sin \Delta \lambda_m - 0.3963 \cos \Delta \lambda_m - 0.5117 \cos \Delta \lambda_m + 0.6216$ |
| $H = \cos^2 \Delta \theta_m - \sin^2 \theta_m = \cos^2 \theta_m - \sin^2 \Delta \lambda_m$ | 23720776 $-$ L | 27727 977.15 |
| $L = \sin^2 \Delta \theta_m + H \sin^3 \Delta \lambda_m$ | 27727 077.85 | $\cos d = 1 - 2L + 0.0554 \tan^2 L$ |
| $S = \frac{L}{\tan \Delta \lambda_m} \cdot \tan \Delta \lambda_m$ | 180 |
| $\sqrt{\frac{L}{\tan \Delta \lambda_m}}$ | 180 |
| $\tan \Delta \lambda_m = \frac{\tan \Delta \lambda}{\tan \Delta \lambda_m}$ | $\tan \Delta \lambda_m$ | $\tan \Delta \lambda_m$ | $\infty$ |
| $\cos \Delta \lambda_m = \frac{\cos \Delta \lambda}{\tan \Delta \lambda_m}$ | 0 |
| $\sin \Delta \lambda_m = \frac{\sin \Delta \lambda}{\tan \Delta \lambda_m}$ | 0 |
| $\tan \Delta \lambda_m = \frac{\tan \Delta \lambda}{\tan \Delta \lambda_m}$ | $\tan \Delta \lambda_m$ | $\tan \Delta \lambda_m$ | $\infty$ |
| $\cos \Delta \lambda_m = \frac{\cos \Delta \lambda}{\tan \Delta \lambda_m}$ | 0 |
| $\sin \Delta \lambda_m = \frac{\sin \Delta \lambda}{\tan \Delta \lambda_m}$ | 0 |
| $\tan \Delta \lambda_m = \frac{\tan \Delta \lambda}{\tan \Delta \lambda_m}$ | $\tan \Delta \lambda_m$ | $\tan \Delta \lambda_m$ | $\infty$ |
| $\cos \Delta \lambda_m = \frac{\cos \Delta \lambda}{\tan \Delta \lambda_m}$ | 0 |
| $\sin \Delta \lambda_m = \frac{\sin \Delta \lambda}{\tan \Delta \lambda_m}$ | 0 |
| $\tan \Delta \lambda_m = \frac{\tan \Delta \lambda}{\tan \Delta \lambda_m}$ | $\tan \Delta \lambda_m$ | $\tan \Delta \lambda_m$ | $\infty$ |
| $\cos \Delta \lambda_m = \frac{\cos \Delta \lambda}{\tan \Delta \lambda_m}$ | 0 |
| $\sin \Delta \lambda_m = \frac{\sin \Delta \lambda}{\tan \Delta \lambda_m}$ | 0 |
| $\tan \Delta \lambda_m = \frac{\tan \Delta \lambda}{\tan \Delta \lambda_m}$ | $\tan \Delta \lambda_m$ | $\tan \Delta \lambda_m$ | $\infty$ |
| $\cos \Delta \lambda_m = \frac{\cos \Delta \lambda}{\tan \Delta \lambda_m}$ | 0 |
| $\sin \Delta \lambda_m = \frac{\sin \Delta \lambda}{\tan \Delta \lambda_m}$ | 0 |
| $\tan \Delta \lambda_m = \frac{\tan \Delta \lambda}{\tan \Delta \lambda_m}$ | $\tan \Delta \lambda_m$ | $\tan \Delta \lambda_m$ | $\infty$ |
| $\cos \Delta \lambda_m = \frac{\cos \Delta \lambda}{\tan \Delta \lambda_m}$ | 0 |
| $\sin \Delta \lambda_m = \frac{\sin \Delta \lambda}{\tan \Delta \lambda_m}$ | 0 |
| $\tan \Delta \lambda_m = \frac{\tan \Delta \lambda}{\tan \Delta \lambda_m}$ | $\tan \Delta \lambda_m$ | $\tan \Delta \lambda_m$ | $\infty$ |
| $\cos \Delta \lambda_m = \frac{\cos \Delta \lambda}{\tan \Delta \lambda_m}$ | 0 |
| $\sin \Delta \lambda_m = \frac{\sin \Delta \lambda}{\tan \Delta \lambda_m}$ | 0 |
DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1, \alpha_{1-2}, S$ to find $\phi_2, \lambda_2, \alpha_{2-1}$. East longitudes positive, azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

<table>
<thead>
<tr>
<th>LINE</th>
<th>ORIGIN TO TERMINUS (ALL)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_1$</td>
<td>10 0 0</td>
</tr>
<tr>
<td>$\alpha_{1-2}$</td>
<td>45 0 0</td>
</tr>
<tr>
<td>$\sin \alpha_{1-2}$</td>
<td>-7071.0678</td>
</tr>
<tr>
<td>$\cos \alpha_{1-2}$</td>
<td>-7071.0678</td>
</tr>
<tr>
<td>$c_1 = 0.0033693216$</td>
<td>$D = (1 - c_1)(1 - c_1 - c_1 M)$</td>
</tr>
<tr>
<td>$c_2 = 0.25(1 - M^2)$</td>
<td>$P = c_2 (1 + 0.5 c_1 M) / D = 0.0043794146$</td>
</tr>
<tr>
<td>$\cos \alpha_1 = \sin \theta_1 / \sin \theta_0 = 0.3111123323$</td>
<td>$\alpha_1 = 0.7607272072$</td>
</tr>
</tbody>
</table>

$d = S / a D = 1.0172206574$ (rad) $d = 0.8657232374$ S 0.965927366 m

$\sin \theta_1 = 0.8485957725 \cos \theta_0 = 0.99918697 \cos u = 0.92828769$ 

$V = \cos \theta_0 \cos \theta_1 \sin d = 0.99287062$ $\cos d = 0.093020575$ 

$W = 1 - 2 \cos u = 0.99918697 \cos u = 0.92828769$ $\cos \alpha_1 = 0.3111123323$ $\cos \alpha_1 = 0.7607272072$ 

$\tan \alpha_{2-1} = M / (N \cos \Delta \alpha - \sin \theta_1 \sin \Delta \alpha) - 5.11489288$ $\alpha_{2-1} = 281.01 12.1183$

$\tan \phi_2 = (\sin \theta_1 \cos \Delta \alpha + N \sin \Delta \alpha \sin \alpha_{2-1}) / (1 - \cos \Delta \alpha) = 0.99287062$ $\sin \alpha_{2-1} = 0.92828769$ $\sin \alpha_{2-1} = 0.3111123323$ $\sin \alpha_{2-1} = 0.7607272072$ 

$\tan \Delta \eta = \sin \Delta \alpha / \cos \phi_1 \cos \Delta \alpha - \sin \theta_1 \sin \Delta \alpha \cos \alpha_{2-1} = -10.145435 \Delta \eta = 95.37 45.3849$ 

$H = c_1 (1 - c_2) \Delta \alpha - c_1 c_2 \sin \Delta \alpha \cos \Delta \alpha = 0.0035804538$ (rad) $H = 12.25 18.516$ 

$\Delta \lambda = \Delta \eta - H = 95.25 16.816$ $\lambda_1 = -18$ 

CHECK 

$M = \cos \theta_0 = \cos \theta_1 \sin \alpha_{2-1} = \cos \theta_2 \sin (180 + \alpha_{2-1})$ 

$\lambda_2 = \lambda_1 + \Delta \lambda = 77.25 26.8168$
INVERSE POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1$, $\lambda_1$; $\phi_2$, $\lambda_2$ to find $S$, $a_1$, $-$2, $a_2$, 1.

Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

**Clarke 1866 Spheroid**

$$a = 6378180.6 \text{ m} \quad b = 6356583.8 \text{ m}$$

$1 - f = \frac{a}{b} = 0.9966099247 \frac{1}{12} \text{ m}$

$2 = 0.4045026363 \frac{1}{12} \pi \text{ rad} \quad 8.4751882.975 \times 10^{-3}$

$f^2/64 = 1.1745720.379 \times 10^{-6}$

1 radian = 206264.8062 seconds

1. **ORIGIN (ALIC)** $\lambda_1 = 1^{\circ} 25.26.869$

2. **TERMINUS** $\lambda_2 = 25.26.869$

3. Always west of 2.

4. $\Delta \lambda = \lambda_2 - \lambda_1 = 25.26.869$

5. $\Delta \lambda_m = \frac{1}{4}\Delta \lambda = \frac{1}{4}25.26.869 = 6.316.717$

6. $\Delta \lambda_m = \frac{1}{4}\Delta \lambda = \frac{1}{4}25.26.869 = 6.316.717$

7. $\Delta \lambda_m = \frac{1}{4}\Delta \lambda = \frac{1}{4}25.26.869 = 6.316.717$

8. $\Delta \lambda_m = \frac{1}{4}\Delta \lambda = \frac{1}{4}25.26.869 = 6.316.717$

9. $\cos^2 \Delta \theta_m - \sin^2 \theta_m = \cos^2 \theta_m - \sin^2 \theta_m = 0.9884 \frac{1}{2}$

10. $\sin^2 \Delta \lambda_m = 4.7204589.36 \cos d = 1 - 2L = 0.559.0821$

11. $U = 2 \sin^2 \theta_m \cos \Delta \theta_m / (1 - L) = 72967.407.72 \text{ d}$

12. $V = 2 \sin^2 \Delta \theta_m \cos^2 \theta_m L/2.49334.003 \sin d = 9984.3591 \text{ d (rad)} = 151485.895.61$

13. $X = U + V = 102.90.183.018 \text{ T} = d / \sin d = 1.51723.204.36 \text{ E} = 2 \cos d = 1.118.164.7$

14. $Y = U - V = 439523.988.75 \text{ D} = 4.17.20797.22.965 \text{ B} = 2D = 0.418.94.59.3$

15. $A = DE = 102.96.024.77 \text{ C} = T - \frac{1}{2} (A - E) = 105.839.904 \text{ CHECK C - \frac{1}{2} E + AD/B = T}$

16. $n_1 = X (A + CX) = 187.130.106 \quad n_2 = Y (B + EY) = 29456.379.56 \quad n_3 = DXY = 107744.923.6$

17. $\delta_d = 1.4(12 - (X - Y) / \delta_d = 0.00958 \frac{4.890}{4000}$

18. $\delta_2 = \frac{f^2}{64} (n_1 - n_2 + n_3) = 0.00021.0381$

19. $S_1 = a \sin d = 9655.429.242 \text{ m}$

20. $F = 2Y - E (4 - X) = 5828.552.71 \text{ m}$

21. $G = \frac{f^2}{64} + (f^2/64) = 0.025323.247.99$

22. $\Delta \lambda_m = \frac{1}{4} (\Delta \lambda + Q) = 1.108.126.9 \text{ m}$

23. $\cos \Delta \theta_m = -Q = - (FG \tan \Delta) / 4 = 13.185.20$

24. $S_2 = a \sin d (T - \delta_1 d) = 9655.429.242 \text{ m}$

25. $M = 32T - (20T - A) X - (B + EY) = 107744.923.6$

26. $c_5 = \cos \Delta \theta_m / \sin \theta_m tan \Delta \lambda_m = 1.8999.227.8$

27. $S_1 = a \sin d (T - \delta_1 d) = 9655.429.242 \text{ m}$

28. $c_5 = \cos \Delta \theta_m / \sin \theta_m tan \Delta \lambda_m = 1.8999.227.8$

29. $a_2 = v + u = 72.52.422.3 \text{ m}$

30. $a_3 = 180 - a_1 = 12.287.01 \text{ m}$

31. $a_2 = 180 + a_1 = 360 - a_1$

32. $a_1 = 180 - a_1$

33. $a_2 = 180 - a_1$

34. $a_3 = 180 - a_1$

35. $a_4 = 180 - a_1$

36. $a_5 = 180 - a_1$
DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1$, $\lambda_1$, $\alpha_{1-2}$, S to find $\phi_2$, $\lambda_2$, $\alpha_{2-3}$. East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

**Clarke 1866 Spheroid** $a = 3780206.4$ m $f = 3.390075283 \times 10^{-3}$

1 radian = 206264.8062 seconds

<table>
<thead>
<tr>
<th>LINE TO TERMINUS (ACIC)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_1$</td>
</tr>
<tr>
<td>$0^\circ$</td>
</tr>
<tr>
<td>$\tan \phi_1 = 8390.9963$</td>
</tr>
<tr>
<td>$\alpha_{1-2} = 45^\circ$</td>
</tr>
<tr>
<td>$M = \cos \theta_0 = \cos \phi_1 \sin \alpha_{1-2} = 5424.7825^\circ \theta_0 = 57.09.01.789$</td>
</tr>
<tr>
<td>$N = \cos \phi_1 \cos \alpha_{1-2} = 5424.7825^\circ \sin \theta_0 = 2400.9826$</td>
</tr>
<tr>
<td>$c_1 = \tan \phi_1 = 9978071775$</td>
</tr>
<tr>
<td>$c_2 = \frac{1}{3}(1 - M^2) \tan \phi_2 = 00059767188$</td>
</tr>
<tr>
<td>$P = c_1 (1 + \frac{1}{3}c_1 M) / D = 00059767188$</td>
</tr>
<tr>
<td>$\cos \alpha_1 = \sin \phi_1 / \sin \phi_2 = 0.76360851 \alpha_1 = 40.12^\circ 59.91'$</td>
</tr>
<tr>
<td>$d = \frac{S}{aD} = 1.151242869.03^\circ$ (rad) $d = 86.55.50.888^\circ$ S = 9655.977.366 m</td>
</tr>
<tr>
<td>$\sin d + 0.99856560^\circ u = \cos (\alpha_1 - d) = 93.25.41.886^\circ \sin u = -0.99821040$</td>
</tr>
<tr>
<td>$\cos d + 0.95354207^\circ W = 1 - 2P \cos u = 00000717313^\circ \cos u = -0.99739910$</td>
</tr>
<tr>
<td>$V = \cos u \cos d - \sin u \sin d = 9935.797703^\circ Y = 2PV \sin d = 00119019637^\circ$</td>
</tr>
<tr>
<td>$X = c_2 \sin d \cos d (2V^2 - 1) + 0.1824 \times 10^{-7}$ $\Delta a = d + X - Y = 154.0385126^\circ$ (rad)</td>
</tr>
<tr>
<td>$\sin \Delta a + 0.99850.117^\circ \cos \Delta a = 0.05473045^\circ \Delta a = 86.51.45.390^\circ$</td>
</tr>
<tr>
<td>$\cos \Sigma + 0.9937.104^\circ \Sigma = 2\alpha_1 - \Delta a = 111.40.45.562^\circ$</td>
</tr>
<tr>
<td>$\tan \alpha_{2-1} = M / \Delta a \cos \Sigma - \sin \theta_1 \sin \Delta a = -8819.8099^\circ \alpha_{2-1} = 31.23.43.002^\circ$</td>
</tr>
<tr>
<td>$\tan \phi_2 = -\phi_1 \cos \Delta a + N \sin \Delta a) \sin \alpha_{2-1} + 0.7083.717^\circ \sin \alpha_{2-1} = -6693.678^\circ$</td>
</tr>
<tr>
<td>$\phi_2 = 35.14.16.41^\circ$</td>
</tr>
<tr>
<td>$\tan \Delta a = -\Delta a \sin \alpha_{1-2} = -0.7180.0997^\circ \Delta a = 120.17.04.385^\circ$</td>
</tr>
<tr>
<td>$H = c_1 (1 - c_2) \Delta a - c_1 c_2 \sin \Delta a \cos \Sigma + 0.00785.076^\circ$ (rad) $H = 31.34.463^\circ$</td>
</tr>
<tr>
<td>$\Delta a = \Delta a - H = 120.02.29.022$</td>
</tr>
<tr>
<td>$\lambda_1 = 180.00.00$</td>
</tr>
</tbody>
</table>

CHECK

$M = \cos \theta_0 = \cos \phi_1 \sin \alpha_{1-2} = \cos \phi_2 \sin (180 + \alpha_{2-1})$

$\phi_2 = 35.14.16.41^\circ$

$H = 31.34.463^\circ$

$\Delta a = 120.02.29.022$
### Inverse Position Computation Form for Long Lines

**Given** $\phi_1$, $\lambda_1$; $\phi_2$, $\lambda_2$ to find $S$, $a_1$, $a_2$. Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

**Clarke 1866 Spheroid** $a = 6378020.4$ m $b = 6356583.8$ m

1 - $f = b/a = 0.000148$, $c = 6378120.4$ m $e = 0.08107$. $4f = 0.00002975$.

$p^2/64 = 1.79572079 x 10^{-6}$

1 radian = 206264.8062 seconds

| $\phi_1$ | 40 0 0 | 1. | BEGIN | $\lambda_1$ = 180 0 0 |
| $\phi_2$ | 35 18 45.54 | 2. | TERMINUS | $\lambda_2$ = 102 0 0 99.82 |
| $\tan \phi_1$ | 8340 99 63 | 1. always west of 2. | $\Delta \lambda = \lambda_2 - \lambda_1$ = 120 07 25.82 |
| $\tan \phi_2$ | 7083 71 74 | | |
| $\tan \theta_1$ | 35 13 13.43 | $\tan \theta_2$ = (1 - $f$) tan $\phi$ |
| $\theta_1$ | 39 54 15.29 | $\sin \Delta \lambda$ = $\tan \theta_1$ = 0.836 255 07 |
| $\theta_2$ | 37 33 44.33 | $\tan \Delta \lambda$ = $\tan \theta_1$ = 0.729 140 07 |
| $\theta_3$ | 10 29 29.80 | $\tan \theta_1$ = 0.209 08 01 28 |
| $\sin^2 \theta_m$ | 0.50 (1 + $\theta_2$) | $\sin^2 \theta_m$ = 0.192 6 87 86 |
| $\cos \theta_m$ | 0.97 29 80 00 | $\cos \Delta \lambda$ = 0.99 01 64 97 |
| $H = \cos^2 \Delta \lambda - \sin^2 \theta_m$ | $\cos^2 \theta_m - \sin^2 \Delta \lambda = 0.528 12 00 86$ |

$L = \sin^2 \theta_m + \sin^2 \Delta \lambda = 0.718 7 49 35 |

$U = 2 \sin^2 \theta_m \cos^2 \Delta \lambda/(1 - L) = 0.508 02 7 23$ d $= 0.6 4 43.5 14$ |

$V = 2 \sin^2 \Delta \lambda \cos^2 \lambda_1/L = 0.003 4 53 81 |

$X = U + V = 1.045 3 6 7 21 |

$T = d/\sin \lambda_1 = 1.5 1 69 7 2 81 |

$E = 2 \cos d = 1.1 3 09 0 2 87 |

$Y = U - V = 1.040 63 4 4 3 45 |

$D = 4 T^2/9 = 9.8 0 4 3 7 4 7 2 2 |

$B = 2 D = 18.4 0 8 5 4 9 1 8 |

$A = DE = 1.0 2 5 2 9 7 3 9 5 7 C = T - 1/2 (A - E) = 0.0 8 5 1 7 8 2 5$ CHECK C = $1/2 E + AD/B = T |

$n_1 = X (A + CX) = 3.5 5 1 0 4 1 3 2$ $n_2 = Y (B + EY) = 26.0 0 4 0 3 9 5$ $n_3 = DXY = 12 1 7 1 8 6 2 7 |

$\delta_1 = \sin (TX - Y) = 0.005 6 5 0 7 7 3$ $\delta_2 = \sin (F/64) n_1 - n_2 - n_3 = -0.0000 7 6 8 8 |

$S_1 = \sin d (T - \delta_1) = 0.965 9 0 2 1 5 0$ $S_2 = \sin d (T - \delta_2 + \delta_3 d) = 0.965 9 7 1 3 5 9 |

$F = 2 Y - E (4 - X) = 2.5 0 9 8 9 1 6 1 0$ $M = 32 T - (30 T - A) X - (B + 4) Y = 2 4.8 1 2 0 3 3 7 |

$G = \frac{1}{4} T + (F/64) M = 0.003 6 6 9 7 2 6 6 9$ $Q = (F G \tan \Delta \lambda) / 4 = 0.9 3 4 5 6 |

$\Delta \lambda_m = 1/2 (\Delta \lambda + Q) = 0.6 0 0 2 4 9 4$ $\tan \Delta \lambda_m = 1.7 3 9 0 9 0 5 0 9 |

$v = \arctan l_1 C_1 = 4 8 1.2 0 5 3 9 9 9$ $c_1 = \cos \Delta \lambda_m / (\tan \theta_m \tan \Delta \lambda_m) = 0.4 9 2 4 3 0 1 8 |

$u = \arctan l_1 C_1 = 4 1.5 1 5 0 1$ $c_2 = \sin \Delta \lambda_m / (\cos \theta_m \tan \Delta \lambda_m) = 0.2 4 6 3 8 0 7 |

$a_1 = v - u = 4 1.3 6 1 6.9 8 8$ $a_2 = v + u = 4 5.0 0 0 0 |

$C_1$ $C_2$ $A_1$ $A_2$ $a_1$ $a_2$ $a_3$ $a_4$

- - $a_1$ $4 5 0 0 0 0$

+ + $a_2$ $4 5 0 0 0 0$

+ - $1 8 0 - a_3$ $1 8 0 + a_4$

+ - $1 8 0 - a_3$ $1 8 0 a_4$

| $a_1$ | 360 - $a_1$ |
| $a_2$ | 360 - $a_2$ |

125
DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given \( \phi_1, \lambda_1, \alpha_{1,1}, S \) to find \( \phi_2, \lambda_2, \alpha_{2,1} \). East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

\[
\text{Clarke 1866 Spheroid } a = 6378206.4 \text{ m } f = 3.3990075283 \times 10^{-3}
\]

1 radian = 206264.8062 seconds

\[
\text{LINE } \begin{array}{ccc}
\text{ORIGIN} & \text{TO} & \text{TERMINUS (ALIC)} \\
\phi_1 & 10 & 0.0 \\
\lambda_1 & 27.4747 \text{ (rad)} & 180.0 \\
\theta_1 & (1 - f) \tan \phi_1 & 1.78 \text{ (rad)} \\
\alpha_{1,1} & 45 & 0.0 \\
\sin \alpha_{1,1} & 0.7071 & 0.679 \\
\cos \alpha_{1,1} & 0.7071 & 0.679 \\
\end{array}
\]

\[
\text{d} = \frac{S}{aD} = \frac{1.5166 \times 10^6}{1.5166 \times 10^6} = 1 \text{ (rad)} \\
\sin d = 0.99852396 \\
\cos d = 0.05417377 \\
W = 1 - 2P \cos d = 1.0008482484 \\
\Delta = 1.5157740933 \text{ (rad)} \\
\sin \Delta = 0.998984166 \\
\cos \Delta = 0.05499448 \\
V = 2PV \sin d = 1.5157740933 \\
\cos \alpha_{2,1} = \sin \alpha_{2,1} = 0.26236438 \\
\tan \alpha_{2,1} = M(N \cos \Delta - \sin \theta_1 \sin \Delta \cos \theta_1) = 1.30428056 \\
\tan \phi_2 = (\sin \theta_1 \cos \Delta + N \sin \theta_1 \sin \Delta) \sin \alpha_{2,1} = 2.38377540 \\
\tan \Delta \eta = \sin \Delta \theta \sin \alpha_{1,1} = -1.09577302 \\
\Delta = \Delta \theta - H = 122.18 \text{ (rad)} \\
M = \cos \theta_1 = \cos \theta_1 \sin \alpha_{1,1} = \cos \theta_1 \sin (180 + \alpha_{1,1}) \\
\lambda_1 = 19.00 \\
\lambda_2 = \lambda_1 + \Delta \lambda = 141.38 \text{ (rad)}
\]

CHECK

\[
M = \cos \theta_1 = \cos \theta_1 \sin \alpha_{1,1} = \cos \theta_1 \sin (180 + \alpha_{1,1}) \\
\]

126
INVERSE POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1$, $\phi_2$, $\lambda_1$ to find $S$, $a_1$, $a_2$. Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

CLARKE 1866 SPHEROID $a = 6378206.3$ m $b = 6356583.8$ m

1 - $f = b/a = 0.000034$ $\lambda_f = 165.002764 / 10^3$ $\lambda_f = 165.002764 / 10^3$

$1^2 / 64 = 9179203.39 \times 10^6$

1 radian = 206264.8062 seconds

$$\tan \phi_1 = 2.34747242$$ $\tan \phi_2 = 3.0848058$ $\tan \theta_1 = 1.97472478$ $\tan \theta_2 = 2.77381326$

$$\Delta \theta_m = \frac{1}{2} (\theta_1 - \theta_2) - 2.65252.167 \sin \Delta \theta_m = -4.4750 \text{ arc sec}$$

$$\Delta \phi = \Delta \lambda - \lambda_2 - \lambda_1$$

$$H = \cos^2 \Delta \theta_m - \sin^2 \Delta \theta_m = \cos^2 \Delta \theta_m - \sin^2 \Delta \theta_m$$

$$L = \sin^2 \Delta \theta_m + H \sin^2 \Delta \theta_m$$

$$U = 2 \sin^2 \Delta \theta_m \cos^2 \Delta \theta_m / (1 - L)$$

$$V = 2 \sin^2 \Delta \theta_m \cos^2 \theta_m / L$$

$$X = U + V \cdot 820475010$$

$$Y = U - V \cdot 1000458322$$

$$A = \text{DE} = 0.10926778$$ $C = T - \frac{1}{2} (A - E)$ $1.08362979$ $\text{CHECK C - DE} + \text{AD} / B = T$

$$n_1 = X (A + CX)$$

$$\delta d = \frac{1}{2} (TX - Y)$$

$$S_x = \sin d (T - \delta d)$$

$$S_y = \sin d (A + B - D)$$

$$F = 2Y - E (A - X)$$

$$G = \frac{1}{2} (T + (A^2 / 6)) M$$

$$\Delta \lambda_1 = \frac{1}{2} (\Delta \lambda + Q)$$

$$\nu = \arctan \frac{c_2}{\nu + u}$$

$$V = \arctan \frac{c_2}{u + v}$$

$$a_1 = v + u$$

$$c_1 = c_2$$

$$c_1 = c_2$$

$$a_1 = a_2$$

$$c_1 = c_2$$

$$a_1 = a_2$$

127
DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1, \alpha_{1-2}, S$ to find $\phi_2, \lambda_2, \alpha_{2-1}$; East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

**Clarke 1866 Spheroid** $a = 6378206.4$ m $f = 3.390075283 \times 10^{-2}$

1 radian = 206264.8062 seconds

<table>
<thead>
<tr>
<th>LINE</th>
<th>ORIGIN</th>
<th>TO</th>
<th>TERMINUS (ARCIC)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_1$</td>
<td>10° 0' 0&quot;</td>
<td>$\tan \phi_1 = 0.17632698$</td>
<td>$\tan \theta_1 = (1-f) \tan \phi_1 = 0.17572927$</td>
</tr>
<tr>
<td>$\alpha_{1-2}$</td>
<td>90° 0' 0&quot;</td>
<td>$\sin \theta_1 = 0.722714$</td>
<td>$\cos \theta_1 = 0.68048279 \theta_1 = 0.009^\circ$</td>
</tr>
<tr>
<td>$\sin \alpha_{1-2}$</td>
<td>1</td>
<td>$M = \cos \theta_0 = \cos \theta_1 \sin \alpha_{1-2} = \cos \Theta$</td>
<td>$\theta_0 = \Theta$</td>
</tr>
<tr>
<td>$\cos \alpha_{1-2}$</td>
<td>0</td>
<td>$N = \cos \theta_1 \cos \alpha_{1-2} = 0$</td>
<td>$\sin \theta_0 = \sin \beta_1$</td>
</tr>
<tr>
<td>$c_1 = fM = 0.00399318$</td>
<td></td>
<td>$D = (1 - c_1)(1 - c_1 - c_1 M)$</td>
<td>$0.976607849$</td>
</tr>
<tr>
<td>$c_2 = \frac{1}{2}(1 - M^2)f = 0.00025390272$</td>
<td></td>
<td>$P = c_2 (1 + \frac{1}{2}c_1 M)/D$</td>
<td>$0.000255514966$</td>
</tr>
<tr>
<td>$\cos \alpha_1 = \sin \theta_1 / \sin \theta_0$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$d = S/aD = 1.5188739576$</td>
<td></td>
<td>(d = 0^\circ 01' 50.847)</td>
<td>$S = 0.6529773.366$ m</td>
</tr>
<tr>
<td>$\sin d = 0.9966751$</td>
<td></td>
<td>(u = 2(\alpha_1 - d) = 12^\circ 03' 4.166\sin u = -0.10345948)</td>
<td></td>
</tr>
<tr>
<td>$\cos d = 0.0579939$</td>
<td></td>
<td>(W = 1 - 2P \cos u + 1.00005075109 \cos u = -0.99463367)</td>
<td></td>
</tr>
<tr>
<td>$V = \cos u \cos d - \sin u \sin d \cdot \cos d$</td>
<td></td>
<td>$Y = 2PVW \sin d$</td>
<td>$+0.002826399$</td>
</tr>
<tr>
<td>$X = c_1 \sin d \cos d (2V^2 - 1) - 38.\cos \left(\frac{10}{\sqrt{29.1}}\right)$</td>
<td></td>
<td>(\Delta d = d + X - Y)</td>
<td>$1.5188732177$ (rad)</td>
</tr>
<tr>
<td>$\sin \Delta d = 0.9965738$</td>
<td></td>
<td>$\cos \Delta d = 0.057061)</td>
<td>$d = 0^\circ 01' 50.294$</td>
</tr>
<tr>
<td>$\cos \Sigma_0 = 0.50510$</td>
<td></td>
<td>$\Sigma_0 = 2\alpha_1 - \Delta d$</td>
<td>$-60^\circ$</td>
</tr>
<tr>
<td>$\tan \alpha_{2-1} = M(N \cos \Sigma_0 - \sin \theta_1 \sin \Delta d) = -5.6922288$</td>
<td></td>
<td>$\alpha_{2-1} = 27.957^\circ 13.198$</td>
<td></td>
</tr>
<tr>
<td>$\tan \phi_2 = -\sin \theta_1 \cos \Delta d + N \sin \Delta d) \sin \alpha_{2-1}$</td>
<td></td>
<td>$\tan \phi_2 = 0.08999659$</td>
<td>$-9849.4785$</td>
</tr>
<tr>
<td>$\sin \phi_2 = 0.300.55670$</td>
<td></td>
<td>$\phi_2 = 0.300.55670$</td>
<td></td>
</tr>
<tr>
<td>$\tan \Delta \eta = \frac{\sin \Delta d \sin \alpha_{2-1}}{\cos \theta_1 \cos \Delta d - \sin \theta_1 \sin \Delta d \cos \alpha_{2-1}} = 19.05279523$</td>
<td></td>
<td>$\Delta \eta = 0^\circ 04' 31.349$</td>
<td></td>
</tr>
<tr>
<td>$H = c_1 (1 - c_1) \Delta d - c_1 c_2 \sin \Delta d \cos \Sigma_0 = 0.0050507987$ (rad)</td>
<td></td>
<td>$H = 17^\circ 26.028$</td>
<td></td>
</tr>
<tr>
<td>$\Delta \lambda = \Delta \eta - H = 0^\circ 01' 50.261$</td>
<td></td>
<td>$\lambda_1 = -18^\circ$</td>
<td></td>
</tr>
<tr>
<td>CHECK</td>
<td></td>
<td>$M = \cos \theta_0 = \cos \theta_1 \sin \alpha_{1-2} = \cos \theta_2 \sin (180 + \alpha_{2-1})$</td>
<td>$\lambda_2 = \lambda_1 + \Delta \lambda = 0^\circ 01' 50.261$</td>
</tr>
</tbody>
</table>

128
Inverse Position Computation Form for Long Lines. Given $\phi_1$, $\lambda_1$; $\phi_2$, $\lambda_2$ to find $S$, $a_1$, $a_2$. Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

### Clarke 1866 Spheroid

\[ a = 6378206.4 \text{ m} \]
\[ b = 6356751.9 \text{ m} \]
\[ 1 - f = b/a = 0.0068703 \]
\[ f^2/64 = 0.00179 

1 radian = 206264.8062 seconds

| $\phi_1$ | 0° | 0° | 1. ORIGIN (Lカンヒ) | $\lambda_1$ | 1° | 0° |
| $\phi_2$ | 0° | 30° 55.39 | 2. TERMINUS | $\lambda_2$ | 42° | 47° 05.259 |
| tan $\phi_1$ | 176.3698 | 1. always west of 2. | $\Delta \lambda = \lambda_3 - \lambda_1$ | 86° 47° 05.259 |
| $\theta_1$ | 0.00899058 | tan $\theta = (1 - f) \tan \phi$ | $\Delta \lambda_3 = \frac{1}{2} \Delta \lambda$ | 42° 23° 31.629 |
| $\theta_2$ | 9° | 0° 49.787 | $\sin \Delta \lambda = \pm 0.68699.019$ |
| $\theta_m = \frac{1}{2} (\theta_1 + \theta_2)$ | 5° 14° 48.23 | $\sin \theta_m = +0.91912.039$ | $\cos \theta_m = +0.3958.049$ |
| $\Delta \lambda_m = \frac{1}{2} (\theta_1 - \theta_2)$ | -4° 35° 57.57 | $\sin \Delta \lambda_m = -0.0824.0011$ | $\cos \Delta \lambda_m = +0.9965.9939$ |
| $H = \cos^2 \Delta \lambda_m - \sin^2 \theta_m = \cos^2 \theta_m - \sin^2 \theta_m = 0.98967.6391$ | 1 - L = 52° 19° 47.28 |

\[ L = \sin^2 \Delta \lambda_m + H \sin^2 \Delta \lambda_m = 42° 02° 31.15 \]

\[ \cos d = 1 - 2L = 0.5678958 \]

\[ V = 2 \sin^2 \Delta \lambda_m \cos \theta_m / (1 - L) = 0.03135.81594 \]

\[ d = 86° 47° 05.259 \]

\[ U = 2 \sin^2 \Delta \lambda_m \cos \Delta \lambda_m / (1 - L) = 0.0478 \]

\[ d = 90° 00° 00'00'' \]

\[ X = U + V = 0.0591.3990 \]

\[ T = d / \sin d = 15842.3927 \]

\[ E = 2 \cos d = 1153.7911 \]

\[ Y = U - V = 0.0380.4679 \]

\[ D = 4T = 9.1896.5694 \]

\[ B = 2D = 18.3763.1388 \]

\[ A = D - E + AD/B \]

\[ a = 6378206.4 \text{ m} \]

\[ b = 6356751.9 \text{ m} \]

\[ 1 - f = b/a = 0.0068703 \]

\[ f^2/64 = 0.00179 \]

\[ 1 \text{ radian} = 206264.8062 \text{ seconds} \]

### Check

\[ C = T - \frac{1}{2} (A - E) = 1.05265.3863 \]

\[ 1 \text{ radian} = 206264.8062 \text{ seconds} \]

\[ n_1 = X / (A + CX) = 006034597 \]

\[ n_2 = Y / (B + EY) = 05185.8541 \]

\[ n_3 = DXY / (A + CX) = 00155.5825 \]

\[ \delta / d = (TX - Y) / (T - X) = 0.000796.239 \]

\[ \delta / d = (f^2/64) / (n_1 - n_3) = 1.32910 \]

\[ S_1 = \sin (T - \delta / d) = 0.965977.115 \]

\[ S_2 = \sin (T - \delta / d) = 0.965977.115 \]

\[ F = 2Y - E (4 - X) = 0.4450.25479 \]

\[ M = 3T - (20T - A) X - (B + 4Y) Y = 0.28726.0057 \]

\[ G = \sin (T - \delta / d) = 0.005378.2128 \]

\[ Q = 17.36.58 \]

\[ \Delta \lambda_m = \frac{1}{2} (\Delta \lambda + Q) = 22° 17' \]

\[ 0.55 \Delta \lambda_m = 90° 07' 15.40 \]

\[ v = \arctan (c_3 / c_2) = \arctan \left( \frac{c_3}{c_2} \right) \]

\[ u = \arctan (c_3 / c_2) = \arctan \left( \frac{c_3}{c_2} \right) \]

\[ a_1 = v - u = 80° 00° 00.000 \]

\[ c_2 = c_1 / c_2 = a_1 \]

\[ a_1 = 360° \]

\[ a_2 = 360° - a_1 \]

\[ a_3 = 279° 57' 13.18' \]

\[ a_4 = 180° + a_1 \]

\[ a_5 = 180° + a_1 \]

129
DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given \( \phi_1, \lambda_1, \alpha_{1-2}, S \) to find \( \phi_2, \lambda_2, \alpha_{2-3} \). East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

**CLARKE 1866 SPHEROID**

<table>
<thead>
<tr>
<th>LINE</th>
<th>ORIGIN</th>
<th>TO</th>
<th>TERMINUS (AC1C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \phi_1 )</td>
<td>( 00,00 )</td>
<td>( \tan \phi_1 )</td>
<td>( 2390.0963 )</td>
</tr>
<tr>
<td>( \alpha_{1-2} )</td>
<td>( 00,00 )</td>
<td>( \sin \theta_1 )</td>
<td>( 0.4458618 )</td>
</tr>
<tr>
<td>( \sin \alpha_{1-2} )</td>
<td>1</td>
<td>( M = \cos \theta_0 - \cos \delta_1 \sin \alpha_{1-2} )</td>
<td>( = \cos \theta_1 \theta_0 - \theta_1 )</td>
</tr>
<tr>
<td>( \cos \alpha_{1-2} )</td>
<td>0</td>
<td>( N = \cos \theta_1 \cos \alpha_{1-2} )</td>
<td>( = \sin \theta_1 )</td>
</tr>
<tr>
<td>( c_1 = N )</td>
<td>( 0026.0058733 )</td>
<td>( D = (1 - c_2)(1 - c_1 - c_2 \sin \theta_1) )</td>
<td>( 0.9970021042 )</td>
</tr>
<tr>
<td>( c_2 = \frac{1}{2}(1 - M^2)f )</td>
<td>( 000248.7757 )</td>
<td>( P = c_2 (1 + \frac{1}{2}c_1 M) / D )</td>
<td>( 0.0035004975 )</td>
</tr>
<tr>
<td>( \cos \alpha_1 = \sin \theta_1 / \sin \theta_0 )</td>
<td>1</td>
<td>( \alpha_1 )</td>
<td>0</td>
</tr>
<tr>
<td>( d = S / D )</td>
<td>151787.76475 (rad)</td>
<td>( d )</td>
<td>86.58 17477</td>
</tr>
<tr>
<td>( \sin d + 2.99260.595 )</td>
<td>( u = 2(\alpha_1 - d) )</td>
<td>( = 17.56 54.884 )</td>
<td></td>
</tr>
<tr>
<td>( \cos d + 0.05778414 )</td>
<td>( W = 1 - 2P \cos u + \frac{1.000696381}{\cos u} )</td>
<td>( = -9947.2767 )</td>
<td></td>
</tr>
<tr>
<td>( V = \cos u \cos d - \sin u \sin d )</td>
<td>( \cos )</td>
<td>( Y = 2PV \sin d )</td>
<td>( = 0000.869304 )</td>
</tr>
<tr>
<td>( X = c_1 \sin d \cos d )</td>
<td>( 0.138 )</td>
<td>( \Delta )</td>
<td>( = 0.157952057 )</td>
</tr>
<tr>
<td>( \sin \Delta )</td>
<td>2.9926 0.595</td>
<td>( \cos \Delta + 0.0578 2.103 )</td>
<td>( \Delta )</td>
</tr>
<tr>
<td>( \cos \sigma_0 = \cos \Delta )</td>
<td>( 0.3050 )</td>
<td>( \Sigma = 2\alpha_1 - \Delta \sigma )</td>
<td>( = -\Delta \sigma )</td>
</tr>
<tr>
<td>( \tan \alpha_{1-1} = M / (N \cos \Delta \sigma - \sin \theta_1 \sin \Delta \sigma) )</td>
<td>( = -1.1974 1141 )</td>
<td>( \alpha_{1-1} )</td>
<td>( 309.51 52.520 )</td>
</tr>
<tr>
<td>( \tan \phi_2 = \frac{1}{\sin \theta_1 \cos \Delta \sigma + \sin \theta_0 \sin \Delta \sigma \cos \alpha_{1-2}} )</td>
<td>( 4.0340 1582 )</td>
<td>( \sin \alpha_{1-2} )</td>
<td>( = -1.7675 5875 )</td>
</tr>
<tr>
<td>( \cos \Sigma = \cos \alpha_{1-1} )</td>
<td>( 4.246.647475 )</td>
<td>( \Delta \eta )</td>
<td>( \theta_1 )</td>
</tr>
<tr>
<td>( \sin \Delta = \sin \theta_0 \sin \alpha_{1-2} )</td>
<td>( = 24.0 35.854 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( H = c_1 (1 - c_2) \Delta \sigma - c_1 c_2 \sin \Delta \sigma \cos \Sigma + 0.003946137 ) (rad)</td>
<td>( H )</td>
<td>( 1.3 33.950 )</td>
<td></td>
</tr>
<tr>
<td><strong>CHECK</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( M = \cos \theta_0 = \cos \theta_1 \sin \alpha_{1-1} = \cos \theta_2 \sin (180 + \alpha_{2-1}) )</td>
<td>( \lambda_1 )</td>
<td>( 18.0 )</td>
<td></td>
</tr>
<tr>
<td>( \lambda_2 = \lambda_1 + \Delta \lambda )</td>
<td>( 18.0 )</td>
<td>( 27.01.14 )</td>
<td></td>
</tr>
</tbody>
</table>

1 radian = 206264.8062 seconds

130
INVERSE POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1$, $\lambda_1$, $\phi_2$, $\lambda_2$ to find $S$, $\alpha_2$, $\alpha_3$. Azimuths clockwise from north, east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

**Clarke 1866 Spheroid:** $a = 6378160.4$ m, $b = 6356753.8$ m

1 - $f = (b/a)^2 = 0.006694379$.

$\lambda_3 = \phi_2 - \lambda_1$.

**Terminus:**

1. **Origin (Halic)**: $\phi_1 - \phi_2$.
2. **Terminus**: $\phi_2 - \phi_1$.

$\Delta \lambda = \lambda_2 - \lambda_1$, 89.27 00.115.

$\Delta \lambda_m = \frac{1}{2} \Delta \lambda$, 44.62 40.558.

$\cos \theta_m = \frac{0.5572}{1.3571}$.

$\cos \theta_m = +0.9240 60.00$.

$\Delta \Delta_m = \frac{1}{2} (\phi_2 - \phi_1)$ 85.75 55.389.

$\Delta \Delta_m = -125.75 55.755$.

$\cos \Delta \Delta_m = +0.9415 55.40$.

$H = \cos^2 \Delta \Delta_m - \sin^2 \Delta \Delta_m = \cos^2 \Delta \Delta_m - \sin^2 \Delta \Delta_m = 0.5372 18.547$.

$L = \sin^2 \Delta \Delta_m + H \sin^2 \Delta \Delta_m = 0.4720 04.53$. $\cos d = 1 - 2L = 0.6554 82.17$.

$U = 2 \sin^2 \Delta \Delta_m \cos^2 \Delta \Delta_m (1 - L) = 48202 73.873$.

$d = 42.27 55.413$.

$V = 2 \sin \Delta \Delta_m \cos \Delta \Delta_m / (L - \sin^2 \Delta \Delta_m / 2) = 1.5492 35.27$.

$\phi = \frac{180}{\pi} \arctan \left( \frac{1}{\tan \alpha} \right)$.

$\phi = \frac{180}{\pi} \arctan \left( \frac{1}{\tan \alpha} \right)$.

$\phi = \frac{180}{\pi} \arctan \left( \frac{1}{\tan \alpha} \right)$.

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$\phi = \frac{180}{\pi} \arctan \left( \frac{1}{\tan \alpha} \right)$.
DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given \( \phi_1, \lambda_1, \alpha_{1,2}, \) to find \( \phi_2, \lambda_2, \alpha_{2,3}. \) East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

### Clarke 1866 Spheroid

![Image of Clarke 1866 Spheroid](image)

1 radian = 206264.8062 seconds

<table>
<thead>
<tr>
<th>( \Phi_1 )</th>
<th>( \Theta_1 )</th>
<th>( \Theta_2 )</th>
<th>( \Theta_3 )</th>
<th>( \Theta_4 )</th>
<th>( \Theta_5 )</th>
<th>( \Theta_6 )</th>
<th>( \Theta_7 )</th>
<th>( \Theta_8 )</th>
<th>( \Theta_9 )</th>
<th>( \Theta_{10} )</th>
<th>( \Theta_{11} )</th>
<th>( \Theta_{12} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

### Example Calculation

\[
\begin{align*}
\phi_1 &= 17.42^\circ \\
\lambda_1 &= 33.25^\circ \\
\alpha_{1,2} &= 0.76^\circ \\
\sin \alpha_{1,2} &= 0.013 \\
\cos \alpha_{1,2} &= 0.999 \\
\sin \lambda_1 &= 0.566 \\
\cos \lambda_1 &= 0.826 \\
\m &= 0.0000 \\
\cos \phi_2 &= 0.9690 \\
\sin \phi_2 &= 0.2530 \\
\Delta \lambda &= 6.00^\circ \\
\Delta \phi &= 0.50^\circ \\
\lambda_2 &= 8.06^\circ \\
\alpha_{2,3} &= 3.50^\circ \\
\sin \alpha_{2,3} &= 0.061 \\
\cos \alpha_{2,3} &= 0.998 \\
\end{align*}
\]

### Check

\[
M = \cos \phi_2 \cos \lambda_1 \sin \alpha_{1,2} = \cos \phi_3 \sin \alpha_{2,3} \quad (180 + \alpha_{2,3})
\]

\[
\lambda_2 = \lambda_1 + \Delta \lambda
\]

\[
\begin{align*}
\text{LINE} &\quad \text{ORIGIN TO TERMINUS (A1C)} \\
\phi_1 &= 17.42^\circ \\
\lambda_1 &= 33.25^\circ \\
\alpha_{1,2} &= 0.76^\circ \\
\sin \alpha_{1,2} &= 0.013 \\
\cos \alpha_{1,2} &= 0.999 \\
\sin \lambda_1 &= 0.566 \\
\cos \lambda_1 &= 0.826 \\
\m &= 0.0000 \\
\cos \phi_2 &= 0.9690 \\
\sin \phi_2 &= 0.2530 \\
\Delta \lambda &= 6.00^\circ \\
\Delta \phi &= 0.50^\circ \\
\lambda_2 &= 8.06^\circ \\
\alpha_{2,3} &= 3.50^\circ \\
\sin \alpha_{2,3} &= 0.061 \\
\cos \alpha_{2,3} &= 0.998 \\
\end{align*}
\]
INVERSE POSITION COMPUTATION FORM FOR LONG LINES. Given \( \phi_1, \lambda_1; \phi_2, \lambda_2 \) to find \( S, \phi, -1, \phi, -2, \phi, -1, \).

Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

\[
\begin{align*}
\text{CLARKE 1866 SPHEROID } & a = 6378206.4 \text{ m } b = 6356583.8 \text{ m} \\
1 - f = b/a &= 0.99664929417 \\
1 + f = b/a &= 1.99602764510^{-3} \\
1 - f^2 &= 0.99573203791 \times 10^{-6} \\
\end{align*}
\]
1 radian = 206264.8062 seconds

\[
\begin{align*}
\phi_1 &= 70.00^\circ \\
\phi_2 &= 55.00^\circ \\
\phi_1 &= \text{ORIGIN (N1K)} \lambda_1 &= 120.00^\circ \\
\tan \phi_1 &= 2.7924 \times 2.7924 \quad 1. \text{ always west of 2.} \\
\tan \phi_2 &= 0.0519 \times 0.0519 \\
\tan \phi &= (1 - f) \tan \phi \\
\theta_2 &= \arctan \left( \frac{\tan \phi_2}{1 + \tan^2 \phi_2} \right) \\
\tan \theta_2 &= \frac{1}{\cos^2 \theta_2} \\
\Delta \lambda &= \lambda_2 - \lambda_1 \\
\Delta \lambda_m &= \frac{1}{2} \Delta \lambda \\
\theta_\text{m} &= \frac{1}{2} (\theta_1 + \theta_2) \\
\tan \theta_\text{m} &= \frac{1}{\cos^2 \theta_\text{m}} \\
\Delta \theta_\text{m} &= \Delta \theta_\text{m} \\
\Delta \theta &= \frac{1}{2} \Delta \theta_\text{m} \\
H &= \cos^2 \Delta \theta - \sin^2 \theta_\text{m} - \cos^2 \theta_\text{m} \quad \text{cos } \phi_1 = 0.9047 \\
L &= \sin^2 \theta_\text{m} + \sin^2 \Delta \theta_\text{m} \\
U &= \sin \theta_\text{m} \cos \Delta \theta_\text{m} (1 - L) \\
V &= \sin \theta_\text{m} \cos \Delta \theta_\text{m} (1 - L) \\
X &= U + \frac{1}{2} \nu_\text{m} \cos \Delta \theta_\text{m} (1 - L) \\
Y &= U - \frac{1}{2} \nu_\text{m} \cos \Delta \theta_\text{m} (1 - L) \\
A = \nu_\text{m} &= 1.76^\circ \times 1.76^\circ \\
B &= 2 \nu \sin \nu_\text{m} \tan \nu_\text{m} \\
\delta_1 &= \nu \sin \nu_\text{m} \tan \nu_\text{m} \\
\delta_2 &= \nu \sin \nu_\text{m} \tan \nu_\text{m} \\
\delta &= \frac{1}{2} (\delta_1 + \delta_2) \\
S_1 &= \sin \delta \sin (T - \delta, d) \\
S_2 &= \sin \delta \sin (T - \delta, d) \\
F &= 2Y - E - X \\
G &= \frac{1}{2} (\nu \sin \nu_\text{m}) \\
\Delta \lambda_\text{m} &= \frac{1}{2} (\Delta \lambda + Q) \\
\tan \Delta \lambda_\text{m} &= \frac{1}{2} (\Delta \lambda + Q) \\
\nu &= \arctan l \nu_\text{m} \\
\beta &= \arctan l \nu_\text{m} \\
\Delta \lambda_\nu &= \frac{1}{2} (\Delta \lambda + Q) \\
\tan \Delta \lambda_\nu &= \frac{1}{2} (\Delta \lambda + Q) \\
\rho &= \nu + u \\
\alpha &= \nu + u \\
\Delta \theta &= \frac{1}{2} (\Delta \lambda + Q) \\
\Delta \theta &= \frac{1}{2} (\Delta \lambda + Q) \\
\end{align*}
\]
CONTROL COMPUTATIONS FOR THE HEMISPHEROIDAL GEODESIC CONTAINING AN ACIC GIVEN ARC

These are the control computations for the geodesic as presented in Figure 26. To completely determine the configuration we need compute only the constants A, B, C, D, E, F from equations (49); \( \Delta \alpha_1, \Delta \alpha_2 \) from equations (48); \( \Delta \lambda_3, \Delta \lambda_4 \) and \( \Delta \lambda_5, \Delta \lambda_6 \) from equations (47); and \( \Delta \alpha_0, \Delta \alpha_4 \) from equations (54). (The equations cited are from Appendix 1). These will provide the check equations:

\[
\begin{align*}
S_4 &= S_3 + S_4 = (1/2)S_0 - S_1 - S_3, \\
\Delta \lambda_4 &= \Delta \lambda_3 - \Delta \lambda_5 - \Delta \lambda_6 = (1/2)\Delta \lambda_0 - \Delta \lambda_1 - \Delta \lambda_2.
\end{align*}
\]

Now

\[
\begin{align*}
f &= 0.003390075283, \sin \theta_0 = 0.97013371, \cos \theta_0 = 0.24257076, c_1 = f \cos \theta_0 = 0.0008223331378, \\
c_2 &= (1/4) \sin^2 \theta_0 = 0.0007976503177, c_3 = 1 + c_1 \cos \theta_0 = 1.001999474, c_4 = c_2 + c_3 \\
&= 1.000997124,
\end{align*}
\]

\[
\begin{align*}
A &= c_1(1 - c_2c_4) = 0.0082167655, B = (1/2)c_1c_2c_3 = 0.3280325 \times 10^{-4}, C = (1/4)c_1c_2^2 \\
&= 0.1308 \times 10^{-4}, \\
D &= 2 + c_2(c_1^2 + c_2^2) - (1 + c_4)c_4 - c_2 = 0.9982060223, E = 0/2c_4[2 + c_2(c_3 - 1) - c_4]\ \\
&= 0.007977296351, \\
F &= (1/4)c_2^2(2c_4 - 1) = 0.1593787 \times 10^{-4}.
\end{align*}
\]

From equations (150) (Appendix 1) we have:

\[
\begin{align*}
\theta_1 &= 69^\circ 56' 14.590, \theta_2 = 17^\circ 05' 21.296, \theta_0 = 75^\circ 57' 42.053 \\
\sin \theta_1 &= 0.93931830, \sin \theta_2 = 0.29386097, \sin \theta_0 = 0.97013371 \\
\cos \theta_1 &= 0.34304686, \cos \theta_2 = 0.95584817, \cos \theta_0 = 0.24257076 \\
\tan \theta_1 &= 2.73816326, \tan \theta_2 = 0.30743478, \tan \theta_0 = 3.99938439 \\
\sin \theta_1 / \sin \theta_0 &= 0.96823591, \sin \theta_2 / \sin \theta_0 = 0.30290770 \\
\tan \theta_1 / \tan \theta_0 &= 0.68464618, \tan \theta_2 / \tan \theta_0 = 0.07687053
\end{align*}
\]

From these last four numbers:

\[
\begin{align*}
\sigma_1 &= \arccos(0.96823591) = 14^\circ 28' 47.231 = 0.2427199475 \text{ radians} \\
\eta_1 &= \arccos(0.68464618) = 46^\circ 47' 31.966 = 0.8166781775 \text{ radians} \\
\beta_1 &= \arcsin(0.96823591) = 75^\circ 31' 12.769 = 1.3180763796 \text{ radians} \\
\gamma_1 &= \arcsin(0.68464618) = 43^\circ 12' 28.034 = 0.7541181497 \text{ radians} \\
\beta_2 &= \arccos(0.30290770) = 77^\circ 37' 56.390 = 0.307422231 \text{ radians} \\
\gamma_2 &= \arccos(0.07687053) = 98^\circ 24' 31.342 = 0.769464374 \text{ radians} \\
\sin 2\sigma_1 &= 0.48419238, \sin 2\beta_1 = 0.48419238, \sin 2\beta_2 = 0.57735414 \\
\sin 4\sigma_1 &= 0.84729944, \sin 4\beta_1 = 0.84729944, \sin 4\beta_2 = 0.94281220
\end{align*}
\]

We can now make the computations, with \( a = 6378206.4 \text{ meters} \) and \( \pi = 3.1415926536 \):

\[
\begin{align*}
\eta_1 &= 0.8166781775 \\
A &= 0.2522665735 \\
-B &= 0.002067541 \\
E/2 &= 0.003862546 \\
D &= 0.8164705234 \\
&= 0.2526526421 \\
F &= 0.8164706822 \\
&= 0.2526526931 \\
C &= 0.8164707123
\end{align*}
\]
\[ \Delta \lambda_1 (\text{rad}) \cdot 0.8164706821 \]
\[ \Delta \lambda_1 \cdot 46^\circ 46' 49.167'' \]
\[ S_1 \cdot 1611471.024 \text{ meters} \]
\[ \gamma_2 \cdot 0.7541181497 \]
\[ - A \delta_2 - 0.0010830332 \]
\[ - E \sin 2\delta_2 - 0.0003862546 \]
\[ 1.3153255254 \]
\[ - B \sin 2\delta_2 - 1.3153256604 \]
\[ - C \sin 4\delta_2 + \]
\[ \Delta \lambda_2 (\text{rad}) \cdot 0.7530349578 \]
\[ \Delta \lambda_2 \cdot 43^\circ 08' 44.610'' \]
\[ S_2 \cdot 8389418.545 \text{ meters} \]
\[ \gamma_3 \cdot 0.0769464374 \]
\[ - A \delta_3 - 0.0002528648 \]
\[ - E \sin 2\delta_3 - 0.0004605725 \]
\[ 0.3071901404 \]
\[ - B \sin 2\delta_3 - 0.3067294176 \]
\[ - C \sin 4\delta_3 + \]
\[ \Delta \lambda_3 (\text{rad}) \cdot 0.7530349578 \]
\[ \Delta \lambda_3 \cdot 4^\circ 23' 39.146'' \]
\[ S_3 \cdot 1956383.534 \text{ meters} \]
\[ \Delta \lambda_4 = \Delta \lambda_2 - \Delta \lambda_3 = 0/2 \lambda \delta_0 - \Delta \lambda_1 - \Delta \lambda_3 = 38^\circ 45' 05.464'' \]
\[ S_4 = S_1 - S_3 = 0/2 \lambda \delta_0 - S_1 - S_3 = 6433035.010 \text{ meters} \]
\[ 1 - A = 0.9991783229, \Delta \lambda_0 = \pi (1 - A) = 3.1390112787 \text{ (rad)} \]
\[ = 179^\circ 51' 07.553'' \]
\[ \Delta \lambda_0/2 = 89^\circ 55' 33.777, S_0 = a \pi D = 20001779.136 \text{ m}, S_0/2 = 10000889.568 \text{ m} \]

As an overall check, we compute from formulae (48), Appendix 1, the values \( \Delta \lambda = \Delta \lambda_1 + \Delta \lambda_4 \).
\[ S = S_1 + S_4 \]

\[ \Delta \lambda = \Delta \lambda_1 + \Delta \lambda_4 = 85^\circ 31' 54.631'' + 4^\circ 23' 39.146'' = 89^\circ 55' 33.777'' = 0/2 \Delta \lambda_0. \]

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\[ S_1 + S_4 + S_3 = 8044506.036 + 1956383.534 = 10000889.570 = \frac{1}{2}S_0, \]
which gives a flat check for longitude, and length within .002 meter.

**Computation of \( N_1, N_2 \) from Figure 26. (Inverse solution)**

Formulae are from equations (149), Appendix 1. We have

\[
f = 0.003390075283, \quad a = 6378206.4 \text{ meters}, \quad \Delta \lambda_0 = 179^\circ 51' 07'' 554 = 3.1390112787 \text{ radians},
\]

\[
\pi = 3.1415926536, \quad D = \frac{1}{f} \left[ 1 + \frac{1}{4}f + \frac{2}{4}f^2 \right] = 295.22912379,
\]

\[
u = \frac{(1/4)f - ((f/4)^2} = 0.0008468005326, \quad v = D(1 - \Delta \lambda_0/a) = 0.2425830253,
\]

\[
\cos \theta_0 = v - uv^3 = 0.24257094, \quad \theta_0 = 75^\circ 57' 42.015, \quad a_{1,2} = 90^\circ - \theta_0 = 14^\circ 02' 17.985,
\]

\[
a_{2,-1} = 270^\circ + \theta_0 = 345^\circ 57' 42.015, \quad A = 1 + \cos^2 \theta_0 = 1.0588406609,
\]

\[
B = (1 + 3 \cos^2 \theta_0)(1 - \cos^2 \theta_0) = 1.1072946516,
\]

\[
C = (1 + 2 \cos^2 \theta_0 + 5 \cos^4 \theta_0)(1 - \cos^2 \theta_0) = 1.0682087334,
\]

\[
S_0 = a_1 \left[ 1 - (f/4)A + ((f/4)^2)B + 2((f/4)^3)C \right] = 20001779.127 \text{ meters}.
\]

Alternatively, when one has \( \cos \theta_0 \), \( S_0 \) may be computed from equations (54) after computing \( D \) from equations (49), Appendix 1. Since there are two reverse solutions for the geodesic, node to node, the azimuths of the second solution are \( a'_{1,2} = 90^\circ + \theta_0 = 165^\circ 57' 42.015, \quad a'_{1,2} = 270^\circ - \theta_0 = 194^\circ 02' 17.985. \)

Now from (150), Appendix 1, \( \theta_0 = 75^\circ 57' 42.053 \); from (152), (154) respectively \( S_0 = 20001779.136 \text{ meters}, \quad a_{1,2} = 14^\circ 02' 17.947. \) Hence the computed values by use of equations (149), Appendix 1, are within the criteria adopted initially.
DIRECT AND INVERSE LINE COMPUTATIONS OVER A HEMISPHEROIDAL
GEODESIC CONTAINING AN ACIC 6000 MILE ARC
(Clarke 1866 ellipsoid—See Appendix 1, Figure 26)
DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1, \alpha_{1-1}, S$ to find $\phi_2, \lambda_2, \alpha_{2-1}$. East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

<table>
<thead>
<tr>
<th>LINE</th>
<th>VERT. 1</th>
<th>TO</th>
<th>P</th>
<th>(V,P)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi$</td>
<td>-76 00 36.41</td>
<td>$\tan \phi_1 = 0.1298$</td>
<td>$\tan \theta_1 = (1 - f) \tan \phi_1 = 0.9992$</td>
<td>$\alpha_{1-1}$</td>
</tr>
<tr>
<td>$\alpha_{1-1}$</td>
<td>90 00 00</td>
<td>$\sin \theta_1 = 0.9701$</td>
<td>$\cos \theta_1 = 0.2767$</td>
<td>$\theta_1 = 25 57 42.053$</td>
</tr>
<tr>
<td>$\sin \alpha_{1-1}$</td>
<td>1</td>
<td>$M = \cos \theta_0 = \cos \theta_1 \sin \alpha_{1-1}$</td>
<td>$= 2425 7076 \theta_0 + 175 57 42 053$</td>
<td></td>
</tr>
<tr>
<td>$\cos \alpha_{1-1}$</td>
<td>0</td>
<td>$N = \cos \theta_1 \cos \alpha_{1-1}$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$c_1 = \sqrt{M}$</td>
<td>0.00022333478</td>
<td>$D = (1 - c_1)(1 - c_1 - c_1 M) = 0.00020607$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$c_2 = \sqrt{(1 - M^2) C_2}$</td>
<td>0.000193650237</td>
<td>$P = c_2 (1 + \frac{1}{2} c_1 M) / D = 0.000193650237$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\cos \alpha_1 = \sin \theta_1 / \sin \theta_0 = 1$</td>
<td>180 00 00</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$d = \frac{S}{D}$</td>
<td>0.25210676.14 (rad)</td>
<td>$d = 0.20020.72$</td>
<td>$S = 161 14 76.024$</td>
<td></td>
</tr>
<tr>
<td>$\sin d = 0.2504 12.94$</td>
<td>$u = 2(\alpha_1 - d) = 280 59 44.96 \sin u = 4478.691$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\cos d = 0.961 / 391.7$</td>
<td>$W = 1 - 2P \cos u + 0.9986202143 \cos u = 0.8755 40.72$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$V = \cos u \cos d - \sin u \sin d = 0.961 23.91$</td>
<td>$Y = 2PVW \sin d = 0.000 356.948$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$X = c_2 \sin d \cos (2V^3 - 1)M = 1.899 \times 10^{-6}$</td>
<td>$\Delta d = d + X - Y = 0.000 23.47$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sin \Delta d = 0.000 38.45$</td>
<td>$\cos \Delta d = 0.961 \times 39.71$</td>
<td>$\Delta d = 14 20 43.23$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\cos \Delta d = 0.654$</td>
<td>$\Delta \alpha = 2\alpha_1 - \Delta d = 3\alpha_1 - \Delta d$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\tan \alpha_{1-1} = M/(N \cos \Delta d - \sin \theta_1 \sin \Delta d)$</td>
<td>$= 3.9900$</td>
<td>$\alpha_{1-1}$</td>
<td>225 00 00.014</td>
<td></td>
</tr>
<tr>
<td>$\tan \phi_2 = (\sin \theta_1 \cos \Delta d + N \sin \Delta d)$</td>
<td>$\sin \alpha_{1-1} = 2.7947 7257$</td>
<td>$\sin \alpha_{1-1}$</td>
<td>270 71 06.82</td>
<td></td>
</tr>
<tr>
<td>$\sin \alpha_{1-1}$</td>
<td>$\frac{\sin \alpha_{1-1}}{\cos \theta_1 \sin \Delta d - \sin \theta_1 \sin \Delta d \cos \alpha_{1-1}}$</td>
<td>$= 1.064 / 0172$</td>
<td>$\Delta \eta = 0.44 47 31.954$</td>
<td></td>
</tr>
<tr>
<td>$\tan \Delta \eta = \sin \Delta d \sin \alpha_{1-1} + 1.064 / 0172$</td>
<td>$\Delta \eta$</td>
<td>96 47 31.954</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$H = c_1 (1 - c_1) \Delta d - c_1 c_2 \sin \Delta d \cos \Sigma = 0.000 107 / 954$</td>
<td>(rad) $H$</td>
<td>23.799</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Delta \lambda = \Delta \eta - H = 46 46 49.166$</td>
<td>$\lambda_1 = 151 07 18.357$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CHECK</td>
<td>$M = \cos \theta_0 = \cos \theta_1 \sin \alpha_{1-1} = \cos \theta_1 \sin (180 + \alpha_{2-1})$</td>
<td>$\lambda_2 = \lambda_1 + \Delta \lambda = 104 17 29.337$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Inverse Position Computation Form for Long Lines. Given \( \phi_1, \lambda_1; \phi_2, \lambda_2 \) to find \( S, \alpha_1, \alpha_2 \).

Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

\[
\text{CLARKE 1866 SPHEROID: } \frac{a}{b} = 0.9966992472 \quad \lambda_1 = \frac{0.0166503765}{b} = \frac{0.000847518205}{a}
\]

\[
f^2/64 = 1795.7204 \times 10^{-6}
\]

1 radian = 206264.8062 seconds

\[
\begin{align*}
\phi_1 & = -26 \quad 00 \quad 36.41^\prime \\
\phi_2 & = -20 \quad 00 \quad 00^\prime \\
\tan \phi_1 & = 1 \quad \text{always west of 2.} \\
\tan \phi_2 & = 1 \\
\theta_1 & = 69 \quad 56 \quad 14.890 \\
\theta_2 & = 75 \quad 57 \quad 42.053 \\
\Delta \theta_m & = 1/4 (\theta_1 + \theta_2) = 72 \quad 56 \quad 37.79 \\
\sin \theta_m & = 0.95600827 \\
\cos \theta_m & = 0.29221291 \\
\sin \Delta \theta_m & = 0.05554769 \\
\cos \Delta \theta_m & = 0.99774947 \\
\tan \Delta \theta_m & = 0.00313543 \\
\sin^2 \Delta \theta_m + \sin^2 \lambda_m & = 0.019377568 \\
\cos d & = 1 - 2L \\
\end{align*}
\]

\[
\begin{align*}
U & = 2 \sin^2 \theta_m \cos \Delta \theta_m / (1 - L) = 1.8524 \times 10^{-3} \\
V & = 2 \sin^2 \Delta \theta_m \cos^2 \theta_m / L = 0.29906923 \sin d = 0.2990971 \quad \text{d (rad)} = 2.5266 \times 96.987 \\
X & = U + V \times 1.8823 \times 1870 \\
Y & = U - V \times 1.8823 \times 5054 \\
A & = DE = 7.913953 \times 15 \quad C = T - \frac{1}{4} (A - E) = 1.971 \times 0.31 \\
B & = 2D = 8.17744 \times 1224 \\
X_1 & = X (A + CX) \times 2.888 \times 151.3 \quad n_2 = Y (B + EY) \times 2.888 \times 127.17 \\
\delta_1 & = \delta_1 = (T^2 / 64) (n_1 - n_2) = 1.6479 \times 6.1 \\
S_1 & = \sin d (T - \delta_1, d) = 0.111470.945 \\
F & = 2Y - E (4 - X) = 4.56827.925 \\
G & = M (T + (F^2 / 64) M) \\
\Delta \alpha_1 & = \frac{\lambda_1}{2} (\Delta \lambda + Q) = 1.2184 \times 2.7 \times 10^{-3} \\
\tan \Delta \alpha_1 & = 0.04326 \times 8.977 \\
\tan \alpha_2 & = 0.04326 \times 8.977 \\
\end{align*}
\]

\[
\begin{align*}
v & = \arctan \tan \alpha_1 \times 47 \quad 54 \quad 59.992 \\
u & = \arctan \tan \alpha_1 \times 52 \quad 39 \quad 59.992 \\
\alpha_1 & = v - u \times 45 \quad 00 \quad 00.006 \\
\alpha_2 & = 360 \quad - \alpha_1 \times 360 \quad - \alpha_2 \\
\alpha_3 & = 180 \quad - \alpha_3 \times 180 \quad - \alpha_3 \\
\end{align*}
\]

\[
\begin{align*}
\text{CHECK: } \Delta \lambda / 4 & = 0.02799 \\
\tan \Delta \alpha_1 & = 0.04326 \times 8.977 \\
c_2 & = \cos \Delta \alpha_1 / \cos \theta_m \tan \Delta \alpha_1 = 2.4477 \times 13.21 \\
c_1 & = \sin \Delta \alpha_1 / \cos \theta_m \tan \Delta \alpha_1 = 0.4477 \times 13.21 \\
\end{align*}
\]
DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1, a_{1.2}, S$ to find $\phi_2, \lambda_2, a_{2.1}$. East
longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables
(as Peters) required for desk work.

**Clarke 1866 Spheroid $a = 6378137, m = 0.02399 0072 383$**

1 radian = 206264.8062 seconds

<table>
<thead>
<tr>
<th>LINE</th>
<th>VERTEX 1 ((\phi_1, \lambda_1))</th>
<th>VERTEX 2 ((\phi_2, \lambda_2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_1$</td>
<td>$-76.00$ to $26.41$</td>
<td>$\phi_2$</td>
</tr>
<tr>
<td>$\lambda_1$</td>
<td>$4.012987$</td>
<td>$\lambda_2$</td>
</tr>
<tr>
<td>$a_{1.2}$</td>
<td>$90.00$</td>
<td>$a_{2.1}$</td>
</tr>
<tr>
<td>$\sin \theta_1$</td>
<td>$-0.2013371$</td>
<td>$\cos \theta_2$</td>
</tr>
<tr>
<td>$\cos \phi_1$</td>
<td>$-0.5476076$</td>
<td>$\theta_1 + 75.57 42.053$</td>
</tr>
<tr>
<td>$\sin a_{1.2}$</td>
<td>$0.5000000$</td>
<td>$N = \cos \theta_1 \cos \phi_1$</td>
</tr>
<tr>
<td>$\cos a_{1.2}$</td>
<td>$0.0000000$</td>
<td>$\sin \theta_0 + 0.92013371$</td>
</tr>
<tr>
<td>$c_1 = FM$</td>
<td>$0.00829 \ 729 \ 128$</td>
<td>$D = (1 - c_1)(1 - c_1)M$</td>
</tr>
<tr>
<td>$c_2 = H(1 - M^2)$</td>
<td>$0.00079650322$</td>
<td>$P = c_1 (1 + \frac{v}{c_1}, M)/D$</td>
</tr>
<tr>
<td>$\cos a_1$</td>
<td>$\sin \phi_1 / \sin \phi_0$</td>
<td>$\cos a_1.1$</td>
</tr>
<tr>
<td>$\sin a_1$</td>
<td>$1$</td>
<td>$\phi_1$</td>
</tr>
<tr>
<td>$\cos a_1$</td>
<td>$0$</td>
<td>$\phi_1$</td>
</tr>
<tr>
<td>$\sin d$</td>
<td>$0$</td>
<td>$d = 180.00$</td>
</tr>
<tr>
<td>$\cos d$</td>
<td>$-1$</td>
<td>$d = 180.00$</td>
</tr>
<tr>
<td>$V = \cos u \cos d - \sin u \sin d$</td>
<td>$0$</td>
<td>$V = 2PV \sin d$</td>
</tr>
<tr>
<td>$Y = 2PVW \sin d$</td>
<td>$0$</td>
<td>$Y = 2PVW \sin d$</td>
</tr>
<tr>
<td>$X = c_1 \sin d \cos (2V^2 - 1)$</td>
<td>$0$</td>
<td>$X = c_1 \sin d \cos (2V^2 - 1)$</td>
</tr>
<tr>
<td>$\sin \Delta a$</td>
<td>$0$</td>
<td>$\sin \Delta a = -1$</td>
</tr>
<tr>
<td>$\cos \Delta a$</td>
<td>$0$</td>
<td>$\Delta a = 180.00$</td>
</tr>
<tr>
<td>$\cos \Sigma a$</td>
<td>$-1$</td>
<td>$\Sigma a = 2a_1 - \Delta a$</td>
</tr>
<tr>
<td>$\tan \alpha_{2.1}$</td>
<td>$M/(N \cos \Delta a - \sin \theta_1 \sin \Delta a)$</td>
<td>$\alpha_{2.1}$</td>
</tr>
<tr>
<td>$\tan \phi_2 = (\sin \theta_1 \cos \Delta a + N \sin \Delta a) \sin a_{1.1}$</td>
<td>$+ 0.012987$</td>
<td>$\sin a_{1.1}$</td>
</tr>
<tr>
<td>$\sin a_{1.1}$</td>
<td>$0$</td>
<td>$a_{1.1}$</td>
</tr>
<tr>
<td>$\tan \Delta h$</td>
<td>$\sin \Delta a \sin a_{1.1}$</td>
<td>$\Delta h$</td>
</tr>
<tr>
<td>$\cos \theta_1 \cos \Delta a - \sin \theta_1 \sin \Delta a \cos a_{1.1}$</td>
<td>$0$</td>
<td>$\cos \theta_1 \cos \Delta a - \sin \theta_1 \sin \Delta a \cos a_{1.1}$</td>
</tr>
<tr>
<td>$H = c_1(1 - c_1) \Delta a - c_1 c_3 \sin \Delta a \cos \Sigma a$</td>
<td>$0.00259 \ 37\ 07$</td>
<td>$H = c_1(1 - c_1) \Delta a - c_1 c_3 \sin \Delta a \cos \Sigma a$</td>
</tr>
<tr>
<td>$\omega = \Delta h - H$</td>
<td>$179.57 10.997$</td>
<td>$\omega = \Delta h - H$</td>
</tr>
<tr>
<td>$\lambda_1 = -151.04 18.307$</td>
<td>$\lambda_1 = -151.04 18.307$</td>
<td></td>
</tr>
</tbody>
</table>

**CHECK**

$M = \cos \theta_1 \cos \theta_1 \sin a_{1.1} = \cos \theta_1 \sin (180 + a_{1.1})$
**Inverse Position Computation: Form for Long Lines.** Given \( \phi_1, \lambda_1; \phi_2, \lambda_2 \) to find \( S, \alpha_1, \alpha_2 \), Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

**Clarke 1866 Spheroid**

\[ a = 6378206.4 \text{ m} \]

<table>
<thead>
<tr>
<th>( f^2/4 )</th>
<th>( 1785.7204 \times 10^{-6} )</th>
</tr>
</thead>
</table>

1 radian = 206264.8062 seconds

<table>
<thead>
<tr>
<th>( \phi_1 )</th>
<th>( \phi_2 )</th>
<th>( \lambda_1 )</th>
<th>( \lambda_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tan \phi_1 )</td>
<td>( \tan \phi_2 )</td>
<td>( \Delta \lambda = \lambda_2 - \lambda_1 )</td>
<td>( 89.55 ) 33.777</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \Delta \lambda_m = \frac{\lambda_2 - \lambda_1}{2} )</td>
<td>57.91887</td>
</tr>
<tr>
<td>( \theta_2 )</td>
<td>( \theta_1 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \Delta \theta_m = \frac{\theta_2 - \theta_1}{2} )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( H = \cos^2 \theta_m - \sin^2 \theta_m )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( L = \sin^2 \Delta \theta_m + H \sin^2 \Delta \lambda_m )</td>
<td></td>
<td></td>
<td></td>
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<tr>
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<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( V = 2 \sin^2 \theta_m \cos^2 \theta_m / L )</td>
<td></td>
<td></td>
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</table>
DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1, \alpha_{1,2}, S$ to find $\phi_2, \lambda_2, \alpha_{2,3}$. East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

**CLARKE 1866 SPHEROID $a = 6378206.4$ m**

1 radian = 206264.8062 seconds

<table>
<thead>
<tr>
<th>LINE</th>
<th>$P$</th>
<th>TO NODE 1</th>
<th>($P$,M)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_1$</td>
<td>$70$</td>
<td>$\tan \phi_1$</td>
<td>$\tan \phi_1 = (1 - f) \tan \phi_1$</td>
</tr>
<tr>
<td>$\alpha_{1,2}$</td>
<td>$95$</td>
<td>$\sin \theta_1 = -9.283 1820 \cos \theta_1$</td>
<td>$\sin \theta_1 = -9.283 1820 \cos \theta_1$</td>
</tr>
<tr>
<td>$\sin \alpha_{1,2}$</td>
<td>$7071 0678$</td>
<td>$M = \cos \theta_0 = \cos \theta_1 \sin \alpha_{1,2}$</td>
<td>$M = \cos \theta_0 = \cos \theta_1 \sin \alpha_{1,2}$</td>
</tr>
<tr>
<td>$\cos \alpha_{1,2}$</td>
<td>$7071 0678$</td>
<td>$N = \cos \theta_1 \cos \alpha_{1,2}$</td>
<td>$N = \cos \theta_1 \cos \alpha_{1,2}$</td>
</tr>
<tr>
<td>$c_1 = \tan M$</td>
<td>$0.00797531 178$</td>
<td>$D = (1 - c_1)(1 - c_2 - c_1 M)$</td>
<td>$D = (1 - c_1)(1 - c_2 - c_1 M)$</td>
</tr>
<tr>
<td>$c_2 = \tan (1 - M^2)$</td>
<td>$0.00797531 271$</td>
<td>$P = c_1 (1 + c_1 M) / D$</td>
<td>$P = c_1 (1 + c_1 M) / D$</td>
</tr>
<tr>
<td>$\cos \alpha_1 = \sin \theta_1 / \sin \theta_0$</td>
<td>$-9.283 2591$</td>
<td>$\alpha_1$</td>
<td>$165.31 12.719$</td>
</tr>
<tr>
<td>$d = (D / 1.317 436 6587 ) (rad) \sin d$</td>
<td>$0.968 1997$</td>
<td>$u = 2(\alpha_1 - d)$</td>
<td>$180.07 3.572 \sin u$</td>
</tr>
<tr>
<td>$\cos d$</td>
<td>$2.524 12.44$</td>
<td>$W = 1 - 2P \cos u$</td>
<td>$1.005 982 267 \cos u$</td>
</tr>
<tr>
<td>$\sin d$</td>
<td>$0.968 1997$</td>
<td>$V = \cos u \cos d - \sin u \sin d$</td>
<td>$-2.996 938 831$</td>
</tr>
<tr>
<td>$\cos d$</td>
<td>$2.524 12.44$</td>
<td>$X = c_1 \sin d \cos (2V^2 - 1) - 1.15 \times 10^{-6}$</td>
<td>$-2.996 938 831$</td>
</tr>
<tr>
<td>$\sin \delta_0$</td>
<td>$-9.283 2591$</td>
<td>$\cos \delta_0$</td>
<td>$0.250 03843$</td>
</tr>
<tr>
<td>$\cos \delta_0$</td>
<td>$-9.283 2591$</td>
<td>$\Sigma = 2a_1 - \delta_0$</td>
<td>$255.12 12.799$</td>
</tr>
<tr>
<td>$\tan \alpha_{1,2} = M / (N \cos \delta_0 - \sin \theta_1 \sin \delta_0)$</td>
<td>$-2.520 03843$</td>
<td>$\alpha_{1,2}$</td>
<td>$184.02 12.946$</td>
</tr>
<tr>
<td>$\tan \phi_2 = -(\sin \theta_1 \cos \delta_0 + N \sin \delta_0) \sin \alpha_{1,2}$</td>
<td>$0.4 \times 10^{-7}$</td>
<td>$\sin \alpha_{1,2}$</td>
<td>$-2.520 03843$</td>
</tr>
<tr>
<td>$\Delta^8 = \frac{\sin \delta_0 \sin \alpha_{1,2}}{\cos \theta_1 \cos \delta_0 + \sin \theta_1 \sin \delta_0 \cos \alpha_{1,2}}$</td>
<td>$0.293 1831$</td>
<td>$\Delta \delta$</td>
<td>$0.07 0.0006$</td>
</tr>
<tr>
<td>$H = c_1 (1 - c_2) \Delta \delta - c_1 c_2 \sin \delta_0 \cos \Sigma$</td>
<td>$0.001 083 142 114 (rad)$</td>
<td>$H$</td>
<td>$2 92.424$</td>
</tr>
<tr>
<td>$\Delta \lambda = \Delta \eta - H$</td>
<td>$0.08 44.411$</td>
<td>$\lambda_1 = \Delta \lambda$</td>
<td>$0.08 44.411$</td>
</tr>
<tr>
<td>CHECK</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$M = \cos \theta_0 = \cos \theta_1 \sin \alpha_{1,2} = \cos \theta_2 \sin (180 + \alpha_{1,2})$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda_2 = \lambda_1 + \Delta \lambda$</td>
<td>$0.08 44.411$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

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INVERSE POSITION COMPUTATION FORM FOR LONG LINES. Given \( \phi_1, \lambda_1; \phi_2, \lambda_2 \) to find \( S, \alpha_1, \alpha_2, \alpha_4 \). Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

\[
\text{Clarke 1866 SPHEROID} \quad b = 6378206.4 \quad \text{m} \\
1 - f = \frac{b}{a} = 0.9966492 \times 10^{-6} \\
f^2/64 = 1.7957204 	imes 10^{-6} \\
1 \text{ radian} = 206264.8062 \text{ seconds}
\]

Given \( \phi_1, \lambda_1; \phi_2, \lambda_2 \) to find \( S, \alpha_1, \alpha_2, \alpha_4 \).

1. \( \phi_1 \) 2. \( \phi_2 \) 3. \( \lambda_1 \) 4. \( \lambda_2 \)

\[
\begin{align*}
\tan \phi_1 & = \tan \phi_2 & \Delta \lambda & = \lambda_2 - \lambda_1 \\
\Delta \phi & = \phi_1 - \phi_2 & \Delta \phi_m & = \Delta \phi \\
\tan \theta_1 & = \frac{1}{\cos \phi_1} & \tan \theta_2 & = \frac{1}{\cos \phi_2} \\
\tan \theta & = \frac{1}{\cos \phi} & \sin \Delta \phi_m & = \sin \Delta \phi \\
\theta_m & = \frac{\Delta \phi}{2} & \Delta \phi_m & = \Delta \phi \\
\Delta \phi_m^2 & = \Delta \phi_m \sin \Delta \phi_m & \cos \Delta \phi_m & = \frac{1}{\cos \Delta \phi_m} \\
\Delta \phi_m & = \sqrt{(\Delta \phi_m)^2 - \Delta \phi_m \sin \Delta \phi_m} & \cos \Delta \phi_m & = \frac{1}{\cos \Delta \phi_m} \\
H & = \cos \Delta \phi_m \sin \theta_m & \tan \theta & = \frac{1}{\cos \theta} \\
X & = \cos \theta \sin \Delta \phi_m & \tan \theta & = \frac{1}{\cos \theta} \\
Y & = \cos \theta \sin \Delta \phi_m & \tan \theta & = \frac{1}{\cos \theta} \\
Z & = \cos \theta \sin \Delta \phi_m & \tan \theta & = \frac{1}{\cos \theta} \\
C & = \cos \Delta \phi_m \sin \theta_m & \tan \theta & = \frac{1}{\cos \theta} \\
T & = \cos \theta \sin \Delta \phi_m & \tan \theta & = \frac{1}{\cos \theta} \\
D & = \cos \theta \sin \Delta \phi_m & \tan \theta & = \frac{1}{\cos \theta} \\
\alpha_1 & = V + U & \alpha_2 & = V + U \\
\alpha_4 & = V + U & \alpha_2 & = V + U \\
\alpha_1 & = V + U & \alpha_2 & = V + U \\
\alpha_4 & = V + U & \alpha_2 & = V + U \\
\end{align*}
\]
DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given \( \phi_1, \lambda_1, \alpha_{1-2}, S \) to find \( \phi_2, \lambda_2, \alpha_{2-3}. \) East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

**CLARKE 1866 SPHEROID** \( a = 6378206.4 \text{ m} \)

1 radian = 206264.8062 seconds

<table>
<thead>
<tr>
<th>LINE</th>
<th>NODE 1</th>
<th>TO</th>
<th>( P_2 )</th>
<th>( (M, P_2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \phi_1 )</td>
<td>0</td>
<td>( \tan \phi_1 )</td>
<td>( \tan \theta_1 = (1 - f) \tan \phi_1 )</td>
<td>( \cos \theta_1 )</td>
</tr>
<tr>
<td>( \alpha_{1-2} )</td>
<td>14 07 17.947</td>
<td>( \sin \theta_1 )</td>
<td>0</td>
<td>( \cos \theta_1 )</td>
</tr>
<tr>
<td>( \sin \alpha_{1-2} )</td>
<td>( \cos \theta_0 = M )</td>
<td>( M = \cos \theta_0 = \cos \theta_1 \sin \alpha_{1-2} )</td>
<td>( 23:35:57.076 )</td>
<td>( \theta_0 )</td>
</tr>
<tr>
<td>( \cos \alpha_{1-2} )</td>
<td>( \sin \alpha_{1-2} )</td>
<td>( N = \cos \theta_1 \cos \alpha_{1-2} )</td>
<td>( 9701 :3371 )</td>
<td>( \sin \theta_0 )</td>
</tr>
<tr>
<td>( c_1 = fM )</td>
<td>( 0.000827 :333178 )</td>
<td>( D = (1 - c_2)(1 - c_1 M) )</td>
<td>( 99820.6057 )</td>
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</tr>
<tr>
<td>( c_2 = 0.4(1 - M^2) )</td>
<td>( \phi_0 = 0.799650227 )</td>
<td>( P = c_2 (1 + \frac{1}{2} c_1 M) / D )</td>
<td>( \phi_0 = 0.799650227 )</td>
<td></td>
</tr>
<tr>
<td>( \cos \alpha_1 = \sin \theta_1 / \sin \theta_0 )</td>
<td>0</td>
<td>( \alpha_1 )</td>
<td>90</td>
<td>0</td>
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<tr>
<td>( d = S / AD )</td>
<td>( 0.0728 :0672.7 ) (rad)</td>
<td>( d = 17 :26 :21.188 )</td>
<td>( S = 1956283.534 ) m</td>
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</tr>
<tr>
<td>( \sin d )</td>
<td>( 0.0244 :6271 )</td>
<td>( u = 2(\alpha_1 - d) )</td>
<td>( 199 :47 :13627 )</td>
<td>( \sin u )</td>
</tr>
<tr>
<td>( \cos d )</td>
<td>( 0.9531 :5960 )</td>
<td>( W = 1 - 2\cos u + \frac{1}{2} \sin^2 0.01338756 )</td>
<td>( \cos u )</td>
<td>( -0.170 :3646 )</td>
</tr>
<tr>
<td>( V = \cos u \cos d - \sin u \sin d )</td>
<td>( -0.9531 :5960 )</td>
<td>( Y = 2PV \sin d )</td>
<td>( \tan \alpha_{2-1} = M / (N \cos \Delta - \sin \theta_1 \sin \Delta \alpha) )</td>
<td>( +:1.949 :001 )</td>
</tr>
<tr>
<td>( \Delta o = d + X - Y )</td>
<td>( +0.714 :222 :17 ) (rad)</td>
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<tr>
<td>( \sin \Delta o )</td>
<td>( 30.9 :0270 )</td>
<td>( \cos \Delta o = 0.9530 :1990 )</td>
<td>( \Delta o = 17 :37 :56 :390 )</td>
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<tr>
<td>( \cos \Sigma o = -0.9530 :1990 )</td>
<td>( \Sigma o = 2\alpha_1 - \Delta o )</td>
<td>( \pi - \Delta o )</td>
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</tr>
<tr>
<td>( \tan \alpha_{2-1} = M / (N \cos \Delta o - \sin \theta_1 \sin \Delta \alpha) )</td>
<td>( +2.627 :1439 )</td>
<td>( \alpha_{2-1} )</td>
<td>( -194 :47 :03 :7274 )</td>
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<tr>
<td>( \tan \phi_2 = -\sin \theta_1 \cos \Delta o + N \sin \sin (\alpha_{2-1}) / (1 - \frac{1}{2} M) )</td>
<td>( +0.089 :0035 )</td>
<td>( \phi_2 )</td>
<td>( -0.2537 :7541 )</td>
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<tr>
<td>( \tan \Delta \alpha = \sin \sin (\alpha_{2-1}) / \cos \theta_1 \cos \Delta o - \sin \theta_1 \sin \Delta o \sin \cos \alpha_{1-2} )</td>
<td>( +0.2720 :9565 )</td>
<td>( \Delta \alpha )</td>
<td>( 17 :08 :36 :316 )</td>
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</tr>
<tr>
<td>( H = c_1 (1 - c_1) \Delta o - c_1 c_2 \sin \Delta o \cos \Sigma o )</td>
<td>( 0.0023530541 ) (rad)</td>
<td>( H = 52.194 )</td>
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<tr>
<td>( \Delta \lambda = \Delta o - H )</td>
<td>( +4 :39 :39 :144 )</td>
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<tr>
<td>( \lambda_1 )</td>
<td>( -61 :08 :44 :610 )</td>
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<tr>
<td>( \lambda_2 = \lambda_1 + \Delta \lambda )</td>
<td>( -56 :45 :05 :466 )</td>
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</tbody>
</table>
INVERSE POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1$, $\lambda_1$; $\phi_2$, $\lambda_2$ to find $S$, $\alpha_1$, $\alpha_2$. Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

<table>
<thead>
<tr>
<th>$\phi_1$</th>
<th>$\phi_2$</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>NODE 1</td>
<td>$\Delta \phi = \lambda_2 - \lambda_1$</td>
<td>$\phi_2$</td>
</tr>
<tr>
<td>2.</td>
<td>$\tan \phi_1$</td>
<td>$\tan \theta_2$</td>
<td>$\sin \Delta \phi$</td>
</tr>
<tr>
<td>$\theta_1$</td>
<td>$\theta_2$</td>
<td>$\Delta \phi = \frac{1}{2}(\theta_1 + \theta_2)$</td>
<td>$\cos \Delta \phi$</td>
</tr>
<tr>
<td>$\theta_3$</td>
<td>$\theta_4$</td>
<td>$\frac{1}{2}(\theta_3 - \theta_4)$</td>
<td>$\cos \theta_m$</td>
</tr>
<tr>
<td>$\Delta \theta_m = \frac{1}{2}(\theta_3 + \theta_4)$</td>
<td>$\Delta \theta_m = \frac{1}{2}(\theta_3 - \theta_4)$</td>
<td>$\cos \theta_m$</td>
<td></td>
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</table>

Azimuths clockwise from north; cast longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

<table>
<thead>
<tr>
<th>$a_1 = \frac{\pi}{2}$</th>
<th>$a_2 = \frac{\pi}{2}$</th>
<th>$a_3 = \frac{\pi}{2}$</th>
<th>$a_4 = \frac{\pi}{2}$</th>
</tr>
</thead>
</table>

1 radian = 206264.8062 seconds

Example:

- $\phi_1 = 40^\circ 20' 31.396''$  
- $\lambda_1 = 21^\circ 41' 42.706''$  
- $\phi_2 = 30^\circ 20' 31.396''$  
- $\lambda_2 = 21^\circ 41' 42.706''$

**Formulae:**

$\tan \phi_1 = \tan \phi_2$  
$\tan \theta_1 = \frac{1}{\tan \phi_2} - \sin \Delta \phi$  
$\sin \Delta \phi = \frac{1}{2}(\theta_1 + \theta_2)$  
$\cos \Delta \phi = \frac{1}{2}(\theta_1 - \theta_2)$

**Example Calculations:**

- $\theta_1 = 40^\circ 20' 31.396''$  
- $\theta_2 = 30^\circ 20' 31.396''$

- $\Delta \phi = \frac{1}{2}(\theta_1 + \theta_2)$

- $\sin \Delta \phi = \frac{1}{2}(\theta_1 - \theta_2)$

- $\cos \Delta \phi = \frac{1}{2}(\theta_1 + \theta_2)$

**Example Results:**

- $\Delta \phi = 5^\circ 40' 31.396''$

- $\sin \Delta \phi = 0.08333333$

- $\cos \Delta \phi = 0.99666667$

- $\sin \theta_m = \frac{1}{2}(\theta_1 + \theta_2)$

- $\cos \theta_m = \frac{1}{2}(\theta_1 - \theta_2)$

- $\Delta \theta_m = \frac{1}{2}(\theta_3 + \theta_4)$

- $\Delta \theta_m = \frac{1}{2}(\theta_3 - \theta_4)$

- $\cos \theta_m = 0.99666667$

- $\sin \theta_m = 0.08333333$

**Check:**

- $\Delta \phi = \lambda_2 - \lambda_1$

- $\phi_2$

**Further Calculations:**

- $\tan \phi_1 = \tan \phi_2$

- $\tan \theta_1 = \frac{1}{\tan \phi_2} - \sin \Delta \phi$

- $\sin \Delta \phi = \frac{1}{2}(\theta_1 + \theta_2)$

- $\cos \Delta \phi = \frac{1}{2}(\theta_1 - \theta_2)$

- $\sin \theta_m = \frac{1}{2}(\theta_1 + \theta_2)$

- $\cos \theta_m = \frac{1}{2}(\theta_1 - \theta_2)$

- $\Delta \theta_m = \frac{1}{2}(\theta_3 + \theta_4)$

- $\Delta \theta_m = \frac{1}{2}(\theta_3 - \theta_4)$

- $\cos \theta_m = 0.99666667$

- $\sin \theta_m = 0.08333333$

**Example Results:**

- $\Delta \phi = 5^\circ 40' 31.396''$

- $\sin \Delta \phi = 0.08333333$

- $\cos \Delta \phi = 0.99666667$

- $\sin \theta_m = 0.08333333$

- $\cos \theta_m = 0.99666667$

- $\Delta \theta_m = 5^\circ 40' 31.396''$

- $\sin \Delta \theta_m = 0.08333333$

- $\cos \Delta \theta_m = 0.99666667$

- $\cos \theta_m = 0.99666667$

- $\sin \theta_m = 0.08333333$

**Further Checks:**

- $\Delta \phi = \lambda_2 - \lambda_1$

- $\phi_2$

**Conclusion:**

The inverse position computation form for long lines is a method to find the position of one point relative to another, given the positions of the two points and their azimuths. This method is particularly useful for surveying and navigation where precise calculations are necessary. The formulas provided allow for the calculation of the necessary angles and distances to determine the position of the second point relative to the first.
DIREC T POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1, \alpha_{1,-2}, \delta$ to find $\phi_2, \lambda_2, \alpha_{2,-1}$. East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural! tables (as Peters) required for desk work.

**Clarke 1866 Spheroid** $a = 6378.137 m \; f = 0.00339981383$

$1 - f = 0.99960982777$

1 radian = 206264.8062 seconds

<table>
<thead>
<tr>
<th>LINE</th>
<th>$P_1$</th>
<th>TO</th>
<th>$P_2$</th>
<th>(PR)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_1$</td>
<td>tan $\phi_1$</td>
<td>tan $\theta_1 = (1 - f) \tan \phi_1$</td>
<td>$a_{1,-2}$</td>
<td>45</td>
</tr>
<tr>
<td>$\alpha_{1,-2}$</td>
<td>$-0.70710657$</td>
<td>$M = \cos \phi_2 = \cos \phi_1 \sin \phi_1 ; -0.70710657 ; \theta_2 = 75 ; \delta = 57.43295$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sin \alpha_{1,-2}$</td>
<td>$-0.70710657$</td>
<td>$N = \cos \phi_2 \cos \phi_1 ; -0.70710657 ; \sin \theta_2 = 0.7071071$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$c_1 = fM$</td>
<td>$0.000812331378$</td>
<td>$D = (1 - c_1)(1 - c_1 - c_1 M) = 0.99960982777$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$c_2 = \frac{1}{2}(1 - M^2) f$</td>
<td>$0.000797650277$</td>
<td>$P = c_2 (1 + \frac{f}{c_1 M}) / D = 0.000797650277$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\cos \alpha_1 = \sin \theta_1 / \sin \theta_2 = 0.99932591 \alpha_1 = 0.81 ; 11.78$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$d = S / d = 1.624702409 \quad $ (rad) $\; \delta = 93.06 \; 14.12^2 \; S = 0.0063022.079 \lambda$ |

$\sin d = 0.91532295 \quad u = 2(\alpha_1 - d) = 1.4444931794 \sin u = 0.87594794$ |

$\cos d = 0.54414742 \quad W = 1 - 2 \cos u = 0.00136287 \cos u = 0.21742328$ |

$V = \cos u \cos d - \sin u \sin d = -0.32076293 \quad Y = 2PV \sin d = -0.02897485$ |

$X = c_2 \sin d \cos d = (2V^2 - 1) \times 1.5 \times 10^{-7} \quad \Delta = d + X - Y = 1.624706049 \quad $ (rad) |

$\sin \Delta = 0.59812666 \cos \Delta = -0.81499457 \Delta = 93.09 \; 09.189$ |

$\cos \Sigma_0 = 0.53157272 \quad \Sigma_0 = 2\alpha_1 - \Delta \; \Sigma_0 = 537.93 \; 11.379$ |

$\tan \alpha_{2,-1} = M(N \cos \Delta + \sin \theta_1 \sin \Delta) = 0.36298185 \; \alpha_{2,-1} = 0.45 \; 02.294$ |

$\tan \phi_2 = -\frac{\sin \theta_1 \cos \Delta + N \sin \theta_1 \sin \Delta_{2,-1}}{(1 - f)M} \phi_2 = 1.7 \; 05 \; 12.328$ |

$\tan \Delta \eta = \frac{\sin \alpha_1 \sin \alpha_{1,-2}}{\cos \theta_1 \cos \Delta - \sin \theta_1 \sin \Delta \cos \alpha_1} \Delta \eta = 97.36 \; 09.326$ |

$H = c_1(1 - c_1) \Delta \eta - c_1 \Delta \sin \cos \Sigma \; 0.00183329427 (\text{rad}) \quad H = \Delta \eta - H = 47.32 \; 23.785$ |

$\lambda_1 = 0.104 \; 17 \; 29.220$ |

**CHECK**

$M = \cos \phi_1 = \cos \phi_1 \sin \phi_1 \sin \phi_1 \sin (180 + \alpha_{2,-1}) |

$\lambda_2 = \lambda_1 + \Delta = 56 \; 45 \; 05.465$
INVERSE POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1; \phi_2, \lambda_2$ to find $S, a_1, a_2, \ldots$. Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

**Clarke 1866 Spheroid**

1 - f = b/a = 0.99664992472

\[ f^2/64 = 0.1957204 \times 10^{-6} \]

1 radian = 206264.8062 seconds

<table>
<thead>
<tr>
<th>$\phi_1$</th>
<th>$\phi_2$</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tan \phi_1$</td>
<td>$\tan \phi_2$</td>
<td>$\Delta \lambda = \lambda_2 - \lambda_1$</td>
<td>$\Delta \lambda / \Delta \lambda_2$</td>
</tr>
<tr>
<td>$\theta_1 = \frac{\pi}{2} (1 - f) \tan \phi_1$</td>
<td>$\theta_2 = \frac{\pi}{2} (1 - f) \tan \phi_2$</td>
<td>$\sin \Delta \lambda_2$</td>
<td>$\cos \theta_2$</td>
</tr>
<tr>
<td>$\theta_1 = \frac{\pi}{2} \tan \phi_1$</td>
<td>$\theta_2 = \frac{\pi}{2} \tan \phi_2$</td>
<td>$\sin \Delta \lambda_2$</td>
<td>$\cos \theta_2$</td>
</tr>
<tr>
<td>$\Delta \theta = \frac{\pi}{2} (1 - f_2) \tan \phi_2$</td>
<td>$\Delta \theta = \frac{\pi}{2} (1 - f_1) \tan \phi_1$</td>
<td>$\sin \Delta \lambda_2$</td>
<td>$\cos \theta_2$</td>
</tr>
<tr>
<td>$H = \cos^2 \Delta \theta - \sin^2 \theta_2 = \cos^2 \theta_2 - \sin^2 \Delta \lambda_2 = \sin^2 \theta_2 - \sin^2 \Delta \lambda_2 = \sin^2 \Delta \lambda_2 - \sin^2 \theta_2$</td>
<td>$H = \cos^2 \Delta \theta - \sin^2 \theta_2 = \cos^2 \theta_2 - \sin^2 \Delta \lambda_2 = \sin^2 \theta_2 - \sin^2 \Delta \lambda_2 = \sin^2 \Delta \lambda_2 - \sin^2 \theta_2$</td>
<td>$H = \cos^2 \Delta \theta - \sin^2 \theta_2 = \cos^2 \theta_2 - \sin^2 \Delta \lambda_2 = \sin^2 \theta_2 - \sin^2 \Delta \lambda_2 = \sin^2 \Delta \lambda_2 - \sin^2 \theta_2$</td>
<td>$H = \cos^2 \Delta \theta - \sin^2 \theta_2 = \cos^2 \theta_2 - \sin^2 \Delta \lambda_2 = \sin^2 \theta_2 - \sin^2 \Delta \lambda_2 = \sin^2 \Delta \lambda_2 - \sin^2 \theta_2$</td>
</tr>
<tr>
<td>$L = \sin \Delta \theta_2 + \sin \Delta \lambda_2$</td>
<td>$L = \sin \Delta \theta_2 + \sin \Delta \lambda_2$</td>
<td>$L = \sin \Delta \theta_2 + \sin \Delta \lambda_2$</td>
<td>$L = \sin \Delta \theta_2 + \sin \Delta \lambda_2$</td>
</tr>
<tr>
<td>$U = 2 \sin^2 \theta_2 \cos \Delta \lambda_2 / (1 - L)$</td>
<td>$U = 2 \sin^2 \theta_2 \cos \Delta \lambda_2 / (1 - L)$</td>
<td>$U = 2 \sin^2 \theta_2 \cos \Delta \lambda_2 / (1 - L)$</td>
<td>$U = 2 \sin^2 \theta_2 \cos \Delta \lambda_2 / (1 - L)$</td>
</tr>
<tr>
<td>$V = 2 \sin \Delta \theta_2 \sin \Delta \lambda_2 / L$</td>
<td>$V = 2 \sin \Delta \theta_2 \sin \Delta \lambda_2 / L$</td>
<td>$V = 2 \sin \Delta \theta_2 \sin \Delta \lambda_2 / L$</td>
<td>$V = 2 \sin \Delta \theta_2 \sin \Delta \lambda_2 / L$</td>
</tr>
<tr>
<td>$X = U + V$</td>
<td>$X = U + V$</td>
<td>$X = U + V$</td>
<td>$X = U + V$</td>
</tr>
<tr>
<td>$Y = U - V$</td>
<td>$Y = U - V$</td>
<td>$Y = U - V$</td>
<td>$Y = U - V$</td>
</tr>
<tr>
<td>$A = D \times 21,301,231$</td>
<td>$A = D \times 21,301,231$</td>
<td>$A = D \times 21,301,231$</td>
<td>$A = D \times 21,301,231$</td>
</tr>
<tr>
<td>$B = D \times 21,301,231$</td>
<td>$B = D \times 21,301,231$</td>
<td>$B = D \times 21,301,231$</td>
<td>$B = D \times 21,301,231$</td>
</tr>
<tr>
<td>$\alpha_1 = a - u = 14.69' 43.729$</td>
<td>$\alpha_2 = a - u = 14.69' 43.729$</td>
<td>$\alpha_2 = a - u = 14.69' 43.729$</td>
<td>$\alpha_2 = a - u = 14.69' 43.729$</td>
</tr>
<tr>
<td>$\gamma_1 = b - x = 25.40' 40.58'</td>
<td>$\gamma_1 = b - x = 25.40' 40.58'</td>
<td>$\gamma_1 = b - x = 25.40' 40.58'</td>
<td>$\gamma_1 = b - x = 25.40' 40.58'$</td>
</tr>
<tr>
<td>$\gamma_2 = b - x = 25.40' 40.58'</td>
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</tr>
<tr>
<td>$\delta_1 = a - u = 14.69' 43.729$</td>
<td>$\delta_1 = a - u = 14.69' 43.729$</td>
<td>$\delta_1 = a - u = 14.69' 43.729$</td>
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</tr>
<tr>
<td>$\delta_2 = a - u = 14.69' 43.729$</td>
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<td>$\delta_2 = a - u = 14.69' 43.729$</td>
<td>$\delta_2 = a - u = 14.69' 43.729$</td>
</tr>
</tbody>
</table>
DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1$, $\lambda_1$, $\alpha_{1-2}$, $S$ to find $\phi_2$, $\lambda_2$, $\alpha_{2-1}$. East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

<table>
<thead>
<tr>
<th>$\phi_1$</th>
<th>$\phi_2$</th>
<th>$\alpha_{1-2}$</th>
<th>$\alpha_{2-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>14.47, 03.725</td>
<td>2837.6097</td>
</tr>
<tr>
<td>sin $\alpha_{1-2}$</td>
<td>0.2577 564</td>
<td>$M = \cos \theta_0 = \cos \theta_1 \sin \alpha_{1-2}$</td>
<td>$24.527076 \theta_0$</td>
</tr>
<tr>
<td>cos $\alpha_{1-2}$</td>
<td>0.9672 6317</td>
<td>$N = \cos \theta_1 \cos \alpha_{1-2}$</td>
<td>0.4745 6673</td>
</tr>
<tr>
<td>$c_1 = fM$</td>
<td>0.00022788327</td>
<td>$D = (1 - c_3)(1 - c_2 - c_1M)$</td>
<td>0.9922666207</td>
</tr>
<tr>
<td>$c_2 = 14(1 - M^2)f$</td>
<td>0.000797650327</td>
<td>$P = c_2 (1 + \frac{c_3}{2}M)/D$</td>
<td>0.00242916565</td>
</tr>
<tr>
<td>$\cos \alpha_1 = \sin \theta_1/\sin \theta_0$</td>
<td>$\phi_2 = 0.0029 0270 \alpha_1$</td>
<td>$74.32 09.68$</td>
<td></td>
</tr>
<tr>
<td>$d = \Delta / \Delta d$</td>
<td>101.040 08.9853</td>
<td>(rad)</td>
<td>57.52 31.95</td>
</tr>
<tr>
<td>sin $d$</td>
<td>$\sin \theta_1/\sin \theta_0$</td>
<td>$\sin \theta_1 = \sin \theta_0$, $\cos \theta_0 = \cos \theta_1 \sin \alpha_{1-2}$</td>
<td>0.4849 615</td>
</tr>
<tr>
<td>$\cos d$</td>
<td>0.57 11.4</td>
<td>$\tan \theta_1 = (1 - f) \tan \phi_1$</td>
<td>1.11 01.0</td>
</tr>
<tr>
<td>$V = \cos \theta_1 \cos d - \sin \theta_1 \sin d \cos \theta_0$</td>
<td>0.0559093363</td>
<td>$Y = 2PV \sin d$</td>
<td>0.9445 17.10</td>
</tr>
<tr>
<td>$X = c_3 \sin d \cos d (2V^2 - 1)$</td>
<td>$-2.8977 10^{-6}$</td>
<td>$\Delta \alpha = d + X - Y$</td>
<td>1.01238459</td>
</tr>
<tr>
<td>$\sin \Delta \alpha = 0.9720 0952 \cos \Delta \alpha = 0.5315 7727 \Delta \alpha = 57.52 46.279$</td>
<td></td>
<td>$\cos \Sigma = 0.8549 9452 \Sigma_0 = 2\alpha_1 - \Delta \alpha$</td>
<td>86.50 53.94</td>
</tr>
<tr>
<td>$\tan \alpha_{2-1} = M(N \cos \Delta \alpha - \sin \theta_1, \sin \Delta \alpha)$</td>
<td>1.0000 02 165</td>
<td>$\alpha_{2-1}$</td>
<td>225.60 06.017</td>
</tr>
<tr>
<td>$\tan \phi_2 = \frac{- \sin \theta_1 \cos \Delta \alpha + N \sin \Delta \alpha}{(1 - N)M}$</td>
<td>$\phi_2 = 20.60 00.005$</td>
<td>$\phi_2 = 20.60 00.005$</td>
<td>$\phi_2 = 20.60 00.005$</td>
</tr>
<tr>
<td>$\tan \Delta \eta = \frac{\sin \Delta \alpha \sin \alpha_{1-2}}{\cos \theta_1 \cos \Delta \alpha - \sin \theta_1 \sin \Delta \alpha \cos \alpha_{1-2}}$</td>
<td>$-0.0094 318$</td>
<td>$\Delta \eta = 28.47 56.310$</td>
<td>$\Delta \eta = 28.47 56.310$</td>
</tr>
<tr>
<td>$H = c_3 (1 - c_1) \Delta \alpha - c_1 c_3 \sin \Delta \alpha \cos \Sigma_0 = 0.0008013999$</td>
<td>(rad)</td>
<td>$H = 0.0008013999$</td>
<td>$H = 0.0008013999$</td>
</tr>
<tr>
<td>$\Delta \lambda = \Delta \eta - H$</td>
<td>$2.51.228$</td>
<td>$\Delta \lambda = \Delta \eta - H$</td>
<td>$2.51.228$</td>
</tr>
<tr>
<td>$\lambda_1 = \Delta \lambda - 11$</td>
<td>18.45 05.472</td>
<td>$\lambda_1 = \Delta \lambda - 11$</td>
<td>18.45 05.472</td>
</tr>
<tr>
<td>CHECK</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$M = \cos \theta_0 = \cos \theta_1 \sin \alpha_{1-2} = \cos \theta_2 \sin (180 + \alpha_{2-1})$</td>
<td>$\lambda_1 = \lambda_1 + \Delta \lambda = 17.59 59.28$</td>
<td>$\lambda_1 = \lambda_1 + \Delta \lambda = 17.59 59.28$</td>
<td></td>
</tr>
</tbody>
</table>
Inverse Position Computation Form for Long Lines. Given $\phi_1, \lambda_1; \phi_2, \lambda_2$ to find $S, a, -a$.

Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

Clarke 1866 Spheroid $a = 63782064$ m

\[1 - \frac{b}{a} = 0.00673947\quad \frac{a}{b} = 0.0084751929\]

$\rho^2/64 = 0.0073249\times10^{-6}$

1 radian = 206264.8062 seconds

\[
\begin{align*}
\phi_1 & \quad 1. \quad P_0 \quad \lambda_1 \\
\phi_2 & \quad 2. \quad \text{INITIAL (P, E)} \quad \lambda_2 \\
\tan \phi_1 & \quad 1. \quad \text{always west of 2.} \\
\tan \phi_2 & \quad \tan \theta = (1 - f) \tan \phi \\
\theta_2 & \quad 49^\circ 56' 14.89^\prime \\
\theta_1 & \quad 17^\circ 05' 31.39^\prime \\
\theta_m & = \frac{1}{2} (\theta_1 + \theta_2) \quad 49^\circ 30' 47.94^\prime \\
\sin \theta_m & = 0.7853938 \cos \theta_m = 0.6176204 \\
\Delta \lambda & = \lambda_2 - \lambda_1 \quad 18^\circ 45' 05.46^\prime \\
\Delta \lambda_m & = \frac{1}{2} \Delta \lambda \quad 9^\circ 22' 32.23^\prime \\
H & = \cos^2 \Delta \lambda_m - \sin^2 \theta_m = \cos^2 \theta_m - \sin^2 \theta_m \\
1 - L & = 74.5874109 \\
L & = \sin^2 \Delta \lambda_m + \sin^2 \Delta \lambda_m \\
U & = 2 \sin^2 \theta_m \Delta \lambda_m / (1 - L) \\
V & = 2 \sin^2 \Delta \lambda_m \cos^2 \theta_m / L \\
X & = U + V \\
Y & = U - V \\
C & = \text{DE-605/122770} \\
T & = \text{CHECK C-HE+AD/EB=T} \\
\delta_1 & = \text{m} \quad \text{m} \\
S_1 & = \text{m} \quad \text{m} \\
F & = \text{Y-E} (4 - X) \\
G & = \frac{1}{4} T + \frac{1}{4} F \\
\Delta \lambda_m & = \frac{1}{2} (\Delta \lambda + Q) \\
\tan \Delta \lambda_m & = 15.8 \times 10^{-5} \\
\alpha_1 & = \text{arc tan} c_1 \quad 14^\circ 42' 03.73^\prime \\
\alpha_2 & = \text{arc tan} c_2 \quad 13^\circ 20' 32.29^\prime \\
\alpha_3 & = \frac{1}{2} \text{arc tan} c_1 \quad 13^\circ 50' 16.26^\prime \\
\alpha_4 & = \text{arc tan} c_2 \quad 14^\circ 42' 03.73^\prime \\
\alpha_5 & = \frac{1}{2} \text{arc tan} c_1 \quad 13^\circ 50' 16.26^\prime \\
\alpha_6 & = \text{arc tan} c_2 \quad 14^\circ 42' 03.73^\prime \\
\alpha_7 & = \frac{1}{2} \text{arc tan} c_1 \quad 13^\circ 50' 16.26^\prime \\
\alpha_8 & = \text{arc tan} c_2 \quad 14^\circ 42' 03.73^\prime \\
\alpha_9 & = \frac{1}{2} \text{arc tan} c_1 \quad 13^\circ 50' 16.26^\prime \\
\alpha_{10} & = \text{arc tan} c_2 \quad 14^\circ 42' 03.73^\prime \\
\alpha_{11} & = \frac{1}{2} \text{arc tan} c_1 \quad 13^\circ 50' 16.26^\prime \\
\alpha_{12} & = \text{arc tan} c_2 \quad 14^\circ 42' 03.73^\prime \\
\alpha_{13} & = \frac{1}{2} \text{arc tan} c_1 \quad 13^\circ 50' 16.26^\prime \\
\alpha_{14} & = \text{arc tan} c_2 \quad 14^\circ 42' 03.73^\prime \\
\alpha_{15} & = \frac{1}{2} \text{arc tan} c_1 \quad 13^\circ 50' 16.26^\prime \\
\alpha_{16} & = \text{arc tan} c_2 \quad 14^\circ 42' 03.73^\prime \\
\alpha_{17} & = \frac{1}{2} \text{arc tan} c_1 \quad 13^\circ 50' 16.26^\prime \\
\alpha_{18} & = \text{arc tan} c_2 \quad 14^\circ 42' 03.73^\prime \\
\alpha_{19} & = \frac{1}{2} \text{arc tan} c_1 \quad 13^\circ 50' 16.26^\prime \\
\alpha_{20} & = \text{arc tan} c_2 \quad 14^\circ 42' 03.73^\prime \\
\alpha_{21} & = \frac{1}{2} \text{arc tan} c_1 \quad 13^\circ 50' 16.26^\prime \\
\alpha_{22} & = \text{arc tan} c_2 \quad 14^\circ 42' 03.73^\prime \\
\alpha_{23} & = \frac{1}{2} \text{arc tan} c_1 \quad 13^\circ 50' 16.26^\prime \\
\alpha_{24} & = \text{arc tan} c_2 \quad 14^\circ 42' 03.73^\prime \\
\alpha_{25} & = \frac{1}{2} \text{arc tan} c_1 \quad 13^\circ 50' 16.26^\prime \\
\alpha_{26} & = \text{arc tan} c_2 \quad 14^\circ 42' 03.73^\prime \\
\alpha_{27} & = \frac{1}{2} \text{arc tan} c_1 \quad 13^\circ 50' 16.26^\prime \\
\alpha_{28} & = \text{arc tan} c_2 \quad 14^\circ 42' 03.73^\prime 
\end{align*}

150
DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given \( \phi_1, \lambda_1, \alpha_{1,2}, S \) to find \( \phi_2, \lambda_2, \alpha_{2,3} \). East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

**LINE** \( P_1 \) \( \rightarrow \) \( P_2 \) \( (P, P_2) \)

\( \phi_1 \) \( \tan \phi_1 \) \( \tan \theta_1 = (1 - n) \tan \phi_1 \) \( \alpha_{1,2} \) \( \frac{45}{\sin \theta_1} = 9893 \) \( \cos \theta_1 \) \( -59 \) \( 56 \) \( 14.570 \)
\( \sin \alpha_{1,2} \) \( 7071 / 0678 \) \( M = \cos \theta_1 \cos \alpha_{1,2} \) \( 7071 \) \( 0678 \) \( N = \cos \theta_1 \cos \alpha_{1,2} \) \( 2445 / 7076 \) \( \sin \theta_0 + 0.701 / 3871 \)
\( c_1 = \frac{\cos \theta_1}{\cos \alpha_{1,2}} \) \( 0.000422 / 333 \) \( 1078 \) \( D = \left( 1 - c_1 \right) \left( 1 - c_2 - c_3 \right) M \) \( 0.9682 / 0207 \)
\( c_2 = \frac{4}{1 - M^2} \) \( 0.000791 / 50277 \) \( P = c_2 \left( 1 + \frac{c_1}{M} \right) D \) \( 0.000789 / 18565 \)
\( \cos \alpha_1 = \sin \theta_1 / \sin \theta_0 \) \( -0.9682 / 355 \) \( \alpha_1 \) \( 165^\circ / 11 \) \( 21.27^\circ \)
\( d = S / a D \) \( 3.4459 / 26558 \) \( \text{rad} \) \( 180^\circ / 0 \) \( 0 \) \( S = 3090 / 729 / 126^\circ \)
\( \sin d \) \( 0 \) \( u = 2 \left( a_1 - d \right) \) \( \sin u \) \( \cos d \) \( -1 \) \( W = 1 - 2 \cos u \) \( \cos u \) \( V = \cos u \cos d + \sin u \sin d \) \( Y = 2PVW \sin d \) \( 0 \)
\( X = c_1^2 \sin d \cos d \left( \frac{2V^2 - 1}{1} \right) \) \( 0 \) \( \Omega = d + X - Y \) \( \Omega = \text{rad} \)
\( \sin \Delta \) \( 0 \) \( \cos \Delta \) \( -1 \) \( \Delta = 180^\circ \)
\( \cos \Sigma \) \( 0 \) \( \Sigma = \Delta_1 - \Delta_2 \)
\( \tan \alpha_{1,2} = \frac{M \left( N \cos \Delta_2 - \sin \theta_1 \sin \Delta_2 \right)}{1 - \cos \Delta_2} \) \( -1 \) \( \alpha_{1,2} \)
\( \tan \phi_2 = \frac{-\left( \sin \theta_1 \cos \Delta_2 + N \sin \Delta_2 \cos \alpha_{1,2} \right)}{1 - \cos \Delta_2} \) \( 2.2474 / 774 \) \( \sin \alpha_{1,2} \)
\( \tan \Delta \eta = \frac{\sin \Delta_2 \sin \alpha_{1,2}}{\cos \theta_1 \cos \Delta_2 - \sin \theta_1 \sin \Delta_2 \cos \alpha_{1,2}} \) \( 0 \) \( \Delta \eta \)
\( H = c_1 \left( 1 - c_2 \right) \Delta_2 - c_3 \sin \Delta_2 \cos \Sigma_2 \) \( 0.0251 / 225 \) \( \text{rad} \) \( H = c_1 \left( 1 - c_2 \right) \Delta_2 - c_3 \sin \Delta_2 \cos \Sigma_2 \) \( 0.0251 \) \( \Delta \lambda = \Delta \eta - H \) \( 19^\circ / 51 \) \( 0.735 \)
\( \lambda_1 - 180^\circ / 17 \) \( 29.72^\circ \)

**CHECK**

\( M = \cos \theta_2 = \cos \theta_1 \sin \alpha_{1,2} = \cos \theta_1 \sin \left( 180^\circ + \alpha_{1,2} \right) \)
\( \lambda_2 = \lambda_1 + \Delta \) \( 15^\circ / 23 \) \( 12.339^\circ \)
DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given \( \phi_1, \lambda_1, \alpha_{1-2}, S \) to find \( \phi_2, \lambda_2, \alpha_{2-3} \). East longitudes positive; azimuth clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

**Clarke 1866 Spheroid** a 6378206.4

1 - f = 0.99660992474

1 radian = 206264.8062 seconds

**Line Vertex** TO **Terminal** \((X, Y, Z)\)

<table>
<thead>
<tr>
<th>( \phi_1 )</th>
<th>( \alpha_{1-2} )</th>
<th>( \alpha_{2-3} )</th>
<th>( \sin \theta_1 )</th>
<th>( \cos \theta_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 90 )</td>
<td>( \sin \theta_1 = \sin \phi_1 )</td>
<td>( \cos \theta_1 = \cos \phi_1 )</td>
<td>( 75.57 \times 10^3 )</td>
<td></td>
</tr>
<tr>
<td>( 90 )</td>
<td>( 1 )</td>
<td>( M = \cos \theta_2 = \cos \theta_1 \sin \alpha_{1-2} )</td>
<td>( 24.5 \times 10^3 \times 75.57 \times 10^3 )</td>
<td></td>
</tr>
<tr>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( N = \cos \theta_1 \cos \alpha_{1-2} )</td>
<td>( 0 )</td>
<td></td>
</tr>
<tr>
<td>( c_1 = \frac{M}{\cos \alpha_{1-2}} )</td>
<td>( D = (1 - c_1)(1 - c_2 - c_1 M) )</td>
<td>( C = \frac{N}{\cos \alpha_{1-2}} )</td>
<td>( 0.00799163565 )</td>
<td></td>
</tr>
</tbody>
</table>

**Depression**

\( d = \frac{M}{\cos \alpha_{1-2}} \)

\( W = 1 - 2P \cos u \)

\( V = \cos u \cos d \sin u \sin d \)

\( X = c_1 \sin d + \cos d (2V^2 - 1) \)

\( \Delta \phi = d - X - Y \)

\( \Delta \lambda = \cos \Delta \sin \alpha_{1-2} \)

\( \Delta \eta = \sin \Delta \sin \alpha_{1-2} \)

\( H = c_1 (1 - c_1) \Delta \eta - c_1 c_2 \sin \Delta \cos \alpha_{1-2} \)

**Check**

\( M = \cos \theta_2 = \cos \theta_1 \sin \alpha_{1-2} \)

\( \lambda_2 + \Delta \Lambda = \lambda_1 \)

152
INVERSE POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \phi_2, \lambda_1, \lambda_2$ to find $S, r, \theta_2$. Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

<table>
<thead>
<tr>
<th>$\phi_1$</th>
<th>$\phi_2$</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>$\theta_1$</th>
<th>$\theta_2$</th>
<th>$\Delta\lambda$</th>
<th>$\Delta\theta_m$</th>
<th>$H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tan \phi_1$</td>
<td>$\tan \phi_2$</td>
<td>$\tan \theta_1$</td>
<td>$\tan \theta_2$</td>
<td>$\sin \Delta\lambda_m$</td>
<td>$\cos \Delta\theta_m$</td>
<td>$\cos \Delta \lambda_m$</td>
<td>$\cos \Delta \theta_m$</td>
<td>$\epsilon$</td>
</tr>
</tbody>
</table>

$S$ and $r$ can be found:

$$S = \frac{r \tan \phi_1}{\tan \phi_2}.$$
DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1$, $\lambda_1$, $\alpha_{1-2}$, $S$ to find $\phi_2$, $\lambda_2$, $\alpha_{2-3}$. East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

<table>
<thead>
<tr>
<th>LINE</th>
<th>INITIAL TO TERMINAL (I T)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_1$</td>
<td>$\tan \phi_1$</td>
</tr>
<tr>
<td>$\alpha_{1-3}$</td>
<td>$\cos \alpha_{1-3}$</td>
</tr>
<tr>
<td>$\sin \alpha_{1-3}$</td>
<td>$\tan \alpha_{1-3}$</td>
</tr>
<tr>
<td>$S$</td>
<td>$M$</td>
</tr>
<tr>
<td>$\cos \alpha_{1-3}$</td>
<td>$N$</td>
</tr>
<tr>
<td>$c_1 = \frac{\pi}{M}</td>
<td>D = (1 - c_2)(1 - c_2 - c_1 M)$</td>
</tr>
<tr>
<td>$c_2 = \frac{\pi}{4}(1 - M^2)$</td>
<td>$P = c_1 (1 + \frac{1}{c_1 M})$</td>
</tr>
<tr>
<td>$\cos \phi_1 = \sin \phi_1 / \sin \theta_0$</td>
<td>$\cos \phi_1 = \sin \phi_1 / \sin \theta_0$</td>
</tr>
<tr>
<td>$d = S / A D$</td>
<td>$d = S / A D$</td>
</tr>
<tr>
<td>$\sin d = \sin \phi_1 \sin \phi_2 + \cos \phi_1 \cos \phi_2 \cos \alpha_{2-3}$</td>
<td>$\sin d = \sin \phi_1 \sin \phi_2 + \cos \phi_1 \cos \phi_2 \cos \alpha_{2-3}$</td>
</tr>
<tr>
<td>$\cos d = \cos \phi_1 \cos \phi_2 - \sin \phi_1 \sin \phi_2 \cos \alpha_{2-3}$</td>
<td>$\cos d = \cos \phi_1 \cos \phi_2 - \sin \phi_1 \sin \phi_2 \cos \alpha_{2-3}$</td>
</tr>
<tr>
<td>$\Delta = d + X - Y$</td>
<td>$\Delta = d + X - Y$</td>
</tr>
<tr>
<td>$\sin \Delta = \sin \phi_1 \sin \phi_2 + \cos \phi_1 \cos \phi_2 \cos \alpha_{2-3}$</td>
<td>$\sin \Delta = \sin \phi_1 \sin \phi_2 + \cos \phi_1 \cos \phi_2 \cos \alpha_{2-3}$</td>
</tr>
<tr>
<td>$\cos \Delta = \cos \phi_1 \cos \phi_2 - \sin \phi_1 \sin \phi_2 \cos \alpha_{2-3}$</td>
<td>$\cos \Delta = \cos \phi_1 \cos \phi_2 - \sin \phi_1 \sin \phi_2 \cos \alpha_{2-3}$</td>
</tr>
<tr>
<td>$\cos \phi_2 = \left(1 - f \right) M$</td>
<td>$\cos \phi_2 = \left(1 - f \right) M$</td>
</tr>
<tr>
<td>$\tan \phi_2 = \frac{\sin \phi_2}{\cos \phi_2}$</td>
<td>$\tan \phi_2 = \frac{\sin \phi_2}{\cos \phi_2}$</td>
</tr>
<tr>
<td>$\sin \alpha_{2-3} = \frac{\sin \phi_2}{\cos \alpha_{2-3}}$</td>
<td>$\sin \alpha_{2-3} = \frac{\sin \phi_2}{\cos \alpha_{2-3}}$</td>
</tr>
<tr>
<td>$\cos \alpha_{2-3} = \sqrt{1 - \sin^2 \phi_2}$</td>
<td>$\cos \alpha_{2-3} = \sqrt{1 - \sin^2 \phi_2}$</td>
</tr>
<tr>
<td>$H = c_1 (1 - c_2) \Delta - c_2 \sin \Delta \cos \Sigma_0$</td>
<td>$H = c_1 (1 - c_2) \Delta - c_2 \sin \Delta \cos \Sigma_0$</td>
</tr>
<tr>
<td>$\Delta \lambda = \Delta \eta - H$</td>
<td>$\Delta \lambda = \Delta \eta - H$</td>
</tr>
<tr>
<td>$\lambda_1 = \lambda_2 + \Delta \lambda$</td>
<td>$\lambda_1 = \lambda_2 + \Delta \lambda$</td>
</tr>
</tbody>
</table>

CHECK

$M = \cos \theta_0 = \cos \phi_1 \sin \alpha_{1-2} = \cos \phi_2 \sin \left(180 + \alpha_{1-2}\right)$

$\lambda_2 = \lambda_1 + \Delta \lambda$
INVERSE POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1; \phi_2, \lambda_2$ to find $S, a_1, a_2$. Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

<table>
<thead>
<tr>
<th>$\phi_1$</th>
<th>$\phi_2$</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. INITIAL (Origin)</td>
<td>2. TERMINAL (ITT)</td>
<td>3.</td>
<td></td>
</tr>
<tr>
<td>$\tan \phi_1$</td>
<td>$\tan \phi_2$</td>
<td>$\tan \theta_1$</td>
<td>$\tan \theta_2$</td>
</tr>
<tr>
<td>always west of 2.</td>
<td>$\tan \theta = (1 - \tan \phi)$</td>
<td>$\sin \Delta \lambda_m$</td>
<td>$\cos \Delta \lambda_m$</td>
</tr>
<tr>
<td>$\theta_2 \geq 0.05 \geq 21.28$</td>
<td>$\theta_1 \geq 0.58 \geq 14.59$</td>
<td>$\sin \Delta \lambda_m$</td>
<td>$\cos \Delta \lambda_m$</td>
</tr>
<tr>
<td>$\theta_m = \frac{1}{2}(\theta_1 + \theta_2) \leq 0.05 \leq 21.28$</td>
<td>$\sin \Delta \lambda_m$</td>
<td>$\cos \Delta \lambda_m$</td>
<td>$\sin \Delta \lambda_m$</td>
</tr>
<tr>
<td>$\Delta \lambda_m = \frac{1}{2}(\theta_2 - \theta_1) \leq 0.25 \leq 16.44$</td>
<td>$\sin \Delta \lambda_m$</td>
<td>$\cos \Delta \lambda_m$</td>
<td>$\sin \Delta \lambda_m$</td>
</tr>
<tr>
<td>$H = \cos^2 \Delta \lambda_m - \sin^2 \theta_m = \cos^2 \lambda_m - \sin^2 \lambda_m$</td>
<td>$\sin \Delta \lambda_m$</td>
<td>$\cos \Delta \lambda_m$</td>
<td>$\sin \Delta \lambda_m$</td>
</tr>
<tr>
<td>$L = \sin^2 \Delta \lambda_m + H \sin^2 \Delta \lambda_m$</td>
<td>$\sin \Delta \lambda_m$</td>
<td>$\cos \Delta \lambda_m$</td>
<td>$\sin \Delta \lambda_m$</td>
</tr>
<tr>
<td>$U = 2 \sin \theta_m \cos^2 \Delta \lambda_m / (1 - L)$</td>
<td>$\sin \Delta \lambda_m$</td>
<td>$\cos \Delta \lambda_m$</td>
<td>$\sin \Delta \lambda_m$</td>
</tr>
<tr>
<td>$V = 2 \sin \Delta \lambda_m \cos^2 \theta_m / L$</td>
<td>$\sin \Delta \lambda_m$</td>
<td>$\cos \Delta \lambda_m$</td>
<td>$\sin \Delta \lambda_m$</td>
</tr>
<tr>
<td>$X = U + V \geq 0.1827 \geq 0.17794751$</td>
<td>$\sin \Delta \lambda_m$</td>
<td>$\cos \Delta \lambda_m$</td>
<td>$\sin \Delta \lambda_m$</td>
</tr>
<tr>
<td>$Y = X + V \geq 0.1827 \geq 0.17794751$</td>
<td>$\cos \Delta \lambda_m$</td>
<td>$\sin \Delta \lambda_m$</td>
<td>$\cos \Delta \lambda_m$</td>
</tr>
<tr>
<td>$F = 2Y - E \geq 1.8779 \geq 1.89063797$</td>
<td>$\cos \Delta \lambda_m$</td>
<td>$\sin \Delta \lambda_m$</td>
<td>$\cos \Delta \lambda_m$</td>
</tr>
<tr>
<td>$G = \frac{1}{2}T + (f/64) M \geq 0.2864 \geq 0.286492714$</td>
<td>$\cos \Delta \lambda_m$</td>
<td>$\sin \Delta \lambda_m$</td>
<td>$\cos \Delta \lambda_m$</td>
</tr>
<tr>
<td>$\Delta \lambda_m = \frac{1}{2}(\Delta \lambda + Q) \geq 0.18 \geq 0.18$</td>
<td>$\cos \Delta \lambda_m$</td>
<td>$\sin \Delta \lambda_m$</td>
<td>$\cos \Delta \lambda_m$</td>
</tr>
<tr>
<td>$V = \arctan \frac{c_1}{c_2} \geq 0.18 \geq 0.18$</td>
<td>$\cos \Delta \lambda_m$</td>
<td>$\sin \Delta \lambda_m$</td>
<td>$\cos \Delta \lambda_m$</td>
</tr>
<tr>
<td>$a_1 = \arctan \frac{c_1}{c_2} \geq 0.18 \geq 0.18$</td>
<td>$\cos \Delta \lambda_m$</td>
<td>$\sin \Delta \lambda_m$</td>
<td>$\cos \Delta \lambda_m$</td>
</tr>
<tr>
<td>$c_1 = \cos \Delta \lambda_m / \sin \Delta \lambda_m$</td>
<td>$\cos \Delta \lambda_m$</td>
<td>$\sin \Delta \lambda_m$</td>
<td>$\cos \Delta \lambda_m$</td>
</tr>
<tr>
<td>$c_2 = \sin \Delta \lambda_m / \cos \Delta \lambda_m$</td>
<td>$\cos \Delta \lambda_m$</td>
<td>$\sin \Delta \lambda_m$</td>
<td>$\cos \Delta \lambda_m$</td>
</tr>
<tr>
<td>$\alpha_2 = u \geq 0.18 \geq 0.18$</td>
<td>$\cos \Delta \lambda_m$</td>
<td>$\sin \Delta \lambda_m$</td>
<td>$\cos \Delta \lambda_m$</td>
</tr>
</tbody>
</table>

1 radian = 206264.8062 seconds

<table>
<thead>
<tr>
<th>$\alpha_2 - 2$</th>
<th>$\alpha_2 - 1$</th>
<th>$\alpha_2$</th>
<th>$\alpha_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>360 - $\alpha_2$</td>
<td>360 - $\alpha_2$</td>
<td>360 - $\alpha_2$</td>
<td>360 - $\alpha_2$</td>
</tr>
<tr>
<td>180 - $\alpha_2$</td>
<td>180 - $\alpha_2$</td>
<td>180 - $\alpha_2$</td>
<td>180 - $\alpha_2$</td>
</tr>
</tbody>
</table>

155
DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given \( \phi_1, \lambda_1, \alpha_{1-2} \), to find \( \phi_2, \lambda_2, \alpha_{2-1} \). East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

**Clarke 1866 Spheroid:** 6378160 m  
1 radian = 206264.8062 seconds

---

**LINE** | **TERMINAL** | **NODE 2** | **(radians)**
--- | --- | --- | ---

\( \phi_1 \) | \( \tan \phi_1 \) | \( \tan \theta_1 = (1-f) \tan \phi_1 \) | 
\( \alpha_{1-2} \) | \( \sin \theta_1 \cdot 2438.6097 \) | \( \cos \theta_1 \cdot 9558.8817 \) | \( 17.05 \) | 21.296
\( \sin \alpha_{1-2} \) | \( 2527.7541 \) | \( M = \cos \theta_0 = \cos \theta_1 \sin \alpha_{1-2} \) | \( 3425.7016 \) | 
\( \cos \alpha_{1-2} \) | \( -2969.6217 \) | \( N = \cos \theta_1 \cos \alpha_{1-2} \) | \( -9245.5673 \) | \( \sin \theta_0 \cdot 9701.2311 \)

\( c_1 = \sin \theta_1 \sin \theta_0 \cdot 3029.0770 \) | \( \alpha_1 \) | \( \alpha_2 \) | \( 32.02.610 \) | \( = 90 - \alpha_1 \) (see computation)

---

**Check**

\( M = \cos \theta_0 = \cos \theta_1 \sin \alpha_{1-2} = \cos \theta_2 \sin (180 + \alpha_{2-1}) \)

\( \lambda_2 = \lambda_1 + \Delta \lambda \) | \( \Delta \lambda \) | \( 42.22.047 \) |
INVERSE POSITION COMPUTATION FORM FOR LONG LINES. Given \( \phi_1, \lambda_1; \phi_2, \lambda_2 \) to find \( S, \alpha_2, \). Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

**Clarke 1866 Spheroid**

\[ a = 6378206.4 \text{ m} \]
\[ b = \frac{a}{\sqrt{2}} \]
\[ f = 1 - \frac{b}{a} = 0.0016953274 \]
\[ f^2/64 = 0.00875188208 \]

1 radian = 206264.8062 seconds

<table>
<thead>
<tr>
<th>( \phi_1 )</th>
<th>1. TERMINAL</th>
<th>( \lambda_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \phi_2 )</td>
<td>2. NODE (I TH)</td>
<td>( \lambda_2 )</td>
</tr>
<tr>
<td>tan ( \phi_1 )</td>
<td>1. always west of 2.</td>
<td>( \Delta \lambda = \lambda_2 - \lambda_1 )</td>
</tr>
<tr>
<td>tan ( \theta_1 )</td>
<td>tan ( \theta_2 )</td>
<td>( \Delta \lambda_m = \frac{\pi}{2} \Delta \lambda )</td>
</tr>
<tr>
<td>( \theta_1 )</td>
<td>17.05, 21.294</td>
<td>( \sin \Delta \lambda_m )</td>
</tr>
<tr>
<td>( \theta_m ) = ( \frac{1}{4} (\theta_1 + \theta_2) )</td>
<td>( \theta_m ) = ( \frac{1}{4} (\theta_1 + \theta_2) )</td>
<td>( \Delta \delta_m = \frac{\pi}{2} (\theta_1 - \theta_2) )</td>
</tr>
<tr>
<td>( \delta_m )</td>
<td>( \cos^2 \delta_m - \sin^2 \delta_m = \cos^2 \theta_m - \sin^2 \delta_m = 0.585788168 )</td>
<td>( L = \sin^2 \delta_m + H \sin^2 \Delta \lambda_m )</td>
</tr>
<tr>
<td>( U = 2 \sin^2 \theta_m \cos^2 \delta_m / (1 - L) )</td>
<td>( d = 2 \tan^{-1} \left( \frac{\sin \theta_1 \cos \delta_m}{\cos \theta_1 \sin \delta_m} \right) )</td>
<td></td>
</tr>
<tr>
<td>( V = 2 \sin^2 \delta_m \cos^2 \theta_m / L )</td>
<td>( \sin \delta_m = \frac{\sin \theta_1 \cos \delta_m}{\cos \theta_1 \sin \delta_m} )</td>
<td></td>
</tr>
<tr>
<td>( X = U + V )</td>
<td>( \sin \theta_1 )</td>
<td></td>
</tr>
<tr>
<td>( Y = U - V )</td>
<td>( \sin \delta_m )</td>
<td></td>
</tr>
<tr>
<td>( A = DE )</td>
<td>( B = 2 \cos \theta_2 )</td>
<td></td>
</tr>
<tr>
<td>( n_3 = X (A + CX) )</td>
<td>( n_3 = X (A + CX) )</td>
<td></td>
</tr>
<tr>
<td>( \delta_d = \frac{1}{4} \left( \frac{1201}{164} \right) )</td>
<td>( \delta_d = \frac{1}{4} \left( \frac{1201}{164} \right) )</td>
<td></td>
</tr>
<tr>
<td>( S_1 = a \sin d (T - \delta_d) )</td>
<td>( S_1 = a \sin d (T - \delta_d) )</td>
<td></td>
</tr>
<tr>
<td>( F = \sin Y = \sin E (A + X) )</td>
<td>( M = 32T - 105 )</td>
<td></td>
</tr>
<tr>
<td>( G = \frac{f}{2} (T + 1) )</td>
<td>( \sin \Delta \lambda = \frac{\pi}{2} (\theta_1 - \theta_2) )</td>
<td></td>
</tr>
<tr>
<td>( \Delta \lambda_m = \frac{\pi}{2} (\lambda_1 + \theta_2) )</td>
<td>( \cos \delta_m = \frac{\cos \theta_1 \sin \delta_m}{\sin \theta_1 \cos \delta_m} )</td>
<td></td>
</tr>
<tr>
<td>( v = \arctan \left( \frac{S_1}{S_2} \right) )</td>
<td>( c_1 = \cos \delta_m / (\sin \theta_m \tan \Delta \lambda_m) )</td>
<td></td>
</tr>
<tr>
<td>( u = \arctan \left( \frac{S_1}{S_2} \right) )</td>
<td>( c_1 = \sin \delta_m / (\cos \theta_m \tan \Delta \lambda_m) )</td>
<td></td>
</tr>
<tr>
<td>( a_1 = v )</td>
<td>( a_2 = u )</td>
<td></td>
</tr>
<tr>
<td>( a_2 - a_1 )</td>
<td>( a_2 - a_1 )</td>
<td></td>
</tr>
</tbody>
</table>

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DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1$, $\lambda_1$, $\alpha_{1,2}$, $S$ to find $\phi_2$, $\lambda_2$, $\alpha_{2,1}$. East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

CLARKE 1866 SPHEROID $a = 6378249.1$ m $f = 0.00668992472$

1 radian = 206264.8062 seconds

$$M = \text{cos } \theta_0 = \text{cos } \phi_1 \text{ sin } \alpha_{1,2} \text{ sin } (180 + \alpha_{2,1})$$

$$\lambda_2 = \lambda_1 + \Delta \lambda = 118.42' \ 32.943$$
APPENDIX 4

SUBROUTINE GEODIST

Fortran statements as prepared by the Earth Sciences Division of Teledyne Industries and based on the inverse (reverse) solution of P.D. Thomas. (The card deck including the arc tangent library function (ATAN2) is available.)
**TITLE:** Subroutine GOODIST

**DATE:** August 1968

GOODIST is a Fortran-63 subroutine used in computing surface distances on a given spheroid. The method used was supplied by Mr. Paul D. Thomas of the Naval Research Laboratory.

The subroutine currently uses the constants for the Clarke 1866 model of the Earth but another model may easily be substituted by replacing the semi-major and semi-minor axes (in kilometers) in the subroutine. The variable names are respectively, AL and BL.

The calling sequence for this subroutine includes the following values in the order listed: (Latitude 1, Longitude 1, Latitude 2, Longitude 2, Azimuth, Back Azimuth, Distance-kilometers, Distance-degrees). The Latitudes and Longitudes are the Geographic coordinates of the two points. The subroutine assumes that positive values are North or East and that negative values are South or West. The forward azimuth, in degrees east of north, is from point one to point two and conversely the back azimuth is from point two to point one. The distance in kilometers is the geodesic distance between the two points on the surface of the spheroid. The fiducial central angle distance in degrees (based on an equivalent mean sphere) is obtained by assuming that one degree equals 111.195 kilometers. All arguments are type REAL.

**CAUTION:** The back azimuth from either pole may be slightly in error.

The subroutine requires 225 CDC 1604 words plus four 1604 words of labeled common/GOODISTC/. There are no alarms, error returns, error stops or printouts. The time required is less than 0.15 seconds per call.

This subroutine uses an arctangent library function (ATAN2) which returns an angle between 0 and 2 Pi radians. If this function is not in the system library, it must be input with the subroutine. The fortran statements for this function are attached.

**REFERENCE:** Mr. Paul D. Thomas, Code 7004, Naval Research Laboratory, Washington, D.C. 20390.
SUBROUTINE GEODISTIEPLATEPLUNSTLAT#STLON#AZogAZoD;TEG
COMMON/6eO0i6TC/AL,9Leo2R#
TYPE REAL L1R,L2R#KLKK#L
DATA (AL86378206.49,eSL263561#83.),(D2e.O~l7453292919)0
il
1pI2"6.28315830716$
2pS86L/AL
S
P1Retp~Ar.p
2
R
P2RSSTLAI*
R
OLR*L2R-.L1
F2RwATAN(9oA eyAh
p2R))
GY"8(T2R-TlR)12,0
CYN*COS(Ihl
CDTH§CSo(Dym)
KL86TM*CDTy
AIXSUTH*UTN
SDLNmeSIl(0Lk
2
oUl
LUSDTN*SDTN5SDLmR*SDLHP.(CDTftCDTeq.STMqST.i,
CD4.100..OL
Dsjh(DL1
I*2
SUKL*KL/I1.eUL)
V5e.,OU*KWE/
Olrt
(AlfV961'e'sI$ltotle~poefplt)eJ( I $o
162
FUNCTION ATAN2 (Y, X)
PI = 3.1415926536
ARG = ATANF (Y/X)
IF (X) 10:14:11
1) ATAN2*PI+ARG
RETURN
11 IF (Y) 12, 13, 13
12 ATAN2=2.0*PI+ARG
RETURN
13 ATAN2 ARG
RETURN
14 IF (Y) 15, 16, 17
15 ATAN2=1.5*PI
RETURN
16 ATAN2=0.0
RETURN
17 ATAN2=0.5*PI
RETURN
END
W/ Subroutine Geodist

CLARKE 1866 CONSTANTS

S

REFERENCE POINT
LATITUDE 59 49 19.5
LONGITUDE 37 34 19.5

OBJECT POINT
LATITUDE 33 56 3.5
LONGITUDE 18 28 41.4

DISTANCE BETWEEN POINTS 10182.069865 KM
FORWARD AZIMUTH 195 49 17.8 DEG
BACK AZIMUTH 16 59 32.3 DEG

REFERENCE POINT
LATITUDE 49 0 0
LONGITUDE 106 0 0

OBJECT POINT
LATITUDE 20 0 0
LONGITUDE 0 0 0

DISTANCE BETWEEN POINTS 9649.171338 KM
FORWARD AZIMUTH 299 17 20.9 DEG
BACK AZIMUTH 24 56 30.7 DEG

REFERENCE POINT
LATITUDE 20 26 6.0
LONGITUDE -190 -1-03.0

OBJECT POINT
LATITUDE -50 25.0
LONGITUDE -79-34-24.0

DISTANCE BETWEEN POINTS 6494.621018 KM
FORWARD AZIMUTH 85 37 10.4 DEG
BACK AZIMUTH 269 57 17.4 DEG
W/ Subroutine Geodist

INTERNATIONAL CONSTANTS

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DISTANCE BETWEEN POINTS: 10142.679988 KM

FORWARD AZIMUTH: 195 46 16.5 DEG
BACK AZIMUTH: 10 39 31.1 DEG

Reference Point

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DISTANCE BETWEEN POINTS: 9649.412604 KM

FORWARD AZIMUTH: 295 17 18.6 DEG
BACK AZIMUTH: 42 56 36.0 DEG

Reference Point

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<td>8 58 25.8</td>
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<td>-15A -1 33.0</td>
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DISTANCE BETWEEN POINTS: 8486.852888 KM

FORWARD AZIMUTH: 85 37 12.3 DEG
BACK AZIMUTH: 28V 37 18.5 DEG