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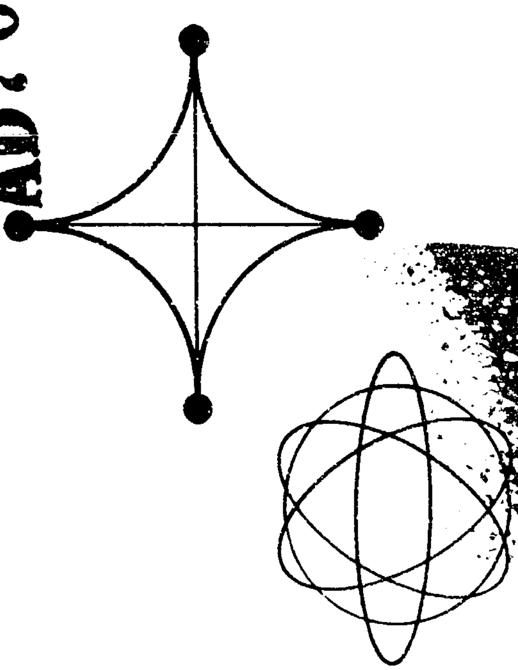

Spheroidal Geodesics, Reference Systems, & Local Geometry

U.S. NAVAL OCEANOGRAPHIC OFFICE
WASHINGTON, D.C. 20390

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ABSTRACT

A discussion of the geodesic on the oblate spheroid (reference ellipsoid) is given with formulae of geodetic accuracy (second order in the flattening, distance and azimuths) for the noniterative direct and inverse solutions over the hemispheroid, requiring no root extraction and no tabular data except 8-place tables of the natural trigonometric functions.

Forms are presented for use with any ellipsoid of reference and the formulae are adaptable to high speed electronic computers. Instructions for use of the forms in desk computations are given with the parameters for ten known ellipsoids of reference and the radii of spherical approximations.

A discussion is included of the computation of a long reference line in stations and of reference systems in the vicinity of a station as may be useful in oceanography, seismology, or other geophysical disciplines.

While the formulae introduced are satisfactory for short as well as long lines, the emphasis is on long lines out to maximum spheroidal geodesic length under the shortest distance property of the geodesic. The use of certain types of map projections for such base line work is also discussed.

The direct and inverse solutions as presented here have been adapted to high speed computers by the Earth Sciences Division of Teledyne, Inc., Alexandria, Virginia under the direction of Dr. E. F. Chiburis. The Fortran statements for the inverse solution are given in Appendix 4.

January 1970

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FOREWORD

This report fills a void in the theory and computation of long geodetic distances on the reference ellipsoid. The results will be particularly useful to long range navigation systems such as the Omega, and to several geophysical disciplines such as oceanography, seismology, and geodesy.

T. K. Treadwell

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PREFACE

The exposition of the computation of geodesics on the reference ellipsoid (oblate spheroid with small eccentricity) is based on the mathematical investigation I have conducted and included as Appendix 1 to this report. The many papers which have appeared on the subject since the early work of Legendre and Bessel are evidence of the dissatisfaction with the classic methods. This paper is no exception. *It is a "fresh" investigation,* but shows the influence of literature search. Where results were identifiable in other treatises, I have made reference to them. All the published works consulted are listed in the bibliography. *Many of the results presented here are new.* The emphasis is on long lines, based upon somewhat arbitrary criteria, i.e., an accuracy of at least 1 meter in position-geodetic length within 1 meter; latitude, longitude, and azimuth within .035 second—over the longest possible hemispheroidal geodesics employing no tables except 8-place natural trigonometric for desk computations—in any case meeting the 1/100,000 distance and 1 second azimuth requirement as specified by ACIC in their special studies (bibliographical reference [22] of this report); easy adaptation to any reference ellipsoid by merely changing the defining parameters; no root extraction or iteration with formulae limited to first and second powers of the flattening and which are compatible with both desk and large electronic computers.

Since the investigation included the longest possible geodesics, the following questions had to be resolved in the evaluation: If we take an arbitrary point on a given nonplanar spheroidal geodesic, *can we find a second limiting point on the geodesic beyond which the unique shortest distance property fails?* While Euler's differential equation is a necessary condition, is it sufficient? For example, in a limiting case, the equator as well as a meridian on the spheroid are geodesics (both satisfy Euler's condition) and both contain a common equatorial diameter—*is there an arc of the equator which satisfies the shortest distance criteria? Are there more than two consecutive geodesic vertices or more than two nodes (equatorial crossings) in a hemispheroid? Are there any antipodal points on nonplanar spheroidal geodesics? What happens antipodally in a family of geodesics each having a vertex in a common meridian? How can we independently check approximation equations for very long geodesics?*

In the 1957 report of the study group No. 2 on long lines, International Association of Geodesy, we find the statement: "Consequently, if two points are situated near the equator and are separated by nearly 180° of longitude there is a certain ambiguity as to what is meant by the geodesic between them." In his paper "The distance between two widely separated points on the surface of the earth" (Bibliographical reference [17] below), Dr. W. D. Lambert stated (concerning the ambiguity): "There appears to be no comprehensive treatment readily available in English. The author hopes to publish one shortly." This was never done.

From my investigation (Appendix 1) it was concluded *that the maximum lengths of all oblate spheroidal geodesics, under the shortest distance property, each having a vertex in a common semimeridian (pole to pole) are contained in a hemispheroid (on the same side of the meridian orthogonal to that containing the vertices).* This permitted determination of maximum distances over which approximation formulae to geodesics need hold under assumed accuracy criteria.

The antipodal zones were investigated for such a family of geodesics (each with a vertex in a common meridian) and formulae developed for determining the axes of the geodesic evolutes (envelopes). A formula for the latitude of the conjugate of an arbitrary point on the spheroidal geodesic (the point beyond which the unique shortest distance property fails) was found.

Formulae were developed in terms of the vertex latitude of the geodesic for longitude difference and length to serve as control checks on approximation formulae, and to check already published lines to be used for comparative purposes. A new direct solution was developed, and the inverse solution (previously published in NAVOCEANO TR-182, 1966) improved in form layout, azimuths to second order in the flattening were added and the quadrant search for azimuths eliminated. Where possible or feasible the formulae presented were developed through at least two different analyses, the details of which are presented in Appendix 1.

No apology is made for including the computations of a large number of numerical results throughout the discourse of Appendix 1, or for those included as a group in Appendix 3. One of the disappointing aspects of the literature review (Bibliography to this report), was the frequency of a single or at most two numerical examples presented in verification of formulae, such formulae being subsequently unacceptable when applied to lines differing considerably from those presented. The numerical results of Appendix 3 are also useful as checks, should individual programming of the equations be attempted, and all the ACIC test lines, already published in reference [22], have been included in Appendix 3 for check purposes. Appendix 2 contains the parameters for ten reference ellipsoids, the radii of spherical approximations, antipodal zone axes and areas, coordinate systems and other useful formulae.

The formulae presented here for the direct and inverse (reverse) solutions of geodesics in terms of parametric latitude have been programmed (Fortran) by the Earth Sciences Division, Teledyne, Inc., Alexandria, Virginia, under the supervision of Dr. E. F. Chiburis. The Fortran statements for the inverse solution are given in Appendix 4, and the card deck is available.

Finally, it seemed desirable to devote a section to a discussion of the use of forms presented for desk computations, and in applications such as the computation of reference lines and local associated geometry in the neighborhood of stations on the base line as may be needed in geophysical surveys and studies.

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HISTORICAL NOTE

The French Mathematician, Legendre, published papers in 1806 and 1811 on the theory of spheroidal geodesics, consolidating and extending his work in the Traité des Fonctions Elliptiques, 1825.

The German astronomer, Bessel, published an approximation solution to the spheroidal geodesic in 1825, [1],* and since that time an almost endless stream of publications on the subject has appeared. Other famous 19th century scientists who studied the problem include Bennet (1850, '51), Christoffel (1868), Hansen (1868), Cayley (1870), Jacobi (posthumous publication), Halphen (1888), Darboux (1894), A.R. Forsyth (1895). Cayley was the first to use the term "parametric latitude" for the eccentric angle of the meridian ellipse, [25], preferring it to Legendre's "reduced latitude." Two outstanding 19th century treatises, in each of which the geodesic problem is presented with approximation solutions (iterative), are those of the British geodesist Clarke and of Helmert, the German contemporary, both volumes appearing in 1880, [2].

The Bessel-Helmert method, which is an iterative type computation of the development of the projection on the sphere of the spheroidal geodesic, has been modified by some investigators to eliminate the iterative process and the use of tables other than natural trigonometric, but usually involving root extraction, [3], [4]. Others have followed Clarke's method which in general involves tables for a particular reference ellipsoid, and may involve root extraction, [5].

Since the difference in length between the elliptic normal sections or the great elliptic section and the geodesic is of the 4th order in the eccentricity of the meridian ellipse, formulae have appeared computing these lengths rather than the geodesic, some using also azimuths of these sections rather than geodesic azimuths and with the option, in some cases, of applying difference or differential correction formulae for finally converting to geodesic length and geodesic azimuths, [6], [7], [8], [9]. Particularly with respect to long geodetic lines, the literature is quite extensive, [10], [11], [12], [13], [14]. Many of these formulae as published were developed to give distances up to a fixed predetermined maximum length with a given accuracy and fail almost immediately on lines in excess of that maximum. Many involve coefficients of many terms in powers of the eccentricity or other associated parameter. None of these examined appeared capable of supplying the versatility required under the criteria adopted for the present study.

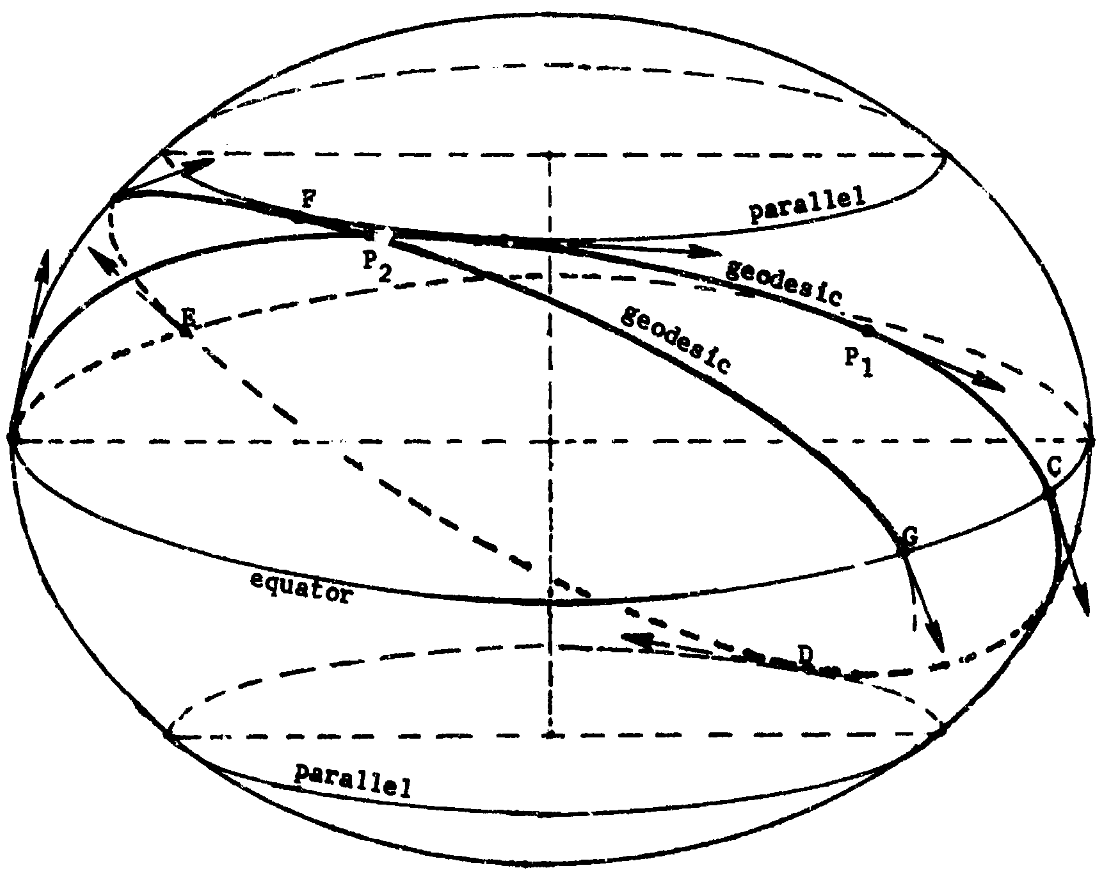
THE GEODESIC ON THE OBLATE SPHEROID

The longest plane closed curve on the oblate ellipsoid of revolution is the circular equator, and the shortest closed curve, which is also a geodesic, is the meridian. The equator and the meridians are the only

*Bracketed numbers refer to the bibliography attached to this report.

plane geodesics and the only closed geodesics. All other geodesics are three dimensional space curves, that is they have at each point two principal radii of curvature (a radius of curvature and a radius of torsion). The nonplanar geodesic oscillates symmetrically between tangencies to its two associated symmetric parallels with respect to the equator, and because of the flattening retrogresses through each revolution about the spheroid and thus cannot close on itself, as shown in Figure 1.

The geodesic is fundamentally defined as the curve of shortest distance between two points on a surface. From the integral for arc length we may, by the calculus of variations, determine the conditions on the integrand for the arc length to be a minimum. Actually maximum or minimum (extrema). That is the distance by way of the geodesic around the back side of the oblate spheroid, between two points within a hemispheroid, would be the longest geodesic distance. In Figure 1, note the geodesic arc P_1P_2 where P_2 is a first point of crossing after one revolution about the spheroid. Around the backside, the geodesic distance from P_1 to P_2 is the long geodesic arc P_1CDEFP_2.



If the geodesic is traced from a point A, a node, on the equator in the direction of the tangents as shown it will not pass again through A after a complete revolution but will cross the equator at a point E as shown. The course is $AP_2P_1CDEFP_2G \ldots$; P_2 is the first point where the geodesic crosses itself.

Figure 1. Pictorial representation of the nonplanar geodesic on the oblate ellipsoid of revolution.

From the results of the extremal conditions may be deduced the property that the osculating plane at each point of a geodesic contains the normal to the surface, or equivalently that at each point of a geodesic the principal normal to the curve must coincide with the normal to the surface, [16]. But from simple mechanics, considering a string stretched under tension between two points on a *smooth* spheroid, we can show that the curve assumed by the string is a geodesic, [15].

Analogy with the subsatellite trace.

The normal projection of the orbit of an earth artificial satellite upon an ellipsoid of reference simulates the geodesic. The normal projection of an equatorial orbit is very near the equator and that of a polar orbit is close to a meridian. For other orbits, the satellite responds in greater degree to the flattening (the equatorial bulge) of the geoid (sea level surface) which is approximated by the reference ellipsoid. This effect on the satellite (sustained by its velocity-falling very slowly back to earth) with the rotation of the earth under the orbit, causes the trace of the trajectory (orbit) as projected normally upon the reference ellipsoid to oscillate between two parallels symmetric with respect to the equator as shown in Figure 2. The symmetric parallels are in latitude ± 48° corresponding to the satellite inclination (the angle between the orbit and the equator). Note also in Figure 2 that the longitude difference between successive equatorial traces is 30°. Hence for each half revolution of the satellite the earth turns 15° to the east under the orbit which is itself in an easterly direction. Hence the longitude difference, node to node (N_1 to N_2) of the continuous trace is 165° as shown. The orbit also retrogresses but only about 3° per day as shown at the injection point of the orbit. Now a geodesic on the Clarke 1866 ellipsoid with vertex parametric latitude 48° has a longitude difference node to node, of about 179° 36′ (see TABLE 8), and no geodesic on it can have a longitude difference, node to node, of less than about 179° 24′ and this is along the equator itself. Hence *the subsatellite trace is not a geodesic* on the reference ellipsoid but it behaves like one, oscillating between two symmetric parallels in latitude equal to the inclination of the orbit, and with no more than two nodes or two vertices (of the trace) within a hemispheroid (on the same side of a meridian). But this digression is useful to remind us that the nonplanar geodesic tries to climb to the nearest pole.

Geodesic antipodal zones.

The behavior of the geodesic, when the geodesic arc end points are nearly antipodal has been discussed in several sources [17], [24], [25]. Clearly if the two points are 180° apart on the equator, then the shortest distance between them on the surface is the meridional semilength. In fact the shortest distance on the surface between the end points of any diameter of the spheroid is either of the two equal arcs of the meridian subtended by the diameter—that is the *meridians are the only antipodal geodesics.* This is clearly so because of all the plane elliptic sections through any diameter of the oblate spheroid, the one with the largest eccentricity and therefore shortest length is the meridian.

Only the circular length πb along the equator belongs to the hemispheroidal family of geodesics (a vertex of each geodesic in a common meridian) and it is the shortest member. *There are no antipodal points on nonplanar spheroidal geodesics.* See Appendix 1 to this report for the proofs.

If the difference in longitude of two points on the equator is not π radians but $\pi(1-k)$ radians, where k is a small quantity, $k < f$ (f is the flattening of the spheroid) then there are two geodesics, symmetric with

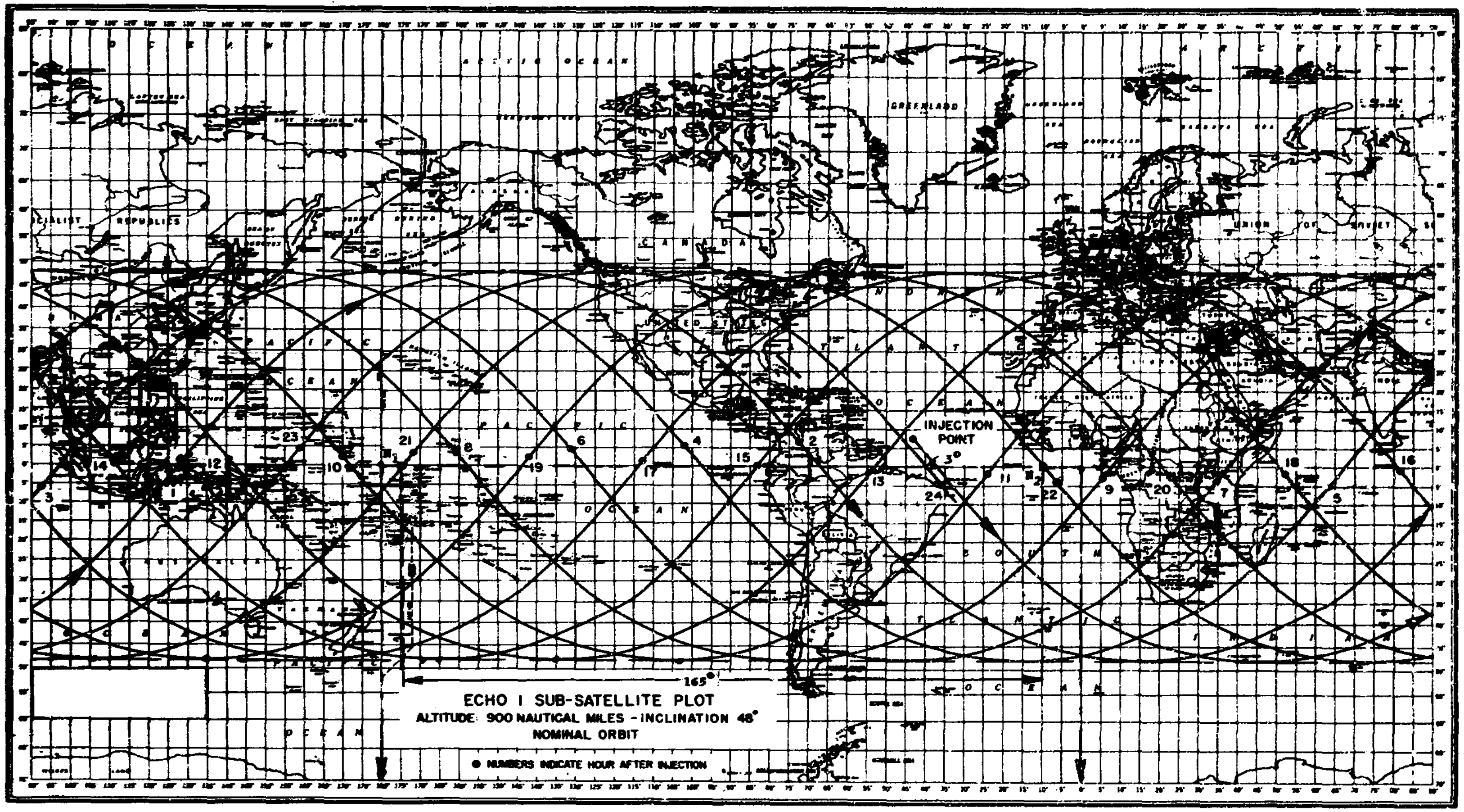

ECHO I SUB-SATELLITE PLOT
ALTITUDE: 900 NAUTICAL MILES - INCLINATION 48°
NOMINAL ORBIT
NUMBERS INDICATE HOUR AFTER INJECTION
INJECTION POINT
3°
165°
GREENLAND
CANADA
UNITED STATES
PACIFIC OCEAN
ARCTIC OCEAN
ATLANTIC OCEAN
SOUTH ATLANTIC OCEAN
INDIAN OCEAN
UNION OF SOVIET SOCIALIST REPUBLICS

Figure 2.

respect to the equator, which take advantage of the flattening and climb toward the poles. Note the geodesics (1) and (2) in Figure 3. If $k = f$, *the geodesic consists of the equatorial arc* $D\,D' = C\,C' = \pi a(1 - e^2)^{1/2} = \pi a(1 - f)$.

Continuing the discussion, with the help of Figure 3, we suppose that $T\,T'$ is an equatorial diameter of the spheroid orthogonal to a fixed meridian as shown. An arbitrary point P on the meridian has the symmetric R′ with respect to the equator, the symmetric R with respect to the polar axis, and the symmetric P′ with respect to the spheroidal center. There are thus four equal geodesics, two each with vertex latitude $\pm\,\theta_0$, determined by every point P and all are orthogonal to the fixed meridian. In the limit as $k \to f$, geodesics (1) and (2) coincide with the arc D D′ of the equator and analogously geodesics (3) and (4) coincide with the arc C C′. When $k \to 0$, $\theta_0 \to \pi/2$, $-\theta_0 \to -\pi/2$ and then geodesics (1) and (3), (2) and (4) respectively coincide with the upper and lower halves of the meridian ABA′B′ (plane of the paper in Figure 3).

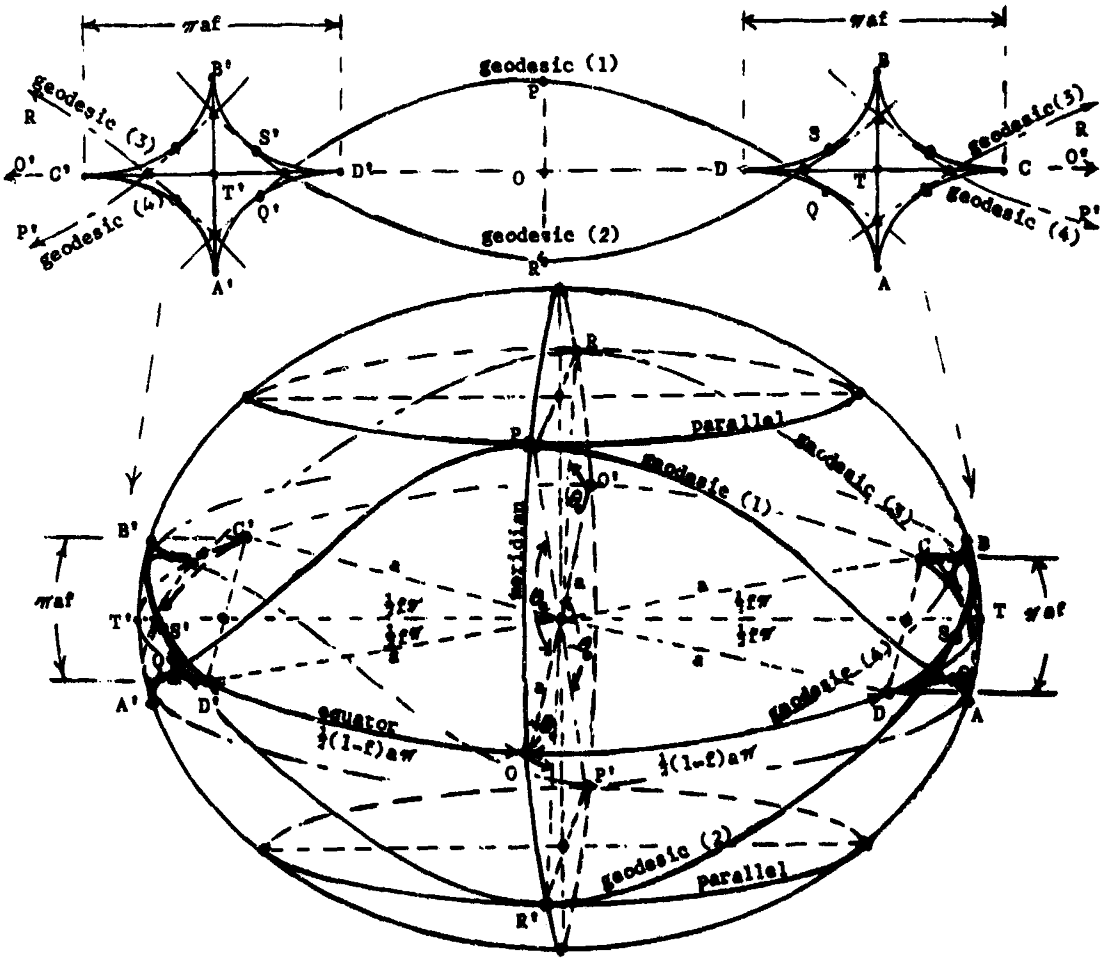


Figure 3. Pictorial representation of the two geodesic antipodal zones with respect to a given meridian of the oblate spheroid.

It has been shown that this family of geodesics, as depicted in Figure 3, has two evolutes. Cayley, [25], called these geodesic evolutes with respect a given meridian. These are shown pictorially in Figure 3 as the figures ABCD, A′B′C′D′ and they resemble the evolute of the meridian ellipse or a hypocycloid of four cusps since the eccentricity of the meridian is small with respect to earth reference ellipsoids. See also Figure 12 of Appendix 1. (The evolute of a given plane curve is the curve tangent to all normals or perpendiculars to the given curve—also called the envelope of the normals. On the spheroid, arcs of geodesics correspond to straight line segments in the plane relative to the shortest distance property.) Determination of the meridional arc axes of the geodesic evolutes (AB = A′B′ of Figure 3) requires the solution of a transcendental equation and is discussed in Appendix 1.

The spheroidal areas enclosed by the geodesic evolutes are called the geodesic antipodal zones with respect to a given meridian. Note from Figure 3 that only two consecutive nodes (equator crossings) occur in a hemispheroid and that they always lie in the geodesic antipodal zones with respect to the meridian containing the geodesic vertex. Because of the symmetry about the equator, the distance between consecutive nodes is the same as between consecutive vertices. Hence we may within a hemispheroid (on the same side of a meridian) have a maximum of two consecutive nodes and the vertex between them; or a maximum of two vertices and the node between them. For proof see Appendix 1 to this report.

Other properties of the geodesic.

The differential equation of the spheroidal geodesics may be found using the property of coincidence of principal normal to the curve and the normal to the surface at an arbitrary common point and it can be shown that the integral arc length depends on the evaluation of an elliptic integral. Since the eccentricity and the flattening are small quantities for earth reference ellipsoids, the series expansion of the integral in terms of eccentricity, flattening, or other associated parameter converges rapidly and evaluation is usually made in this way rather than by interpolation in elliptic integral tables.

An easily demonstrated but very important well known property of the geodesic on the oblate spheroid (or of the geodesic on any revolute) is ***that at each point of the geodesic the product of the radius of the parallel and the sine of the angle which the geodesic makes with the meridian is constant***. The mathematical demonstration is found in Appendix 1.

The problem of determining azimuths or geographic position of an end point of a geodesic arc involves solution of a polar spheroidal triangle and is usually approximated by solution of a corresponding spherical triangle or a sequence of them (iteration).

ACCURACY CRITERIA FOR COMPUTATIONS

While sophisticated computer systems are becoming more available universally, there is a need additionally or alternatively to have some computing forms which will give a reasonable geodetic accuracy over hemispheroidal geodesics for both direct or inverse (reverse) solutions with minimum requirements of a desk computer, only 8-place tables of natural trigonometric functions—no iteration or root extraction.

Accordingly the following criteria were adopted relative to the mathematical study included as Appendix I to this report:

1. An accuracy of 1 meter in position-geodetic length within 1 meter; latitude, longitude, and azimuth within .035 second-over the longest possible hemispheroidal geodesics, but in any case equalling the 1/100,000 distance and 1 second azimuth requirement adopted by ACIC, [22].

2. No tabular data required except 8-place natural trigonometric for desk computations.

3. No iteration or root extraction with formulae also adaptable to large electronic computing systems.

4. Easy adaptation to any reference ellipsoid by merely changing the scale parameters a, f, etc.

DIRECT SOLUTION

All direct solutions of the spheroidal triangle involve approximations by one or more spherical triangles. They differ with respect to the variables, parameters, required tabular data, arithmetic operations and subsequent accuracy. The formulae to be presented here involve corrections to a single spherical triangle. The variables are longitude, λ, parametric latitude, θ. Parameters are a, f, θ_0 where a, f are the semimajor axis and flattening of the reference ellipsoid and θ_0 is the parametric latitude of the geodesic vertex. The only tabular data required is a table, such as Peters, of the natural trigonometric functions. No root extraction or iteration is required in arithmetic operations.

We are given the point P_1 (ϕ_1, λ_1) on the spheroid, where ϕ_1, λ_1 are geodetic latitude and longitude (geographic coordinates); the forward azimuth $\alpha_{1\text{-}2}$ and distance S to a second point P_2 (ϕ_2, λ_2); and from these we are to find the geographic coordinates ϕ_2, λ_2 and the back azimuth $\alpha_{2\text{-}1}$. The given quantities are $\phi_1, \lambda_1, \alpha_{1\text{-}2}$, S.

Formulae. (The derivations are given in Appendix I)

Second Order in f.

$\tan\theta_1 = (1 - f)\tan\phi_1$, $M = \cos\theta_0 = \cos\theta_1 \sin\alpha_{1\text{-}2}$,

$N = \cos\theta_1 \cos\alpha_{1\text{-}2}$, $c_1 = fM$, $c_2 = (1/4) f(1 - M^2)$, $D = (1 - c_2)(1 - c_2 - c_1 M)$,

$P = c_2 [1 + (1/2)c_1 M]/D$, $\cos\sigma_1 = \sin\theta_1/\sin\theta_0$, $d = S/a\,D$, $u = 2(\sigma_1 - d)$,

$W = 1 - 2P\cos u$, $V = \cos(u + d) = \cos u \cos d - \sin u \sin d$, $X = c_2^2 \sin d \cos d\,(2V^2 - 1)$,

$Y = 2PVW \sin d$, $\Delta\sigma = d + X - Y$, $\Sigma\sigma = 2\sigma_1 - \Delta\sigma$,

$\tan\alpha_{2\text{-}1} = M/(N\cos\Delta\sigma - \sin\theta_1 \sin\Delta\sigma)$,

$\tan\phi_2 = -(\sin\theta_1 \cos\Delta\sigma + N\sin\Delta\sigma)\sin\alpha_{2\text{-}1}/(1 - f)M$,

$\tan\Delta\eta = \sin\Delta\sigma \sin\alpha_{1\text{-}2}/(\cos\theta_1 \cos\Delta\sigma - \sin\theta_1 \sin\Delta\sigma \cos\alpha_{1\text{-}2})$,

$H = c_1 (1 - c_2)\Delta\sigma - c_1 c_2 \sin\Delta\sigma \cos\Sigma\sigma$, $\Delta\lambda = \Delta\eta - H$, $\lambda_2 = \lambda_1 + \Delta\lambda$

First Order in f ($f^2 = 0$)

We place terms in f^2 equal to zero in the above equations which will remain the same except for the following:

$D = 1 - 2c_2 - c_1 M, P = c_2/D, X = 0, \Delta\sigma = d - Y, H = c_1 \Delta\sigma.$

Spherical ($f = 0$)

If we place $f = 0$ in the above equations we have $\tan \phi_1 = \tan \theta_1, \phi_1 = \theta_1,$

$M = \cos \theta_1 \sin \alpha_{1\text{-}2} = \cos \theta_0, N = \cos \theta_1 \cos \alpha_{1\text{-}2},$

$\Delta\sigma = d = S/a, \tan \alpha_{2\text{-}1} = M/(N \cos d - \sin \theta_1 \sin d),$

$\tan \phi_2 = -(\sin \theta_1 \cos d + N \sin d) \sin \alpha_{2\text{-}1}/M,$

$\tan \Delta\lambda = \sin d \sin \alpha_{1\text{-}2}/(\cos \theta_1 \cos d - \sin \theta_1 \sin d \cos \alpha_{1\text{-}2})$

$\lambda_2 = \lambda_1 + \Delta\lambda$; *a may be the radius of a spherical approximation such as given in Appendix 2.*

Sign Conventions for Azimuth and Longitude

We take the initial point to be west of the terminus in the direct solution and then always $180° \leq \alpha_{2\text{-}1} \leq 360°$. We also have $0 \leq \Delta\eta \leq \Delta\lambda \leq \pi$. If two arbitrary points are both in the southern hemispheroid (both in negative latitude), we solve as though both were in the northern hemispheroid and write symmetric elements with respect to the equator. While not necessary, these conventions simplify somewhat the determination of azimuth and longitude difference in desk computing.

From the quantities above in the formulae we find the first quadrant angles u and v given by $\tan u = |\tan \alpha_{2\text{-}1}|, \tan v = |\tan \Delta\eta|.$

If $\tan \alpha_{2\text{-}1} > 0$, *then* $\alpha_{2\text{-}1} = 180° + u$; *if* $\tan \alpha_{2\text{-}1} < 0$, *then* $\alpha_{2\text{-}1} = 360° - u$. *If* $\tan \Delta\eta > 0$, *then* $\Delta\eta = v$; *if* $\tan \Delta\eta < 0$, *then* $\Delta\eta = 180° - v$.

The conventions are sufficient, under the assumptions, as demonstrated by the following:

Always $0 \leq \alpha_{1\text{-}2} \leq 180°$. When $\tan \alpha_{2\text{-}1} > 0$, then $\alpha_{2\text{-}1}$ is in the third quadrant and is of the form $180° + u$, since $\tan(180° + u) = \tan u$. When $\tan \alpha_{2\text{-}1} < 0$, then $\alpha_{2\text{-}1}$ is in the fourth quadrant and is of the form $360° - u$, since $\tan(360° - u) = -\tan u$.

Since always $0 \leq \Delta\eta \leq \Delta\lambda \leq \pi$; when $\tan \Delta\eta > 0$, $\Delta\eta$ is in the first quadrant and $\Delta\eta = v$. When $\tan \Delta\eta < 0$, $\Delta\eta$ is in the second quadrant and is of the form $180 - v$, since $\tan(180 - v) = -\tan v$.

The arrangement of the direct formulae into a computing form is shown in Figure 18, Appendix 1.

INVERSE (REVERSE) SOLUTION

The published inverse solutions have been more varied than the direct. The series expansion for the geodesic length in the flattening f, spherical length d (with reference to the geodetic latitude of the vertex of the great elliptic arc) in the form

$$S = a[d - F_1(d)f + F_2(d)f^2 + \ldots]$$

was published by A.R. Forsyth in 1895, [20]. Errors in $F_2(d)$, making untenable the use of the second order term, remained undetected until 1965, [21]. The more recent examinations also revealed that the

Andoyer-Lambert expansions to first order in the flattening are merely those of Forsyth to first order in f, [18]. An independent verification of the corrections to Forsyth's equations was found in the work of Gougenheim, [23]. *Gougenheim's work has apparently gone unnoticed although he has had a correct expansion in terms of geodetic latitude to second order in the flattening since 1950.*

Forsyth had the expansion in parametric latitude to first order in the flattening. This was extended to second order as reported in [18]. The formulae for distance to be used here are basically those from [18]. The azimuth formulae are adaptations of those presented by Gougenheim in [23]. See Appendix 1, Equations (143).

We are given the points P_1 (ϕ_1, λ_1), P_2 (ϕ_2, λ_2) on the spheroid and are to find the distance S between the points and the forward and back azimuths, $\alpha_{1\text{-}2}$ and $\alpha_{2\text{-}1}$. Given quantities are ϕ_1, λ_1, ϕ_2, λ_2. It is assumed that east longitudes are positive and that P_1 is west of P_2.

Formulae

Second Order in f

$$\tan\theta_i = (1-f)\tan\phi_i,\ i = 1, 2$$
$$\theta_m = (1/2)(\theta_1 + \theta_2),\ \Delta\theta_m = (1/2)(\theta_2 - \theta_1),\ \Delta\lambda = \lambda_2 - \lambda_1,$$
$$\Delta\lambda_m = (1/2)\Delta\lambda,\ H = \cos^2\Delta\theta_m - \sin^2\theta_m = \cos^2\theta_m - \sin^2\Delta\theta_m,$$
$$L = \sin^2\Delta\theta_m + H\sin^2\Delta\lambda_m = \sin^2(1/2)d,\ 1 - L = \cos^2(1/2)d,\ \cos d = 1 - 2L,$$
$$U = 2\sin^2\theta_m\cos^2\Delta\theta_m/(1-L),\ V = 2\sin^2\Delta\theta_m\cos^2\theta_m/L,\ X = U + V,$$
$$Y = U - V,\ T = d/\sin d,\ D = 4T^2,\ E = 2\cos d,\ A = DE,\ B = 2D,$$
$$C = T - (1/2)(A - E); \qquad \text{check: } C - ½E + AD/B = T.$$
$$n_1 = X(A + CX),\ n_2 = Y(B + EY),\ n_3 = DXY,\ \delta_1 d = (1/4)f(TX - Y),$$
$$\delta_2 d = (f^2/64)(n_1 - n_2 + n_3),\ S_1 = a\sin d\,(T - \delta_1 d),\ S_2 = a\sin d\,(T - \delta_1 d + \delta_2 d),$$
$$F = 2Y - E(4 - X),\ M = 32T - (20T - A)X - (B + 4)Y,$$
$$G = (1/2)fT + (f^2/64)M,\ Q = -(FG\tan\Delta\lambda)/4,\ \Delta\lambda'_m = (1/2)(\Delta\lambda + Q),$$
$$c_1 = -\sin\Delta\theta_m/\cos\theta_m\tan\Delta\lambda'_m,\ u = \arctan|c_1|,\ a_1 = v - u,$$
$$c_2 = \cos\Delta\theta_m/\sin\theta_m\tan\Delta\lambda'_m,\ v = \arctan|c_2|,\ a_2 = v + u,$$

c_1	c_2	$\alpha_{1\text{-}2}$	$\alpha_{2\text{-}1}$
-	+	a_1	$360 - a_2$
+	+	a_2	$360 - a_1$
-	-	$180 - a_2$	$180 + a_1$
+	-	$180 - a_1$	$180 + a_2$

First Order in f ($f^2 = 0$)

$$\tan\theta_i = (1-f)\tan\phi_i,\ i = 1, 2;\ \theta_m = (1/2)(\theta_1 + \theta_2),\ \Delta\theta_m = (1/2)(\theta_2 - \theta_1),$$
$$\Delta\lambda = \lambda_2 - \lambda_1,\ \Delta\lambda_m = (1/2)\Delta\lambda,\ H = \cos^2\Delta\theta_m - \sin^2\theta_m = \cos^2\theta_m - \sin^2\Delta\theta_m,$$
$$L = \sin^2\Delta\theta_m + H\sin^2\Delta\lambda_m = \sin^2 ½d,\ 1 - L = \cos^2 ½d,\ \cos d = 1 - 2L,$$

$U = 2\sin^2\theta_m \cos^2\Delta\theta_m/(1-L)$, $V = 2\sin^2\Delta\theta_m \cos^2\theta_m/L$,

$X = U + V$, $Y = U - V$, $T = d/\sin d$, $\delta_1 d = (1/4)f(TX - Y)$,

$S = a\sin d\,(T - \delta_1 d)$, $F = 2\,[Y - (1-2L)(4-X)]$, $G = (1/2)fT$,

$Q = -(FG\tan\Delta\lambda)/4$, $\Delta\lambda'_m = (1/2)(\Delta\lambda + Q)$; the rest of the azimuth solution is the same as for the original formulae above.

Spherical (f = 0)

With f = 0 in the above formulae we have:

$\tan\phi_1 = \tan\theta_1$, $\phi_1 = \theta_1$, $\theta_m = (1/2)(\theta_1 + \theta_2)$, $\Delta\theta_m = (1/2)(\theta_2 - \theta_1)$,

$\Delta\lambda = \lambda_2 - \lambda_1$, $\Delta\lambda_m = (1/2)\Delta\lambda$, $H = \cos^2\Delta\theta_m - \sin^2\theta_m = \cos^2\theta_m - \sin^2\Delta\theta_m$,

$L = \sin^2\Delta\theta_m + H\sin^2\Delta\lambda_m$, $\cos d = 1 - 2L$, $S = ad$, $Q = 0$, $\Delta\lambda'_m = \Delta\lambda_m$,

$c_1 = -\sin\Delta\theta_m/\cos\theta_m \tan\Delta\lambda_m$, $c_2 = \cos\Delta\theta_m/\sin\theta_m \tan\Delta\lambda_m$; the rest of the azimuth solution is the same as above.

Azimuth Determination—Elimination of Quadrant Search

In the above formulae we ***find two first quadrant angles*** given by $u = \arc\tan|c_1|$, $v = \arc\tan|c_2|$. We then form $\alpha_1 = v - u$, $\alpha_2 = v + u$ and determine the azimuths according to the signs of c_1 and c_2 from the array:

c_1	c_2	$\alpha_{1\text{-}2}$	$\alpha_{2\text{-}1}$
-	+	α_1	$360 - \alpha_2$
+	+	α_2	$360 - \alpha_1$
-	-	$180 - \alpha_2$	$180 + \alpha_1$
+	-	$180 - \alpha_1$	$180 + \alpha_2$

This, in effect, eliminates the quadrant search since it has been done in advance. For the development of these expressions see Appendix 1.

The arrangement of the inverse formulae into a computing form is shown in figure 25, Appendix 1.

DESK COMPUTATIONS OF DIRECT AND INVERSE SOLUTIONS

For a demonstration of the direct and inverse forms, Figures 18, 25–Appendix 1, the long line published in reference [4] will be used. Its elements as given there are:

ORIGIN $\phi_1 = 20°$, $\lambda_1 = 0$; $S = 9649412.505$ meters

TERMINUS $\phi_2 = 45°$, $\lambda_2 = 106°$; $\alpha_{1\text{-}2} = 42°\ 56'\ 30''.035$.

$f = .003367003367$, $a = 6378388$ meters, $\alpha_{2\text{-}1} = 295°\ 17'\ 18''.600$

To provide a check for this line we use equations (49), (50) of Appendix 1 to make an independent computation as follows:

$f = .003367003367$, $a = 6378388$ m, $\cos\theta_0 = .64042078$, $\sin\theta_0 = .76802423$,

$c_1 = f\cos\theta_0 = .2156299892 \times 10^{-2}$, $A = c_1(1 - c_2 c_4) = .215522628 \times 10^{-2}$

$c_2 = (1/4)f \sin^2 \theta_0 = .49651618 \times 10^{-3}$. $B = (1/2)c_1c_2c_3 = .53606 \times 10^{-6}$

$c_3 = 1 + c_1 \cos \theta_0 = 1.0013809386$, $D = 2 + c_2(c_3^2 + c_4^2) - (1 + c_2)\, c_4 - c_2 = .9976269631$

$c_4 = c_2 + c_3 = 1.0018774548$, $E = ½c_2\,[2 + c_3(c_3 - 1) - c_2^2] = .49685942 \times 10^{-3}$

$\eta_1 = 72° \; 23' \; 36''.933$, $F = ⅛c_2^2(2c_4 - 1) = .6186 \times 10^{-7}$

$\eta_2 = 33° \; 47' \; 36''.695$, $\Sigma\eta = \eta_1 + \eta_2 = 106° \; 11' \; 13''.628 = 1.8533148482$ rad.

$\sigma_1 = 63° \; 38' \; 26''.269$, $\Sigma\sigma = \sigma_1 + \sigma_2 = 86° \; 50' \; 29''.583 = 1.5156709899$ rad.

$\sigma_2 = 23° \; 12' \; 03''.314$, $\sin \Sigma\sigma = .99848098$, $\Delta\sigma = \sigma_1 - \sigma_2 = 40° \; 26' \; 22''.955$,

$\cos \Delta\sigma = .76108893$, $\sin 2\Sigma\sigma = .11002746$, $p = 2 \sin \Sigma\sigma \cos \Delta\sigma = 1.51986564$,

$\cos 2\Delta\sigma = .15851273$, $q = 2 \sin 2\Sigma\sigma \cos 2\Delta\sigma = .034881506$.

$\Sigma\eta$	1.8533148482	$D\Sigma\sigma$	1.5120742467
$-A\Sigma\sigma$	$-$ .0032666139	$+Ep$	+ .0007551596
	1.8500482343		1.5128294063
$+Bp$	+ .0000008147	$-Fq$	$-$. 22
$\Delta\lambda$(rad)	1.8500490490	S/a	1.5128294041
$\Delta\lambda$	$106° \; 00' \; 00''.009$	S	9649412.917 m

Figures 4 to 9 are respectively the direct and inverse solutions of this line – second order in f, first order in f(f^2 = 0), and spherical (f = 0). The crosses in spaces of the first order and spherical examples indicate values to be omitted in the computation.

Computation of the direct solution.

Second order in f. We first identify the reference ellipsoid to be used at the top of the form and enter the indicated spheroidal constants from Appendix 2. The given quantities ϕ_1, α_{1-2}, S, λ_1 are then entered in the spaces provided with heavy underline as shown in figure 4; $\sin \alpha_{1-2}$, $\cos \alpha_{1-2}$, $\tan \phi_1$ are found from the Peters (or other) 8-place tables of natural trigonometric functions; $\tan \phi_1$ is multiplied by 1 – f to get $\tan \theta_1$ as shown, and then $\sin \theta_1$, $\cos \theta_1$ are found from the tables. In using linear interpolation in the Peters Tables, always take the tabular difference at the particular second in the table unless the difference is constant for the particular column as marked top and bottom. For instance, at 41° 52' the tabular difference is a constant 361 for the sine column as indicated top and bottom, but this is seldom so. Convenient checks are provided by the identities $\sin \theta/\cos \theta = \tan \theta$, $\sin^2 \theta + \cos^2 \theta = 1$.

After M, N, θ_0, $\sin \theta_0$ have been found we compute the constants c_1, c_2, D, P. We may compute c_2 in two ways since $1 - M_2 = \sin^2 \theta_0$. We next find σ_1; then d = S/aD which is in radians. At the top of the form find 1 radian = 206264.8062 seconds. This factor is multiplied by d (radians) and then divided by 3600 seconds (1 degree) which will give an integral number of degrees plus a decimal part of a degree. This decimal part is multiplied by 60 to get an integral number of minutes plus a decimal part of a minute. The decimal part of a minute is multiplied by 60 to get seconds retaining three decimals. If the total number of seconds is less than 3600, but more than 60, we divide by 60 to get minutes and then continue as above. Always check by reversing the process to get the radians d.

With σ_1 and d in degrees we form $u = 2(\sigma_1 - d)$ and find sin d, cos d, sin u, cos u. These are checked by $\sin^2 x + \cos^2 x = 1$. V and W may now be computed and then X and Y. Note that X may be ignored if it is less than $.3 \times 10^{-8}$. Next $\Delta\sigma$ is computed from the radian values of d, X, Y and converted to degrees. $\Delta\sigma$ and d always differ by only a few minutes. $\Sigma\sigma$ is formed in degrees and then $\sin \Delta\sigma$, $\cos \Delta\sigma$, $\cos \Sigma\sigma$ found from the tables. We are then able to compute $\tan \alpha_{2-1}$. The first quadrant solution for $\tan u = 2.11661579$ is $u = 64° 42' 41.''399$. Since the sign of $\tan \alpha_{2-1}$ as computed is negative, we have $\alpha_{2-1} = 360° - u = 295° 17' 18.''601$; $\sin \alpha_{2-1} = \sin (360° - u) = -\sin u = -\sin 64° 42' 41.''399 = -.90416831$. We may now compute $\tan \phi_2$ which is found to be $1.00000003 = 1 + (3 \times 10^{-8})$ and from the table $\phi_2 = 45° 00' 00.''003$. Next find $\tan \Delta\eta = -3.4449133$. From the Peters tables we find for $\tan v = +3.4449133$, that $v = 73° 48' 46.''375$, but since the sign of $\tan \Delta\eta$ is negative, $\Delta\eta = 180° - v = 106° 11' 13.''625$. Now the computation for H is in radians and converting to angular value, $H = 11' 13.''620$. We subtract H from $\Delta\eta$ and add the difference, $\Delta\lambda$, to λ_1 to get $\lambda_2 = 106° 00' 00.''005$ as shown.

First order in f($f^2 = 0$). The input quantities are the same as shown in figure 4. We "cross out" the quantities to be omitted as shown in Figure 6, and the computational procedure is then the same.

Spherical ($f = 0$). We must adopt a spherical radius. For figure 8 we have adopted the great normal radius for $\phi = 20°$, see Appendix 2, equations (11) and (22). The quantities to be omitted are then "crossed out" and the simplified computations made as shown.

Computation of the inverse solution.

Second order in f. We enter the name of the reference ellipsoid to be used and the corresponding spheroidal constants from Appendix 2. The given quantities $\phi_1, \phi_2, \lambda_1, \lambda_2$ are entered in the spaces with heavy underline as shown in Figure 5. We find $\tan \phi_1$, and ϕ_2 from the tables and compute $\tan \theta_1$, $\tan \theta_2$ as shown; then back to the tables to find θ_1, θ_2. We then form θ_m and $\Delta\theta_m$ and check by adding, since $\theta_m + \Delta\theta_m = \theta_2$. Next find $\Delta\lambda$, $\Delta\lambda_m$ and then from the tables $\sin \theta_m$, $\cos \theta_m$; $\sin \Delta\theta_m$, $\cos \Delta\theta_m$; $\sin \Delta\lambda_m$, $\tan \Delta\lambda$. We next compute two values of H as shown which should agree within 5 in the 9th place of decimals. Take the mean and retain 8 decimals. L is then computed retaining 8 decimals.

With the value of L, we form $1 - L$, $\cos d = 1 - 2L$ as shown; then find d, sin d from the tables. Now compute U, V, X, Y, T, E, D, B, A, C. Note that $B = 2D$, $A = DE$, $D = 4T^2$, $C = T - (1/2)(A - E)$, so that these are relatively easy to compute. The check is given by $T = C - ½E + AD/B = d/\sin d$.

Compute n_1, n_2, n_3, and then $\delta_1 d$, $\delta_2 d$. We can now compute S_1 (first order for comparison) or go directly to S_2 for the second order distance as shown in figure 5.

For the azimuths, we compute in order F, M, G, Q, $\Delta\lambda'_m$, $\tan \Delta\lambda'_m$. Then c_1, c_2, u, v and in that order. We add and subtract the quantities u, v to get $\alpha_1 = v - u$, $\alpha_2 = v + u$. Now the signs of c_1, c_2 are $-$, $+$ as shown in figure 5. Hence the azimuths are $\alpha_{1-2} = \alpha_1$, $\alpha_{2-1} = 360 - \alpha_2$ as shown.

First order in f. The heading information to first order in f and input quantities are the same as in figure 5. The quantities to be omitted are "crossed out" as shown in figure 7 and then the computations are done as before computing S_1 after finding the first order correction $\delta_1 d$.

Spherical ($f = 0$). We need a radius approximation to the ellipsoid and use that determined for the spherical direct solution which is the great normal length for $\phi = 20°$, $r = 6380897.5$ meters (international ellipsoid). The omitted quantities are then "crossed out" as shown in figure 9, and the simplified computation made analogously as shown.

DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1, \alpha_{1\text{-}2}$, S to find $\phi_2, \lambda_2, \alpha_{2\text{-}1}$. East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

INTERNATIONAL SPHEROID a 6378388 m f .003367 003367

1 - f .9966 32 9966 1 radian = 206264.8062 seconds

LINE INITIAL TO TERMINUS

ϕ_1 20° 0' 0" tan ϕ_1 .3639 7023 tan $\theta_1 = (1 - f)\tan\phi_1$ .36274474

$\alpha_{1\text{-}2}$ 42 56 30.035 sin θ_1 .3410 0267 cos θ_1 .94006233 θ_1 19° 56' 16.706"

sin $\alpha_{1\text{-}2}$ .6812 5353 $M = \cos\theta_0 = \cos\theta_1 \sin\alpha_{1\text{-}2}$ .64042078 θ_0 50 10 36.468

cos $\alpha_{1\text{-}2}$ .73204755 $N = \cos\theta_1 \cos\alpha_{1\text{-}2}$ .6881 7033 sin θ_0 .7680 2423

$c_1 = fM$ .2156299x10^-2 $D = (1 - c_2)(1 - c_2 - c_1 M)$ .997626961 2

$c_2 = \frac{1}{4}(1 - M^2)f$ .4965162x10^-3 $P = c_2(1 + \frac{1}{2}c_1 M)/D$ .498041x10^-3

$\cos\sigma_1 = \sin\theta_1/\sin\theta_0$ .4439 9989 σ_1 63° 38' 26.271"

$d = S/aD$ 1.51642 7946 (rad) d 86 53 05.716 S 9649412.912 m

sin d .9985 2240 $u = 2(\sigma_1 - d)$ -46 29 18.890 sin u -.7252 3716

cos d .0543 4160 $W = 1 - 2P\cos u$ .999314199 cos u .6884 9914

$V = \cos u \cos d - \sin u \sin d$ .761579694 $Y = 2PVW \sin d$ .0007569553

$X = c_2^2 \sin d \cos d (2V^2 - 1)$ .214x10^-8 (ignore) $\Delta\sigma = d + X - Y$ 1.515670991 (rad)

sin $\Delta\sigma$ .9984 8098 cos $\Delta\sigma$ .0550 9742 $\Delta\sigma$ 86° 50' 29.583"

cos $\Sigma\sigma$ +.7610 8892 $\Sigma\sigma = 2\sigma_1 - \Delta\sigma$ 40° 26' 22.959"

$\tan\alpha_{2\text{-}1} = M/(N\cos\Delta\sigma - \sin\theta_1 \sin\Delta\sigma)$ -2.1166 1579 $\alpha_{2\text{-}1}$ 295 17 18.601

$\tan\phi_2 = \dfrac{-(\sin\theta_1 \cos\Delta\sigma + N \sin\Delta\sigma)\sin\alpha_{2\text{-}1}}{(1 - f)M}$ 1+(3x10^-8) sin $\alpha_{2\text{-}1}$ -.9041 6831

ϕ_2 45° 00' 00.003"

$\tan\Delta\eta = \dfrac{\sin\Delta\sigma \sin\alpha_1}{\cos\theta_1 \cos\Delta\sigma - \sin\theta_1 \sin\Delta\sigma \cos\alpha_{1\text{-}2}}$ -3.4449133 $\Delta\eta$ 106° 11' 13.625"

$H = c_1(1 - c_2)\Delta\sigma - c_1 c_2 \sin\Delta\sigma \cos\Sigma\sigma$ .003265803 (rad) H 11 13.620

$\Delta\lambda = \Delta\eta - H$ 106 00 00.005

λ_1 0 0 0

CHECK

$M = \cos\theta_0 = \cos\theta_1 \sin\alpha_{1\text{-}2} = \cos\theta_2 \sin(180 + \alpha_{2\text{-}1})$ $\lambda_2 = \lambda_1 + \Delta\lambda$ 106° 00' 00.005"

Figure 4. Direct computation—second order in f.

INVERSE POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1; \phi_2, \lambda_2$ to find $S, \alpha_{1-2}, \alpha_{2-1}$. Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

INTERNATIONAL SPHEROID a 6378388 m   b 6356911.945 m

$1 - f = b/a$ .9966329966 $\tfrac{1}{2}f$ .0016835017 $\tfrac{1}{4}f$ .00084175085

$f^2/64$ .17713612 x10⁻⁶ → $.17713612 \times 10^{-6}$ 1 radian = 206264.8062 seconds

ϕ_1 20° 0′ 0″	1. INITIAL	λ_1 0° 0′ 0″
ϕ_2 45 0 0	2. TERMINUS	λ_2 106 0 0
$\tan\phi_1$ .36397023	1. always west of 2.	$\Delta\lambda = \lambda_2 - \lambda_1$ 106°
$\tan\phi_2$ 1	$\tan\theta = (1-f)\tan\phi$	$\Delta\lambda_m = \tfrac{1}{2}\Delta\lambda$ 53
θ_2 44° 54′ 12.″168	$\tan\theta_2$ .99663300	$\sin\Delta\lambda_m$ .79863551
θ_1 19 56 16.706	$\tan\theta_1$ .36274475	$\tan\Delta\lambda$ −3.4874144
$\theta_m = \tfrac{1}{2}(\theta_1+\theta_2)$ 32° 25′ 14.″437	$\sin\theta_m$ .53613146	$\cos\theta_m$ .84413450
$\Delta\theta_m = \tfrac{1}{2}(\theta_2-\theta_1)$ 12 28 57.731	$\sin\Delta\theta_m$ .21614487	$\cos\Delta\theta_m$ .97636130

$H = \cos^2\Delta\theta_m - \sin^2\theta_m = \cos^2\theta_m - \sin^2\Delta\theta_m$ .66584445 $1 - L$ .52859337

$L = \sin^2\Delta\theta_m + H\sin^2\Delta\lambda_m$ .47140663 $\cos d = 1 - 2L$ +.05718674

$U = 2\sin^2\theta_m\cos^2\Delta\theta_m/(1-L)$ 1.036745079 d 86° 43′ 17.″950

$V = 2\sin^2\Delta\theta_m\cos^2\theta_m/L$ .141236672 $\sin d$ .99836350 d (rad) 1.5135783741

$X = U + V$ 1.1779811751	$T = d/\sin d$ 1.5160594053	$E = 2\cos d$ .11437348
$Y = U - V$ .8955084070	$D = 4T^2$ 9.193744487	$B = 2D$ 18.38748896
$A = DE$ 1.051520551	$C = T - \tfrac{1}{2}(A-E)$ 1.04748586	CHECK $C - \tfrac{1}{2}E + AD/B = T$
$n_1 = X(A+CX)$ 2.692204325	$n_2 = Y(B+EY)$ 16.55787107	$n_3 = DXY$ 9.698407923

$\delta_1 d = \tfrac{1}{4}f(TX - Y)$ .000749479 $\delta_2 d = (f^2/64)(n_1 - n_2 + n_3)$ $-.738 \times 10^{-6}$

$S_1 = a\sin d\,(T - \delta_1 d)$ 9649417.494 m $S_2 = a\sin d\,(T - \delta_1 d + \delta_2 d)$ 9649412.793 m

$F = 2Y - E(4 - X)$ 1.4682527 $M = 32T - (20T - A)X - (B+4)Y$ −6.01340099

$G = \tfrac{1}{2}fT + (f^2/64)M$ .00255122337 $Q = -(FG\tan\Delta\lambda)/4$ + 0° 11′ 13.″625

$\Delta\lambda'_m = \tfrac{1}{2}(\Delta\lambda + Q)$ 53° 05′ 36.″813 $\tan\Delta\lambda'_m$ 1.33156317

$v = \arctan|c_2|$ 53° 49′ 35.″717 $c_2 = \cos\Delta\theta_m/(\sin\theta_m\tan\Delta\lambda'_m)$ +1.36765794

$u = \arctan|c_1|$ 10 53 05.683 $c_1 = -\sin\Delta\theta_m/(\cos\theta_m\tan\Delta\lambda'_m)$ −.19229656

$\alpha_1 = v - u$ 42° 56′ 30.″034 $\alpha_2 = v + u$ 64° 42′ 41.″400

c_1	c_2	α_{1-2}		α_{2-1}	
−	+	α_1	42° 56′ 30.″034	$360 - \alpha_2$	295° 17′ 18.″600
+	+	α_2		$360 - \alpha_1$	
−	−	$180 - \alpha_2$		$180 + \alpha_1$	
+	−	$180 - \alpha_1$		$180 + \alpha_2$	

Figure 5. Inverse computation—second order in f.

DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1, \alpha_{1\text{-}2}$, S to find $\phi_2, \lambda_2, \alpha_{2\text{-}1}$. East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

INTERNATIONAL SPHEROID a 6378388 m f .003367 003367

1 - f .9966329966 1 radian = 206264.8062 seconds

LINE INITIAL TO TERMINUS

ϕ_1 20° 0' 0" tan ϕ_1 .3639 7023 tan θ_1 = (1 - f) tan ϕ_1 .3627 4474

$\alpha_{1\text{-}2}$ 42 56 30.035 sin θ_1 .3410 0267 cos θ_1 .9400 6233 θ_1 19° 56' 16.706"

sin $\alpha_{1\text{-}2}$ .6812 5353 M = cos θ_0 = cos θ_1 sin $\alpha_{1\text{-}2}$ .6404 2078 θ_0 50 10 36.468

cos $\alpha_{1\text{-}2}$ .7320 4755 N = cos θ_1 cos $\alpha_{1\text{-}2}$ .6881 7033 sin θ_0 .7680 2423

c_1 = fM .2156299×10^{-2} D = ~~(1 - c₂)(1 - c₁ - c₁M)~~ 1 - 2c₂ - c₁M .99762603

c_2 = ¼(1 - M²)f .4965162×10^{-3} P = c_2 (1 + ~~½c₁M~~)/D = c_2/D .497698×10^{-3}

cos σ_1 = sin θ_1/sin θ_0 .4439 9989 σ_1 63° 38' 26.271"

d=S/aD 1.516429361 (rad) d 86 53 06.008 S 9649412.912 m

sin d .9985 2248 u = 2(σ_1 - d) -46 29 19.474 sin u -.7252 3911

cos d .0543 4019 W = 1 - 2P cos u .999314673 cos u +.6884 9708

V = cos u cos d - sin u sin d .761580617 Y = 2PVW sin d .000756435

X = c_2^2 sin d cos d (2V² - 1) X Δσ = d ~~+X~~ - Y = d-Y 1.515672926 (rad)

sin Δσ .9984 8109 cos Δσ .0550 9549 Δσ 86° 50' 29.982"

cos Σσ X Σσ = 2σ_1 - Δσ X

tan $\alpha_{2\text{-}1}$ = M/(N cos Δσ - sin θ_1 sin Δσ) -2.11660624 $\alpha_{2\text{-}1}$ 295° 17' 18.960"

$$\tan\phi_2 = \frac{-(\sin\theta_1 \cos\Delta\sigma + N \sin\Delta\sigma)\sin\alpha_{2\text{-}1}}{(1-f)M}$$ .99999835 sin $\alpha_{2\text{-}1}$ .9041 6756

ϕ_2 44° 59' 59.830"

$$\tan\Delta\eta = \frac{\sin\Delta\sigma \sin\alpha_{1\text{-}2}}{\cos\theta_1 \cos\Delta\sigma - \sin\theta_1 \sin\Delta\sigma \cos\alpha_{1\text{-}2}}$$ -3.44488816 Δη 106° 11' 14.135"

H = c_1(1 - ~~c_2~~) Δσ - ~~$c_1 c_2$ sin Δσ cos Σσ~~ = c_1Δσ .003268243 (rad) H 11 14.124

Δλ = Δη - H 106 00 00.011

λ_1 0 0 0

CHECK

M = cos θ_0 = cos θ_1 sin $\alpha_{1\text{-}2}$ = cos θ_2 sin (180 + $\alpha_{2\text{-}1}$) λ_2 = λ_1 + Δλ 106° 00' 00.011"

Figure 6. Direct computation--first order in f.

INVERSE POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1; \phi_2, \lambda_2$ to find $S, \alpha_{1-2}, \alpha_{2-1}$. Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

INTERNATIONAL SPHEROID a 6378388 m b 6356911.945 m

$1 - f = b/a$ .9966329966 $\frac{1}{2}f$ .0016835017 $\frac{1}{4}f$ .00084175085

$f^2/64$ 0 1 radian = 206264.8062 seconds

ϕ_1 20° 0′ 0″ 1. INITIAL λ_1 0° 0′ 0″

ϕ_2 45 0 0 2. TERMINUS λ_2 106

$\tan \phi_1$ .36397023 1. always west of 2. $\Delta\lambda = \lambda_2 - \lambda_1$ 106

$\tan \phi_2$ 1 $\tan \theta = (1 - f) \tan \phi$ $\Delta\lambda_m = \frac{1}{2}\Delta\lambda$ 53

θ_2 44° 54′ 12.″168 $\tan \theta_2$ .99663300 $\sin \Delta\lambda_m$ .79863551

θ_1 19 56 16.706 $\tan \theta_1$ .36274475 $\tan \Delta\lambda$ −3.4874144

$\theta_m = \frac{1}{2}(\theta_1 + \theta_2)$ ° ′ ″ $\sin \theta_m$ .53613146 $\cos \theta_m$ .84413450

$\Delta\theta_m = \frac{1}{2}(\theta_2 - \theta_1)$ $\sin \Delta\theta_m$ .21614487 $\cos \Delta\theta_m$ .97636130

$H = \cos^2 \Delta\theta_m - \sin^2 \theta_m = \cos^2 \theta_m - \sin^2 \Delta\theta_m$ .66584445 $1 - L$ .52859337

$L = \sin^2 \Delta\theta_m + H \sin^2 \Delta\lambda_m$ .47140663 $\cos d = 1 - 2L$ +.05718674

$U = 2 \sin^2 \theta_m \cos^2 \Delta\theta_m / (1 - L)$ 1.036745079 d 86° 43′ 17.″950

$V = 2 \sin^2 \Delta\theta_m \cos^2 \theta_m / L$ .141236672 $\sin d$ .99836350 d (rad) 1.5135783741

$X = U + V$ 1.177981175 $T = d/\sin d$ 1.516059405 $E = 2 \cos d$ .11437348

$Y = U - V$ .895508407 $D = 4T^2$ X $B = 2D$ X

$A = DE$ X $C = T - \frac{1}{2}(A - E)$ X CHECK $C - \frac{1}{2}E + AD/B = T$

$n_1 = X(A + CX)$ X $n_2 = Y(B + EY)$ X $n_3 = DXY$ X

$\delta_1 d = \frac{1}{4}f(TX - Y)$ .000749479 $\delta_2 d = (f^2/64)(n_1 - n_2 + n_3)$ X

$S_1 = a \sin d (T - \delta_1 d)$ 9649417.494 m $S_2 = a \sin d (T - \delta_1 d + \delta_2 d)$ X m

$F = 2Y - E(4 - X)$ 1.4682527 $M = 32T - (20T - A)X - (B + 4)Y$ X

$G = \frac{1}{2}fT + (\cancel{f^2/64})M$ .0025522886 $Q = -(FG \tan \Delta\lambda)/4$ 11′ 13.″906

$\Delta\lambda_m' = \frac{1}{2}(\Delta\lambda + Q)$ 53° 05′ 36.″953 $\tan \Delta\lambda_m'$ 1.33156506

$v = \arctan |c_2|$ 53° 49′ 35.″578 $c_2 = \cos \Delta\theta_m / (\sin \theta_m \tan \Delta\lambda_m')$ +1.36765600

$u = \arctan |c_1|$ 10° 53′ 05.″627 $c_1 = -\sin \Delta\theta_m / (\cos \theta_m \tan \Delta\lambda_m')$ −.19229628

$\alpha_1 = v - u$ 42° 56′ 29.″951 $\alpha_2 = v + u$ 64° 42′ 41.″205

c_1	c_2	α_{1-2}		α_{2-1}	
−	+	α_1	42° 56′ 29.″951	$360 - \alpha_2$	295° 17′ 18.″795
+	+	α_2		$360 - \alpha_1$	
−	−	$180 - \alpha_2$		$180 + \alpha_1$	
+	−	$180 - \alpha_1$		$180 + \alpha_2$	

Figure 7. Inverse solution—first order in f.

DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1, \alpha_{1\text{-}2}$, S to find $\phi_2, \lambda_2, \alpha_{2\text{-}1}$. East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

SPHERE ~~SPHE~~ROID a = N₀ 6380892.5 m f 0

1 - f 0 1 radian = 206264.8062 seconds

LINE INITIAL TO TERMINUS

ϕ_1 20° 0' 0" tan ϕ_1 X tan $\theta_1 = (1 - f) \tan \phi_1$ X

$\alpha_{1\text{-}2}$ 42 56 30.035 sin θ_1 .34202014 cos θ_1 .93969262 $\theta_1 = \phi$ 20° 0' 0"

sin $\alpha_{1\text{-}2}$ .68175353 $M = \cos\theta_0 = \cos\theta_1 \sin\alpha_{1\text{-}2}$ .64016891 θ_0 X

cos $\alpha_{1\text{-}2}$ .73204755 $N = \cos\theta_1 \cos\alpha_{1\text{-}2}$ .68789968 sin θ_0 X

$c_1 = fM$ X $D = (1 - c_2)(1 - c_2 - c_1 M)$ X

$c_2 = \frac{1}{4}(1 - M^2)f$ X $P = c_2 (1 + \frac{1}{2}c_1 M)/D$ X

$\cos\sigma_1 = \sin\theta_1 / \sin\theta_0$ X σ_1 ° ' " X

d = ~~S/aD~~ = S/N₀ 1.5122344 (rad) d X S 9649412.912 m

sin d X $u = 2(\sigma_1 - d)$ X sin u X

cos d X $W = 1 - 2P \cos u$ X cos u X

$V = \cos u \cos d - \sin u \sin d$ X $Y = 2PVW \sin d$ X

$X = c_2^2 \sin d \cos d (2V^2 - 1)$ X $\Delta\sigma = d + X - Y$ X (rad)

sin ~~Δσ~~ d .99828575 cos ~~Δσ~~ d .05852843 Δσ = d 86° 38' 40.742"

cos $\Sigma\sigma$ X $\Sigma\sigma = 2\sigma_1 - \Delta\sigma$ X

$\tan\alpha_{2\text{-}1} = M/(N \cos\Delta\sigma - \sin\theta_1 \sin\Delta\sigma)$ -2.12559137 $\alpha_{2\text{-}1}$ 295° 11' 41.935"

$\tan\phi_2 = \dfrac{-(\sin\theta_1 \cos\Delta\sigma + N \sin\Delta\sigma)\sin\alpha_{2\text{-}1}}{(1 - f)M}$ +.99896073 sin $\alpha_{2\text{-}1}$ -.90486614

ϕ_2 44° 58' 12.238"

$\tan \Delta\cancel{\eta}\lambda = \dfrac{\sin\Delta\sigma \sin\alpha_{1\text{-}2}}{\cos\theta_1 \cos\Delta\sigma - \sin\theta_1 \sin\Delta\sigma \cos\alpha_{1\text{-}2}}$ -3.4885659 $\Delta\eta$ ° ' " X

$H = c_1(1 - c_2)\Delta\sigma - c_1 c_2 \sin\Delta\sigma \cos\Sigma\sigma$ 0 (rad) H X

$\Delta\lambda =$ ~~Δη - H~~ 105 59 41.961

λ_1 0 0 0

CHECK

$M = \cos\theta_0 = \cos\theta_1 \sin\alpha_{1\text{-}2} = \cos\theta_2 \sin(180 + \alpha_{2\text{-}1})$ $\lambda_2 = \lambda_1 + \Delta\lambda$ 105° 59' 41.961"

Figure 8. Direct Computation–spherical.

INVERSE POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1; \phi_2, \lambda_2$ to find S, $\alpha_{1-2}, \alpha_{2-1}$. Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

SPHERE ~~SPHEROID~~ a = N 6380891.5 m b X m

1 − f = b/a X ½f X ¼f X

$f^2/64$ X 1 radian = 206264.8062 seconds

ϕ_1 20° 0′ 0″	1. INITIAL	λ_1 0° 0′ 0″
ϕ_2 45 0 0	2. TERMINUS	λ_2 106
$\tan\phi_1$ X	1. always west of 2.	$\Delta\lambda = \lambda_2 - \lambda_1$ 106
$\tan\phi_2$ X	$\tan\theta = (1-f)\tan\phi$	$\Delta\lambda_m = ½\Delta\lambda$ 53
θ_2 X	$\tan\theta_2$ X	$\sin\Delta\lambda_m$ .79863551
θ_1 X	$\tan\theta_1$ X	$\tan\Delta\lambda$ X
$\theta_m = ½(\theta_1+\theta_2)$ 32° 30′	$\sin\theta_m$ .53729961	$\cos\theta_m$ .84339145
$\Delta\theta_m = ½(\theta_2-\theta_1)$ 12 30	$\sin\Delta\theta_m$ .21643961	$\cos\Delta\theta_m$ .97629601

$H = \cos^2\Delta\theta_m - \sin^2\theta_m = \cos^2\theta_m - \sin^2\Delta\theta_m$ .66446303 1 − L X

$L = \sin^2\Delta\theta_m + H\sin^2\Delta\lambda_m$ .47065304		$\cos d = 1 - 2L$ .05869393
$U = 2\sin^2\theta_m\cos^2\Delta\theta_m/(1-L)$ X		d 86° 38′ 06.545″
$V = 2\sin^2\Delta\theta_m\cos^2\theta_m/L$ X	sin d X	d (rad) 1.51206864
X = U + V X	T = d/sin d X	E = 2 cos d X
Y = U − V X	$D = 4T^2$ X	B = 2D X
A = DE X	C = T − ½ (A − E) X	CHECK C − ½ E + AD/B = T
$n_1 = X(A + CX)$ X	$n_2 = Y(B + EY)$ X	$n_3 = DXY$ X

$\delta_1 d = ¼f(TX - Y)$ X $\delta_2 d = (f^2/64)(n_1 - n_2 + n_3)$ X

$S_1 = \cancel{a \sin d (T - \delta_1 d)}$ = dN 9648355 m $S_2 = a \sin d (T - \delta_1 d + \delta_2 d)$ X m

F = 2Y − E (4 − X) X M = 32T − (20 T − A) X − (B + 4) Y X

$G = ½fT + (f^2/64)M$ X $Q = -(FG\tan\Delta\lambda)/4$ X

$\Delta\lambda'_m = ½(\Delta\lambda + Q)$ 53° $\tan\Delta\lambda'_m$ = tan $\Delta\lambda_m$ 1.32704482

v = arctan $|c_2|$ 53° 51′ 29.273″ $c_2 = \cos\Delta\theta_m/(\sin\theta_m\tan\Delta\lambda'_m)$ +1.36923943

u = arctan $|c_1|$ 10 56 42.068 $c_1 = -\sin\Delta\theta_m/(\cos\theta_m\tan\Delta\lambda'_m)$ −.19338463

α_1 = v − u 42° 54′ 47.205″ α_2 = v + u 64° 48′ 11.341″

c_1	c_2	α_{1-2}		α_{2-1}	
−	+	α_1	42° 54′ 47.205″	360 − α_2	295° 11′ 48.659″
+	+	α_2		360 − α_1	
−	−	180 − α_2		180 + α_1	
+	−	180 − α_1		180 + α_2	

Figure 9. Inverse computation—spherical.

Table 1 gives a comparison of the given line elements, the control computation, second order, first orders, and spherical computations. See Appendix 3 for more examples of direct and inverse solutions for several line lengths and in several azimuths. Also see the evaluation and comparison in Appendix 1. NOTE: *Appendix 4 gives the Fortran statements for the inverse solution as presented here.* The card deck including the arctangent library function (ATAN2) is available.

DISCUSSION OF PROBLEMS INVOLVING LONG GEODETIC LINES, LOCAL COORDINATE SYSTEMS, AND ASSOCIATED GEOMETRY

General Remarks.

If we wish to compute reference lines connecting islands, continents, shoals in ocean areas, there are several alternatives available depending on the purpose for which the reference is needed and the accuracy required. Direct scaling from a large accurate globe may be used. If a mean spherical representation of the reference ellipsoid can be tolerated, then a plot of computed great circle intervals on an authalic (equal area), autogonal (true angles about points), or aphylactic (neither authalic nor autogonal) projection may suffice. Within a radius of 10 n.m. of a station, simple plane coordinates, appropriately scaled, will be adequate for most geodetic work, and small relative errors will be incurred as far as 100 n.m. See Table 14, Appendix 2, for errors in distance from the origin associated with plane coordinates involving several types of geometric projection. Also included there is a discussion of plane coordinates. See also reference [9].

For track line reference, the azimuthal equidistant or doubly equidistant projection may be useful, although both are aphylactic. Appendix 2 has a discussion of the doubly equidistant projection with its equations. The Department of Scientific and Industrial Research, New Zealand, has found the azimuthal equidistant projections useful in their South Pacific studies, see reference [34].

		ϕ_2 ° ' "	λ_2 ° ' "	S(meters)	α_{1-2} ° ' "	α_{2-1} ° ' "
Given, Reference [4]		45	106	9649412.505	42 56 30.035	295 17 18.600
Control	Order f^3		106 00 00.009	.917		
Direct	f^2	45 00 00.003	00.005			.601
Inverse	f^2			.793	.034	.600
Direct	f	44 59 59.830	.021			.960
Inverse	f			17.494	29.951	.795
Direct (sphere)	0	44 58 12.238	105 59 41.961			11 41.935
Inverse (sphere)	0			9648355.0	54 47.205	11 48.659

Table 1. Comparison of direct and inverse desk computations.

The diagonal (skew, oblique) Mercator (cylindrical) projection, is often useful, since the base line (the track) is one of the reference axes for rectangular coordinates, the scale may be held true along the base line or along two parallels symmetric to the base line, and the projection is autogonal. A mathematical development is given in reference [16]. General tables exist to provide coverage for route charting, see reference [33].

For detail and greater accuracy in local area surveys connected with a base line, rectangular spherical coordinates may be more convenient, particularly for point to point computation away from the base line. The formulas for this kind of computation are included in Appendix 2. Appendix 2 also includes transformations from local rectangular spherical coordinates to space rectangular at a point of the ellipsoid referred to the normal, great normal section tangent, and meridional tangent, and this system in turn referred to the rectangular system at ellipsoid center, with the axis of rotation a coordinate axis. These may be useful relative to the adoption of the World Geodetic Reference System, 1967, see Appendix 2.

For oceanographic surveys, the positioning problem may not be essentially different from the navigation track plot. The *gnomonic linear plot, with projection center on the track*, gives the geographical coordinates of the great circle which can then be transferred to any suitable projection, the resulting curve being the great circle track. Distances may then be scaled from the map or chart, azimuths or bearings measured directly, if the map is autogonal, etc. Where accuracy requirements are not high, the possibility of using existing maps and charts should be considered since U. S. Government agencies such as AMS, GIMRADA, ACIS, C&GS, USGS, NAVOCEANO; the National Geographic Society; the State governments; mapping and charting agencies of other countries, collectively publish large numbers of maps, charts and grids on various projections and at several scales. *Direct scaling from a large globe may suffice.*

For world reference, positions may be expressed in terms of the *Universal Transverse Mercator coordinate system*, reference [35]. See also an extensive study of world plane coordinate reference systems and recommendations as given in reference [36]. Positions may also be referenced in rectangular coordinates at ellipsoid center, see Appendix 2.

Long spheroidal geodesics—partitioning.

If the end points are in triangulation nets on different spheroids, one station can be transferred to the ellipsoid of the other or both can be transferred to a third. The equations as used in the NASA tracking system will be found in reference [37]. See also references [9] and [38].

With the end point coordinates on the same ellipsoid, an inverse computation will give the distance and azimuths. This may be done by use of a form such as Figure 5. The distance is partitioned according to a preplot of the base line on a globe, into stations to fit islands, shoal areas, etc. Beginning with the first distance and forward azimuth of the base line, the coordinates of the first station and back azimuth are computed by a direct solution using the form of Figure 4. For best accuracy use the order f^4 computation, keep the initial azimuth and position but increase the distance incrementally as partitioned until the terminal point is reached. The desk computing could be formidable if the line is very long and the stations numerous. Use of a large scale computer is then indicated if available.

Alternatively one may compute from station to station along the base line, but this requires additional computation, even if first order in f suffices, since all input elements change for each succeeding computation.

Spherical case.

A method of computing stations along a great circle and parallels to the great circle simultaneously is given in reference [18]. Alternatively the forms as given in Figures 8 and 9, can be used. See also reference [39]. The best spherical radius to use is probably the ellipsoidal mean radius computed for the mean latitude of the base line terminals, see Appendix 2, equation (12).

Problems in local geometry.

Problem. To compute the geographic coordinates of a point at distance S from a base line station and at angle a with the base line. The geographic coordinates ϕ_i, λ_i, and azimuth α_i at the particular station are known, which with given S and a, provide the input ϕ_i, λ_i, $\alpha_i + a$, S for a direct solution from the form as shown in Figures 4, 6 or 8, depending on the magnitude of S and accuracy required. For a point at distance S on the perpendicular to the base line, $a = 90°$. If S is constant and $a = 90°$ at each base station, the direct computation at each station provides points on a parallel at a given distance S from the geodetic base line (this geodesic parallel is not itself a geodesic). If the base line is a great circle, a circle parallel to it is generated. If geographic coordinates along a partitioned spherical base line with corresponding coordinates along two symmetric parallels are required, the method as given in reference [18] may be used.

Problem. Given the geographic coordinates $Q_1(\phi_1, \lambda_1)$, $Q_2(\phi_2, \lambda_2)$ of two stations of a spherical base line, to find the perpendicular distance s from an arbitrary third point $p(\phi_k, \lambda_k)$ to the base line.

From equations (3) and (4), page (23), reference [18] solve for ϕ_0, λ_0:

$$\tan\lambda_0 = (\tan\phi_2 \cos\lambda_1 - \tan\phi_1 \cos\lambda_2)/(\tan\phi_1 \sin\lambda_2 - \tan\phi_2 \sin\lambda_1)$$

$$\cot\phi_0 = \cot\phi_1 \cos(\lambda_0 - \lambda_1) = \cot\phi_2 \cos(\lambda_0 - \lambda_2).$$

From the two figures, page (27) of reference [18], using the spherical formula $\cos a = \cos b \cos c + \sin b \cdot \sin c \cos A$, with $a = s$, find

$$\sin s = \pm\, [\sin\phi_k \cos\phi_0 - \cos\phi_k \sin\phi_0 \cos(\lambda_0 - \lambda_k)],$$

where the + sign corresponds to $k = p$, the − sign to $k = p'$, relative to the points $p(\phi_p, \lambda_p)$, $p'(\phi_p', \lambda_p')$ respectively as shown in Figure 3, page 26, reference [18].

Note also the solution in Appendix 2 following equations (47), with reference to the distance s of Figure 34. Additionally $s \to$ y-coordinate of the doubly-equidistant projection, see the discussion following equations (56), Appendix 2.

Problem. An observer at the known station Q (ϕ_0, λ_0); h_0 meters above the spherical surface (assumed sea level), Figure 31, measures a linear distance D to S_0 (target on a hill, island mountain peak, etc.) at a measured angle of elevation δ, and in measured or known azimuth α. If the spheroid at Q is approximated with a sphere of radius N_0 (the great normal length for ϕ_0, equation 11, Appendix 2) find the rectangular space coordinates of S_0 referred to the normal and tangents to the parallel and meridian at Q, the geographic coordinates of the normal projection P of S_0 upon the sphere, the spherical distance d = PQ and the height h of S_0 above the sphere (sea level). We have a, D, N_0, δ, ϕ_0, λ_0, h_0, *to find* X, Y, Z, h, d, ϕ, λ. From Figure 31, and some trigonometric identities we have $D_2 = D\cos\delta$, $X = D_2\cos a$, $Y = D_2 \sin a$, $Z = h_0 + D\sin\delta$, $\tan\tau = D_2/(N_0 + Z)$, $d = N_0\tau$ (radians), $h = (N_0 + Z)\sec\tau - N_0$, $h = D_2\csc\tau - N_0$, $\sin\phi = \cos d \sin\phi_0 + \sin d\cos\phi_0\cos a$, $\cot\Delta\lambda = (\cos\phi_0\cos d - \sin\phi_0\sin d\cos a)/\sin d\sin a$, $\lambda = \lambda_0 - \Delta\lambda$.

These problems illustrate the use of the geodetic line computing forms, and the formulae of Appendix 2, for solving local problems of computation for a station configuration. For very long base lines, it may be desirable to compute the positions of the stations along them very accurately, but in the vicinity of a particular station, a spherical approximation or plane coordinate configuration may suffice. Additional formulae such as for dip; maximum separation, chord-arc; geographic coordinates of point of maximum separation, etc. will be found in reference [18]. Other coordinate problems are discussed in Appendix 2. For uniform high accuracy over a considerable extent of the spheroid, a plane rectangular coordinate system based on one of the autogonal projections as used for geodesy may be more appropriate, see references [9], [16], [36].

BIBLIOGRAPHY

[1] Bessel, F.W. Uber die Berechnungen der geographischen längen and Breiten aus geodätischen vermessungen: Astronomische Nachrichten No. 86 (1825), Vol. 4.

[2] Clarke, A.R. Geodesy, Oxford, 1880, Chapter VI; Helmert, F.R. Die mathematischen and physikalischen Theorian der höheren geodäsie, Leipzig, 1880, Vol. 1, ch. 5.

[3] Bodemuller, H. Die geodätischen Linien des Rotations–ellipsoides and die Losung der geodatischen Hauptaufgaben fur grosse Strecken unter besonderer Berucksichtigung der Bessel-Helmertschen Losungs-methode; Deutsche Geodätische Komission, Reihe B: Angewandte Geodäsie-Heft, Nr. 13, 1954.

[4] Sodano, E.M. General Non-iterative solution of the Inverse and Direct Geodetic Problems, USAEGIMRADA Research Note No. 11, April 1963.

[5] Robbins, A.R. Long Lines on the Spheroid, Survey Review, Vol. XVI, No. 125, 1963. Also, length and azimuths of long lines on the Earth, Empire Survey Review, Vol. XI, No. 84, 1952.

[6] Tobey, W.M. Geodesy, Geodetic Survey of Canada Publication No. 11, 1928. Note on page 24, that if we replace D, N, l, m with their functions of latitude and azimuth, the expression for the difference in length of the normal section and the geodesic may be written:

$$S_n - S = (b^4 S^5 \sin^2 a \cos^2 a \cos^4 \phi/90)e^4 + F(b, S, a, \phi)e^6 + \ldots$$

which is the 4th order in the eccentricity, e.

[7] Ward, L.E. Geodesics and Plane Arcs on an Oblate Spheroid, American Mathematical Monthly, Aug.-Sept. 1943. Note that the separate expansions for the geodesic and the great elliptic arc in the same parameters, as found on pages 426 and 428 respectively, are identical to terms in e^4.

[8] Thomas, P.D. Inverse computation for long lines, transactions American Geophysical Union, Vol. 29, No. 6, 1948. Note–this form, based largely on Ward's work, uses the great elliptic arc for distance and the normal section azimuths.

[9] Bomford, G. Geodesy, Sound Edition, Clarendon Press, 1962.

[10] Rainsford, H.F. Long Lines on the Earth, Various Formulae, Empire Survey Review, Vol. X, No. 71, Vol. X, No. 72, 1949; Long Geodesics on the Ellipsoid, Bulletin Geodesique, No. 37, Sept. 1955. Rainsford, H.F.; Brazier, H.H. Long Lines on the Earth, A new and easier solution, Survey Conference of 1951; Rainsford, H.F., et al, Long Lines on the Earth, Photogrammetria, No. 3, 1950-1951.

[11] Cole, J.H. Computation of Distances for Long Arcs, Empire Survey Review, Vol. No. 59, 1946; Vol. No. 83, 1952; Vol. No. 84, 1952.

[12] Andersen, E. Practical formulas for accurate calculations by relative long distances, of geographical coordinates or distances and azimuths on the International Ellipsoid of Rotation, Memorial Institute of Geodesy, Denmark, Vol. 16, No. 3, 1953.

[13] McCaw, G.T. Long Lines on the Earth, Empire Survey Review, Vol. 1, Vol. 11.

[14] Dufour, H.M. Resolutions Practiques du Problem des Grandes Géodésiques par l'Emploi d'Une Sphere Auxilliare, Institute Geographique National, Paris, 1956; Calcul Electronic des Grandes Géodésiques Realise par l'Institute Geographique National.

[15] Jordan, Vermessungskunde, Vol. 3, p. 371, 1890.

[16] Thomas, P.D. Conformal projections in geodesy and cartography, C&GS Special Publication No. 251, 1952, pages 63-66.

[17] Lambert, W.D. The distance between two widely separated points on the surface of the earth, Journal of the Washington Academy of Sciences, Vol. 32, No. 5, 1942.

[18] Thomas, P.D. Mathematical models for navigation systems, U. S. Naval Oceanographic Office TR-182, October 1965.

[19] Chauvenet, W. A treatise on plane and spherical trigonometry, 9th Edition, Philadelphia, J.P. Lippincott, 1871, pages 180-181.

[20] Forsyth, A.R. Geodesics on an oblate spheroid, Messenger of Mathematics, Vol. XXV, 1895.

[21] Thomas, P.D. Another note on the method of Forsyth. Bulletin Géodésique, No. 76, 1965; Geodesic arc length on the reference ellipsoid to second order terms in the flattening, Journal of Geophysical Research, Vol. 70, No. 14, July 1965.

[22] U. S. Aeronautical Chart and Information Center (ACIC), Technical Report No. 59, Geodetic distance and azimuth computations for lines under 500 miles, September, 1960; Technical Report No. 80, Geodetic distance and azimuth computations for lines over 500 miles (to 6000 miles), Dec. 1959.

[23] Gougenheim, André. Note sur la methode de Forsyth, Bulletin Géodésique, No. 15, March 1950.

[24] Fichot, M.E. Sur les systèmes géodésiques équilatères à la surface un sphéroide terrestre, annels hydrographiques: 3rd series, 4th Volume, No. 707, 1921, pgs. 99-192; la zone géodésique antipode, 3rd series, 5th Volume, 1937, pgs. 23-76.

[25] Cayley. On the geodesic lines on an oblate spheroid, the London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, Vol. XI.—4th Series, July-December 1870.

[26] Any of the following give a lucid introduction to elliptic integrals and elliptic functions: Integral Calculus, Byerly, Ginn, 1888, pgs. 215-282; Advanced Calculus, Woods, Ginn, 1934, pgs. 365-382; Modern Analysis, Whittaker and Watson, Cambridge, 1962, pgs. 429-535.

[27] Carll, L.B. Calculus of Variations, Wiley, 1881, page 182.

[28] Forsyth, A.R. On conjugate points of geodesics on an oblate spheroid, Messenger of Mathematics, Vol. XXV, 1895.

[29] Bliss, G.A. Lectures on the calculus of variations, University of Chicago Press, Chicago, 1946.

[30] Moritz, Helmut. The Geodetic Reference System 1967, Allgen.ine Vermessungnachrichten, 1/1968, pp. 2-7.

[31] Lambert, W.D. The International Gravity Formula, American Journal of Science, Vol. 243-A, Daly Volume 1945.

[32] Crandall, C.L. Geodesy and Least Squares, John Wiley & Sons, New York, 1907.

[33] Oblique Mercator Projection Tables, U. S. Department of Commerce, Coast & Geodetic Survey, Washington, D.C. 1951.

[34] Notes on Azimuths, Distances, and Equidistant Azimuthal Projections in the South Pacific, New Zealand Journal of Science and Technology, Vol. 29, No. 1 (Sec. B), pp. 325-330, 1948.

[35] Universal Transverse Mercator Grid, Army Map Service Technical Manual No. 19, Washington, D.C. 1948.

[36] Colvocoresses, A.P. A Unified Plane Coordinate Reference System, Doctoral Dissertation, Ohio State University, 1965.

[37] Goddard Directory of Tracking Station Locations, Goddard Space Flight Center Report X-554-67-54, Greenbelt, Maryland, August 1966.

[38] Vincenty, T. Transformation of Geodetic Data between Reference Ellipsoids, Journal of Geophysical Research, Vol. 71, No. 10, May 15, 1966.

[39] Carver, H.C. Distance and Azimuth Computations (Spherical) with Tables, Engineering Research Institute, University of Michigan, for Air Research and Development Command. 1954.

[40] Bagratuni, G.V. Course in spheroidal geodesy, Moscow, 1962. (Defense documentation center translation AD 650 520, 1967).

[41] Fischer, Irene. A Modification of the Mercury Datum, Army Map Service Technical Report No. 67, June 1968.

APPENDIX I. MATHEMATICAL DISCUSSION OF THE SPHEROIDAL GEODESIC

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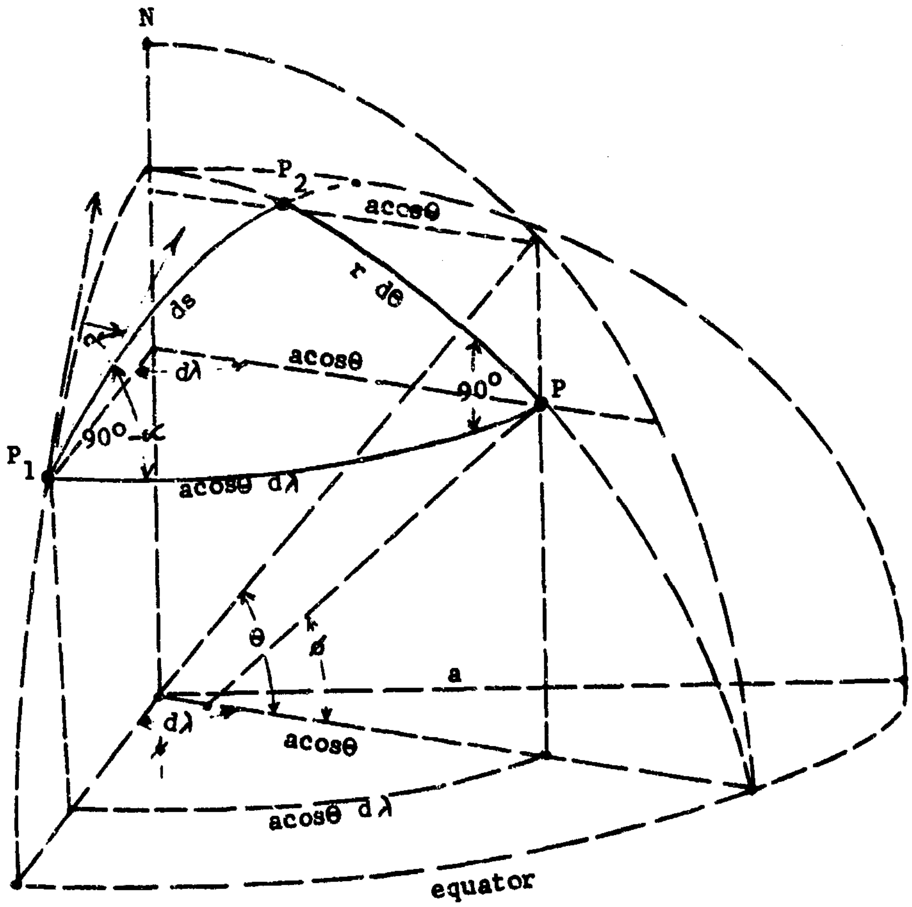
MATHEMATICAL DISCUSSION OF THE SPHEROIDAL GEODESIC

In Figure 10, α is the angle which the differential arc length, ds, makes with the meridian at P_1. The radius of the parallel in parametric latitude θ is $a \cdot \cos\theta$. Then $a \cdot \cos\theta \cdot d\lambda$ is the differential arc length along the parallel in latitude θ. Now the element of arc length along the meridian is defined as $Rd\phi$ where R is the radius of curvature in the meridian given by $R = a(1-e^2)/(1-e^2\sin^2\phi)^{3/2}$, see reference [16], page 59. The transformation between geodetic and parametric latitude is $\tan\phi = \tan\theta/(1-e^2)^{1/2}$,



From the differential right triangle PP_1P_2 we have $ds^2 = a^2\cos^2\theta\, d\lambda^2 + r^2 d\theta^2$ where $r^2 = a^2(1-e^2\cos^2\theta)$.

Figure 10. Differential arc length on the oblate spheroid as obtained from a differential right triangle.

whence

$$1/(1-e^2\sin^2\phi)^{3/2} = \frac{(1-e^2\cos^2\theta)^{3/2}}{(1-e^2)^{3/2}},\ d\phi = (1-e^2)^{1/2}d\theta/(1-e^2\cos^2\theta),$$

and

$$Rd\phi = a(1-e^2)d\phi/(1-e^2\sin^2\phi)^{3/2} = \frac{a(1-e^2)(1-e^2\cos^2\theta)^{3/2}}{(1-e^2)^{3/2}}\ \frac{(1-e^2)^{1/2}d\theta}{(1-e^2\cos^2\theta)}$$

or

$$Rd\phi = rd\theta, \text{ where } r = a(1-e^2\cos^2\theta)^{1/2} = R(1-e^2)^{1/2}/(1-e^2\cos^2\theta).$$

NOTE that r is not the radius of curvature in the spheroidal meridian, but $rd\theta$ is the differential arc length along the meridian in terms of parametric latitude and applying the pythagorean theorem to the right differential triangle $P_1 P P_2$ we have at once the formula for the general differential arc length on the spheroid in terms of parametric latitude:

$$ds^2 = a^2[(1-e^2\cos^2\theta)d\theta^2 + \cos^2\theta d\lambda^2]. \tag{1}$$

Differential equation from Euler's Condition

We may write (1) as

$$s = \int H\, d\theta \tag{2}$$

where $H = a[1-e^2\cos^2\theta + \cos^2\theta\lambda'^2]^{1/2}$, $\lambda' = d\lambda/d\theta$.

Now along geodesics, the Euler equation $d(\partial H/\partial\lambda')/d\theta - \partial H/\partial\lambda = 0$ must be satisfied.

Since $\partial H/\partial\lambda = 0$, the equation is $d(\partial H/\partial\lambda')/d\theta = 0$, a first integral being then

$$\partial H/\partial\lambda' = c \text{ (constant)}. \tag{3}$$

From (2) $\partial H/\partial\lambda' = (a\lambda'\cos^2\theta)/H$ and this value placed in (3) gives

$$a\lambda'\cos^2\theta = cH = ac[1-e^2\cos^2\theta + \cos^2\theta\lambda'^2]^{1/2} \tag{4}$$

Solving (4) for λ' and then placing $\lambda' = d\lambda/d\theta$ gives

$$d\lambda = \frac{c}{\cos\theta}\cdot\frac{(1-e^2\cos^2\theta)^{1/2}}{(\cos^2\theta - c^2)^{1/2}}\,d\theta. \tag{5}$$

From (2), $H = ds/d\theta$ and this value place in (4) gives

$$a\cos^2\theta\, d\lambda/ds = c, \text{ or } a^2\cos^2\theta d\lambda/ds = ac. \tag{6}$$

From the differential right triangle $P P_1 P_2$ of Figure 10

$$\cos(90^\circ - \alpha) = a\cos\theta d\lambda/ds = \sin\alpha. \tag{7}$$

The value from (7) placed in (6) gives

$$\cos\theta\sin\alpha = c, \text{ or } a\cos\theta\sin\alpha = ac. \tag{8}$$

Since $a\cos\theta$ is the radius of the parallel in latitude θ and α is the angle which the geodesic makes with the meridian as shown in Figure 10, equation (8) states *that the product of the radius of the parallel and the sine of the azimuth, α, is a constant along the geodesic.*

Now the geodesic will be orthogonal to a meridian when $\alpha = 90^\circ$, and using this value in (8) we have $c = \cos\theta_0$, where *θ_0 is the parametric latitude of the vertex of the geodesic.* With this value of c, equation (5) becomes

$$d\lambda = \frac{\cos\theta_0}{\cos\theta}\ \frac{(1-e^2\cos^2\theta)^{1/2}}{(\cos^2\theta - \cos^2\theta_0)^{1/2}}\,d\theta, \tag{9}$$

where always

$$\cos\theta\sin\alpha = \cos\theta_0. \tag{10}$$

With the differential equation to the geodesics in this form, equation (9), we can make several observations concerning the behavior of the geodesic. The substitution of $\pm\theta$ does not alter the coefficient of $d\theta$, since $\cos \pm\theta = \cos\theta$ and *therefore the curve is symmetric about the equatorial plane.* When $\theta = \pm\theta_0$, $d\theta = 0$, which means that the *geodesic is tangent to the parallels $\theta = \pm\theta_0$ and hence undulates alternately between tangencies to them.*

From (10), with $\theta = 0$, we have $\sin a_0 = \cos\theta_0$. *That is at a node (a point where the geodesic crosses the equator) the sine of the angle which the geodesic makes with the meridian is equal to the cosine of the parametric latitude of the vertex, or $a_0 = 90 - \theta_0$.* (11)

For reference in the developments to follow we include here a short resume of elliptic integrals and functions to be used, [26].

Elliptic Integrals (Legendre Forms) <u>*CLASS*</u>

$$S = F(k,\sigma) = \int_0^\sigma \frac{d\sigma}{(1-k^2\sin^2\sigma)^{1/2}} = \int_0^\sigma \frac{d\sigma}{\Delta\sigma}, k < 1 \qquad 1$$

$$E(k,\sigma) = \int_0^\sigma (1-k^2\sin^2\sigma)^{1/2} d\sigma = \int_0^\sigma \Delta\sigma\, d\sigma \qquad 2 \qquad (12)$$

$$\Pi(n,k,\sigma) = \int_0^\sigma \frac{d\sigma}{(1+n\sin^2\sigma)(1-k^2\sin^2\sigma)^{1/2}} = \int_0^\sigma \frac{d\sigma}{\delta\sigma\Delta\sigma} \qquad 3$$

Complete Elliptic Integrals <u>*CLASS*</u>

$$K = F(k,\pi/2) = \int_0^{\pi/2} \frac{d\sigma}{(1-k^2\sin^2\sigma)^{1/2}} = \int_0^{\pi/2} \frac{d\sigma}{\Delta\sigma} \qquad 1 \qquad (12)a$$

$$E = E(k,\pi/2) = \int_0^{\pi/2} (1-k^2\sin^2\sigma)^{1/2} d\sigma = \int_0^{\pi/2} \Delta\sigma d\sigma \qquad 2$$

Elliptic Functions

In the elliptic integral of the first class, σ is called *the amplitude* of $S = F(k,\sigma)$ and $k < 1$ is the modulus. *$\sin\sigma$, $\cos\sigma$, $\Delta\sigma$* are called the *sine, cosine, delta of the amplitude of S* and we have the following:

Definitions (Jacobi)

$$\sigma = \text{am}S, \sin\sigma = \text{sn}S, \cos\sigma = \text{cn}S, \tan\sigma = \text{tn}S,$$
$$\Delta\sigma = (1-k^2\sin^2\sigma)^{1/2} = \text{dn}S. \qquad (13)$$

Identities

$$\text{sn}^2S + \text{cn}^2S = \text{dn}^2S + k^2\text{sn}^2S = 1, \text{tn}S = \text{sn}S/\text{cn}S,$$
$$\text{dn}^2S - k^2\text{cn}^2S = k'^2 = 1 - k^2, k < 1\ (k^2 = c, c' = 1 - c = k'^2) \qquad (13)a$$
$$\text{sn}(S' \pm S) = (\text{sn}S'\text{cn}S\text{dn}S \pm \text{cn}S'\text{sn}S\text{dn}S')/(1 - k^2\text{sn}^2S'\text{sn}^2S),$$

$$E(k,\sigma_1) \pm E(k,\sigma_2) = E(k,\sigma_1 \pm \sigma_2) \pm k^2 \,\mathrm{sn}\sigma_1 \,\mathrm{sn}\sigma_2 \,\mathrm{sn}(\sigma_1 \pm \sigma_2)$$
$$\mathrm{cn}(S+S') = (\mathrm{cn}S' \,\mathrm{cn}S - \mathrm{sn}S' \,\mathrm{dn}S' \,\mathrm{sn}S\mathrm{dn}S)/(1 - \mathrm{csn}^2 S' \mathrm{sn}^2 S)$$
$$E(x_1 + 2K, k) = E(x_1, k) + 2E,\ E - c'K = 2cc'dK/dc \qquad (13)a$$
$$E(x_1, k) - c'x_1 = c\int_0^{x_1} \mathrm{cn}^2\, x\, dx$$

Special Values:

$S = 0$, $\mathrm{am}(0) = 0$, $\mathrm{cn}(0) = \mathrm{dn}(0) = 1$, $\mathrm{sn}(0) = \mathrm{tn}(0) = 0$;

for $S = K$; $\mathrm{sn}K = 1$, $\mathrm{cn}K = 0$, $\mathrm{dn}K = k' = (1-k^2)^{1/2}$; (13)b

for $S = 2K$; $\mathrm{sn}2K = \mathrm{sn}(0) = 0$, $\mathrm{cn}(2K) = -\mathrm{cn}(0) = -1$, $\mathrm{dn}(2K) = \mathrm{dn}(0) = 1$;

$\mathrm{cn}(S + 2K) = -\mathrm{cn}S$.

Differentials:

$$d\,\mathrm{am}S = d\sigma = (1 - k^2 \sin^2 \sigma)^{1/2}\, dS = \mathrm{dn}S\, dS$$
$$d\,\mathrm{sn}S = \cos\sigma\, d\,\sigma = \mathrm{cn}S\,\mathrm{dn}S\, dS$$
$$d\,\mathrm{cn}S = -\sin\sigma\, d\,\sigma = -\mathrm{sn}S\,\mathrm{dn}S\, dS$$
$$d\,\mathrm{dn}S = d\Delta\sigma = -k^2 \sin\sigma \cos\sigma\, (1 - k^2 \sin^2 \sigma)^{-1/2} d\sigma \qquad (13)c$$
$$= -k^2 \mathrm{sn}S\,\mathrm{cn}S\,(\mathrm{dn}S)^{-1} \mathrm{dn}S\, dS$$
$$d\,\mathrm{dn}S = -k^2 \mathrm{sn}S\,\mathrm{cn}S\, dS$$
$$d\,\mathrm{tn}S = \sec^2\sigma\, d\sigma = d\sigma/\cos^2\sigma = \mathrm{dn}S\, dS/\mathrm{cn}^2 S$$

Note that the elliptic functions as determined by (13) *have an analogy with trigonometric functions but S is not an angle as is clear from its integral definition,* (12). Like trigonometric functions they have a real period and like exponential functions have a pure imaginary period and are thus doubly periodic.

If we define $K' = F(k', \pi/2)$ where $k' = (1-k^2)^{1/2}$, that is K' is the complete integral K of (12)a with the modulus k replaced by k' then the periods of the elliptic functions snS, cnS, dnS are:

	Periods	
snS	4K, 2iK'	
cnS	4K, 2k + 2iK'	(14)
dnS	2K, 4iK'	

where $i = \sqrt{-1}$, $K = F(k, \pi/2)$, $K' = F(k', \pi/2)$, $k' = (1-k^2)^{1/2}$, $k < 1$.

Expression of longitude and arc length in elliptic integrals.

If we let $\cos\sigma = \mathrm{cn}S = \sin\theta/\sin\theta_0$ we have then:

$$1 - e^2\cos^2\theta = (1 - e^2\cos^2\theta_0)(1 - k^2\sin^2\sigma) = (1 - e^2\cos^2\theta_0)\,\Delta\sigma^2$$
$$\cos^2\theta - \cos^2\theta_0 = \sin^2\theta_0 \sin^2\sigma$$
$$d\theta^2 = \sin^2\theta_0 \sin^2\sigma\, d\sigma^2/\cos^2\theta$$
$$\cos^2\theta = 1 - \sin^2\theta = 1 - \sin^2\theta_0\cos^2\sigma = \cos^2\theta_0\,(1 + n\sin^2\sigma) = \delta\sigma\cos^2\theta_0$$
$$\cos^2\sigma = \csc^2\theta_0 \sin^2\theta = \csc^2\theta_0\,(1 - \cos^2\theta) \qquad (15)$$
$$= \csc^2\theta_0 - \cot^2\theta_0\,(1 + n\sin^2\sigma) = \csc^2\theta_0 - \delta\sigma\cot^2\theta_0$$
$$\sin^2\sigma = 1 - \cos^2\sigma = -\cot^2\theta_0 + \cot^2\theta_0\,(1 + n\sin^2\sigma) = -\cot^2\theta_0\,(1 - \delta\sigma)$$
$$k^2 = e^2\sin^2\theta_0/(1 - e^2\cos^2\theta_0),\ n = \tan^2\theta_0.$$

Eliminating $d\lambda$ between equations (1) and (9) we have

$$ds = \frac{a(1 - e^2 \cos^2 \theta)^{1/2} \cos\theta}{(\cos^2\theta - \cos^2\theta_0)^{1/2}}\, d\theta \tag{16}$$

Applying the transformation equations (15) to (16) and (9) we get

$$S = \frac{ea \sin\theta_0}{k} \cdot \int_0^{\sigma} (1 - k^2 \sin^2\sigma)^{1/2}\, d\sigma = \frac{ea \sin\theta_0}{k} \int_0^{\sigma} \Delta\sigma\, d\sigma,$$

$$\Delta\lambda = \frac{e \tan\theta_0}{k} \int_0^{\sigma} \frac{(1 - k^2 \sin^2\sigma)^{1/2}}{1 + n \sin^2\sigma}\, d\sigma = \frac{e \tan\theta_0}{k} \int_0^{\sigma} \frac{\Delta\sigma}{\delta\sigma}\, d\sigma. \tag{17}$$

In the second of (17), multiply numerator and denominator of the integrand by $(1 - k^2 \sin^2\sigma)^{1/2}$ and in the resulting numerator replace $\sin^2\sigma$ with its value from (15) which then allows the integral to be written in the form

$$\Delta\lambda = (e \tan\theta_0/k) \left[(1 + k^2 \cot^2\theta_0) \int_0^{\sigma} \frac{d\sigma}{\delta\sigma\, \Delta\sigma} - k^2 \cot^2\theta_0 \int_0^{\sigma} \frac{d\sigma}{\Delta\sigma}\right]. \tag{18}$$

Now comparing the first integral of (17) and the integrals of (18) with the elliptic integrals (12) we can then write

$$S = \frac{ea \sin\theta_0}{k} E(k, \sigma), \tag{19}$$

$$\Delta\lambda = \frac{e \tan\theta_0}{k} [(1 + k^2 \cot^2\theta_0)\, \Pi(n, k, \sigma) - k^2 \cot^2\theta_0 F(k, \sigma)],$$

Where the modulus is $k = e \sin\theta_0/(1 - e^2 \cos^2\theta_0)^{1/2}$; $n = \tan^2\theta_0$; and the *amplitude is $\sigma = \text{arc} \cos(\sin\theta/\sin\theta_0)$ or the spherical length from the vertex of the geodesic in parametric latitude θ_0 to a point in parametric latitude θ on the geodesic as shown in Figure 11, $|\theta| \leq |\theta_0|$. That is, the formulae (19) give longitude and distance along the geodesic measured from the geodesic vertex in terms of the spherical distance.*

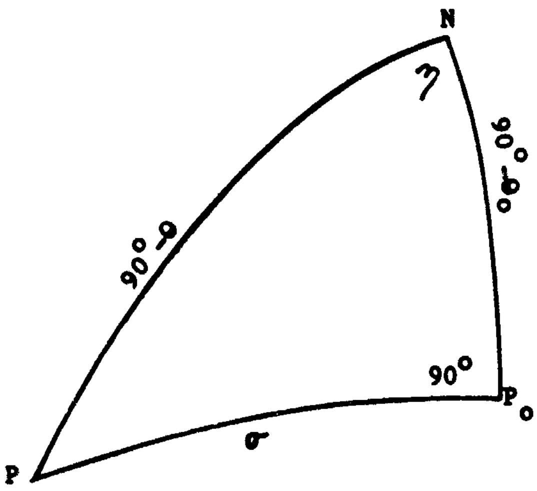
The elliptic functions in terms of the amplitude σ and modulus k; $\sigma = \text{arc} \cos(\sin\theta/\sin\theta_0)$, $k = e \sin\theta_0/(1 - e^2 \cos^2\theta_0)^{1/2}$.

From the definitions (13) we have:

$$\begin{aligned}
&\text{cnS} = \cos\sigma = \sin\theta/\sin\theta_0;\ \text{snS} = \sin\sigma = (\sin^2\theta_0 - \sin^2\theta)^{1/2}/\sin\theta_0;\\
&\text{tnS} = \tan\sigma = \text{snS}/\text{cnS} = (\sin^2\theta_0 - \sin^2\theta)^{1/2}/\sin\theta;\\
&\Delta\sigma = \text{dnS} = (1 - k^2 \sin^2\sigma)^{1/2} = (1 - e^2 \cos^2\theta)^{1/2}/(1 - e^2 \cos^2\theta_0)^{1/2};\\
&\delta\sigma = 1 + n \sin^2\sigma = 1 + n\, \text{sn}^2 S = \sec^2\theta_0 \cos^2\theta,\ n = \tan^2\theta_0.
\end{aligned} \tag{19a}$$

Since $\sigma = \text{arc} \cos(\sin\theta/\sin\theta_0)$, we have the correspondences $\theta = 0, \sigma = \pi/2; \theta = \theta_0, \sigma = 0$. From (13)c, (17) and (19)a, we may write for the geodesic, vertex to vertex or node to node:

$$\begin{aligned}
S_0 &= 2a(1 - e^2 \cos^2\theta_0)^{1/2} \int_0^{\pi/2} \text{dn}^2 S\, dS\\
\Delta\lambda_0 &= \frac{2(1 - e^2 \cos^2\theta_0)^{1/2}}{\cos\theta_0} \int_0^{\pi/2} \frac{\text{dn}^2 S\, dS}{1 + n\, \text{sn}^2 S}
\end{aligned} \tag{19b}$$



P is an arbitrary point on the geodesic, P_0 is the geodesic vertex, and σ is the amplitude of the elliptic functions. In the right spherical triangle P_0PN as shown we have:

$\sin\theta_0 = \sin\theta_0 \cos\sigma$, $\sigma = \arccos(\sin\theta/\sin\theta_0)$, $\cos\eta = \tan\theta/\tan\theta_0$, $\eta = \arccos(\tan\theta/\tan\theta_0)$, $\tan\sigma = \cos\theta_0 \tan\eta$, $\eta = \arctan(\sec\theta_0 \tan\sigma)$.

Figure 11. The amplitude of elliptic functions expressed as spherical distance from the geodesic vertex to an arbitrary point on the geodesic.

When $\theta = \theta_0 = 0$, we have from (19)a that $1 + n\,\mathrm{sn}^2 S = \mathrm{dn} S = 1$, and from (19)b,

$$\Delta\lambda_0 = 2(1-e^2)^{1/2} \int_0^{\frac{1}{2}\pi} ds = \pi(1-e^2)^{1/2} = \pi b/a,$$

$S_0 = a\pi(1-e^2)^{1/2} = \pi b$; where a, b are the semimajor, semiminor axes of the spheroid. *This shows that an arc of the equator of length πb is a limiting position of spheroidal geodesics and that there are no antipodal points on nonplanar spheroidal geodesics.*

Since the vertex, θ_0, may be negative and internal or external to a segment S_{1-2} of the geodesic, all alternatives are included from the first of (19) by writing

$$S_{1-2} = \frac{ea}{k} \left| \sin\theta_0 [E(k, \sigma_1) \pm E(k, \sigma_2)] \right|, \tag{20}$$

and by use of the addition formula for elliptic integrals of the second class with the same modulus, from (13)a, we may write (20) as

$$S_{1-2} = \frac{ea}{k} \left| \sin\theta_0 \left\{ E(k, \sigma_1 \pm \sigma_2) \pm k^2 \sin\sigma_1 \sin\sigma_2 \sin(\sigma_1 \pm \sigma_2) \right\} \right| \tag{21}$$

where $\sigma_1 = \arccos(\sin\theta_1/\sin\theta_0)$, $\sigma_2 = \arccos(\sin\theta_2/\sin\theta_0)$, $k = e\sin\theta_0/(1 - e^2\cos^2\theta_0)^{1/2}$.

Similar expressions may be written for the longitude difference from the second of (19).

Integration of differential equations

Since many tables of the elliptic integrals exist it would appear that evaluation of expressions like (21) would be simple. But (21) is in terms of θ_0, the parametric latitude of the vertex of the geodesic, and not obtainable directly from the geographic coordinates of two given points on the nonplanar geodesic. Interpolation in the tables is not easy. Since the eccentricity and flattening of oblate spheroids, as used for the earth representation, are small, series expansions in them converge rapidly and numerical evaluation is then relatively simple. Now the elliptic integrals themselves can be expanded in series of e or f since the modulus k is a function of e—see equations (19)—but we will first expand the differential equations (9) and (16) in powers of e and of f and integrate term by term. The eccentricity, e, and flattening, f, are connected by the relation $1 - f = (1 - e^2)^{1/2}$, or $e^2 = 2f - f^2$.

From (9) and (16) we write again for reference

$$d\lambda = \frac{\cos\theta_0}{\cos\theta}\,\frac{(1 - e^2\cos^2\theta)^{1/2}\,d\theta}{(\cos^2\theta - \cos^2\theta_0)^{1/2}}$$

$$ds = \frac{a\cos^2\theta}{\cos\theta_0}\,d\lambda = \frac{a\cos\theta(1 - e^2\cos^2\theta)^{1/2}\,d\theta}{(\cos^2\theta - \cos^2\theta_0)^{1/2}} \tag{22}$$

The expansion by the binomial formula of $(1 - e^2\cos^2\theta)^{1/2}$ to sixth order in e is

$$(1 - e^2\cos^2\theta)^{1/2} = 1 - (1/2)e^2\cos^2\theta - (1/8)e^4\cos^4\theta - (1/16)e^6\cos^6\theta - \ldots \tag{23}$$

If we place $e^2 = 2f - f^2$, $e^4 = 4f^2 - 4f^3$, $e^6 = 8f^3$, then

$$(1 - e^2\cos^2\theta)^{1/2} = 1 - f\cos^2\theta + (f^2/2)(\cos^2\theta - \cos^4\theta) + (f^3/2)(\cos^4\theta - \cos^6\theta) + \ldots \tag{24}$$

Substituting from (23) and (24), in (22) we find

$$\begin{aligned}\Delta\lambda &= I_1 - (e^2/2)\cos\theta_0\, I_2 - (e^4/8)\cos\theta_0\, I_3 - (e^6/16)\cos\theta_0\, I_4 - \ldots\\ &= I_1 - f\cos\theta_0\, I_2 + (f^2/2)\cos\theta_0\,(I_2 - I_3) + (f^3/2)(I_3 - I_4)\cos\theta_0 + \ldots\\ S &= a[I_2 - (e^2/2)I_3 - (e^4/8)I_4 - (e^6/16)I_5 - \ldots]\\ &= a[I_2 - fI_3 + (f^2/2)(I_3 - I_4) + (f^3/2)(I_4 - I_5) + \ldots]\end{aligned} \tag{25}$$

Where

$$I_1 = \int \frac{\cos\theta_0}{\cos\theta}\,\frac{d\theta}{(\cos^2\theta - \cos^2\theta_0)^{1/2}} = \int \frac{\left(\dfrac{\sec^2\theta}{\tan\theta_0}\right)d\theta}{\left(1 - \dfrac{\tan^2\theta}{\tan^2\theta_0}\right)^{1/2}} = \arcsin\left(\frac{\tan\theta}{\tan\theta_0}\right) = \gamma,$$

$$I_2 = \int \frac{\cos\theta\,d\theta}{(\cos^2\theta - \cos^2\theta_0)^{1/2}} = \int \frac{\left(\dfrac{\cos\theta}{\sin\theta_0}\right)d\theta}{\left(1 - \dfrac{\sin^2\theta}{\sin^2\theta_0}\right)^{1/2}} = \arcsin\left(\frac{\sin\theta}{\sin\theta_0}\right) = \beta. \tag{26}$$

$$\begin{aligned}I_3 &= \int \frac{\cos^3\theta\,d\theta}{(\cos^2\theta - \cos^2\theta_0)^{1/2}} = \int \frac{(1 - \sin^2\theta)\cos\theta\,d\theta}{(\sin^2\theta_0 - \sin^2\theta)^{1/2}} = \int \frac{(1 - x^2)\,dx}{(c^2 - x^2)^{1/2}}\\ I_4 &= \int \frac{\cos^5\theta\,d\theta}{(\cos^2\theta - \cos^2\theta_0)^{1/2}} = \int \frac{(1 - \sin^2\theta)^2\cos\theta\,d\theta}{(\sin^2\theta_0 - \sin^2\theta)^{1/2}} = \int \frac{(1 - x^2)^2\,dx}{(c^2 - x^2)^{1/2}}\\ I_5 &= \int \frac{\cos^7\theta\,d\theta}{(\cos^2\theta - \cos^2\theta_0)^{1/2}} = \int \frac{(1 - \sin^2\theta)^3\cos\theta\,d\theta}{(\sin^2\theta_0 - \sin^2\theta)^{1/2}} = \int \frac{(1 - x^2)^3\,dx}{(c^2 - x^2)^{1/2}}\end{aligned} \tag{27}$$

and where $x = \sin\theta$, $c = \sin\theta_0$.

We let $x = c \sin\beta$ in the three last integrals of (27), whence $dx = c\cos\beta\, d\beta$, $(c^2 - x^2)^{1/2} = c\cos\beta$ and the integrals may be written

$$I_3 = \int(1 - c^2 \sin^2 \beta)d\beta\,,\ I_4 = \int(1 - c^2 \sin^2 \beta)^2 d\beta,\ I_5 = \int(1 - c^2 \sin^2 \beta)^3 d\beta \tag{28}$$

where $c = \sin\theta_0$ and β is the integral I_2 of (26).

Now
$$\sin^2\beta = (1/2)(1 - \cos 2\beta)$$
$$\sin^4\beta = (1/8)(3 - 4\cos 2\beta + \cos 4\beta)$$
$$\sin^6\beta = (1/32)(10 - 15\cos 2\beta + 6\cos 4\beta - \cos 6\beta) \tag{29}$$

By expanding the integrands in equations (28) and using the identities (29) we are able to integrate term by term and we find that the integrals (28) are

$$I_3 = (1/4)[2(1 + \cos^2\theta_0)\beta + \sin^2\theta_0 \sin 2\beta]\,,$$
$$I_4 = (1/32)[4(8\cos^2\theta_0 + 3\sin^4\theta_0)\beta + 8\sin^2\theta_0(1 + \cos^2\theta_0)\sin 2\beta + \sin^4\theta_0 \sin 4\beta]\,,$$
$$I_5 = (1/192)\begin{bmatrix} 12(1 + \cos^2\theta_0)(8\cos^2\theta_0 + 5\sin^4\theta_0)\beta \\ + 9(16\cos^2\theta_0 + 5\sin^4\theta_0)\sin^2\theta_0 \sin 2\beta \\ + 9(1 + \cos^2\theta_0)\sin^4\theta_0 \sin 4\beta + \sin^6\theta_0 \sin 6\beta \end{bmatrix}, \tag{30}$$

where
$$\beta = I_2 = \arc\sin\left(\frac{\sin\theta}{\sin\theta_0}\right).$$

Formulae referred to a node

If we place the values of the integrals I_1, I_2, I_3, I_4, I_5 from (26) and (30) in (25) we may write in terms of e

$$\Delta\lambda = \gamma - \frac{e^2}{2}\cos\theta_0\beta - \frac{e^4}{32}\cos\theta_0[2(1 + \cos^2\theta_0)\beta + \sin^2\theta_0 \sin 2\beta]$$
$$- \frac{e^6}{512}\cos\theta_0 \begin{bmatrix} 4(8\cos^2\theta_0 + 3\sin^4\theta_0)\beta + 8\sin^2\theta_0(1 + \cos^2\theta_0)\sin 2\beta \\ + \sin^4\theta_0 \sin 4\beta \end{bmatrix},$$

$$S/a = \beta - \frac{e^2}{8}[2(1 + \cos^2\theta_0)\beta + \sin^2\theta_0 \sin 2\beta] \tag{31}$$
$$- \frac{e^4}{256}\begin{bmatrix} 4(8\cos^2\theta_0 + 3\sin^4\theta_0)\beta + 8\sin^2\theta_0(1 + \cos^2\theta_0)\sin 2\beta \\ + \sin^4\theta_0 \sin 4\beta \end{bmatrix}$$
$$- \frac{e^6}{3072}\begin{bmatrix} 12(1 + \cos^2\theta_0)(8\cos^2\theta_0 + 5\sin^4\theta_0)\beta \\ + 9(16\cos^2\theta_0 + 5\sin^4\theta_0)\sin^2\theta_0 \sin 2\beta \\ + 9(1 + \cos^2\theta_0)\sin^4\theta_0 \sin 4\beta + \sin^6\theta_0 \sin 6\beta \end{bmatrix},$$

and in terms of f

$$\Delta\lambda = \gamma - f\cos\theta_0\beta + \frac{f^2}{8}\cos\theta_0 \sin^2\theta_0(2\beta - \sin 2\beta) \tag{32}$$
$$+ \frac{f^3}{64}\cos\theta_0 \sin^2\theta_0[4(1 + 3\cos^2\theta_0)\beta - 8\cos^2\theta_0 \sin 2\beta - \sin^2\theta_0 \sin 4\beta]\,,$$

$$
\begin{aligned}
S/a = {} & \beta - (f/4)[2(1+\cos^2\theta_0)\beta + \sin^2\theta_0 \sin 2\beta] \\
& + \frac{f^2}{64}\sin^2\theta_0 [4(1+3\cos^2\theta_0)\beta - 8\cos^2\theta_0 \sin 2\beta - \sin^2\theta_0 \sin 4\beta] \\
& + \frac{f^3}{384}\sin^2\theta_0 \begin{bmatrix} 12(1+2\cos^2\theta_0 + 5\cos^4\theta_0)\beta \\ -3(1+3\cos^2\theta_0)(5\cos^2\theta_0 - 1)\sin 2\beta \\ -3(1+3\cos^3\theta_0)\sin^2\theta_0 \sin 4\beta - \sin^4\theta_0 \sin 6\beta \end{bmatrix}
\end{aligned}
$$

Where $\quad \gamma = \arcsin \dfrac{\tan\theta}{\tan\theta_0}$, $\beta = \arcsin \dfrac{\sin\theta}{\sin\theta_0}$.

Limiting cases of integral equations

We first make some preliminary evaluations of equations (32). First we find the values of $\Delta\lambda$, S between $\theta = 0$, $\theta = \theta_0$ or from a node to the first vertex. For $\theta = 0$, $\beta = \gamma = \arcsin 0 = 0$. For $\theta = \theta_0$, $\beta = \gamma = \arcsin 1 = \pi/2$, and from (32) we have (doubling the result)

$$
\begin{aligned}
\Delta\lambda_0 &= \pi[1 - f\cos\theta_0 + (f^2/4)\cos\theta_0 \sin^2\theta_0 + (f^3/16)\cos\theta_0 \sin^2\theta_0(1+3\cos^2\theta_0)], \\
S_0 &= a\pi[1 - (f/2)(1+\cos^2\theta_0) + (f^2/16)\sin^2\theta_0\,(1+3\cos^2\theta_0) \\
&\qquad + (f^3/32)\sin^2\theta_0\,(1+2\cos^2\theta_0 + 5\cos^4\theta_0)],
\end{aligned}
\tag{33}
$$

which will subsequently be shown to give all hemispheroidal geodesics, vertex to vertex or node to node; compare (19)b.

The expressions (33) are even functions of θ_0, $f(-\theta_0) = f(\theta_0)$, which would be expected from the discussion of symmetry following equation (10). Therefore, the expressions (33) *give longitude and distance between successive vertices and also between successive nodes.*

From the first of equations (33) we have $\pi - \Delta\lambda_0 = \pi f\cos\theta_0 \cdot [1 - (f/4)\sin^2\theta_0 - (f^2/16)\sin^2\theta_0(1+3\cos^2\theta_0)]$, which *shows again that except for the meridian ($\theta_0 = \pi/2$), two consecutive vertices of the geodesic on the oblate spheroid cannot be antipodal (end points of a diameter).*

From equations (32) and (33) we have with

$$
\begin{aligned}
\theta_0 = 0:\quad & \Delta\lambda_0 = \pi(1-f) = \pi(1-e^2)^{1/2} = \pi b/a \\
& S_0 = a\pi(1-f) = \pi b = a\Delta\lambda_0\,; \\
\theta_0 = \pi/2:\quad & \Delta\lambda_{\pi/2} = \pi, \\
& S_{\pi/2} = a\pi(1 - f/2 + f^2/16 + f^3/32 + \ldots).
\end{aligned}
\tag{34}
$$

If we take the derivative of S_0 with respect to θ_0 and place equal to zero we obtain

$$\sin\theta_0 \cos\theta_0 [15 f^2 \cos^4\theta_0 + 6f(2-f)\cos^2\theta_0 + 16 - 4f - f^2] = 0.$$

The discriminant of the quadratic factor in $\cos^2\theta_0$ is $48f^2[2f(1+f) - 17] < 0$, since $f < 1$, hence the only real values are given by $\sin\theta_0 = 0$, $\cos\theta_0 = 0$, or by $\theta_0 = 0$, $\theta_0 = \pi/2$; equations (34) *are actually the upper and lower limits to hemispheroidal geodesic length* (vertex to next vertex or node to next node.) Along the equator, only the arc πb satisfies the fundamental definition of the geodesic, i.e. the *longest hemispheroidal geodesic is the semimeridian, the shortest is the spherical arc πb.* The values of S_0, $\Delta\lambda_0$ from (33) satisfy the inequalities

$$a\pi(1 - f/2 + f^2/16 + f^3/32) \geq S_0 \geq \pi b;\ \pi \geq \Delta\lambda_0 \geq \pi b/a \tag{35}$$

If derivatives of the second and third order terms in equations (33) are placed equal to zero, we find that for the Clarke 1866 ellipsoid:

$$
\begin{aligned}
\Delta\lambda_0:&\quad \Delta\lambda_0(f^2)(\max) \text{ occurs at } \theta_0 = 54^\circ\ 44'\ 08''.197\\
&\quad \Delta\lambda_0(f^3)(\max) \text{ occurs at } \theta_0 = 43^\circ\ 28'\ 31''\\
S_0:&\quad S_0(f^2)(\max) \text{ occurs at } \theta_0 = 54^\circ\ 44'\ 08''.197\\
&\quad S_0(f^3)(\max) \text{ occurs at } \theta_0 = 43^\circ\ 28'\ 31''
\end{aligned}
\tag{36}
$$

With the values of θ_0 from (36) placed in (33) we find the maximum contribution of second and third order terms over the Clarke 1866 hemispheroid:

$$
\begin{aligned}
\Delta\lambda_0(f^2)(\max.) &= 3474.2 \times 10^{-9} \text{ radians} \approx 3.5 \text{ seconds}\\
\Delta\lambda_0(f^3)(\max.) &= 6.8 \times 10^{-9} \text{ radians} \approx .0012 \text{ seconds}\\
S_0(f^2)(\max.) &= 19.190 \text{ meters}\\
S_0(f^3)(\max.) &= .040 \text{ meters}
\end{aligned}
\tag{37}
$$

Formulae referred to a vertex

Now equations (31) and (32) are referred to a node, (equator crossing) of the geodesic.

If we substract, respectively, the equations for longitude and distance in (32) from those of (33), than place $\gamma = (\pi/2) - \eta$, $\sigma = (\pi/2) - \beta$ we have:

$$
\begin{aligned}
\Delta\lambda = \eta - f\cos\theta_0\sigma &+ (f^2/8)\cos\theta_0\sin^2\theta_0(2\sigma + \sin 2\sigma)\\
&+ (f^3/64)\cos\theta_0\sin^2\theta_0[4(1+3\cos^2\theta_0)\sigma + 8\cos^2\theta_0\sin 2\sigma - \sin^2\theta_0\sin 4\sigma],\\
\frac{S}{a} = \sigma - (f/4)[2(1&+\cos^2\theta_0)\sigma - \sin^2\theta_0\sin 2\sigma]\\
&+ (f^2/64)\sin^2\theta_0[4(1+3\cos^2\theta_0)\sigma + 8\cos^2\theta_0\sin 2\sigma - \sin^2\theta_0\sin 4\sigma]\\
&+ (f^3/384)\sin^2\theta_0\begin{bmatrix}12(1+2\cos^2\theta_0+5\cos^4\theta_0)\sigma\\ +3(1+3\cos^2\theta_0)(5\cos^2\theta_0-1)\sin^2\sigma\\ -3(1+3\cos^2\theta_0)\sin^2\theta_0\sin 4\sigma + \sin^4\theta_0\sin 6\sigma\end{bmatrix}
\end{aligned}
\tag{38}
$$

where now $\sigma = \text{arc cos}(\sin\theta/\sin\theta_0)$, $\eta = \text{arc cos}(\tan\theta/\tan\theta_0)$, *and the formulae (38) give longitude and distance from the vertex of the geodesic to a point on the geodesic in parametric latitude θ, where* $|\theta| < |\theta_0|$.

Note that η and σ are spherical longitude and spherical distance from the geodesic vertex, see figure 11. To show that $\Delta\lambda$ and S of equations (38) are in fact the expansions of equations (19), we write from the first of (17) using the binomial formula,

$$
\begin{aligned}
S &= \frac{ea\sin\theta_0}{k}\int_0^\sigma (1-k^2\sin^2\sigma)^{1/2}\,d\sigma\\
&= \frac{ea\sin\theta_0}{k}\int_0^\sigma \left[1-(1/2)k^2\sin^2\sigma-(1/8)k^4\sin^4\sigma-(1/16)k^6\sin^6\sigma\ldots\right)d\sigma\Big]\\
&= a\int_0^\sigma\left[\frac{e\sin\theta_0}{k}-(1/2)ek\sin\theta_0\sin^2\sigma-(1/8)ek^3\sin\theta_0\sin^4\sigma-(1/16)ek^5\sin\theta_0\sin^6\sigma\right]d\sigma
\end{aligned}
\tag{39}
$$

From (15), $k = e\sin\theta_0/(1-e^2\cos^2\theta_0)^{1/2}$, and

$$
\begin{aligned}
(e/k)\sin\theta_0 &= (1-e^2\cos^2\theta_0)^{1/2} = 1-(1/2)e^2\cos^2\theta_0-(1/8)e^4\cos^4\theta_0-(1/16)e^6\cos^6\theta_0\\
ek\sin\theta_0 &= e^2\sin^2\theta_0[1+(1/2)e^2\cos^2\theta_0+(3/8)e^4\cos^4\theta_0]
\end{aligned}
\tag{40}
$$

$$ek^3 \sin\theta_0 = e^4 \sin^4\theta_0 [1 + (3/2)e^2 \cos^2\theta_0]$$

$$ek^5 \sin\theta_0 = e^6 \sin^6\theta_0 .$$

Now from (29), with β replaced by σ, we have

$$\int \sin^2\sigma \, d\sigma = \int (1/2)(1 - \cos 2\sigma)\, d\sigma = (1/4)(2\sigma - \sin 2\sigma)$$

$$\int \sin^4\sigma \, d\sigma = \int (1/8)(3 - 4\cos 2\sigma + \cos 4\sigma)\, d\sigma = (1/8)[3\sigma - 2\sin 2\sigma + (1/4)\sin 4\sigma] \tag{41}$$

$$\int \sin^6\sigma \, d\sigma = \int (1/32)(10 - 15\cos 2\sigma + 6\cos 4\sigma - \cos 6\sigma)\, d\sigma$$
$$= (1/32)[10\sigma - (15/2)\sin 2\sigma + (3/2)\sin 4\sigma - (1/6)\sin 6\sigma]$$

With the values from (40) and (41) we may evaluate (39) and we have then

$$\begin{aligned} S/a = {} & [1 - (1/2)e^2\cos^2\theta_0 - (1/8)e^4\cos^4\theta_0 - (1/16)e^6\cos^6\theta_0]\sigma \\ & - (1/8)e^2\sin^2\theta_0[1 + (1/2)e^2\cos^2\theta_0 + (3/8)e^4\cos^4\theta_0](2\sigma - \sin 2\sigma) \\ & - (1/64)e^4\sin^4\theta_0[1 + (3/2)e^2\cos^2\theta_0][3\sigma - 2\sin 2\sigma + (1/4)\sin 4\sigma] \\ & - (1/512)e^6\sin^6\theta_0[10\sigma - (15/2)\sin 2\sigma + (3/2)\sin 4\sigma - (1/6)\sin 6\sigma] \end{aligned} \tag{42}$$

Collecting the coefficients of the terms in like powers of e, letting $e^2 = 2f - f^2$, $e^4 = 4f^2 - 4f^3$, $e^6 = 8f^3$, and using some elementary trigonometric identities in the coefficients of the powers of f, we find that equation (42) becomes exactly the second of equations (38).

Similarly from the second of equations (17) we have:

$$\Delta\lambda = \frac{e\tan\theta_0}{k}\int_0^\sigma \frac{(1 - k^2\sin^2\sigma)^{1/2}\, d\sigma}{1 + n\sin^2\sigma} = \frac{e\tan\theta_0}{k} I_1 - (1/2)ek\tan\theta_0 I_2$$
$$- (1/8)ek^3\tan\theta_0 I_3 - (1/16)ek^5\tan\theta_0 I_4 \tag{43}$$

Where

$$I_1 = \int_0^\sigma \frac{d\sigma}{1 + n\sin^2\sigma} = \cos\theta_0 \arctan(\sec\theta_0 \tan\sigma) = \cos\theta_0\, \eta, \text{ (see Figure 11)}$$

$$I_2 = \int_0^\sigma \frac{\sin^2\sigma\, d\sigma}{1 + n\sin^2\sigma} = \cot^2\theta_0(\sigma - I_1) \tag{44}$$

$$I_3 = \int_0^\sigma \frac{\sin^4\sigma\, d\sigma}{1 + n\sin^2\sigma} = \cot^4\theta_0[(1/2)(\tan^2\theta_0 - 2)\sigma - (1/4)\tan^2\theta_0\sin 2\sigma + I_1]$$

$$I_4 = \int_0^\sigma \frac{\sin^6\sigma\, d\sigma}{1 + n\sin^2\sigma} = \cot^6\theta_0\left[\begin{array}{l}(1/8)(3\tan^4\theta_0 - 4\tan^2\theta_0 + 8)\sigma - I_1 \\ +(1/4)\tan^2\theta_0(1 - \tan^2\theta_0)\sin 2\sigma + (1/32)\tan^4\theta_0\sin 4\sigma\end{array}\right]$$

Now $k = e\sin\theta_0/(1 - e^2\cos^2\theta_0)^{1/2}$, $e\tan\theta_0/k = \sec\theta_0(1 - e^2\cos^2\theta_0)^{1/2}$ and expanding by the binomial formula to sixth order terms in e we have

$$\frac{e\tan\theta_0}{k} = [\sec\theta_0 - (1/2)e^2\cos\theta_0 - (1/8)e^4\cos^3\theta_0 - (1/16)e^6\cos^5\theta_0]$$

$$\begin{aligned} & - (1/2)ek\tan\theta_0 = -(e^2/2)\sin\theta_0\tan\theta_0[1 + (1/2)e^2\cos^2\theta_0 + (3/8)e^4\cos^4\theta_0] \\ & - (1/8)ek^3\tan\theta_0 = -(1/8)e^4\sin^3\theta_0\tan\theta_0[1 + (3/2)e^2\cos^2\theta_0] \\ & - (1/16)ek^5\tan\theta_0 = -(1/16)e^6\sin^5\theta_0\tan\theta_0 \end{aligned} \tag{45}$$

Placing the values from (44) and (45) in (43), collecting like terms and employing some trigonometric identities we have

$$\Delta\lambda = \eta - \frac{e^2}{2}\cos\theta_0\,\sigma - \frac{e^4}{32}\cos\theta_0\,[2(1+\cos^2\theta_0)\sigma - \sin^2\theta_0\sin 2\sigma]$$
$$- \frac{e^6}{512}\cos\theta_0\begin{bmatrix}4(8\cos^2\theta_0 + 3\sin^4\theta_0)\sigma \\ -8\sin^2\theta_0\,(1+\cos^2\theta_0)\sin 2\sigma + \sin^4\theta_0\sin 4\sigma\end{bmatrix} \tag{46}$$

Placing $e^2 = 2f - f^2$, $e^4 = 4f^2 - 4f^3$, $e^6 = 8f^3$ in (46) find

$$\Delta\lambda = \eta - f\cos\theta_0\sigma + \frac{f^2}{8}\cos\theta_0\sin^2\theta_0\,(2\sigma + \sin 2\sigma)$$
$$+ \frac{f^3}{64}\cos\theta_0\sin^2\theta_0\,[4(1+3\cos^2\theta_0)\sigma + 8\cos^2\theta_0\sin 2\sigma - \sin^2\theta_0\sin 4\sigma]$$

which is exactly the first of equations (38).

Collecting like terms in β and σ, equations (32) and (38) may be written with *longitude and arc length measured from the geodesic node;*

$$\begin{aligned}&\Delta\lambda = \gamma - A\beta - B\sin 2\beta - C\sin 4\beta,\ \gamma = \arcsin(\tan\theta/\tan\theta_0),\\ &S/a = D\beta - E\sin 2\beta - F\sin 4\beta - G\sin 6\beta,\ \beta = \arcsin(\sin\theta/\sin\theta_0),\end{aligned} \tag{47}$$

longitude and arc length measured from the geodesic vertex;

$$\begin{aligned}&\Delta\lambda = \eta - A\sigma + B\sin 2\sigma - C\sin 4\sigma,\ \eta = \arccos(\tan\theta/\tan\theta_0)\\ &S/a = D\sigma + E\sin 2\sigma - F\sin 4\sigma + G\sin 6\sigma,\ \sigma = \arccos(\sin\theta/\sin\theta_0)\end{aligned} \tag{48}$$

and where in both cases with $c_1 = f\cos\theta_0$, $c_2 = (1/4)f\sin^2\theta_0$, $c_3 = 1 + c_1\cos\theta_0$, $c_4 = c_2 + c_3$, we have

$$\begin{aligned}&A = c_1\,(1 - c_2c_4),\ B = (1/2)c_1c_2c_3,\ C = (1/4)c_1c_2^{\,2},\\ &D = 2 + c_2(c_4^{\,2} + c_2^{\,2}) - (1+c_2)\,c_4 - c_2,\ E = (1/2)c_2\,[2 + c_3(c_3 - 1) - c_2^{\,2}],\\ &F = (1/4)c_2^{\,2}\,(2c_4 - 1),\ G = (1/6)c_2^{\,3},\end{aligned} \tag{49}$$

and c_1, c_2, c_3 satisfy $c_1^{\,2} - 4c_2(c_3 - 1) + c_3(2 - c_3) = 1$.

Formulae for longitude and arc length between two arbitrary points on the hemispheroidal geodesic

From (48), for a geodesic arc containing a vertex

$$\begin{aligned}&\Delta\lambda = \Sigma\eta - A\Sigma\sigma + Bp - Cq && \Sigma\eta = \eta_1 + \eta_2,\ \Sigma\sigma = \sigma_1 + \sigma_2,\ \Delta\sigma = \sigma_2 - \sigma_1,\\ &S/a = D\Sigma\sigma + Ep - Fq - Gr && p = 2\sin\Sigma\sigma\cos\Delta\sigma,\ q = 2\sin 2\Sigma\sigma\cos 2\Delta\sigma,\\ &r = 2\sin 3\Sigma\sigma\cos 3\Delta\sigma,\ \eta_i = \arccos(\tan\theta_i/\tan\theta_0),\ \sigma_i = \arccos(\sin\theta_i/\sin\theta_0)\end{aligned} \tag{50}$$

Also from (48) for a geodesic arc containing neither node nor vertex

$$\begin{aligned}&\Delta\lambda = \Delta\eta - A\Delta\sigma + Bp - Cq && \Delta\eta = \eta_2 - \eta_1,\ \Delta\sigma = \sigma_2 - \sigma_1,\ \Sigma\sigma = \sigma_1 + \sigma_2\\ &S/a = D\Delta\sigma + Ep - Fq - Gr && p = 2\cos\Sigma\sigma\sin\Delta\sigma,\ q = 2\sin 2\Delta\sigma\cos 2\Sigma\sigma\\ &r = 2\cos 3\Sigma\sigma\sin 3\Delta\sigma,\ \eta_i = \arccos(\tan\theta_i/\tan\theta_0),\ \sigma_i = \arccos(\sin\theta_i/\sin\theta_0)\end{aligned} \tag{51}$$

From (47) for a geodesic arc containing a node

$$\begin{aligned}&\Delta\lambda = \Sigma\gamma - A\Sigma\beta - Bp - Cq, && \Sigma\gamma = \gamma_1 + \gamma_2,\ \Sigma\beta = \beta_1 + \beta_2,\ \Delta\beta = \beta_2 - \beta_1\\ &S/a = D\Sigma\beta - Ep - F1 - Gr, && p = 2\sin\Sigma\beta\cos\Delta\beta,\ q = 2\sin 2\Sigma\beta\cos 2\Delta\beta,\\ &r = 2\sin 3\Sigma\beta\cos 3\Delta\beta && \gamma_i = \arcsin(\tan\theta_i/\tan\theta_0),\ \beta_i = \arcsin(\sin\theta_i/\sin\theta_0)\end{aligned} \tag{52}$$

Also from (47) for a geodesic arc containing neither node nor vertex

$$\Delta\lambda = \Delta\gamma - A\Delta\beta - Bp - Cq \qquad \Delta\gamma = \gamma_1 - \gamma_2,\ \Delta\beta = \beta_1 - \beta_2,\ \Sigma\beta = \beta_1 + \beta_2,$$
$$S/a = D\Delta\beta - Ep - Fq - Gr \qquad p = 2\cos\Sigma\beta\sin\Delta\beta,\ q = 2\cos 2\Sigma\beta\sin 2\Delta\beta \tag{53}$$
$$r = 2\cos 3\Sigma\beta\sin 3\Delta\beta, \qquad \gamma_i = \arcsin(\tan\theta_i/\tan\theta_0),\ \beta_i = \arcsin(\sin\theta_i/\sin\theta_0)$$

The constants A, B, C, D, E, F, G, of formulae (50), (51), (52), (53) are given by (49). Since (51) and (53) should give the same results one should transform into the other if we make the substitutions respectively from $\sigma + \beta = \pi/2$, $\eta + \gamma = \pi/2$. For instance in (53)

$$\Delta\gamma = \gamma_1 - \gamma_2 = (\pi/2) - \eta_1 - (\pi/2) + \eta_2 = \eta_2 - \eta_1 = \Delta\eta,$$
$$\Delta\beta = \beta_1 - \beta_2 = (\pi/2) - \sigma_1 - (\pi/2) + \sigma_2 = \sigma_2 - \sigma_1 = \Delta\sigma,\ \Sigma\beta = \beta_1 + \beta_2 = \pi - \Sigma\sigma.$$

These substitutions in $\Delta\lambda$ and S/a of (53) give

$$\Delta\lambda = \Delta\eta - A\Delta\sigma + Bp - Cq \qquad p = 2\cos\Sigma\sigma\sin\Delta\sigma,\ q = 2\cos 2\Sigma\sigma\sin 2\Delta\sigma$$
$$S/a = D\Delta\sigma + Ep - Fq - Gr \qquad r = 2\cos 3\Sigma\sigma\sin 3\Delta\sigma$$

Which are formulae (51).

Now in (50) with $\theta_1 = \theta_2 = 0$, we have $\Sigma\sigma = \Sigma\eta = \pi$, $p = q = r = 0$. Analogously for (52) with $\theta_1 = \theta_2 = \theta_0$ we have $\Sigma\gamma = \Sigma\beta = \pi$, $p = q = r = 0$ and both therefore give for length and longitude of hemispheroidal geodesics, node to node or vertex to vertex,

$$\Delta\lambda_0 = \pi(1 - A),\ S_0 = \pi a D. \tag{54}$$

Equations (54) are thus a shorter version of equations (33). Referring to equations (49), (54), when $\theta_0 = \pi/2$, $c_1 = 0$, $c_2 = (1/4)f$, $c_3 = 1$, $c_4 = 1 + (1/4)f$, $A = 0$, $D = 1 - f/2 + (1/16)f^2 + (1/32)f^3$, and *again for the semi-meridian* $\Delta\lambda = \pi$, $S = a\pi[1 - f/2 + (1/16)f^2 + (1/32)f^3]$. When $\theta_0 = 0$, $c_1 = f$, $c_2 = 0$, $c_3 = c_4 = 1 + f$, $A = f$, $D = 2 - (1 + f) = 1 - f$ and *we have again the equatorial limiting arc* $\Delta\lambda = \pi(1 - f)$, $S = a\pi(1 - f)$.

Throughout this discussion $\Delta\sigma$ *has been used to represent two quantities.* When dealing with elliptic integrals and functions, $\Delta\sigma = (1 - k^2\sin^2\sigma)^{1/2}$, see equations (12). When dealing with computational formulae for distance and longitude, $\Delta\sigma = \sigma_2 - \sigma_1$, see equations (50), (51). The usage is clearly indicated in each case, and no ambiguity occurs.

We now have equations to third order in the flattening which may be used to check approximation formulae to the geodesic and to check known or published geodetic lines. After a discussion of the spheroidal triangle, some of these formulae will be used in the derivation of the direct solution for the long geodetic line. But we next examine the antipodal zones and conjugate points with respect to the nonplanar geodesic.

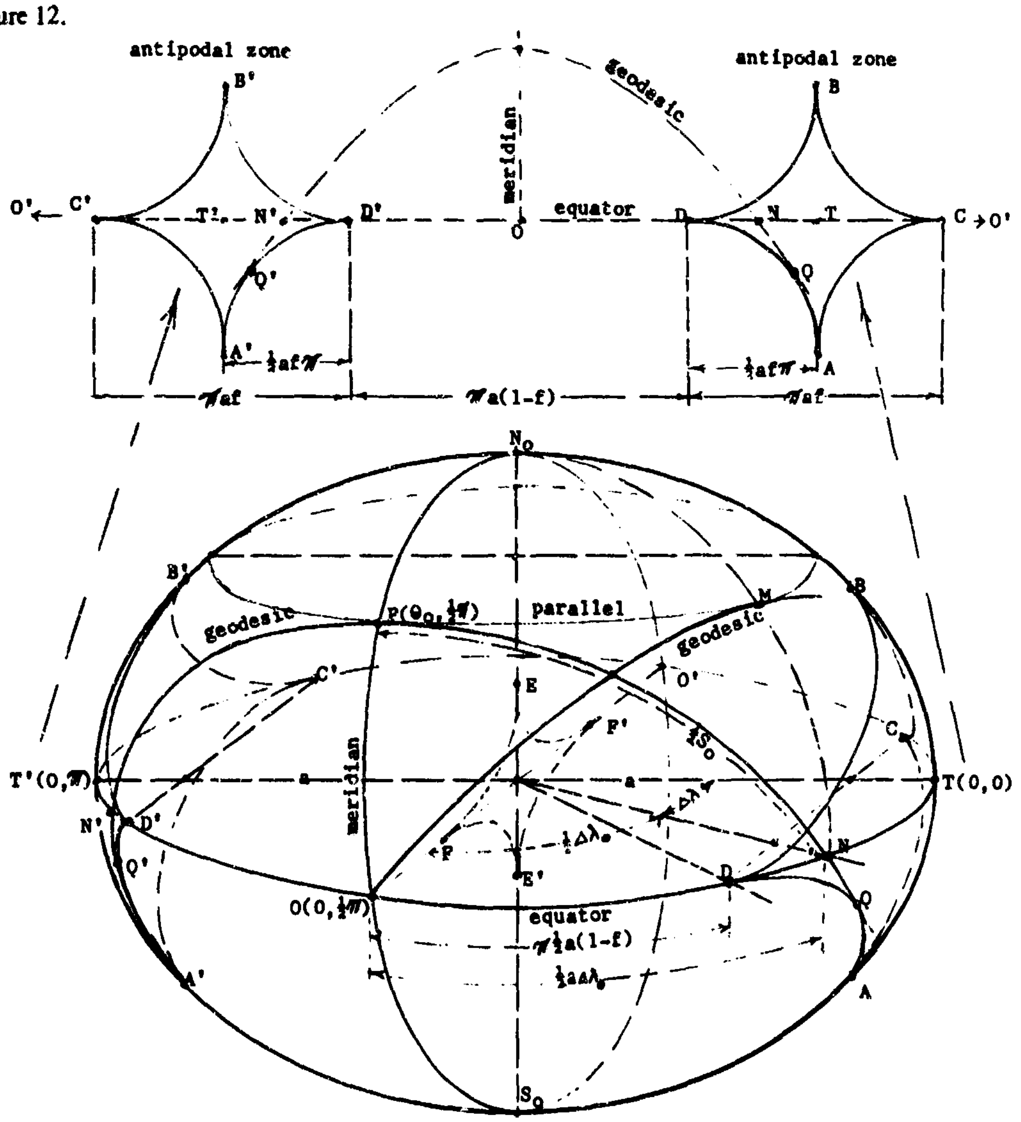
Antipodal zones

The hemispheroidal geodesic is that part included between two consecutive vertices or two consecutive nodes since no more than two consecutive of either nodes or vertices can be contained in the same hemispheroid (on the same side of a meridian).

The antipodal zones are the two equal areas bounded by the two symmetric geodesic evolutes (envelopes) of all oblate spheroidal geodesics which have a vertex in a common fixed meridian. Cayley, reference [25].

NOTE: The evolute of a given curve is the curve tangent to all normals (perpendiculars) of the given curve, or the envelope of the normals. The normal to the meridian ellipse in terms of parametric latitude

θ (eccentric angle of the ellipse) is $F(\theta) = ax/\cos\theta - by/\sin\theta - (a^2 - b^2) = 0$, where a, b are semimajor, semiminor axes of the spheroid, x and y are rectangular coordinates in the plane of the meridian, the y-axis coinciding with the ellipsoidal polar axis. The evolute (envelope) is obtained by eliminating θ between $F(\theta) = F'(\theta) = 0$ where the prime denotes differentiation with respect to θ. The result is the equation $a^{2/3} x^{2/3} + b^{2/3} y^{2/3} = (a^2 - b^2)^{2/3}$, its graph resembling the geodetic evolutes as displayed in Figure 12.



The geodesic evolutes are the figures ADBC, A′D′B′C′ which resemble the meridional evolute EFE′F′. Geodesic arcs PN, OM are equal. Location of the nodes N, N′ within the antipodal zones is known from equations (33). When $P(\theta_0, \pi/2) \to O(0, \pi/2)$, then $\theta_0 \to 0$, and $Q \to N \to D$, $Q' \to N' \to D'$; when $P \to N_0$, then $N \to T$, $N' \to T'$, and $Q \to A$, $Q' \to A'$.

Figure 12. Geodesic evolutes and antipodal zones on the oblate spheroid (pictorial).

Two consecutive nodes are in the geodesic antipodal zones with respect to the meridian containing the included vertex of the geodesic. From the first of the inequalities (35) we have, when $\theta_0 = 0$,

$S_0 = \pi b = a\pi(1 - f)$. Hence the *equatorial arc axis of the geodesic evolute* is then $a\pi - a\pi(1 - f) = a\pi f$ as shown in Figure 12.

The distance from node to node (N to N$'$ in the diametrically opposite antipodal zone) is given by equations (33) and by symmetry this is the same distance as that between two consecutive vertices. *Is the geodesic distance thus obtained the maximum under the shortest distance property of the geodesic?* Apparently this is so from the limits given by inequalities (35). But *for any point P on a given geodesic, is there a point P$'$ on the geodesic beyond which the unique shortest distance property does not hold?* Before we attempt to answer this question we find the length of the meridional arc axis of the geodetic evolute (antipodal zone), the segments AB = A$'$B$'$ of Figure 12.

In Figure 12, note that for the geodesic with vertex $P(\theta_0, \pi/2)$ we have

$$\Delta\lambda = (\pi/2) - (1/2)\Delta\lambda_0. \tag{55}$$

As $\theta_0 \to \pi/2$, N $\to$ T, Q $\to$ A, and there exists the value $+\theta$ which is the parametric latitude of A as given by (55). From equations (32), (33) and (55) we have to terms in f^2:

$$\gamma - f\cos\theta_0\beta + (f^2/8)\cos\theta_0\sin^2\theta_0\,(2\beta - \sin 2\beta) = \pi/2 - \pi/2 + (\pi/2)f\cos\theta_0 - (\pi/2)(f^2/4)\cos\theta_0\sin^2\theta_0$$

or

$$F(\theta, \theta_0) = (\gamma/\cos\theta_0) - f(\beta + \pi/2) + (f^2/8)\sin^2\theta_0\,(\pi + 2\beta - \sin 2\beta) = 0. \tag{56}$$

Where $\quad \gamma = \arcsin(\tan\theta\cot\theta_0),\ \beta = \arcsin(\sin\theta\csc\theta_0)$.

We must therefore solve for θ in the equation remaining by taking

$$\lim_{\theta_0 \to \pi/2} F(\theta, \theta_0) = 0.$$

Only the first term of (56) is bothersome in determining the required limit:

$$\lim_{\theta_0 \to \pi/2} \frac{\gamma}{\cos\theta_0} = \lim_{\theta_0 \to \pi/2} \frac{\arcsin(\tan\theta\cot\theta_0)}{\cos\theta_0} = \lim_{\theta_0 \to \pi/2} \frac{(d/d\theta_0)\arcsin(\tan\theta\cot\theta_0)}{(d/d\theta_0)\cos\theta_0}$$
$$= \lim_{\theta_0 \to \pi/2} \frac{\tan\theta}{\sin^3\theta_0\,(1 - \tan^2\theta\cot^2\theta_0)^{1/2}} = \tan\theta. \tag{57}$$

We have then from (56) and (57)

$$\lim_{\theta_0 \to \pi/2} F(\theta, \theta_0) = \tan\theta - (f/2)(\pi + 2\theta) + (f^2/8)(\pi + 2\theta - \sin 2\theta) = 0,$$

or

$$\tan\theta - B\sin 2\theta = A(\pi + 2\theta) \tag{58}$$
$$B = f^2/8,\ A = (f/2) - B$$

In (58) let $\tan\theta = \theta$, $B = 0$, to get the approximation

$$2\theta = \pi f/(1 - f). \tag{59}$$

With the value f = .003390075283 (Clarke 1866 ellipsoid–Appendix 2), π = 3.1415926536, (59) gives $\theta = 18'\,22''.121$ which fails to satisfy (58) by .00000457, i.e.

$$\tan\theta - 2A\theta - B\sin 2\theta > A\pi \text{ by } .00000457.$$

Now the tangent different for 1 second at 18$'$ 22$''$ is .00000485 (Peters Tables). For 1 second change, $2A\theta$ changes by 2×10^{-8} but there is no change in $B\sin 2\theta$. Hence we take 459/485 = .946 second and reduce the first estimate by that amount since $\tan\theta > \theta > \sin\theta$, i.e. $\theta = 18'\,22''.121 - ''.946 = 18'\,21''.175$. This last values checks (58) to 1 in the 8th decimal.

Since the flattening does not vary much among the 10 reference ellipsoids of Appendix 2, we may alter the approximation (59) to give a solution for any reference ellipsoid. This was accomplished by changing .946 second to radians, factoring πf, writing $1/(1-f) = 1 + f + \ldots$ and then adjusting for the variation in f among the values as given in Appendix 2. The resulting solution to terms in f^2 is

$$2\theta = \pi f(1 + .7495f). \tag{60}$$

Seven of the values of θ computed from (60) checked (58) exactly to 8 decimals and 3 were within 1 in the 8th decimal. The computations are included in Appendix 2, where the axes and approximate areas of the antipodal zones for the 10 spheroids are also given.

Conjugate points on spheroidal geodesics

For an arbitrary point P_1 on a spheroidal geodesic there exists a second point P_2 on that geodesic beyond which the unique shortest distance property fails. Forsyth, citing Jacobi, called such pairs conjugate points, reference [28].

Because of symmetry, the distance, node to node is the same as vertex to vertex, or point P_1 to P_2 in numerically equal but opposite signed latitudes when *the longitude difference is the same as node to node or vertex to vertex.* But *consecutive nodes are not* conjugate since there exist two equal geodesics symmetric with respect to the equator with these common nodes, see Figure 12. Again, but in the meridian, any diameter bisects the meridianal arc length, hence the diametral end points are both antipodal and conjugate. *Hence by inference two consecutive vertices should be conjugate.*

That this is so may be demonstrated in Figure 13. The equal symmetric nodal hemispheroidal geodesics are N_1QN_2, N_1RN_2. Arc lengths N_1P_1, N_2P_2, P_1T are equal, hence the hemispheroidal geodesics P_1QP_2, P_1SP_2 are both equal to N_1QN_2 or N_1RN_2. By symmetry the longitude difference, $\Delta\lambda_0$, node to node, is equal to that from P_1 to P_2. Again, the arc lengths $V_1P'_1$, $V_2P'_2$, P'_1U are equal and therefore the geodesics $P'_1OP'_2$, $P'_1MP'_2$, V_1OV_2, N_1QN_2 are all equal and the longitude difference, P'_1 to P'_2, is $\Delta\lambda_0$, the same as from N_1 to N_2.

For the mathematical demonstration we will *maximize the equation for longitude difference between two points on the geodesic.* As a preliminary, note from the inequalities (35), that for hemispheroidal geodesics we have the longitude difference and length satisfying

$$\pi \geqslant \Delta\lambda_0 \geqslant \pi b/a$$

$$a\pi[1 - (f/2) + (f^2/16) + f^3/32] \geqslant S_0 \geqslant \pi b,$$

i.e. along the equator from a given point one can extend the length to πb before two equal and symmetric geodesics of length shorter than the subtended equatorial arc exist. In Figure 13 we can extend the distance along the equator from N_1 to T_1, $N_1T_1 = \pi b$, before the two symmetric geodesics N_1QN_2, N_1RN_2 exist. That is $N_1N_2 > N_1QN_2 = N_1RN_2 > N_1T_1 = \pi b$ and *the points N_1, T_1 are conjugate.*

From equations (17) we may write

$$\Delta\lambda = IF,\ F = \sec\theta_0\,(1 - e^2\cos^2\theta_0)^{1/2} = e\tan\theta_0/k,$$

$$I = \int_{\sigma_2}^{\sigma_1} \frac{\Delta\sigma}{\delta\sigma}\,d\sigma,\ \Delta\sigma = (1 - c\sin^2\sigma)^{1/2},\ \delta\sigma = 1 + n\sin^2\sigma, \tag{61}$$

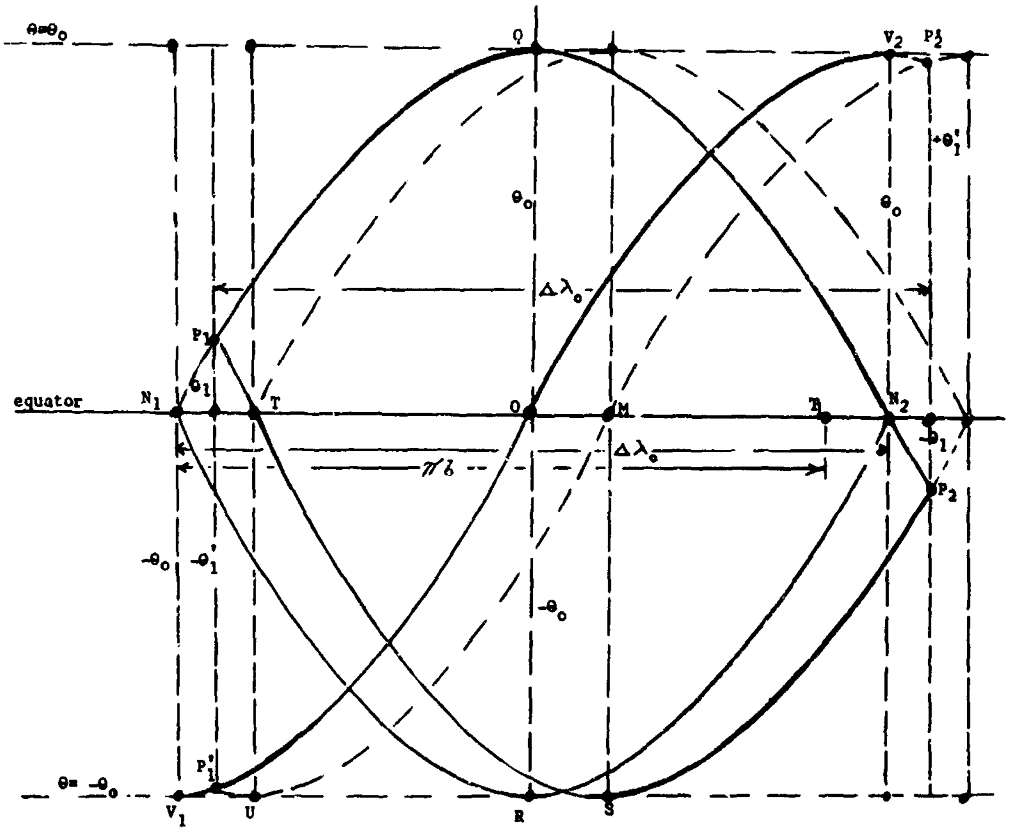


Figure 13. Conjugate points on the oblate spheroid.

$$n = \tan^2 \theta_0,\ c = e^2 \sin^2 \theta_0/(1 - e^2 \cos^2 \theta_0) = k^2,$$

$$\sigma_i = \arccos(\sin \theta_i/\sin \theta_0),\ |\theta_i| \leq |\theta_0|.$$

Now
$$\frac{d\Delta\lambda}{d\theta_0} = F\frac{dI}{d\theta_0} + I\frac{dF}{d\theta_0} = 0,$$

or equivalently

$$\frac{1}{F}\frac{d\Delta\lambda}{d\theta_0} = \frac{dI}{d\theta_0} + \frac{I}{F}\frac{dF}{d\theta_0} = 0 \tag{62}$$

Since $\sigma_1, \sigma_2, \Delta\sigma, \delta\sigma$ are all functions of θ_0, we have

$$\frac{dI}{d\theta_0} = \int_{\sigma_2}^{\sigma_1} \frac{\partial}{\partial\theta_0}\frac{\Delta\sigma}{\delta\sigma}\, d\sigma + \frac{\Delta\sigma_1}{\delta\sigma_1}\frac{d\sigma_1}{d\theta_0} - \frac{\Delta\sigma_2}{\delta\sigma_2}\frac{d\sigma_2}{d\theta_0} \tag{63}$$

$$\frac{\partial}{\partial\theta_0}\left(\frac{\Delta\sigma}{\delta\sigma}\right) = \frac{1}{\delta\sigma^2}\left(\delta\sigma\frac{d\Delta\sigma}{d\theta_0} - \Delta\sigma\frac{d\delta\sigma}{d\theta_0}\right). \tag{64}$$

From (61) we have

$$\frac{d\Delta\sigma}{d\theta_0} = -\frac{\sin^2\sigma}{2\Delta\sigma}\frac{dc}{d\theta_0},\ \frac{dc}{d\theta_0} = \frac{2c^2(1-e^2)}{e^2}\cot\theta_0 \csc^2\theta_0,$$

$$\frac{d\delta\sigma}{d\theta_0}=\sin^2\sigma\,\frac{dn}{d\theta_0},\;\frac{dn}{d\theta_0}=2\tan\theta_0\sec^2\theta_0,$$
$$\frac{d\sigma_i}{d\theta_0}=\cot\sigma_i\cot\theta_0,\;(1/F)\,\frac{dF}{d\theta_0}=(c/e^2)\csc\theta_0\sec\theta_0 \tag{65}$$

With the values of $d\sigma_i/d\theta_0$ from (65) the last two terms of (63) may be written

$$\frac{\Delta\sigma_1}{\delta\sigma_1}\frac{d\sigma_1}{d\theta_0}-\frac{\Delta\sigma_2}{\delta\sigma_2}\frac{d\sigma_2}{d\theta_0}=\cot\theta_0\left(\frac{\Delta\sigma_1}{\delta\sigma_1}\cot\sigma_1-\frac{\Delta\sigma_2}{\delta\sigma_2}\cot\sigma_2\right)$$
$$=\cot\theta_0\int_{\sigma_2}^{\sigma_1}\frac{d}{d\sigma}\left(\frac{\Delta\sigma}{\delta\sigma}\cot\sigma\right)d\sigma \tag{66}$$

With the values of $\Delta\sigma, \delta\sigma$ from (61) we find

$$\frac{d}{d\sigma}\frac{(\Delta\sigma\cot\sigma)}{\delta\sigma}=-\frac{\Delta\sigma}{\delta\sigma\sin^2\sigma}-\frac{c\cos^2\sigma}{\Delta\sigma\,\delta\sigma}-\frac{2\tan^2\theta_0\cos^2\sigma\,\Delta\sigma}{\delta\sigma^2} \tag{67}$$

With the values of $d\Delta\sigma/d\theta_0$, $d\delta\sigma/d\theta_0$ from (65), we may write (64) as

$$\frac{\partial}{\partial\theta_0}\left(\frac{\Delta\sigma}{\delta\sigma}\right)=-\frac{c^2(1-e^2)}{e^2\,\Delta\sigma\,\delta\sigma}\cot\theta_0\csc^2\theta_0\sin^2\sigma-2\tan\theta_0\sec^2\theta_0\frac{\Delta\sigma}{\delta\sigma^2}\sin^2\sigma \tag{68}$$

With the value of $(1/F)\,dF/d\theta_0$ from (65) and the value of I from (61) we have

$$\frac{I}{F}\frac{dF}{d\theta_0}=\int_{\sigma_2}^{\sigma_1}(c/e^2)\sec\theta_0\csc\theta_0\frac{\Delta\sigma}{\delta\sigma}\,d\sigma. \tag{69}$$

Now with the value of (67) placed in (66) and the result returned to (63), together with the value of $\partial/\partial\theta_0\,(\Delta\sigma/\delta\sigma)$ from (68) for the first term of (63), we may, with the resulting value of $dI/d\theta_0$ and the value of $(I/F)dF/d\theta_0$ from (69) write the condition (62) as

$$\cot\theta_0\int_{\sigma_2}^{\sigma_1}\frac{d\sigma}{e^2\,\Delta\sigma\,\delta\sigma^2\sin^2\sigma}\left\{\begin{array}{l}\overset{(1)}{c^2(1-e^2)\csc^2\theta_0\,\delta\sigma\sin^4\sigma}+\overset{(2)}{2e^2n\sec^2\theta_0\,\Delta\sigma^2\sin^4\sigma}\\ +\overset{(3)}{e^2\delta\sigma\,\Delta\sigma^2}+\overset{(4)}{ce^2\delta\sigma\sin^2\sigma\cos^2\sigma}\\ +\overset{(5)}{2e^2n\,\Delta\sigma^2\sin^2\sigma\cos^2\sigma}-\overset{(6)}{c\sec^2\theta_0\,\Delta\sigma^2\,\delta\sigma\sin^2\sigma}\end{array}\right\}=0 \tag{70}$$

where $n=\tan^2\theta_0$, $\Delta\sigma^2=1-c\sin^2\sigma$, $\delta\sigma=1+n\sin^2\sigma$, $c=e^2\sin^2\theta_0/(1-e^2\cos^2\theta_0)$.

In (70), within the braces, we first combine the terms (2) and (5) to get

$$(2)+(5)=2e^2n\,\Delta\sigma^2\,\delta\sigma\sin^2\sigma. \tag{71}$$

We next combine terms (1), (4), and (6) to get analogously

$$(1)+(4)+(6)=-ne^2\delta\sigma\sin^2\sigma+cne^2\,\delta\sigma\sin^4\sigma. \tag{72}$$

With the values from (71) and (72) returned to (70), we now have for the quantity within the braces

$$\{\delta\sigma(e^2n\,\Delta\sigma^2\sin^2\sigma+e^2\,\Delta\sigma^2\,\delta\sigma-e^2n\,\Delta\sigma^2\sin^2\sigma)\}=\{e^2\,\Delta\sigma^2\,\delta\sigma^2\}, \tag{73}$$

where in the reductions we have used the identities given with equation (70). The value from (73) placed in (70) gives

$$(1/F)\, d\Delta\lambda/d\theta_0 = (dI/d\theta_0) + (I/F)(dF/d\theta_0) = -\cot\theta_0 \int_{\sigma_2}^{\sigma_1} \frac{e^2\, \Delta\sigma^2\, \delta\sigma^2\, d\sigma}{e^2\, \Delta\sigma\, \delta\sigma^2 \sin^2\sigma}$$

$$= -\cot\theta_0 \int_{\sigma_2}^{\sigma_1} \frac{\Delta\sigma d\sigma}{\sin^2\sigma} = 0. \tag{74}$$

Since the equation to the geodesic evolute will be given by the elimination of θ_0 between $d\Delta\lambda/d\theta_0 = 0$, and $\Delta\lambda = FI$, equations (61) and (74), *we should be able to get an equation for determining the parametric latitude of the meridional vertex of the geodesic evolute and thus provide a check for equation (58).* In fact the equation should be given by (74), that is from

$$\lim_{\theta_0 \to \pi/2} \int_{-(\pi/2+\theta)}^{\pi/2+\theta} \Delta\sigma d\sigma/\sin^2\sigma = 0.$$

With $\theta_0 = \pi/2$, $c = e^2$, $\Delta\sigma = (1 - e^2 \sin^2\sigma)^{1/2}$, we have

$$\Delta\sigma/\sin^2\sigma = \csc^2\sigma[1 -(1/2)e^2\sin^2\sigma - (1/8)e^4 \sin^4\sigma - ..] = \csc^2\sigma - f(1-f/4) + (1/4)f^2 \cos 2\sigma.$$

Integrating this last expression with respect to σ and evaluating for the limits $\pi/2 + \theta$, $-(\pi/2 + \theta)$ we obtain equation (58).

In equation (74), the factor $\cot\theta_0 = 0$ implies the meridian, $\theta_0 = (1/2)\pi$. Now from (13) and (13)c, $\Delta\sigma = \mathrm{dnS}$, $\sin^2\sigma = \mathrm{sn}^2\mathrm{S}$, $\sigma = \mathrm{amS}$, $d\sigma = d\,\mathrm{amS} = \mathrm{dnS}\, d\mathrm{S}$ and the integral (74) may be written

$$\int_{S_2}^{S_1} \mathrm{dn}^2 S\, dS/\mathrm{sn}^2 S = 0. \tag{75}$$

By manipulation of the identities (13)a and differentials (13)c, we can write the integral (75), indefinite, as

$$\int dS\,(c' - \mathrm{dn}^2 S + \mathrm{dn}^2 S + c\,\mathrm{cn}^2 S + \mathrm{cn}^2 S\, \mathrm{dn}^2 S/\mathrm{sn}^2 S) = \int c' dS - \int \mathrm{dn}^2 S\, dS - \int d\,(\mathrm{cnS}\, \mathrm{dnS}/\mathrm{snS}), \tag{76}$$

$$\int \mathrm{dn}^2 S\, dS/\mathrm{sn}^2 S = c'S - \int \mathrm{dn}^2 S\, dS - \mathrm{cnS}\, \mathrm{dnS}/\mathrm{snS}. \tag{77}$$

From (12), (13), (13)c we have

$$E(k, \sigma) = \int_0^{\sigma} \Delta\sigma\, d\sigma = \int_0^{S} \mathrm{dn}^2 S\, dS = E(S, k). \tag{78}$$

From (77) and (78) the definite integral (75) may be written

$$\int_{S_2}^{S_1} \mathrm{dn}^2 S\, dS/\mathrm{sn}^2 S = [c'S - E(S, k) - \mathrm{cnS}\, \mathrm{dnS}/\mathrm{snS}]_{S_2}^{S_1} = 0. \tag{79}$$

We expand (79) and write the result in the form

$$c'(S_1 - S_2) = E(S_1, k) - E(S_2, k) - [(\mathrm{cnS}_2\, \mathrm{dnS}_2/\mathrm{snS}_2) - (\mathrm{cnS}_1\, \mathrm{dnS}_1/\mathrm{snS}_1)]. \tag{80}$$

Using the difference formula for two elliptic integrals of the second class with the same modulus, and the difference formula for the sine amplitude from (13)a, we can write the right members of (80) respectively as

$$\begin{aligned} &E(S_1, k) - E(S_2, k) = E(S_1 - S_2, k) - c\, \mathrm{snS}_1\, \mathrm{snS}_2\, \mathrm{sn}(S_1 - S_2) \\ &[-(\mathrm{cnS}_1\, \mathrm{dnS}_1/\mathrm{snS}_1) + (\mathrm{cnS}_2\, \mathrm{dnS}_2/\mathrm{snS}_2)] = \mathrm{sn}(S_1 - S_2)(1 - c\, \mathrm{sn}^2 S_1\, \mathrm{sn}^2 S_2)/\mathrm{snS}_1\, \mathrm{snS}_2. \end{aligned} \tag{81}$$

Placing the values from (81) in (80) and solving for $\mathrm{sn}(S_1 - S_2)$ we find

$$\mathrm{sn}(S_1 - S_2) = \mathrm{snS}_1\, \mathrm{snS}_2\, [E(S_1 - S_2, k) - c'(S_1 - S_2)], \tag{82}$$

where $c' = 1 - c = 1 - k^2 = (1 - e^2)/(1 - e^2 \cos^2\theta_0)$, $c = e^2 \sin^2\theta_0/(1 - e^2\cos^2\theta_0)$.

If we place $S_1 = S_2$ in equation (82), the equation is satisfied since from (12) and (13)b, $\mathrm{sn}(0) = E(0, k) = 0$. If we place, in equation (82), $S_1 = 2K$, $S_2 = 0$, the equation is satisfied since $\mathrm{sn}2K = \mathrm{sn}(0) = 0$. Hence the value of S_1 required is the root of (82) next greater than S_2 where $0 < S_2 < 2K$. Note that K is a complete elliptic integral, see (12)a.

For an approximation we write x for S_1 in (82) and consider the intersection of the functions (curves)

$$
\begin{aligned}
y = \mathrm{sn}(x - S_2)/\mathrm{sn}(x)\mathrm{sn}S_2 &= E(x - S_2, k) - c'(x - S_2),\ c' = 1 - k^2 \\
&= (1 - e^2)/(1 - e^2 \cos^2 \theta_0) < 1.
\end{aligned} \tag{83}
$$

As shown in Figure 14, the next value of x for which equations (83) are satisfied is S_1 and we have

$$
0 < x_1 < 2K - S_2,\ \text{where } x_1 = S_1 - (S_2 + 2K). \tag{84}
$$

If we solve (84) for S_1 and place this value, $x = S_1 = x_1 + S_2 + 2K$, in (83), we may write, using the values from (13)b

$$
\begin{aligned}
y_1 = \mathrm{sn}(x_1 + 2K)/\mathrm{sn}(x_1 + S_2 + 2K)\mathrm{sn}S_2 &= \mathrm{sn}x_1/\mathrm{sn}(x_1 + S_2)\mathrm{sn}S_2 \\
&= E(x_1 + 2K, k) - c'(x_1 + 2K).
\end{aligned} \tag{85}
$$

Using the appropriate identities from (13)a, we transform the right member of (85) as follows:

$$
\begin{aligned}
E(x_1 + 2K, k) - c'(x_1 + 2K) = E(x_1, k) - c'x_1 + 2(E - c'K) &= E(x_1, k) - c'x_1 + 4cc'dK/dc \\
&= c \int_0^{x_1} \mathrm{cn}^2 x\, dx + 4cc'dK/dc.
\end{aligned} \tag{86}
$$

Using the value of K from (12)a, we have

$$
dK/dc = \int_0^{\pi/2} \frac{\partial}{\partial c} (1 - c \sin^2 x)^{-1/2}\, dx = (1/2) \int_0^{\pi/2} (1 - c \sin^2 x)^{-3/2} \sin^2 x\, dx. \tag{87}
$$

From (85), (86), and (87) we have, for the determination of x_1, the equation

$$
\begin{aligned}
y_1 = \mathrm{sn}x_1/\mathrm{sn}(x_1 + S_2)\mathrm{sn}S_2 = c &\int_0^{x_1} \mathrm{cn}^2 x\, dx \\
&+ 2c(1 - c) \int_0^{\pi/2} (1 - c \sin^2 x)^{-3/2} \sin^2 x\, dx.
\end{aligned} \tag{88}
$$

Since $c = e^2 \sin^2 \theta_0/(1 - e^2 \cos^2 \theta_0) = 2f \sin^2 \theta_0 + \ldots$, and $2f = e^2 + f^2$, then always $2f > e^2 \geqslant c$. For earth reference ellipsoids $2f - e^2 = f^2 \approx 1 \times 10^{-5}$. We consider here that $2f, e^2, c$ are of the same order and reject terms of second and higher order in c or f. Since x_1, $\mathrm{sn}x_1$ are of the same order as c, we place $\mathrm{sn}x_1 = x_1$, $\mathrm{sn}(x_1 + S_2) = \mathrm{sn}S_2$ and write (88) as

$$
y_1 = x_1/\mathrm{sn}^2 S_2 = c \int_0^{x_1} (1 - ..)\, dx + 2c \int_0^{\pi/2} (1/2 - ..)\, dx = 0 + (1/2)c\pi,
$$

and we find

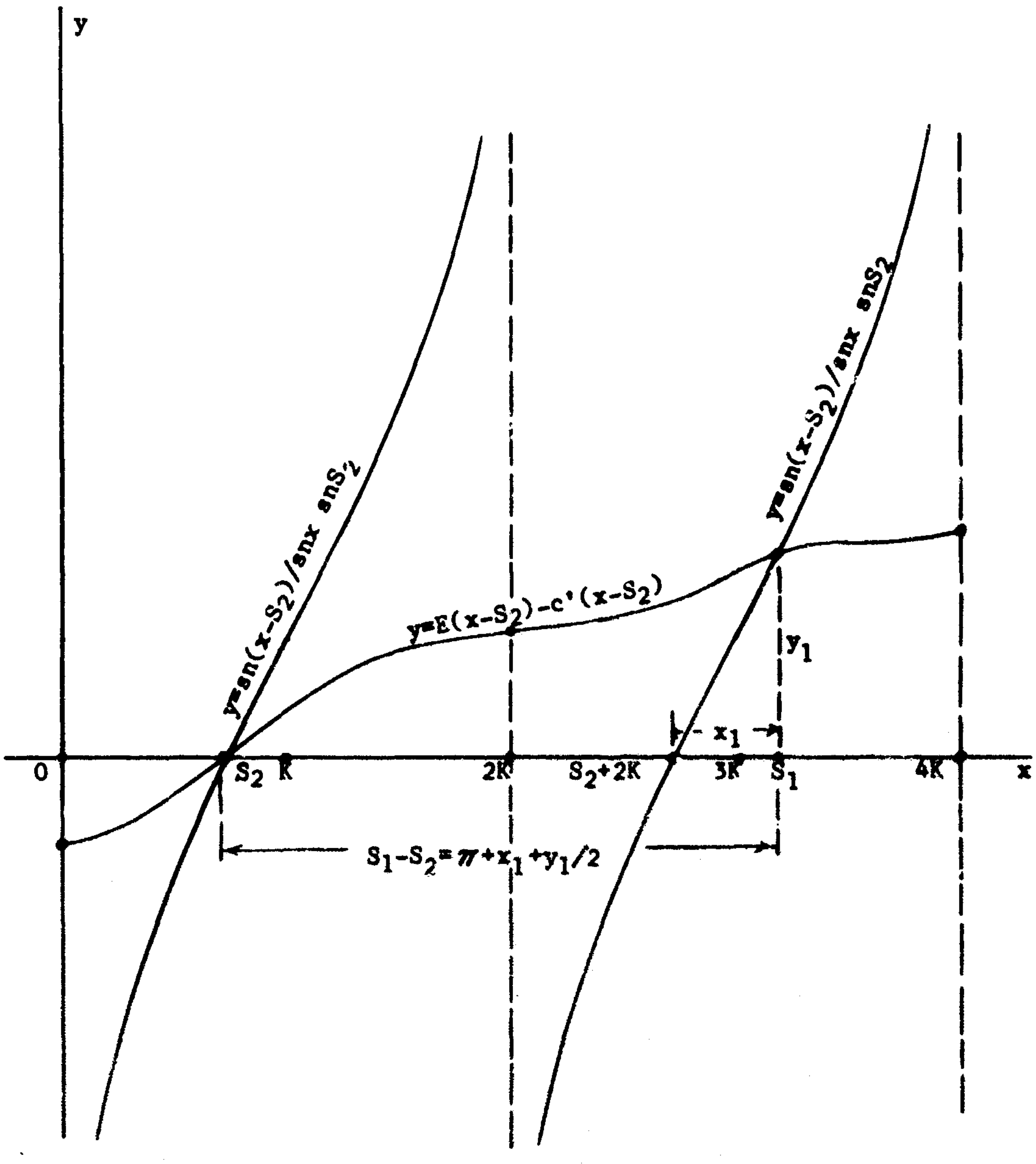
$$
x_1 = (1/2)c\pi\ \mathrm{sn}^2 S_2. \tag{89}
$$

Placing $c = 2f \sin^2 \theta_0$ in (89) we may write

$$
x_1 = \pi f \sin^2 \theta_0\ \mathrm{sn}^2 S_2 = y_1 \mathrm{sn}^2 S_2,\ y_1 = \pi f \sin^2 \theta_0. \tag{90}
$$

From (84) and (90) we obtain

$$
S_1 = 2K + S_2 + x_1 = 2K + S_2 + \pi f \sin^2 \theta_0\ \mathrm{sn}^2 S_2. \tag{91}
$$



$$x_1 = y_1 sn^2 S_2 = \pi f \sin^2\theta_0\, sn^2 S_2,\ y_1 = \pi f \sin^2\theta_0$$

Figure 14. Graphical solution of equations (83).

Note that the value given by (91) is analogous to that obtained by Forsyth, see reference [28]. By direct integration of the integral for K, equations (12)a, we find to first order in f that $2K = \pi(1 + (1/2)f \sin^2\theta_0) = \pi + (1/2)y_1$. Substitution in (91) gives $S_1 - S_2 = \pi + x_1 + (1/2)y_1$ as shown in Figure 14.

From (61), (13), (13)a, (13)b we have

$$cnS_1 = \cos\sigma_1 = \sin\theta_1/\sin\theta_0,\ cn(2K + S) = -cnS,\ sn^2 S = 1 - cn^2 S. \tag{92}$$

We write from (91), with the help of (92)

$$\text{cn}S_1 = \text{cn}[2K + (S_2 + x_1)] = -\text{cn}(S_2 + x_1),$$

$$\sin\theta_1 = -\sin\theta_0 \,\text{cn}(S_2 + x_1),\ x_1 = \pi f \sin^2\theta_0 \,\text{sn}^2 S_2 = \pi f(\sin^2\theta_0 - \sin^2\theta_2), \tag{93}$$

which to first order in f relates the parametric latitudes θ_1, θ_2 of conjugate points on spheroidal geodesics.

When $\theta_2 = \theta_0$, a geodesic vertex, then $x_1 = 0$, $\text{cn}S_2 = 1$, and we have $\sin\theta_1 = -\sin\theta_0$, or $\theta_1 = -\theta_0$. That is the conjugate of a vertex of the geodesic is the next vertex, a result obtained by another argument. See the discussion following equations (33) and the geometric demonstration, Figure 13. A special case of this last is given by $\theta_0 = (1/2)\pi$, whence $\sin\theta_1 = -1$, $\theta_1 = -(1/2)\pi$, i.e. the poles are conjugate as well as antipodal for the meridian, a known result. When $\theta_0 = \theta_2 = 0$, then $\theta_1 = 0$ and we have the end points of the equatorial limiting geodesic arc πb, the segment N_1T_1 of Figure 13.

Since along the geodesic $|\theta_2| \leq |\theta_0|$, the range of x_1 for a particular geodesic is $0 \leq x_1 \leq \pi f \sin^2\theta_0 = y_1$ and over the spheroid is $0 \leq x_1 \leq \pi f$. *Note that πf is the equatorial central angle subtended by the equatorial arc axis of the geodesic evolute, see Figure 12, or to first order in f, πf is the meridional central angle subtended by the meridional arc axis, see equation (60).* Consistent with the approximations used to obtain (89), we have

$$\text{sn}x_1 = x_1,\ \text{cn}x_1 = \text{dn}x_1 = 1,\ 2f\sin^2\theta_0 \,\text{sn}^2 x_1 = \text{c}x_1^2 = 0. \tag{94}$$

From (13)a, the addition formula for the cosine amplitude, we may write (93) as

$$\sin\theta_1 = -\sin\theta_0\,(\text{cn}S_2\,\text{cn}x_1 - \text{sn}S_2\,\text{dn}S_2\,\text{sn}x_1\,\text{dn}x_1)/(1 - 2f\sin^2\theta_0\,\text{sn}^2x_1\,\text{sn}^2S_2)$$

and using (94)

$$\sin\theta_1 = -\sin\theta_0(\text{cn}S_2 - x_1\,\text{sn}S_2\,\text{dn}S_2). \tag{95}$$

Now from (19)a, with $e^2 = 2f$ and retaining terms in f, we have

$$\text{sn}S_2 = (\sin^2\theta_0 - \sin^2\theta_2)^{1/2}/\sin\theta_0,\ \text{dn}S_2 = [1 - 2f(\sin^2\theta_0 - \sin^2\theta_2)]^{1/2},$$

$$\text{cn}S_2 = \sin\theta_2/\sin\theta_0. \tag{96}$$

The values from (96), and the value of x_1 from (93), placed in (95), retaining terms of first order in f, give

$$\sin\theta_1 = -\sin\theta_2 + \pi f(\sin^2\theta_0 - \sin^2\theta_2)^{3/2},\ |\theta_2| \leq |\theta_0|, \tag{97}$$

which to first order in *f is the equation relating the parametric latitudes θ_1, θ_2 of conjugate points on spheroidal geodesics but free of elliptic functions.* Note that (97) also gives the special cases discussed following equations (93), as it should.

Discussion. We have demonstrated mathematically and geometrically (pictorially in Figure 13) that along the equator, the end points of the segment πb are conjugate. We have proved that consecutive vertices are conjugate from both (93) and (97). Now if we ignore the term in f in equation (97), we have $\theta_1 = -\theta_2$ with the longitude difference that of the hemispheroidal geodesic in vertex parametric latitude θ_0, whence we get two equal geodesics as demonstrated in Figure 13. Hence to test approximation formulae to the geodesic we need not exceed the length of the hemispheroidal geodesic (node to node or vertex to vertex) since it is maximum under the unique shortest distance criterion.

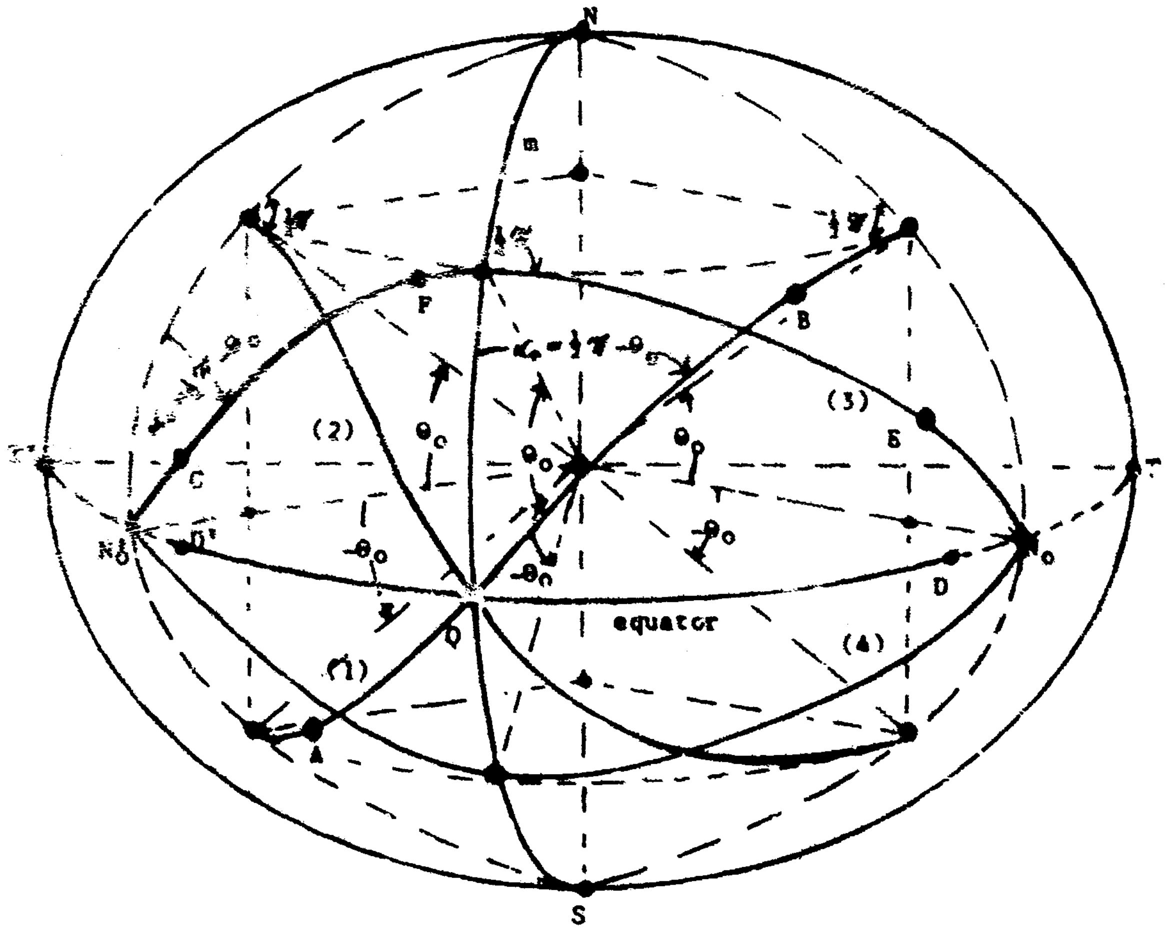
Note that equation (74) provides through the subsequent discussion, the sufficient condition for maximum geodetic length under the shortest distance property, the Euler equation, equation (3) above, being the necessary condition.

Also note that the parallels $\theta = \pm\,\theta_0$ are envelopes of all the geodesics whose vertex latitudes are $\pm\,\theta_0$, and the points $V_1\,V_2$ (vertices), Figure 13, are points of tangency to the envelopes. But any conjugate point, according to the analytic definition, is a contact point of an envelope, reference [29] page 34. Finally note that two types of envelopes are involved. The envelope of all the geodesics having the same vertex parametric latitude $|\,\theta_0\,|$ are the parallels $\theta = \pm\,\theta_0$; the envelopes of all geodesics with a vertex in a common meridian are the two symmetric geodesic evolutes as shown in Figure 12.

Hemispheroidal geodesics under the shortest distance property

With the help of equations (33), (34), (35), (97) we establish that the nonplanar geodesic distance between two given spheroidal points, under the shortest distance property of the spheroidal geodesic, lies on an arc of one of the four equivalent spheroidal geodesics, as shown in Figure 15, where θ_0 is the vertex parametric latitude of the geodesic through the two given points.

Graphically, if a wire frame were constructed connecting the semiequator, the semimeridian m, the meridian NTST' and the four hemispheroidal geodesics with vertex latitude θ_0 as shown in Figure 15, then



$\cos\theta_0 = \cos\theta \sin\alpha$. The arc $DD' = a\pi(1 - f) = b\pi$.

Figure 15. The four equal nonplanar hemispheroidal geodesics determined by a given vertex parametric latitude.

the rotation of the ellipsoid about the polar axis under this frame would bring the two given points into coincidence with one of these four equal nonplanar spheroidal geodesics. Note that when $\theta_0 \to 0$, all four geodesics coincide with the equatorial limiting distance DD′ and when $|\theta_0| \to (1/2)\pi$ both spheroidal geodesics (1) and (2) coincide with the semimeridian m while the spheroidal geodesics (3) and (4) coincide with the meridian NTST′.

We may then construct a table of possible cases of hemispheroidal geodesics to be considered in the testing of approximation formulae to the geodesic. With the help of equations (32), (33), (35), (38), (50)-(54), (97) the possible cases are listed in Table 2.

Note in Figure 15 that hemispheroidal geodesics (2) and (4) are the reflections in the equator of (1) and (3). Also the meridian NQS bisects all four hemispheroidal geodesics with the same vertex latitude θ_0. We can treat a geodesic in the southern hemispheroid as though it were in the northern and translate computed elements symmetrically with respect to the equator. Thus all possible cases required to test approximation formulae to a geodesic with vertex latitude θ_0 are as shown in Figure 15. Note that arc AB contains a node, arc CE contains a vertex, and arc CF contains neither vertex nor node.

Table 2. Hemispheroidal geodesics.

$\Delta\lambda_0$ is the longitude difference, node to node or vertex to vertex, of the spheroidal geodesic whose vertex parametric latitude is θ_0, see equations (33) and Table 8.

CASE	
I. $|\theta_2| \neq |\theta_1|, \Delta\lambda_1 \pm \Delta\lambda_2 < \Delta\lambda_0$	*General* – the geodesic arc may not include either a vertex or a node; may include one vertex or one node.
a. $|\theta_0| > \theta_2 > \theta_1 \geqslant 0$	
b. $|\theta_0| > \theta_1 > \theta_2 \geqslant 0$	
c. $\theta_2 < 0, \theta_1 > 0$	
1. $|\theta_0| > |\theta_2| > \theta_1 \geqslant 0$	
2. $|\theta_0| > \theta_1 > |\theta_2| \geqslant 0$	
d. $\theta_1 < 0, \theta_2 > 0$	
1. $|\theta_0| > |\theta_1| > \theta_2 \geqslant 0$	
2. $|\theta_0| > \theta_1 > |\theta_2| \geqslant 0$	
II. $|\theta_2| = |\theta_1|, \Delta\lambda_1 = \pm \Delta\lambda_2 < (1/2)\Delta\lambda_0$	*Symmetric* – with respect to a node or vertex but not maximum; contains one node or one vertex. Special case of I.
a. $|\theta_0| > \theta_1 > 0, \theta_2 = -\theta_1$	
b. $\theta_1 < 0, \theta_2 = |\theta_1| < |\theta_0|$	
c. $\theta_2 = \theta_1, |\theta_0| > \theta_1$	
III. $\Delta\lambda_1 = \pm \Delta\lambda_2 = (1/2)\Delta\lambda_0$	*Maximum* – between two consecutive vertices or two consecutive nodes or between two points as in IIIc. Special case of II.
a. $|\theta_2| = |\theta_1| = |\theta_0| \neq 0$	
b. $\theta_1 = \theta_2 = 0, \theta_0 \neq 0$	
c. $\theta_1 = -\theta_2, \Delta\lambda = \Delta\lambda_0$	

Some numerical considerations

Since there are 206264.8062 seconds in one radian, a 6 in the ninth decimal place of one radian represents .001 second. Maximum hemispheroidal radian geodesic length is the semimeridian which is slightly under π radians. Table 3 shows the effect of radian decimal places and significant figures in computing geodetic distances over the hemispheroid. Note that with 10 decimal places of radians there will be some uncertainty in the third decimal of meters at maximum hemispheroidal geodetic length.

The spheroidal triangle

We first indicate some analogies between spherical and spheroidal right triangles. From the definitions, equations (13), we have $\sin\sigma = \text{snS}$, $\cos\sigma = \text{cnS}$, $\tan\sigma = \text{tnS}$; where $\sigma = \text{amS}$, amplitude of the elliptic integral of the first kind, $S = F(k, \sigma)$, and where the modulus is $k = e \sin \theta_0/(1 - e^2 \cos^2 \theta)^{1/2}$—see equations

Table 3. Effect of radian decimal places and significant figures in computation of geodetic distances over the hemispheroid

Clarke 1866 ellipsoid, a = 6378206.4 meters

radians	*decimals*	*significant figures*	*meters (a·radians)*
$\pi/32$ = .0981748	7	6	626179
.77	8	7	8.9
...04	10	9	.949
.....25	12	11	.94904
$\pi/4$ = .7853982	7	7	5009432
.16	8	8	1.6
...3	9	9	.59
....4	10	10	.592
$\pi/2$ = 1.5707963	7	8	10018863
.3	8	9	3.2
.27	9	10	.19
..68	10	11	.185
π = 3.1415927	7	8	20037727
.65	8	9	6.3
...4	9	10	.37
...36	10	11	.369

(15) and (19)s. Hence for the spherical and spheroidal triangles $N'P_0'P'$, NP_0P as shown in Figure 16, we have the following analogies:

	Spherical		*Spheroidal*		
A.1	$\cos\theta_0 = \cos\theta \sin\alpha$	$\leftarrow$ equation (10) $\rightarrow$	$\cos\theta_0 = \cos\theta\sin\alpha$	B.1	
A.2	$\cot\theta_0 = \tan\alpha\sin\sigma$		$\cot\theta_0 = \tan\alpha \,\mathrm{snS}$	B.2	(98)
A.3	$\sin\theta = \sin\theta_0\cos\sigma$		$\sin\theta = \sin\theta_0 \,\mathrm{cnS}$	B.3	
A.4	$\tan\sigma = \cos\theta_0 \tan\eta$		$\mathrm{tnS} = \cos\theta_0 \tan\Delta\lambda$		
			$= \cot\theta\cos\alpha$	B.4	

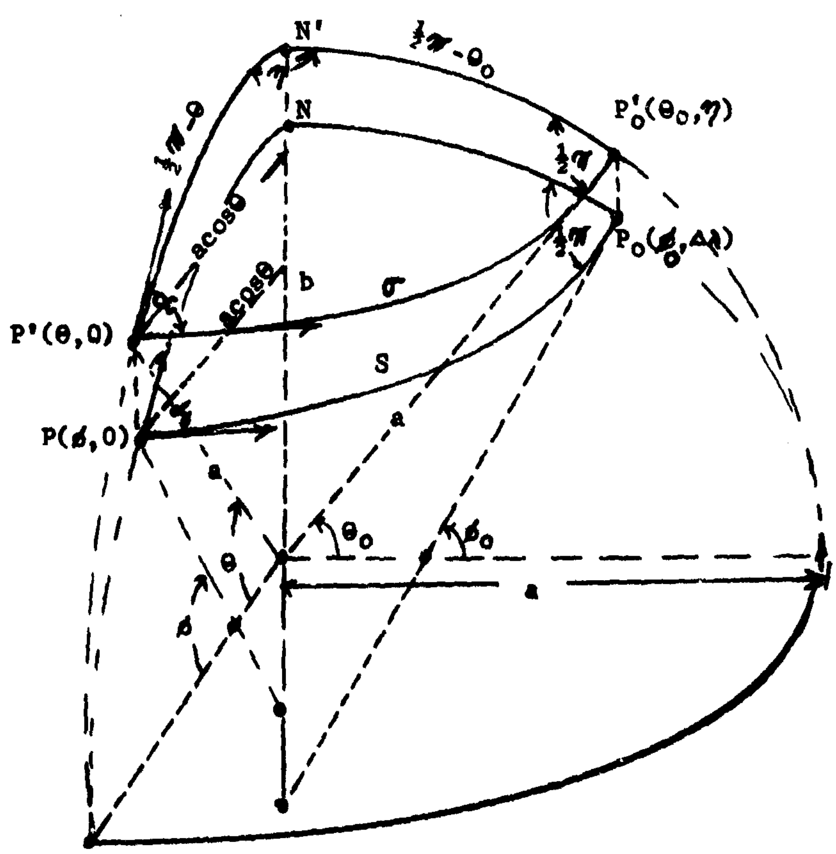
From Figure 10 and the identity (98) (A.1 = B.1) we have respectively:

$$\cos\alpha = \sin\left(\frac{\pi}{2} - \alpha\right) = a(1 - e^2\,\theta)^{1/2}\left(-\frac{d\theta}{ds}\right),$$

$$\sin\alpha = \cos\theta_0 \sec\theta,$$

whence

$$\tan\alpha = \frac{\cos\theta_0}{a(1 - e^2\cos^2\theta)^{1/2}} \cdot \frac{ds}{-\cos\theta\, d\theta} \qquad (99)$$



θ is geocentric latitude for the sphere, parametric for the ellipsoid; φ is geodetic latitude

Figure 16. Illustrating analogies between spherical and spheroidal right triangles.

From (98), B.3 and (13)c

$$-\cos\theta\, d\theta = \sin\theta_0\, \mathrm{snS}\, \mathrm{dnS}\, dS \tag{100}$$

From (13) and (15)

$$(1 - e^2\cos^2\theta)^{1/2} = (1 - e^2\cos^2\theta_0)^{1/2}\, \mathrm{dnS} \tag{101}$$

From (13)c and (17)

$$ds = \frac{ea\sin\theta_0\, \mathrm{dn}^2 S\, dS}{k} \tag{102}$$

From (100) and (102), with $k = e\sin\theta_0/(1 - e^2\cos^2\theta_0)^{1/2}$,

$$\frac{ds}{-\cos\theta\, d\theta} = \frac{a(1 - e^2\cos^2\theta_0)^{1/2}}{\sin\theta_0}\,\frac{\mathrm{dnS}}{\mathrm{snS}} \tag{103}$$

Substituting from (101) and (103) in (99) find

$$\tan a = \frac{\cos\theta_0}{a}\cdot\frac{1}{(1 - e^2\cos^2\theta_0)^{1/2}\,\mathrm{dnS}}\cdot\frac{a(1 - e^2\cos^2\theta_0)^{1/2}}{\sin\theta_0}\,\frac{\mathrm{dnS}}{\mathrm{snS}} = \frac{\cot\theta_0}{\mathrm{snS}},$$

or

$$\cot\theta_0 = \tan a\, \mathrm{snS} \tag{104}$$

which is (98) B.2 and becomes (98) A.2 when $e \to 0$, $k \to 0$, $S \to \int_0^\sigma d\sigma = \sigma$, see equations (12), and $\mathrm{snS} \to \sin\sigma$, $\mathrm{cnS} \to \cos\sigma$, where σ is then the spherical distance from the vertex of the geodesic (parametric latitude θ_0) to a point on the geodesic in parametric latitude, θ, i.e. $\sigma = \arccos(\sin\theta/\sin\theta_0)$ or (98) A.3; see also Figure 11.

To find an expression for $\Delta\lambda$ in (98) B.4, we have from equation (17)

$$(k/e)\cot\theta_0\, d\lambda = (1 - k^2\sin^2\sigma)^{1/2}\, d\sigma/(1 + n\sin^2\sigma),\ n = \tan^2\theta_0 \tag{105}$$

and from (13)c and (19)a find

$$(1 - k^2\sin^2\sigma)^{1/2} = \mathrm{dnS},\ \sin^2\sigma = \mathrm{sn}^2 S,\ d\sigma = \mathrm{dnS}\, dS,$$

whence (105) becomes

$$(k/e)\cot\theta_0\, d\lambda = \mathrm{dn}^2 S\, dS/(1 + n\,\mathrm{sn}^2 S) \tag{106}$$

If we let $\tan U = \sec\theta_0\, \mathrm{tnS}$, then

$$\sec^2 U\, dU = \sec\theta_0\, d(\mathrm{tnS}) = \sec\theta_0\, \mathrm{dnS}\, dS/\mathrm{cn}^2 S \tag{107}$$

where we have used $d(\mathrm{tnS}) = \mathrm{dnS}\, dS/\mathrm{cn}^2 S$ from (13)c.

Now

$$\begin{aligned}\sec^2 U &= 1 + \tan^2 U = 1 + \sec^2\theta_0\, \mathrm{tn}^2 S\\ &= 1 + (1 + \tan^2\theta_0)\, \mathrm{tn}^2 S\\ &= (\mathrm{cn}^2 S + \mathrm{sn}^2 S + \tan^2\theta_0\, \mathrm{sn}^2 S)/\mathrm{cn}^2 S\end{aligned}$$

$$\sec^2 U = (1 + n\,\mathrm{sn}^2 S)/\mathrm{cn}^2 S. \tag{108}$$

(from the identities (13)a, $\mathrm{sn}^2 S + \mathrm{cn}^2 S = 1$, $\mathrm{tnS} = \mathrm{snS}/\mathrm{cnS}$)

From (107) and (108) we have

$$\cos\theta_0\, dU = \mathrm{dnS}\, dS/(1 + n\,\mathrm{sn}^2 S),\ n = \tan^2\theta_0. \tag{109}$$

Subtracting respective members of (106) from (109), find

$$d\lambda = (e/k)\tan\theta_0\left[\cos\theta_0\, dU - \frac{\mathrm{dnS} - \mathrm{dn}^2 S}{1 + n\,\mathrm{sn}^2 S}\, dS\right],$$

or

$$\Delta\lambda = (e\sin\theta_0/k)U - (e\tan\theta_0/k)\int_0^S \frac{\mathrm{dnS} - \mathrm{dn}^2 S}{1 + n\,\mathrm{sn}^2 S}\, dS \tag{110}$$

where $U = \text{arc tan}(\sec\theta_0 \text{ tnS})$, $k = e \sin\theta_0/(1 - e^2 \cos^2\theta_0)^{1/2}$, $n = \tan^2\theta_0$. Solving for U, (110) may be written

$$\text{tnS} = \cos\theta_0 \tan\left[(k/e \sin\theta_0)\Delta\lambda + \sec\theta_0 \int_0^S \frac{\text{dnS} - \text{dn}^2 S}{1 + n\,\text{sn}^2 S}\,dS\right]. \tag{111}$$

When $e \to 0$, $k/e \sin\theta_0 \to 1 \to \text{dnS}$, $\text{tnS} \to \tan\sigma$ and (111) becomes the spherical formula $\tan\sigma = \cos\theta_0 \tan\Delta\lambda$ where $\Delta\lambda = \eta$, the spherical longitude, i.e. (111) becomes (98) A.4 when $e \to 0$. *Thus the analogies (98) are implicit in the spherical approximation to the spheroidal triangle as demonstrated in Figure 11.*

The approximate solution for geodesy

Direct Solution

For the direct solution we are given the geodesic length S from a given point $P_1(\phi_1, \lambda_1)$ in given azimuth α_{1-2} to find the geographic coordinates ϕ_2, λ_2 of a point $P_2(\phi_2, \lambda_2)$ and the azimuth α_{2-1}. A solution, reliable over the hemispheroid, will be sought consistent with the following criteria:

1. An accuracy of 1 meter in position—geodetic distance within 1 meter: latitude, longitude, azimuth within .035 second over the longest possible hemispheroidal geodesics; at least, in the limiting case, equalling the 1/100,000 distance and 1 second azimuth requirement adopted by ACIC, reference [22].
2. No tables required in the computations except natural trigonometric as Peters 8-place for desk computing.
3. Easy adaptation to any reference ellipsoid by merely changing the ellipsoid defining parameters.
4. No root calculation or iteration and formulae adaptable to both desk computation and large electronic computers with terms no higher than second order in the flattening.

Now the parametric latitude θ_1 of P_1 may be computed from $\tan\theta_1 = (1 - f)\tan\phi_1$ and from equation (10) or (98) we have

$$\cos\theta_0 = \sin\alpha_{1-2}\cos\theta_1 = -\sin\alpha_{2-1}\cos\theta_2. \tag{112}$$

We place $d = S/aD$, (113)

and from equations (48) write

$$\xi_1 = S_1/aD = \sigma_1 + P\sin 2\sigma_1 - Q\sin 4\sigma_1 + R\sin 6\sigma_1 \tag{114}$$

$$\xi_2 = d - \xi_1 = S_2/aD = \sigma_2 + P\sin 2\sigma_2 - Q\sin 4\sigma_2 + R\sin 6\sigma_2 \tag{115}$$

where $\sigma_1 = \text{arc cos}(\sin\theta_1/\sin\theta_0)$, $\sigma_2 = \text{arc cos}(\sin\theta_2/\sin\theta_0)$

and $P = E/D$, $Q = F/D$, $R = G/D$,

with D, E, F, G from equations (49).

Since we have $\cos\theta_0$ from (112), the constants c_1, c_2, c_3, c_4 and A, B, C, D, E, F, G may be computed from (49). Since we have θ_1 and θ_0 we can compute σ_1 and then ξ_1 from (114), ξ_2 from (115), i.e. from $\xi_2 = d - \xi_1$. But we need σ_2 and therefore the series (115) must be reversed. Figure 17 shows the spherical triangle being used.

Now the ranges of c_1, c_2, c_3, c_4 are for $0 < |\theta_0| < \pi/2$; $f > c_1 > 0$, $0 < c_2 < f/4$, $1 + f > c_3 > 1$, $1 + f > c_4 > 1 + f/4$. Since the maximum hemispheroidal geodesic length under the shortest distance criterion is the semimeridian, given when $\theta_0 = (1/2)\pi$, we have for this value of θ_0:

$$A = B = C = 0,\ D = .9983056819,\ E = .8475185 \times 10^{-3},\ F = .1799 \times 10^{-6},$$
$$G = 1 \times 10^{-10},\ P = E/D = .8489569 \times 10^{-3},\ Q = F/D = .1802 \times 10^{-6}, \tag{116}$$
$$R = G/D = 1.001 \times 10^{-9}, \text{ where } f = .3390075283 \times 10^{-2} \text{ (Clarke 1866).}$$

The maximum contributions of the terms $E \sin 2\sigma$, $F \sin 4\sigma$, $G \sin 6\sigma$ are: $2aE = 10{,}811.296$m, $2aF = 2.295$m, $2aG = .0013$m. An examination of Table 3 shows that there will be a maximum angular error of .001 second in holding 8 decimals of radians. We arbitrarily reject all decimal radian terms of the order $.3 \times 10^{-8}$ or less in the analysis to follow. We have at once from (116) that $G = R = 0$.

We write (115) as

$$\xi_2 = \sigma_2 + \Sigma,\ \Sigma = P \sin 2\sigma_2 - Q \sin 4\sigma_2 \tag{117}$$

whence

$$\sin 2\xi_2 = \sin 2\sigma_2 \cos 2\Sigma + \cos 2\sigma_2 \sin 2\Sigma$$
$$\sin 4\xi_2 = \sin 4\sigma_2 \cos 4\Sigma + \cos 4\sigma_2 \sin 4\Sigma \tag{118}$$

We use the series approximations, $\sin X = X - X^3/6$,

$\cos X = 1 - X^2/2$, and find, rejecting terms of the order of $.3 \times 10^{-8}$

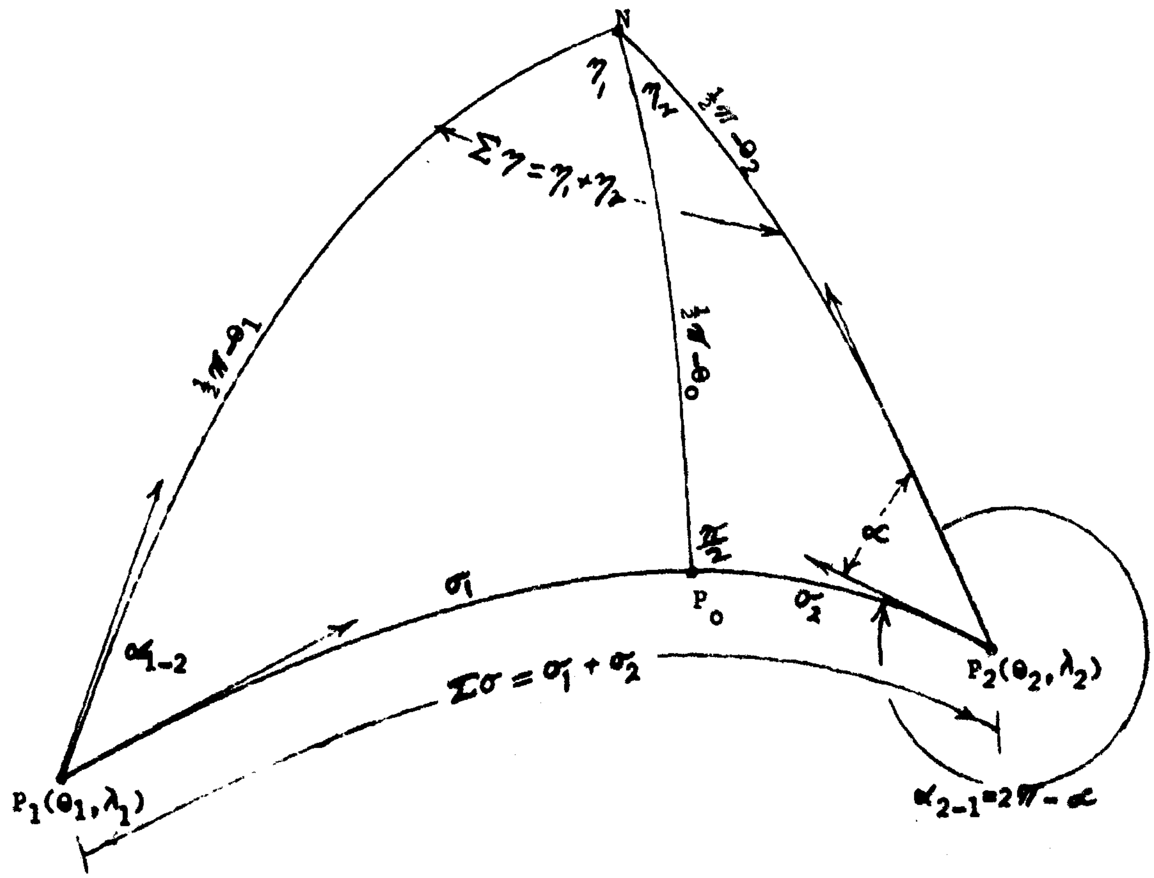


Figure 17. Spherical triangle used in approximating the spheroidal triangle. Azimuths are from north with east longitudes positive.

or less (using the values of P and Q from (116)):

$$
\begin{aligned}
\sin 2\Sigma &= 2(P \sin 2\sigma_2 - Q \sin 4\sigma_2) \\
\cos 2\Sigma &= 1 - 2P^2 \sin^2 2\sigma_2 \\
\sin 4\Sigma &= 4(P \sin 2\sigma_2 - Q \sin 4\sigma_2) \\
\cos 4\Sigma &= 1 - 8P^2 \sin^2 2\sigma_2
\end{aligned}
\tag{119}
$$

The values from (119) returned to (118), with the help of some trigonometric identities, give

$$\sin 2\xi_2 = [1-(3/2)P^2 - Q] \sin 2\sigma_2 + P \sin 4\sigma_2 + [(1/2)P^2 - Q] \sin 6\sigma_2 \tag{120}$$

$$\sin 4\xi_2 = -2P \sin 2\sigma_2 + (1-4P^2) \sin 4\sigma_2 + 2P \sin 6\sigma_2 + 2(P^2 - Q) \sin 8\sigma_2 \tag{121}$$

If we multiply (120) by P, rejecting terms of the order $.3 \times 10^{-8}$ or less, with the values of P and Q from (116), we get

$$P \sin 2\xi_2 = P \sin 2\sigma_2 + P^2 \sin 4\sigma_2 \tag{122}$$

Now subtract respective members of (122) from (117) to get

$$\xi_2 - P \sin 2\xi_2 = \sigma_2 - (Q + P^2) \sin 4\sigma_2 \tag{123}$$

Next multiply (121) by $Q + P^2$, rejecting terms of order $.3 \times 10^{-8}$ or less, to get

$$(Q + P^2) \sin 4\xi_2 = (Q + P^2) \sin 4\sigma_2 \tag{124}$$

From (123) and (124) we have then

$$\sigma_2 = \xi_2 - P \sin 2\xi_2 + M \sin 4\xi_2, \tag{125}$$

where $M = P^2 + Q$.

Now from (116), $Q - F = .3 \times 10^{-9}$, hence we may place $Q = F$ and we write from (114), (115), and (125) with $G = R = 0$, $Q = F$, $P = E/D$, $M = P^2 + F$, $d = S/aD$,

$$
\begin{aligned}
\sigma_2 &= U - P \sin 2U + M \sin 4U \\
U &= d - \xi_1 = d - \sigma_1 - P \sin 2\sigma_1 + F \sin 4\sigma_1 .
\end{aligned}
\tag{126}
$$

We next examine our fundamental coefficients for exclusion of terms of order $.3 \times 10^{-8}$ *or smaller.*

From (49),

$$C = (1/4)\, c_1 c_2^{\,2},\ c_1 = f \cos\theta_0,\ c_2 = (1/4)\, f \sin^2\theta_0,$$

$$\frac{dC}{d\theta_0} = (1/4)\, c_2 \left(c_2 \frac{dc_1}{d\theta_0} + 2c_1 \frac{dc_2}{d\theta_0} \right) = 0,\ \frac{dc_1}{d\theta_0} = -f \sin\theta_0,\ \frac{dc_2}{d\theta_0} = (1/2) f \sin\theta_0 \cos\theta_0,$$

and we find

$$\frac{dC}{d\theta_0} = (1/4)\, c_2 f \sin\theta_0 \,(-c_2 + c_1 \cos\theta_0) = 0,$$

whence the minimum is given by $\theta_0 = 0$ (the equatorial limiting arc), and the maximum by $c_2 = c_1 \cos\theta_0$ or $(1/4) f \sin^2\theta_2 = f \cos^2\theta_0$, whence $\tan\theta_0 = 2$, $\theta_0 = 63^\circ\, 26'\, 05''.816$ and maximum value of C is $C = (1/4)\, c_1 c_2^{\,2} = .174 \times 10^{-9}$. Hence we place $C = 0$ with respect to our rejection criterion $.3 \times 10^{-8}$. Since $c_2^{\,3} = (1/64) f^3 \sin^6\theta_0$, the maximum value is at $\theta_0 = \pi/2$, when $c_2^{\,3} = .6 \times 10^{-9}$ (Clarke 1866). Hence we neglect terms of the order $c_2^{\,3}$ in the coefficients D and E and write:

$$
\begin{aligned}
&A = c_1(1 - c_2 c_4),\ B = (1/2) c_1 c_2 c_3,\ D = 2 + c_2 c_4 (c_4 - 1) - (c_2 + c_4), \\
&E = (1/2) c_2\ [2 + c_3(c_3 - 1)],\ F = (1/4) c_2^{\,2} (2c_4 - 1), \\
&c_1 = f \cos\theta_0,\ c_2 = (1/4) f \sin^2\theta_0,\ c_3 = 1 + c_1 \cos\theta_0,\ c_4 = c_2 + c_3 .
\end{aligned}
\tag{127}
$$

We next consider the effect of omitting the terms in f^3 in the coefficients. If this is done they become

$$A = c_1(1 - c_2),\ B = (1/2)c_1c_2,\ D = (1 - c_2)(2 - c_2 - c_3),\ E = (1/2)c_2(1 + c_3),$$
$$F = (1/4)c_2^{\,2},\ c_1 = fN,\ c_2 = (1/4)f(1 - N^2),\ c_3 = 1 + fN^2,\ N = \cos\theta_0, \tag{128}$$

identity: $c_1^{\,2} - 4c_2(c_3 - 1) + c_3(2 - c_3) = 1.$

Now we form the differences of coefficient values from (127) and (128) and examine for maximum values:

$$|\Delta A| = c_1c_2\,|(c_4 - 1)| = (1/16)f^3N(1 - N^2)(1 + 3N^2) = \frac{4N}{1 + 3N^2}|\Delta D|$$

$$|\Delta B| = (1/2)c_1c_2\,|(1 - c_3)| = (1/8)f^3N^3(1 - N^2) = \frac{2N^2}{1 + 3N^2}|\Delta A| = \frac{2N^2}{1 + 3N^2}\cdot\frac{4N}{1 + 3N^2}|\Delta D|$$

$$|\Delta D| = c_2\,|\,[c_4(2 - c_4) - 1]\,| = (1/64)f^3(1 + 3N^2)^2(1 - N^2) \tag{129}$$

$$|\Delta E| = (1/2)c_2\,|(2c_3 - 1 - c_3^{\,2})| = (1/8)f^3N^4(1 - N^2) = N|\Delta B| = N\cdot\frac{2N^2}{1 + 3N^2}\cdot\frac{4N}{1 + 3N^2}\cdot|\Delta D|$$

$$|\Delta F| = (1/2)c_2^{\,2}\,|(1 - c_4)| = (1/128)f^3(1 - N^2)^2(1 + 3N^2) = \frac{1 - N^2}{2(1 + 3N^2)}|\Delta D|$$

Since $|\theta_0| \leq \pi/2,\ 0 \leq N = \cos\theta_0 \leq 1,\ 0 \leq \dfrac{1 - N^2}{2(1 + 3N^2)} \leq .5,$

$$0 \leq \frac{2N^2}{1 + 3N^2} \leq .5,\ 0 \leq \frac{4N}{1 + 3N^2} \leq 1.2, \text{ we have}$$

from (129) that

$|\Delta A| \leq (1.2)\,|\Delta D|\max,\ |\Delta B| \leq (0.6)\,|\Delta D|\max,\ |\Delta E| \leq (.6)\,|\Delta D|\max,$

$|\Delta F| \leq (0.5)\,|\Delta D|\max.$

Thus we have only to find $|\Delta D|$ max and show that both it and $|\Delta A|$ are less than the rejection criterion, $.3 \times 10^{-8}$. We find

$$\frac{d\,|\Delta D|}{d\,\theta_0} = (f^3/64)\cdot 2(1 + 3N^2)NN'(5 - 9N^2) = 0,$$

whence $N = 0, (\theta_0 = \pi/2), N' = 0, (\theta_0 = 0), N^2 \max = 5/9$. With this last value $|\Delta D|\max = .2 \times 10^{-8}$, $|\Delta A| \leq (1.2)\,|\Delta D|\max = .24 \times 10^{-8}$ *which is sufficient to justify the values (128).*

Now $\quad \sin\theta_2 = \sin\theta_0\cos\sigma_2,\ \tan\phi_2 = \tan\theta_2/(1 - f),$

$$\sin\alpha_{2\text{-}1} = \cos\theta_0/\cos\theta_2 = \cos\theta_1\,\sin\alpha_{1\text{-}2}/\cos\theta_2 \tag{130}$$

From (50)

$$\begin{aligned}&\Sigma\eta = \eta_1 + \eta_2,\ \Sigma\sigma = \sigma_1 + \sigma_2,\ \Delta\sigma = \sigma_1 - \sigma_2,\\ &P = 2\sin\Sigma\sigma\cos\Delta\sigma,\ \Delta\lambda = \Sigma\eta - A\Sigma\sigma + BP,\ \lambda_2 = \lambda_1 + \Delta\lambda,\\ &\eta_i = \arccos(\tan\theta_i/\tan\theta_0) = \arccos(\cos\theta_0\cos\sigma_i/\cos\theta_i)\\ &\sigma_i = \arccos(\sin\theta_i/\sin\theta_0).\end{aligned} \tag{131}$$

Summary of first direct solution, given $\phi_1, \lambda_1, S, \alpha_{1\text{-}2}$.

1. Convert ϕ_1 to parametric latitude from $\tan\theta_1 = (1 - f)\tan\phi_1$

2. Compute $\cos\theta_0 = \cos\theta_1 \sin a_{1\text{-}2}$ (geodesic vertex)
3. Compute $\sigma_1 = \text{arc cos}(\sin\theta_1/\sin\theta_0)$, $\sin 2\sigma_1$, $\sin 4\sigma_1$
4. Compute A, B, D, E, F from (128) and (132)
   $P = E/D$, $M = F + D^2$, $d = S/aD$
5. From (126), $U = d - \sigma_1 - P\sin 2\sigma_1 + F\sin 4\sigma_1$, $\sin 2U$, $\sin 4U$
6. $\sigma_2 = U - P\sin 2U + M\sin 4U$, $\cos\sigma_2$
7. $\theta_2 = \text{arc sin}(\sin\theta_0 \cos\sigma_2)$, $a_{2\text{-}1} = 2\pi - \text{arc sin}(\cos\theta_0/\cos\theta_2)$
   $\cos\theta_2$, $\tan\theta_2$, $\tan\phi_2 = \tan\theta_2/(1-f)$
8. $\eta_1 = \text{arc cos}(\tan\theta_1/\tan\theta_0)$, $\eta_2 = \text{arc cos}(\cos\theta_0\cos\sigma_2/\cos\theta_2)$
9. From (131), $\Sigma\eta = \eta_1 + \eta_2$, $\Sigma\sigma = \sigma_1 + \sigma_2$, $\Delta\sigma = \sigma_1 - \sigma_2$,
   $p = 2\sin\Sigma\sigma\cos\Delta\sigma$, $\Delta\lambda = \Sigma\eta - A\Sigma\sigma + Bp$, $\lambda_2 = \lambda_1 + \Delta\lambda$

Alternative trigonometric formulae, reference [19].

When $\Sigma\sigma = \sigma_1 + \sigma_2$ has been found

$$\begin{aligned}
\tan a_{2\text{-}1} &= \cos\theta_0/(\sin\Sigma\sigma\sin\theta_1 - N\cos\Sigma\sigma),\ N = \sin\theta_0\sin\sigma_1 = \cos\theta_0\cos a_{1\text{-}2}\\
\tan\phi_2 &= (\cos\Sigma\sigma\sin\theta_1 + N\sin\Sigma\sigma)\sin a_{2\text{-}1}/(1-f)\cos\theta_0\\
\Sigma\eta &= \text{arc tan}\ [\sin\Sigma\sigma\cos\theta_0/(\cos\Sigma\sigma - \sin\theta_1\sin\theta_2)]\\
&= \text{arc tan}\ [\sin\Sigma\sigma\sin a_{1\text{-}2}/(\cos\theta_1\cos\Sigma\sigma - \sin\theta_1\sin\Sigma\sigma\cos a_{1\text{-}2})]
\end{aligned}\tag{133}$$

We make the following changes for a geodesic arc that will *contain no vertex, but will contain a node:*

$$\begin{aligned}
a_{2\text{-}1} &= \pi + a = \pi + \text{arc sin}(\cos\theta_0/\cos\theta_2),\\
U &= \sigma_1 - d + P\sin 2\sigma_1 - F\sin 4\sigma_1\\
\tan\Delta\eta &= \tan(\eta_1 - \eta_2) = \sin\Delta\sigma\sin a_{1\text{-}2}/(\cos\theta_1\cos\Delta\sigma - \sin\theta_1\sin\Delta\sigma\cos a_{1\text{-}2})\\
\Delta\lambda &= \Delta\eta - A\Delta\sigma + 2B\sin\Delta\sigma\cos\Sigma\sigma, \text{ from (51).}
\end{aligned}\tag{134}$$

General hemispheroidal direct solution. (First form).

Now the formulae (132), (134) respectively, suggest the following general direct solution over the hemispheroid:

from
$$\begin{aligned}
U &= \sigma_1 - d + P\sin 2\sigma_1 - F\sin 4\sigma_1\\
\sigma_1 &= U + d - P\sin 2\sigma_1 + F\sin 4\sigma_1\\
\sigma_2 &= U - P\sin 2U + M\sin 4U
\end{aligned}$$

we have $\Delta\sigma = \sigma_1 - \sigma_2 = d + P(\sin 2U - \sin 2\sigma_1) + F\sin 4\sigma_1 - M\sin 4U$ (135)

Now
$$\begin{aligned}
\sin 2U &= \sin 2(\sigma_1 - d)\cos 2\Sigma + \cos 2(\sigma_1 - d)\sin 2\Sigma\\
\sin 4U &= \sin 4(\sigma_1 - d)\cos 4\Sigma + \cos 4(\sigma_1 - d)\sin 4\Sigma,
\end{aligned}\tag{136}$$

where $\Sigma = P\sin 2\sigma_1 - F\sin 4\sigma_1$.

With the approximations $\sin x = x - x^3/6$, $\cos x = 1 - x^2/2$, where $x = 2\Sigma, 4\Sigma$ and rejecting terms whose coefficients are $.3 \times 10^{-8}$ or less in using the values of P, F, $M = P^2 + Q$ from (116), we find

$$\begin{aligned}
P\sin 2U &= P\sin 2(\sigma_1 - d) + 2P^2\sin 2\sigma_1\cos 2(\sigma_1 - d)\\
M\sin 4U &= (F + P^2)\sin 4(\sigma_1 - d)
\end{aligned}\tag{137}$$

The values from (137) placed in (135) and use of some trigonometric identitiks enable us to write

$$\Delta\sigma = \sigma_1 - \sigma_2 = d - 2P \sin d \cos(2\sigma_1 - d)\,[1 - 2P\cos 2(\sigma_1 - d)] + 2F \sin 2d \cos 2(2\sigma_1 - d)$$
$$\Sigma\sigma = \sigma_1 + \sigma_2 = 2\sigma_1 - \Delta\sigma. \qquad (138)$$

From equations (128), (133), and (138) we assemble *the formulae for the general direct hemispheroidal solution:*

$$\tan\theta_1 = (1-f)\tan\phi_1,\ M = \cos\theta_0 = \cos\theta_1 \sin\alpha_{1-2},\ N = \cos\theta_1\cos\alpha_{1-2},$$
$$c_1 = fM,\ c_2 = (1/4)f(1-M^2),\ A = c_1 - 2B,\ B = (1/2)c_1c_2,\ D = (1-c_2)^2 - AM,$$
$$E = c_2 + BM,\ F = (1/4)c_2^2,\ P = E/D,\ \text{Check: } AM - 2BM + D + 2E - 4F = 1.$$
$$\sigma_1 = \operatorname{arc}\cos(\sin\theta_1/\sin\theta_0),\ d = S/aD,$$
$$\Delta\sigma = d - 2P\sin d\cos(2\sigma_1 - d)[1 - 2P\cos 2(\sigma_1 - d)] + 2F\sin 2d\cos 2(2\sigma_1 - d)$$
$$\cos(2\sigma_1 - d) = \cos 2(\sigma_1 - d)\cos d - \sin 2(\sigma_1 - d)\sin d, \qquad (139)$$
$$\cos 2(2\sigma_1 - d) = 2\cos^2(2\sigma_1 - d) - 1,$$
$$\Sigma\sigma = 2\sigma_1 - \Delta\sigma,\ \tan\alpha_{2-1} = M/(N\cos\Delta\sigma - \sin\theta_1\sin\Delta\sigma)$$
$$\tan\phi_2 = -(\sin\theta_1\cos\Delta\sigma + N\sin\Delta\sigma)\sin\alpha_{2-1}/(1-f)M,$$
$$\tan\Delta\eta = \sin\Delta\sigma\sin\alpha_{1-2}/(\cos\theta_1\cos\Delta\sigma - \sin\theta_1\sin\Delta\sigma\cos\alpha_{1-2})$$
$$\Delta\lambda = \Delta\eta - A\Delta\sigma + 2B\sin\Delta\sigma\cos\Sigma\sigma,\ \text{Check: } M = \cos\theta_1\sin\alpha_{1-2} = \cos\theta_2\sin(\pi + \alpha_{2-1})$$

We arrange equations (139) as follows for construction of a computing form:

$$\tan\theta_1 = (1-f)\tan\phi_1,\ M = \cos\theta_0 = \cos\theta_1\sin\alpha_{1-2},\ N = \cos\theta_1\cos\alpha_{1-2},$$
$$c_1 = fM,\ c_2 = (1/4)f(1-M^2),\ D = (1-c_2)(1-c_2-c_1M),\ P = c_2[1 + (1/2)c_1M]/D,$$
$$\cos\sigma_1 = \sin\theta_1/\sin\theta_0,\ d = S/aD,\ u = 2(\sigma_1 - d),\ W = 1 - 2P\cos u,$$
$$V = \cos(u + d) = \cos u\cos d - \sin u\sin d, \qquad (140)$$
$$X = c_2^2\sin d\cos d\,(2V^2 - 1),\ Y = 2PVW\sin d,$$
$$\Delta\sigma = d + X - Y,\ \Sigma\sigma = 2\sigma_1 - \Delta\sigma,\ \tan\alpha_{2-1} = M/(N\cos\Delta\sigma - \sin\theta_1\sin\Delta\sigma),$$
$$\tan\phi_2 = -(\sin\theta_1\cos\Delta\sigma + N\sin\Delta\sigma)\sin\alpha_{2-1}/(1-f)M,$$
$$\tan\Delta\eta = \sin\Delta\sigma\sin\alpha_{1-2}/(\cos\theta_1\cos\Delta\sigma - \sin\theta_1\sin\Delta\sigma\cos\alpha_{1-2}),$$
$$H = c_1(1-c_2)\Delta\sigma - c_1c_2\sin\Delta\sigma\cos\Sigma\sigma,\ \Delta\lambda = \Delta\eta - H,\ \lambda_2 = \lambda_1 + \Delta\lambda,$$

Check: $M = \cos\theta_0 = \cos\theta_1\sin\alpha_{1-2} = \cos\theta_2\sin(\pi + \alpha_{2-1})$.

Figure 18 shows equations (140) arranged in a computing form.

General hemispheroidal direct solution. (Second form)

With the hope of reducing the number of trigonometric functions involved, a second solution was developed which involves successive solutions on two spheres. The formulae are identical in some instances to those of the first solution. The quantities are the same in some cases but appear in different form with respect to formulae. The principal difference is in obtaining $\Delta\sigma$. The solution from there on is identical. The formulas are:

$$\tan\theta_1 = (1-f)\tan\phi_1,\ M = \cos\theta_0 = \cos\theta_1\sin\alpha_{1-2},\ N = \cos\theta_1\cos\alpha_{1-2},\ c_1 = fM,\ c_2 = (1/4)f(1-M^2),$$
$$\cos\sigma_1 = \sin\theta_1/\sin\theta_0,\ d = S/b,\ T = d/\sin d,\ V = 1 + h\sin^2\theta_1,\ h = (1/2)[1/(1-f)^2 - 1],\ A = V(1-M^2),$$
$$\sin\theta_2' = \sin\theta_1\cos d + N\sin d,\ B = V\sin\theta_1\sin\theta_2',\ C = T - \cos d,\ L = AC + 2B,\ D = 4(B + L) - A\cos d,$$
$$E = 8B(B + 2L)\cos d,\ P = 2AD\sin^2 d,\ Q = 3CA^2 + E,\ \Delta\sigma = \sin d[T - (1/2)hL + (h^2/16)(P + Q)], \qquad (141)$$

DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1, \alpha_{1\text{-}2}, S$ to find $\phi_2, \lambda_2, \alpha_{2\text{-}1}$. East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

______ SPHEROID a ______ m f ______

1 - f ______ 1 radian = 206264.8062 seconds

LINE ______ TO ______

° ′ ″

ϕ_1 ______ $\tan\phi_1$ ______ $\tan\theta_1 = (1 - f)\tan\phi_1$ ______ ° ′ ″

$\alpha_{1\text{-}2}$ ______ $\sin\theta_1$ ______ $\cos\theta_1$ ______ θ_1 ______

$\sin\alpha_{1\text{-}2}$ ______ $M = \cos\theta_0 = \cos\theta_1 \sin\alpha_{1\text{-}2}$ ______ θ_0 ______

$\cos\alpha_{1\text{-}2}$ ______ $N = \cos\theta_1 \cos\alpha_{1\text{-}2}$ ______ $\sin\theta_0$ ______

$c_1 = fM$ ______ $D = (1 - c_2)(1 - c_2 - c_1 M)$ ______

$c_2 = \frac{1}{4}(1 - M^2)f$ ______ $P = c_2\,(1 + \frac{1}{2}c_1 M)/D$ ______

° ′ ″

$\cos\sigma_1 = \sin\theta_1/\sin\theta_0$ ______ σ_1 ______

$d = S/aD$ ______ (rad) d ______ S ______ m

$\sin d$ ______ $u = 2(\sigma_1 - d)$ ______ $\sin u$ ______

$\cos d$ ______ $W = 1 - 2P\cos u$ ______ $\cos u$ ______

$V = \cos u \cos d - \sin u \sin d$ ______ $Y = 2PVW \sin d$ ______

$X = c_2^2 \sin d \cos d\,(2V^2 - 1)$ ______ $\Delta\sigma = d + X - Y$ ______ (rad)

° ′ ″

$\sin\Delta\sigma$ ______ $\cos\Delta\sigma$ ______ $\Delta\sigma$ ______

$\cos\Sigma\sigma$ ______ $\Sigma\sigma = 2\sigma_1 - \Delta\sigma$ ______ ° ′ ″

$\tan\alpha_{2\text{-}1} = M/(N\cos\Delta\sigma - \sin\theta_1 \sin\Delta\sigma)$ ______ $\alpha_{2\text{-}1}$ ______

$$\tan\phi_2 = \frac{-(\sin\theta_1 \cos\Delta\sigma + N\sin\Delta\sigma)\sin\alpha_{2\text{-}1}}{(1 - f)M}$$ ______ $\sin\alpha_{2\text{-}1}$ ______

° ′ ″

ϕ_2 ______

° ′ ″

$$\tan\Delta\eta = \frac{\sin\Delta\sigma \sin\alpha_{1\text{-}2}}{\cos\theta_1 \cos\Delta\sigma - \sin\theta_1 \sin\Delta\sigma \cos\alpha_{1\text{-}2}}$$ ______ $\Delta\eta$ ______

$H = c_1(1 - c_2)\Delta\sigma - c_1 c_2 \sin\Delta\sigma \cos\Sigma\sigma$ ______ (rad) H ______

$\Delta\lambda = \Delta\eta - H$ ______

λ_1 ______

CHECK

° ′ ″

$M = \cos\theta_0 = \cos\theta_1 \sin\alpha_{1\text{-}2} = \cos\theta_2 \sin(180 + \alpha_{2\text{-}1})$ $\lambda_2 = \lambda_1 + \Delta\lambda$ ______

Figure 18. First direct solution computing form.

$$\Sigma\sigma = 2\sigma_1 - \Delta\sigma, \tan \alpha_{2\text{-}1} = M/(N \cos \Delta\sigma - \sin \theta_1 \sin \Delta\sigma), \qquad (141)$$

$$\tan \phi_2 = -(\sin \theta_1 \cos \Delta\sigma + N \sin \Delta\sigma) \sin \alpha_{2\text{-}1}/(1 - f)M,$$

$$\tan \Delta\eta = \sin \Delta\sigma \sin \alpha_{1\text{-}2}/(\cos \theta_1 \cos \Delta\sigma - \sin \theta_1 \sin \Delta\sigma \cos \alpha_{1\text{-}2})$$

$$H = c_1(1 - c_2)\Delta\sigma - c_1 c_2 \sin \Delta\sigma \cos \Sigma\sigma, \Delta\lambda = \Delta\eta - H, \lambda_2 = \lambda_1 + \Delta\lambda.$$

Check: $\quad M = \cos \theta_0 \sin \alpha_{1\text{-}2} = \cos \theta_2 \sin (\pi + \alpha_{2\text{-}1}).$

In essence, one solves for $\Delta\sigma$ through two spherical triangles. With $\alpha_{1\text{-}2}$, θ_1, and $d = S/b$ one solves for θ_2' in the triangle of Figure 19, by the formula $\sin \theta_2' = \sin \theta_1 \cos d + N \sin d$. With this value of θ_2', one computes the several quantities including $\Delta\sigma$ and then one solves for $\alpha_{2\text{-}1}$, θ_2, $\Delta\eta$ in the triangle of Figure 20 as was done in the first general direct solution, equations (140).

The second method appears to be slightly less accurate than the first, and little if anything is saved in computation. Figure 21 shows equations (141) arranged in a computing form.

Conventions for azimuth and longitude

We assume the *initial is west of the terminus* in the direct solution and then always $0 \leq \alpha_{1\text{-}2} \leq 180°$, $0 \leq \Delta\eta \leq \Delta\lambda \leq \pi$. We *find the first quadrant angles* u and v given by $\tan u = |\tan \alpha_{2\text{-}1}|$, $\tan v = |\tan \Delta\eta|$.

If $\tan \alpha_{2\text{-}1} > 0$, then $\alpha_{2\text{-}1} = 180° + u$; if $\tan \alpha_{2\text{-}1} < 0$, then $\alpha_{2\text{-}1} = 360° - u$. If $\tan \Delta\eta > 0$, then $\Delta\eta = v$; if $\tan \Delta\eta < 0$, then $\Delta\eta = 180° - v$.

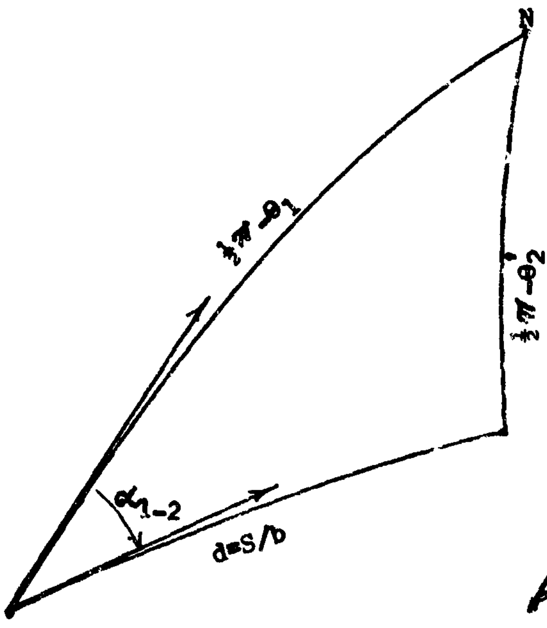


Figure 19. First spherical solution-second direct solution method.

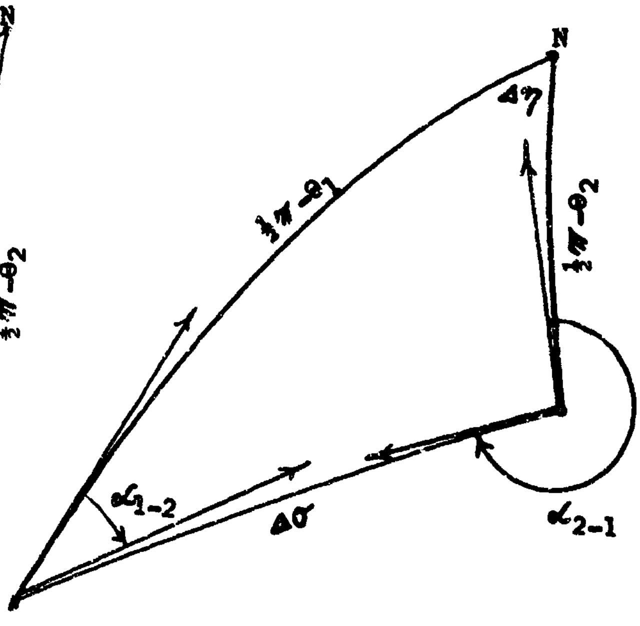


Figure 20. Second spherical solution-second direct solution method.

DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1, \alpha_{1\text{-}2}$, S to find $\phi_2, \lambda_2, \alpha_{2\text{-}1}$. East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

________ SPHEROID a ________ m b ________ m

$1 - f = b/a$ ________ f ________ $h = \frac{1}{2}\,[1/(b/a)^2 - 1]$ ________

1 radian = 206264.8062 seconds

LINE ________ TO ________

° ′ ″

ϕ_1 ________ $\tan\phi_1$ ________ $\tan\theta_1 = (1 - f)\tan\phi_1$ ________

° ′ ″

S ________ m θ_1 ________ $\sin\theta_1$ ________ $\cos\theta_1$ + ________

$\alpha_{1\text{-}2}$ ________ $\sin\alpha_{1\text{-}2}$ ________ $\cos\alpha_{1\text{-}2}$ ________

$d(\text{rad}) = S/b$ + ________ d ________ sin d + ________

$M = \cos\theta_1 \sin\alpha_{1\text{-}2}$ ________ $T = d/\sin d$ + ________ cos d ________

$N = \cos\theta_1 \cos\alpha_{1\text{-}2}$ ________ $V = 1 + h\sin^2\theta_1$ + ________

$A = V(1 - M^2)$ + ________ $B = V\sin\theta_1\,(N\sin d + \sin\theta_1\cos d)$ ________

$C = T - \cos d$ ________ $L = AC + 2B$ ________

$D = 4(L + B) - A\cos d$ ________ $E = 8B(2L + B)\cos d$ ________

$P = 2AD\sin^2 d$ ________ $Q = 3A^2C + E$ ________ P + Q ________

° ′ ″

$\Delta\sigma = \sin d\,[T - (h/2)L + (h^2/16)(P + Q)]$ + ________ σ_1 ________

° ′ ″

$\sin\Delta\sigma$ ________ $\cos\Delta\sigma$ ________ $\Delta\sigma$ ________

$\cos\Sigma\sigma$ ________ $\Sigma\sigma = 2\sigma_1 - \Delta\sigma$ ________

° ′ ″

$\tan\alpha_{2\text{-}1} = M/(N\cos\Delta\sigma - \sin\theta_1\sin\Delta\sigma)$ ________ $\alpha_{2\text{-}1}$ ________

$$\tan\phi_2 = \frac{-(\sin\theta_1\cos\Delta\sigma + N\sin\Delta\sigma)\sin\alpha_{2\text{-}1}}{(1 - f)M}$$ ________ $\sin\alpha_{2\text{-}1}$ ________

° ′ ″

ϕ_2 ________

° ′ ″

$$\tan\Delta\eta = \frac{\sin\Delta\sigma\sin\alpha_{1\text{-}2}}{\cos\theta_1\cos\Delta\sigma - \sin\theta_1\sin\Delta\sigma\cos\alpha_{1\text{-}2}}$$ ________ $\Delta\eta$ ________

$H = c_1(1 - c_2)\Delta\sigma - c_1c_2\sin\Delta\sigma\cos\Sigma\sigma$ ________ (rad) H ________

$c_1 = fM$ ________ $\Delta\lambda =$ $\Delta\lambda = \Delta\eta - H$ ________

$c_2 = \frac{1}{4}f(1 - M^2)$ ________ λ_1 ________

CHECK

° ′ ″

$M = \cos\theta_0 = \cos\theta_1\sin\alpha_{1\text{-}2} = \cos\theta_2\sin(180 + \alpha_{2\text{-}1})$ $\lambda_2 = \lambda_1 + \Delta\lambda$ ________

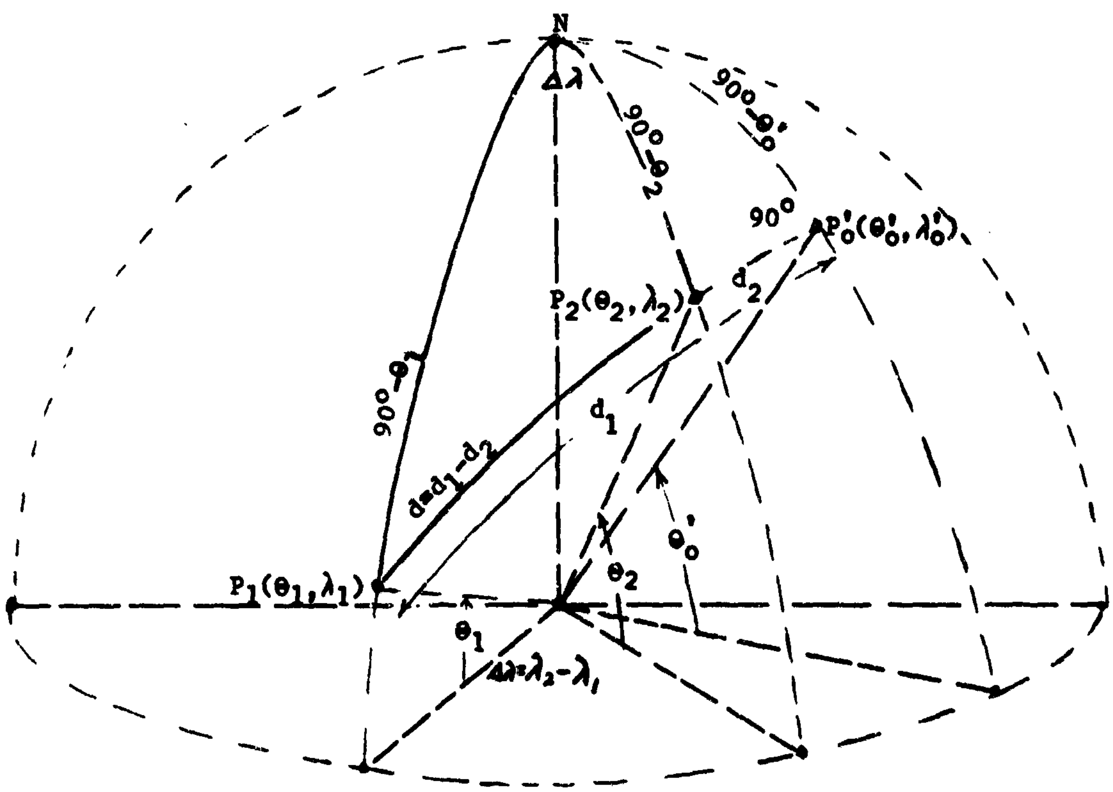
Figure 21. Second direct solution computing form.

General hemispheroidal inverse (reverse) solution

The following geodetic length approximation for the inverse (reverse) solution between two points $P_1(\theta_1, \lambda_1)$, $P_2(\theta_2, \lambda_2)$ of the reference ellipsoid, was developed by the author, following the method of Forsyth [20], and published in [18]:

$$
\begin{aligned}
&S = a[d - (f/4)(Xd - Y \sin d) + (f^2/64)(AX - BY + CX^2 + DXY - EY^2)], \\
&\cos d = \sin\theta_1 \sin\theta_2 + \cos\theta_1 \cos\theta_2 \cos\Delta\lambda, \ \Delta\lambda = \lambda_2 - \lambda_1, \\
&B = 8d^2/\sin d, A = B\cos d, D = B/2, E = 2\sin d \cos d, C = d + (1/2)(E - A), \\
&X = (\sin\theta_1 + \sin\theta_2)^2/(1 + \cos d) + (\sin\theta_1 - \sin\theta_2)^2/(1 - \cos d) = 2\sin^2\theta_0', \\
&Y = (\sin\theta_1 + \sin\theta_2)^2/(1 - \cos d) - (\sin\theta_1 - \sin\theta_2)^2/(1 - \cos d) = X\cos(d_1 + d_2),
\end{aligned}
\tag{142}
$$

where θ_0' *is the vertex of the great elliptic section* through P_1, P_2 (contains the center of the ellipsoid) and d_1, d_2 are the spherical distances from this vertex to the points P_1, P_2; ($d = d_1 - d_2$). Other trigonometric formulae may be used to obtain the most accurate value of d. Figure 22 shows the spherical elements involved.



θ_0' is the parametric latitude of the vertex of the great elliptic section. In the spherical triangle NP_1P_2 we have $\cos d = \sin\theta_1 \sin\theta_2 + \cos\theta_1 \cos\theta_2 \cos\Delta\lambda$. In the right spherical triangles $NP_0'P_1$, $NP_0'P_2$ we have respectively $\cos d_1 = \sin\theta_1/\sin\theta_0'$, $\cos d_2 = \sin\theta_2/\sin\theta_0'$. Thus d_1 and d_2 are analogous to σ_1 and σ_2, equations (114) and Figure 11, where θ_0 is the parametric latitude of the geodesic vertex.

Figure 22. The spherical triangles used in the inverse approximation.

To assure the best trigonometric solution for d, we adapt mid-latitude formulae, reference [18], page 87. We factor sin d out of each term and write equations (142) in the following form for computing (east longitudes considered positive):

Inverse (Reverse) Solution Formulae

$$
\begin{aligned}
&\tan\theta_i = (1-f)\tan\phi_i,\ i = 1, 2.\ \ \theta_m = (1/2)(\theta_1+\theta_2),\ \Delta\theta_m = (1/2)(\theta_2-\theta_1),\\
&\Delta\lambda = \lambda_2-\lambda_1,\ \Delta\lambda_m = (1/2)\Delta\lambda,\ H = \cos^2\Delta\theta_m - \sin^2\theta_m = \cos^2\theta_m - \sin^2\Delta\theta_m,\\
&L = \sin^2\Delta\theta_m + H\sin^2\Delta\lambda_m = \sin^2(1/2)d,\ 1-L = \cos^2(1/2)d,\ \cos d = 1-2L,\\
&U = 2\sin^2\theta_m\cos^2\Delta\theta_m/(1-L),\ V = 2\sin^2\Delta\theta_m\cos^2\theta_m/L,\ X = U+V,\\
&Y = U-V,\ T = d/\sin d,\ D = 4T^2,\ E = 2\cos d,\ A = DE,\ B = 2D,\\
&C = T-(1/2)(A-E);\ \text{Check: } C-(1/2)E+AD/B = T. \qquad\qquad (143)\\
&n_1 = X(A+CX),\ n_2 = Y(B+EY),\ n_3 = DXY,\ \delta_1 d = (1/4)f(TX-Y),\\
&\delta_2 d = (f^2/64)(n_1-n_2+n_3),\ S_1 = a\sin d(T-\delta_1 d),\ S_2 = a\sin d(T-\delta_1 d+\delta_2 d),\\
&F = 2Y-E(4-X),\ M = 32T-(20T-A)X-(B+4)Y,\\
&G = (1/2)fT+(f^2/64)M,\ Q = -(FG\tan\Delta\lambda)/4,\ \Delta\lambda'_m = (1/2)(\Delta\lambda+Q),\\
&c_1 = -\sin\Delta\theta_m/(\cos\theta_m\tan\Delta\lambda'_m),\ u = \arctan|c_1|,\ \alpha_1 = v-u,\\
&c_2 = \cos\Delta\theta_m/(\sin\theta_m\tan\Delta\lambda'_m),\ v = \arctan|c_2|,\ \alpha_2 = v+u,
\end{aligned}
$$

c_1	c_2	$\alpha_{1\text{-}2}$	$\alpha_{2\text{-}1}$
−	+	α_1	$360-\alpha_2$
+	+	α_2	$360-\alpha_1$
−	−	$180-\alpha_2$	$180+\alpha_1$
+	−	$180-\alpha_1$	$180+\alpha_2$

The principal difference in equations (143) and those of reference [18] page 87, is the arrangement for *P_1 to be always west of P_2, east longitudes positive, and the addition of azimuth equations to second order in f.* The azimuths are an adaptation of Guggenheim's equations, reference [23], where conversion has been made to parametric latitude and terms transformed into the parameters used in the length computations. The arrangement for identifying the azimuths without the quadrant search, as displayed in the last of (143), will be generated in a discussion of azimuths to follow.

Azimuth determination in the inverse solution

With the point P_1 always west of P_2, east longitudes positive, we must establish some conventions in order to determine the azimuths from north. In a spherical triangle P_1NP_2, as shown in Figure 23, we have the corresponding parts as indicated: $B = \alpha_{1\text{-}2}$, $A = 360-\alpha_{2\text{-}1}$, $a = 90-\theta_1$, $b = 90-\theta_2$, $C = \Delta\lambda'$ and

$$
\begin{aligned}
&(1/2)(A+B) = 180^\circ + (1/2)(\alpha_{1\text{-}2}-\alpha_{2\text{-}1}),\ (1/2)(A-B) = 180^\circ-(1/2)(\alpha_{1\text{-}2}+\alpha_{2\text{-}1}),\\
&(1/2)(a-b) = (1/2)(\theta_2-\theta_1) = \Delta\theta_m,\ (1/2)(a+b) = 90^\circ-(1/2)(\theta_1+\theta_2) = 90-\theta_m, \qquad (144)\\
&C/2 = (1/2)\Delta\lambda' = \Delta\lambda'_m.
\end{aligned}
$$

From Gauss's equations, reference [19] page 162:

$$
\left.
\begin{aligned}
&\tan[(1/2)(A+B)] = \cos[(1/2)(a-b)]/\cos[(1/2)(a+b)]\ \tan[(1/2)C],\\
&\tan[(1/2)(A-B)] = \sin[(1/2)(a-b)]/\sin[(1/2)(a+b)]\ \tan[(1/2)C].
\end{aligned}
\right\} \qquad (145)
$$

The values from (144) placed in (145) give

$$\tan\left[(1/2)(\alpha_{1\text{-}2} + \alpha_{2\text{-}1})\right] = -\sin\Delta\theta_m/\cos\theta_m \tan\Delta\lambda'_m = c_1,$$
$$\tan\left[(1/2)(\alpha_{1\text{-}2} - \alpha_{2\text{-}1})\right] = \cos\Delta\theta_m/\sin\theta_m \tan\Delta\lambda'_m = c_2. \qquad (146)$$

The formulae (146) were given with equations (143) *where $\Delta\lambda'_m$ is the mean longitude difference as corrected to account for the ellipsoid.*

Since $|\theta_m| = |(1/2)(\theta_1 + \theta_2)| \leq 90°$ and $|\Delta\theta_m| = |(1/2)(\theta_2 - \theta_1)| \leq 90°$, then always $\cos(\pm\theta_m) > 0$, $\cos(\pm\Delta\theta_m) > 0$. Always, since east longitudes are positive, with P_1 west of P_2 $\Delta\lambda > 0$, $\Delta\lambda_m > 0$, $\Delta\lambda'_m > 0$. Hence the signs of c_1 and c_2, in equations (146) depend only on the signs of $\sin\Delta\theta_m$ and of $\sin\theta_m$ respectively. Now Figure 24 shows all the possible azimuth situations, $\theta_1 \neq \theta_2$, from which the corresponding signs of $\sin\Delta\theta_m = \sin[(1/2)(\theta_2 - \theta_1)]$, $\sin\theta_m = \sin[(1/2)(\theta_1 + \theta_2)]$ can be determined. A summary of sign conventions as obtained from Figure 24 and equations (146) is given in Table 4.

If we find the first quadrant angles u and v corresponding to $\tan u = |c_1|$, $\tan v = |c_2|$ and then form $\alpha_1 = v - u$, $\alpha_2 = v + u$, we may determine all azimuths from Table 5.

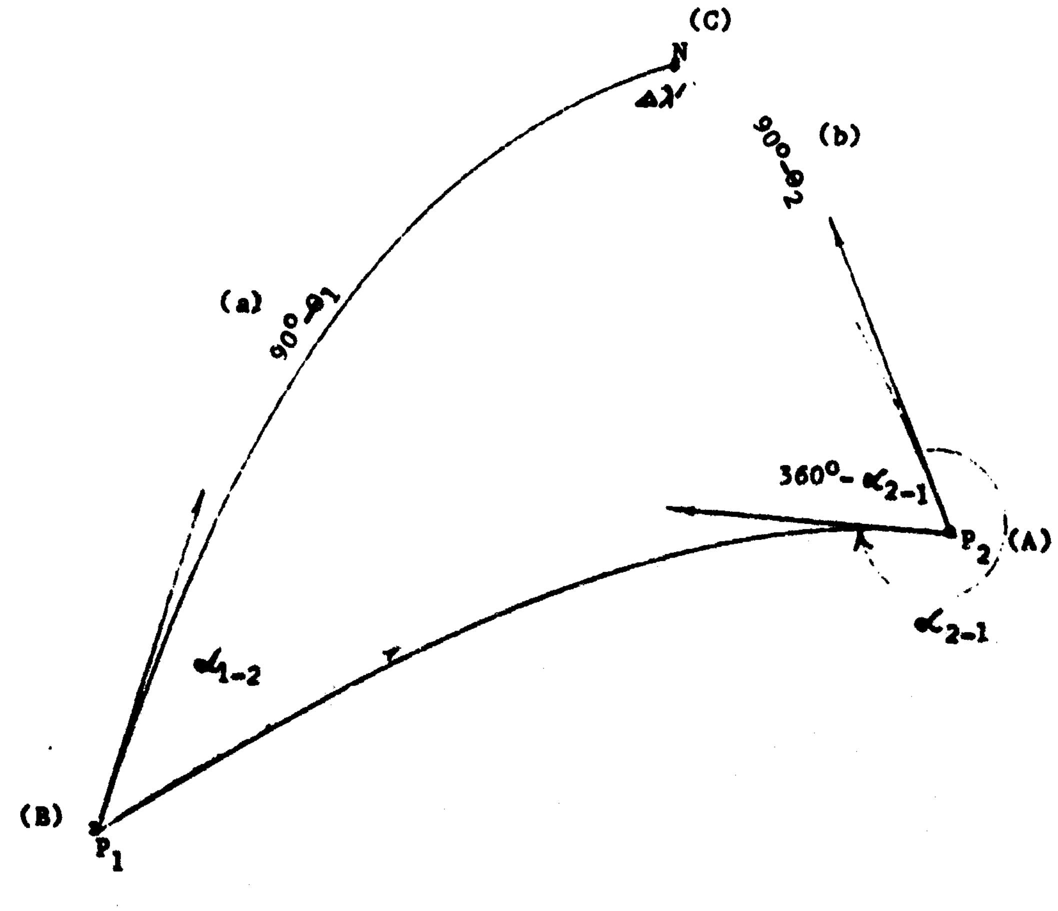


Figure 23. Azimuths in the equivalent spherical triangle.

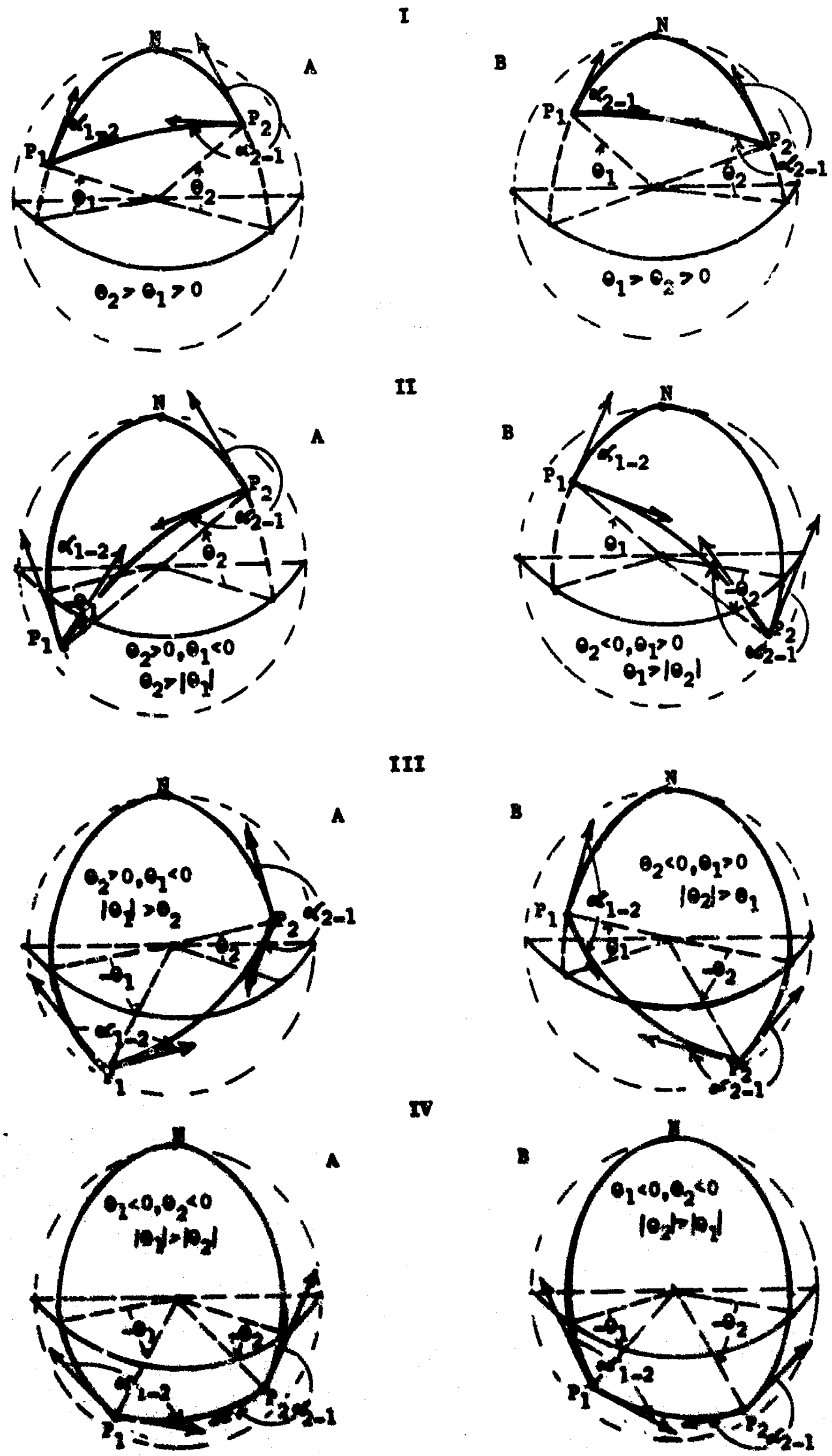


Figure 34. Azimuth situations over the hemispheroid.

Table 4. Summary of azimuth sign conventions.

Figure	*Sin* $\Delta\theta_m$	*Sin* θ_m	*Latitude Conditions*	c_1	c_2
IA, IIA	+	+	$\theta_2 > 0, \theta_2 > \vert\theta_1\vert$	-	+
IB, IIB	-	+	$\theta_1 > 0, \theta_1 > \vert\theta_2\vert$	+	+
IIIA, IVA	+	-	$\theta_1 < 0, \vert\theta_1\vert > \theta_2$	-	-
IIIB, IVB	-	-	$\theta_2 < 0, \vert\theta_2\vert > \theta_1$	+	-

Table 5. Azimuth determination in the inverse solution.

Figure	*Latitude Conditions*	c_1	c_2	α_{1-2}	α_{2-1}
IA,IIA	$\theta_2 > 0, \theta_2 > \vert\theta_1\vert$	-	+	α_1	$360 - \alpha_2$
IB, IIB	$\theta_1 > 0, \theta_1 > \vert\theta_2\vert$	+	+	α_2	$360 - \alpha_1$
IIIA, IVA	$\theta_1 < 0, \vert\theta_1\vert > \theta_2$	-	-	$180 - \alpha_2$	$180 + \alpha_1$
IIIB, IVB	$\theta_2 < 0, \vert\theta_2\vert > \theta_1$	+	-	$180 - \alpha_1$	$180 + \alpha_2$

The last four columns of Table 5 are given with equations (143) and in effect eliminate the quadrant search since it has been done in advance. Figure 25 shows equations (143) arranged in a computing form.

Direct and inverse solutions of maximum spheroidal geodesics, node to node, vertex to vertex

Vertex to vertex. The direct and inverse are identical since the end points of the arc are the vertices and the longitude difference and length are given by equations (33) or (54). Azimuths are 90° and 270°.

Node to node. For the direct, $\theta_0 = 90° - \alpha_{1-2}$, longitude and length are then given by equations (33) or (54). The back azimuth is given by $\alpha_{2-1} = 270° + \theta_0$. For the inverse, we are given $\Delta\lambda_0 = \lambda_2 - \lambda_1$, i.e. the end points are $P_1(0, \lambda_1)$, $P_2(0, \lambda_2)$ on the equator, and we have two cases:

1. $\Delta\lambda_0 \leq \pi(1-f)$. The distance P_1P_2 is $S = a\Delta\lambda_0$ and azimuths are 90° and 270°.
2. $\pi(1-f) < \Delta\lambda_0 < \pi$. The nodes are in the respective antipodal zones. In the first of equations (33) we place $\sin^2\theta_0 = 1 - \cos^2\theta_0$ and write

$$(1 - \Delta\lambda_0/\pi) = f(1 - f/4 - f^2/16)\cos\theta_0 + (1/4)f^2(1 - f/2)\cos^3\theta_0 + 3f^2\cos^5\theta_0/16. \qquad (147)$$

Using $1/(1-x) = 1 + x + x^2 + \ldots$, we may write

$$D = (1/f)(1 - f/4 - f^2/16) = (1/f)[1 + f/4 + 2(f/4)^2 + 3(f/4)^3 + 4(f/4)^4 + \ldots]$$

We then write (147) as

$$\begin{aligned} &\cos\theta_0 + u\cos^3\theta_0 = v = D(1 - \Delta\lambda_0/\pi),\ u = D(1/4)f^2(1 - f/2) = f/4 - (f/4)^2, \\ &D = (1/f)[1 + f/4 + 2(f/4)^2], \end{aligned} \qquad (148)$$

where unnecessary terms have been omitted.

Finally the formula for v is reversed in (148) and with the equation for S_0 from (33) we write for the inverse solution

$$\begin{aligned} &\cos\theta_0 = v - uv^3,\ v = D(1 - \Delta\lambda_0/\pi),\ u = f/4 - (f/4)^2,\ D = (1/f)[1 + f/4 + 2(f/4)^2], \\ &\alpha_{1-2} = 90° - \theta_0,\ \alpha_{2-1} = 270° + \theta_0,\ S_0 = a\pi[1 - 2(f/4)A + (f/4)^2B + 2(f/4)^3C], \\ &A = 1 + \cos^2\theta_0,\ B = (1 + 3\cos^2\theta_0)(1 - \cos^2\theta_0),\ C = (1 + 2\cos^2\theta_0 + 5\cos^4\theta_0)\cdot \\ &\qquad (1 - \cos^2\theta_0). \end{aligned} \qquad (149)$$

INVERSE POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1; \phi_2, \lambda_2$ to find S, $\alpha_{1\text{-}2}, \alpha_{2\text{-}1}$. Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

____ SPHEROID a ____ m b ____ m

$1 - f = b/a$ ____ $\frac{1}{2}f$ ____ $\frac{1}{4}f$ ____

$f^2/64$ ____ 1 radian = 206264.8062 seconds

° ′ ″ ° ′ ″

ϕ_1 ____ 1. ____ λ_1 ____

ϕ_2 ____ 2. ____ λ_2 ____

$\tan\phi_1$ ____ 1. always west of 2. $\Delta\lambda = \lambda_2 - \lambda_1$ ____

$\tan\phi_2$ ____ $\tan\theta = (1-f)\tan\phi$ $\Delta\lambda_m = \frac{1}{2}\Delta\lambda$ ____

° ′ ″

θ_2 ____ $\tan\theta_2$ ____ $\sin\Delta\lambda_m$ ____

θ_1 ____ $\tan\theta_1$ ____ $\tan\Delta\lambda$ ____

° ′ ″

$\theta_m = \frac{1}{2}(\theta_1 + \theta_2)$ ____ $\sin\theta_m$ ____ $\cos\theta_m$ ____

$\Delta\theta_m = \frac{1}{2}(\theta_2 - \theta_1)$ ____ $\sin\Delta\theta_m$ ____ $\cos\Delta\theta_m$ ____

$H = \cos^2\Delta\theta_m - \sin^2\theta_m = \cos^2\theta_m - \sin^2\Delta\theta_m$ ____ $1 - L$ ____

$L = \sin^2\Delta\theta_m + H\sin^2\Delta\lambda_m$ ____ $\cos d = 1 - 2L$ ____

° ′ ″

$U = 2\sin^2\theta_m\cos^2\Delta\theta_m/(1-L)$ ____ d ____

$V = 2\sin^2\Delta\theta_m\cos^2\theta_m/L$ ____ $\sin d$ ____ d (rad) ____

$X = U + V$ ____ $T = d/\sin d$ ____ $E = 2\cos d$ ____

$Y = U - V$ ____ $D = 4T^2$ ____ $B = 2D$ ____

$A = DE$ ____ $C = T - \frac{1}{2}(A - E)$ ____ CHECK $C - \frac{1}{2}E + AD/B = T$

$n_1 = X(A + CX)$ ____ $n_2 = Y(B + EY)$ ____ $n_3 = DXY$ ____

$\delta_1 d = \frac{1}{4}f(TX - Y)$ ____ $\delta_2 d = (f^2/64)(n_1 - n_2 + n_3)$ ____

$S_1 = a\sin d(T - \delta_1 d)$ ____ m $S_2 = a\sin d(T - \delta_1 d + \delta_2 d)$ ____ m

$F = 2Y - E(4 - X)$ ____ $M = 32T - (20T - A)X - (B + 4)Y$ ____

° ′ ″

$G = \frac{1}{2}fT + (f^2/64)M$ ____ $Q = -(FG\tan\Delta\lambda)/4$ ____

° ′ ″

$\Delta\lambda_m' = \frac{1}{2}(\Delta\lambda + Q)$ ____ $\tan\Delta\lambda_m'$ ____

° ′ ″

$v = \arctan|c_2|$ ____ $c_2 = \cos\Delta\theta_m/(\sin\theta_m\tan\Delta\lambda_m')$ ____

$u = \arctan|c_1|$ ____ $c_1 = -\sin\Delta\theta_m/(\cos\theta_m\tan\Delta\lambda_m')$ ____

° ′ ″ ° ′ ″

$\alpha_1 = v - u$ ____ $\alpha_2 = v + u$ ____

c_1	c_2	$\alpha_{1\text{-}2}$ ° ′ ″	$\alpha_{2\text{-}1}$ ° ′ ″
−	+	α_1 ____	$360 - \alpha_2$ ____
+	+	α_2 ____	$360 - \alpha_1$ ____
−	−	$180 - \alpha_2$ ____	$180 + \alpha_1$ ____
+	−	$180 - \alpha_1$ ____	$180 + \alpha_2$ ____

Figure 25. Inverse position computation form.

Now when $\theta_0 = 0$ in (149) we get $B = C = 0$, $A = 2$, $S_0 = a\pi(1 - f)$, $v = Df = 1 + f/4 + 2(f/4)^2$, $uv^3 = f/4 + 2(f/4)^2$, $\cos\theta_0 = v - uv^3 = 1$, $\alpha_{1\text{-}2} = 90^\circ$, $\alpha_{2\text{-}1} = 270^\circ$, which is case 1 with the equality sign and $\Delta\lambda_0 = \pi(1 - f)$. When $\theta_0 = \pi/2$, $\Delta\lambda_{\pi/2} = \pi$, $v = 0$, $uv^3 = 0$, $\cos\theta_0 = 0$, $A = B = C = 1$, $S_{\pi/2} = a\pi(1 - f/2 + f^2/16 + f^3/32)$, the meridian semilength, see equations (34).

Direct and inverse computation of the ACIC 6000 mile lines

To begin the evaluation of equations (140) and (143), as arranged in the forms of Figures 18 and 25, the nine ACIC 6000 mile lines were computed. The results are compared in Table 6 and the actual computations displayed in Appendix 3. Note, for the meridional limiting case of the direct solution, that when $\theta_0 = 90^\circ$, then $\cos\sigma_1 = \sin\theta_1 = \cos(90^\circ - \theta_1)$ or $\sigma_1 = 90^\circ - \theta_1$, $\sigma_2 = 90^\circ - \theta_2$, $N = \cos\theta_1$, $\Delta\sigma = \sigma_1 - \sigma_2 = \theta_2 - \theta_1$. Using the identity $\cos\theta_1 \sin\alpha_{1\text{-}2} = -\cos\theta_2 \sin\alpha_{2\text{-}1}$, we have

$$\tan\phi_2 = -\frac{(\sin\theta_1 \cos\Delta\sigma + \cos\theta_1 \sin\Delta\sigma)\sin\alpha_{2\text{-}1}}{-(1-f)\cos\theta_2 \sin\alpha_{2\text{-}1}} = \frac{\sin(\theta_1 + \Delta\sigma)}{(1-f)\cos\theta_2} = \frac{\sin\theta_2}{(1-f)\cos\theta_2} = \frac{\tan\theta_2}{(1-f)},$$

and $\alpha_{1\text{-}2} = 0$, $\alpha_{2\text{-}1} = 360^\circ$, $\theta_2 = 90^\circ + \sigma_1 - \Delta\sigma$. Hence in the limit $\tan\phi_2 = \tan(90^\circ + \sigma_1 - \Delta\sigma)/(1 - f)$.

Table 6 shows that good results were obtained using only 8-place tables, (Peters). *The maximum difference in length for the control value of 9655977.366 meters is -.189 meter, the minimum difference is +.004 meter, and the mean difference for the nine line positions is -.044 meter. All the angular values are flat checks or at most .003 second from the control value. These results are better, at 6000 miles, than the adopted criteria (1 meter, .035 sec.) by a factor of 10 for both distance and angular quantities.*

Complete check of direct and reverse solutions over a hemispheroidal geodesic

In order to test for all the cases as delineated in Table 2, we construct a geodesic model as given in Figure 26 containing the given initial and terminal points of the ACIC 6000 mile check line having the largest vertex parametric latitude (excluding the meridian), i.e.

I (initial) $\quad \phi_1 = 70^\circ$, $\theta_1 = 69^\circ\ 56'\ 14''.590$, $\lambda_1 = -18^\circ$

T (terminus) $\quad \phi_2 = 17^\circ\ 08'\ 38''.317$, $\theta_2 = 17^\circ\ 05'\ 21''.296$, $\lambda_2 = 114^\circ\ 18'\ 43''.800$

$$\phi_0 = 76^\circ\ 00'\ 26''.641,\ \theta_0 = 75^\circ\ 57'\ 42''.053,\ S = 9655977.366 \text{ meters} \tag{150}$$

$$\alpha_{1\text{-}2} = 45^\circ,\ \alpha_{2\text{-}1} = 345^\circ\ 17'\ 56''.277$$

From our geodesic model, Figure 26, we choose the arcs:

V_1P_1	$\Delta\lambda_1$	S_1	A vertex end point	
V_1V_2	$\Delta\lambda_0$	S_0	Contains two vertices (end points)	
P_1N_1	$\Delta\lambda_2$	S_2	A node end point	
N_1P_2	$\Delta\lambda_3$	S_3	A node end point	
P_1P_2	$\Delta\lambda_2 + \Delta\lambda_3$	$S_2 + S_3$	Contains a node	(151)
P_2I	$\Delta\lambda_4$	S_4	Contains neither node nor vertex	
P_1P_3	$\Delta\lambda_0$	S_0	Contains a node and a vertex	
V_2T	$\Delta\lambda_1 + \Delta\lambda_4$	$S_1 + S_4$	A vertex end point	
IT	$\lambda_2 - \lambda_1 = 2\Delta\lambda_1 + \Delta\lambda_4$	$2S_1 + S_4$	Given ACIC line—contains a vertex	
TN_2	$\Delta\lambda_3$	S_3	A node end point	
$N_1N_2 = V_1N_1 + N_1V_2$	$\Delta\lambda_0$	S_0	Contains two nodes (end points)	

First listed values are those given for the ACIC 6000 mile lines, Appendix 3. D and I indicate direct and inverse values from the computations, Appendix 3.

ORIGIN N	W		TERMINUS (lat.)	TERMINUS (long.)	S(meters)	ΔS(m)	α_{1-2}	α_{2-1}	
10°	18°		83° 11' 48".545N	162°	E 9655977.366		0°	360°	
		D	.546	162				360	D
(1)		I			.177	-.189	0	360	I
40	18		53 23 45.785	162				360	
		D	.786	162				360	D
(2)		I			.351	-.015	0	360	I
70	18		23 18 44.908	162				360	
		D	.908	162				360	D
(3)		I			.516	+.152	0	360	I
10	18		44 54 05.381	77 25 26.869			45	281 01 12.685	
		D	.381	.868				.683	D
(4)		I			.312	-.054	44° 59' 59".999	.687	I
40	18		35 18 45.644	102 02 29.821			45	318 23 43.000	
		D	.642	.822				.002	D
(5)		I			.259	-.107	45	.002	I
70	18		17 08 38.317	114 18 45.800			45	345 17 56.277	
		D	.318	.800				.277	D
(6)		I			.304	-.062	45	.276	I
10	18		0 30 55.629	68 47 05.259			90	279 57 13.199	
		D	.630	.261				.198	D
(7)		I			.345	-.021	90 00 00.002	.198	I
40	18		1 56 54.386	69 27 01.115			90	309 51 53.419	
		D	.387	.114				.420	D
(8)		I			.370	+.004	89 59 59.998	.420	I
70	18		2 55 17.426	70 50 04.891			90	339 54 37.211	
		D	.426	.892				.209	D
(9)		I			.266	-.100	89 59 59.997	.211	I

Table 6. Computation summary—ACIC 6000 mile lines.

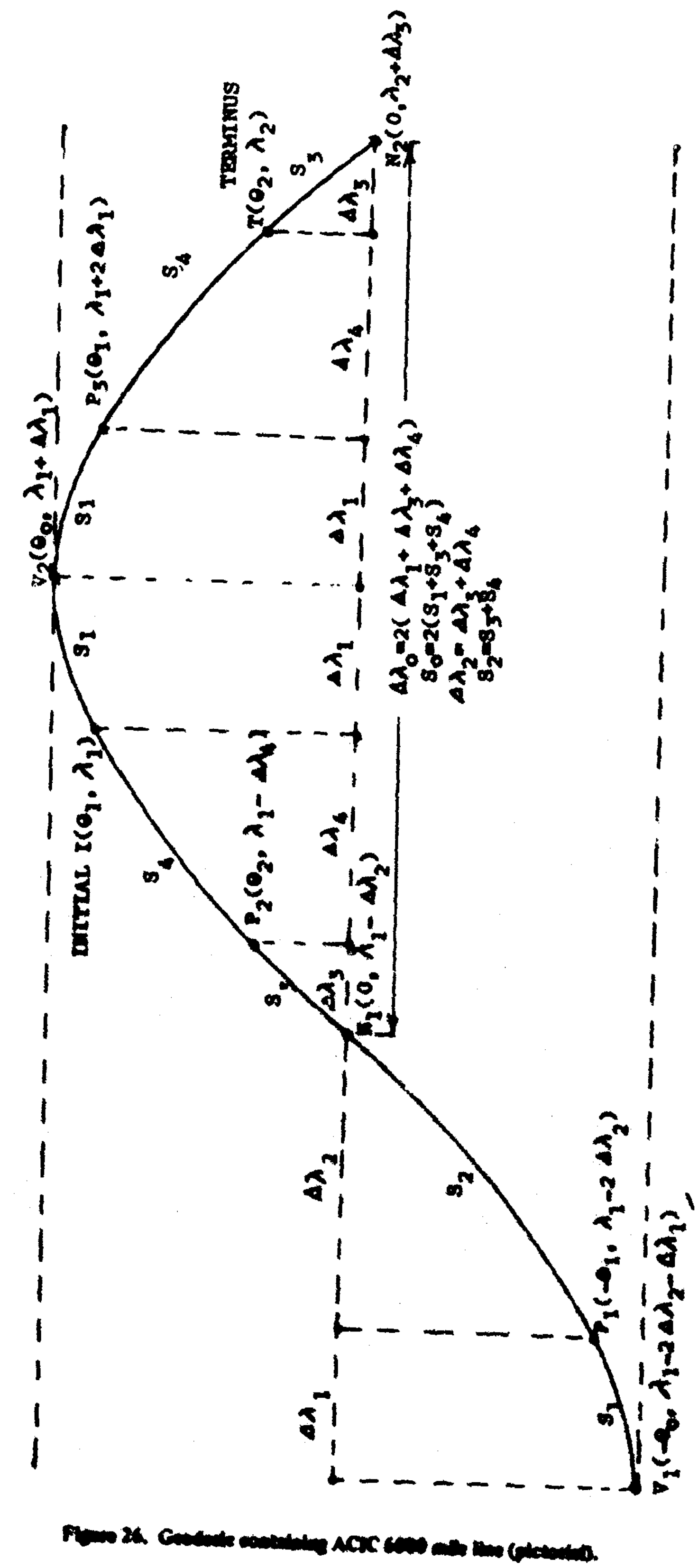


Figure 26. Geodesic containing ACIC 6000 mile line (pictorial).

For control we compute $\Delta\lambda_0, \Delta\lambda_1, \Delta\lambda_2, \Delta\lambda_3, \Delta\lambda_4$ and S_0, S_1, S_2, S_3, S_4 from equations (47)-(54). This provides incidentally a check for the ACIC line (150). The computations are included in Appendix 3. The values obtained are:

	°	'	"		meters	
$\Delta\lambda_0$	179	51	07.553	S_0	20001779.136	
$\Delta\lambda_1$	46	46	49.167	S_1	1611471.024	
$\Delta\lambda_2$	43	08	44.610	S_2	8389418.545	(152)
$\Delta\lambda_3$	4	23	39.146	S_3	1956383.534	
$\Delta\lambda_4$	38	45	05.464	S_4	6433035.010	

From the longitude values of (152) and the given line (150) we have the coordinates of the points:

Point	θ °	'	"	λ °	'	"	
V_1	- 75	57	42.053	- 151	04	18.387	
P_1	- 69	56	14.590	- 104	17	29.220	
N_1	0			- 61	08	44.610	
P_2	+ 17	05	21.296	- 56	45	05.464	
I	+ 69	56	14.590	- 18	0	0	(153)
V_2	+ 75	57	42.053	+ 28	46	49.167	
P_3	+ 69	56	14.590	+ 75	33	38.334	
T	+ 17	05	21.296	+ 114	18	43.798	
N_2	0			+ 118	42	22.944	

From (150), (152) we may write the values for (151) including azimuths:

Line	$\Delta\lambda$ °	'	"	*S (meters)*	$a_{1\text{-}2}$ °	'	"	$a_{2\text{-}1}$ °	'	"	
V_1P_1	46	46	49.167	1611471.024	90			225			
$\underline{V_1V_2}$	179	51	07.553	20001779.136	90			270			
P_1N_1	43	08	44.610	8389418.545	45			194	02	17.947	
N_1P_2	4	23	39.146	1956383.534	14	02	17.947	194	42	03.723	
P_1P_2	47	32	23.756	10345802.079	45			194	42	03.723	
P_2I	38	45	05.464	6433035.010	14	42	03.723	225			(154)
$\underline{P_1P_3}$	179	51	07.553	20001779.136	45			315			
V_2T	85	31	54.631	8044506.034	90			345	17	56.277	
IT	132	18	43.798	9655977.058	45			345	17	56.277	
TN_2	4	23	39.146	1956383.534	165	17	56.277	345	57	42.053	
$\underline{N_1N_2}$	179	51	07.553	20001779.136	14	02	17.947	345	57	42.053	

By comparing the properties of the lines, as delineated in (151), with Table 2, it is seen that the computation of hemispheroidal geodesics and arcs of (154) is sufficient. Note that the lines V_1V_2, P_1P_3, N_1N_2 are maximum hemispheroidal geodesics under the unique shortest distance property, i.e. node to node; vertex to vertex; points in equal but opposite signed latitudes separated by maximum longitude (that between successive nodes or successive vertices).

We first dispose of the computation of the equal maximum hemispheroidal geodesics V_1V_2, N_1N_2, P_1P_3.

V_1V_2. Since the direct and inverse are identical, the end points of the arc are the vertices $\pm\theta_0$; one may compute A and D from (49) and then $\Delta\lambda_0$ and S_0 from (54). This has already been done and the computations are given in Appendix 3. Azimuths are always $\alpha_{1-2} = 90°$, $\alpha_{2-1} = 270°$ (second vertex always east of the first).

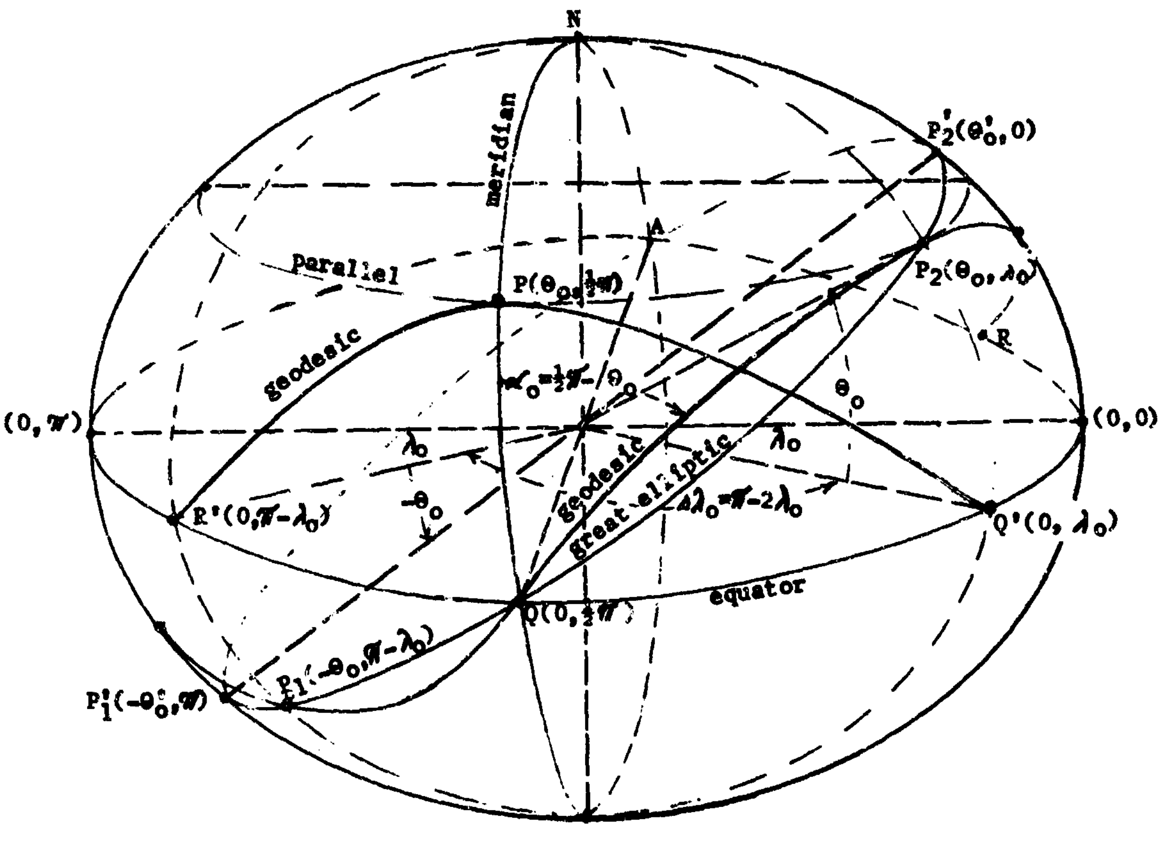
Note in equations (143) that the term $T = d/\sin d$ grows very large when $d \to \pi$. Now from equations (142) with $\theta_2 = -\theta_1 = -\theta_0$, $\Delta\lambda_0 = \pi - 2\lambda_0$ we have:

$$-Y = X = 2\sin^2\theta_0' = 4\sin^2\theta_0/(1 - \cos d) = 2\sin^2\theta_0/\sin^2(d/2).$$

$$\tan\theta_0' = \sin\theta_0/\cos\theta_0\cos\lambda_0,\ \cos(d/2) = \cos\theta_0\sin\lambda_0,\ \cos d = 2\cos^2(d/2) - 1,$$

$$S = a\left[d - (f/4)X(d + \sin d) + (f^2/64)X\left\{X(d - \sin d\cos d) + 4d^2(2 - X)\cot(d/2)\right\}\right] \quad (155)$$

where θ_0' is the vertex parametric latitude of the great elliptic section, see Figure 27.



The great elliptic section containing the geodesic vertices P_1, P_2 has the antipodal vertices P_1', P_2' and passes through the point Q as shown. Its plane has the equation $\tan\theta = \tan\theta_0 \sec\lambda_0 \cos\lambda$. $\Delta\lambda_0$, as given by equations (33), is related to λ_0 by $\Delta\lambda_0 = \pi - 2\lambda_0$, and $\alpha_0 = \pi/2 - \theta_0$. The arc lengths P_1QP_2, QP_2R, $Q'PR'$ are all equal maximum hemispheroidal geodesics under the shortest distance property.

Figure 27. The great elliptic section containing two consecutive vertices of the geodesic.

From the control computations, Appendix 3, $\Delta\lambda_0 = 179° 51' 07''.554$, and hence $\lambda_0 = (1/2)(\pi - \Delta\lambda_0)$ $= 4' 26''.223$;

$$\sin\lambda_0 = .00129069, \sin\theta_0 = .97013371, \cos\theta_0 = .24257076, \cos(d/2) = \cos\theta_0 \sin\lambda_0 = .00031308,$$

$$\sin(d/2) = .99999995, \sin d = 2\sin(d/2)\cos(d/2) = .00062616, \cos d = 2\cos^2(d/2) - 1 = -.99999980,$$

$$\cot(d/2) = \cos(d/2)/\sin(d/2) = .00031308, d = 179° 57' 50''.846 = 3.140966498 \text{ radians},$$

$$d - \sin d \cos d = d + \sin d = 3.141592658 = \pi, a = 6378206.4 \text{ meters}$$

$$S = a(3.140966498 - .005011785 + .000001999) = (6378206.4)(3.135956712) = 20001779.171 \text{ m},$$

which is within .035 meter of the control, Appendix 3.

N_1N_2. For the direct solution, $\theta_0 = 90° - \alpha_{1-2}$. A, D are computed from (49) and $\Delta\lambda_0$, S_0 from (54). For the inverse we are given $\Delta\lambda_0$, whence we have two possible cases as described in (147). For our case the second solution is appropriate and we solve for θ_0 and then S_0 from equations (149). The calculations are given in Appendix 3. *Note that there are two solutions symmetric with respect to the equator for this reverse problem.*

P_1P_3. For the direct solution we are given θ_1, α_{1-2} and we have θ_0 from equation (10), $\cos\theta_0 = \cos\theta_1 \sin\alpha_{1-2}$. $\Delta\lambda_0$, S_0 are then given by (54) after computing A and D from (49). $\theta_2 = -\theta_1$, $\alpha_{2-1} = 360° - \alpha_{1-2}$. For the reverse solution we are given $\theta_1, \theta_2, \lambda_1, \lambda_2$ where $\theta_2 = -\theta_1$, $\Delta\lambda = \lambda_2 - \lambda_1 = \Delta\lambda_0$. From $\Delta\lambda_0$ we may solve for $\cos\theta_0$ and then S_0 from equations (149). Then $\sin\alpha_{1-2} = \cos\theta_0/\cos\theta_1$, $\alpha_{2-1} = 360° - \alpha_{1-2}$. *Since there are two solutions* (see Figure 13) the alternative azimuths are $\alpha'_{1-2} = 180° - \alpha_{1-2}$, $\alpha'_{2-1} = 180° + \alpha_{1-2}$.

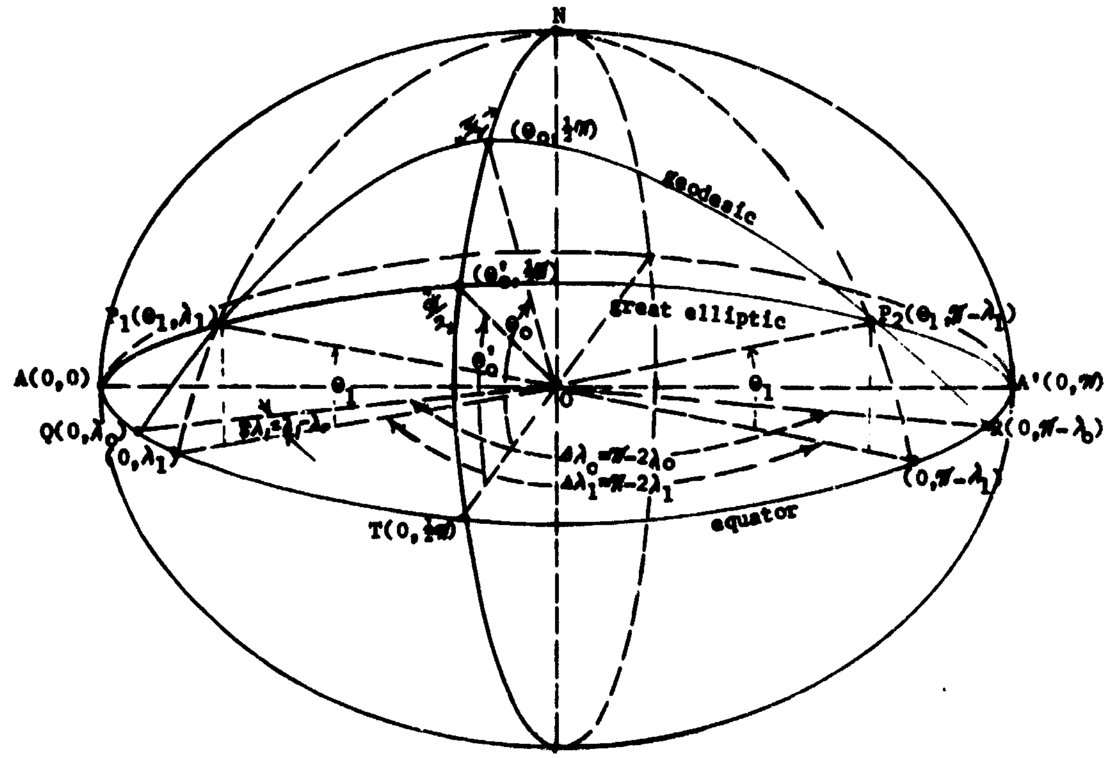
Comparison of direct and inverse computations of the geodesic line segments of (154) are given in Table 7, and the computations are included in Appendix 3. Over lengths of 1.5, 2, 6, 8, 9.5, 10, 20 megameters, maximum length error was .26 m, and maximum angular error was .018 second. *All values were a factor of 2 to 10 better than the assumed criteria.*

A geometric limitation in the inverse solution

Since $T = d/\sin d$ grows large when $d \to \pi$, some increase in accuracy is made for long almost antipodal geodesic arcs by returning sin d to the formulae, that is using them in the form of equations (142). However, when two spheroidal points are in nearly the same small latitude, and separated by maximum hemispheroidal geodetic longitude difference, as shown in Figure 28, a limitation is imposed which is purely geometric. An examination of Figure 28 shows that the separation of geodesic and great elliptic vertices may be large where P_1, P_2 are in the same latitude and near the equator (in the antipodal zones), because the great elliptic section through P_1, P_2 always contains the diameter AA′, while the geodesic does only in the limiting case of the meridian. In fact for the complete hemispheroidal geodesic, node to node, the great elliptic section coincides with the equator, and $S = a\Delta\lambda$, for all such hemispheroidal geodesics as given by equations (142) but which is true for only the limiting case of the geodesic equatorial limiting arc when $\Delta\lambda = \pi(1 - f)$, see equations (34).

Arc	ϕ_1 ° ' "	λ_1 ° ' "	ϕ_2 ° ' "	λ_2 ° ' "	S(meters)	ΔS m	α_{1-2} ° ' "	α_{2-1} ° ' "
(V_1P_1) Control	-76 00 26.641	-151 04 18.387	-70 00 00.000	-104 17 29.220	1611471.024		90 00 00.000	225 00 00.000
Direct			-70 00 00.004	29.232				225 00 00.012
Inverse					.110	+.09	90 00 00.008	225 00 00.006
(V_1V_2) Control	-76 00 26.641	-151 04 18.387	+76 00 26.641	+ 28 46 49.167	20001779.136		90	270
Direct			26.641	49.166				270
(P_1N_1) Control	-70	-104 17 29.220	0	- 61 08 44.610	8389418.545		45	194 02 17.947
Direct			0 00 00.010	44.609				17.946
Inverse					.341	-.20	45 00 00.001	17.945
(N_1P_2) Control	0	- 61 08 44.610	+17 08 38.317	- 56 45 05.464	1956383.534		14 02 17.947	194 42 03.723
Direct			.316	.466				.724
Inverse					.521	-.01	.947	.725
(P_1P_2) Control	-70	-104 17 29.220	+17 08 38.317	- 56 45 05.464	10345802.079		45	194 42 03.723
Direct			.328	.465				.724
Inverse					1.820	-.26	45 00 00.001	.723
(P_2I) Control	+17 08 38.317	- 56 45 05.464	+70	- 18	6433035.010		14 42 03.723	225
Direct			+70 00 00.005	- 17 59 59.982				225 00 00.017
Inverse					4.783	-.23	.722	225
(P_1P_3) Control	-70	-104 17 29.220	+70	+ 75 33 38.334	20001777.136		45	315
Direct			+70	.333				315
(V_2T) Control	+76 00 26.641	+ 28 46 49.167	+17 08 38.317	+114 18 43.798	8044506.034		90	345 17 56.277
Direct			.317	.800				.276
Inverse					.076	+.04	89 59 59.999	.277
$(I\ T)$ Control	+70	- 18	+17 08 38.317	+114 18 43.798	9655977.058		45	345 17 56.277
Direct			.326	.798				.276
Inverse					.304	+.25	45	.276
$(T\ N_2)$ Control	+17 08 38.317	+114 18 43.798	0	+118 42 22.944	1956383.534		165 17 56.277	345 57 42.053
Direct			0	.942				.052
Inverse					.521	-.01	.275	.053
(N_1N_2) Control	0	- 61 08 44.610	0	+118 42 22.944	20001779.136		14 02 17.947	345 57 42.053
Direct			0	.943				.054

Table 7. Summary of computations for the hemispheroidal geodesic containing an ACIC 6000 mile arc.



The plane of the great elliptic now has for equation $\tan\theta = \tan\theta_1 \sin\lambda/\sin\lambda_1$. Hence the vertex of the great elliptic is given by $\tan\theta_0' = \tan\theta_1/\sin\lambda_1$.

Figure 28. The geodesic and great elliptic section through two points in the same latitude.

This geometric limitation applies also, unfortunately, to the inverse solution as given in reference [4]. This geometric singularity is also inherent in any solution based on the normal section for when two points on the geodesic are near the equator (same latitude) separated by maximum hemispheroidal geodesic longitude difference, the plane common to the normals at the geodesic arc end points, containing the common plane section vertex, lies near the equator, while the geodesic vertex is near the pole.

To obtain some estimates of this limitation, hemispheroidal geodesics, vertex to vertex, were computed from equations (33) and (155) simultaneously for several geodesic vertex parametric latitudes as shown in the summary, Table 8. From Figure 28 we have the vertex parametric latitude of the great

Top line: S_c from equations (33)
Lower line: S from equations (155)

θ_o ° ' "	$\Delta\lambda_o$ ° ' "	(1)	(2)(f)	(3)(f^2)	(4)(f^3)	Σ (meters)	$S-S_o$ m	$\theta'_o-\theta_o$ sec
90 00 00.000	180 00 00.000	20037726.369 20037726.369	-33964.700 -33964.700	+14.393 +14.393	+.024	20003776.086 20003776.062	-.02	0
85	179 56 48.700	20037726.369 20037210.774	-34222.700 -33706.701	+14.609 +14.175	+.025	20003518.303 20003518.248	-.06	+.003
75	179 50 31.884	20037726.369 20033179.542	-36239.904 -31689.501	+16.127 +12.533	+.026	20001502.618 20001502.574	-.04	+.048
60	179 41 42.315	20037726.369 20020754.853	-42455.876 -25473.570	+18.891 + 8.141	+.033	19995289.417 19995289.424	+.01	+.316
54 44 08.197	179 38 52.412	20037726.369 20015095.865	-45286.267 -22643.205	+19.190 + 6.468	+.036	19992459.328 19992459.128	-.20	+.458
45	179 34 07.309	20037726.369 20003775.971	-50947.051 -16982.470	+17.991 + 3.718	+.040	19986797.349 19986797.219	-.13	+.731
30	179 28 17.946	20037726.369 19986790.226	-59438.226 - 8491.310	+11.694 1.034	+.032	19978299.869 19978299.950	+.08	+.951
15	179 24 38.205	20037726.369 19974351.003	-65654.198 2275.260	+ 3.663 + 0.120	+.012	19972075.846 19972075.863	+.02	+.683
5	179 23 31.605	20037726.369 19970313.340	-67671.401 - 258.007	+ [illegible].435 + 0.003	+.001	19970055.404 19970055.341	-.06	+.252
0	179 23 23.231	20037726.369 20037726.369	-67929.401 -67929.401			19969796.968 19969796.968	0	0

Clarke 1866 Ellipsoid

Table 8. Summary of hemispheroidal geodesic computation (vertex to vertex).

elliptic section given by $\tan\theta_0' = \tan\theta_1/\sin\lambda_1$ and with $\theta_2 = \theta_1$ we have from (47), (128), and (142) the following formulae:

$$\begin{gathered}
\theta_0' = \arcsin(\sin\theta_1/\cos d/2) = \arctan(\tan\theta_1/\sin\lambda_1),\ X = Y = 2\sin^2\theta_0', \\
\sin d/2 = \cos\theta_1 \cos\lambda_1, \\
\cos d = 1 - 2\sin^2 d/2,\ \lambda_1 = \delta\lambda_1 + \pi/2 - \Delta\lambda_0/2,\ \Delta\lambda_1 = \pi - 2\lambda_1,\ \Delta\lambda_0 = \theta_0 - \theta_0', \\
S_1 = a[d - (f/4)X(d - \sin d) + (f^2/64)X\{X(d - \sin d\cos d) - 4d^2(1 - \cos d)(2 - X)/\sin d\}], \\
\delta\lambda_1 = \gamma - A\beta - B\sin 2\beta,\ \gamma = \arcsin(\tan\theta_1/\tan\theta_0),\ \beta = \arcsin(\sin\theta_1/\sin\theta_0), \\
\delta S_1/a = D\beta - E\sin 2\beta - F\sin 4\beta,\ M = \cos\theta_0,\ c_1 = fM,\ c_2 = (1/4)f(1 - M^2),\ A = c_1 - 2B, \\
B = c_1c_2/2,\ D = (1 - c_2)^2 - AM,\ E = c_2 + BM,\ F = c_2^2/4,\ \alpha_{1\text{-}2} = \arcsin(\cos\theta_0/\cos\theta_1),
\end{gathered} \tag{156}$$

From Table 8 we have S_0 and $\Delta\lambda_0$ for the hemispheroidal geodesics with vertex parametric latitudes $\theta_0 = 5, 15, 30, 45, 60, 75, 85$ degrees. The values of $\Delta\theta = \theta_0' - \theta_0$ given there are for the hemispheroidal geodesic, vertex to vertex. Since the length of the geodesic, node to node, is the same and longitude difference is the same, distances and longitude differences were computed between $P_1(\theta_1, \lambda_1)$ and $P_2(\theta_1, \pi - \lambda_1)$, Figure 28, as follows:

With the values of $\theta_1 = 30', 1°, 5°, 10°$ for each value of θ_0, the values of $\delta\lambda_1, \delta S_1$ were computed from their formulae as given in (156). Thus $\lambda_1 = \delta\lambda_1 + \pi/2 - \Delta\lambda_0/2$, $\Delta\lambda_1 = \pi - 2\lambda_1$, $S = S_0 - 2\delta S_1$ were correspondingly determined which define the control for each geodetic line P_1P_2. Then θ_0' and S_1 were computed from (156) and the corresponding values of $\Delta S = S_1 - S$, $\Delta\theta_0 = \theta_0 - \theta_0'$, $S' = S_1/\Delta S$ obtained. *Only geodesic arcs with end points in the same latitude and separated by maximum geodesic longitude were thus obtained.*

Table 9 gives the results of the computations. Figure 29 shows the graphs of θ_0 versus $\Delta\theta_0 = \theta_0 - \theta_0'$ for $\theta_1 = 30', 1°, 5°, 10°$ and corresponding distance errors over maximum geodetic lengths, 10.6 to 19.9 megameters. Some conclusions may be drawn from these results. Under the distance criterion of one meter, when two points are in about the same latitude, $\theta_1 > 10°$, separated by maximum hemispheroidal longitude difference for that common latitude and particular geodesic, the inverse solution holds for geodesic vertex latitude range $10° < \theta_0 < 90°$. Under the ACIC criterion 1/100000 for distance, the inverse is satisfactory for two points in the same latitude $\theta_1 > 1°$, for values of geodesic vertex latitude $1° < \theta_0 < 90°$, and with longitude separation maximum for a given geodesic. The formulae will also hold under the ACIC distance criterion for $\theta_1 = 30'$ at maximum longitude separation for $0 < \theta_0 < 30°$, $82° < \theta_0 < 90°$. All these values are approximations as deduced from Table 9. Obviously if the longitude separation between two points in the same latitude is less than the maximum possible for hemispheroidal geodesics, the formulae will give better results since the separation between geodesic and great elliptic vertices will be less, see Figure 28.

Θ_1		Θ_0 (deg) 5	15	30	45	60	75	85
30'	S_1 (m)	18695003.58	19543379.19	19756609.79	19830125.31	19866795.23	19885806.13	19892003.95
	$\Delta\lambda_1$ (o)	167° 50' 53.544"	175 41 25.636	177 44 43.306	178 34 19.376	179 07 10.838	179 34 18.146	179 51 34.808
	ΔS (m)	+1.15	+31.44	+205.08	+227.59	-387.31	-839.32	-147.25
	$\Delta\Theta_0$ (o)	14' 05.376"	1 55 57.936	6 04 45.237	9 59 46.819	11 21 33.366	8 12 07.969	3 00 48.134
	$1/S'$	1/16256525	1/621609	1/96336	1/87131	1/51294	1/23692	1/135090
1°	S_1	17406993.12	19114172.30	19534521.60	19673027.09	19738922.14	19771610.42	19780776.65
	$\Delta\lambda_1$	156 27 11.722	171 57 56.834	176 01 06.300	177 34 30.622	178 32 38.998	179 18 28.986	179 46 28.852
	ΔS	+1.09	+8.47	+62.89	+40.85	-150.14	-177.32	-7.15
	$\Delta\Theta_0$	6 37.559	1 00 31.723	3 19 18.952	5 28 45.147	6 02 50.161	4 05 11.371	1 25 40.630
	$1/S'$	1/15925885	1/2255625	1/310622	1/481650	1/131475	1/111502	1/2766156
5°	S_1	$\Theta_1 = 2°$ 16732443.96	15605615.24	17750853.98	18615805.41	18713665.31	18852812.00	18889824.75
	$\Delta\lambda_1$	152 30 06.892	141 25 30.236	162 05 55.802	169 33 51.642	173 55 26.450	177 09 52.968	179 06 21.734
	ΔS	+.36	+.22	+3.02	+1.44	-5.71	-6.02	-2.14
	$\Delta\Theta_0$	2 39.182	9 53.038	38 58.756	1 06 31.943	1 12 22.498	47 23.553	17 06.400
	$1/S'$	1/40923495	1/70934619	1/5360758	1/12788754	1/3277349	1/3131696	1/8827021
10°	S_1	$\Theta_1 = 3°$ 11781332.53	10621760.50	15468954.82	16832380.38	17428663.56	17703436.32	17776011.83
	$\Delta\lambda_1$	106 02 28.922	97 22 55.984	144 01 32.298	159 19 30.260	168 02 54.080	174 26 39.398	178 11 05.870
	ΔS	+.52	-.42	+.44	-.34	-1.24	+.70	-.14
	$\Delta\Theta_0$	1 14.522	2 40.719	16 25.198	30 07.994	33 31.980	22 05.484	7 58.068
	$1/S'$	1/22656418	1/25289906	1/33385337	1/49507001	1/14055374	1/25290623	1/126971513

Table 9. Summary of computations made with formulae (156).

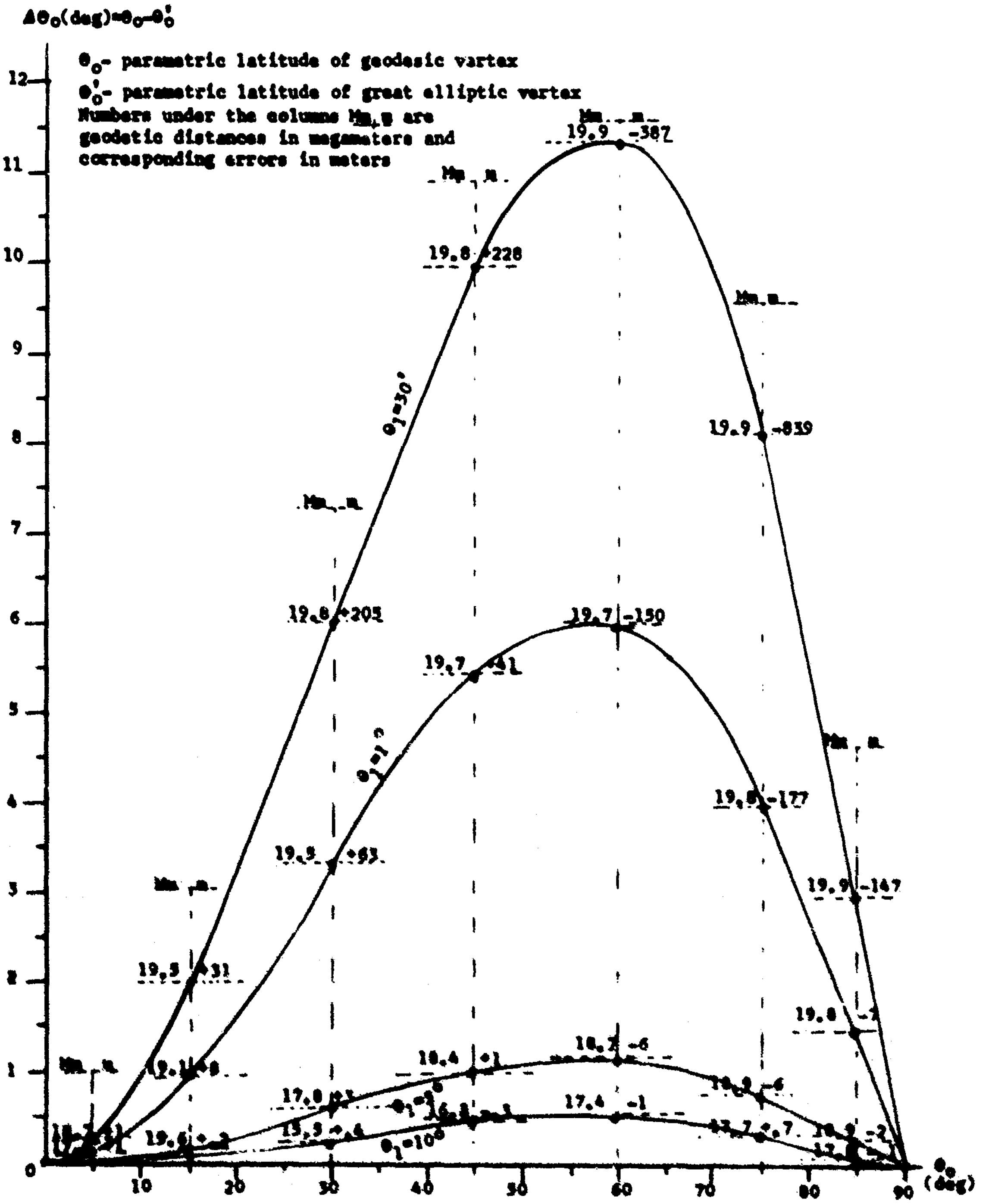


Figure 29. Graphs of θ_0 versus $\Delta\theta_0$ for θ_1 = 30', 1°, 5°, 10° and corresponding distance errors (maximum geodetic longitude separation for a given θ_0).

DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1, \alpha_{1\text{-}2}$, S to find $\phi_2, \lambda_2, \alpha_{2\text{-}1}$. East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

________ SPHEROID a ________ m f ________

$1 - f$ ________ 1 radian = 206264.8062 seconds

LINE ________ TO ________

° ′ ″

ϕ_1 ________ $\tan \phi_1$ ________ $\tan \theta_1 = (1 - f) \tan \phi_1$ ________ ° ′ ″

$\alpha_{1\text{-}2}$ ________ $\sin \theta_1$ ________ $\cos \theta_1$ ________ θ_1 ________

$\sin \alpha_{1\text{-}2}$ ________ $M = \cos \theta_0 = \cos \theta_1 \sin \alpha_{1\text{-}2}$ ________ θ_0 ________

$\cos \alpha_{1\text{-}2}$ ________ $N = \cos \theta_1 \cos \alpha_{1\text{-}2}$ ________ $\sin \theta_0$ ________

$c_1 = fM$ ________ $D = (1 - c_2)(1 - c_2 - c_1 M)$ ________

$c_2 = \frac{1}{4}(1 - M^2)f$ ________ $P = c_2 (1 + \frac{1}{2} c_1 M)/D$ ________

° ′ ″

$\cos \sigma_1 = \sin \theta_1 / \sin \theta_0$ ________ σ_1 ________

$d = S/aD$ ________ (rad) d ________ S ________ m

$\sin d$ ________ $u = 2(\sigma_1 - d)$ ________ $\sin u$ ________

$\cos d$ ________ $W = 1 - 2P \cos u$ ________ $\cos u$ ________

$V = \cos u \cos d - \sin u \sin d$ ________ $Y = 2PVW \sin d$ ________

$X = c_2^2 \sin d \cos d (2V^2 - 1)$ ________ $\Delta\sigma = d + X - Y$ ________ (rad)

° ′ ″

$\sin \Delta\sigma$ ________ $\cos \Delta\sigma$ ________ $\Delta\sigma$ ________

$\cos \Sigma\sigma$ ________ $\Sigma\sigma = 2\sigma_1 - \Delta\sigma$ ________ ° ′ ″

$\tan \alpha_{2\text{-}1} = M/(N \cos \Delta\sigma - \sin \theta_1 \sin \Delta\sigma)$ ________ $\alpha_{2\text{-}1}$ ________

$$\tan \phi_2 = \frac{-(\sin \theta_1 \cos \Delta\sigma + N \sin \Delta\sigma) \sin \alpha_{2\text{-}1}}{(1 - f)M}$$ ________ $\sin \alpha_{2\text{-}1}$ ________ ° ′ ″

ϕ_2 ________ ° ′ ″

$$\tan \Delta\eta = \frac{\sin \Delta\sigma \sin \alpha_{1\text{-}2}}{\cos \theta_1 \cos \Delta\sigma - \sin \theta_1 \sin \Delta\sigma \cos \alpha_{1\text{-}2}}$$ ________ $\Delta\eta$ ________

$H = c_1(1 - c_2)\Delta\sigma - c_1 c_2 \sin \Delta\sigma \cos \Sigma\sigma$ ________ (rad) H ________

$\Delta\lambda = \Delta\eta - H$ ________

λ_1 ________

CHECK

° ′ ″

$M = \cos \theta_0 = \cos \theta_1 \sin \alpha_{1\text{-}2} = \cos \theta_2 \sin (180 + \alpha_{2\text{-}1})$ $\lambda_2 = \lambda_1 + \Delta\lambda$ ________

INVERSE POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1; \phi_2, \lambda_2$ to find S, $\alpha_{1\text{-}2}$, $\alpha_{2\text{-}1}$. Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

______ SPHEROID a ______ m   b ______ m

$1 - f = b/a$ ______ $\frac{1}{2}f$ ______ $\frac{1}{4}f$ ______

$f^2/64$ ______   1 radian = 206264.8062 seconds

° ′ ″		° ′ ″
ϕ_1 ______	1. ______	λ_1 ______
ϕ_2 ______	2. ______	λ_2 ______
$\tan \phi_1$ ______	1. always west of 2.	$\Delta\lambda = \lambda_2 - \lambda_1$ ______
$\tan \phi_2$ ______	$\tan \theta = (1 - f) \tan \phi$	$\Delta\lambda_m = \frac{1}{2}\Delta\lambda$ ______
θ_2 ______ (° ′ ″)	$\tan \theta_2$ ______	$\sin \Delta\lambda_m$ ______
θ_1 ______	$\tan \theta_1$ ______	$\tan \Delta\lambda$ ______
$\theta_m = \frac{1}{2}(\theta_1 + \theta_2)$ ______ (° ′ ″)	$\sin \theta_m$ ______	$\cos \theta_m$ ______
$\Delta\theta_m = \frac{1}{2}(\theta_2 - \theta_1)$ ______	$\sin \Delta\theta_m$ ______	$\cos \Delta\theta_m$ ______

$H = \cos^2\Delta\theta_m - \sin^2\theta_m = \cos^2\theta_m - \sin^2\Delta\theta_m$ ______ $1 - L$ ______

$L = \sin^2\Delta\theta_m + H \sin^2\Delta\lambda_m$ ______ $\cos d = 1 - 2L$ ______

$U = 2 \sin^2\theta_m \cos^2\Delta\theta_m/(1 - L)$ ______   d (° ′ ″) ______

$V = 2 \sin^2\Delta\theta_m \cos^2\theta_m/L$ ______ $\sin d$ ______   d (rad) ______

$X = U + V$ ______ $T = d/\sin d$ ______ $E = 2 \cos d$ ______

$Y = U - V$ ______ $D = 4T^2$ ______ $B = 2D$ ______

$A = DE$ ______ $C = T - \frac{1}{2}(A - E)$ ______   CHECK $C - \frac{1}{2}E + AD/B = T$

$n_1 = X(A + CX)$ ______ $n_2 = Y(B + EY)$ ______ $n_3 = DXY$ ______

$\delta_1 d = \frac{1}{4}f(TX - Y)$ ______ $\delta_2 d = (f^2/64)(n_1 - n_2 + n_3)$ ______

$S_1 = a \sin d (T - \delta_1 d)$ ______ m $S_2 = a \sin d (T - \delta_1 d + \delta_2 d)$ ______ m

$F = 2Y - E(4 - X)$ ______ $M = 32T - (20T - A)X - (B + 4)Y$ ______

$G = \frac{1}{2}fT + (f^2/64) M$ ______ $Q = -(FG \tan \Delta\lambda)/4$ ______

$\Delta\lambda'_m = \frac{1}{2}(\Delta\lambda + Q)$ ______ (° ′ ″) $\tan \Delta\lambda'_m$ ______

$v = \arctan |c_2|$ ______ (° ′ ″) $c_2 = \cos \Delta\theta_m/(\sin \theta_m \tan \Delta\lambda'_m)$ ______

$u = \arctan |c_1|$ ______ (° ′ ″) $c_1 = -\sin \Delta\theta_m/(\cos \theta_m \tan \Delta\lambda'_m)$ ______

$\alpha_1 = v - u$ ______ $\alpha_2 = v + u$ ______

c_1	c_2	$\alpha_{1\text{-}2}$ (° ′ ″)	$\alpha_{2\text{-}1}$ (° ′ ″)
−	+	α_1 ______	$360 - \alpha_2$ ______
+	+	α_2 ______	$360 - \alpha_1$ ______
−	−	$180 - \alpha_2$ ______	$180 + \alpha_1$ ______
+	−	$180 - \alpha_1$ ______	$180 + \alpha_2$ ______

Appendix 2.

SPHEROID PARAMETERS; SPHERICAL APPROXIMATIONS; SPACE COORDINATES AT A POINT OF THE SPHEROID; OTHER USEFUL FORMULAE

Ellipsoid	a meters	1/f	b meters	1-f	append $x10^{-3}$ f	f/2	f/4	append $x10^{-6}$ $f^2/64$
AUSTRALIAN	6378160	298.25	6356774.7192	.996647108131	3.352891869	1.6764459345	.83822296725	.1756544357
FISCHER (MERCURY)	6378166 6378150 *	298.3	6356784.28361 6356768.33724*	.996647670131	3.352329869	1.6761649345	.83808246725	.1755955555
KRASSOVSKY	6378245	298.3	6356863.01877	.996647670131	3.352329869	1.6761649345	.83808246725	.1755955555
INTERNATIONAL	6378388	297	6356911.94613	.996632996633	3.367003367	1.6835016835	.84175084175	.1771361199
HOUGH	6378270	297	6356794.34344	.996632996633	3.367003367	1.6835016835	.84175084175	.1771361199
CLARKE 1866	6378206.4	294.9787	6356583.8	.996609924717	3.390075283	1.6950376415	.84751882075	.1795720379
CLARKE 1880	6378249.1	293.4663	6356514.92098	.996592453716	3.407546284	1.703773142	.851886571	.1814276825
EVEREST	6377276.3	300.8017	6356075.36029	.99667550703	3.324449297	1.6622246485	.83111232425	.1726869239
AIRY	6377563.4	299.325	6356256.91575	.996659149754	3.340850246	1.670425123	.8352125615	.1743950057
BESSEL	6377397.2	299.1528	6356079.00676	.996657226675	3.342773325	1.6713866625	.83569333125	.174595836

NOTE: The parameters a and 1/f were held fixed in the above derivations. The AIRY, CLARKE 1880, and EVEREST are revised ellipsoids. The star values are for the MERCURY ellipsoid which was changed in 1968, see reference[41].

Table 10. Spheroid parameters.

APPROXIMATING SPHERES FOR THE OBLATE SPHEROID

Equivalent area or volume

The area and volume of the oblate spheroid are given by

$$A = 4\pi(a/b^2)\int_0^b [b^4 + (a^2 - b^2)y^2]^{1/2}dy$$

$$= 2\pi\left[a^2 + (b^2/e)(1/2)\ln\left(\frac{1+e}{1-e}\right)\right] = 2\pi[a^2 + (b^2/e)\,\mathrm{arc\,tanh}\,(e)] \qquad (1)$$

$$V = 2\pi(b/a)\int_0^a (a^2 - x^2)^{1/2}x\,dx = (4/3)\pi a^2 b,$$

where the meridian ellipse (y-axis polar), is

$$b^2x^2 + a^2y^2 = a^2b^2,\ b^2 = a^2(1 - e^2)$$

The area and volume of the sphere are

$$A_s = 4\pi R_A^2,\ V_s = (4/3)\pi R_V^3. \qquad (2)$$

From: (1) and (2), the equalities $A_s = A$, $V_s = V$ lead to

$$2R_A^2 = a^2 + (b^2/e)\,\mathrm{arc\,tanh}\,(e),\ b^2 = a^2(1 - e^2), \qquad (3)$$

$$R_V^3 = a^2 b,\ b = a(1 - e^2)^{1/2}. \qquad (4)$$

Now

$$(1/e)\,\mathrm{arc\,tanh}\,(e) = (1/2e)\ln\left(\frac{1+e}{1-e}\right) = 1 + e^2/3 + e^4/5 + e^6/7 + ..$$

and this substitution in (3) gives

$$2R_A^2 = a^2 + a^2(1 - e^2)(1 + e^2/3 + e^4/5 + e^6/7 + ..)$$

which may be written, after expanding and combining like terms as

$$R_A = a[1 - e^2(1/3 + e^2/15 + e^4/35 + ..)]^{1/2}. \qquad (5)$$

Expanding the radical in (5) to 6th order terms in e leads to

$$R_A = a(1 - e^2/6 - 17e^4/360 - 67e^6/3024 - ..), \qquad (6)$$

From (4) we have

$$R_V^3 = a^3(1 - e^2)^{1/2} \text{ or } R_V = a(1 - e^2)^{1/6}$$

and expanding the radical to 6th order terms in e we find

$$R_V = a(1 - e^2/6 - 5e^4/72 - 55e^6/1296 - ..). \qquad (7)$$

From (6) and (7) we have

$$\Delta R = R_A - R_V = a(e^4/45 + 23e^6/1134 + ..). \qquad (8)$$

With $e^2 = 2f - f^2,\ e^4 = 4f^2 - 4f^3,\ e^6 = 8f^3$ we may write from (7) and (8)

$$\left.\begin{aligned} R_V &= a(1 - f/3 - f^2/9 - 5f^3/81 - ..)\\ \Delta R &= (4/45)af^2(1 + 52f/63)\\ R_A &= R_V + \Delta R \end{aligned}\right\} \qquad (9)$$

Mean spherical approximations

$$r_A = (1/2)(a+b),\ r_G = (ab)^{1/2},\ r_H = 2ab/(a+b) = ab/r_A,\ (r_G^2 = r_A \cdot r_H) \tag{10}$$

are respectively the radii equal to the arithmetic, geometric, and harmonic means of the ellipsoid semiaxes and $r_A > r_G > r_H$. Since a and b differ very little, $r_G = (1/2)(r_A + r_H)$ is a satisfactory formula for reference ellipsoids.

Principal radii of curvature

The radii of curvature of the meridian and the normal section perpendicular to the meridian at a given point of the reference ellipsoid are the principal radii of curvature, i.e.

Meridian: $R = a(1-e^2)/(1-e^2\sin^2\phi)^{3/2} = a(1-e^2\cos^2\theta)^{3/2}/(1-e^2)^{1/2}$

Great Normal: $N = a/(1-e^2\sin^2\phi)^{1/2} = a(1-e^2\cos^2\theta)^{1/2}/(1-e^2)^{1/2}$

$$\begin{aligned} &= a(1 + e^2\sin^2\phi/2 + 3e^4\sin\phi/8 + 5e^6\sin^6\phi/16) \\ &= a[1 + f\sin\phi - (1/2)f^2\sin^2\phi(1-3\sin^2\phi) - (1/2)f^3\sin^4\phi(3-5\sin^2\phi) - \ldots] \\ &= a\{1 + (1/2)e^2\sin^2\theta + (1/8)e^4[4-(1+\cos^2\theta)^2] + (1/16)e^6\sin^2\theta[4+(1+\cos^2\theta)^2] \\ &\quad + \ldots\} \end{aligned} \tag{11}$$

$$\sin\phi = \sin\theta/(1-e^2\cos^2\theta)^{1/2},\ \cos\phi = (1-e^2)^{1/2}\cos\theta/(1-e^2\cos^2\theta)^{1/2},$$
$$\tan\phi = \tan\theta/(1-e^2)^{1/2} = \tan\theta/(1-f),\ e^2 = 2f - f^2.$$

Mean radius of the spheroid at a given point of the surface

The mean radius of the spheroid at a given point of its surface is the geometric mean of the principal radii of curvature. From (11) we have

$$R_m = (RN)^{1/2} = a(1-e^2)^{1/2}/(1-e^2\sin^2\phi) = b/(1-e^2\sin^2\phi),\ e^2 = 2f - f^2,$$

where ϕ is geodetic latitude, or in terms of parametric latitude

$$R_m = \{a/(1-e^2)^{1/2}\}(1-e^2\cos^2\theta) = (a^2/b)(1-e^2\cos^2\theta), \tag{12}$$

see references [6], [9], or [16].

Table 11 gives the corresponding radii R_A, R_V, r_A, r_G, r_H for each of the 10 reference ellipsoids included here. Equations (9) and (10) above were used for the computations.

Meridional and equatorial arc axes and area of antipodal zones

From equations (58), (60)—Appendix I—with the constants for the 10 given ellipsoids, the parametric latitudes of the endpoints of the meridional arc axes of the antipodal zones were computed as shown in Table 12.

From the second of equations (32)—Appendix 1—with $\theta_0 = \pi/2$, and from Figure 12, we have for the arc length of the antipodal zone axes:

$$\begin{aligned} S_M &= a[2\theta - (f/2)(2\theta + \sin 2\theta)]\ \text{(meridional)}, \\ S_E &= a\pi f\ \text{(equatorial)},\ 2\theta = \pi f(1 + .7495f). \end{aligned} \tag{13}$$

An approximation to the area of the antipodal zone is that of the hypocycloid of four cusps, i.e.

$$A = (3/8)\pi t^2,\ t = (1/4)(S_M + S_E). \tag{14}$$

With the values of θ from Table 12, S_M, S_E were computed from (13) and then A from (14) for each of the 10 given spheroids. The computations are displayed in Table 13.

SPHEROID	R_A (AREA) meters	R_V (VOLUME) meters	$r_A = (1/2)(a+b)$ meters	$r_G = (1/2)(r_A + r_H)$ meters	$r_H = ab/r_A$ meters
FISCHER (MERCURY) a = 6378166 b = 6356784.28361 m f = 3.352329869x10^{-3}	6371037.171	6371030.782	6367475.1418	6367466.1669	6367457.1921
KRASSOVSKY a = 6378245 m b = 6356863.01877 m f = 3.352329869x10^{-3}	6371116.083	6371109.694	6367554.0094	6367545.0344	6367536.0594
AUSTRALIAN a = 6378160 m b = 6356774.7192 m f = 3.352891869x10^{-3}	6371029.982	6371023.591	6367467.3596	6367458.3818	6367449.4039
INTERNATIONAL a = 6378388 m b = 6356911.94613 m f = 3.367003367x10^{-3}	6371227.712	6371221.266	6367649.9731	6367640.9191	6367631.8651
HOUGH a = 6378270 b = 6356794.34344 f = 3.367003367x10^{-3}	6371109.844	6371103.399	6367532.1717	6367523.1179	6367514.0641
CLARKE 1866 a = 6378206.4 m b = 6356583.8 m f = 3.390075283x10^{-3}	6370997.241	6370990.707	6367395.1000	6367385.9216	6367376.7433
CLARKE 1880 a = 6378249.1 m b = 6356514.92098 m f = 3.407546284x10^{-3}	6371002.731	6370996.129	6367382.0105	6367372.7372	6367363.4638
EVEREST a = 6377276.3 m b = 6356075.36829 m f = 3.324449297x10^{-3}	6370207.759	6370201.477	6366675.8341	6366667.0093	6366658.1845
AIRY a = 6377563.4 m b = 6356256.91575 m f = 3.340850246x10^{-3}	6370459.660	6370453.315	6366910.1579	6366901.2452	6366892.3326
BESSEL a = 6377397.2 m b = 6356079.00676 m f = 3.342773325x10^{-3}	6370289.555	6370283.203	6366738.1034	6366729.1808	6366720.2581

Table 11. Spheroid approximating spheres.

SPHEROID	f	$2\theta = \pi f(1 + .7495f)$	$\tan\theta$, $\sin 2\theta$	$B = f^2/8$, $A = (f/2)-B$	Check: $\tan\theta - B \sin 2\theta = A(\pi + 2\theta)$
Fischer (Mercury) Krassovsky	.00335232987	rad 2θ .010558116 2θ 0:36 17.768 θ 0:18 08.884	$\tan\theta$ .00527911 $\sin 2\theta$ .01055792	B $.1405 \times 10^{-5}$ A .0016747602	.00527910 .00527910
Australian	.00335289187	rad 2θ .010559891 2θ 0:36 18.134 θ 0:18 09.067	$\tan\theta$ .00527999 $\sin 2\theta$ .01055970	B $.1405 \times 10^{-5}$ A .0016750407	.00527998 .00527998
International Hough	.00336700337	rad 2θ .010604447 2θ 0:36 27.324 θ 0:18 13.662	$\tan\theta$ .00530227 $\sin 2\theta$ .01060425	B $.1417 \times 10^{-5}$ A .0016820846	.00530225 .00530226
Clarke 1866	.00339007528	rad 2θ .010677296 2θ 0:36 42.350 θ 0:18 21.175	$\tan\theta$ .00533870 $\sin 2\theta$ .01067709	B $.1437 \times 10^{-5}$ A .0016936011	.00533868 .00533869
Clarke 1880 (revised)	.00340754628	rad 2θ .010732463 2θ 0:36 53.730 θ 0:18 26.865	$\tan\theta$ .00536629 $\sin 2\theta$ .01073226	B $.1451 \times 10^{-5}$ A .001702322	.00536627 .00536627
Everest (revised)	.00332444930	rad 2θ .010470089 2θ 0:35 59.610 θ 0:17 59.805	$\tan\theta$ .00523509 $\sin 2\theta$ .01046989	B $.1381 \times 10^{-5}$ A .001660843	.00523508 .00523508
Airy (revised)	.00334085025	rad 2θ .010521871 2θ 0:36 10.292 θ 0:18 05.146	$\tan\theta$ .00526099 $\sin 2\theta$ .01052168	B $.1395 \times 10^{-5}$ A 001669030	.00526098 .00526097
Bessel	.00334277333	rad 2θ .010527943 2θ 0:36 11.544 θ 0:18 05.772	$\tan\theta$ .00526402 $\sin 2\theta$ .01052775	B $.1397 \times 10^{-5}$ A .00166999	.00526401 .00526401

Table 12. Computation—end point latitudes of antipodal zone meridional axes.

SPHEROID	f, a	2θ, $\sin 2\theta$	MERIDIONAL AXIS $S_M = a[2\theta - \frac{f}{2}(2\theta + \sin 2\theta)]$	EQUATORIAL AXIS $S_E = a\pi f$	MEAN SEMIAXIS $t = (1/4)(S_M + S_E)$	AREA $A = (3/8)\pi t^2$
					Statute Miles	Sq. Stat. Miles
Mercury	.0033523299 6378166 m	2θ .010558116	67115.67 m	67172.64 m	20.86068	512.670
Krassovsky*	6378245*	$\sin 2\theta$ .01055792	67116.50*	67173.48*	20.86094*	512.683*
Australian	.0033528919 6378160	.010559891 .01055970	67126.85	67183.84	20.86416	512.841
International	.0033670034 6378388	.01060[illegible]	67411.54	67469.01	20.95268	517.202
Hough*	6378270	.01060425	67410.29*	67467.76*	20.95229*	517.183*
Clarke 1866	.00339007[illegible] 6378206.4	.010677296 .01067709	67871.13	67929.40	21.09559	524.281
Clarke 1880 (revised)	.0034075463 6378249.1	.010732463 .01073226	68221.06	68279.94	21.20440	529.704
Everest (revised)	0033244493 6377276.3	.010470088 .01046989	66548.[illegible]7	66604.63	20.68438	504.041
Airy (revised)	.0033408502 6377563.4	.010521871 .01052168	66879.72	66936.29	20.78731	509.070
Bessel	.0033427733	.010527943 .01052775	66916.44	66973.08	20.79873	509.[illegible]30

Table 13. Computation of antipodal zone arc axes and areas.

A space coordinate system referred to the normal and tangents to the meridian and parallel through a given point of the reference ellipsoid

In Figure 30, note that a change of coordinates from the center, 0, of the ellipsoid with axes x_1, y_1, z_1 to the point Q on the surface with axes X, Y, Z involves a translation from 0 to Q in the $x_1 z_1$-plane, and then a rotation about Q in that plane through the angle ϕ_0. If we are interested in the slant range, D, from a point S_0 at a height h *above or below the ellipsoidal surface,* to a point Q_0 at a height h_0 *above or below* Q then the following derivation will give D.

From Figure 30, the parametric representation of the point $P(x_1, y_1, z_1)$ on the ellipsoidal surface relative to the rectangular system with origin, 0, the ellipsoid center, is

$$x_1 = N \cos\phi \cos\Delta\lambda,\ y_1 = N \cos\phi \sin\Delta\lambda,\ z_1 = N(1 - e^2) \sin\phi \tag{15}$$

where ϕ is geodetic latitude, $\Delta\lambda$ is the longitude computed from the meridian through Q, $N = a/(1 - e^2 \sin^2 \phi)^{1/2}$ is the great normal, see equation (11) above.

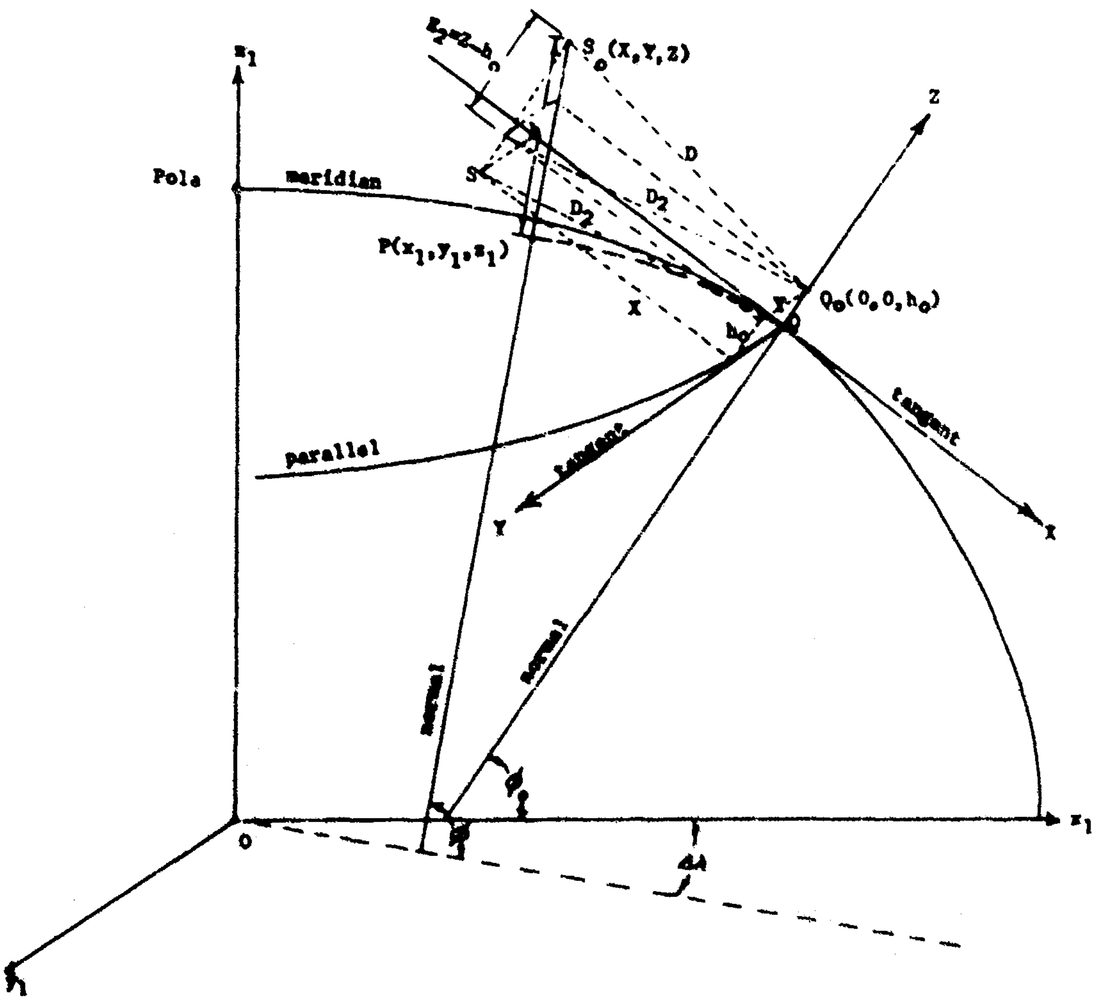


Figure 30. Space coordinate system referred to the normal and tangents to the meridian and parallel at an arbitrary point of the ellipsoid.

The coordinates of any point S_0 at a height h above or below $P(x_1, y_1, z_1)$ on the normal to the surface at P are

$$x_2 = (N \pm h)\cos\phi\cos\Delta\lambda,\ y_2 = (N \pm h)\cos\phi\sin\Delta\lambda,\ z_2 = [(1-e^2)N \pm h]\sin\phi. \tag{16}$$

Now the transformation equations which give the coordinates of S_0 referred to the normal and tangents to the meridian and parallel through a point Q in latitude ϕ_0 (translation from 0 to Q and rotation about Q through ϕ_0 in the x_1z_1-plane) are

$$\begin{aligned} X &= (x_2 - N_0\cos\phi_0)\sin\phi_0 - [z_2 - N_0(1-e^2)\sin\phi_0]\cos\phi_0 \\ Y &= y_2 \\ Z &= (x_2 - N_0\cos\phi_0)\cos\phi_0 + [z_2 - N_0(1-e^2)\sin\phi_0]\sin\phi_0. \end{aligned} \tag{17}$$

Placing the values of x_2, y_2, z_2 from (16) in (17) we have

$$\begin{aligned} X &= u_1\cos\phi\cos\Delta\lambda - u_2\sin\phi - c_1 \\ Y &= (N \pm h)\cos\phi\sin\Delta\lambda \\ Z &= v_1\cos\phi\cos\Delta\lambda + v_2\sin\phi - c_2 \\ u_1 &= (N \pm h)\sin\phi_0,\ u_2 = [N(1-e^2) \pm h]\cos\phi_0,\ c_1 = N_0 e^2\sin\phi_0\cos\phi_0 \\ v_1 &= (N \pm h)\cos\phi_0,\ v_2 = [N(1-e^2) \pm h]\sin\phi_0,\ c_2 = N_0(1 - e^2\sin^2\phi_0) \end{aligned} \tag{18}$$

With the coordinates from (18) we have then, as seen from Figure 30,

$$D_2^2 = X^2 + Y^2,\ E_2 = Z \mp h_0,\ D = (D_2^2 + E_2^2)^{1/2} = [X^2 + Y^2 + (Z \mp h_0)^2]^{1/2}. \tag{19}$$

In the computation of the coordinates (18), the values of N, N_0 may be taken from tables, if available, or computed from the series given above in equations (11).

Now the coordinates (16), with h = o, represent a point on the ellipsoid. Hence if we solve (17) for x_2, y_2, z_2 we obtain

$$\begin{aligned} x_2 &= Z\cos\phi_0 + X\sin\phi_0 + N_0\cos\phi_0 \\ y_2 &= Y \\ z_2 &= Z\sin\phi_0 - X\cos\phi_0 + N_0(1-e^2)\sin\phi_0 \end{aligned} \tag{20}$$

and x_2, y_2, z_2, with h = o, must satisfy the ellipsoid equation

$$(x_2^2 + y_2^2)/a^2 + z_2^2/b^2 = 1, \text{ or since } b^2 = a^2(1-e^2),$$

$$(1-e^2)(x_2^2 + y_2^2) + z_2^2 = a^2(1-e^2). \tag{21}$$

Now (21) may be written as $x_2^2 + y_2^2 + z_2^2 - e^2(x_2^2 + y_2^2 - a^2) = a^2$ which, when e = o, represents the sphere of radius a. Analogously, if we place x_2, y_2, z_2 from (20) in (21), we obtain *the equation of the ellipsoid referred to the point Q as origin (See Figure 30).*

$$X^2 + Y^2 + (Z + N_0)^2 = N_0^2 - \frac{e^2}{1-e^2}(X\cos\phi - Z\sin\phi)^2. \tag{22}$$

Now when e = o, equation (22) becomes the equation to a sphere tangent to the ellipsoid at Q with radius N_0, the great normal length at Q. *Hence the justification for using the great normal radius at the initial point when the spherical forms of the direct and inverse geodetic line solutions are used. See Figures 8 and 9.*

The spherical case

If we place $e = o$ in equations (18), we get $u_1 = v_2 = (N_0 \pm h) \sin \phi_0$, $v_1 = u_2 = (N_0 \pm h) \cos \phi_0$, $c_1 = o$, $c_2 = N_0$, where N_0 is the great normal radius at Q, see equation (22) above, and then

$$
\begin{aligned}
X &= (N_0 \pm h)(\cos \phi \sin \phi_0 \cos \Delta\lambda - \sin \phi \cos \phi_0) \\
Y &= (N_0 \pm h) \cos \phi \sin \Delta\lambda \\
Z &= (N_0 \pm h)(\cos \phi \cos \phi_0 \cos \Delta\lambda + \sin \phi \sin \phi_0) - N_0 \\
D_2^2 &= X^2 + Y^2,\ E_2 = Z \mp h_0,\ D = (D_2^2 + E_2^2)^{1/2} = [X^2 + Y^2 + (Z \mp h_0)^2]^{1/2},
\end{aligned}
\tag{23}
$$

see Figure 31. Also note in Figure 31, the quantities, $u, v, c, \alpha, \delta, \tau$. α is the azimuth of P from Q. δ is the angle of elevation of S_0 above the horizontal at Q_0, τ is the central angle subtended by the arc distance $s = PQ$, $\tau(\text{rad}) = s/N_0$. We may write the following formulae involving the spherical triangle P'PQ and other quantities as indicated in Figure 31.

$$
\begin{aligned}
&\cot \alpha = (\cos \phi \sin \phi_0 \cos \Delta\lambda - \sin \phi \cos \phi_0)/\cos \phi \sin \Delta\lambda \\
&\cos \tau = \sin \phi \sin \phi_0 + \cos \phi \cos \phi_0 \cos \Delta\lambda = N_0/(N_0 + c) \\
&\sin \phi = \cos \tau \sin \phi_0 + \sin \tau \cos \phi_0 \cos \alpha \\
&\cot \Delta\lambda = (\cos \phi_0 \cos \tau - \sin \phi_0 \sin \tau \cos \alpha)/\sin \tau \sin \alpha \\
&u + v = D_2 = D \cos \delta,\ X = D_2 \cos \alpha,\ Y = D_2 \sin \alpha,\ Z = h_0 + D \sin \delta \\
&\tan \tau = D_2/(N_0 + Z),\ h = (N_0 + Z) \sec \tau - N_0 = D_2 \csc \tau - N_0, \\
&u = Z \tan \tau,\ v = N_0 \tan \tau,\ c = N_0(1 - \cos \tau)/\cos \tau, \\
&u/v = Z/N_0 = (h - c)/(N_0 + c),\ u + v = D_2 = (N_0 + Z) \tan \tau = (h + N_0) \sin \tau.
\end{aligned}
\tag{24}
$$

Now h, h_0 in equations (24) can have opposite or like signs, negative signs indicating below the surface of the sphere, see Figure 31. Note that further simplification of this type local reference system is possible for $d = PQ \leq 8$ minutes ≈ 8 nautical miles, for then $\tau \approx \sin \tau \approx \tan \tau$, $\cos \tau \approx 1$.

Rectangular spherical coordinates

In Figure 32, we have the space rectangular coordinate system X, Y, Z with axes the normal and the tangents to the parallel and meridian through a point Q of the surface. Now the tangent at Q to the parallel is also tangent to the great circle containing the poles C, C' of the meridian through Q. A rectangular spherical system on the surface may be used where x-coordinates are measured along the circular meridian from Q and y coordinates are measured along the great circles through the poles C, C' of the meridian through Q. The points P and T as shown have the spherical coordinates x, y and x', y' respectively. The angles β, β' at P, T respectively are measured from the line $PT = s$ to parallels through P, T having the same poles C, C' as the meridian through Q.

Now in the spherical triangle PTC we have

$$
\begin{aligned}
&P = 90^\circ - \beta,\ T = 90^\circ + \beta',\ C = (x' - x)/N_0,\ a = 90^\circ - y'/N_0, \\
&b = 90^\circ - y/N_0,\ c = s/N_0,\ P + T = \pi - (\beta - \beta'), \\
&a - b = -(y' - y)/N_0,\ a + b = \pi - (y + y')/N_0.
\end{aligned}
\tag{25}
$$

To solve for x', y', β' we need the following three spherical formulae (Napier's fourth analogy, sine, and cosine laws)

$$
\begin{aligned}
&\tan \tfrac{1}{2}(P + T) = \cos \tfrac{1}{2}(a - b) \sec \tfrac{1}{2}(a + b) \cot \tfrac{1}{2}c, \\
&\cos a = \cos b \cos c + \sin b \sin c \cos P,\ \sin C = \sin c \sin P/\sin a
\end{aligned}
\tag{26}
$$

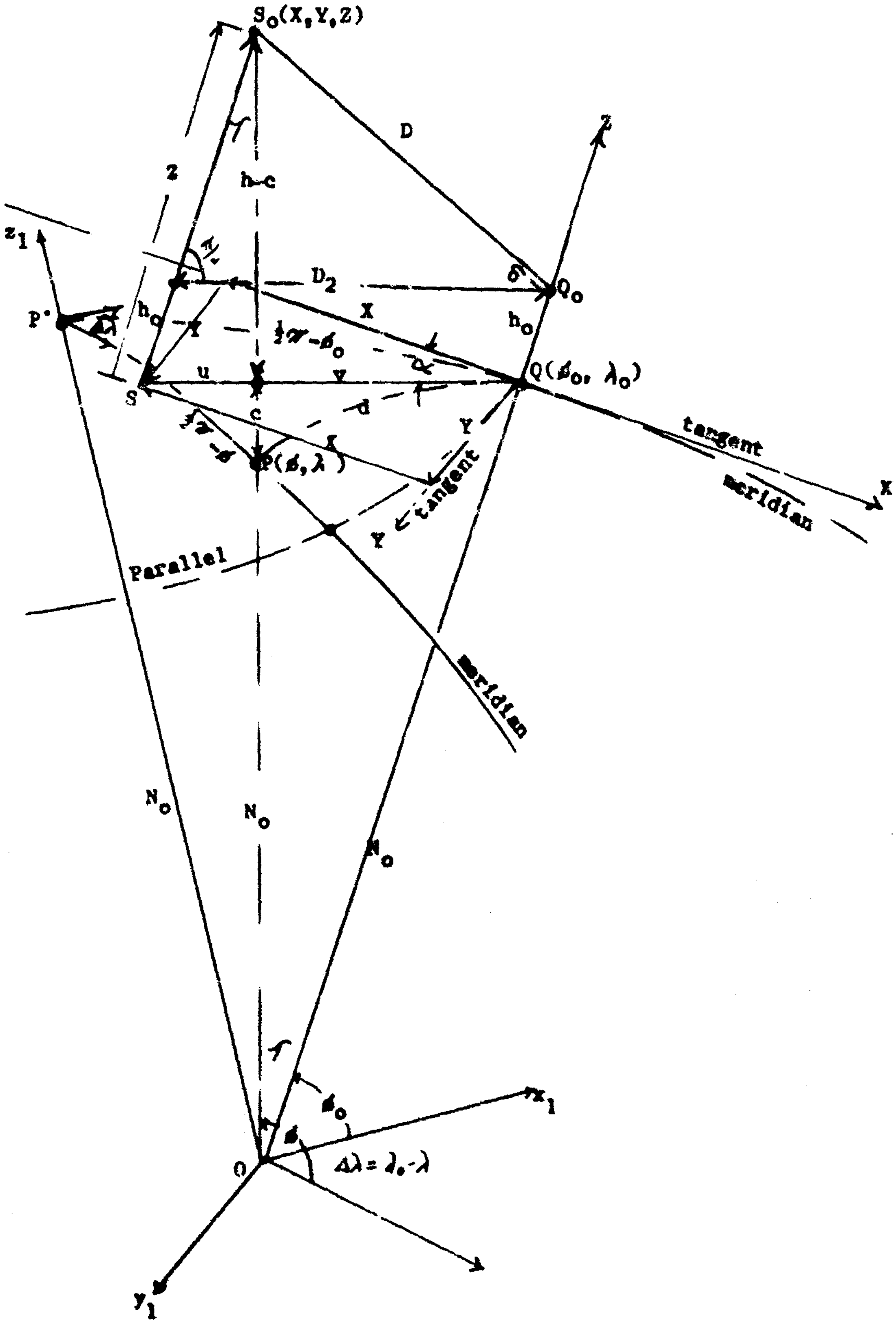


Figure 31. Local space coordinate system at a point of the sphere.

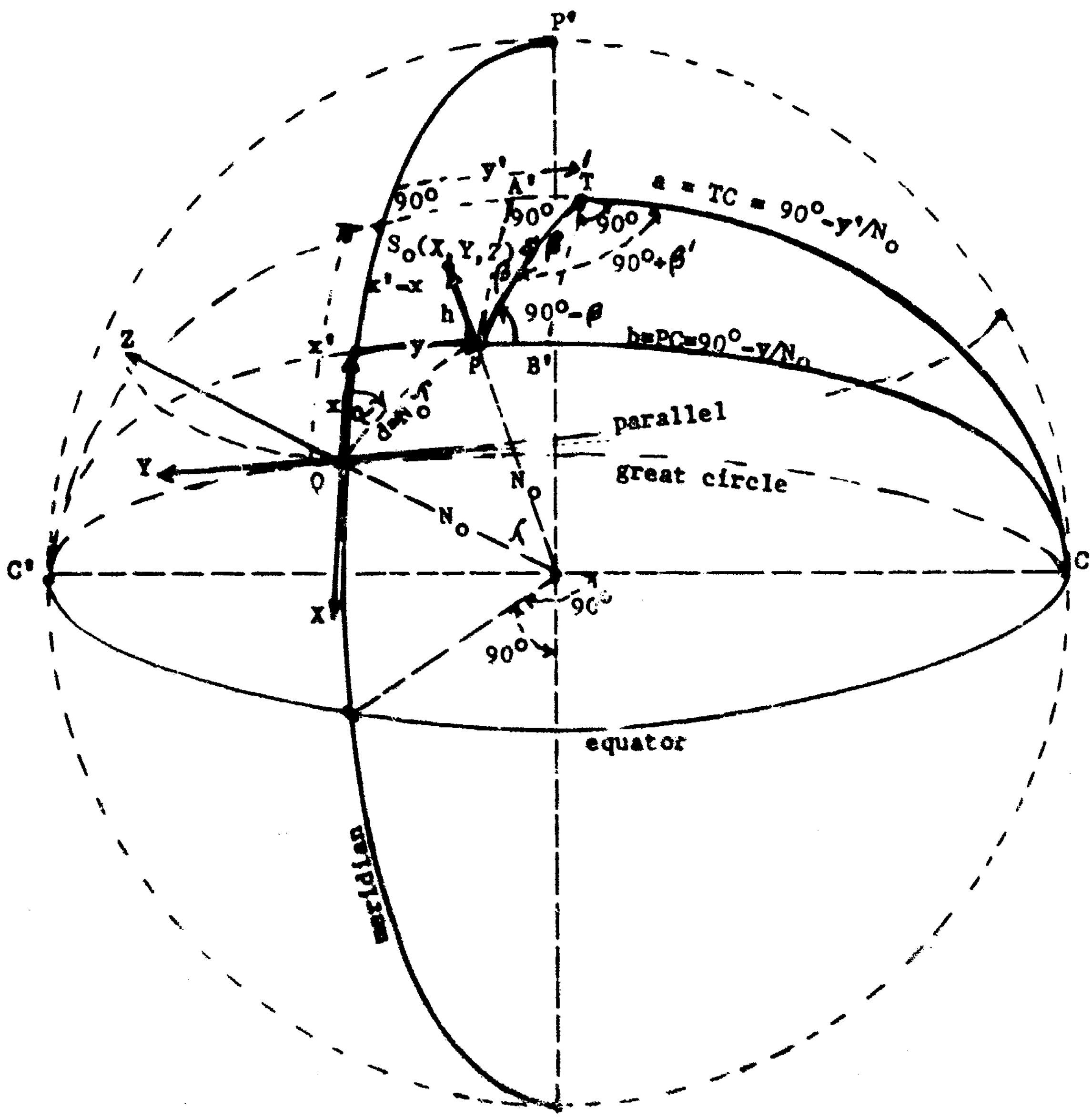


Figure 32. A spherical rectangular space coordinate system x, y, h associated with the rectangular space coordinate system X, Y, Z at a point Q of the surface.

The values from (25) placed in (26) give

$$\tan \tfrac{1}{2}(\beta - \beta') = \tan \frac{1}{2N_o}(x' - x) \sec \frac{1}{2N_o}(y' - y) \sin \frac{1}{2N_o}(y' + y)$$

$$\sin (y'/N_o) = \cos (s/N_o) \sin (y/N_o) + \sin (s/N_o) \cos (y/N_o) \sin \beta \qquad (27)$$

$$\sin \frac{1}{N_o}(x' - x) = \sin (s/N_o) \cos \beta \sec (y'/N_o)$$

For *local coordinate systems* the angles s/N_o, y'/N_o, y/N_o, $(x' - x)/N_o$, $(y' - y)/2N_o$, $(x' - x)/2N_o$, $(y' + y)/2N_o$ are small and we may use the first *two terms of their series expansion, i.e.*

$\sin x = x - x^3/6$, $\cos x = 1 - x^2/2$, $\tan x = x + x^3/3$. Hence we have

$$\sin(y'/N_0) = y'/N_0 - y'^3/6N_0^3,\ \sin(y/N_0) = y/N_0 - y^3/6N_0^3, \tag{28}$$
$$\sin(s/N_0) = s/N_0 - s^3/6N_0^3,\ \cos(s/N_0) = 1 - s^2/2N_0^2,\ \cos(y/N_0) = 1 - y^2/2N_0^2.$$

The values from (28) placed in the second of (27) give

$$y' - y'^3/6N_0^2 = (1 - s^2/2N_0^2)(y - y^3/6N_0^2) + (s - s^3/6N_0^2)(1 - y^2/2N_0^2)\sin\beta$$

$$y' - y'^3/6N_0^2 = y + s\sin\beta - \frac{1}{6N_0^2}(y^3 + 3y^2 s\sin\beta + 3ys^2 + s^3\sin\beta) \tag{29}$$

where the terms in $s^2y^2/12N_0^4$ have been ignored.

Now in (29), if we ignore the terms in $1/6N_0^2$ we have the first term of the series which is $y' = y + s\sin\beta$. We now place this value of y' in the term in y'^3 and write (29) as

$$y' = y + s\sin\beta + \frac{1}{6N_0^2}[(y + s\sin\beta)^3 - y^3 - 3y^2 s\sin\beta - 3ys^2 - s^3\sin\beta]$$

$$= y + s\sin\beta + \frac{1}{6N_0^2}(3ys^2\sin^2\beta - 3ys^2 + s^3\sin^3\beta - s^3\sin\beta)$$

$$y' = y + s\sin\beta - s^2\cos^2\beta(3y + s\sin\beta)/6N_0^2. \tag{30}$$

Similarly from the last of equations (27) we have

$$(x' - x) = \frac{(x' - x)^3}{6N_0^2} + \left(s + \frac{sy'^2}{2N_0^2} - \frac{s^3}{6N_0^2}\right)\cos\beta \tag{31}$$

and if we ignore the terms in $1/N_0^2$ we get as first approximation $x' - x = s\cos\beta$. This value returned to the term in $(x' - x)^3$ in (31) allows us to write

$$x' - x = s\cos\beta + \frac{s\cos\beta}{6N_0^2}(s^2\cos^2\beta + 3y'^2 - s^2)$$

$$= s\cos\beta + \frac{s\cos\beta}{6N_0^2}(3y'^2 - s^2\sin^2\beta) \tag{32}$$

From the first of (27), since

$$\sin\frac{1}{2N_0}(y' + y) = \sin\left[\frac{1}{2N_0}(y' - y) + \frac{y}{N_0}\right],$$

and

$$\sec\frac{1}{2N_0}(y' - y)\sin\frac{1}{2N_0}(y' + y) = \tan\frac{1}{2N_0}(y' - y)\cos\frac{y}{N_0} + \sin\frac{y}{N_0},$$

we may write

$$\tan \tfrac{1}{2}(\beta - \beta') = \tan\frac{1}{2N_0}(x' - x)\left[\tan\frac{1}{2N_0}(y' - y)\cos\frac{y}{N_0} + \sin\frac{y}{N_0}\right] \tag{33}$$

Using $\tan x = x + x^3/3$ and the values of $\sin y/N_0$, $\cos y/N_0$ from (28) we can write the right member of (33) as

$$\left[\frac{x' - x}{2N_0} + \frac{(x' - x)^3}{24N_0^3}\right]\left[\left(\frac{y' - y}{2N_0} + \frac{(y' - y)^3}{24N_0^3}\right)\left(1 - \frac{y^2}{2N_0^2}\right) + \frac{y}{N_0} - \frac{y^3}{6N_0^3}\right]. \tag{34}$$

Retaining terms in $1/N_0^2$, we, from (34), write (33) as

$$\tan \tfrac{1}{2}(\beta - \beta') = \frac{1}{4N_0^2}[(x' - x)(y' - y) + 2(x' - x)y] \tag{35}$$

With $x' - x = s\cos\beta$, $y' - y = s\sin\beta$ from (32) and (30), equation (35) becomes

$$\tan \tfrac{1}{2}(\beta - \beta') = s\cos\beta(2y + s\sin\beta)/4N_0^2 \qquad (36)$$

or

$$\beta' = \beta - 2\arctan\left[s\cos\beta(2y + s\sin\beta)/4N_0^2\right]$$

With $\arctan u = u - u^3/3$, we may write (36) as

$$\beta' = \beta - s\cos\beta(2y + s\sin\beta)/2N_0^2. \qquad (37)$$

Finally, equations (30), (32), and (37) may be written as

$$\left.\begin{aligned} y' &= y + v - u^2(3y + v)/6N_0^2 \\ x' &= x + u\left[1 + (3y'^2 - v^2)/6N_0^2\right] \\ \beta' &= \beta - u(2y + v)/2N_0^2 \sin 1'', \quad u = s\cos\beta,\ v = s\sin\beta \\ &= \beta - u(y + y')/2N_0^2 \sin 1'', \end{aligned}\right\} \qquad (38)$$

since

$$\frac{u}{2N_0^2}(y + y') = \frac{u}{2N_0^2}\left[y + y + v - u^2\frac{(3y + v)}{6N_0^2}\right] = \frac{u}{2N_0^2}(2y + v)$$

If we place $x = y = o$, $\beta = a$, $s = d$, $P \to Q$ and equations (38) become

$$x = d\cos a(1 + d^2\sin^2 a/3N_0^2),\ y = d\sin a(1 - d^2\cos^2 a/6N_0^2) \qquad (38)a$$

$$\beta' = a - d^2\cos a\sin a/2N_0^2 \sin 1''.$$

The terms in $1/N_0^2$ are corrections to plane coordinates. If the ellipsoid is to be taken into account one uses instead of $1/N_0^2$, the value $1/R_0N_0$ which is the square of the mean radius in latitude ϕ_0, see equations (12). Equations (38), in equivalent form, are found in references [15], [32]. *Note that β, β' are not azimuths as usually defined,* that is the lines PA', TB' in Figure 32 are parallels to the meridian.

Transformations between rectangular space coordinates X, Y, Z and local spherical space coordinates x, y, h

In figure 32, if $S_0(X, Y, Z)$ is a point at altitude h above or below the point $P(x, y)$, where x, y are the spherical coordinates as shown, we may from (24) and some formulae for right spherical triangles establish some transformations between the x, y, h system and the X, Y, Z system.

We have

$$\cos\tau = \cos\left(\frac{x}{N_0}\right)\cos\left(\frac{y}{N_0}\right) \approx \left(1 - \frac{x^2}{2N_0^2}\right)\left(1 - \frac{y^2}{2N_0^2}\right) \approx 1 - (x^2 + y^2)/2N_0^2,$$

$$D_2 = (h + N_0)\sin\tau,\ \sin a = \sin\left(\frac{y}{N_0}\right)\Big/\sin\tau,\ \cos a = \tan\left(\frac{x}{N_0}\right)\Big/\tan\tau$$

$$\underline{X = D_2\cos a = (h + N_0)\tan\left(\frac{x}{N_0}\right)}\cos\tau \approx (h + N_0)\left(\frac{x}{N_0} + \frac{x^3}{3N_0^3}\right)\left[1 - (x^2 + y^2)/2N_0^2\right]$$

$$\left.\begin{aligned} &\underline{X \approx x\left[1 + h/N_0 - (x^2 + 3y^2)/6N_0^2\right]} \approx x - (2/3)\frac{x^3}{N_0^2} \\ &Y = D_2\sin a = (h + N_0)\sin\left(\frac{y}{N_0}\right) \approx (h + N_0)\left(\frac{y}{N_0} - \frac{y^3}{6N_0^3}\right) \\ &\underline{Y \approx y\left[1 + h/N_0 - y^2/6N_0^2\right]} \approx y - y^3/6N_0^2 \\ &Z = h\cos\tau - N_0(1 - \cos\tau) \approx h - (N_0 + h)(x^2 + y^2)/2N_0^2 \\ &\underline{Z \approx h - (x^2 + y^2)/2N_0} \end{aligned}\right\} \qquad (39)$$

Now

$$\sin\left(\frac{x}{N_0}\right) = \frac{X}{h+N_0}\sec\left(\frac{y}{N_0}\right)$$

$$\frac{x}{N_0} - \frac{x^3}{6N_0^3} \approx \frac{X}{h+N_0}\left(1+\frac{y^2}{2N_0^2}\right)$$

$$x \approx \frac{X^3}{6N_0^2} + X\left(1+\frac{X^2}{2N_0^2}\right) \approx X + (2/3)\frac{X^3}{N_0^2} \tag{40}$$

$$\sin\left(\frac{y}{N_0}\right) \approx \frac{1}{h+N_0} Y$$

$$\frac{y}{N_0} - \frac{y^3}{6N_0^3} \approx \frac{1}{N_0} Y,\ y \approx Y + Y^3/6N_0^2$$

$$h \approx Z + (x^2+y^2)/2N_0 \approx Z + (X^2+Y^2)/2N_0$$

this last implies $x \approx X \approx D_2 \cos a$, $y \approx Y \approx D_2 \sin a$

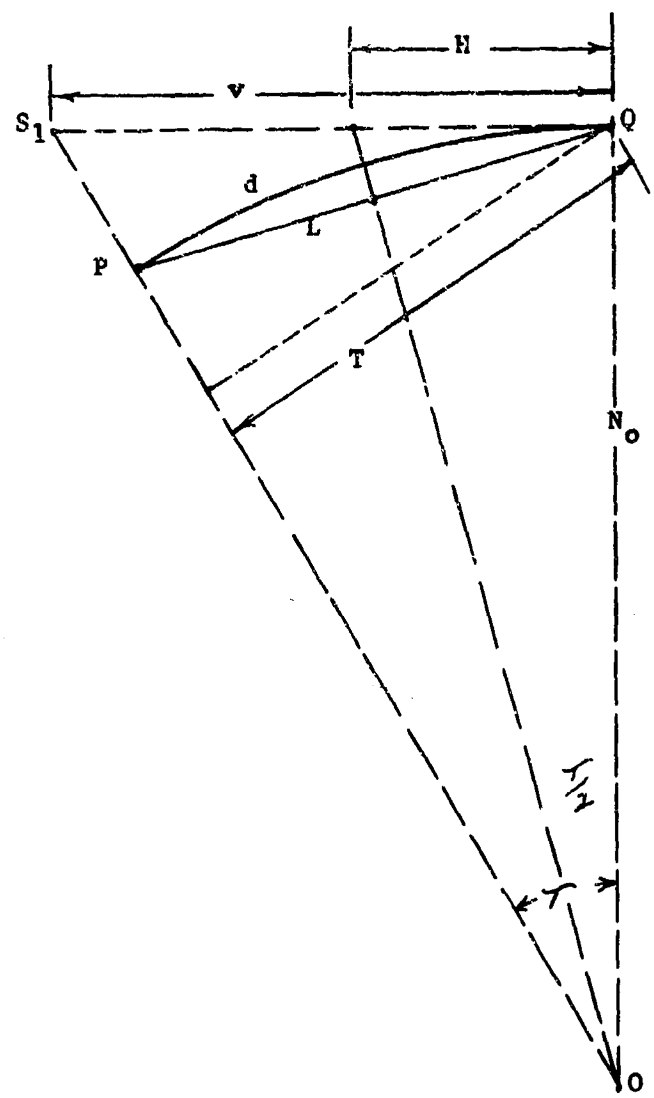


Figure 33. Tangent-arc-chord.

Plane coordinates and map projections

Figure 33 shows the familiar tangent-arc-chord relationship as inherent in the spherical approximation as given in Figure 31. We have the following formulae relating T, v, d, L, H, τ, N_0 as shown in Figure 33:

$$
\begin{aligned}
d &= \tau N_0 = L + L^3/24N_0^2 = v - v^3/3N_0^2 \\
L &= 2N_0 \sin(\tau/2) = N_0(\tau - \tau^3/24) = d - d^3/24N_0^2 = v - 3v^3/8N_0^2 \\
v &= N_0 \tan\tau = N_0(\tau + \tau^3/3) = d + d^3/3N_0^2 = L + 3L^2/8N_0^2 \\
\tau &= d/N_0 = L/N_0 + L^3/24N_0^3 = v/N_0 - v^3/3N_0^3 \\
H &= N_0 \tan \tfrac{1}{2}\tau = \tfrac{1}{2}N_0(\tau + \tau^3/12) = \tfrac{1}{2}(d + d^3/12N_0^2) \\
T &= N_0 \sin\tau = N_0(\tau - \tau^3/6) = d - d^3/6N_0^2 \\
d - L &= d^3/24N_0^2,\ 2H - d = d^3/12N_0^2 = 2(d - L),\ d - T = d^3/6N_0^2 = 4(d - L), \\
v - 2H &= d^3/4N_0^2 = 6(d - L),\ v - d = d^3/3N_0^2 = 8(d - L),\ v - L = (3/8)d^3/N_0^2, \\
v - T &= d^3/2N_0^2 = 12(d - L).
\end{aligned}
\tag{41}
$$

Table 14 gives the differences, the last of equations (41), for arc distances from 10 to 100 n.m. in 5 n.m. increments.

Now in Figure 33 note that S_1 is the linear projection of the point P upon the tangent plane at Q from the spherical center 0. Such projection is *called gnomonic*. Since the tangent, v, is the projection of the great circle arc, d, upon the tangent plane, any straight line through Q in the tangent plane represents a great circle on the sphere.

From equations (24), (26) we have, with Z = 0, h = c, see Figure 31,

$$
\begin{aligned}
v &= D_2 = N_0 \tan\tau,\ \sin\alpha = \cos\phi \sin\Delta\lambda/\sin\tau, \\
\cos\alpha &= (\sin\phi - \cos\tau \sin\phi_0)/\cos\phi_0 \sin\tau, \\
X &= D_2 \cos\alpha = \sin\phi \sec\tau - \sin\phi_0,\ Y = D_2 \sin\alpha = \cos\phi \sin\Delta\lambda \sec\tau, \\
\sec\tau &= 1/(\sin\phi \sin\phi_0 + \cos\phi \cos\phi_0 \cos\Delta\lambda), \\
\cos\tau &= N_0/(X^2 + Y^2 + N_0^2)^{1/2},\ \sin\phi = (X + \sin\phi_0)\cos\tau \\
\sin\Delta\lambda &= Y \cos\tau/\cos\phi,\ \lambda = \lambda_0 - \Delta\lambda.
\end{aligned}
\tag{42}
$$

Equations (42) thus give the plane coordinates X, Y as functions of the geographic coordinates of the points P, Q, that is of ϕ, ϕ_0, $\Delta\lambda = \lambda_0 - \lambda$. The last of equations (42) show the solution for the geographic coordinates ϕ, λ of P when the plane coordinates X, Y of S, referenced to the tangents to the meridian and parallel at Q, *are given, assuming S is gnomonically projected.*

Now if we let $v = D_2 = d = N_0 \tau$, the resulting plane coordinates map a *Lambert azimuthal equidistant projection* on the tangent plane; if $v = D_2 = L = 2N_0 \sin \tau/2$ the resulting projection is the *Lambert azimuthal authalic (equal area)*; if $v = D_2 = T = N_0 \sin\tau$ the projection *is orthographic*, points P are projected on the tangent plane at Q by lines parallel to OQ, see Figure 33; if $D_2 = 2H = 2N_0 \tan \tau/2$ the projection *is stereographic*, angles are preserved about each point of the projection (conformal or autogonal).

The last four columns of Table 14 show the error in the radius $v = D_2$ about Q when we allow $v = D_2$ to be 2H, d, L, or T, i.e. the point P to be projected upon the tangent plane at Q stereographically, equidistantly, equal-areally, or orthographically.

d n.m.	m	d - L m	2H - d m	d - T m	v - 2H m	v - d m	v - L m	v - T m
10	18520	.01	.02	.04	.06	.08	.09	.12
15	27780	.02	.04	.08	.12	.16	.18	.24
20	37040	.05	.10	.20	.30	.40	.45	.60
25	46300	.10	.20	.40	.60	.80	.90	1.20
30	55560	.17	.34	.72	1.06	1.44	1.61	2.12
35	64820	.27	.54	1.08	1.62	2.16	2.43	3.24
40	74080	.41	.82	1.64	2.46	3.28	3.69	4.92
45	83340	.58	1.16	2.32	3.48	4.64	5.22	6.96
50	92600	.79	1.58	3.16	4.74	6.32	7.11	9.48
55	101860	1.06	2.12	4.24	6.36	8.48	9.54	12.72
60	111120	1.37	2.74	5.48	8.22	10.96	12.33	16.44
65	120380	1.74	3.48	6.96	10.44	13.92	15.66	20.88
70	129640	2.18	4.36	8.72	13.08	17.44	19.62	26.16
75	138900	2.68	5.36	10.72	16.08	21.44	24.12	32.16
80	148160	3.25	6.50	13.00	19.50	26.00	29.25	39.00
85	157420	3.90	7.80	15.60	23.40	31.20	35.10	46.80
90	166680	4.63	9.26	18.52	27.78	37.04	41.67	55.56
95	175940	5.45	10.90	21.80	32.70	43.60	49.05	65.40
100	185200	6.35	12.70	25.40	38.10	50.80	57.15	76.20

Table 14. Differences for d, L, 2H, T, v from equations (41).

Now from equations (24), (26), (41), (42) we have

$$U = \cos\tau = \sin\phi \sin\phi_0 + \cos\phi \cos\phi_0 \cos\Delta\lambda$$

$$V = \sin\phi \cos\phi_0 - \cos\phi \sin\phi_0 \cos\Delta\lambda, \; W = \cos\phi \sin\Delta\lambda, \tag{43}$$

$$\sin a = W/\sin\tau, \; \cos a = V/\sin\tau$$

With the help of (43) we can express the rectangular plane coordinates of the several projections as functions of U, V, W and hence of the geographical coordinates of P and Q, i.e. as functions of ϕ, ϕ_0, $\Delta\lambda = \lambda_0 - \lambda$.

Gnomonic. $D_2 = N_0 \tan\tau$

$$X = D_2 \cos a = N_0 \tan\tau \cos a = \frac{N_0 \sin\tau}{\cos\tau} \cdot \frac{V}{\sin\tau} = N_0 V/U$$

$$Y = D_2 \sin a = N_0 \tan\tau \sin a = \frac{N_0 \sin\tau}{\cos\tau} \cdot \frac{W}{\sin\tau} = N_0 W/U$$

Azimuthal equidistant. $D_2 = d = N_0\tau$

$$X = D_2 \cos a = s\cos a = N_0\tau\cos a = N_0\tau V/\sin\tau$$
$$= N_0 V \operatorname{arc}\cos U/\sin(\operatorname{arc}\cos U)$$
$$Y = D_2 \sin a = s\sin a = N_0\tau\sin a = N_0\tau W/\sin\tau$$
$$= N_0 W \operatorname{arc}\cos U/\sin(\operatorname{arc}\cos U)$$

Azimuthal equal area (authalic). $D_2 = L = 2N_0 \sin \tau/2 = N_0 \sin\tau/[\tfrac{1}{2}(1+\cos\tau)]^{1/2}$

$$X = D_2 \cos a = \frac{N_0 \sin\tau}{[\frac{1}{2}(1+\cos\tau)]^{1/2}} \cdot \frac{V}{\sin\tau} = N_0 V/[\tfrac{1}{2}(1+U)]^{1/2}$$

$$Y = D_2 \sin a \frac{N_0 \sin\tau}{[\frac{1}{2}(1+\cos\tau)]^{1/2}} \cdot \frac{W}{\sin\tau} = N_0 W/[\tfrac{1}{2}(1+U)]^{1/2}$$

Orthographic. $D_2 = T = N_0 \sin\tau$

$$X = N_0 \sin\tau\cos a = N_0 \sin\tau \frac{V}{\sin\tau} = N_0 V \tag{44}$$

$$Y = N_0 \sin\tau \sin a = N_0 \sin\tau \frac{W}{\sin\tau} = N_0 W$$

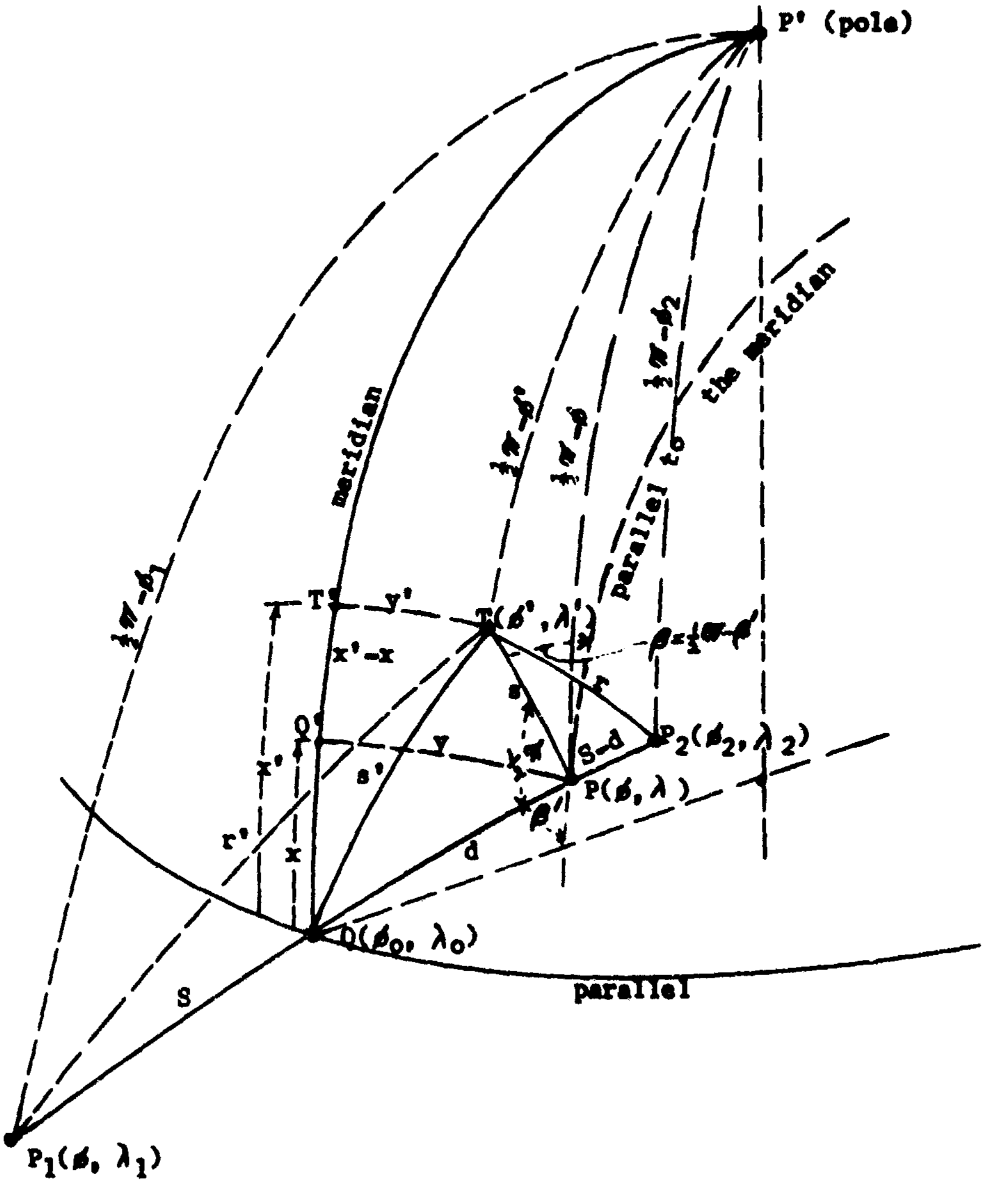
Stereographic. $D_2 = 2H = 2N_0 \tan\tfrac{1}{2}\tau = 2N_0 \sin\tau/(1+\cos\tau)$

$$X = \frac{2N_0 \sin\tau}{1+\cos\tau}\cos a = \frac{2N_0 \sin\tau}{1+\cos\tau} \; \frac{V}{\sin\tau} = 2N_0 V/(1+U)$$

$$Y = \frac{2N_0 \sin\tau}{1+\cos\tau}\sin a = \frac{2N_0 \sin\tau}{1+\cos\tau} \; \frac{W}{\sin\tau} = 2N_0 W/(1+U)$$

Spherical coordinates relative to a great circle arc determined by two given points of the sphere

In Figure 34, Q is the midpoint (but may be any point) of the great circle arc determined by two given points $P_1(\phi_1, \lambda_1)$, $P_2(\phi_2, \lambda_2)$ of length 2S. The azimuth at Q is a and Q is taken as origin of the



Q is the midpoint of the great circle arc P_1P_2.

Figure 34. Spherical coordinates referred to the meridian through the midpoint of a general great circle arc.

spherical coordinate system as shown. At an arbitrary point P of P_1P_2, at distance d from Q, a perpendicular PT = s is constructed. Note that $\beta = \pi/2 - \beta_1$, hence the spherical rectangular coordinates x', y' of T may be computed using the value of x and y from (38)a and the value $\beta = \pi/2 - \beta'$, where β' is given by the expression in (38)a, i.e.

$$y' = y + v - u^2(3y + v)/6N_0^2, \quad x' = x + u[1 + (3y'^2 - v^2)]/6N_0^2, \tag{45}$$

where $u = s\cos\beta$, $v = s\sin\beta$, $\beta = \pi/2 - \alpha + d^2\cos\alpha\sin\alpha/2N_0^2\sin 1''$, $x = d\cos\alpha(1 + d^2\sin^2\alpha/3N_0^2)$, $y = d\sin\alpha(1 - d^2\cos^2\alpha/6N_0^2)$.

Note that in *the mapping of spherical coordinates*, the y-coordinates are laid off perpendicular to the central meridian, which causes an increase in the latitude scale as the distance from the central meridian increases. *The magnification is given by* $K = 1 + [y + y')^2 - yy'] \cos^2 \beta/6N_0^2$ and for short lines we may let $y = y'$, giving $K = 1 + y^2 \cos^2 \beta/2N_0^2$. When $\beta = \pi/2$, $K = 1$, and the map gives then true longitude differences. When $\beta = 0$, K is maximum with the value $K = 1 + y^2/2N_0^2$.

Formulae relating spherical coordinates to geographic coordinates

The given reference line P_1P_2 of Figure 34, having been already established, we may wish to compute the distance s to P_1P_2 from an arbitrary point $T(\phi', \lambda')$, or given s, find geographical coordinates of $T(\phi', \lambda')$ and $P(\phi, \lambda)$ at a given distance d from Q along P_1P_2. From the right spherical triangles TPP_1, QTT', QPT, QPQ', $TT'P'$, TPP_2, we have

$$\cos s' = \cos x' \cos y' = \cos s \cos d = \cos s \cos x \cos y \tag{46}$$

$$\tan x = \tan d \cos \alpha, \ \sin y = \sin d \sin \alpha, \ \tan s = \tan (\alpha - \alpha') \sin d,$$

$$\sin \phi' = \cos y' \sin (x' + \phi_0), \ \sin y' = \cos \phi' \sin (\lambda' - \lambda_0),$$

$$\sin s = \sin (\alpha - \alpha') \sin s',$$

$$\cos (\phi_0 + x') = \tan y' \cot (\lambda' - \lambda_0), \ \tan x' = \tan s' \cos \alpha',$$

$$\tan d = \cos (\alpha - \alpha') \tan s',$$

$$\sin y' = \sin s' \sin \alpha', \ \cos r = \cos s \cos (S - d), \ \cos r' = \cos S \cos (S + d).$$

Since P_1, Q, P_2 are fixed, the constants $2S = P_1P_2, \alpha_{1\text{-}2}, \alpha_{2\text{-}1}, \alpha, \phi_1, \phi_2, \phi_0, \lambda_1, \lambda_0, \lambda_2$, are known. Some of the oblique spherical triangles involving these known parameters and the coordinates of T and P are, P_1TP_2, QTP_2, $P'QT$, $P'QP$, $P_1P'T$, $P'TP_2$, $PP'P_2$, $PP'P_1$, $TP'P$. From these we obtain the following spherical formulae from the sine and cosine laws for spherical triangles:

$P'TP_2$: $\quad \cos r = \sin \phi' \sin \phi_2 + \cos \phi' \cos \phi_2 \cos (\lambda_2 - \lambda')$

$P'QT$: $\quad \cos s' = \sin \phi_0 \sin \phi' + \cos \phi_0 \cos \phi' \cos (\lambda' - \lambda_0)$

$\quad \sin s' \sin \alpha' = \cos \phi' \sin (\lambda' - \lambda_0)$

$\quad \sin \phi' = \cos s' \sin \phi_0 + \sin s' \cos \phi_0 \cos \alpha'$

$\quad \sin \phi = \sin \phi_0 \cos d + \sin d \cos \phi_0 \cos \alpha$

$P'QP$: $\quad \cos d = \sin \phi_0 \sin \phi + \cos \phi_0 \cos \phi \cos (\lambda - \lambda_0)$

$\quad \sin d \sin \alpha = \cos \phi \sin (\lambda - \lambda_0)$

QTP_2: $\quad \cos r = \cos s' \cos S + \sin s' \sin S \cos (\alpha - \alpha')$ (47)

P_1TP_2: $\quad \cos 2S = \cos r \cos r' + \sin r \sin r' \cos (\lambda_2 - \lambda_1)$

$PP'P_1$: $\quad \cos (S + d) = \sin \phi_1 \sin \phi + \cos \phi_1 \cos \phi \cos (\lambda - \lambda_1)$

$\quad \sin \alpha_{1\text{-}2} \sin (S + d) = \cos \phi \sin (\lambda - \lambda_1)$

$PP'P_2$: $\quad \cos (S - d) = \sin \phi_2 \sin \phi + \cos \phi_2 \cos \phi \cos (\lambda_2 - \lambda)$

$\quad \cos \phi \sin (\lambda_2 - \lambda) = -\sin \alpha_{2\text{-}1} \sin (S - d)$

$P_1P'T$: $\quad \cos r' = \sin \phi_1 \sin \phi' + \cos \phi_1 \cos \phi' \cos (\lambda' - \lambda_1)$

$TP'P$: $\quad \cos s = \sin \phi \sin \phi' + \cos \phi \cos \phi' \cos (\lambda - \lambda')$

From (38), (38)a, (45), (46), (47) we have the formulae to handle most of the geometric problems which may occur in the local geometry of a given base line. *For instance if* s is constant but d varies, then β varies, and the rectangular coordinates x', y', as given by (45), give points T on the parallel at distance s from the base line P_1P_2.

Suppose that we are given the geographic coordinates ϕ', λ' of an arbitrary point T to *find the perpendicular distance s to the base line, the geographic coordinates ϕ, λ of the foot, P, of the perpendicular, and the distance d from the origin Q to P.* Now the known constants are $S = (1/2)P_1P_2$, a_{1-2}, a_{2-1}, a, ϕ', ϕ_1, ϕ_2, ϕ_0, λ_1, λ_0, λ_2, λ' and we are to find s, d, ϕ, λ. From (46) we have

$$\begin{aligned} \cos(S+d) &= \cos S \cos d - \sin S \sin d = \cos r'/\cos s \\ \cos(S-d) &= \cos S \cos d + \sin S \sin d = \cos r/\cos s \end{aligned} \qquad (48)$$

Adding and subtracting respective members of (48) get

$$\begin{aligned} \cos s \sin d &= (\cos r - \cos r')/2 \sin S \\ \cos s \cos d &= (\cos r + \cos r')/2 \cos S \end{aligned} \qquad (49)$$

Dividing respective members of (49) we find

$$\underline{\tan d} = \cot S (\cos r - \cos r')/(\cos r + \cos r') \qquad (50)$$

where from (47), triangles $P'TP_2$, $P'TP_1$

$$\cos r = \sin\phi' \sin\phi_2 + \cos\phi' \cos\phi_2 \cos(\lambda_2 - \lambda')$$

$$\cos r' = \sin\phi' \sin\phi_1 + \cos\phi' \cos\phi_1 \cos(\lambda' - \lambda_1)$$

From (46) and triangle P'QT of (47) we have

$$\begin{aligned} &\underline{\cos s} = \cos s'/\cos d, \\ &\cos s' = \sin\phi' \sin\phi_0 + \cos\phi' \cos\phi_0 \cos(\lambda' - \lambda_0). \end{aligned} \qquad (51)$$

From triangle P'QP of (47) we have

$$\begin{aligned} &\underline{\sin\phi} = \sin\phi_0 \cos d + \cos\phi_0 \sin d \cos a \\ &\underline{\sin(\lambda - \lambda_0)} = \sin d \sin a/\cos\phi \text{ or } \underline{\lambda} = \arcsin[\sin d \sin a/\cos\phi] + \lambda_0. \end{aligned} \qquad (52)$$

Note also a type of spherical rectangular coordinate system *referenced to the base line and a great circle orthogonal to the base line at its midpoint as presented in Reference [18].*

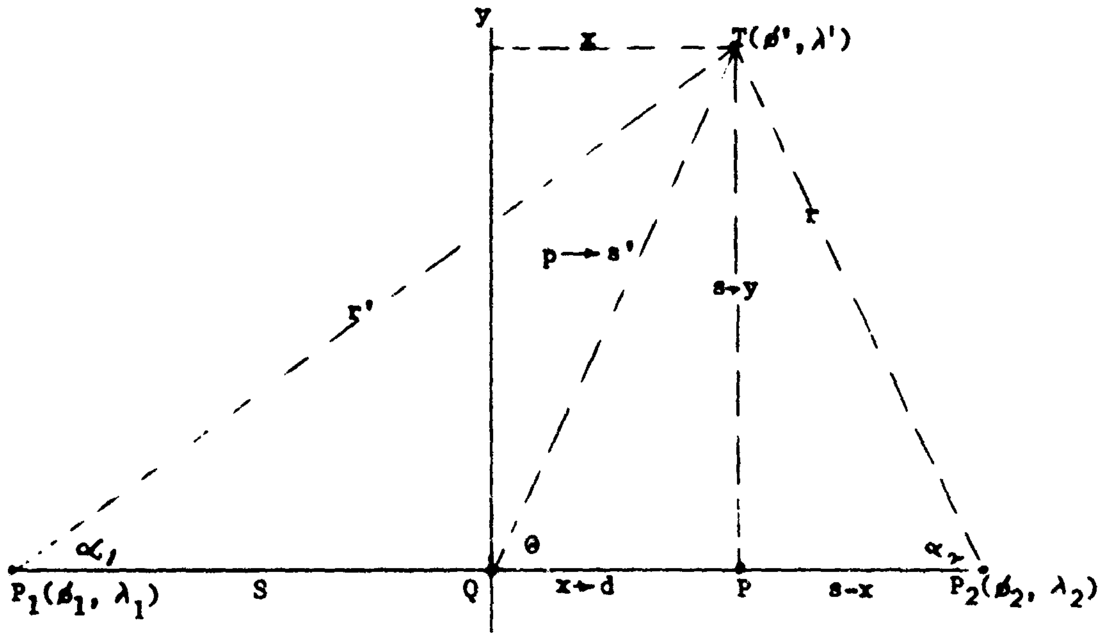
The doubly equidistant projection

This is a useful projection for investigators in fields such as seismology (earthquakes or microseisms), meteorology (long range location of cyclone trajectories), electronic distance measuring systems as Hiran or Shiran, location of aurorae or of meteors, studies of water waves, tsunamis or swell, oceanography. It is obtained by constructing the spheroidal (spherical) triangle P_1P_2T, of Figure 34, in the plane as shown in Figure 35. The true length (to scale) of the base line $P_1P_2 = 2S$ is drawn as a straight line in the plane. Points T are located with respect to the base line from the intersection of circular arcs about P_1, P_2 with radii the true lengths (scaled) of r', r. Either spheroidal or spherical distances for S, r', r may be used. The projection is not conformal, that is angles are not preserved about every point of the projection.

The equations relating the several parameters as shown in Figure 35 are:

$$x = p\cos\theta = \frac{1}{2S}(S^2 + p^2 - r^2) = \frac{1}{4S}(r'^2 - r^2) = r'\cos a_1 - S = S - r\cos a_2,$$

$$y = x\tan\theta = p\sin\theta = \pm(p^2 - x^2)^{1/2} = r'\sin a_1 = r\sin a_2 = \pm\frac{1}{4S}[16S^2r'^2 - (r'^2 - r^2 + 4S^2)^2]^{1/2}.$$



Given distances on the spheroid (sphere) are S, r', r.

Figure 35. The doubly equidistant projection.

$$p^2 = \tfrac{1}{2}(r'^2 + r^2) - S^2, \cos\theta = x/P = \frac{1}{4pS}(r'^2 - r^2) = \frac{1}{2pS}(S^2 + p^2 - r^2),$$

$$\cos\alpha_1 = (4S^2 + r'^2 - r^2)/4r'S, \cos\alpha_2 = (4S^2 - r'^2 + r^2)/4rS \tag{53}$$

and where for the spherical case, we have from (47)

$$r = \arc\cos[\sin\phi' \sin\phi_2 + \cos\phi' \cos\phi_2 \cos(\lambda_2 - \lambda')]$$

$$r' = \arc\cos[\sin\phi' \sin\phi_1 + \cos\phi' \cos\phi_1 \cos(\lambda' - \lambda_1)]$$

$$2S = \arc\cos[\sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 \cos(\lambda_2 - \lambda_1)]$$

Discussion. The doubly equidistant projection may also be called bi-polar, since we have two radii r', r from two foci P_1, P_2 and the angles α_1, α_2. We have given, or compute, the values S, r', r and then from (53)

$$\begin{aligned} &\cos\alpha_1 = (4S^2 + r'^2 - r^2)/4r'S, x = r'\cos\alpha_1 - S, y = r'\sin\alpha_1, \\ &\cos\alpha_2 = (4S^2 - r'^2 + r^2)/4rS, x = S - r\cos\alpha_2, y = r\sin\alpha_2 \end{aligned} \tag{54}$$

which provide a check for rectangular coordinate computation.

Given the rectangular coordinates x, y of the point T on the doubly equidistant projection and the constants S, α, ϕ_0, λ_0 of the base line as shown in Figure 34: find the geographic coordinates, ϕ', λ', of the point T on the sphere.

From x, y, S we have

$$r' = [(x + S)^2 + y^2]^{1/2}, r = [(S - x)^2 + y^2]^{1/2},$$

From (50) we have

$$\tan d = \cot S\,(\cos r - \cos r')/(\cos r + \cos r'), \tag{55}$$

From (46) find

$$\cos s' = \cos r' \cos d/\cos(S + d) = \cos r \cos d/\cos(S - d),$$

From (46) (QTP) we have

$$\alpha' = \alpha - \text{arc cos}\,(\tan d/\tan s'),$$

Finally from P'QT of (47) find

$$\phi' = \text{arc sin}\,(\cos s' \sin \phi_0 + \sin s' \cos \phi_0 \cos \alpha'),$$

and

$$\lambda' = \lambda_0 + \text{arc sin}\,(\sin s' \sin \alpha'/\cos \phi').$$

From equation (50) we have

$$\tan d = \frac{\cos r - \cos r'}{\tan S\,(\cos r + \cos r')} = \frac{\cos^2 r - \cos^2 r'}{\tan S\,(\cos r + \cos r')^2} = \frac{\sin^2 r' - \sin^2 r}{\tan S\,(\cos r + \cos r')^2}. \tag{56}$$

Considering d/N_0, r/N_0, r'/N_0, S/N_0 to be small enough to place $\tan d/N_0 = d/N_0$, $\tan S/N_0 = S/N_0$, $\cos r = \cos r' = 1$, $\sin r' = r'$, $\sin r = r$, then (56) becomes $d = (r'^2 - r^2)/4S$ = x-coordinate of the doubly equidistant projection, equations (53). From (46) (QTP) we have $\tan s = \tan(\alpha - \alpha') \sin d$, and if we place $\sin d/N_0 = d/N_0$, $\tan s/N_0 = s/N_0$ then $s = \tan(\alpha - \alpha')d$. But then d is x of Figure 35, $\alpha - \alpha' = \theta$, and s is the y-coordinate of the doubly equidistant projection, $y = x \tan \theta$, see figure 35 and equations (53).

The world geodetic reference system 1967

The International Union of Geodesy and Geophysics has tentatively adopted a new geodetic reference system, see reference [30]. It is defined by the three constants: equatorial radius $a = 6378160$ *meters;* earth geocentric gravitational constant including the atmosphere $GM = 398603 \times 10^9 m^3 s^{-2}$; earth dynamical form factor $J_2 = 10827 \times 10^{-7}$.

Now the earth's rotational velocity is given by

$$\omega = \{2\pi(1 + s_1/86400)/[s_1 + p \cos \epsilon/1500]\}\ s^{-1}, \text{ where}$$

$s_1 = 31556925.9747$ (ephermeris seconds in one tropical year, 1900)

$p = 5025.64$ (seconds general precession in longitude per tropical century, 1900)

$\epsilon = 23^\circ\ 27'\ 08''\!.26$ (obliquity of the ecliptic, 1900)

$$\omega = 7.292115144 \times 10^{-5}\ s^{-1}.$$

Since $J_2 = e'^2(1 - 2me'/15q_0)/3(1 + e'^2)$, where

$$m = \omega^2 a^3/GM(1 + e'^2)^{1/2},\quad q_0 = \tfrac{1}{2}\{(1 + 3/e'^2)\,\text{arc tan}\, e' - 3/e'\},$$

and a, GM, J_2, ω are known we may solve for e',

$$e' = 0.08209582892 \text{ (the second eccentricity).}$$

We can then solve for b, 1/f, and e^2:

$$b = a/(1 + e'^2)^{1/2} = 6356774.516\ m;\quad 1/f = a/(a - b) = 298.2471675.$$

$$e^2 = (a^2 - b^2)/a^2 = 2f - f^2 = .006694605326$$

The formulae for gravity at the pole; equator; general (normal) are:

$$g_p = GM(1 + me'q_0'/3q_0)/a^2,\quad g_e = GM(1 - m - me'q_0'/6q_0)/ab$$

$$g = (ag_e \cos^2 \phi + bg_p \sin^2 \phi)/(a^2 \cos^2 \phi + b^2 \sin^2 \phi)^{1/2},$$

where $q'_0 = 3(1 + 1/e'^2)\ (1 - \frac{1}{e'} \text{ arc tan } e') - 1.$

With the above values of constants, these become:

$g_p = 983.2177279$ *gal*, $g_e = 978.0318456$ *gal*

$g = 978.0318\ (1 + .0053024 \sin^2\phi - .0000058 \sin^2 2\phi)$ gal

(this last the series expansion of the above general formula).

The numerical values are preliminary. The official values will be published by the International Association of Geodesy.

Summary Values

Given $a = 6378160$ m, $GM = 398603 \times 10^9\ m^3 s^{-2}$, $J_2 = 10827 \times 10^{-7}$

Computed $\omega = 7.292115144 \times 10^{-5} s^{-1}$, $e^2 = 0.006694605326$,

$e'^2 = 0.006739725126$, $g_p = 983.2177279$ gal,

$g_e = 978.0318456$ gal, $1/f = 298.2471675$, $b = 6356774.516$ m

$g = 978.0318\ (1 + .0053024 \sin^2\phi - .0000058 \sin^2 2\phi)$ gal

Comment. A comparison with the AUSTRALIAN ellipsoidal constants, Table 10, shows that, for practical geodetic purposes, using 8-place tables, *we may use the AUSTRALIAN ellipsoid.* Tables of meridian arc length, principal radii of curvature, and latitude functions have been published for several spheroids by the Department of the Army as technical manuals. For instance, CLARKE 1866 (TM5-241-18), INTERNATIONAL (AMS TM-67); AUSTRALIAN (TM5-241-33); FISCHER (MERCURY) (TM5-241-35).

If we take $a = 6378160$ m, $\omega = 7.292115 \times 10^{-5} s^{-1}$, $1/f = 298.25$,

$f = .3352892 \times 10^{-2}$, $g_e = 978.0318$ cm s^{-2}, we find $m = \omega^2 a/g_e = .3467753 \times 10^{-2}$,

$fm = .1162703 \times 10^{-4}$. From reference [31], page 366, we have

$g = g_e[1 + \beta \sin^2\phi - \beta_1 \sin^2 2\phi]$, where

$\beta = (5/2)m - f - (17/14)mf$, $\beta_1 = (5/8)mf - (1/8)f^2$

With the stated values of f, m, and mf we have

$\beta = .0053024$, $\beta_1 = .00000586$, and

$g = 978.0318[1 + .0053024 \sin^2\phi - .00000586 \sin^2 2\phi]$ gal.

This gravity formula gives only $.06 \sin^2 2\phi$ mgal less than the recommended formula above, the maximum difference being .06 mgal at $\phi = 45°$.

Appendix 3.

THE ACIC CHECK LINES, 50-6000 MILES, AND GEODETIC LINE COMPUTATIONS

ACIC CHECK LINES

Taken directly from ACIC Technical Reports 59, 80. See bibliographical reference [22]. The 500 mile lines are repeated since they were given in both publications.

ORIGIN AND TERMINAL POSITIONS OF LINES

AZIMUTH	ORIGIN		50 MILES		100 MILES		200 MILES	
	LAT	LONG	LATITUDE	LONGITUDE	LATITUDE	LONGITUDE	LATITUDE	LONGITUDE
0°	10°	18°	10°43'39".078	18°	11°27'18".032	18°	12°54'35".538	18°
	40°	18°	40°43'28".790	18°	41°26'57".248	18°	42°53'53".164	18°
	70°	18°	70°43'16".379	18°	71°26'32".550	18°	72°53'04".295	18°
45°	10°	18°	10°30'50".497	17°28'48".777	11°01'37".857	16°57'31".358	12°03'02".498	15°54'36".649
	40°	18°	40°30'37".757	17°19'43".280	41°01'01".097	16°38'49".777	42°01'02".610	15°15'08".672
	70°	18°	70°30'12".925	16°28'22".844	70°59'37".295	14°52'09".888	71°55'44".745	11°25'02".986
90°	10°	18°	9°59'57".087	17°15'57".926	9°59'48".349	16°31'55".877	9°59'13".405	15°03'51".963
	40°	18°	39°59'46".211	17°03'27".942	39°59'04".850	16°06'56".642	39°56'19".507	14°13'59".336
	70°	18°	69°59'15".149	15°53'37".449	69°57'00".764	13°47'32".949	69°48'05".702	9°37'28".707

	ORIGIN		300 MILES		400 MILES		500 MILES	
	LAT	LONG	LATITUDE	LONGITUDE	LATITUDE	LONGITUDE	LATITUDE	LONGITUDE
0°	10°	18°	14°21'52".456	18°	15°49'08".725	18°	17°16'24".286	18°
	40°	18°	44°20'47".740	18°	45°47'40".974	18°	47°14'32".867	18°
	70°	18°	74°19'35".289	18°	75°46'05".589	18°	77°12'35".253	18°
45°	10°	18°	13°04'12".564	14°51'13".283	14°05'06".663	13°47'18".635	15°05'43".367	12°42'50".044
	40°	18°	43°00'00".556	13°48'49".111	43°57'50".690	12°19'43".420	44°54'28".506	10°47'43".884
	70°	18°	72°47'48".242	07°36'58".487	73°35'09".210	3°26'35".190	74°17'05".184	-1°06'51".561
90°	10°	18°	9°58'15".192	13°35'48".467	9°56'53".751	12°07'45".595	9°55'09".138	10°39'43".554
	40°	18°	39°51'44".295	12°21'14".090	39°45'19".750	10°28'46".813	39°37'06".613	8°36'43".277
	70°	18°	69°33'22".562	5°32'01".822	69°13'03".648	01°33'11".478	68°47'25".009	-2°17'23".583

Latitudes north; longitudes west, except those prefixed by a minus sign.

ACTUAL DISTANCE OF LINES

CLARKE 1866 ELLIPSOID

Latitude	Azimuth	50 Miles	100 Miles	200 Miles	300 Miles	400 Miles	500 Miles
10°	0°	80,466.478	160,932.956	321,865.912	482,798.868	643,731.824	804,664.780
40°	0°	80,466.478	160,932.956	321,865.912	482,798.868	643,731.824	804,664.780
70°	0°	80,466.478	160,932.956	321,865.912	482,798.868	643,731.824	804,664.780
10°	45°	80,466.477	160,932.956	321,865.912	482,798.868	643,731.824	804,664.780
40°	45°	80,466.478	160,932.955	321,865.911	482,798.868	643,731.824	804,664.780
70°	45°	80,466.478	160,932.956	321,865.912	482,798.868	643,731.824	804,664.780
10°	90°	80,466.476	160,932.956	321,865.912	482,798.868	643,731.824	804,664.780
40°	90°	80,466.477	160,932.955	321,865.912	482,798.868	643,731.824	804,664.780
70°	90°	80,466.478	160,932.956	321,865.912	482,798.868	643,731.824	804,664.780

ACTUAL FORWARD AZIMUTH OF LINES

Latitude	Azimuth	50 Miles	100 Miles	200 Miles	300 Miles	400 Miles	500 Miles
10°	0°	00°00'00".000	00°00'00".000	00°00'00".000	00°00'00".000	00°00'00".000	00°00'00".000
40°	0°	00°00'00".000	00°00'00".000	00°00'00".000	00°00'00".000	00°00'00".000	00°00'00".000
70°	0°	00°00'00".000	00°00'00".000	00°00'00".000	00°00'00".000	00°00'00".000	00°00'00".000
10°	45°	44°59'59".999	45°00'00".000	45°00'00".000	45°00'00".000	45°00'00".000	45°00'00".000
40°	45°	45°00'00".000	45°00'00".000	44°59'59".999	44°59'59".999	45°00'00".000	45°00'00".000
70°	45°	45°00'00".000	45°00'00".000	45°00'00".000	45°00'00".000	45°00'00".000	45°00'00".000
10°	90°	90°00'00".000	90°00'00".001	90°00'00".000	90°00'00".000	90°00'00".000	90°00'00".000
40°	90°	90°00'00".000	90°00'00".000	90°00'00".000	90°00'00".000	90°00'00".000	90°00'00".000
70°	90°	90°00'00".001	90°00'00".000	90°00'00".000	90°00'00".000	90°00'00".000	90°00'00".000

ACTUAL BACK AZIMUTH OF LINES

Latitude	Azimuth	50 Miles	100 Miles	200 Miles	300 Miles	400 Miles	500 Miles
10°	0°	180°00'00".000	180°00'00".000	180°00'00".000	180°00'00".000	180°00'00".000	180°00'00".000
40°	0°	180°00'00".000	180°00'00".000	180°00'00".000	180°00'00".000	180°00'00".000	180°00'00".000
70°	0°	180°00'00".000	180°00'00".000	180°00'00".000	180°00'00".000	180°00'00".000	180°00'00".000
10°	45°	225°05'33".200	225°11'24".056	225°23'59".176	225°37'46".346	225°52'46".641	226°09'01".224
40°	45°	225°26'01".695	225°52'43".715	226°48'12".147	227°46'32".222	228°47'59".982	229°52'15".525
70°	45°	226°26'13".935	227°57'04".162	231°13'26".981	234°50'49".050	238°50'31".359	243°13'18".356
10°	90°	270°07'38".786	270°15'17".480	270°30'34".337	270°45'49".945	271°01'03".684	271°16'14".933
40°	90°	270°36'20".315	271°12'39".796	272°25'12".925	273°37'32".768	274°49'32".801	276°01'06".777
70°	90°	271°58'45".079	273°57'12".072	277°52'01".046	281°42'12".088	285°25'45".725	289°01'02".923

ORIGIN AND TERMINAL POSITIONS OF ALL TEST LINES

AZIMUTH	ORIGIN		500 MILES		1000 MILES	
	LAT	LONG	LATITUDE	LONGITUDE	LATITUDE	LONGITUDE
0°	10° N	18° W	17° 16' 24".286N	18° W	24° 32' 29".539N	18° W
	40° N	18° W	47° 14' 32".867N	18° W	54° 28' 32".474N	18° W
	70° N	18° W	77° 12' 35".253N	18° W	84° 24' 56".178N	18° W
45°	10° N	18° W	15° 05' 43".367N	12° 42' 50".044W	20° 03' 33".190N	7° 10' 22".015W
	40° N	18° W	44° 54' 28".506N	10° 47' 43".884W	49° 16' 35".187N	2° 19' 56".359W
	70° N	18° W	74° 17' 05".184N	1° 06' 51".561E	76° 00' 26".593N	28° 42' 03".634E
90°	10° N	18° W	9° 55' 09".138N	10° 39' 43".554W	9° 40' 41".618N	3° 19' 52".797W
	40° N	18° W	39° 37' 06".613N	8° 36' 43".277W	38° 29' 31".652N	0° 34' 31".140E
	70° N	18° W	68° 47' 25".009N	2° 17' 23".583E	65° 30' 59".633N	18° 55' 21".211E

AZIMUTH	ORIGIN		3000 MILES		6000 MILES	
	LAT	LONG	LATITUDE	LONGITUDE	LATITUDE	LONGITUDE
0°	10° N	18° W	53° 32' 00".497N	18° W	83° 11' 48".545N	162° E
	40° N	18° W	83° 20' 01".540N	18° W	53° 23' 45".785N	162° E
	70° N	18° W	66° 45' 22".460N	162° E	23° 18' 44".908N	162° E
45°	10° N	18° W	37° 18' 49".295N	19° 34' 07".117E	44° 54' 05".381N	77° 25' 26".869E
	40° N	18° W	57° 06' 00".851N	45° 08' 40".841E	35° 18' 45".644N	102° 02' 29".821E
	70° N	18° W	58° 13' 05".486N	95° 02' 29".439E	17° 08' 38".317N	114° 18' 43".800E
90°	10° N	18° W	7° 14' 05".521N	25° 48' 13".908E	0° 30' 55".629N	68° 47' 05".259E
	40° N	18° W	27° 49' 42".130N	32° 54' 13".184E	1° 56' 54".386N	69° 27' 01".115E
	70° N	18° W	43° 07' 36".475N	52° 01' 00".626E	2° 55' 17".426N	70° 50' 04".891E

DISTANCE AND AZIMUTH

ϕ	α	S	Distance (meters)	Forward Azimuth	Back Azimuth
10°	0°	500	804664.780	00° 00' 00".000	180° 00' 00".000
40°	0°	500	804664.780	00° 00' 00".000	180° 00' 00".000
70°	0°	500	804664.780	00° 00' 00".000	180° 00' 00".000
10°	45°	500	804664.780	45° 00' 00".000	226° 09' 01".224
40°	45°	500	804664.780	45° 00' 00".000	229° 52' 15".525
70°	45°	500	804664.780	45° 00' 00".000	243° 13' 18".356
10°	90°	500	804664.780	90° 00' 00".000	271° 16' 14".933
40°	90°	500	804664.780	90° 00' 00".000	276° 01' 06".634
70°	90°	500	804664.780	90° 00' 00".000	289° 01' 02".923
10°	0°	1000	1609329.561	00° 00' 00".000	180° 00' 00".000
40°	0°	1000	1609329.561	00° 00' 00".000	180° 00' 00".000
70°	0°	1000	1609329.561	00° 00' 00".000	180° 00' 00".000
10°	45°	1000	1609329.561	45° 00' 00".000	227° 49' 35".353
40°	45°	1000	1609329.561	45° 00' 00".000	236° 04' 46".580
70°	45°	1000	1609329.561	45° 00' 00".000	269° 55' 22".938
10°	90°	1000	1609329.561	90° 00' 00".000	272° 31' 12".316
40°	90°	1000	1609329.561	90° 00' 00".000	281° 48' 53".917
70°	90°	1000	1609329.561	90° 00' 00".000	304° 22' 03".656
10°	0°	3000	4827988.683	00° 00' 00".000	180° 00' 00".000
40°	0°	3000	4827988.683	00° 00' 00".000	180° 00' 00".000
70°	0°	3000	4827988.683	00° 00' 00".000	360° 00' 00".000
10°	45°	3000	4827988.683	45° 00' 00".000	240° 59' 37".859
40°	45°	3000	4827988.683	45° 00' 00".000	274° 57' 29".108
70°	45°	3000	4827988.683	45° 00' 00".000	332° 38' 58".143
10°	90°	3000	4827988.683	90° 00' 00".000	276° 53' 56".283
40°	90°	3000	4827988.683	90° 00' 00".000	299° 54' 41".259
70°	90°	3000	4827988.683	90° 00' 00".000	332° 00' 43".685
10°	0°	6000	9655977.366	00° 00' 00".000	360° 00' 00".000
40°	0°	6000	9655977.366	00° 00' 00".000	360° 00' 00".000
70°	0°	6000	9655977.366	00° 00' 00".000	360° 00' 00".000
10°	45°	6000	9655977.366	45° 00' 00".000	281° 01' 12".685
40°	45°	6000	9655977.366	45° 00' 00".000	318° 23' 43".000
70°	45°	6000	9655977.366	45° 00' 00".000	345° 17' 56".277
10°	90°	6000	9655977.366	90° 00' 00".000	279° 57' 13".199
40°	90°	6000	9655977.366	90° 00' 00".000	309° 51' 53".419
70°	90°	6000	9655977.366	90° 00' 00".000	339° 54' 37".211

CLARKE 1866 ELLIPSOID

DIRECT AND REVERSE COMPUTATIONS OF ALL ACIC 6000 MILE CHECK LINES –
Clarke 1866 Ellipsoid

DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1, \alpha_{1\text{-}2}$, S to find $\phi_2, \lambda_2, \alpha_{2\text{-}1}$. East longitudes positive; azimuths clockwise from north; no root extraction; only 8 place trigonometric natural tables (as Peters) required for desk work.

CLARKE 1866 SPHEROID a 6378206.4 m f 3.390075283x10^{-3}

1 - f ____________ 1 radian = 206264.8062 seconds

LINE ORIGIN TO TERMINUS (ACIC)

ϕ_1 10° 0′ 0″ tan ϕ_1 0.17632698 tan θ_1 = (1 - f) tan ϕ_1 .1757 2972

$\alpha_{1\text{-}2}$ 0 sin θ_1 .1730 7716 cos θ_1 .98490827 θ_1 9° 58′ 00.408″

sin $\alpha_{1\text{-}2}$ 0 M = cos θ_0 = cos θ_1 sin $\alpha_{1\text{-}2}$ 0 θ_0 90 0 0

cos $\alpha_{1\text{-}2}$ 1 N = cos θ_1 cos $\alpha_{1\text{-}2}$ = cos θ_1 sin θ_0 1

c_1 = fM 0 D = $(1 - c_2)(1 - c_2 - c_1 M)$ .9983056806

$c_2 = \frac{1}{4}(1 - M^2)f$ .8475188821x10^{-3} P = $c_2 (1 + \frac{1}{2} c_1 M)/D$ .0008489572254

cos σ_1 = sin θ_1/sin θ_0 = cos(90 - θ_1) σ_1 80° 01′ 59.592″

d=S/aD 1.5164710635 (rad) d 86 53 14.610 S 9655977.366 m

sin d +.9985 2475 u = $2(\sigma_1 - d)$ -13 42 30.036 sin u -.2369 7962

cos d +.05429855 W = 1 - 2P cos u +.9983504513 cos u +.9715 1462

V = cos u cos d - sin u sin d +.28938 1851 Y = 2PVW sin d +.4898 1147x10^{-3}

X = c_2^2 sin d cos d $(2V^2 - 1)$ -.32422x10^{-7} $\Delta\sigma$ = d + X - Y 1.515981 2196 (rad)

sin $\Delta\sigma$ ____ cos $\Delta\sigma$ ____ $\Delta\sigma$ 86° 51′ 33.572″

cos $\Sigma\sigma$ ____ $\Sigma\sigma = 2\sigma_1 - \Delta\sigma$ ____ ° ′ ″

tan $\alpha_{2\text{-}1}$ = M/(N cos $\Delta\sigma$ - sin θ_1 sin $\Delta\sigma$) 0 $\alpha_{2\text{-}1}$ 360

$$\tan \phi_2 = \frac{-(\sin \theta_1 \cos \Delta\sigma + N \sin \Delta\sigma) \sin \alpha_{2\text{-}1}}{(1 - f)M}$$ 8.3822927 sin $\alpha_{2\text{-}1}$ ____

= tan (90 + σ_1 - $\Delta\sigma$)/(1-f) ϕ_2 83° 11′ 48.546″

$$\tan \Delta\eta = \frac{\sin \Delta\sigma \sin \alpha_{1\text{-}2}}{\cos \theta_1 \cos \Delta\sigma - \sin \theta_1 \sin \Delta\sigma \cos \alpha_{1\text{-}2}}$$ 0 $\Delta\eta$ 180° ′ ″

H = $c_1(1 - c_2)\Delta\sigma - c_1 c_2 \sin \Delta\sigma \cos \Sigma\sigma$ 0 (rad) H 0

$\Delta\lambda = \Delta\eta$ - H 180

λ_1 -18

CHECK

M = cos θ_0 = cos θ_1 sin $\alpha_{1\text{-}2}$ = cos θ_2 sin (180 + $\alpha_{2\text{-}1}$) $\lambda_2 = \lambda_1 + \Delta\lambda$ 162° ′ ″

INVERSE POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1; \phi_2, \lambda_2$ to find S, $\alpha_{1\text{-}2}$, $\alpha_{2\text{-}1}$. Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

CLARKE 1866 SPHEROID a 6378206.4 m b 6356583.8 m

$1 - f = b/a$ .9966099247 17 $\frac{1}{2}f$ .1695037645 15 $\times 10^{-2}$ $\frac{1}{4}f$ .8475188207 5 $\times 10^{-3}$

$f^2/64$ .1795770379 $\times 10^{-6}$ 1 radian = 206264.8062 seconds

	° ′ ″			° ′ ″
ϕ_1	10 0 0	1. ORIGIN (ALIC)	λ_1	−18 0 0
ϕ_2	83 11 48.545	2. TERMINUS	λ_2	162 0 0
$\tan\phi_1$	0.1763 2698	1 always west of 2.	$\Delta\lambda = \lambda_2 - \lambda_1$	180 0 0
$\tan\phi_2$	8.3822925	$\tan\theta = (1 - f)\tan\phi$	$\Delta\lambda_m = \frac{1}{2}\Delta\lambda$	90 0 0
θ_2	83 10 26.018	$\tan\theta_2$ 8.3538760	$\sin\Delta\lambda_m$	1
θ_1	9 58 00.408	$\tan\theta_1$ .17572922	$\tan\Delta\lambda$	0
$\theta_m = \frac{1}{2}(\theta_1 + \theta_2)$	46 34 13.213	$\sin\theta_m$ +.72621887	$\cos\theta_m$	+.68746357
$\Delta\theta_m = \frac{1}{2}(\theta_2 - \theta_1)$	36 36 12.805	$\sin\Delta\theta_m$ +.59627471	$\cos\Delta\theta_m$	+.80278046

$H = \cos^2\Delta\theta_m - \sin^2\theta_m = \cos^2\theta_m - \sin^2\Delta\theta_m$ +.117062625 $1 - L$ +.527393845

$L = \sin^2\Delta\theta_m + H\sin^2\Delta\lambda_m$ .472606155 $\cos d = 1 - 2L$ +.05477876 9

$U = 2\sin^2\theta_m\cos^2\Delta\theta_m/(1 - L)$ +1.288912937 d 86° 51′ 33.566″

$V = 2\sin^2\Delta\theta_m\cos^2\theta_m/L$ +.711087067 $\sin d$ +.99849803 d (rad) 1.515981138 3

$X = U + V$ +2 $T = d/\sin d$ 1.518261571 6 $E = 2\cos d$ +.10957538

$Y = U - V$ +.577825872 $D = 4T^2$ 9.220472799 $B = 2D$ 18.440945598

$A = DE$ +1.010336811 $C = T - \frac{1}{2}(A - E)$ +1.067880856 CHECK $C - \frac{1}{2}E + AD/B = T$

$n_1 = X(A + CX)$ +6.292197046 $n_2 = Y(B + EY)$ +10.6922408 $n_3 = DXY$ +10.655655471

$\delta_1 d = \frac{1}{4}f(TX - Y)$ +.002083792 2 $\delta_2 d = (f^2/64)(n_1 - n_2 + n_3)$ .00000011233

$S_1 = a\sin d(T - \delta_1 d)$ 9655970.023 m $S_2 = a\sin d(T - \delta_1 d + \delta_2 d)$ 9655972.177 m

$F = 2Y - E(4 - X)$ $M = 32T - (20T - A)X - (B + 4)Y$

$G = \frac{1}{2}fT + (f^2/64)M$ $Q = -(FG\tan\Delta\lambda)/4$

$\Delta\lambda'_m = \frac{1}{2}(\Delta\lambda + Q)$ = $\Delta\lambda_m$ $\tan\Delta\lambda'_m$ = ~~tan~~ $\Delta\lambda_m$ → +∞

$v = \arctan|c_2|$ 0 $c_2 = \cos\Delta\theta_m/(\sin\theta_m\tan\Delta\lambda'_m)$ → 0

$u = \arctan|c_1|$ 0 $c_1 = -\sin\Delta\theta_m/(\cos\theta_m\tan\Delta\lambda'_m)$ → 0

$\alpha_1 = v - u$ 0 $\alpha_2 = v + u$ 0

c_1	c_2	$\alpha_{1\text{-}2}$ ° ′ ″	$\alpha_{2\text{-}1}$ ° ′ ″
−	+	α_1 0	$360 - \alpha_2$ 360
+	+	α_2	$360 - \alpha_1$
−	−	$180 - \alpha_2$	$180 + \alpha_1$
+	−	$180 - \alpha_1$	$180 + \alpha_2$

DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1, \alpha_{1-2}$, S to find $\phi_2, \lambda_2, \alpha_{2-1}$. East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

CLARKE 1866 SPHEROID a 6378206.4 m f $3.39007528 3\times10^{-3}$

1 - f .9966099 24717 1 radian = 206264.8062 seconds

LINE ORIGIN TO TERMINUS (CALC)

ϕ_1 40° 0′ 0″ tan ϕ_1 0.83909963 tan θ_1 = (1 - f) tan ϕ_1 .83625502

α_{1-2} 0 sin θ_1 .64150618 cos θ_1 .76711787 θ_1 39° 54′ 15.203″

sin α_{1-2} 0 M = cos θ_0 = cos θ_1 sin α_{1-2} 0 θ_0 90 0 0

cos α_{1-2} 1 N = cos θ_1 cos α_{1-2} = cos θ_1 sin θ_0 1

c_1 = fM 0 D = $(1 - c_2)(1 - c_2 - c_1 M)$ .9983056806

c_2 = ¼$(1 - M^2)$f $.8475188207 5\times10^{-3}$ P = c_2 (1 + ½c_1M)/D .00084895 72254

cos σ_1 = sin θ_1/sin θ_0 = cos(90 - θ_1) σ_1 50° 05′ 44.797″

d=S/aD 1.5164710635 (rad) d 86° 53′ 14.610″ S 9655977.366 m

sin d +.99852475 u = 2(σ_1 - d) -73 34 59.626 sin u -.95972129

cos d +.05429855 W = 1 - 2P cos u +.9995201316 cos u +.28262224

V = cos u cos d - sin u sin d +.9731621619 Y = 2PVW sin d +.0016491167

X = c_2^2 sin d cos d $(2V^2 - 1)$ $+.3487\times10^{-7}$ $\Delta\sigma$ = d + X - Y 1.5148219816 (rad)

sin $\Delta\sigma$ cos $\Delta\sigma$ $\Delta\sigma$ 86° 47′ 34.462″

cos $\Sigma\sigma$ $\Sigma\sigma$ = $2\sigma_1 - \Delta\sigma$ ° ′ ″

tan α_{2-1} = M/(N cos $\Delta\sigma$ - sin θ_1 sin $\Delta\sigma$) 0 α_{2-1} 360

tan ϕ_2 = $\dfrac{-(\sin\theta_1 \cos\Delta\sigma + N \sin\Delta\sigma)\sin\alpha_{2-1}}{(1-f)M}$ 1.3463 0731 sin α_{2-1}

= tan(90+σ_1-$\Delta\sigma$)/(1-f) ϕ_2 53° 23′ 45.786″

tan $\Delta\eta$ = $\dfrac{\sin\Delta\sigma \sin\alpha_{1-2}}{\cos\theta_1 \cos\Delta\sigma - \sin\theta_1 \sin\Delta\sigma \cos\alpha_{1-2}}$ 0 $\Delta\eta$ 180° ′ ″

H = $c_1(1 - c_2)\Delta\sigma - c_1 c_2 \sin\Delta\sigma \cos\Sigma\sigma$ 0 (rad) H 0

$\Delta\lambda = \Delta\eta$ - H 180

λ_1 -18

CHECK

M = cos θ_0 = cos θ_1 sin α_{1-2} = cos θ_2 sin (180 + α_{2-1}) $\lambda_2 = \lambda_1 + \Delta\lambda$ 162° ′ ″

INVERSE POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1, \phi_2, \lambda_2$ to find S, α_{1-2}, α_{2-1}. Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

CLARKE 1866 SPHEROID a 6378206.4 m b 6356583.8 m

1 - f = b/a .99660992 717 ½f 1.6950376415×10⁻³ ¼f .8475188 2075×10⁻³

f²/64 .1795 720379×10⁻⁶ 1 radian = 206264.8062 seconds

	° ′ ″			° ′ ″
ϕ_1	40 0 0	1. ORIGIN (ALIC)	λ_1	-18 0 0
ϕ_2	53 23 45.785	2. TERMINUS	λ_2	162 0 0

tan ϕ_1 .8390 9963 — 1. always west of 2. — $\Delta\lambda = \lambda_2 - \lambda_1$ 180 0 0

tan ϕ_2 1.3463 0730 — $\tan\theta = (1-f)\tan\phi$ — $\Delta\lambda_m = \frac{1}{2}\Delta\lambda$ 90 0 0

θ_2 53 18 10.335 — tan θ_2 1.3417 4322 — sin $\Delta\lambda_m$ 1

θ_1 39 54 15.203 — tan θ_1 .8362 5502 — tan $\Delta\lambda$ 0

$\theta_m = \frac{1}{2}(\theta_1 + \theta_2)$ 46° 36′ 12.″769 — sin θ_m +.7266 1721 — cos θ_m +.6870 4253

$\Delta\theta_m = \frac{1}{2}(\theta_2 - \theta_1)$ 6 41 57.566 — sin $\Delta\theta_m$ +.1166 5902 — cos $\Delta\theta_m$ +.9931 7203

$H = \cos^2\Delta\theta_m - \sin^2\theta_m = \cos^2\theta_m - \sin^2\Delta\theta_m$ +.45841811 02 — 1 - L +.52797 2562

$L = \sin^2\Delta\theta_m + H\sin^2\Delta\lambda_m$ .4720 2 7438 — cos d = 1 - 2L +.0559 4512

$U = 2\sin^2\theta_m\cos^2\Delta\theta_m/(1-L)$ +1.972781392 — d 86° 47′ 34.″463

$V = 2\sin^2\Delta\theta_m\cos^2\theta_m/L$ +.02721865 4 — sin d .9984 3385 — d (rad) 1.5148 2 1984 2

X = U + V 2.000000046 — T = d/sin d 1.5171981441 — E = 2 cos d +.1118 9024

Y = U - V 1.945562738 — D = 4T² 9.2075608338 — B = 2D 18.415121668

A = DE 1.030 2 361915 — C = T - ½(A - E) 1.058025168 — CHECK C - ½E + AD B = T

$n_1 = X(A + CX)$ +6.2925733 — $n_2 = Y(B + EY)$ +36.25130308 — $n_3 = DXY$ +35.82777536

$\delta_1 d = \frac{1}{4}f(TX - Y)$ +.000922807 — $\delta_2 d = (f^2/64)(n_1 - n_2 + n_3)$ +.0000010539

$S_1 = a\sin d(T - \delta_1 d)$ 9655970.639 m — $S_2 = a\sin d(T - \delta_1 d + \delta_2 d)$ 9655977.351 m

F = 2Y - E(4 - X) — M = 32T - (20T - A)X - (B + 4)Y

G = ½fT + (f²/64) M — Q = -(FG tan Δλ)/4 0

$\Delta\lambda'_m = \frac{1}{2}(\Delta\lambda + Q)$ = $\Delta\lambda_m$ ° ′ ″ — $\tan\Delta\lambda'_m$ = $\tan\Delta\lambda_m \rightarrow +\infty$

v = arctan (c₂) 0 — $c_2 = \cos\Delta\theta_m / (\sin\theta_m \tan\Delta\lambda'_m)$ → 0

u = arctan (c₁) 0 ° ′ ″ — $c_1 = -\sin\Delta\theta_m / (\cos\theta_m \tan\Delta\lambda'_m)$ → 0 ° ′ ″

α_1 = v - u 0 — α_2 = v + u 0

c_1	c_2	α_{1-2} ° ′ ″	α_{2-1}
-	+	α_1 0	360 - α_2 360
+	+	α_2	360 - α_1
-	-	180 - α_2	180 + α_1
+	-	180 - α_1	180 + α_2

DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1, \alpha_{1-2}$, S to find $\phi_2, \lambda_2, \alpha_{2-1}$. East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

CLARKE 1866 SPHEROID a 6378206.4 m f $3.390075283 \times 10^{-3}$

1 - f .9966099924717 1 radian = 206264.8062 seconds

LINE ORIGIN TO TERMINUS (CALC)

ϕ_1 70° 0' 0" $\tan\phi_1$ 2.7474 7747 $\tan\theta_1 = (1-f)\tan\phi_1$ 2.7381 6326

α_{1-2} 0 $\sin\theta_1$ ____ $\cos\theta_1$ .34304686 θ_1 69° 56' 14.590"

$\sin\alpha_{1-2}$ 0 $M = \cos\theta_0 = \cos\theta_1 \sin\alpha_{1-2}$ 0 θ_0 90 0 0

$\cos\alpha_{1-2}$ 1 $N = \cos\theta_1 \cos\alpha_{1-2}$ $= \cos\theta_1$ $\sin\theta_0$ 1

$c_1 = fM$ 0 $D = (1-c_2)(1-c_2-c_1M)$ .9983056806

$c_2 = \frac{1}{4}(1-M^2)f$ $.8475188721 \times 10^{-3}$ $P = c_2(1+\frac{1}{2}c_1M)/D$ .000848957225

$\cos\sigma_1 = \sin\theta_1/\sin\theta_0$ $= \cos(90-\theta_1)$ σ_1 20° 03' 45.410"

$d = S/aD$ 1.5164710635 (rad) d 86 53 14.610 S 9655977.366 m

$\sin d$ +.99852475 $u = 2(\sigma_1 - d)$ -133 38 58.400 $\sin u$ -.72357513

$\cos d$ +.05429855 $W = 1 - 2P\cos u$ +1.001171978 $\cos u$ -.69024562

$V = \cos u \cos d - \sin u \sin d$ +.6850283395 $Y = 2PVW \sin d$ +.00116276476

$X = c_2^2 \sin d \cos d(2V^2-1)$ $-.2394 \times 10^{-8}$ (ignore) $\Delta\sigma = d + X - Y$ 1.5153082987 (rad)

$\sin\Delta\sigma$ ____ $\cos\Delta\sigma$ ____ $\Delta\sigma$ 86° 49' 14.773"

$\cos\Sigma\sigma$ ____ $\Sigma\sigma = 2\sigma_1 - \Delta\sigma$ ____

$\tan\alpha_{2-1} = M/(N\cos\Delta\sigma - \sin\theta_1 \sin\Delta\sigma)$ 0 α_{2-1} 360

$$\tan\phi_2 = \frac{-(\sin\theta_1 \cos\Delta\sigma + N\sin\Delta\sigma)\sin\alpha_{2-1}}{(1-f)M}$$ +.43092616 $\sin\alpha_{2-1}$ ____

$= \tan(90+\sigma_1-\Delta\sigma)/(1-f)$ ϕ_2 23° 18' 44.908"

$$\tan\Delta\eta = \frac{\sin\Delta\sigma \sin\alpha_{1-2}}{\cos\theta_1 \cos\Delta\sigma - \sin\theta_1 \sin\Delta\sigma \cos\alpha_{1-2}}$$ 0 $\Delta\eta$ 180

$H = c_1(1-c_2)\Delta\sigma - c_1c_2 \sin\Delta\sigma \cos\Sigma\sigma$ 0 (rad) H 0

$\Delta\lambda = \Delta\eta - H$ 180

λ_1 -18

CHECK

$M = \cos\theta_0 = \cos\theta_1 \sin\alpha_{1-2} = \cos\theta_2 \sin(180+\alpha_{2-1})$ $\lambda_2 = \lambda_1 + \Delta\lambda$ 162

INVERSE POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1; \phi_2, \lambda_2$ to find $S, \alpha_{1\text{-}2}, \alpha_{2\text{-}1}$. Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

CLARKE 1866 SPHEROID a 6378206.4 m b 6356583.8 m

$1 - f = b/a$ .99660992471 7 $\frac{1}{2}f$ 1.6950376415 $\times 10^{-3}$ $\frac{1}{4}f$ .8475188 2075 $\times 10^{-3}$

$f^2/64$ .1795720379 $\times 10^{-6}$ 1 radian = 206264.8062 seconds

	°	′	″			°	′	″
ϕ_1	70	0	0	1. ORIGIN (ACIC)	λ_1	−18	0	0
ϕ_2	23	18	44.908	2. TERMINUS	λ_2	162	0	0

$\tan \phi_1$ 2.7474 7742 — 1. always west of 2. — $\Delta\lambda = \lambda_2 - \lambda_1$ 180

$\tan \phi_2$ .4309 2616 — $\tan \theta = (1 - f) \tan \phi$ — $\Delta\lambda_m = \frac{1}{2}\Delta\lambda$ 90

θ_2 23° 14′ 30.″638 — $\tan \theta_2$ .42946529 — $\sin \Delta\lambda_m$ 1

θ_1 69 56 14.590 — $\tan \theta_1$ 2.7381 6326 — $\tan \Delta\lambda$ 0

$\theta_m = \frac{1}{2}(\theta_1 + \theta_2)$ 46° 35′ 22.″614 — $\sin \theta_m$ +.7264 5012 — $\cos \theta_m$ +.6872 1920

$\Delta\theta_m = \frac{1}{2}(\theta_2 - \theta_1)$ −23 20 51.976 — $\sin \Delta\theta_m$ −.3963 1114 — $\cos \Delta\theta_m$ +.9181 1627

$H = \cos^2 \Delta\theta_m - \sin^2 \theta_m = \cos^2 \theta_m - \sin^2 \Delta\theta_m$ .3152077091 — $1 - L$ +.5277297715

$L = \sin^2 \Delta\theta_m + H \sin^2 \Delta\lambda_m$ .4722702285 — $\cos d = 1 - 2L$ +.0554 5954

$U = 2 \sin^2 \theta_m \cos^2 \Delta\theta_m / (1 - L)$ 1.6858749876 — d 86° 49′ 14.″777

$V = 2 \sin^2 \Delta\theta_m \cos^2 \theta_m / L$ 3.1412503 96 — $\sin d$ .9984 6094 — d (rad) 1.5153083202

$X = U + V$ 2.0000000272 — $T = d/\sin d$ 1.5176440655 — $E = 2 \cos d$ +.1109 1908

$Y = U - V$ 1.3717499948 — $D = 4T^2$ 9.2129740382 — $B = 2D$ 18.425948076

$A = DE$ 1.0218946044 — $C = T - \frac{1}{2}(A - E)$ 1.0621563033 — CHECK $C - \frac{1}{2}E + AD/B = T$

$n_1 = X(A + CX)$ 6.2924145654 — $n_2 = Y(B + EY)$ 25.484509517 — $n_3 = DXY$ 25.275793659

$\delta_1 d = \frac{1}{4}f(TX - Y)$ +.0014098800 — $\delta_2 d = (f^2/64)(n_1 - n_2 + n_3)$ +.0000010925

$S_1 = a \sin d (T - \delta_1 d)$ 9655970.560 m — $S_2 = a \sin d (T - \delta_1 d + \delta_2 d)$ 9655977.518 m

$F = 2Y - E(4 - X)$ ______ — $M = 32T - (20T - A)X - (B + 4)Y$ ______

$G = \frac{1}{2}fT + (f^2/64)M$ ______ — $Q = -(FG \tan \Delta\lambda)/4$ 0

$\Delta\lambda_m' = \frac{1}{2}(\Delta\lambda + Q)$ $= \Delta\lambda_m$ — $\tan \Delta\lambda_m'$ $= \tan \Delta\lambda_m \to \infty$

$v = \arctan |c_2|$ 0 — $c_2 = \cos \Delta\theta_m / (\sin \theta_m \tan \Delta\lambda_m')$ $\to 0$

$u = \arctan |c_1|$ 0 — $c_1 = -\sin \Delta\theta_m / (\cos \theta_m \tan \Delta\lambda_m')$ $\to 0$

$\alpha_1 = v - u$ 0 — $\alpha_2 = v + u$ 0

c_1	c_2	$\alpha_{1\text{-}2}$	° ′ ″	$\alpha_{2\text{-}1}$	° ′ ″
−	+	α_1	0	$360 - \alpha_2$	360
+	+	α_2		$360 - \alpha_1$	
−	−	$180 - \alpha_2$		$180 + \alpha_1$	
+	−	$180 - \alpha_1$		$180 + \alpha_2$	

DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given ϕ_1, λ_1, $\alpha_{1\text{-}2}$, S to find ϕ_2, λ_2, $\alpha_{2\text{-}1}$. East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

CLARKE 1866 SPHEROID a 6378206.4 m f $3.39007528 3 \times 10^{-3}$

1 - f .9966 0992 4717 1 radian = 206264.8062 seconds

LINE ORIGIN TO TERMINUS (CALC)

ϕ_1 10° 0′ 0″ tan ϕ_1 0.1763 2698 tan θ_1 = (1 - f) tan ϕ_1 .1757 2922

$\alpha_{1\text{-}2}$ 45 0 0 sin θ_1 .1730 7716 cos θ_1 .9849 0827 θ_1 9° 58′ 00.408″

sin $\alpha_{1\text{-}2}$ .7071 0678 M = cos θ_0 = cos θ_1 sin $\alpha_{1\text{-}2}$ .6964 3532 θ_0 45 51 29.865

cos $\alpha_{1\text{-}2}$ .7071 0678 N = cos θ_1 cos $\alpha_{1\text{-}2}$ .6964 3532 sin θ_0 .7176 1957

c_1 = fM .0023609 6816 D = $(1 - c_2)(1 - c_2 - c_1 M)$ .99748 37397

$c_2 = \frac{1}{4}(1 - M^2)f$ .0004364 5348 P = $c_2(1 + \frac{1}{2}c_1 M)/D$ .0004379 14146

cos σ_1 = sin θ_1/sin θ_0 .2411 8233 σ_1 76° 02′ 37.202″

d=S/aD 1.51772 06574 (rad) d 86 57 32.357 S 9655977.366 m

sin d + .9985 9182 u = $2(\sigma_1 - d)$ -21 49 50.310 sin u - .3718 6434

cos d + .0530 5075 W = 1 - 2P cos u .9991 8698 cos u + .9282 8709

V = cos u cos d - sin u sin d .4205 8701 44 Y = 2PVW sin d .0003675 4422

X = c_2^2 sin d cos d ($2V^2$ - 1) $-.652 \times 10^{-8}$ $\Delta\sigma$ = d + X - Y 1.51735 31067 (rad)

sin $\Delta\sigma$ + .9985 7225 cos $\Delta\sigma$ +.0534 1779 $\Delta\sigma$ 86° 56′ 16.544″

cos $\Sigma\sigma$ + .4202 5353 $\Sigma\sigma = 2\sigma_1 - \Delta\sigma$ 65° 08′ 57.860″

tan $\alpha_{2\text{-}1}$ = M/(N cos $\Delta\sigma$ - sin θ_1 sin $\Delta\sigma$) -5.1348 9288 $\alpha_{2\text{-}1}$ 281 01 12.683

$\tan\phi_2 = \dfrac{-(\sin\theta_1 \cos\Delta\sigma + N \sin\Delta\sigma)\sin\alpha_{2\text{-}1}}{(1-f)M}$ +.9965 6741 sin $\alpha_{2\text{-}1}$ -.9815 5988

ϕ_2 44° 54′ 05.381″

$\tan\Delta\eta = \dfrac{\sin\Delta\sigma \sin\alpha_{1\text{-}2}}{\cos\theta_1 \cos\Delta\sigma - \sin\theta_1 \sin\Delta\sigma \cos\alpha_{1\text{-}2}}$ -10.1454135 $\Delta\eta$ 95° 37′ 45.384″

H = $c_1(1 - c_2)\Delta\sigma - c_1 c_2 \sin\Delta\sigma \cos\Sigma\sigma$ .0035804 2638 (rad) H 12 18.516

$\Delta\lambda = \Delta\eta$ - H 95 25 26.868

λ_1 -18

CHECK

M = cos θ_0 = cos θ_1 sin $\alpha_{1\text{-}2}$ = cos θ_2 sin (180 + $\alpha_{2\text{-}1}$) $\lambda_2 = \lambda_1 + \Delta\lambda$ 77° 25′ 26.868″

INVERSE POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1; \phi_2, \lambda_2$ to find S, $\alpha_{1-2}, \alpha_{2-1}$. Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

CLARKE 1866 SPHEROID a 6378206.4 m b 6356583.8 m

$1 - f = b/a$ .9966099 24717 $\frac{1}{2}f$ 1.69503764155×10⁻³ $\frac{1}{4}f$ .84751882075×10⁻³

$f^2/64$ .1795720379×10⁻⁶ 1 radian = 206264.8062 seconds

	°	′	″			°	′	″
ϕ_1	10	0	0	1. ORIGIN (ALIC)	λ_1	-18	0	0
ϕ_2	44	54	05.381	2. TERMINUS	λ_2	77	25	26.869

$\tan\phi_1$ .1763 2698 1. always west of 2. $\Delta\lambda = \lambda_2 - \lambda_1$ 95 25 26.869

$\tan\phi_2$ .9965 6741 $\tan\theta = (1 - f)\tan\phi$ $\Delta\lambda_m = \frac{1}{2}\Delta\lambda$ 47 42 43.434

θ_2 44° 48′ 15.164″ $\tan\theta_2$ .9931 8897 $\sin\Delta\lambda_m$ +.7397 7279

θ_1 9 58 00.408 $\tan\theta_1$ .1757 2922 $\tan\Delta\lambda$ -10.531552

$\theta_m = \frac{1}{2}(\theta_1 + \theta_2)$ ° ′ ″ $\sin\theta_m$ +.4599 7502 $\cos\theta_m$ +.8879 3185

$\Delta\theta_m = \frac{1}{2}(\theta_2 - \theta_1)$ $\sin\Delta\theta_m$ +.2993 5249 $\cos\Delta\theta_m$ +.9541 4259

$H = \cos^2\Delta\theta_m - \sin^2\theta_m = \cos^2\theta_m - \sin^2\Delta\theta_m$ .6988110561 $1 - L$ .52795 41064

$L = \sin^2\Delta\theta_m + H\sin^2\Delta\lambda_m$ .4720458936 $\cos d = 1 - 2L$ +.0559 0821

$U = 2\sin^2\theta_m\cos^2\Delta\theta_m/(1 - L)$ .7296740478 d 86° 47′ 42″.089

$V = 2\sin^2\Delta\theta_m\cos^2\theta_m/L$ .2993441603 $\sin d$ .9984 3591 d (rad) 1.514858 9561

$X = U + V$ 1.029018 2081 $T = d/\sin d$ 1.5172320436 $E = 2\cos d$ .1118 1642

$Y = U - V$ .4303298875 $D = 4T^2$ 9.2079722965 $B = 2D$ 18.415944593

$A = DE$ 1.0296024977 $C = T - \frac{1}{2}(A - E)$ 1.0583390048 CHECK $C - \frac{1}{2}E + AD/B = T$

$n_1 = X(A + CX)$ 2.180132106 $n_2 = Y(B + EY)$ 7.945637956 $n_3 = DXY$ 4.077449336

$\delta_1 d = \frac{1}{4}f(TX - Y)$ .000958 4840 $\delta_2 d = (f^2/64)(n_1 - n_2 + n_3)$ -.0000003031

$S_1 = a\sin d(T - \delta_1 d)$ 9655979.242 m $S_2 = a\sin d(T - \delta_1 d + \delta_2 d)$ 9655977.312 m

$F = 2Y - E(4 - X)$ +.52845 52271 $M = 32T - (20T - A)X - (B + 4)Y$ +8.7394 66222

$G = \frac{1}{2}fT + (f^2/64)M$ .0025 7333479 $Q = -(FG\tan\Delta\lambda)/4$ + 12″ 18.520

$\Delta\lambda_m' = \frac{1}{2}(\Delta\lambda + Q)$ 47° 48′ 52″.694 $\tan\Delta\lambda_m'$ 1.1034 1269

$v = \arctan|c_2|$ 61° 59′ 23″.656 $c_2 = \cos\Delta\theta_m/(\sin\theta_m\tan\Delta\lambda_m')$ +1.8799 2728

$u = \arctan|c_1|$ 16 59 23.657 $c_1 = -\sin\Delta\theta_m/(\cos\theta_m\tan\Delta\lambda_m')$ -.3055 3802

$\alpha_1 = v - u$ 44° 59′ 59″.999 $\alpha_2 = v + u$ 78° 58′ 47″.313

c_1	c_2	α_{1-2}		α_{2-1}	
-	+	α_1	44° 59′ 59″.999	$360 - \alpha_2$	281° 01′ 12″.687
+	+	α_2		$360 - \alpha_1$	
-	-	$180 - \alpha_2$		$180 + \alpha_1$	
+	-	$180 - \alpha_1$		$180 + \alpha_2$	

DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1, \alpha_{1-2}$, S to find $\phi_2, \lambda_2, \alpha_{2-1}$. East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

CLARKE 1866 SPHEROID a 6378206.4 m f $3.390075283 \times 10^{-3}$

1 - f .99660992 4717 1 radian = 206264.8062 seconds

LINE ORIGIN TO TERMINUS (ACIC)

ϕ_1 40° 0′ 0″ tan ϕ_1 .8390 9963 tan θ_1 = (1 - f) tan ϕ_1 .8362 5502

α_{1-2} 45 0 0 sin θ_1 .6415 0618 cos θ_1 .7671 1787 θ_1 39° 54′ 15.203″

sin α_{1-2} .7071 0678 M = cos θ_0 = cos θ_1 sin α_{1-2} .5424 3425 θ_0 57 09 01.789

cos α_{1-2} .7071 0678 N = cos θ_1 cos α_{1-2} .5424 3425 sin θ_0 .84009826

c_1 = fM .0018389294 D = $(1 - c_2)(1 - c_2 - c_1M)$ .9978071775

$c_2 = \frac{1}{4}(1 - M^2)f$ .000598149194 P = $c_2(1 + \frac{1}{2}c_1M)/D$ .000599762688

cos σ_1 = sin θ_1/sin θ_0 .7636 0851 σ_1 40° 12′ 59.914″

d=S/aD 1.51772 86903 (rad) d 86 55 50.882 S 9655977.366 m

sin d +.9985 6560 u = $2(\sigma_1 - d)$ -93 25 41.936 sin u - .9982 1040

cos d +.0535 4202 W = 1 - 2P cos u 1.0000717313 cos u - .05979970

V = cos u cos d - sin u sin d .9935767703 Y = 2PVW sin d .0011 9019637

X = c_2^2 sin d cos d $(2V^2 - 1)$ $+.1864 \times 10^{-7}$ $\Delta\sigma$ = d + X - Y 1.5160385126 (rad)

sin $\Delta\sigma$ + .9985 0117 cos $\Delta\sigma$ +.0547 3045 $\Delta\sigma$ 86° 51′ 45.390″

cos $\Sigma\sigma$ + .9937 1074 $\Sigma\sigma = 2\sigma_1 - \Delta\sigma$ - 6° 25′ 45.562″

tan α_{2-1} = M/(N cos $\Delta\sigma$ - sin θ_1 sin $\Delta\sigma$) -.8879 8892 α_{2-1} 218 23 43.002

$$\tan \phi_2 = \frac{-(\sin\theta_1 \cos\Delta\sigma + N \sin\Delta\sigma)\sin\alpha_{2-1}}{(1 - f)M}$$ +.7083 7173 sin α_{2-1} -.6639 8783

ϕ_2 35° 18′ 45.642″

$$\tan \Delta\eta = \frac{\sin\Delta\sigma \sin\alpha_{1-2}}{\cos\theta_1 \cos\Delta\sigma - \sin\theta_1 \sin\Delta\sigma \cos\alpha_{1-2}}$$ -1.71808992 $\Delta\eta$ 120° 12′ 04.285″

H = $c_1(1 - c_2)\Delta\sigma - c_1c_2 \sin\Delta\sigma \cos\Sigma\sigma$ +.0027850236 (rad) H 9 34.463

$\Delta\lambda = \Delta\eta$ - H 120 02 29.822

λ_1 -18 0 0

CHECK

M = cos θ_0 = cos θ_1 sin α_{1-2} = cos θ_2 sin $(180 + \alpha_{2-1})$ $\lambda_2 = \lambda_1 + \Delta\lambda$ 102° 02′ 29.822″

INVERSE POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1; \phi_2, \lambda_2$ to find S, α_{1-2}, α_{2-1}. Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

CLARKE 1866 SPHEROID a 6378206.4 m b 6356583.8 m

$1 - f = b/a$ .99660992 4717 $\frac{1}{2}f$ 1.6950376415 × 10^{-3} $\frac{1}{4}f$ .8475188 2075 × 10^{-3}

$f^2/64$ .1795720379 × 10^{-6} 1 radian = 206264.8062 seconds

	° ′ ″				° ′ ″
ϕ_1	40 0 0	1.	ORIGIN	λ_1	−18 0 0
ϕ_2	35 18 45.644	2.	TERMINUS	λ_2	102 02 29.821
$\tan\phi_1$	.83909963	1. always west of 2.		$\Delta\lambda = \lambda_2 - \lambda_1$	120 02 29.821
$\tan\phi_2$	.70837174	$\tan\theta = (1-f)\tan\phi$		$\Delta\lambda_m = \frac{1}{2}\Delta\lambda$	60 01 14.910
θ_2	35° 13′ 15.″443	$\tan\theta_2$	.70697031	$\sin\Delta\lambda_m$	+.86620693
θ_1	39 54 15.203	$\tan\theta_1$	.83625502	$\tan\Delta\lambda$	−1.72914904
$\theta_m = \frac{1}{2}(\theta_1 + \theta_2)$	37° 33′ 45.″323	$\sin\theta_m$	+.60962772	$\cos\theta_m$	+.79268786
$\Delta\theta_m = \frac{1}{2}(\theta_2 - \theta_1)$	−2 20 29.880	$\sin\Delta\theta_m$	−.04085784	$\cos\Delta\theta_m$	+.99916497

$H = \cos^2\Delta\theta_m - \sin^2\theta_m = \cos^2\theta_m - \sin^2\Delta\theta_m$ .62684[illegible] $1 - L$ .52812 00685

$L = \sin^2\Delta\theta_m + H\sin^2\Delta\lambda_m$ .47187 99315 $\cos d = 1 - 2L$ .05624014

$U = 2\sin^2\theta_m\cos^2\Delta\theta_m/(1-L)$ 1.40508 02733 d 86° 46′ 33.″514

$V = 2\sin^2\Delta\theta_m\cos^2\theta_m/L$ .00444[illegible]58388 sin d .99841727 d (rad) 1.5145264951

$X = U + V$ 1.4095261121 $T = d/\sin d$ 1.5169273816 $E = 2\cos d$ .11248028

$Y = U - V$ 1.40063 44345 $D = 4T^2$ 9.2042 74724 $B = 2D$ 18.4085 49448

$A = DE$ 1.035 2993952 $C = T - \frac{1}{2}(A - E)$ 1.0555178225 CHECK $C - \frac{1}{2}E + AD/B = T$

$n_1 = X(A + CX)$ 3.55634620 $n_2 = Y(B + EY)$ 26.0043095 $n_3 = DXY$ 18.1718627

$\delta_1 d = \frac{1}{4}f(TX - Y)$ .0006250573 $\delta_2 d = (f^2/64)(n_1 - n_2 + n_3)$ −.000000768

$S_1 = a\sin d(T - \delta_1 d)$ 9655982.150 m $S_2 = a\sin d(T - \delta_1 d + \delta_2 d)$ 9655977.359 m

$F = 2Y - E(4 - X)$ 2.5098916408 $M = 32T - (20T - A)X - (B + 4)Y$ −24.14820333

$G = \frac{1}{2}fT + (f^2/64)M$ .0025669 12669 $Q = -(FG\tan\Delta\lambda)/4$ + 9″ 34.″465

$\Delta\lambda'_m = \frac{1}{2}(\Delta\lambda + Q)$ 60° 06′ 02.″142 $\tan\Delta\lambda'_m$ 1.73909509

$v = \arctan|c_2|$ 43° 18′ 08.″499 $c_2 = \cos\Delta\theta_m/(\sin\theta_m\tan\Delta\lambda'_m)$ +.94243013

$u = \arctan|c_1|$ 1 41 51.501 $c_1 = -\sin\Delta\theta_m/(\cos\theta_m\tan\Delta\lambda'_m)$ +.02963807

$\alpha_1 = v - u$ 41° 36′ 16.″998 $\alpha_2 = v + u$ 45° 0′ 0″

c_1	c_2	α_{1-2}	° ′ ″	α_{2-1}	° ′ ″
−	+	α_1		$360 - \alpha_2$	
+	+	α_2	45 0 0	$360 - \alpha_1$	318 23 43.002
−	−	$180 - \alpha_2$		$180 + \alpha_1$	
+	−	$180 - \alpha_1$		$180 + \alpha_2$	

DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1, \alpha_{1\text{-}2}$, S to find $\phi_2, \lambda_2, \alpha_{2\text{-}1}$. East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

CLARKE 1866 SPHEROID a 6378206.4 m f 3.390075283 $\times 10^{-3}$

1 - f .99660992 4717 1 radian = 206264.8062 seconds

LINE ORIGIN TO TERMINUS (ALIC)

ϕ_1 20° 0′ 0″ tan ϕ_1 2.7474 7742 tan θ_1 = (1 - f) tan ϕ_1 2.7381 6326

$\alpha_{1\text{-}2}$ 45 0 0 sin θ_1 .9393 1830 cos θ_1 .3430 4686 θ_1 69° 56′ 14.590″

sin $\alpha_{1\text{-}2}$ .7071 0678 M = cos θ_0 = cos θ_1 sin $\alpha_{1\text{-}2}$ .2425 7076 θ_0 75 57 42.053

cos $\alpha_{1\text{-}2}$ .7071 0678 N = cos θ_1 cos $\alpha_{1\text{-}2}$ .2425 7076 sin θ_0 .9701 3371

c_1 = fM .000822333138 D = $(1 - c_2)(1 - c_2 - c_1 M)$ .99820 60207

$c_2 = \frac{1}{4}(1 - M^2)f$ .000797650327 P = $c_2 (1 + \frac{1}{2} c_1 M)/D$ .000799163565

cos σ_1 = sin θ_1/sin θ_0 .96823591 σ_1 14° 28′ 47.231″

d=S/aD 1.5166224664 (rad) d 86 53 45.839 S 9655977.366 m

sin d +.9985 3296 u = 2(σ_1 - d) -144 49 57.216 sin u -.5759 6785

cos d +.05414737 W = 1 - 2P cos u 1.0013065882 cos u -.8174 7234

V = cos u cos d - sin u sin d +.53085 89049 Y = 2PVW sin d +.0008483484

X = c_2^2 sin d cos d $(2V^2 - 1)$ -.2471 $\times 10^{-7}$ $\Delta\sigma$ = d + X - Y 1.51577409 33 (rad)

sin $\Delta\sigma$ +.9984 8666 cos $\Delta\sigma$ +.0549 9448 $\Delta\sigma$ 86° 50′ 50.850″

cos $\Sigma\sigma$ +.5315 7768 $\Sigma\sigma = 2\sigma_1 - \Delta\sigma$ -57° 53′ 16.388″

tan $\alpha_{2\text{-}1}$ = M/(N cos $\Delta\sigma$ - sin θ_1 sin $\Delta\sigma$) -.2623 6438 $\alpha_{2\text{-}1}$ 345 17 56.277

$$\tan \phi_2 = \frac{-(\sin \theta_1 \cos \Delta\sigma + N \sin \Delta\sigma) \sin \alpha_{2\text{-}1}}{(1 - f)M}$$ +.3084 8056 sin $\alpha_{2\text{-}1}$ -.2537 7540

ϕ_2 17° 08′ 38.318″

$$\tan \Delta\eta = \frac{\sin \Delta\sigma \sin \alpha_{1\text{-}2}}{\cos \theta_1 \cos \Delta\sigma - \sin \theta_1 \sin \Delta\sigma \cos \alpha_{1\text{-}2}}$$ -1.09577302 $\Delta\eta$ 132° 23′ 00.626″

H = $c_1(1 - c_2)\Delta\sigma - c_1 c_2 \sin \Delta\sigma \cos \Sigma\sigma$ +.0012451 2887 (rad) H 4 16.826

$\Delta\lambda = \Delta\eta - H$ 132 18 43.800

λ_1 -18 0 0

CHECK

M = cos θ_0 = cos θ_1 sin $\alpha_{1\text{-}2}$ = cos θ_2 sin (180 + $\alpha_{2\text{-}1}$) $\lambda_2 = \lambda_1 + \Delta\lambda$ 114° 18′ 43.800″

INVERSE POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1; \phi_2, \lambda_2$ to find S, $\alpha_{1\text{-}2}$, $\alpha_{2\text{-}1}$. Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

CLARKE 1866 SPHEROID a 6378206.4 m b 6356583.8 m

$1 - f = b/a$ .99660992 4717 $\frac{1}{2}f$ 1.6950376415 $\times 10^{-3}$ $\frac{1}{4}f$ .8475188 2075 $\times 10^{-3}$

$f^2/64$ .179572037 9 $\times 10^{-6}$ 1 radian = 206264.8062 seconds

	°	′	″			°	′	″
ϕ_1	70	0	0	1. ORIGIN (CALIC)	λ_1	−18	0	0
ϕ_2	17	08	38.317	2. TERMINUS	λ_2	114	18	43.800

$\tan\phi_1$ 2.7474 7742 1. always west of 2. $\Delta\lambda = \lambda_2 - \lambda_1$ 132 18 43.800

$\tan\phi_2$ .3084 8055 $\tan\theta = (1 - f)\tan\phi$ $\Delta\lambda_m = \frac{1}{2}\Delta\lambda$ 66 09 21.900

θ_2 17° 05′ 21.″296 $\tan\theta_2$ .3074 3478 $\sin\Delta\lambda_m$ +.9146 5008

θ_1 69 56 14.590 $\tan\theta_1$ 2.7381 6326 $\tan\Delta\lambda$ −1.0985 1694

$\theta_m = \frac{1}{2}(\theta_1 + \theta_2)$ 43° 30′ 47.″943 $\sin\theta_m$ +.6885 2316 $\cos\theta_m$ +.7252 1435

$\Delta\theta_m = \frac{1}{2}(\theta_2 - \theta_1)$ −26 25 26.647 $\sin\Delta\theta_m$ −.4450 1141 $\cos\Delta\theta_m$ +.8955 2490

$H = \cos^2\Delta\theta_m - \sin^2\theta_m = \cos^2\theta_m - \sin^2\Delta\theta_m$ .3279007015 $1 - L$ .52764 81124

$L = \sin^2\Delta\theta_m + H\sin^2\Delta\lambda_m$ .47235 18876 $\cos d = 1 - 2L$ .0552 9622

$U = 2\sin^2\theta_m\cos^2\Delta\theta_m/(1 - L)$ 1.4410 466666 d 86° 49′ 48.″515

$V = 2\sin^2\Delta\theta_m\cos^2\theta_m/L$ .4410008344 $\sin d$ .9984 6999 d (rad) 1.5154718866

$X = U + V$ 1.8820475010 $T = d/\sin d$ 1.5177941268 $E = 2\cos d$ .11059244

$Y = U - V$ 1.0000458322 $D = 4T^2$ 9.2147960 45 $B = 2D$ 18.4295920 9

$A = DE$ 1.0190867788 $C = T - \frac{1}{2}(A - E)$ 1.0683546954 CHECK $C - \frac{1}{2}E + AD/B = T$

$n_1 = X(A + CX)$ 5.6851162277 $n_2 = Y(B + EY)$ 18.5410393 4 $n_3 = DXY$ 17.3434 78722

$\delta_1 d = \frac{1}{4}f(TX - Y)$ .00157 34317 $\delta_2 d = (f^2/64)(n_1 - n_2 + n_3)$ +.0000008058

$S_1 = a\sin d(T - \delta_1 d)$ 9655972.172 m $S_2 = a\sin d(T - \delta_1 d + \delta_2 d)$ 9655977.304 m

$F = 2Y - E(4 - X)$ +1.7658621297 $M = 32T - (20T - A)X - (B + 4)Y$ −39.07445117

$G = \frac{1}{2}fT + (f^2/64)M$ .25674 9872 $\times 10^{-2}$ $Q = -(FG\tan\Delta\lambda)/4$ 4′ 16.″226

$\Delta\lambda'_m = \frac{1}{2}(\Delta\lambda + Q)$ 66° 11′ 30.″213 $\tan\Delta\lambda'_m$ 2.2664 1999

$v = \arctan|c_2|$ 29° 51′ 01.″862 $c_2 = \cos\Delta\theta_m/(\sin\theta_m\tan\Delta\lambda'_m)$ +.5728 2687

$u = \arctan|c_1|$ 15 08 58.138 $c_1 = -\sin\Delta\theta_m/(\cos\theta_m\tan\Delta\lambda'_m)$ +.27074744

$\alpha_1 = v - u$ 14° 42′ 03.″724 $\alpha_2 = v + u$ 45° 0′ 0″

c_1	c_2	$\alpha_{1\text{-}2}$	°	′	″	$\alpha_{2\text{-}1}$	°	′	″
−	+	α_1				$360 - \alpha_2$			
+	+	α_2	45	0	0	$360 - \alpha_1$	345	17	56.276
−	−	$180 - \alpha_2$				$180 + \alpha_1$			
+	−	$180 - \alpha_1$				$180 + \alpha_2$			

DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1, \alpha_{1\cdot2}$, S to find $\phi_2, \lambda_2, \alpha_{2\cdot1}$. East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

CLARKE 1866 SPHEROID a 6378206.4 m f 3.390075283 X 10⁻³

$1 - f$ .99660992471 7

1 radian = 206264.8062 seconds

LINE ORIGIN TO TERMINUS (ACIC)

ϕ_1 10° 0′ 0″ $\tan \phi_1$ 0.17632698 $\tan \theta_1 = (1 - f) \tan \phi_1$ .17572922

$\alpha_{1\cdot2}$ 90 0 0 $\sin \theta_1$ .17307716 $\cos \theta_1$ .98490827 θ_1 9° 58′ 00.408″

$\sin \alpha_{1\cdot2}$ 1 $M = \cos \theta_0 = \cos \theta_1 \sin \alpha_{1\cdot2}$ $= \cos\theta_1$ θ_0 $= \theta_1$

$\cos \alpha_{1\cdot2}$ 0 $N = \cos \theta_1 \cos \alpha_{1\cdot2}$ 0 $\sin \theta_0$ $= \sin\theta_1$

$c_1 = fM$ .003389131 8 $D = (1 - c_2)(1 - c_2 - c_1 M)$ .9966607849

$c_2 = \frac{1}{4}(1 - M^2)f$ .0000253802 2 $P = c_2 (1 + \frac{1}{2} c_1 M)/D$ .000025514966

$\cos \sigma_1 = \sin \theta_1 / \sin \theta_0$ 1 σ_1 0° 0′ 0″

$d = S/aD$ 1.5189738576 (rad) d 87 01 50.848 S 9655977.366 m

$\sin d$ + .99865751 $u = 2(\sigma_1 - d)$ −174 03 41.696 $\sin u$ − .10345948

$\cos d$ + .05179928 $W = 1 - 2P \cos u$ +1.000050756 09 $\cos u$ − .99463367

$V = \cos u \cos d - \sin u \sin d$ $= \cos d$ $Y = 2PVW \sin d$ +.0000026399

$X = c_2^3 \sin d \cos d (2V^2 - 1)$ −.33 × 10⁻¹⁰ (ignore) $\Delta\sigma = d + X - Y$ 1.5189712177 (rad)

$\sin \Delta\sigma$ +.99865738 $\cos \Delta\sigma$ +.05180191 $\Delta\sigma$ 87° 01′ 50.304″

$\cos \Sigma\sigma$ $= \cos\Delta\sigma$ $\Sigma\sigma = 2\sigma_1 - \Delta\sigma$ $= \Delta\sigma$

$\tan \alpha_{2\cdot1} = M/(N \cos \Delta\sigma - \sin \theta_1 \sin \Delta\sigma)$ −5.69922388 $\alpha_{2\cdot1}$ 279° 57′ 13.198″

$\tan \phi_2 = \dfrac{-(\sin \theta_1 \cos \Delta\sigma + N \sin \Delta\sigma) \sin \alpha_{2\cdot1}}{(1 - f)M}$ +.00899159 $\sin \alpha_{2\cdot1}$ −.98494785

ϕ_2 0° 30′ 55.630″

$\tan \Delta\eta = \dfrac{\sin \Delta\sigma \sin \alpha_{1\cdot2}}{\cos \theta_1 \cos \Delta\sigma - \sin \theta_1 \sin \Delta\sigma \cos \alpha_{1\cdot2}}$ +19.57279153 $\Delta\eta$ 87° 04′ 31.349″

$H = c_1(1 - c_2)\Delta\sigma - c_1 c_2 \sin \Delta\sigma \cos \Sigma\sigma$ +.005071157987 (rad) H 17 26.088

$\Delta\lambda = \Delta\eta - H$ 86 47 05.261

λ_1 −18

CHECK

$M = \cos \theta_0 = \cos \theta_1 \sin \alpha_{1\cdot2} = \cos \theta_2 \sin (180 + \alpha_{2\cdot1})$

$\lambda_2 = \lambda_1 + \Delta\lambda$ 68° 47′ 05.261″

INVERSE POSITION COMPUTATION FORM FOR LONG LINES. Given ϕ_1, λ_1; ϕ_2, λ_2 to find S, α_{1-2}, α_{2-1}. Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

CLARKE 1866 SPHEROID a 6378206.4 m b 6356583.8 m

$1 - f = b/a$ .99660992 4717 $\frac{1}{2}f$ 1.695 0376415 $\times 10^{-3}$ $\frac{1}{4}f$ .8475 1882075 $\times 10^{-3}$

$f^2/64$ .1795 720379 $\times 10^{-6}$ 1 radian = 206264.8062 seconds

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ϕ_1 10° 0′ 0″	1. ORIGIN (ACK)	λ_1 −18° 0′ 0″
ϕ_2 0 30 55.629	2. TERMINUS	λ_2 68 47 05.259
$\tan\phi_1$ .1763 2698	1. always west of 2.	$\Delta\lambda = \lambda_2 - \lambda_1$ 86 47 05.259
$\tan\phi_2$ .0089 9658	$\tan\theta = (1 - f)\tan\phi$	$\Delta\lambda_m = \frac{1}{2}\Delta\lambda$ 43 23 32.629
θ_2 0° 30′ 49.337″	$\tan\theta_2$ .0089 6608	$\sin\Delta\lambda_m$ + .6869 9109
θ_1 9 58 00.408	$\tan\theta_1$ .1757 2422	$\tan\Delta\lambda$ +17.801543
$\theta_m = \frac{1}{2}(\theta_1 + \theta_2)$ 5° 14′ 24.873″	$\sin\theta_m$ + .0913 3203	$\cos\theta_m$ + .9958 2050
$\Delta\theta_m = \frac{1}{2}(\theta_2 - \theta_1)$ −4 43 35.535	$\sin\Delta\theta_m$ − .0824 0011	$\cos\Delta\theta_m$ + .9965 9933
$H = \cos^2\Delta\theta_m - \sin^2\theta_m = \cos^2\theta_m - \sin^2\Delta\theta_m$ .98486868		$1 - L$ .52839 47885
$L = \sin^2\Delta\theta_m + H\sin^2\Delta\lambda_m$ .47160 52115	$\cos d = 1 - 2L$ .0567 8958	
$U = 2\sin^2\theta_m\cos^2\Delta\theta_m/(1 - L)$ .03135 82594	d 86° 44′ 40.002″	
$V = 2\sin^2\Delta\theta_m\cos^2\theta_m/L$ .02855 41415	$\sin d$ .9983 8617	d (rad) 1.5139761734
$X = U + V$ .05991 24009	$T = d/\sin d$ 1.51164234230	$E = 2\cos d$ .1135 7916
$Y = U - V$ .00280 46179	$D = 4T^2$ 9.1403 1599914	$B = 2D$ 18.2963199828
$A = DE$ 1.0447192854	$C = T - \frac{1}{2}(A - E)$ 1.050953403	CHECK $C - \frac{1}{2}E + AD/B = T$
$n_1 = X(A + CX)$ .0663642597	$n_2 = Y(B + EY)$ .0515955417	$n_3 = DXY$ .0015455925
$\delta_1 d = \frac{1}{4}f(TX - Y)$ + .0000746229		$\delta_2 d = (f^2/64)(n_1 - n_2 + n_3)$ + .29 $\times 10^{-8}$
$S_1 = a\sin d(T - \delta_1 d)$ 9655977.326 m		$S_2 = a\sin d(T - \delta_1 d + \delta_2 d)$ 9655977.345 m
$F = 2Y - E(4 - X)$ − .44190 25472		$M = 32T - (20T - A)X - (B + 4)Y$ 46.70826 2054
$G = \frac{1}{2}fT + (f^2/64)M$ .0025787 8228		$Q = -(FG\tan\Delta\lambda)/4$ + 12 26.078
$\Delta\lambda_m' = \frac{1}{2}(\Delta\lambda + Q)$ 43 37 15.668		$\tan\Delta\lambda_m'$ .9502 1540
$v = \arctan\lvert c_2\rvert$ 85° 01′ 33.402″		$c_2 = \cos\Delta\theta_m/(\sin\theta_m\tan\Delta\lambda_m')$ + 11.4835302
$u = \arctan\lvert c_1\rvert$ 4 58 36.600		$c_1 = -\sin\Delta\theta_m/(\cos\theta_m\tan\Delta\lambda_m')$ + .08708125
$\alpha_1 = v - u$ 80° 02′ 46.802″		$\alpha_2 = v + u$ 90° 00′ 00.002″

c_1	c_2	α_{1-2}	° ′ ″	α_{2-1}	° ′ ″
−	+	α_1		$360 - \alpha_2$	
+	+	α_2	90 00 00.002	$360 - \alpha_1$	279 57 13.198
−	−	$180 - \alpha_2$		$180 + \alpha_1$	
+	−	$180 - \alpha_1$		$180 + \alpha_2$	

DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1, \alpha_{1\text{-}2}$, S to find $\phi_2, \lambda_2, \alpha_{2\text{-}1}$. East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

CLARKE 1866 SPHEROID a 6378206.4 m f $3.39007528 3 \times 10^{-3}$

1 - f .99660992 4717 1 radian = 206264.8062 seconds

LINE ORIGIN TO TERMINUS (ACIC)

ϕ_1 40° 0′ 0″ $\tan\phi_1$ .8390 9963 $\tan\theta_1 = (1 - f)\tan\phi_1$ .8362 5502

$\alpha_{1\text{-}2}$ 90 0 0 $\sin\theta_1$ .6415 0618 $\cos\theta_1$ .7671 1787 θ_1 39° 54′ 15.″203

$\sin\alpha_{1\text{-}2}$ 1. $M = \cos\theta_0 = \cos\theta_1 \sin\alpha_{1\text{-}2}$ $= \cos\theta_1$ θ_0 $= \theta_1$

$\cos\alpha_{1\text{-}2}$ 0 $N = \cos\theta_1 \cos\alpha_{1\text{-}2}$ 0 $\sin\theta_0$ $= \sin\theta_1$

$c_1 = fM$ .0026 0058 733 $D = (1 - c_2)(1 - c_2 - c_1 M)$ .9973 083013

$c_2 = \frac{1}{4}(1 - M^2)f$ .0003 4877957 $P = c_2(1 + \frac{1}{2}c_1 M)/D$ .0003 5206975

$\cos\sigma_1 = \sin\theta_1/\sin\theta_0$ 1 σ_1 0° 0′ 0″

$d = S/aD$ 1.51798 76425 (rad) d 86 58 27.427 S 9655977.366 m

$\sin d$ + .9986 0595 $u = 2(\sigma_1 - d)$ -173 56 54.854 $\sin u$ - .1054 2111

$\cos d$ + .0527 8414 $W = 1 - 2P\cos u$ +1.000696238 1 $\cos u$ - .9944 2767

$V = \cos u \cos d - \sin u \sin d$ $= \cos d$ $Y = 2PVW \sin d$ .0000 369304

$X = c_2^2 \sin d \cos d\,(2V^2 - 1)$ $-.638 \times 10^{-8}$ $\Delta\sigma = d + X - Y$ 1.51795 07057 (rad)

$\sin\Delta\sigma$ + .9986 0399 $\cos\Delta\sigma$ + .0528 2103 $\Delta\sigma$ 86° 58′ 19.″808

$\cos\Sigma\sigma$ $= \cos\Delta\sigma$ $\Sigma\sigma = 2\sigma_1 - \Delta\sigma$ $= -\Delta\sigma$

$\tan\alpha_{2\text{-}1} = M/(N\cos\Delta\sigma - \sin\theta_1 \sin\Delta\sigma)$ -1.1974 7914 $\alpha_{2\text{-}1}$ 309° 51′ 53.″420

$\tan\phi_2 = \dfrac{-(\sin\theta_1 \cos\Delta\sigma + N\sin\Delta\sigma)\sin\alpha_{2\text{-}1}}{(1 - f)M}$ +.0340 1982 $\sin\alpha_{2\text{-}1}$ -.7675 5865

ϕ_2 1° 56′ 54.″387

$\tan\Delta\eta = \dfrac{\sin\Delta\sigma \sin\alpha_{1\text{-}2}}{\cos\theta_1 \cos\Delta\sigma - \sin\theta_1 \sin\Delta\sigma \cos\alpha_{1\text{-}2}}$ +24.644745 $\Delta\eta$ 87° 40′ 35.″064

$H = c_1(1 - c_2)\Delta\sigma - c_1 c_2 \sin\Delta\sigma \cos\Sigma\sigma$ +.0039461387 (rad) H 13 33.950

$\Delta\lambda = \Delta\eta - H$ 87 27 01.114

λ_1 -18 0 0

CHECK

$M = \cos\theta_0 = \cos\theta_1 \sin\alpha_{1\text{-}2} = \cos\theta_2 \sin(180 + \alpha_{2\text{-}1})$ $\lambda_2 = \lambda_1 + \Delta\lambda$ 69° 27′ 01.″114

INVERSE POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1; \phi_2, \lambda_2$ to find S, $\alpha_{1-2}, \alpha_{2-1}$. Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

CLARKE 1866 SPHEROID a 6378206.4 m b 6356583.8 m

$1 - f = b/a$ .99660992717 $\frac{1}{2}f$ 1.6950376415 $\frac{1}{4}f$ .8475188207$5 \times 10^{-3}$

$f^2/64$ .179572037$9 \times 10^{-6}$ 1 radian = 206264.8062 seconds

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ϕ_1 40° 0′ 0″	1. ORIGIN (CALC)	λ_1 −18° 0′ 0″
ϕ_2 1 56 54.326	2. TERMINUS	λ_2 69 27 01.115
$\tan\phi_1$ .8390 9963	1. always west of 2.	$\Delta\lambda = \lambda_2 - \lambda_1$ 87 27 01.115
$\tan\phi_2$ .0340 1082	$\tan\theta = (1-f)\tan\phi$	$\Delta\lambda_m = \frac{1}{2}\Delta\lambda$ 43 43 30.558
θ_2 1° 56′ 30″.625	$\tan\theta_2$ .0339 0449	$\sin\Delta\lambda_m$ +.6911 9975
θ_1 39 54 15.203	$\tan\theta_1$ .8362 5502	$\tan\Delta\lambda$ +22.456827
$\theta_m = \frac{1}{2}(\theta_1+\theta_2)$ 20° 55′ 22″.414	$\sin\theta_m$ +.3571 1350	$\cos\theta_m$ +.9340 6100
$\Delta\theta_m = \frac{1}{2}(\theta_2-\theta_1)$ −18 58 52.289	$\sin\Delta\theta_m$ −.3252 5775	$\cos\Delta\theta_m$ +.9456 3540
$H = \cos^2\Delta\theta_m - \sin^2\theta_m = \cos^2\theta_m - \sin^2\Delta\theta_m$ .7667 7345	$1 - L$ .52792 18547	
$L = \sin^2\Delta\theta_m + H\sin^2\Delta\lambda_m$ .47207 81453	$\cos d = 1 - 2L$ .0558 4371	
$U = 2\sin^2\theta_m\cos^2\Delta\theta_m/(1-L)$ .43202 72583	d 86° 47′ 55″.413	
$V = 2\sin^2\Delta\theta_m\cos^2\theta_m/L$ .39104 06316	sin d .9984 3952	d (rad) 1.51492 35527
$X = U + V$ .82306 78899	$T = d/\sin d$ 1.51729 12554	$E = 2\cos d$ .11168742
$Y = U - V$ .04098 66267	$D = 4T^2$ 9.20869 .2143	$B = 2D$ 18.41738 20286
$A = DE$ 1.0284 94941	$C = T - \frac{1}{2}(A - E)$ 1.0588 874949	CHECK $C - \frac{1}{2}E + AD/B = T$
$n_1 = X(A + CX)$ 1.5638 54201	$n_2 = Y(B + EY)$ .75505 3986	$n_3 = DXY$ .3106 53132
$\delta_1 d = \frac{1}{4}f(TX - Y)$ + .0021 0236 731	$\delta_2 d = (f^2/64)(n_1 - n_2 + n_3)$ +.00000 0201	
$S_1 = a\sin d(T - \delta_1 d)$ 9655977.090 m	$S_2 = a\sin d(T - \delta_1 d + \delta_2 d)$ 9655977.370 m	
$F = 2Y - E(4 - X)$ −.27285 00975	$M = 32T - (20T - A)X - (B + 4)Y$ 23.50435423	
$G = \frac{1}{2}fT + (f^2/64)M$ .0025 7608 652	$Q = -(FG\tan\Delta\lambda)/4$ + 13′ 33″.951	
$\Delta\lambda_m' = \frac{1}{2}(\Delta\lambda + Q)$ 43° 52′ 17″.533	$\tan\Delta\lambda_m'$ .9602 4630	
$v = \arctan\lvert c_2\rvert$ 70 04 03″.289	$c_2 = \cos\Delta\theta_m/(\sin\theta_m\tan\Delta\lambda_m')$ +2.7575 9332	
$u = \arctan\lvert c_1\rvert$ 19 55 56.709	$c_1 = -\sin\Delta\theta_m/(\cos\theta_m\tan\Delta\lambda_m')$ + .3626 3504	
$\alpha_1 = v - u$ 50° 08′ 06″.580	$\alpha_2 = v + u$ 89° 59′ 59″.998	

c_1	c_2	α_{1-2} ° ′ ″	α_{2-1} ° ′ ″
−	+	α_1	$360 - \alpha_2$
+	+	α_2 89 59 59.998	$360 - \alpha_1$ 309 51 53.420
−	−	$180 - \alpha_2$	$180 + \alpha_1$
+	−	$180 - \alpha_1$	$180 + \alpha_2$

DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1, \alpha_{1\text{-}2}$, S to find $\phi_2, \lambda_2, \alpha_{2\text{-}1}$. East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

CLARKE 1866 SPHEROID a 6378206.4 m f 3.390075283 x 10⁻³

1 - f .9966099 24717 1 radian = 206264.8062 seconds

LINE ORIGIN TO TERMINUS (ALIC)

ϕ_1 70° 0′ 0″ $\tan \phi_1$ 2.7474 7742 $\tan \theta_1 = (1 - f) \tan \phi_1$ 2.7381 6326

$\alpha_{1\text{-}2}$ 90 0 0 $\sin \theta_1$ .9393 1830 $\cos \theta_1$ .3430 4686 θ_1 69° 56′ 14.590″

$\sin \alpha_{1\text{-}2}$ 1 $M = \cos \theta_0 = \cos \theta_1 \sin \alpha_{1\text{-}2}$ = $\cos \theta_1$ θ_0 = θ_1

$\cos \alpha_{1\text{-}2}$ 0 $N = \cos \theta_1 \cos \alpha_{1\text{-}2}$ 0 $\sin \theta_0$ = $\sin \theta_1$

$c_1 = fM$ .00116295468 $D = (1 - c_2)(1 - c_2 - c_1 M)$ .99810 63459

$c_2 = \frac{1}{4}(1 - M^2) f$ .00074778185 $P = c_2 (1 + \frac{1}{2} c_1 M)/D$ .00074935002

$\cos \sigma_1 = \sin \theta_1 / \sin \theta_0$ 1 σ_1 0° 0′ 0″

$d = S/aD$ 1.5167739223 (rad) d 86 54 17.079 S 9655977.366 m

$\sin d$ .99854114 $u = 2(\sigma_1 - d)$ -173 48 34.158 $\sin u$ - .10782472

$\cos d$ .05399614 $W = 1 - 2P \cos u$ 1.00148996 1 $\cos u$ - .99416883

$V = \cos u \cos d - \sin u \sin d$ = $\cos d$ $Y = 2PVW \sin d$ +.0000809264

$X = c_2^2 \sin d \cos d (2V^2 - 1)$ -.2997 x 10⁻⁷ $\Delta\sigma = d + X - Y$ 1.5166929659 (rad)

$\sin \Delta\sigma$ +.9985 3677 $\cos \Delta\sigma$ +.0540 7697 $\Delta\sigma$ 86° 54′ 00.381″

$\cos \Sigma\sigma$ = $\cos \Delta\sigma$ $\Sigma\sigma = 2\sigma_1 - \Delta\sigma$ = $-\Delta\sigma$

$\tan \alpha_{2\text{-}1} = M/(N \cos \Delta\sigma - \sin \theta_1 \sin \Delta\sigma)$ -.3657 4349 $\alpha_{2\text{-}1}$ 339° 54′ 31.209″

$\tan \phi_2 = \dfrac{-(\sin \theta_1 \cos \Delta\sigma + N \sin \Delta\sigma) \sin \alpha_{2\text{-}1}}{(1 - f)M}$ +.0510 3416 $\sin \alpha_{2\text{-}1}$ -.3434 9028

ϕ_2 2° 55′ 17.426″

$\tan \Delta\eta = \dfrac{\sin \Delta\sigma \sin \alpha_{1\text{-}2}}{\cos \theta_1 \cos \Delta\sigma - \sin \theta_1 \sin \Delta\sigma \cos \alpha_{1\text{-}2}}$ +53.8267631 $\Delta\eta$ 88° 56′ 08.429″

$H = c_1 (1 - c_2) \Delta\sigma - c_1 c_2 \sin \Delta\sigma \cos \Sigma\sigma$ +.0017624 7925 (rad) H 6 03.537

$\Delta\lambda = \Delta\eta - H$ 88 50 04.892

λ_1 -18 0 0

CHECK

$M = \cos \theta_0 = \cos \theta_1 \sin \alpha_{1\text{-}2} = \cos \theta_2 \sin (180 + \alpha_{2\text{-}1})$ $\lambda_2 = \lambda_1 + \Delta\lambda$ 70° 50′ 04.892″

INVERSE POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1; \phi_2, \lambda_2$ to find S, $\alpha_{1\text{-}2}, \alpha_{2\text{-}1}$. Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

CLARKE 1866 SPHEROID a 6378206.4 m b 6356583.8 m

$1 - f = b/a$ .9966099247 17 $\tfrac{1}{2}f$ 1.6950376415×10⁻³ → $1.6950376415\times10^{-3}$ $\tfrac{1}{4}f$ $.8475188 2075\times10^{-3}$

$f^2/64$ $.1795720379\times10^{-6}$ 1 radian = 206264.8062 seconds

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ϕ_1 70° 0′ 0″ 1. ORIGIN (ALIC) λ_1 −18° 0′ 0″

ϕ_2 2 55 12.426 2. TERMINUS λ_2 70 50 04.891

$\tan\phi_1$ 2.7474 7742 1. always west of 2. $\Delta\lambda = \lambda_2 - \lambda_1$ 88 50 04.891

$\tan\phi_2$ .0510 3416 $\tan\theta = (1 - f)\tan\phi$ $\Delta\lambda_m = \tfrac{1}{2}\Delta\lambda$ 44 25 02.446

θ_2 2° 54′ 41″.833 $\tan\theta_2$ .0508 6115 $\sin\Delta\lambda_m$ + .6998 7961

θ_1 69 56 14.590 $\tan\theta_1$ 2.7381 6326 $\tan\Delta\lambda$ +49.16114

$\theta_m = \tfrac{1}{2}(\theta_1 + \theta_2)$ 36° 25′ 28″.212 $\sin\theta_m$ +.5937 6306 $\cos\theta_m$ + .8046 3994

$\Delta\theta_m = \tfrac{1}{2}(\theta_2 - \theta_1)$ −33 30 46.378 $\sin\Delta\theta_m$ −.5521 2447 $\cos\Delta\theta_m$ + .8337 6170

$H = \cos^2\Delta\theta_m - \sin^2\theta_m = \cos^2\theta_m - \sin^2\Delta\theta_m$ .342604[illegible] $1 - L$ .52734 03482

$L = \sin^2\Delta\theta_m + H\sin^2\Delta\lambda_m$ .47265 96518 $\cos d = 1 - 2L$ .0546 8070

$U = 2\sin^2\theta_m\cos^2\Delta\theta_m/(1 - L)$ .92949 96425 d 86° 51′ 55″.667

$V = 2\sin^2\Delta\theta_m\cos^2\theta_m/L$ .835138735[illegible] $\sin d$ .99850389 d (rad) 1.5160883269

$X = U + V$ 1.7646383777 $T = d/\sin d$ 1.5183599704 $E = 2\cos d$ .1093 6140

$Y = U - V$ .09436 09073 $D = 4T^2$ 9.2216679992 $B = 2D$ 18.4433359985

$A = DE$ 1.0084945227 $C = T - \tfrac{1}{2}(A - E)$ 1.0687793409 CHECK $C - \tfrac{1}{2}E + AD/B = T$

$n_1 = X(A + CX)$ 5.1077958824 $n_2 = Y(B + EY)$ 1.7413036703 $n_3 = DXY$ 1.5355264872

$\delta_1 d = \tfrac{1}{4}f(TX - Y)$ +.0021908322 $\delta_2 d = (f^2/64)(n_1 - n_2 + n_3)$ +.0000008803

$S_1 = a\sin d(T - \delta_1 d)$ 9655971.659 m $S_2 = a\sin d(T - \delta_1 d + \delta_2 d)$ 9655977.266 m

$F = 2Y - E(4 - X)$ −.0557404619 2 $M = 32T - (20T - A)X - (B + 4)Y$ −5.3377 5185

$G = \tfrac{1}{2}fT + (f^2/64)M$ .0025727 1879 $Q = -(FG\tan\Delta\lambda)/4$ + 6′ 03″.538

$\Delta\lambda'_m = \tfrac{1}{2}(\Delta\lambda + Q)$ 44° 28′ 04″.215 $\tan\Delta\lambda'_m$ .9815 9445

$v = \arctan|c_2|$ 55° 02′ 41″.393 $c_2 = \cos\Delta\theta_m/(\sin\theta_m\tan\Delta\lambda'_m)$ + 1.43052902

$u = \arctan|c_1|$ 34 57 18.604 $c_1 = -\sin\Delta\theta_m/(\cos\theta_m\tan\Delta\lambda'_m)$ + .69904207

$\alpha_1 = v - u$ 20° 05′ 22″.789 $\alpha_2 = v + u$ 89° 59′ 59″.997

c_1	c_2	$\alpha_{1\text{-}2}$ ° ′ ″	$\alpha_{2\text{-}1}$ ° ′ ″
−	+	α_1	$360 - \alpha_2$
+	+	α_2 89 59 59.997	$360 - \alpha_1$ 339 54 37.211
−	−	$180 - \alpha_2$	$180 + \alpha_1$
+	−	$180 - \alpha_1$	$180 + \alpha_2$

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CONTROL COMPUTATIONS FOR THE HEMISPHEROIDAL GEODESIC CONTAINING AN ACIC GIVEN ARC

These are the control computations for the geodesic as presented in Figure 26. To completely determine the configuration we need compute only the constants A, B, C, D, E, F from equations (49); $\Delta\lambda_1$, S_1 from equations (48); $\Delta\lambda_2$, S_2 and $\Delta\lambda_3$, S_3 from equations (47); and $\Delta\lambda_0$, S_0 from equations (54). (The equations cited are from Appendix 1). These will provide the check equations:

$$S_4 = S_2 - S_3 = (1/2)S_0 - S_1 - S_3,\ \Delta\lambda_4 = \Delta\lambda_2 - \Delta\lambda_3 = (1/2)\Delta\lambda_0 - \Delta\lambda_1 - \Delta\lambda_3.$$

Now $f = .003390075283, \sin\theta_0 = .97013371, \cos\theta = .24257076, c_1 = f\cos\theta_0 = .0008223331378,$

$c_2 = (1/4)f\sin^2\theta_0 = .0007976503177, c_3 = 1 + c_1\cos\theta_0 = 1.000199474, c_4 = c_2 + c_3$
$= 1.000997124,$

$A = c_1(1 - c_2c_4) = .00082167655, B = (1/2)c_1c_2c_3 = .3280325 \times 10^{-6}, C = (1/4)c_1c_2^2$
$= .1308 \times 10^{-9},$

$D = 2 + c_2(c_2^2 + c_4^2) - (1 + c_2)c_4 - c_2 = .9982060223, E = (1/2)c_2[2 + c_3(c_3 - 1) - c_2^2]$
$= .0007977296351,$

$F = (1/4)c_2^2(2c_4 - 1) = .1593787 \times 10^{-6}.$

From equations (150) (Appendix 1) we have:

$\theta_1 = 69°\ 56'\ 14''.590, \theta_2 = 17°\ 05'\ 21''.296, \theta_0 = 75°\ 57'\ 42''.053$

$\sin\theta_1 = .93931830$	$\sin\theta_2 = .29386097$	$\sin\theta_0 = .97013371$
$\cos\theta_1 = .34304686$	$\cos\theta_2 = .95584817$	$\cos\theta_0 = .24257076$
$\tan\theta_1 = 2.73816326$	$\tan\theta_2 = .30743478$	$\tan\theta_0 = 3.99938439$
$\sin\theta_1/\sin\theta_0 = .96823591$	$\sin\theta_2/\sin\theta_0 = .30290770$	
$\tan\theta_1/\tan\theta_0 = .68464618$	$\tan\theta_2/\tan\theta_0 = .07687053$	

From these last four numbers:

$\sigma_1 = \text{arc cos } .96823591 = 14°\ 28'\ 47''.231 = .2427199475$ radians

$\eta_1 = \text{arc cos } .68464618 = 46°\ 47'\ 31''.966 = .8166781775$ radians

$\beta_2 = \text{arc sin } .96823591 = 75°\ 31'\ 12''.769 = 1.3180763796$ radians

$\gamma_2 = \text{arc sin } .68464618 = 43°\ 12'\ 28''.034 = .7541181497$ radians

$\beta_3 = \text{arc sin } .30290770 = 17°\ 37'\ 56''.390 = .3077422231$ radians

$\gamma_3 = \text{arc sin } .07687053 = 4°\ 24'\ 31''.342 = .0769464374$ radians

$\sin 2\sigma_1 = .48419238$	$\sin 2\beta_2 = .48419238$	$\sin 2\beta_3 = .57735414$
$\sin 4\sigma_1 = .84729944$	$\sin 4\beta_2 = -.84729944$	$\sin 4\beta_3 = .94281220$

We can now make the computations, with $a = 6378206.4$ meters and $\pi = 3.1415926536$:

η_1	.8166781775		$D\sigma_1$	.2522665735
$-A\sigma_1$	$-$ .0002076541		$+E\sin 2\sigma_1$	+ .0003862546
	.8164705234			.2526528281
$+B\sin 2\sigma_1$	+ 1588		$-F\sin 4\sigma_1$	$-$ 1350
	.8164706822		S_1/a	.2526526931
$-C\sin 4\sigma_1$	$-$ 1			

$\Delta\lambda_1$(rad)	.8164706821		
$\Delta\lambda_1$	$46^\circ\ 46'\ 49''.167$	S_1	1611471.024 meters
γ_2	.7541181497	$D\beta_2$	1.3157117800
$-A\beta_2$ −	.0010830332	$-E\sin 2\beta_2$ −	.0003862546
	.7530351165		1.3153255254
$-B\sin 2\beta_2$ −	1588	$-F\sin 4\beta_2$ +	1350
	.7530349577	S_2/a	1.3153256604
$-C\sin 4\beta_2$ +			
$\Delta\lambda_2$(rad)	.7530349578		
$\Delta\lambda_2$	$43^\circ\ 08'\ 44''.610$	S_2	8389418.545 meters
γ_3	.0769464374	$D\beta_3$	.3071901404
$-A\beta_3$ −	.0002528648	$-E\sin 2\beta_3$ −	.0004605725
	.0766935726		.3067295679
$-B\sin 2\beta_3$ −	1894	$-F\sin 4\beta_3$ −	1503
$\Delta\lambda_3$(rad)	.0766933832	S_3/a	.3067294176
$\Delta\lambda_3$	$4^\circ\ 23'\ 39''.146$	S_3	1956383.534 meters

$$\Delta\lambda_4 = \Delta\lambda_2 - \Delta\lambda_3 = (1/2)\Delta\lambda_0 - \Delta\lambda_1 - \Delta\lambda_3 = 38^\circ\ 45'\ 05''.464$$

$$S_4 = S_2 - S_3 = (1/2)S_0 - S_1 - S_3 = 6433035.010 \text{ meters}$$

$$1 - A = .9991783229,\ \Delta\lambda_0 = \pi(1 - A) = 3.1390112787 \text{ (rad)} = 179^\circ\ 51'\ 07''.553$$

$$\Delta\lambda_0/2 = 89^\circ\ 55'\ 33''.777,\ S_0 = a\pi D = 20001779.136 \text{ m},\ S_0/2 = 10000889.568 \text{ m}$$

As an overall check, we compute from formulae (48), Appendix 1, the values $\Delta\lambda = \Delta\lambda_1 + \Delta\lambda_4$, $S = S_1 + S_4$. We have

$$\sigma_2 = \text{arc cos } .30290770 = 72^\circ\ 22'\ 03''.610 = 1.2630541041 \text{ radians}$$

$$\eta_2 = \text{arc cos } .07687053 = 85^\circ\ 35'\ 28''.658 = 1.4938498897 \text{ radians}$$

$$\sin 2\sigma_2 = .57735414,\ \sin 4\sigma_2 = -.94281220$$

η_2	1.4938498897	$D\sigma_2$	1.2607882132
$-A\sigma_2$ −	.0010378227	$+E\sin 2\sigma_2$ +	.0004605725
	1.4928120670		1.2612487857
$+B\sin 2\sigma_2$ +	1894	$-F\sin 4\sigma_2$ +	1503
	1.4928122564		
$-C\sin 4\sigma_2$ +	1		
$\Delta\lambda$(rad)	1.4928122565	S/a	1.2612489360
$\Delta\lambda = \Delta\lambda_1 + \Delta\lambda_4$	$85^\circ\ 31'\ 54''.631$	$S = S_1 + S_4$	8044506.036 meters

$$\Delta\lambda_1 + \Delta\lambda_4 + \Delta\lambda_3 = 85^\circ\ 31'\ 54''.631 + 4^\circ\ 23'\ 39''.146 = 89^\circ\ 55'\ 33''.777 = (1/2)\Delta\lambda_0.$$

$S_1 + S_4 + S_3 = 8044506.036 + 1956383.534 = 10000889.570 = (1/2)S_0$, which gives a flat check for longitude, and length within .002 meter.

Computation of N_1N_2 from Figure 26. (Inverse solution)

Formulae are from equations (149), Appendix 1. We have

$f = .003390075283$, $a = 6378206.4$ meters, $\Delta\lambda_0 = 179° \; 51' \; 07''.554 = 3.1390112787$ radians,

$\pi = 3.1415926536$, $D = (1/f)[1 + (1/4)f + 2(f/4)^2] = 295.22912379$,

$u = (1/4)f - (f/4)^2 = .0008468005326$, $v = D(1 - \Delta\lambda_0/\pi) = .2425830253$,

$\cos\theta_0 = v - uv^3 = .24257094$, $\theta_0 = 75° \; 57' \; 42''.015$, $\alpha_{1\text{-}2} = 90° - \theta_0 = 14° \; 02' \; 17''.985$,

$\alpha_{2\text{-}1} = 270° + \theta_0 = 345° \; 57' \; 42''.015$, $A = 1 + \cos^2\theta_0 = 1.0588406609$,

$B = (1 + 3\cos^2\theta_0)(1 - \cos^2\theta_0) = 1.1072946516$,

$C = (1 + 2\cos^2\theta_0 + 5\cos^4\theta_0)(1 - \cos^2\theta_0) = 1.0682087334$,

$S_0 = a\pi[1 - 2(f/4)A + (f/4)^2B + 2(f/4)^3C] = 20001779.127$ meters.

Alternatively, when one has $\cos\theta_0$, S_0 may be computed from equations (54) after computing D from equations (49), Appendix 1. Since there are two reverse solutions for the geodesic, node to node, the azimuths of the second solution are $\alpha'_{1\text{-}2} = 90° + \theta_0 = 165° \; 57' \; 42''.015$, $\alpha'_{2\text{-}1} = 270° - \theta_0 = 194° \; 02' \; 17''.985$. Now from (150), Appendix 1, $\theta_0 = 75° \; 57' \; 42''.053$; from (152), (154) respectively $S_0 = 20001779.136$ meters, $\alpha_{1\text{-}2} = 14° \; 02' \; 17''.947$. Hence the computed values by use of equations (149), Appendix 1, are within the criteria adopted initially.

DIRECT AND INVERSE LINE COMPUTATIONS OVER A HEMISPHEROIDAL GEODESIC CONTAINING AN ACIC 6000 MILE ARC

(Clarke 1866 ellipsoid–See Appendix 1, Figure 26)

DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1, \alpha_{1\text{-}2}, S$ to find $\phi_2, \lambda_2, \alpha_{2\text{-}1}$. East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

CLARKE 1866 SPHEROID a 6378206.4 m f .003390075283

1 - f .99660992472 1 radian = 206264.8062 seconds

LINE VERTEX 1 TO P_1 (V_1P_1)

ϕ_1 -76° 00′ 26.641″ $\tan\phi_1$ -4.01298872 $\tan\theta_1 = (1-f)\tan\phi_1$ -3.99938439

$\alpha_{1\text{-}2}$ 90 0 0 $\sin\theta_1$ -.97013372 $\cos\theta_1$ .24257076 θ_1 -75° 57′ 42.053″

$\sin\alpha_{1\text{-}2}$ 1 $M = \cos\theta_0 = \cos\theta_1 \sin\alpha_{1\text{-}2}$ .24257076 θ_0 ±75 57 42.053

$\cos\alpha_{1\text{-}2}$ 0 $N = \cos\theta_1 \cos\alpha_{1\text{-}2}$ 0 $\sin\theta_0$ ±.97013371

$c_1 = fM$ .000822331378 $D = (1-c_2)(1-c_2-c_1M)$ .99820602 07

$c_2 = \frac{1}{4}(1-M^2)f$ .000797650327 1 $P = c_2(1+\frac{1}{2}c_1M)/D$ .000799163565 1

$\cos\sigma_1 = \sin\theta_1/\sin\theta_0$ -1 σ_1 180° 0′ 0″

$d = S/aD$ .25310676 14 (rad) d 14 30 07.017 S 1611471.024 m

$\sin d$ +.25041294 $u = 2(\sigma_1 - d)$ 330 59 45.966 $\sin u$ -.48486912

$\cos d$ +.96812912 $W = 1 - 2P\cos u$ +.998602 1243 $\cos u$ +.87458672

$V = \cos u \cos d - \sin u \sin d$ +.96813 91143 $Y = 2PVW \sin d$ +.000386948 1

$X = c_2^2 \sin d \cos d (2V^2 - 1)$ +.1349 x 10⁻⁶ $\Delta\sigma = d + X - Y$ .25271 99482 (rad)

$\sin\Delta\sigma$ .25003845 $\cos\Delta\sigma$ .96822591 $\Delta\sigma$ 14° 28′ 47.231″

$\cos\Sigma\sigma$ = cos Δσ $\Sigma\sigma = 2\sigma_1 - \Delta\sigma$ = Δσ

$\tan\alpha_{2\text{-}1} = M/(N\cos\Delta\sigma - \sin\theta_1 \sin\Delta\sigma)$ +1.00000012 $\alpha_{2\text{-}1}$ 225 00 00.012

$\tan\phi_2 = \dfrac{-(\sin\theta_1 \cos\Delta\sigma + N\sin\Delta\sigma)\sin\alpha_{2\text{-}1}}{(1-f)M}$ -2.74747757 $\sin\alpha_{2\text{-}1}$ -.70710682

ϕ_2 -70° 00′ 00.004″

$\tan\Delta\eta = \dfrac{\sin\Delta\sigma \sin\alpha_{1\text{-}2}}{\cos\theta_1 \cos\Delta\sigma - \sin\theta_1 \sin\Delta\sigma \cos\alpha_{1\text{-}2}}$ +1.06460172 $\Delta\eta$ 46° 47′ 31.954″

$H = c_1(1-c_2)\Delta\sigma - c_1c_2 \sin\Delta\sigma \cos\Sigma\sigma$ .000207 4954 (rad) H 42.799

$\Delta\lambda = \Delta\eta - H$ 46 46 49.155

λ_1 -151 04 18.392

CHECK

$M = \cos\theta_0 = \cos\theta_1 \sin\alpha_{1\text{-}2} = \cos\theta_2 \sin(180 + \alpha_{2\text{-}1})$ $\lambda_2 = \lambda_1 + \Delta\lambda$ -104° 17′ 29.237″

INVERSE POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1; \phi_2, \lambda_2$ to find S, $\alpha_{1\text{-}2}$, $\alpha_{2\text{-}1}$. Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

CLARKE 1866 SPHEROID a 6378206.4 m b ______ m

$1 - f = b/a$ .99660992472 ½f .00169503765 ¼f .000847518820(8)

$f^2/64$ .17957204×10⁻⁶ 1 radian = 206264.8062 seconds

ϕ_1 -26° 00′ 26.641″	1. VERTEX 1	λ_1 ______
ϕ_2 -70° 0′ 0″	2. P_1 (V, P_1)	λ_2 ______
$\tan \phi_1$ ______	1. always west of 2.	$\Delta\lambda = \lambda_2 - \lambda_1$ 46° 46′ 49.167″
$\tan \phi_2$ ______	$\tan \theta = (1 - f) \tan \phi$	$\Delta\lambda_m = ½\Delta\lambda$ 23° 23′ 24.584″
θ_2 -69° 56′ 14.590″	$\tan \theta_2$ -2.73816326	$\sin \Delta\lambda_m$ .39699031
θ_1 -75° 57′ 42.053″	$\tan \theta_1$ -3.99938439	$\tan \Delta\lambda$ 1.06415928
$\theta_m = ½(\theta_1 + \theta_2)$ -72° 56′ 58.322″	$\sin \theta_m$ -.95604687	$\cos \theta_m$ .29321391
$\Delta\theta_m = ½(\theta_2 - \theta_1)$ +3° 00′ 43.221″	$\sin \Delta\theta_m$ +.05254768	$\cos \Delta\theta_m$ .99861842
$H = \cos^2\Delta\theta_m - \sin^2\theta_m = \cos^2\theta_m - \sin^2\Delta\theta_m$ .083213134	$1 - L$ .984124243	
$L = \sin^2\Delta\theta_m + H \sin^2\Delta\lambda_m$ .0158757568	$\cos d = 1 - 2L$ +.96824849	
$U = 2 \sin^2\theta_m \cos^2\Delta\theta_m/(1 - L)$ 1.85241197	d 14° 28′ 36.852″	
$V = 2 \sin^2\Delta\theta_m \cos^2\theta_m/L$ .02990693	$\sin d$ .24998971	d (rad) .2526696287
$X = U + V$ 1.88231890	$T = d/\sin d$ 1.010720116	$E = 2 \cos d$ 1.93649698
$Y = U - V$ 1.82250504	$D = 4T^2$ 4.086220612	$B = 2D$ 8.172441224
$A = DE$ 7.912953875	$C = T - ½(A - E)$ -1.977508231	CHECK $C - ½E + AD/B = T$
$n_1 = X(A + CX)$ 7.888144573	$n_2 = Y(B + EY)$ 21.326437717	$n_3 = DXY$ 14.817925615
$\delta_1 d = ¼f(TX - Y)$ .67795×10⁻⁴	$\delta_2 d = (f^2/64)(n_1 - n_2 + n_3)$ .124×10⁻⁶	
$S_1 = a \sin d (T - \delta_1 d)$ 1611470.945 m	$S_2 = a \sin d (T - \delta_1 d + \delta_2 d)$ 1611471.110 m	
$F = 2Y - E(4 - X)$ -.456872975	$M = 32T - (20T - A)X - (B + 4)Y$ -17.996540673	
$G = ½fT + (f^2/64)M$ .00171087483	$Q = -(FG \tan \Delta\lambda)/4$ + 42.799	
$\Delta\lambda_m' = ½(\Delta\lambda + Q)$ 23° 23′ 45.984″	$\tan \Delta\lambda_m'$ .43265297	
$v = \arctan \lvert c_2 \rvert$ 67° 29′ 59.999″	$c_2 = \cos \Delta\theta_m/(\sin \theta_m \tan \Delta\lambda_m')$ -2.41421353	
$u = \arctan \lvert c_1 \rvert$ 22° 29′ 59.993″	$c_1 = -\sin \Delta\theta_m/(\cos \theta_m \tan \Delta\lambda_m')$ -.41421357	
$\alpha_1 = v - u$ 45° 00′ 00.006″	$\alpha_2 = v + u$ 89° 59′ 59.992″	

c_1	c_2	$\alpha_{1\text{-}2}$	° ′ ″	$\alpha_{2\text{-}1}$	° ′ ″
-	+	α_1		$360 - \alpha_2$	
+	+	α_2		$360 - \alpha_1$	
-	-	$180 - \alpha_2$	90 00 00.008	$180 + \alpha_1$	225 00 00.006
+	-	$180 - \alpha_1$		$180 + \alpha_2$	

DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1, \alpha_{1\text{-}2}$, S to find $\phi_2, \lambda_2, \alpha_{2\text{-}1}$. East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

CLARKE 1866 SPHEROID a 6378206.4 m f .00339007528 3

1 − f .99660992472 1 radian = 206264.8062 seconds

LINE VERTEX 1 TO VERTEX 2 ($V_1 V_2$)

ϕ_1 −76° 00′ 26.141″ $\tan\phi_1$ −4.0129887 $\tan\theta_1 = (1-f)\tan\phi_1$ −3.99938439

$\alpha_{1\text{-}2}$ 90 0 0 $\sin\theta_1$ −.97013371 $\cos\theta_1$ .24257076 θ_1 −75° 57′ 42.053″

$\sin\alpha_{1\text{-}2}$ 1 $M = \cos\theta_0 = \cos\theta_1 \sin\alpha_{1\text{-}2}$ .24257076 θ_0 +75 57 42.053

$\cos\alpha_{1\text{-}2}$ 0 $N = \cos\theta_1 \cos\alpha_{1\text{-}2}$ 0 $\sin\theta_0$ +.97013371

$c_1 = fM$ .0008223331378 $D = (1-c_2)(1-c_2-c_1M)$ .9982060207

$c_2 = \frac{1}{4}(1-M^2)f$ .00079765032271 $P = c_2(1 + \frac{1}{2}c_1M)/D$ .0007991635651

$\cos\sigma_1 = \sin\theta_1/\sin\theta_0$ −1 σ_1 180° 0′ 0″

$d = S/aD$ 3.1415926588 (rad) d 180 0 0 S 20001779.136 m

$\sin d$ 0 $u = 2(\sigma_1 - d)$ 0 $\sin u$ 0

$\cos d$ −1 $W = 1 - 2P\cos u$ $\cos u$ +1

$V = \cos u \cos d - \sin u \sin d$ −1 $Y = 2PVW \sin d$ 0

$X = c_2^2 \sin d \cos d (2V^2 - 1)$ 0 $\Delta\sigma = d + X - Y$ 3.1415926588 (rad)

$\sin\Delta\sigma$ 0 $\cos\Delta\sigma$ −1 $\Delta\sigma$ 180° 0′ 0″

$\cos\Sigma\sigma$ −1 $\Sigma\sigma = 2\sigma_1 - \Delta\sigma$ 180° 0′ 0″

$\tan\alpha_{2\text{-}1} = M/(N\cos\Delta\sigma - \sin\theta_1 \sin\Delta\sigma)$ → −∞ $\alpha_{2\text{-}1}$ 270 0 0

$\tan\phi_2 = \dfrac{-(\sin\theta_1 \cos\Delta\sigma + N\sin\Delta\sigma)\sin\alpha_{2\text{-}1}}{(1-f)M}$ +4.0129887 $\sin\alpha_{2\text{-}1}$ −1

ϕ_2 76° 00′ 26.141″

$\tan\Delta\eta = \dfrac{\sin\Delta\sigma \sin\alpha_{1\text{-}2}}{\cos\theta_1 \cos\Delta\sigma - \sin\theta_1 \sin\Delta\sigma \cos\alpha_{1\text{-}2}}$ 0 $\Delta\eta$ 179° 59′ 60.000″

$H = c_1(1-c_2)\Delta\sigma - c_1c_2 \sin\Delta\sigma \cos\Sigma\sigma$ .002581837507 (rad) H 8 52.447

$\Delta\lambda = \Delta\eta - H$ 179 51 07.553

λ_1 −151 04 18.387

CHECK

$M = \cos\theta_0 = \cos\theta_1 \sin\alpha_{1\text{-}2} = \cos\theta_2 \sin(180 + \alpha_{2\text{-}1})$ $\lambda_2 = \lambda_1 + \Delta\lambda$ 28° 46′ 49.166″

INVERSE POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1; \phi_2, \lambda_2$ to find S, $\alpha_{1\text{-}2}$, $\alpha_{2\text{-}1}$. Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

CLARKE 1866 SPHEROID a 6378206.4 m b ______ m

$1 - f = b/a$ ______ $\frac{1}{2}f$ .001695037642 $\frac{1}{4}f$ .000847518820

$f^2/64$ .17957204 $\times 10^{-6}$ 1 radian = 206264.8062 seconds

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	° ′ ″				° ′ ″
ϕ_1		1.	VERTEX 1	λ_1	
ϕ_2		2.	NODE 1 (V,N₁)	λ_2	
$\tan\phi_1$		1. always west of 2.		$\Delta\lambda = \lambda_2 - \lambda_1$	89 55 33.777
$\tan\phi_2$		$\tan\theta = (1-f)\tan\phi$		$\Delta\lambda_m = \frac{1}{2}\Delta\lambda$	44 57 46.888
θ_2	0	$\tan\theta_2$		$\sin\Delta\lambda_m$	.70665031
θ_1	-75 57 42.053	$\tan\theta_1$		$\tan\Delta\lambda$	774.78549

$\theta_m = \frac{1}{2}(\theta_1 + \theta_2)$ −37 57 51.027 $\sin\theta_m$ −.61539793 $\cos\theta_m$ +.78821658

$\Delta\theta_m = \frac{1}{2}(\theta_2 - \theta_1)$ +37 58 51.027 $\sin\Delta\theta_m$ +.61539793 $\cos\Delta\theta_m$ +.78821658

$H = \cos^2\Delta\theta_m - \sin^2\theta_m = \cos^2\theta_m - \sin^2\Delta\theta_m$ .24257076 $1 - L$ .50015648

$L = \sin^2\Delta\theta_m + H\sin^2\Delta\lambda_m$ .49984352 $\cos d = 1 - 2L$ +.00031310

$U = 2\sin^2\theta_m\cos^2\Delta\theta_m/(1-L)$ .94086422 d 89° 58′ 55.″419

$V = 2\sin^2\Delta\theta_m\cos^2\theta_m/L$ .94145468 $\sin d$ .99999995 d (rad) 1.57048323

$X = U + V$ 1.88231899 $T = d/\sin d$ 1.570483309 $E = 2\cos d$ .00062620

$Y = U - V$ −.000589346 $D = 4T^2$ 9.865671295 $B = 2D$ 19.73134259

$A = DE$ .006177883 4 $C = T - \frac{1}{2}(A - E)$ 1.567707467 CHECK $C - \frac{1}{2}E + AD/B = T$

$n_1 = X(A + CX)$ 5.566211922 $n_2 = Y(B + EY)$ −.011628588 $n_3 = DXY$ −.010944356

$\delta_1 d = \frac{1}{4}f(TX - Y)$ .002505893 $\delta_2 d = (f^2/64)(n_1 - n_2 + n_3)$ .9997 $\times 10^{-6}$

$S_1 = a\sin d(T - \delta_1 d)$ 10 000 383.090 m $S_2 = a\sin d(T - \delta_1 d + \delta_2 d)$ 10 000 889.466 m

$F = 2Y - E(4 - X)$ −.002504785 85 $M = 32T - (20T - A)X - (B + 4)Y$ −8.841930513

$G = \frac{1}{2}fT + (f^2/64)M$ .002660440 56 $Q = -(FG\tan\Delta\lambda)/4$ + 4′ 26.″238

$\Delta\lambda_m' = \frac{1}{2}(\Delta\lambda + Q)$ 45° 00′ 00.″007 $\tan\Delta\lambda_m'$ 1.00000007

$v = \arctan|c_2|$ 52° 01′ 08.″967 $c_2 = \cos\Delta\theta_m/(\sin\theta_m\tan\Delta\lambda_m')$ −1.28082414

$u = \arctan|c_1|$ 37 58 51.019 $c_1 = -\sin\Delta\theta_m/(\cos\theta_m\tan\Delta\lambda_m')$ −.78074720

$\alpha_1 = v - u$ 14° 02′ 17.″948 $\alpha_2 = v + u$ 89° 59′ 59.″986

c_1	c_2	$\alpha_{1\text{-}2}$	° ′ ″	$\alpha_{2\text{-}1}$	° ′ ″
−	+	α_1		$360 - \alpha_2$	
+	+	α_2		$360 - \alpha_1$	
−	−	$180 - \alpha_2$	90 00 00.014	$180 + \alpha_1$	194 02 17.948
+	−	$180 - \alpha_1$		$180 + \alpha_2$	

DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1, \alpha_{1\text{-}2}$, S to find $\phi_2, \lambda_2, \alpha_{2\text{-}1}$. East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

CLARKE 1866 SPHEROID a 6378206.4 m f

1 - f .99660992472   1 radian = 206264.8062 seconds

LINE P_1 TO NODE 1 (P_1N_1)

ϕ_1 -70 $\tan \phi_1$ $\tan \theta_1 = (1 - f) \tan \phi_1$

$\alpha_{1\text{-}2}$ 45 $\sin \theta_1$ -.9393 1830 $\cos \theta_1$ .3430 4686 θ_1 -69° 56′ 14.590″

$\sin \alpha_{1\text{-}2}$ .7071 0678 $M = \cos \theta_0 = \cos \theta_1 \sin \alpha_{1\text{-}2}$ .2425 7076 θ_0 75 57 42.053

$\cos \alpha_{1\text{-}2}$ .7071 0678 $N = \cos \theta_1 \cos \alpha_{1\text{-}2}$ .2425 7076 $\sin \theta_0$ .9701 2371

$c_1 = fM$ .0008223331378 $D = (1 - c_2)(1 - c_2 - c_1 M)$ .9982060207

$c_2 = \frac{1}{4}(1 - M^2)f$ .0007976503271 $P = c_2 (1 + \frac{1}{2} c_1 M)/D$ .0007991635651

$\cos \sigma_1 = \sin \theta_1 / \sin \theta_0$ -.9682 3591 σ_1 165° 31′ 12.769″

$d = S/aD$ 1.3176895682 (rad)   d 75 29 52.983   S 8389418.546 m

$\sin d$ .96813912 $u = 2(\sigma_1 - d)$ 180 02 39.572 $\sin u$ - .00077362

$\cos d$ .25041294 $W = 1 - 2P \cos u$ 1.0015983267 $\cos u$ - .99999970

$V = \cos u \cos d - \sin u \sin d$ -.24966 38931 $Y = 2PVW \sin d$ -.0003869481

$X = c_2^2 \sin d \cos d (2V^2 - 1)$ $-.135 \times 10^{-6}$ $\Delta\sigma = d + X - Y$ 1.3180763813 (rad)

$\sin \Delta\sigma$ .9682 3591 $\cos \Delta\sigma$ .2500 3843 $\Delta\sigma$ 75° 31′ 12.769″

$\cos \Sigma\sigma$ -.2500 3843 $\Sigma\sigma = 2\sigma_1 - \Delta\sigma$ 255° 31′ 12.769″

$\tan \alpha_{2\text{-}1} = M/(N \cos \Delta\sigma - \sin \theta_1 \sin \Delta\sigma)$ +.2500 2848 $\alpha_{2\text{-}1}$ 194 02 17.946

$\tan \phi_2 = \dfrac{-(\sin \theta_1 \cos \Delta\sigma + N \sin \Delta\sigma) \sin \alpha_{2\text{-}1}}{(1 - f)M}$ $+.5 \times 10^{-7}$ $\sin \alpha_{2\text{-}1}$ -.2425 7076

ϕ_2 0° 0′ 00.010″

$\tan \Delta\eta = \dfrac{\sin \Delta\sigma \sin \alpha_{1\text{-}2}}{\cos \theta_1 \cos \Delta\sigma - \sin \theta_1 \sin \Delta\sigma \cos \alpha_{1\text{-}2}}$ +.9393 1831 $\Delta\eta$ 43° 12′ 28.035″

$H = c_1(1 - c_2)\Delta\sigma - c_1 c_2 \sin \Delta\sigma \cos \Sigma\sigma$ .0010831 92114 (rad)   H 3 43.424

$\Delta\lambda = \Delta\eta - H$ 43 08 44.611

λ_1 - 104 17 29.220

CHECK

$M = \cos \theta_0 = \cos \theta_1 \sin \alpha_{1\text{-}2} = \cos \theta_2 \sin (180 + \alpha_{2\text{-}1})$ $\lambda_2 = \lambda_1 + \Delta\lambda$ 61° 08′ 44.609″

INVERSE POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1; \phi_2, \lambda_2$ to find S, α_{1-2}, α_{2-1}. Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

CLARKE 1866 SPHEROID a 6378206.4 m b ______ m

$1 - f = b/a$ .9966099 2472 ½f .001695 037642 ¼f .0008475 188208

$f^2/64$ .17957204 x 10^{-6} 1 radian = 206264.8062 seconds

	° ′ ″				° ′ ″
ϕ_1		1.	P_1	λ_1	
ϕ_2		2.	NODE 1 (P_1N_1)	λ_2	
$\tan\phi_1$		1. always west of 2.		$\Delta\lambda = \lambda_2 - \lambda_1$	43 08 44.610
$\tan\phi_2$		$\tan\theta = (1-f)\tan\phi$		$\Delta\lambda_m = ½\Delta\lambda$	21 34 22.305
θ_2	0	$\tan\theta_2$		$\sin\Delta\lambda_m$	.3676 8414
θ_1	-69 56 14.590	$\tan\theta_1$		$\tan\Delta\lambda$	.9372 8147
$\theta_m = ½(\theta_1+\theta_2)$	-34 58 07.295	$\sin\theta_m$	-.57312876	$\cos\theta_m$	+.81946533
$\Delta\theta_m = ½(\theta_2-\theta_1)$	+34 58 07.295	$\sin\Delta\theta_m$	+.57312876	$\cos\Delta\theta_m$	+.81946533

$H = \cos^2\Delta\theta_m - \sin^2\theta_m = \cos^2\theta_m - \sin^2\Delta\theta_m$ .3430468515 $1 - L$ .625146363

$L = \sin^2\Delta\theta_m + H\sin^2\Delta\lambda_m$ .37485 3637 $\cos d = 1 - 2L$ .2502 9273

$U = 2\sin^2\theta_m\cos^2\Delta\theta_m/(1-L)$ .70568 9831 d 75° 30′ 18″.596

$V = 2\sin^2\Delta\theta_m\cos^2\theta_m/L$ 1.17688449 sin d .9681 7021 d (rad) 1.317813742

$X = U + V$ 1.882574321 $T = d/\sin d$ 1.361138494 $E = 2\cos d$ .50058546

$Y = U - V$ -.471194659 $D = 4T^2$ 7.410791999 $B = 2D$ 14.821583999

$A = DE$ 3.709734722 $C = T - ½(A - E)$ -.243436137 CHECK $C - ½E + AD/B = T$

$n_1 = X(A + CX)$ 6.121092702 $n_2 = Y(B + EY)$ -6.872709029 $n_3 = DXY$ -6.573809482

$\delta_1 d = ¼f(TX - Y)$ +.002571066 $\delta_2 d = (f^2/64)(n_1 - n_2 + n_3)$ +.1153 x 10^{-5}

$S_1 = a\sin d(T - \delta_1 d)$ 8289411.221 m $S_2 = a\sin d(T - \delta_1 d + \delta_2 d)$ 8289418.341 m

$F = 2Y - E(4 - X)$ -2.00234183 $M = 32T - (20T - A)X - (B + 4)Y$ +8.160075465

$G = ½fT + (f^2/64)M$ .002308 6463 $Q = -(FG\tan\Delta\lambda)/4$ + 3′ 43″.424

$\Delta\lambda'_m = ½(\Delta\lambda + Q)$ 21° 36′ 14″.017 $\tan\Delta\lambda'_m$ .3960 0662

$v = \arctan|c_2|$ 74° 31′ 08″.972 $c_2 = \cos\Delta\theta_m/(\sin\theta_m\tan\Delta\lambda'_m)$ -3.6105714

$u = \arctan|c_1|$ 60 28 51.027 $c_1 = -\sin\Delta\theta_m/(\cos\theta_m\tan\Delta\lambda'_m)$ -1.7661578

$\alpha_1 = v - u$ 14° 02′ 17″.945 $\alpha_2 = v + u$ 134° 59′ 59″.999

c_1	c_2	α_{1-2}	° ′ ″	α_{2-1}	° ′ ″
-	+	α_1		$360 - \alpha_2$	
+	+	α_2		$360 - \alpha_1$	
-	-	$180 - \alpha_2$	45 00 00.001	$180 + \alpha_1$	194 02 17.945
+	-	$180 - \alpha_1$		$180 + \alpha_2$	

DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1, \alpha_{1-2}$, S to find $\phi_2, \lambda_2, \alpha_{2-1}$. East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

CLARKE 1866 SPHEROID a 6378206.4 m f

1 - f .99660992472 1 radian = 206264.8062 seconds

LINE NODE 1 TO P_2 (N, P_2)

ϕ: 0 ° ′ ″ $\tan\phi_1$ $\tan\theta_1 = (1 - f)\tan\phi_1$

α_{1-2} 14 02 17.947 $\sin\theta_1$ 0 $\cos\theta_1$ 1 θ_1 0 ° ′ ″

$\sin\alpha_{1-2}$ = $\cos\theta_0$ = M $M = \cos\theta_0 = \cos\theta_1 \sin\alpha_{1-2}$ .2425 7076 θ_0 75 57 42.053

$\cos\alpha_{1-2}$ = $\sin\theta_0$ = N $N = \cos\theta_1 \cos\alpha_{1-2}$ .9701 3371 $\sin\theta_0$ .9701 3371

$c_1 = fM$ .0008222331378 $D = (1 - c_2)(1 - c_2 - c_1 M)$ .9982060207

$c_2 = \frac{1}{4}(1 - M^2)f$ .000796503271 $P = c_2(1 + \frac{1}{2}c_1 M)/D$ .000799163565 1

$\cos\sigma_1 = \sin\theta_1/\sin\theta_0$ 0 σ_1 90° 0′ 0″

$d = S/aD$ .30728 06727 (rad) d 17 36 21.188 S 1956383.534 m

$\sin d$ .3024 6781 $u = 2(\sigma_1 - d)$ 144 47 17.624 $\sin u$ + .5766 0018

$\cos d$ .9531 5960 $W = 1 - 2P\cos u$ +1.0013058756 $\cos u$ − .8170 2646

$V = \cos u \cos d - \sin u \sin d$ −.9531596075 $Y = 2PVW \sin d$ −.00046 13996

$X = c_2^2 \sin d \cos d\,(2V^2 - 1)$ + .1494 x 10^{-6} $\Delta\sigma = d + X - Y$ .30774 22217 (rad)

$\sin\Delta\sigma$ .3029 0770 $\cos\Delta\sigma$ .9530 1990 $\Delta\sigma$ 17° 37′ 56.390″

$\cos\Sigma\sigma$ −.9530 1990 $\Sigma\sigma = 2\sigma_1 - \Delta\sigma$ $\pi - \Delta\sigma$

$\tan\alpha_{2-1} = M/(N\cos\Delta\sigma - \sin\theta_1 \sin\Delta\sigma)$ +.2623 6439 α_{2-1} 194° 42′ 03.724″

$\tan\phi_2 = \dfrac{-(\sin\theta_1 \cos\Delta\sigma + N\sin\Delta\sigma)\sin\alpha_{2-1}}{(1 - f)M}$ +.3084 8055 $\sin\alpha_{2-1}$ −.2537 7541

ϕ_2 17° 08′ 38.316″

$\tan\Delta\eta = \dfrac{\sin\Delta\sigma \sin\alpha_{1-2}}{\cos\theta_1 \cos\Delta\sigma - \sin\theta_1 \sin\Delta\sigma \cos\alpha_{1-2}}$ +.0770 9865 $\Delta\eta$ 4° 24′ 31.340″

$H = c_1(1 - c_2)\Delta\sigma - c_1 c_2 \sin\Delta\sigma \cos\Sigma\sigma$ .00025305415 92 (rad) H 52.196

$\Delta\lambda = \Delta\eta - H$ 4 23 39.144

λ_1 −61 08 44.610

CHECK

$M = \cos\theta_0 = \cos\theta_1 \sin\alpha_{1-2} = \cos\theta_2 \sin(180 + \alpha_{2-1})$ $\lambda_2 = \lambda_1 + \Delta\lambda$ −56° 45′ 05.466″

INVERSE POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1; \phi_2, \lambda_2$ to find S, $\alpha_{1-2}, \alpha_{2-1}$. Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

CLARKE 1866 SPHEROID a 6378206.4 m b ______ m

$1 - f = b/a$ .99660992472 ½f .001695037642 ¼f .0008475188208

$f^2/64$ .17957204 $\times 10^{-6}$ 1 radian = 206264.8062 seconds

° ′ ″		° ′ ″
ϕ_1 ______	1. NODE 1	λ_1 ______
ϕ_2 ______	2. P2 (N, P2)	λ_2 ______
$\tan\phi_1$ ______	1. always west of 2.	$\Delta\lambda = \lambda_2 - \lambda_1$ 4 23 39.146
$\tan\phi_2$ ______	$\tan\theta = (1 - f)\tan\phi$	$\Delta\lambda_m = ½\Delta\lambda$ 2 11 49.573
θ_2 17° 05′ 21.″296	$\tan\theta_2$ ______	$\sin\Delta\lambda_m$ .0383 3729
θ_1 0	$\tan\theta_1$ ______	$\tan\Delta\lambda$ .0768 4410
$\theta_m = ½(\theta_1 + \theta_2)$ 8° 32′ 40.″648	$\sin\theta_m$ .1485 7965	$\cos\theta_m$ .9889 0044
$\Delta\theta_m = ½(\theta_2 - \theta_1)$ 8 32 40.648	$\sin\Delta\theta_m$ .1485 7965	$\cos\Delta\theta_m$ .9889 0044

$H = \cos^2\Delta\theta_m - \sin^2\theta_m = \cos^2\theta_m - \sin^2\Delta\theta_m$ .955848178 $1 - L$ .97651 92319

$L = \sin^2\Delta\theta_m + H\sin^2\Delta\lambda_m$ .0234807681 4 $\cos d = 1 - 2L$ .9530 3846

$U = 2\sin^2\theta_m\cos^2\Delta\theta_m/(1 - L)$ .0442153429 d 17° 37′ 43.″748

$V = 2\sin^2\Delta\theta_m\cos^2\theta_m/L$ 1.838829644 $\sin d$ .30284930 d (rad) .307680933

$X = U + V$ 1.883044992 $T = d/\sin d$ 1.015953918 $E = 2\cos d$ 1.906076692

$Y = U - V$ -1.794614306 $D = 4T^2$ 4.128649454 $B = 2D$ 8.257298908

$A = DE$ 7.869523435 $C = T - ½(A - E)$ -1.96576934 CHECK $C - ½E + AD/B = T$

$n_1 = X(A + CX)$ 7.848376885 $n_2 = Y(B + EY)$ -8.679878211 $n_3 = DXY$ -13.952108105

$\delta_1 d = ¼f(TX - Y)$ .003142347 $\delta_2 d = (f^2/64)(n_1 - n_2 + n_3)$ +.463×10^{-6}

$S_1 = a\sin d(T - \delta_1 d)$ 1956382.627 m $S_2 = a\sin d(T - \delta_1 d + \delta_2 d)$ 1956383.521 m

$F = 2Y - E(4 - X)$ -7.624307693 $M = 32T - (20T - A)X - (B + 4)Y$ 31.0645773

$G = ½fT + (f^2/64)M$ .001727658463 $Q = -(FG\tan\Delta\lambda)/4$ 52.″196

$\Delta\lambda_m' = ½(\Delta\lambda + Q)$ 2° 12′ 15.″671 $\tan\Delta\lambda_m'$ .0384 9221

$v = \arctan|c_2|$ 89° 40′ 07.″111 $c_2 = \cos\Delta\theta_m/(\sin\theta_m\tan\Delta\lambda_m')$ +172.91011

$u = \arctan|c_1|$ 75 37 49.164 $c_1 = -\sin\Delta\theta_m/(\cos\theta_m\tan\Delta\lambda_m')$ -3.9033178

$\alpha_1 = v - u$ 14° 02′ 17.″947 $\alpha_2 = v + u$ 165° 17′ 56.″275

c_1	c_2	α_{1-2}		α_{2-1}	
-	+	α_1	14° 02′ 17.″947	$360 - \alpha_2$	194° 42′ 03.″725
+	+	α_2		$360 - \alpha_1$	
-	-	$180 - \alpha_2$		$180 + \alpha_1$	
+	-	$180 - \alpha_1$		$180 + \alpha_2$	

DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1, \alpha_{1\text{-}2}$, S to find $\phi_2, \lambda_2, \alpha_{2\text{-}1}$. East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

CLARKE 1866 SPHEROID a 6378206.4 m f .003390075283

1 - f .99660992472 1 radian = 206264.8062 seconds

LINE P1 TO P2 (P1P2)

ϕ_1 ____ ° ′ ″ $\tan\phi_1$ ____ $\tan\theta_1 = (1 - f)\tan\phi_1$ ____

$\alpha_{1\text{-}2}$ 45 $\sin\theta_1$ -.9393 1830 $\cos\theta_1$ .3430 4686 θ_1 -69° 56′ 14.590″

$\sin\alpha_{1\text{-}2}$ .7071 0678 $M = \cos\theta_0 = \cos\theta_1 \sin\alpha_{1\text{-}2}$ .2425 7076 θ_0 75° 57′ 42.053″

$\cos\alpha_{1\text{-}2}$ .7071 0678 $N = \cos\theta_1 \cos\alpha_{1\text{-}2}$ .2425 7076 $\sin\theta_0$ .9701 3371

$c_1 = fM$ .00082233313 78 $D = (1 - c_2)(1 - c_2 - c_1 M)$ .9992060207

$c_2 = \frac{1}{4}(1 - M^2)f$ .00079765032 71 $P = c_2(1 + \frac{1}{2}c_1 M)/D$ .00079916356 51

$\cos\sigma_1 = \sin\theta_1/\sin\theta_0$ -.9682 3591 σ_1 165° 31′ 12.769″

$d = S/aD$ 1.6249702409 (rad) d 93° 06′ 14.172″ S 10346802.079 m

$\sin d$ .9985 3295 $u = 2(\sigma_1 - d)$ 144° 49′ 57.194″ $\sin u$ .5759 4794

$\cos d$ -.0541 4742 $W = 1 - 2P\cos u$ 1.0013065 88 $\cos u$ -.8174 7228

$V = \cos u \cos d - \sin u \sin d$ -.53085 89514 $Y = 2PVW \sin d$ -.0008 48 3485

$X = c_2^2 \sin d \cos d\,(2V^2 - 1)$ +.15 × 10^-7 $\Delta\sigma = d + X - Y$ 1.6258186044 (rad)

$\sin\Delta\sigma$ .9984 8666 $\cos\Delta\sigma$ -.0549 9452 $\Delta\sigma$ 93° 09′ 09.159″

$\cos\Sigma\sigma$ -.5315 7772 $\Sigma\sigma = 2\sigma_1 - \Delta\sigma$ 237° 53′ 16.379″

$\tan\alpha_{2\text{-}1} = M/(N\cos\Delta\sigma - \sin\theta_1 \sin\Delta\sigma)$ +.2623 3 6439 $\alpha_{2\text{-}1}$ 194° 42′ 03.724″

$\tan\phi_2 = \dfrac{-(\sin\theta_1 \cos\Delta\sigma + N \sin\Delta\sigma)\sin\alpha_{2\text{-}1}}{(1 - f)M}$ +.3084 8061 $\sin\alpha_{2\text{-}1}$ -.2537 7541

ϕ_2 17° 08′ 38.338″

$\tan\Delta\eta = \dfrac{\sin\Delta\sigma \sin\alpha_{1\text{-}2}}{\cos\theta_1 \cos\Delta\sigma - \sin\theta_1 \sin\Delta\sigma \cos\alpha_{1\text{-}2}}$ +1.0957 7304 $\Delta\eta$ 47° 36′ 59.376″

$H = c_1(1 - c_2)\Delta\sigma - c_1 c_2 \sin\Delta\sigma \cos\Sigma\sigma$ .0013362 4634 (rad) H 4′ 35.621″

$\Delta\lambda = \Delta\eta - H$ 47° 32′ 23.755″

λ_1 -104° 17′ 29.220″

CHECK

$M = \cos\theta_0 = \cos\theta_1 \sin\alpha_{1\text{-}2} = \cos\theta_2 \sin(180 + \alpha_{2\text{-}1})$ $\lambda_2 = \lambda_1 + \Delta\lambda$ -56° 45′ 05.465″

INVERSE POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1; \phi_2, \lambda_2$ to find S, $\alpha_{1-2}, \alpha_{2-1}$. Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

CLARKE 1866 SPHEROID a 6378206.4 m b ______ m

$1 - f = b/a$ .99660992472 ½f .001695037642 ¼f .000 8475188208

$f^2/64$ .17957204 ×10⁻⁶ 1 radian = 206264.8062 seconds

	° ′ ″			° ′ ″
ϕ_1		1. P_1	λ_1	
ϕ_2		2. P_2 (P_1P_2)	λ_2	

$\tan\phi_1$ ______ 1. always west of 2. $\Delta\lambda = \lambda_2 - \lambda_1$ 47 32 23.756

$\tan\phi_2$ ______ $\tan\theta = (1-f)\tan\phi$ $\Delta\lambda_m = ½\Delta\lambda$ 23 46 11.878

θ_2 17° 05′ 21″.296 $\tan\theta_2$ ______ $\sin\Delta\lambda_m$ .4030 6563

θ_1 -69 56 14.590 $\tan\theta_1$ ______ $\tan\Delta\lambda$ 1.09283664

$\theta_m = ½(\theta_1 + \theta_2)$ -26° 25′ 26″.647 $\sin\theta_m$ -.4450 1141 $\cos\theta_m$ .8955 2490

$\Delta\theta_m = ½(\theta_2 - \theta_1)$ 43 30 47.943 $\sin\Delta\theta_m$ .6885 2316 $\cos\Delta\theta_m$ .7252 1435

$H = \cos^2\Delta\theta_m - \sin^2\theta_m = \cos^2\theta_m - \sin^2\Delta\theta_m$ .327900703 $1 - L$ .47266 4486

$L = \sin^2\Delta\theta_m + H\sin^2\Delta\lambda_m$ .52733 5514 $\cos d = 1 - 2L$ -.0546 7103

$U = 2\sin^2\theta_m\cos^2\Delta\theta_m/(1-L)$ .4407091768 d 93° 08′ 02″.335

$V = 2\sin^2\Delta\theta_m\cos^2\theta_m/L$ 1.4419009024 sin d .9985 0442 d (rad) 1.625494631

$X = U + V$ 1.882610079 $T = d/\sin d$ 1.627929329 $E = 2\cos d$ -.10934206

$Y = U - V$ -1.001191726 $D = 4T^2$ 10.6006156 $B = 2D$ 21.2012312

$A = DE$ -1.159093147 $C = T - ½(A - E)$ 2.157804873 CHECK $C - ½E + AD/B = T$

$n_1 = X(A + CX)$ 5.447895173 $n_2 = Y(B + EY)$ -21.336100085 $n_3 = DXY$ -19.98060884

$\delta_1 d = ¼f(TX - Y)$ .003445967 $\delta_2 d = (f^2/64)(n_1 - n_2 + n_3)$ +.1222×10⁻⁵

$S_1 = a\sin d(T - \delta_1 d)$ 10345724.038 m $S_2 = a\sin d(T - \delta_1 d + \delta_2 d)$ 10345801.820 m

$F = 2Y - E(4 - X)$ -1.770863676 $M = 32T - (20T - A)X - (B + 4)Y$ +13.84775896

$G = ½fT + (f^2/64)M$ .00276188816 $Q = -(FG\tan\Delta\lambda)/4$ 4′ 35″.621

$\Delta\lambda'_m = ½(\Delta\lambda + Q)$ 23° 48′ 29″.688 $\tan\Delta\lambda'_m$ .4412 2449

$v = \arctan|c_2|$ 74° 51′ 01″.861 $c_2 = \cos\Delta\theta_m/(\sin\theta_m\tan\Delta\lambda'_m)$ -3.6934790

$u = \arctan|c_1|$ 60 08 58.138 $c_1 = -\sin\Delta\theta_m/(\cos\theta_m\tan\Delta\lambda'_m)$ -1.74258407

$\alpha_1 = v - u$ 14° 42′ 03″.723 $\alpha_2 = v + u$ 134° 59′ 59″.999

c_1	c_2	α_{1-2}	° ′ ″	α_{2-1}	° ′ ″
-	+	α_1		$360 - \alpha_2$	
+	+	α_2		$360 - \alpha_1$	
-	-	$180 - \alpha_2$	45 00 00.001	$180 + \alpha_1$	194 42 03.723
+	-	$180 - \alpha_1$		$180 + \alpha_2$	

DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1, \alpha_{1-2}$, S to find $\phi_2, \lambda_2, \alpha_{2-1}$. East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

CLARKE 1866 SPHEROID a 6378206.4 m f ______

1 - f .99660992477 1 radian = 206264.8062 seconds

LINE P2 TO INITIAL (P2I)

ϕ_1 ______ $\tan\phi_1$ ______ $\tan\theta_1 = (1-f)\tan\phi_1$ ______

α_{1-2} 14 42 03.723 $\sin\theta_1$ .2938 6097 $\cos\theta_1$ .9558 4817 θ_1 17 05 21.296

$\sin\alpha_{1-2}$ .2537 7541 $M = \cos\theta_0 = \cos\theta_1 \sin\alpha_{1-2}$ .2425 7076 θ_0 ______

$\cos\alpha_{1-2}$ .9672 6317 $N = \cos\theta_1 \cos\alpha_{1-2}$ .9245 5673 $\sin\theta_0$ .9701 3371

$c_1 = fM$ .000822333137 D = $(1-c_2)(1-c_2-c_1M)$ .99820602 07

$c_2 = \frac{1}{4}(1-M^2)f$ .0007976503271 $P = c_2(1+\frac{1}{2}c_1M)/D$ .0007991635651

$\cos\sigma_1 = \sin\theta_1/\sin\theta_0$ .3029 0770 σ_1 72 22 03.610

d=S/aD 1.0104088953 (rad) d 57 53 31.795 S 6433035.010 m

sin d .8470 4925 $u = 2(\sigma_1 - d)$ 28 57 03.630 sin u .4840 6159

cos d .5315 1441 W = 1 - 2P cos u .9986014095 cos u .8750 3393

V = cos u cos d - sin u sin d .0550691363 Y = 2PVW sin d .744517×10⁻⁴

$X = c_2^2 \sin d \cos d (2V^2 - 1)$ -.2847×10⁻⁶ $\Delta\sigma = d + X - Y$ 1.0103341589 (rad)

$\sin\Delta\sigma$ .8470 0952 $\cos\Delta\sigma$ .5315 7772 $\Delta\sigma$ 57 53 16.379

$\cos\Sigma\sigma$ .0549 9452 $\Sigma\sigma = 2\sigma_1 - \Delta\sigma$ 86 50 50.841

$\tan\alpha_{2-1} = M/(N\cos\Delta\sigma - \sin\theta_1\sin\Delta\sigma)$ 1.000000165 α_{2-1} 225 00 00.017

$\tan\phi_2 = \dfrac{-(\sin\theta_1\cos\Delta\sigma + N\sin\Delta\sigma)\sin\alpha_{2-1}}{(1-f)M}$ +2.7474 7782 $\sin\alpha_{2-1}$ -.7071 0684

ϕ_2 70 00 00.005

$\tan\Delta\eta = \dfrac{\sin\Delta\sigma\sin\alpha_{1-2}}{\cos\theta_1\cos\Delta\sigma - \sin\theta_1\sin\Delta\sigma\cos\alpha_{1-2}}$ +.80399438 $\Delta\eta$ 38 47 56.710

$H = c_1(1-c_2)\Delta\sigma - c_1c_2\sin\Delta\sigma\cos\Sigma\sigma$ .0000830137 99 (rad) H 2 51.228

$\Delta\lambda = \Delta\eta - H$ 38 45 05.482

λ_1 -56 45 05.464

CHECK

$M = \cos\theta_0 = \cos\theta_1 \sin\alpha_{1-2} = \cos\theta_2 \sin(180 + \alpha_{2-1})$ $\lambda_2 = \lambda_1 + \Delta\lambda$ -17 59 59.982

INVERSE POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1; \phi_2, \lambda_2$ to find S, $\alpha_{1-2}, \alpha_{2-1}$. Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

CLARKE 1866 SPHEROID a 6378206.4 m b ______ m

1 - f = b/a ______ ½f .0016950371647 ¼f .00084751858208

$f^2/64$ .17957204×10⁻⁶ 1 radian = 206264.8062 seconds

ϕ_1 ______ 1. P2 λ_1 ______

ϕ_2 ______ 2. INITIAL (P. I) λ_2 ______

$\tan\phi_1$ ______ 1. always west of 2. $\Delta\lambda = \lambda_2 - \lambda_1$ 38° 45′ 05.464″

$\tan\phi_2$ ______ $\tan\theta = (1 - f)\tan\phi$ $\Delta\lambda_m = \frac{1}{2}\Delta\lambda$ 19 22 32.732

θ_2 69° 56′ 14.590″ $\tan\theta_2$ ______ $\sin\Delta\lambda_m$ .33176204

θ_1 17 05 21.296 $\tan\theta_1$ ______ $\tan\Delta\lambda$ .80262841

$\theta_m = \frac{1}{2}(\theta_1 + \theta_2)$ 43° 30′ 47.943″ $\sin\theta_m$ .68852316 $\cos\theta_m$ .72521435

$\Delta\theta_m = \frac{1}{2}(\theta_2 - \theta_1)$ 26 25 26.647 $\sin\Delta\theta_m$ .44501141 $\cos\Delta\theta_m$ .89552490

$H = \cos^2\Delta\theta_m - \sin^2\theta_m = \cos^2\theta_m - \sin^2\Delta\theta_m$ .327900703 1 - L .76587409

$L = \sin^2\Delta\theta_m + H\sin^2\Delta\lambda_m$ .234125891 $\cos d = 1 - 2L$ .53174822

$U = 2\sin^2\theta_m\cos^2\Delta\theta_m/(1 - L)$ .992807492 d 57° 52′ 34.856″

$V = 2\sin^2\Delta\theta_m\cos^2\theta_m/L$ .889724651 sin d .84690249 d (rad) 1.010132847

X = U + V 1.882532143 T = d/sin d 1.192738077 E = 2 cos d 1.06349644

Y = U - V .103082841 $D = 4T^2$ 5.690496482 B = 2D 11.380992964

A = DE 6.051822750 $C = T - \frac{1}{2}(A - E)$ -1.301425078 CHECK $C - \frac{1}{2}E + AD/B = T$

$n_1 = X(A + CX)$ 6.780596028 $n_2 = Y(B + EY)$ 1.184485878 $n_3 = DXY$ 1.104279319

$\delta_1 d = \frac{1}{4}f(TX - Y)$ .001815627 $\delta_2 d = (f^2/64)(n_1 - n_2 + n_3)$ .1203×10⁻⁵

$S_1 = a\sin d(T - \delta_1 d)$ 6433028.285 m $S_2 = a\sin d(T - \delta_1 d + \delta_2 d)$ 6433034.783 m

F = 2Y - E(4 - X) -2.04575 3846 M = 32T - (20T - A)X - (B + 4)Y 3.0674975

$G = \frac{1}{2}fT + (f^2/64)M$ .002022868 $Q = -(FG\tan\Delta\lambda)/4$ 2′ 51.228″

$\Delta\lambda_m' = \frac{1}{2}(\Delta\lambda + Q)$ 19° 23′ 58.346″ $\tan\Delta\lambda_m'$ .35214658

$v = \arctan|c_2|$ 74 51 01.861 $c_2 = \cos\Delta\theta_m/(\sin\theta_m\tan\Delta\lambda_m')$ +3.6934790

$u = \arctan|c_1|$ 60 08 58.139 $c_1 = -\sin\Delta\theta_m/(\cos\theta_m\tan\Delta\lambda_m')$ -1.74253409

$\alpha_1 = v - u$ 14° 42′ 03.722″ $\alpha_2 = v + u$ 135° 0′ 0″

c_1	c_2	α_{1-2}		α_{2-1}	
-	+	α_1	14° 42′ 03.722″	$360 - \alpha_2$	225° 0′ 0″
+	+	α_2		$360 - \alpha_1$	
-	-	$180 - \alpha_2$		$180 + \alpha_1$	
+	-	$180 - \alpha_1$		$180 + \alpha_2$	

DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1, \alpha_{1\text{-}2}$, S to find $\phi_2, \lambda_2, \alpha_{2\text{-}1}$. East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

CLARKE 1866 SPHEROID a 6378206.4 m f

1 - f .996609924772 — 1 radian = 206264.8062 seconds

LINE ° ′ ″ P1 TO P3 (P1P3)

ϕ_1 — $\tan\phi_1$ — $\tan\theta_1 = (1-f)\tan\phi_1$ ° ′ ″

$\alpha_{1\text{-}2}$ 45 — $\sin\theta_1$ -.9393 1830 — $\cos\theta_1$ +.3430 4686 — θ_1 -69 56 14.590

$\sin\alpha_{1\text{-}2}$ .7071 0678 — $M = \cos\theta_0 = \cos\theta_1 \sin\alpha_{1\text{-}2}$ .2425 7076 — θ_0

$\cos\alpha_{1\text{-}2}$ .7071 0678 — $N = \cos\theta_1 \cos\alpha_{1\text{-}2}$ .2425 7076 — $\sin\theta_0$ +.9701 3371

$c_1 = fM$ .0008223331378 — $D = (1-c_2)(1-c_2-c_1M)$ .9982060207

$c_2 = \frac{1}{4}(1-M^2)f$ .0007976503271 — $P = c_2(1+\frac{1}{2}c_1M)/D$ .0007991635651

$\cos\sigma_1 = \sin\theta_1/\sin\theta_0$ -.9682 3591 — σ_1 165° 31′ 12.″769

$d = S/aD$ 3.1415926588 (rad) — d 180 0 0 — S 20001779.126 m

$\sin d$ 0 — $u = 2(\sigma_1 - d)$ — $\sin u$

$\cos d$ -1 — $W = 1 - 2P\cos u$ — $\cos u$

$V = \cos u \cos d - \sin u \sin d$ — $Y = 2PVW \sin d$ 0

$X = c_2^2 \sin d \cos d (2V^2 - 1)$ 0 — $\Delta\sigma = d + X - Y$ = d (rad)

$\sin\Delta\sigma$ 0 — $\cos\Delta\sigma$ -1 — $\Delta\sigma$ 180° ′ ″

$\cos\Sigma\sigma$ — $\Sigma\sigma = 2\sigma_1 - \Delta\sigma$ ° ′ ″

$\tan\alpha_{2\text{-}1} = M/(N\cos\Delta\sigma - \sin\theta_1 \sin\Delta\sigma)$ -1 — $\alpha_{2\text{-}1}$ 225

$\tan\phi_2 = \dfrac{-(\sin\theta_1 \cos\Delta\sigma + N\sin\Delta\sigma)\sin\alpha_{2\text{-}1}}{(1-f)M}$ 2.7474774 — $\sin\alpha_{2\text{-}1}$ -.7071 0678

ϕ_2 70° 0′ 0″

$\tan\Delta\eta = \dfrac{\sin\Delta\sigma \sin\alpha_{1\text{-}2}}{\cos\theta_1 \cos\Delta\sigma - \sin\theta_1 \sin\Delta\sigma \cos\alpha_{1\text{-}2}}$ 0 — $\Delta\eta$ 180° ′ ″

$H = c_1(1-c_2)\Delta\sigma - c_1c_2 \sin\Delta\sigma \cos\Sigma\sigma$ .0025813751 (rad) — H 8 52.447

$\Delta\lambda = \Delta\eta - H$ 179 51 07.553

λ_1 -104 17 29.220

CHECK

$M = \cos\theta_0 = \cos\theta_1 \sin\alpha_{1\text{-}2} = \cos\theta_2 \sin(180 + \alpha_{2\text{-}1})$ — $\lambda_2 = \lambda_1 + \Delta\lambda$ 75° 33′ 38.″333

DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1, \alpha_{1\text{-}2}$, S to find $\phi_2, \lambda_2, \alpha_{2\text{-}1}$. East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

CLARKE 1866 SPHEROID a 6378206.4 m. f

1 - f .99660992472 1 radian = 206264.8062 seconds

LINE VERTEX V TO TERMINAL (V, T)

ϕ_1 ______ $\tan\phi_1$ ______ $\tan\theta_1 = (1-f)\tan\phi_1$ ° ′ ″

$\alpha_{1\text{-}2}$ 90 $\sin\theta_1$ = sin θ₀ $\cos\theta_1$ = cos θ₀ θ_1 75 57 42.053

$\sin\alpha_{1\text{-}2}$ 1 $M = \cos\theta_0 = \cos\theta_1 \sin\alpha_{1\text{-}2}$ .2425 7076 θ_0 75 57 42.053

$\cos\alpha_{1\text{-}2}$ 0 $N = \cos\theta_1 \cos\alpha_{1\text{-}2}$ 0 $\sin\theta_0$ .9701 3371

$c_1 = fM$ .000822331278 $D = (1-c_2)(1-c_2-c_1M)$ .9982060207

$c_2 = \frac{1}{4}(1-M^2)f$ .000797650227 $P = c_2(1+\frac{1}{2}c_1M)/D$ .00079916356

$\cos\sigma_1 = \sin\theta_1/\sin\theta_0$ 1 ° ′ ″ σ_1 0

d=S/aD 1.2635156567 (rad) d 72 23 38.817 S 8044506.034 m

sin d .9531 5960 $u = 2(\sigma_1 - d)$ = -144 47 17.634 sin u = -.5766 0018

cos d .3024 6781 W = 1 - 2P cos u 1.0013058756 cos u = -.8170 2646

V = cos u cos d - sin u sin d .3024 677929 Y = 2PVW sin d +.00046139 96

$X = c_2^2 \sin d \cos d (2V^2 - 1)$ = -.1499 × 10⁻⁶ $\Delta\sigma = d + X - Y$ 1.2630541077 (rad)

° ′ ″

$\sin\Delta\sigma$ .9530 1990 $\cos\Delta\sigma$ .3029 0770 $\Delta\sigma$ 72 22 03.611

$\cos\Sigma\sigma$ .3029 0770 $\Sigma\sigma = 2\sigma_1 - \Delta\sigma$ -Δσ

° ′ ″

$\tan\alpha_{2\text{-}1} = M/(N\cos\Delta\sigma - \sin\theta_1 \sin\Delta\sigma)$ -.2623 6439 $\alpha_{2\text{-}1}$ 345 17 56.276

$\tan\phi_2 = \dfrac{-(\sin\theta_1 \cos\Delta\sigma + N \sin\Delta\sigma)\sin\alpha_{2\text{-}1}}{(1-f)M}$ +.3084 8055 $\sin\alpha_{2\text{-}1}$ -.2537 7541

° ′ ″

ϕ_2 17 08 38.317

° ′ ″

$\tan\Delta\eta = \dfrac{\sin\Delta\sigma \sin\alpha_{1\text{-}2}}{\cos\theta_1 \cos\Delta\sigma - \sin\theta_1 \sin\Delta\sigma \cos\alpha_{1\text{-}2}}$ cot θ .0770 9865 $\Delta\eta$ 85 35 28.660

$H = c_1(1-c_2)\Delta\sigma - c_1c_2 \sin\Delta\sigma \cos\Sigma\sigma$ .0010387633 444 (rad) H 3 34.027

$\Delta\lambda = \Delta\eta - H$ 85 31 54.633

λ_1 28 46 49.167

CHECK

° ′ ″

$M = \cos\theta_0 = \cos\theta_1 \sin\alpha_{1\text{-}2} = \cos\theta_2 \sin(180 + \alpha_{2\text{-}1})$ $\lambda_2 = \lambda_1 + \Delta\lambda$ 114 18 43.800

INVERSE POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1; \phi_2, \lambda_2$ to find S, $\alpha_{1\text{-}2}$, $\alpha_{2\text{-}1}$. Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

CLARKE 1866 SPHEROID a 6378206.4 m b ______ m

1 - f = b/a ______ ½f .0016950377642 ¼f .0008475188208

$f^2/64$ .17957204 × 10^{-6} 1 radian = 206264.8062 seconds

ϕ_1 ______ 1. VERTEX 2 λ_1 ______

ϕ_2 ______ 2. TERMINAL λ_2 ______

tan ϕ_1 ______ 1. always west of 2. $\Delta\lambda = \lambda_2 - \lambda_1$ 85 31 54.631

tan ϕ_2 ______ $\tan\theta = (1-f)\tan\phi$ $\Delta\lambda_m = \frac{1}{2}\Delta\lambda$ 42 45 57.316

θ_2 17 05 21.296 tan θ_2 ______ sin $\Delta\lambda_m$ .67900477

θ_1 75 57 42.053 tan θ_1 ______ tan $\Delta\lambda$ 12.797127

$\theta_m = \frac{1}{2}(\theta_1 + \theta_2)$ 46 31 31.675 sin θ_m .72568024 cos θ_m .68803211

$\Delta\theta_m = \frac{1}{2}(\theta_2 - \theta_1)$ -29 26 10.378 sin $\Delta\theta_m$ -.49145435 cos $\Delta\theta_m$ .87090334

$H = \cos^2\Delta\theta_m - \sin^2\theta_m = \cos^2\theta_m - \sin^2\Delta\theta_m$ .23186811 1 - L .65157378

$L = \sin^2\Delta\theta_m + H\sin^2\Delta\lambda_m$ .34842622 cos d = 1 - 2L +.30314756

$U = 2\sin^2\theta_m\cos^2\Delta\theta_m/(1-L)$ 1.226018161 d 72 21 11.695

$V = 2\sin^2\Delta\theta_m\cos^2\theta_m/L$ .65630082 sin d .95294363 d (rad) 1.262802413

X = U + V 1.882318981 T = d/sin d 1.325159614 E = 2 cos d .60629512

Y = U - V .569717341 D = 4T^2 7.024192201 B = 2D 14.048384402

A = DE 4.258733337 C = T - ½ (A - E) -.5021059495 CHECK C - ½ E + AD/B = T

n_1 = X (A + CX) 6.240978299 n_2 = Y (B + EY) 8.200397955 n_3 = DXY 7.532671617

$\delta_1 d$ = ¼f (TX - Y) .00163118 7 $\delta_2 d = (f^2/64)(n_1 - n_2 + n_3)$ .1001 × 10^{-5}

S_1 = a sin d (T - δ_1d) 8044499.992 m S_2 = a sin d (T - δ_1d + δ_2d) 8044506.076 m

F = 2Y - E (4 - X) -.144504986 M = 32T - (20 T - A) X - (B + 4) Y -9.748536996

G = ½fT + (f^2/64) M .0022444449 Q = - (FG tan $\Delta\lambda$)/4 3 34.027

$\Delta\lambda'_m$ = ½ ($\Delta\lambda$ + Q) 42 47 44.330 tan $\Delta\lambda'_m$ .92586905

v = arctan $|c_2|$ 52 51 01.861 $c_2 = \cos\Delta\theta_m/(\sin\theta_m \tan\Delta\lambda'_m)$ +1.29620917

u = arctan $|c_1|$ 37 38 58.138 $c_1 = -\sin\Delta\theta_m/(\cos\theta_m \tan\Delta\lambda'_m)$ +.77148042

α_1 = v - u 14 42 03.723 α_2 = v + u 89 59 59.999

c_1	c_2	$\alpha_{1\text{-}2}$ ° ′ ″	$\alpha_{2\text{-}1}$ ° ′ ″
-	+	α_1 ______	360 - α_2 ______
+	+	α_2 89 59 59.999	360 - α_1 345 17 56.277
-	-	180 - α_2 ______	180 + α_1 ______
+	-	180 - α_1 ______	180 + α_2 ______

DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1, \alpha_{1\text{-}2}$, S to find $\phi_2, \lambda_2, \alpha_{2\text{-}1}$. East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

CLARKE 1866 SPHEROID a 6378206.4 m f ______

1 - f .99660992477 1 radian = 206264.8062 seconds

LINE INITIAL TO TERMINAL (IT)

ϕ_1 ______ $\tan\phi_1$ ______ $\tan\theta_1 = (1-f)\tan\phi_1$ ______

$\alpha_{1\text{-}2}$ 45° $\sin\theta_1$ .9393 1830 $\cos\theta_1$ .3430 4686 θ_1 69 56 14.590

$\sin\alpha_{1\text{-}2}$ .7071 0678 $M = \cos\theta_0 = \cos\theta_1 \sin\alpha_{1\text{-}2}$ .2425 7096 θ_0 75 57 42.053

$\cos\alpha_{1\text{-}2}$ .7071 0678 $N = \cos\theta_1 \cos\alpha_{1\text{-}2}$ = M $\sin\theta_0$ .9701 3371

$c_1 = fM$ .000822333137 $D = (1-c_2)(1-c_2-c_1M)$ .9982060207

$c_2 = \frac{1}{4}(1-M^2)f$ .00079765082271 $P = c_2(1 + \frac{1}{2}c_1M)/D$ .000799163565 1

$\cos\sigma_1 = \sin\theta_1/\sin\theta_0$ .9682 3591 σ_1 14° 28′ 47.231″

$d = S/aD$ 1.5166224181 (rad) d 86 53 45.829 S 9655977.058 m

$\sin d$ .99853295 $u = 2(\sigma_1 - d)$ = 144 49 57.196 $\sin u$ − .5759 6793

$\cos d$ .05414743 $W = 1 - 2P\cos u$ 1.0013065881 $\cos u$ − .8174 7229

$V = \cos u \cos d - \sin u \sin d$ .5308589326 $Y = 2PVW \sin d$ + .0008483485

$X = c_2^2 \sin d \cos d (2V^2 - 1)$ − .15 x 10⁻⁷ $\Delta\sigma = d + X - Y$ 1.5157740546 (rad)

$\sin\Delta\sigma$ .99848666 $\cos\Delta\sigma$ .0549 9451 $\Delta\sigma$ 86° 50′ 50.842″

$\cos\Sigma\sigma$ .5315 7771 $\Sigma\sigma = 2\sigma_1 - \Delta\sigma$ − 57° 53′ 16.380″

$\tan\alpha_{2\text{-}1} = M/(N\cos\Delta\sigma - \sin\theta_1 \sin\Delta\sigma)$ − .2623 6439 $\alpha_{2\text{-}1}$ 345° 17′ 56.276″

$\tan\phi_2 = \dfrac{-(\sin\theta_1 \cos\Delta\sigma + N\sin\Delta\sigma)\sin\alpha_{2\text{-}1}}{(1-f)M}$ + .3084 8060 $\sin\alpha_{2\text{-}1}$ − .2537 7541

ϕ_2 17° 08′ 38.326″

$\tan\Delta\eta = \dfrac{\sin\Delta\sigma \sin\alpha_{1\text{-}2}}{\cos\theta_1 \cos\Delta\sigma - \sin\theta_1 \sin\Delta\sigma \cos\alpha_{1\text{-}2}}$ − 1.0957 7304 $\Delta\eta$ 132° 23′ 00.624″

$H = c_1(1-c_2)\Delta\sigma - c_1c_2 \sin\Delta\sigma \cos\Sigma\sigma$ .0012451288 3 (rad) H 4′ 16.826″

$\Delta\lambda = \Delta\eta - H$ 132 18 43.798

λ_1 − 18

CHECK

$M = \cos\theta_0 = \cos\theta_1 \sin\alpha_{1\text{-}2} = \cos\theta_2 \sin(180 + \alpha_{2\text{-}1})$ $\lambda_2 = \lambda_1 + \Delta\lambda$ 114° 18′ 43.798″

INVERSE POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1; \phi_2, \lambda_2$ to find S, $\alpha_{1\text{-}2}$, $\alpha_{2\text{-}1}$. Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

CLARKE 1866 SPHEROID a 6378206.4 m b ______ m

1 - f = b/a ______ ½f .0016950237642 ¼f .0008475188208

$f^2/64$ .17957204 × 10^{-6} 1 radian = 206264.8062 seconds

	° ′ ″			° ′ ″
ϕ_1		1. INITIAL (Origin)	λ_1	
ϕ_2		2. TERMINAL (IT)	λ_2	

$\tan \phi_1$ ______ 1. always west of 2. $\Delta\lambda = \lambda_2 - \lambda_1$ 132 18 43.798

$\tan \phi_2$ ______ $\tan \theta = (1 - f) \tan \phi$ $\Delta\lambda_m = ½\Delta\lambda$ 66 09 21.899

θ_2 17° 05′ 21.″296 $\tan \theta_2$ ______ $\sin \Delta\lambda_m$ .9146 5008

θ_1 69 56 14.590 $\tan \theta_1$ ______ $\tan \Delta\lambda$ -1.0985 1696

$\theta_m = ½(\theta_1 + \theta_2)$ 43° 30′ 47.″943 $\sin \theta_m$ .6885 2316 $\cos \theta_m$ .7252 1435

$\Delta\theta_m = ½(\theta_2 - \theta_1)$ -26 25 26.647 $\sin \Delta\theta_m$ -.4450 1141 $\cos \Delta\theta_m$ .8955 2490

$H = \cos^2 \Delta\theta_m - \sin^2 \theta_m = \cos^2 \theta_m - \sin^2 \Delta\theta_m$ .327900303 $1 - L$ .52764 81124

$L = \sin^2 \Delta\theta_m + H \sin^2 \Delta\lambda_m$ .47235 18876 $\cos d = 1 - 2L$ .0552 9632

$U = 2 \sin^2 \theta_m \cos^2 \Delta\theta_m / (1 - L)$ 1.441046667 d 86° 49′ 48.″515

$V = 2 \sin^2 \Delta\theta_m \cos^2 \theta_m / L$ .441000834 $\sin d$ .9984 6999 d (rad) 1.515471887

$X = U + V$ 1.882047501 $T = d/\sin d$ 1.517794127 $E = 2 \cos d$ .1105 9264

$Y = U - V$ 1.000045832 $D = 4T^2$ 9.214796045 $B = 2D$ 18.42959209

$A = DE$ 1.019086779 $C = T - ½(A - E)$ 1.063546957 CHECK $C - ½E + AD/B = T$

$n_1 = X(A + CX)$ 5.685162377 $n_2 = Y(B + EY)$ 18.54103934 $n_3 = DXY$ 17.34347872

$\delta_1 d = ¼f(TX - Y)$ .00157343 12 $\delta_2 d = (f^2/64)(n_1 - n_2 + n_3)$ .806 × 10^{-6}

$S_1 = a \sin d (T - \delta_1 d)$ 9655972.172 m $S_2 = a \sin d (T - \delta_1 d + \delta_2 d)$ 9655977.304 m

$F = 2Y - E(4 - X)$ +1.765862130 $M = 32T - (20T - A) X - (B + 4) Y$ -29.07445117

$G = ½fT + (f^2/64) M$ .25674987 × 10^{-2} $Q = -(FG \tan \Delta\lambda)/4$ 4′ 16.″826

$\Delta\lambda'_m = ½(\Delta\lambda + Q)$ 66° 11′ 30.″313 $\tan \Delta\lambda'_m$ 2.26641999

$v = \arctan |c_2|$ 29° 51′ 01.″862 $c_2 = \cos \Delta\theta_m / (\sin \theta_m \tan \Delta\lambda'_m)$ + .5738 7687

$u = \arctan |c_1|$ 15 08 58.138 $c_1 = -\sin \Delta\theta_m / (\cos \theta_m \tan \Delta\lambda'_m)$ + .2707 4744

$\alpha_1 = v - u$ 14° 42′ 03.″724 $\alpha_2 = v + u$ 45° 0′ 0″

c_1	c_2	$\alpha_{1\text{-}2}$	° ′ ″	$\alpha_{2\text{-}1}$	° ′ ″
-	+	α_1		$360 - \alpha_2$	
+	+	α_2	45 0 0	$360 - \alpha_1$	345 17 56.276
-	-	$180 - \alpha_2$		$180 + \alpha_1$	
+	-	$180 - \alpha_1$		$180 + \alpha_2$	

DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1, \alpha_{1\text{-}2}$, S to find $\phi_2, \lambda_2, \alpha_{2\text{-}1}$. East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

CLARKE 1866 SPHEROID a 6378206.4 m f

1 - f .99660992472   1 radian = 206264.8062 seconds

LINE TERMINAL TO NODE 2 (TN2)

ϕ_1 ° ′ ″ $\tan\phi_1$ $\tan\theta_1 = (1-f)\tan\phi_1$

$\alpha_{1\text{-}2}$ 165 17 56.272 $\sin\theta_1$ .2938 6097 $\cos\theta_1$ .95584817 θ_1 17° 05′ 21.296″

$\sin\alpha_{1\text{-}2}$ .2537 7541 $M = \cos\theta_0 = \cos\theta_1 \sin\alpha_{1\text{-}2}$ .2425 7076 θ_0

$\cos\alpha_{1\text{-}2}$ −.96726317 $N = \cos\theta_1 \cos\alpha_{1\text{-}2}$ −.9245 5673 $\sin\theta_0$ .9701 3371

$c_1 = fM$ .000822333 1378 $D = (1-c_2)(1-c_2-c_1M)$ .9982060207

$c_2 = \frac{1}{4}(1-M^2)f$ .000797650327 $P = c_2(1 + \frac{1}{2}c_1 M)/D$ .000799163 5651

$\cos\sigma_1 = \sin\theta_1/\sin\theta_0$ .3029 0770 σ_1 72° 22′ 03.610″ = 90° − Δσ (see computation NODE 1 TO P2)

d=S/aD (rad)   d   S m

sin d $u = 2(\sigma_1 - d)$ sin u

cos d $W = 1 - 2P\cos u$ cos u

$V = \cos u \cos d - \sin u \sin d$ $Y = 2PVW \sin d$

$X = c_2^2 \sin d \cos d\,(2V^2 - 1)$ $\Delta\sigma = d + X - Y$ (rad)

$\sin\Delta\sigma$ .3029 0770 $\cos\Delta\sigma$ .9530 1990 $\Delta\sigma$ 17° 37′ 56.390″

$\cos\Sigma\sigma$ −.9530 1990 $\Sigma\sigma = 2\sigma_1 - \Delta\sigma$ π − Δσ ° ′ ″

$\tan\alpha_{2\text{-}1} = M/(N\cos\Delta\sigma - \sin\theta_1 \sin\Delta\sigma)$ −.2500 3848 $\alpha_{2\text{-}1}$ 345 57 42.052

$$\tan\phi_2 = \frac{-(\sin\theta_1 \cos\Delta\sigma + N\sin\Delta\sigma)\sin\alpha_{2\text{-}1}}{(1-f)M}$$ 0 $\sin\alpha_{2\text{-}1}$ −.2425 7077

ϕ_2 0° 0′ 0″

$$\tan\Delta\eta = \frac{\sin\Delta\sigma \sin\alpha_{1\text{-}2}}{\cos\theta_1 \cos\Delta\sigma - \sin\theta_1 \sin\Delta\sigma \cos\alpha_{1\text{-}2}}$$ +.0770 9865 $\Delta\eta$ 4° 24′ 31.340″

$H = c_1(1-c_2)\Delta\sigma - c_1 c_2 \sin\Delta\sigma \cos\Sigma\sigma$ .000253054 14 (rad)   H 52.196

$\Delta\lambda = \Delta\eta - H$ 4 23 39.144

λ_1 114 18 43.798

CHECK

$M = \cos\theta_0 = \cos\theta_1 \sin\alpha_{1\text{-}2} = \cos\theta_2 \sin(180 + \alpha_{2\text{-}1})$ $\lambda_2 = \lambda_1 + \Delta\lambda$ 118° 42′ 22.942″

INVERSE POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1; \phi_2, \lambda_2$ to find S, $\alpha_{1\text{-}2}, \alpha_{2\text{-}1}$. Azimuths clockwise from north; east longitudes positive; no tables except 8-place natural trigonometric (Peters); no root extraction.

CLARKE 1866 SPHEROID a 6378206.4 m b ____ m

1 - f = b/a ____ ½f .0016950337642 ¼f .0008475188208

$f^2/64$ .1795 7204 x 10⁻⁶ 1 radian = 206264.8062 seconds

	° ′ ″			° ′ ″
ϕ_1		1. TERMINAL	λ_1	
ϕ_2		2. NODE 2 (TN2)	λ_2	

$\tan\phi_1$ ____	1. always west of 2.	$\Delta\lambda = \lambda_2 - \lambda_1$ 4 23 39.146
$\tan\phi_2$ ____	$\tan\theta = (1-f)\tan\phi$	$\Delta\lambda_m = ½\Delta\lambda$ 2 11 49.573
θ_2 0	$\tan\theta_2$ ____	$\sin\Delta\lambda_m$ .0383 3729
θ_1 17 05 21.296	$\tan\theta_1$ ____	$\tan\Delta\lambda$ .0768 4410
$\theta_m = ½(\theta_1+\theta_2)$ 8 32 40.648	$\sin\theta_m$ +.1485 7965	$\cos\theta_m$ .9889 0044
$\Delta\theta_m = ½(\theta_2-\theta_1)$ -8 32 40.648	$\sin\Delta\theta_m$ -.1485 7965	$\cos\Delta\theta_m$ .9889 0044

$H = \cos^2\Delta\theta_m - \sin^2\theta_m = \cos^2\theta_m - \sin^2\Delta\theta_m$ .9558481681 1 - L .9765192319

$L = \sin^2\Delta\theta_m + H\sin^2\Delta\lambda_m$ .0234 8077 $\cos d = 1 - 2L$ .9530 3846

$U = 2\sin^2\theta_m\cos^2\Delta\theta_m/(1-L)$ .044215343 d 17 37 43.748

$V = 2\sin^2\Delta\theta_m\cos^2\theta_m/L$ 1.838829649 sin d .3028 4930 d (rad) .307680933

$X = U + V$ 1.883044992 $T = d/\sin d$ 1.015953918 $E = 2\cos d$ 1.90607692

$Y = U - V$ -1.794614306 $D = 4T^2$ 4.128649454 $B = 2D$ 8.257298908

$A = DE$ 7.869523435 $C = T - ½(A - E)$ -1.965969934 CHECK $C - ½E + AD/B = T$

$n_1 = X(A + CX)$ 7.648326885 $n_2 = Y(B + EY)$ -8.679878211 $n_3 = DXY$ -13.952610811

$\delta_1 d = ¼f(TX - Y)$ .003142347 $\delta_2 d = (f^2/64)(n_1 - n_2 + n_3)$ +.463x10⁻⁶

$S_1 = a\sin d(T - \delta_1 d)$ 1956382.627 m $S_2 = a\sin d(T - \delta_1 d + \delta_2 d)$ 1956383.521 m

$F = 2Y - E(4 - X)$ -7.624307693 $M = 32T - (20T - A)X - (B + 4)Y$ ____

$G = ½fT + (f^2/64)M$ .001722765846 $Q = -(FG\tan\Delta\lambda)/4$ ° ′ ″ 52.196

$\Delta\lambda'_m = ½(\Delta\lambda + Q)$ 2 12 15.671 $\tan\Delta\lambda'_m$ .0384 9221

$v = \arctan|c_2|$ 89 40 07.111 $c_2 = \cos\Delta\theta_m/(\sin\theta_m\tan\Delta\lambda'_m)$ +172.91011

$u = \arctan|c_1|$ 75 37 49.164 $c_1 = -\sin\Delta\theta_m/(\cos\theta_m\tan\Delta\lambda'_m)$ +3.9033178

$\alpha_1 = v - u$ 14 02 17.947 $\alpha_2 = v + u$ 165 17 56.275

c_1	c_2	$\alpha_{1\text{-}2}$	° ′ ″	$\alpha_{2\text{-}1}$	° ′ ″
-	+	α_1		$360 - \alpha_2$	
+	+	α_2	165 17 56.275	$360 - \alpha_1$	345 57 42.053
-	-	$180 - \alpha_2$		$180 + \alpha_1$	
+	-	$180 - \alpha_1$		$180 + \alpha_2$	

DIRECT POSITION COMPUTATION FORM FOR LONG LINES. Given $\phi_1, \lambda_1, \alpha_{1\text{-}2}$, S to find $\phi_2, \lambda_2, \alpha_{2\text{-}1}$. East longitudes positive; azimuths clockwise from north; no root extraction; only 8-place trigonometric natural tables (as Peters) required for desk work.

CLARKE 1866 SPHEROID a 6378206.4 m f ______

1 - f .996609924772 1 radian = 206264.8062 seconds

LINE NODE 1 TO NODE 2 ($N_1 N_2$)

ϕ_1 ______ $\tan\phi_1$ ______ $\tan\theta_1 = (1 - f)\tan\phi_1$ ______

$\alpha_{1\text{-}2}$ 14 02 17.947 $\sin\theta_1$ 0 $\cos\theta_1$ 1 θ_1 0

$\sin\alpha_{1\text{-}2}$ = M $M = \cos\theta_0 = \cos\theta_1 \sin\alpha_{1\text{-}2}$ .2425 7076 θ_0 75 57 42.053

$\cos\alpha_{1\text{-}2}$ = N $N = \cos\theta_1 \cos\alpha_{1\text{-}2}$ .9701 3371 $\sin\theta_0$ .9701 3371

$c_1 = fM$ .00082233331378 $D = (1 - c_2)(1 - c_2 - c_1 M)$ ______

$c_2 = \frac{1}{4}(1 - M^2)f$ .000796503271 $P = c_2(1 + \frac{1}{2}c_1 M)/D$ ______

$\cos\sigma_1 = \sin\theta_1/\sin\theta_0$ 0 σ_1 90

$d = S/aD$ 3.1415926588 (rad) d 180 S 20001779.136 m

$\sin d$ 0 $u = 2(\sigma_1 - d)$ -180 $\sin u$ 0

$\cos d$ -1 $W = 1 - 2P\cos u$ ______ $\cos u$ -1

$V = \cos u \cos d - \sin u \sin d$ 1 $Y = 2PVW \sin d$ 0

$X = c_2^2 \sin d \cos d (2V^2 - 1)$ 0 $\Delta\sigma = d + X - Y$ = d (rad)

$\sin\Delta\sigma$ 0 $\cos\Delta\sigma$ -1 $\Delta\sigma$ 180

$\cos\Sigma\sigma$ 1 $\Sigma\sigma = 2\sigma_1 - \Delta\sigma$ 0

$\tan\alpha_{2\text{-}1} = M/(N\cos\Delta\sigma - \sin\theta_1 \sin\Delta\sigma)$ -.2500 3848 $\alpha_{2\text{-}1}$ 345 57 42.054

$\tan\phi_2 = \dfrac{-(\sin\theta_1 \cos\Delta\sigma + N\sin\Delta\sigma)\sin\alpha_{2\text{-}1}}{(1 - f)M}$ 0 $\sin\alpha_{2\text{-}1}$ -.2425 7076

ϕ_2 0 0 0

$\tan\Delta\eta = \dfrac{\sin\Delta\sigma \sin\alpha_{1\text{-}2}}{\cos\theta_1 \cos\Delta\sigma - \sin\theta_1 \sin\Delta\sigma \cos\alpha_{1\text{-}2}}$ 0 $\Delta\eta$ 180

$H = c_1(1 - c_2)\Delta\sigma - c_1 c_2 \sin\Delta\sigma \cos\Sigma\sigma$ .002581378033 (rad) H 8 52.447

$\Delta\lambda = \Delta\eta - H$ 179 51 07.553

λ_1 -61 08 44.610

CHECK

$M = \cos\theta_0 = \cos\theta_1 \sin\alpha_{1\text{-}2} = \cos\theta_2 \sin(180 + \alpha_{2\text{-}1})$ $\lambda_2 = \lambda_1 + \Delta\lambda$ 118 42 22.943

APPENDIX 4

SUBROUTINE GEODIST

Fortran statements as prepared by the Earth Sciences Division of Teledyne Industries and based on the inverse (reverse) solution of P.D. Thomas. (The card deck including the arc tangent library function (ATAN2) is available).

SEISMIC DATA LABORATORY
EARTH SCIENCES
A TELEDYNE COMPANY
ALEXANDRIA, VIRGINIA

TITLE: Subroutine GEODIST

DATE: August 1968

GEODIST is a Fortran-63 subroutine used in computing surface distances on a given spherc .d. The method used was supplied by Mr. Paul D. Thomas of the Naval Research Laboratory.

The subroutine currently uses the constants for the Clarke 1866 model of the Earth but another model may easily be substituted by replacing the semi-major and semi-minor axes (in kilometers) in the subroutine. The variable names are respectively, AL and BL.

The calling sequence for this subroutine includes the following values in the order listed: (Latitude 1, Longitude 1, Latitude 2, Longitude 2, Azimuth, Back Azimuth, Distance-kilometers, Distance-degrees). The Latitudes and Longitudes are the Geographic coordinates of the two points. The subroutine assumes that positive values are North or East and that negative values are South or West. The forward azimuth, in degrees east of north, is from point one to point two and conversely the back azimuth is from point two to point one. The distance in kilometers is the geodesic distance between the two points on the surface of the spheroid. The fiducial central angle distance in degrees (based on an equivalent mean sphere) is obtained by assuming that one degree equals 111.195 kilometers. All arguments are type REAL.

CAUTION: The back azimuth from either pole may be slightly in error.

The subroutine requires 225 CDC 1604 words plus four 1604 words of labeled common/GEODISTC/. There are no alarms, error returns, error stops or printouts. The time required is less than 0.15 seconds per call.

This subroutine uses an arctangent library function (ATAN2) which returns an angle between 0 and 2 Pi radians. If this function is not in the system library, it must be input with the subroutine. The fortran statements for this function are attached.

REFERENCE: Mr. Paul D. Thomas, Code 7004, Naval Research Laboratory, Washington, D.C. 20390.

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```
      SUBROUTINE GEODIST(EPLAT,EPLON,STLAT,STLON,AZ,BAZ,DIST,DEG)
      COMMON/GEODISTC/AL,BL,D2R,PI2
      TYPE REAL L1R,L2R,KL,KK,L
      DATA (AL=6378206.4),(BL=6356583.8),(D2R=.017453292519),
     1(PI2=6.28318530716)
      BOA=BL/AL
      F=1.0-BOA
      P1R=EPLAT*D2R
      P2R=STLAT*D2R
      L1R=EPLON*D2R
      L2R=STLON*D2R
      DLR=L2R-L1R
      T1R=ATAN(BOA*TAN(P1R))
      T2R=ATAN(BOA*TAN(P2R))
      TM=(T1R+T2R)/2.0
      DTM=(T2R-T1R)/2.0
      STM=SIN(TM)
      CTM=COS(TM)
      SDTM=SIN(DTM)
      CDTM=COS(DTM)
      KL=STM*CDTM
      KK=SDTM*CTM
      SDLMR=SIN(DLR/2.0)
      L=SDTM*SDTM+SDLMR*SDLMR*(CDTM*CDTM-STM*STM)
      CD=1.0-2.0*L
      DL=ACOS(CD)
      SD=SIN(DL)
      T=DL/SD
      U=2.0*KL*KL/(1.0-L)
      V=2.0*KK*KK/L
      D=4.0*T*T
      X=U+V
      E=-2.0*CD
      Y=U-V
      A=-D*E
      FF64=F*F/64.0
      DIST=AL*SD*(T-(F/4.0)*(T*X-Y)+FF64*(X*(A+(T-(A+E)/2.0)*X)+Y*(-2.0*
     1D+E*Y)+D*X*Y))/1000.0
      DEG = DIST/111.195
      TDLPM=TAN((DLR+(-((E*(4.0-X)+2.0*Y)*((F/2.0)*T+FF64*(32.0*T+(A-20.
     10*T)*X-2.0*(D+2.0)*Y))/4.0)*TAN(DLR)))/2.0)
      HAPBR=ATAN2(SDTM,(CTM*TDLPM))
      HAMBR=ATAN2(CDTM,(STM*TDLPM))
      A1M2=PI2+HAMBR-HAPBR
      A2M1=PI2-HAMBR-HAPBR
    1 IF((A1M2.GE.0.0).AND.(A1M2.LT.PI2)) 5,2
    2 IF(A1M2.GE.PI2) 3,4
    3 A1M2=A1M2-PI2
      GO TO 1
    4 A1M2=A1M2+PI2
      GO TO 1
    5 IF((A2M1.GE.0.0).AND.(A2M1.LT.PI2)) 9,6
    6 IF(A2M1.GE.PI2) 7,8
    7 A2M1=A2M1-PI2
      GO TO 5
    8 A2M1=A2M1+PI2
      GO TO 5
    9 AZ=A1M2/D2R
      BAZ=A2M1/D2R
      RETURN
      END
```

```
      FUNCTION  ATAN2 (Y,X)
      PI=3.1415926536
      ARG=ATANF (Y/X)
      IF (X) 10,14,11
   10 ATAN2=PI+ARG
      RETURN
   11 IF (Y) 12,13,13
   12 ATAN2=2.0*PI+ARG
      RETURN
   13 ATAN2=ARG
      RETURN
   14 IF (Y) 15,16,17
   15 ATAN2=1.5*PI
      RETURN
   16 ATAN2=0.0
      RETURN
   17 ATAN2=0.5*PI
      RETURN
      END
```

W/ Subroutine Geodist

```
CLARKE 1866 CONSTANTS
                                              6378206.4000    6356583.8000

   REFERENCE POINT
                                              OBJECT POINT
        LATITUDE    55 45 19.5
       LONGITUDE    37 34 15.5                  LATITUDE   -33-56 -3.5
                                               LONGITUDE    18 28 41.4
             DISTANCE BETWEEN POINTS  10102.069865 KM
                     FORWARD AZIMUTH    195 48 17.8 DEG
                        BACK AZIMUTH     10 39 32.3 DEG


   REFERENCE POINT
                                              OBJECT POINT
        LATITUDE    45  0   0
       LONGITUDE   106  0   0                   LATITUDE    20  0   0
                                               LONGITUDE     0  0   0
             DISTANCE BETWEEN POINTS   9649.171338 KM
                     FORWARD AZIMUTH    295 17 20.9 DEG
                        BACK AZIMUTH     42 56 30.7 DEG


   REFERENCE POINT
                                              OBJECT POINT
        LATITUDE    21 26  6.0
       LONGITUDE  -158 -1-33.0                  LATITUDE     8 58 25.0
                                               LONGITUDE   -79-34-24.0
             DISTANCE BETWEEN POINTS   8406.621818 KM
                     FORWARD AZIMUTH     85 37 10.6 DEG
                        BACK AZIMUTH    289 57 17.4 DEG
```

W/ Subroutine Geodist

```
INTERNATIONAL CONSTANTS                          6378388.0000   6356911.9462


    REFERENCE POINT                                 OBJECT POINT

          LATITUDE    55 45 19.5                      LATITUDE  -33-56 -3.5
         LONGITUDE    37 34 12.5                     LONGITUDE    18 28 41.4

                 DISTANCE BETWEEN POINTS   10102.670988 KM
                         FORWARD AZIMUTH    195 46 16.5 DEG
                            BACK AZIMUTH     10 39 31.1 DEG



    REFERENCE POINT                                 OBJECT POINT

          LATITUDE    45   0    0                     LATITUDE    28   0    0
         LONGITUDE   108   0    0                    LONGITUDE     0   0    0

                 DISTANCE BETWEEN POINTS    9649.412804 KM
                         FORWARD AZIMUTH    295 17 18.6 DEG
                            BACK AZIMUTH     42 56 30.0 DEG



    REFERENCE POINT                                 OBJECT POINT

          LATITUDE    21 26   6.0                     LATITUDE     8 58 25.0
         LONGITUDE  -158 -1-33.0                     LONGITUDE  -79-34-24.0

                 DISTANCE BETWEEN POINTS    8466.858288 KM
                         FORWARD AZIMUTH     85 37 12.3 DEG
                            BACK AZIMUTH    289 37 18.5 DEG
```


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