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DETERMINATION OF THE SPIN OF THE PROJECTILE

by

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Abstract

The motion of the projectile during the first period of the yaw is studied, to determine a criterion for the spin required for a real rate of precession. M. d'Adhémar first derives the general criterion*:

$$S\delta + 4\,B^2(\delta'' + \theta) + 4\,AB\Omega_1 M \gtreqless 0. \tag{30}$$

He refers to M. de Sparre's criteria, derived in different ways for the case of "perfectly regular departure", viz.:

$$(A\Omega)^2 > 4\,B(Rk\,l)_o, \tag{31}$$

$$(A\Omega)^2 > \frac{16B}{3}(Rk\,l)_o. \tag{32}$$

The departure is defined as "perfectly regular" if

$$\delta_o = \psi_o = p_o = q_o = r_o = 0.$$

Normally, the yaw is slightly different from zero and its initial derivative δ_o' is slightly different from the regular value $|\tau_o'|$. The departure is then defined as "almost perfect" if

$$\delta_o = \delta_2^o,$$

$$\psi_o = 0,$$

$$p_o = \delta_o' + \theta_o', \tag{62}$$

$$q_o = \psi_o' \sin \delta_o, \tag{63}$$

* See the List of Symbols below.

$$r_o = \psi_o' \cos \delta_o, \tag{64}$$

$$p_o = \Omega + r_o. \tag{65}$$

In this case, the criterion (30) takes the form:

$$S > 4 \; B s (Rkl)_o, \tag{80}$$

in which s may be greater than 1/3, which is M. de Sparre's value for the case of "perfectly regular" departure, as indicated by equation (32).

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LIST OF SYMBOLS

a	caliber, or diameter of the projectile.
α	trace of GA on the plane OHN.
b	drag coefficient.
c	French ballistic coefficient (American $C \approx .00114/c$).
f_1	$\omega P\delta$.
f_2	$\omega\Lambda\delta$.
g	gravitational acceleration.
j	slope of the rifling.
k	coefficient of obliquity: γ/δ.
l	distance from G to C.
m	mass of projectile.
n	numerical coefficient.
p	component of angular velocity of GUVz or Gxyz on Gx.
q	component of angular velocity of GUVz or Gxyz on Gy.
r	component of angular velocity of Gxyz on Gz.
s	M. de Sparre's coefficient.
t	time.
u	component of v on OX.
v	velocity of G.
A_1	f_1/ω (if constant).
A_2	f_2/ω (if constant).
$A(t)$	$p + \lambda_o \cos \omega t + \mu_o \sin \omega t$.
$B(t)$	$q + \lambda_o \sin \omega t - \mu_o \cos \omega t$.
C	center of pressure of projectile.

F_1	$f_1/\omega = P\delta$.
F_2	$f_2/\omega = \Lambda\delta$.
$F(v)$	French resistance law.
G	center of gravity of projectile.
GA	axis of projectile, directed towards the point.
GT	tangent of the <u>true</u> trajectory, directed in the direction of motion of G.
GAT	plane of yaw; plane of resistance.
GU	axis $\perp$ GT, related to the projectile.
GV	axis $\perp$ GTU, related to the projectile.
GUVz	"related trihedron".
GY	intersection of GAT and Gx_1y_1.
Gx_1	horizontal axis, directed to the left of GT.
Gy_1	axis in vertical plane containing GT, directed downwards.
Gz_1	axis on GT, in opposite direction.
$Gx_1y_1z_1$	trihedron with origin at G.
Gx	axis $\perp$ GAT.
Gy	axis $\perp$ GA in GAT.
Gz	axis on GA, directed towards the base.
Gxyz	"free trihedron".
H_1	F_1' (if constant).
H_2	F_2' (if constant).
K	radius of gyration of projectile about GA.
L	$\delta_o' t/\delta_o$.
M	$\omega_3\delta + \omega_4$.
N	$\omega_3 - \omega_4\delta$.
O	point on GT: distance GO = 1.

OH	horizontal axis, directed to the right of GT.
ON	axis in vertical plane containing GT, directed upwards.
OHNT	trihedron with origin at point O, and axes ∥ to those of $Gx_1y_1z_1$ but oppositely directed.
O	origin of trajectory.
OX	horizontal axis in plane of fire, directed towards the target.
OY	vertical axis, directed upwards.
OZ	horizontal axis ⊥ OXY, directed to the right.
OXYZ	absolute trihedron.
P	$Rkl/A\Omega = M_x/A\Omega\delta$.
Q	stability function: $P/\lvert\tau'\rvert$.
R	resistance; air pressure.
S	$(A\Omega_1)^2 - 4BRkl$
T	ωt.
U_1	$\lambda_o \cos \omega t$.
U_2	$\lambda_o \sin \omega t$.
V_o	initial velocity of G.
V_m	minimum velocity of G.
W_1	$\lambda_o \cos \omega t + \mu_o \sin \omega t$.
W_2	$\lambda_o \sin \omega t - \mu_o \cos \omega t$.
Z_o	zone of trajectory from $t = 0$ to $t = 2\pi/\omega$.
A	axial moment of inertia of projectile.
B	transverse moment of inertia of projectile.
M_x	moment of external forces about Gx; $Rkl\delta$.
M_y	moment of external forces about Gy.
M_z	moment of external forces about Gz.

P	instantaneous center of precession.
R	retardation: R/m.
α	angle of projection: $\tau_o = \theta_o$.
γ	angle between the direction of resistance and Gz.
δ	yaw; $\angle z_1 Gz$; $\tan\delta = O\alpha$.
δ_1	component of δ on OH; $-\delta\sin\psi$.
δ_2	component of δ on ON; $\delta\cos\psi$.
η	drift angle; $\tan\eta = -\frac{dZ}{dX}$.
θ	angle of inclination of the tangent of the <u>true</u> trajectory, GT.
λ	$p\cos\omega t + q\sin\omega t$.
μ	$p\sin\omega t - q\cos\omega t$.
ρ	component of angular velocity of GUVz on Gz.
τ	angle of inclination of the tangent of the <u>plane</u> trajectory; $\tan\tau = \frac{dY}{dX}$.
υ	ratio: $\lvert\bar{\eta}'\rvert/\lvert\bar{\theta}'\rvert$.
φ	$\angle xGU$.
ψ	orientation of plane of yaw; precession; $\angle NO\alpha = \angle x_1 Gx$.
ω	$A\Omega/B$.
ω_1	component of θ' and η' on Gx_1.
ω_2	component of θ' and η' on Gy_1.
ω_3	component of θ' and η' on Gz_1.
ω_4	component of ω_1 and ω_2 on GY.
ω_5	component of ω_1 and ω_2 on Gx.
ω_6	component of ω_3 and ω_4 on Gy.
ω_7	component of ω_3 and ω_4 on Gz.

Θ	$\theta'' \cos\psi + \eta'' \cos\theta \sin\psi$.
Λ	$M y / A \Omega \delta$.
Φ	$\theta'' \cos\psi + \eta'' \cos\theta \sin\psi - \eta'\theta' \sin\theta \sin\psi$.
Ω	Spin of projectile, impressed by the rifling.
Ω_1	ρ_o; initial angular velocity of projectile about GA.

THE GYROSCOPIC MOTION OF STABLE PROJECTILES

(THIRD REPORT)*

DETERMINATION OF THE SPIN OF THE PROJECTILE

by

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Engineer of Arts and Manufactures, Doctor of Sciences.

(Mémorial de l'artillerie française, Vol. 12, 1933, p. 249).

1. INTRODUCTION. Let Ω be the spin proper, impressed by the rifling; we have:

$$\Omega = 2jV_o/a;$$

V_o is the initial velocity on the trajectory,

a is the diameter of the projectile,

j is the tangent of the angle of inclination of the grooves.

For example, if $a = j = 1/10$, then $\Omega = 2V_o$.

If the number Ω is not very large, the pendular or gyroscopic phenomenon does not exist; then, the drifts are not regular. However, if the number Ω is too large, there is an inconvenience**: for, on the descending branch of the trajectory, the stability function Q may assume too small values.

I shall not study this question in all its amplitude, and I shall only seek a lower limit of Ω.

M. de Sparre obtained a lower limit, in his Report *** of 1904, by using his formulas, which represent, approximately, the gyroscopic motion.

* See: First report, Mém. de l'art. fr. Vol. VIII, No. 3, 1929 (Trans.: Report No.72, Ballistic Lab.). Second report, Mém. de l'art. fr. Vol. XI, No.3, 1932 (Trans.: A-IV-42, Ballistic Sec. file).

** P. Charbonnier. The gyroscopic motion of the projectile, Mém. de l'art. fr., Vol.VI No.3, 1927.

*** M. de Sparre. The motion of oblong projectiles about their centers of gravity , Arkiv for Matematik, Astronomi och Fysik, Stockholm, 1904.

M. Esclangon found the true basis of this discussion*, by starting from the fundamental equations of gyroscopic motion.

This method offers perfect safety. On the contrary, there exists some uncertainty in the use of an approximate solution, which carries all the weight of an accumulation of simplifications; the simplifications may engender deformations. However, M. Esclangon's remarkable analysis must be completed, because he adopted the ballisticians' first method of approximations, as M. de Sparre did otherwise in his classic report of 1904. This first approximation, which is always useful if one does not ask himself what it gives, consists in the substitution of the conventional yaw for the true yaw. Roughly, it neglects the curvature of the horizontal projection of the trajectory; I must give a precise indication on this subject.

2. THE ROLE OF THE CURVATURE OF THE HORIZONTAL PROJECTION OF THE TRAJECTORY. I keep the notations of my former reports**: GT is the tangent of the true trajectory, GA is the axis of the projectile. The plane OHN is perpendicular to the tangent. The axis OH is horizontal and directed towards the rear of the plane of the figure; the axis ON is in the vertical plane containing the tangent and directed upwards. Let a be the trace of GA on this plane of reference, so that:

$$Oa = \tan \delta \approx \delta, \quad GO = 1.$$

The coordinates of a are δ_1 on OH and δ_2 on ON. I designate by ψ the angle NOa, this angle being counted counter-clockwise, so that OH is brought to ON by a rotation of 90°. Then:

$$\delta_1 = -\delta \sin \psi, \quad \delta_2 = \delta \cos \psi.$$

I assume that the rifling is left handed: for a well-designed projectile, the drift will be to the left, i.e., in front of the plane of the figure. This is the plane containing the plane trajectory, which we know how to calculate.

The drift angle is η (Fig. 2). Let η' be the derivative of η with respect to time, t. The sign of η and that of η' depend on the sense of orientation that is chosen; but we have, as the absolute value:

$$u\,|\eta'| = R\,(k-1)\,|\delta_1|. \tag{1}$$

* E. Esclangon. Motion of projectiles about their center of gravity, Mém. de l'art. fr., Vol. VI, 1927, No. 3.

** R. d'Adhémar. On the gyroscopic motion of stable projectiles, Mém. de l'art. fr., 1929 and 1932.

The resistance is designated by R and the retardation by $\mathcal{R}$, so that:

$$R = m\mathcal{R}.$$

Replacing δ_1 by its mean value Δ_1, or else by $1/Q$ (see my previous reports), we have:

$$v^2 \, |\eta'| = g \frac{A\,\Omega}{m} \frac{k - 1}{k\,l}. \tag{2}$$

It is evident that formulas (1) and (2) are not identical. For example, according to (2), η' could not vanish, while, according to (1), η' vanishes at the origin if δ_1 does. This shows the deformation which an element may undergo as the result of taking an average.

In certain cases, it is necessary to make a distinction between an <u>exact element x</u>, and the same <u>element smoothed by averaging</u>. The notations x and $\bar{x}$ might then be used. We shall write, then:

$$v^2 \, |\bar{\eta}'| = g \frac{A\,\Omega}{m} \frac{k - 1}{k\,l}. \tag{3}$$

In every question that demands great precision, it is the element x which must be studied, and not $\bar{x}$. Here, formula (3) will be sufficient.

Let us consider now the angle of inclination of the tangent to the trajectory. For the true trajectory, this angle is θ. For the plane trajectory, this angle, at the same instant t, is τ. If the yaw δ is quite small,

$$\theta \approx \tau, \qquad \theta' \approx \tau'.$$

I do not actually insist on the degree of approximation of these formulas, and I recall that :

$$\frac{d\tau}{dt} = \tau' = -\frac{g \cos \tau}{v}, \tag{4}$$

so that we can write:

$$\bar{\theta}' = -\frac{g \cos \tau}{v}. \tag{5}$$

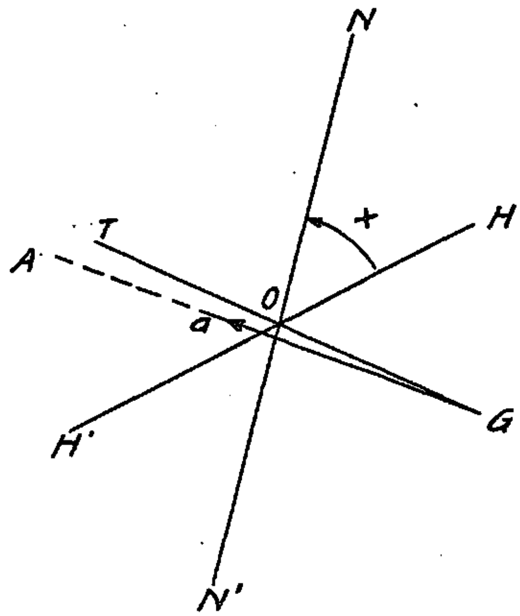

FIG. 1

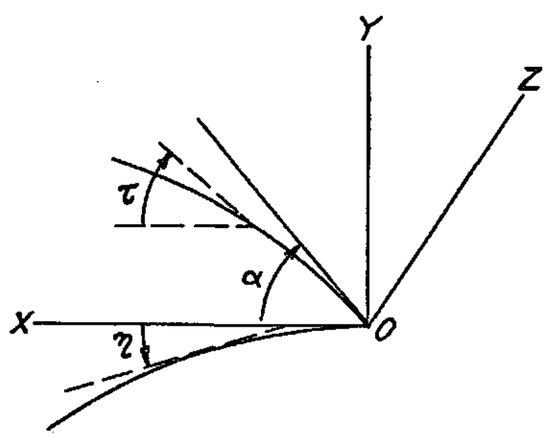

FIG. 2.

Let us form the ratio:

$$\upsilon = \frac{|\overline{\eta}'|}{|\overline{\theta}'|} = \frac{A}{m} \frac{1}{\cos\tau} \frac{2j}{a} \frac{k-1}{kl} \frac{V_o}{v} .$$

The ratio A/m is very small.

The parameter k is poorly known; but we may assume that its value remains near 1 or 2 for very small yaws. This value could be 3, 4, 5, ... for large yaws, in the dangerous zone.

The parameter l is the distance from the center of gravity to the center of pressure C. If this point C is at the height of the middle of the ogive, then, in the formula l = na, n can assume the values 2, 3, ...

Altho the ratio A/m is very small, if $\cos\tau$ is very small, and if the ratio V_o/v becomes quite large, it is quite possible that υ is not negligible; the derivatives η' and θ' may have the same order of magnitude.

In his classic report of 1904 (p. 287), M. Sparre assumed intuitively that the ratio υ is always negligible. However, that is not exact for an almost horizontal trajectory with a very high velocity. What ballisticians call the "Garnier effect" is, we might say, the fact that the ratio υ is not negligible, in general. This remark has been made previously by Engineer General Maurice Garnier.

In the recent works of M. de Sparre and Engineer General Charbonnier, this ratio was no longer taken as negligible *a priori*.

Prudence demands that we take account of the curvature of the horizontal projection of the trajectory, altho its effect may be, in certain circumstances, very small.

3. FIVE INSTANTANEOUS ROTATIONS. I take, as the sense of orientation of trihedrons, the direct sense of trigonometry, which is counterclockwise. Gz_1 is the tangent, in the direction opposite to that of the motion of G. Gx_1 or GH_1 is parallel to the axis OH, which has been defined. Gz is the axis of the projectile, towards the base; the point would be on the prolongation of Gz (Fig. 4).

The moment $\mathcal{M}_x$ of the resistance defines the axis Gx (see my previous reports). The axis Gy carries the second moment $\mathcal{M}_y$ of the forces of resistance. If it is the Magnus effect which is considered, $\mathcal{M}_y < 0$. If it is the Esclangon effect, $\mathcal{M}_y > 0$. According to M. Esclangon, this couple $\mathcal{M}_y$ is caused by lateral frictions and its existence appears certain for a well designed projectile. We have:

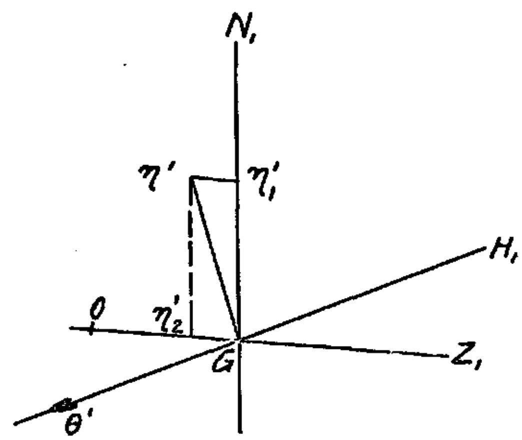

FIG. 3

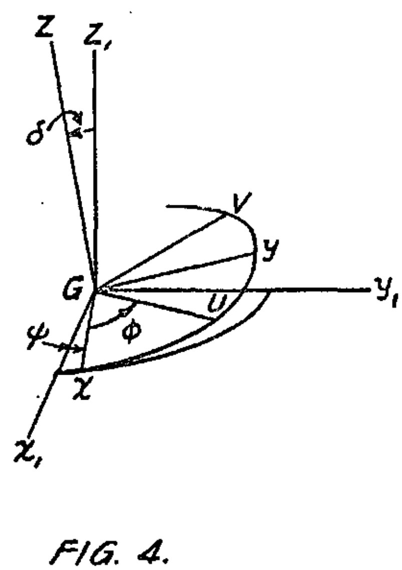

FIG. 4.

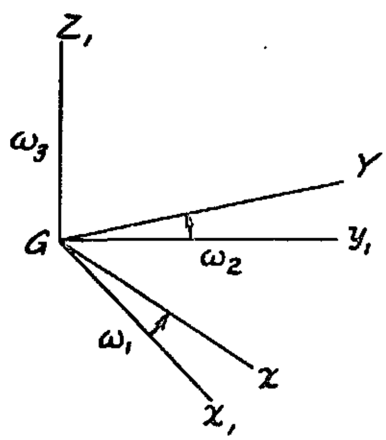

FIG. 5a.

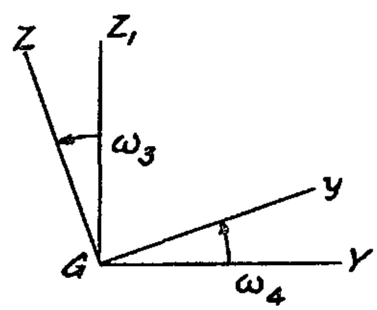

FIG. 5b.

$$\mathcal{M}_x = \mathrm{Rk}\,l\,\delta.$$

We may write:

$$\mathcal{M}_x = \mathcal{A}\Omega P\delta.$$

The symbol P represents a known element. To avoid all discussion on the physical definition of $\mathcal{M}_y$, I shall write:

$$\mathcal{M}_y = \mathcal{A}\Omega\Lambda\delta,$$

and I shall make the apparently natural suppositions:

1° Λ/P is constant and very small if δ is very small; for example, this ratio is equal to 2% or 3% ....

2° Λ/P may assume a more appreciable value if δ becomes relatively quite large; for example, for yaws of about 12° or 15°.

So, at the origin, we suppose Λ/P constant and very small.

We have thus defined two trihedrons, $Gx_1y_1z_1$ and Gxyz. Let GU and GV be two axes in the plane Gxy, related to the projectile. We have a third trihedron GUVz, related to the projectile. I call it the "related trihedron", and the trihedron Gxyz shall be the "free trihedron" (Fig. 4).

If $Gx_1y_1z_1$ were fixed, we would simply have to consider the three Eulerian angles φ, ψ, δ (I must substitute δ for the classic notation θ, since I have already used the symbol θ). Furthermore, we would have only the three instantaneous rotations:

$$\frac{d\varphi}{dt} \text{ or } \varphi', \frac{d\psi}{dt} \text{ or } \psi', \frac{d\delta}{dt} \text{ or } \delta'.$$

We have, besides, two instantaneous rotations:

1° The rate of depression of the tangent, θ',

2° The angular velocity of drift, η' (Figs. 2 and 3).

I suppose essentially that the true trajectory and the plane trajectory are very close, in size and form, so that, at each instant t:

$$\theta \approx \tau,\ \theta' \approx \tau',\ \theta'' \approx \tau''.$$

I do not actually discuss this approximation, which is based on physical intuition of the phenomenon. I shall always designate the initial velocity, on the trajectory, by V_o, and the minimum velocity by V_m.

It must be remarked that:

$$|\tau'| = g \frac{\cos \tau}{v},$$

and

$$\tau'' = |\tau'| \left(\frac{v' - g \sin \tau}{v}\right) . \qquad (6)$$

In the dangerous zone of the trajectory, i.e., in the neighborhood of the point where Q is a minimum and in the neighborhood of the point where v is a minimum, the ratio $|\tau''|/|\tau'|$ will be extremely small. On the other hand, at the origin, this ratio is not very small. In fact, the ratio $|v'|$ /v can take, at the origin, values such as 1/10 or 2/10 ... So, at the origin, the ratio $|\tau''| / |\tau'|$ may be about 1/10 or 2/10 ...., it may not be negligible.

The sense of orientation is counterclockwise (Fig. 2); so, the axis OZ is directed towards the rear of the plane of the figure, the plane OXY. For an observer placed on OZ, the angle τ is positive on the ascending branch, and negative on the descending branch.

An observer placed on OZ sees the tangent turn in a clockwise direction. So, this rotation is represented by a vector $|\theta'|$, directed forwards. Consequently, the component on the axis Gx_1 is negative; it is θ', or approximately τ'.

When I supposed that the true trajectory is very near the plane trajectory, I implicitly assumed that the projectile is stable, that δ remains small. Consequently, the drift will be normal; η and η' will be positive (Figs. 2 and 3).

Moreover, the passage from (2) to (3) would also imply that the projectile is stable. If V_o is large and if V_m, the minimum of v, is not small, the drift will remain feeble, (k - 1) will not become large, and η' will remain small like τ'.

I shall not actually discuss the second derivative,

$$\frac{d^2\eta}{dt^2} = \eta''.$$

If the departure of the projectile is quite regular, $\eta'_o = 0$.

I designate an initial value by x_o or x^o.

4. SOME INDICATIONS ON THE NUMERICAL DATA. Before going farther, I shall give some indications, some figures, whose knowledge is indispensable, if we wish to avoid vague discussions. We designate by A the axial moment of inertia; in the formula $A = na^5$, n may vary, e.g., between 200 and 300*. We designate by B the transverse moment of inertia; in the formula $B = nA$, n may vary from 4 to 8, approximately.

In the formula $A = mK^2$, we see intuitively that K must remain between a/2 and a/3, nearly.

If we take j = 1/10, that corresponds to an angle of inclination of the rifling of 5°43'; this is a probable datum.

For an angle of 7°, j = 0.123. For an angle of 10°, j = 0.176.

The symbol P, which plays an interesting role, represents an approximate mean value of the angular velocity of the point g about the instantaneous center of precession P (see my previous reports). We have:

$$P = \frac{m}{A\Omega} Rkl \quad , \text{ and } R = cF(v).$$

Let us seek an indication of the numerical values of P, at the origin, supposing a = 1/10 (100 mm caliber).

Let us take the resistance law in the form bv^2.

If the velocity v varies from 400 m/s to 1500 m/s, we can set b = 1/3**. This is a rough mean, but it is sufficient for us. Let us take c = 1/2000***; we are at the origin, so we do not have to consider the variations of altitude. We may suppose, at the origin, that k is about 1 or 2. Also, we can take $l = 2a$ or 3a.

So, if we set

$$kl = 4/10, \; K^2 = a^2/5;$$

*Here a is the caliber in meters, and A is the axial moment of inertia in kg.m. divided by the gravitational acceleration in m/sec.^{2} --- H.P.H.

**Hence, bv^2 is the drag, in kg.m/sec.^{2}, of a 1 m. projectile, in air with a density of 1 kg/m^3. This value of b corresponds to a resistance coefficient $C_R = 12 \times 10^{-5}$. ---H.P.H.

***This value of the French ballistic coefficient c corresponds approximately to an American ballistic coefficient C = 2.28. --- H.P.H.

we will have

$$P_o = V_o/60.$$

For $V_o = 600$, $P_o = 10$.

If we take $c = 1/4000$, keeping the values of all the other parameters, we will have $P_o = V_o/120$. For $V_o = 1000$, $P_o \approx 8$.

Let us also seek an indication of the numerical values of P'/P at the origin.

We suppose $V_o > 400$ m/s. We can neglect the variations of b, k, l and take $b = 1/3$, since we are not seeking any great precision here.

I recall the general formula:

$$\frac{dv}{dt} = v' = -g(\sin \tau + R/g).$$

When V_o is a large number, the ratio R/g can assume values such as 40 or 50, e.g., or even 60; then

$$\frac{dv}{dt} \approx -R.$$

Then, we can write:

$$(P'/P)_o \approx 2(v'/v)_o \approx -2bcV_o,$$

and the approximation is good if V_o is large.

For $c = 1/2000$ and $V_o = 500$, $(P'/P)_o = 1/6$.

For $c = 1/4000$ and $V_o = 800$, $(P'/P)_o = 8/60$.

For $c = 1/10,000$ and $V_o = 1500$, $(P'/P)_o = 1/10$.

So:

$$(P'/P)_o = n/10,$$

with a coefficient n about unity.

Let us return to the formula for Ω. For the usual artillery calibers, if $\Omega = nV_o$, the coefficient n is between 1 and 4.

It is often said that Ω is a large number. To be precise: Ω is large with respect to unity, with respect to P, and also with respect to $|\tau'|$, if the minimum velocity V_m, on the trajectory, is not very small; this exceptional case is excluded.

In the theory which is going to be developed, the approximation is very good if the number

$$\omega = A\Omega / B ,$$

which is between 1/6 and 1/8 of Ω , is still quite large with respect to P_o.

5. THE INSTANTANEOUS ROTATION OF THE RELATED TRIHEDRON AND THE INSTANTANEOUS ROTATION OF THE FREE TRIHEDRON. Let us consider the instantaneous rotations of the two trihedrons which have been defined, and let us seek their components on the axes of the free trihedron Gxyz. Let us first take the related trihedron. The components of the rotational velocities δ', ψ', and φ' are:

$$\delta', \text{ on Gx}; \quad \psi' \sin \delta, \text{ on Gy}; \quad \psi' \cos \delta + \varphi', \text{ on Gz}.$$

The components of the rotational velocities θ' and η' are:

$$\omega_1 = \theta' < 0, \text{ on } Gx_1;$$

$$\omega_2 = \eta' \cos \theta, \text{ on } Gy_1;$$

$$\omega_3 = - \eta' \sin \theta, \text{ on } Gz_1.$$

Let us consider the trace GY of the plane of resistance on the plane Gx_1y_1. The axes Gx and GY are orthogonal, as are Gx_1 and Gy_1 (Fig. 5). Let us take the components of ω_1 and ω_2 on these new axes Gx and GY. They are:

$$\omega_5 = \omega_1 \cos \psi + \omega_2 \sin \psi , \text{ on Gx};$$

$$\omega_4 = - \omega_1 \sin \psi + \omega_2 \cos \psi , \text{ on GY}.$$

Now, we must take the components of ω_3 and ω_4 on the axes Gz and Gy; this gives:

$$\omega_6 = \omega_3 \sin \delta + \omega_4 \cos \delta, \text{ on Gy};$$

$$\omega_7 = \omega_3 \cos \delta - \omega_4 \sin \delta, \text{ on Gz}.$$

So, the components of the instantaneous rotation of the related trihedron are:

on Gx: $p = \delta' + \omega_5$,

or $p = \delta' + \theta' \cos\psi + \eta' \cos\theta \sin\psi$;

on Gy: $q = \psi' \sin\delta + \omega_6$,

or $q = \psi' \sin\delta - \eta' \sin\theta \sin\delta$

$+ \cos\delta\,(-\theta' \sin\psi + \eta' \cos\theta \cos\psi)$;

on Gz: $\rho = \varphi' + \psi' \cos\delta + \omega_7$,

or $\rho = \varphi' + \psi' \cos\delta - \eta' \sin\theta \cos\delta$

$- \sin\delta\,(-\theta' \sin\psi + \eta' \cos\theta \cos\psi)$.

If the yaw is quite small, we can substitute δ for $\sin\delta$ and 1 for $\cos\delta$. Then, let

$$M \equiv \omega_3\delta + \omega_4 \tag{7}$$
$$= -\eta'\delta\sin\theta + (-\theta'\sin\psi + \eta'\cos\theta\cos\psi),$$

$$N \equiv \omega_3 - \omega_4\delta \tag{8}$$

$$= -\eta'\sin\theta - \delta\,(-\theta'\sin\psi + \eta'\cos\theta\cos\psi).$$

We will have the following expressions:

$$p = \delta' + \theta'\cos\psi + \eta'\cos\theta\sin\psi, \tag{9}$$

$$q \approx \psi'\delta + M, \tag{10}$$

$$\rho \approx \varphi' + \psi' + N. \tag{11}$$

To pass from the <u>related trihedron</u> to the <u>free trihedron</u>, it is sufficient to set

$$\varphi \equiv 0,\ \varphi' \equiv 0;$$

whence

$$r \approx \psi' + N. \tag{12}$$

The components of the instantaneous rotation of the <u>free trihedron</u> are p, q, r.

I substitute M for ω_6 and N for ω_7, to simplify the explanation; but these approximations are not indispensable. Notice that:

$$M \approx \omega_6 \quad \text{and} \quad \frac{dM}{dt} \approx \frac{d\omega_6}{dt};$$

$$N \approx \omega_7 \quad \text{and} \quad \frac{dN}{dt} \approx \frac{d\omega_7}{dt}.$$

In fact,

$$\frac{d}{dt} \sin \delta = \cos \delta \frac{d\delta}{dt} \approx \frac{d\delta}{dt},$$

$$\frac{d}{dt} \cos \delta = - \sin \delta \frac{d\delta}{dt} \approx 0,$$

since δ remains quite small, and the smaller δ is, the more satisfactory the approximation is. As far as p is concerned, equation (9) is exact, so that we obtain $\frac{dp}{dt}$ immediately by differentiation.

Now, I am going to define "perfectly regular departure". I shall say that the departure is perfectly regular if

$$\delta_o = \Psi_o = p_o = q_o = r_o = 0.$$

I exclude the case of vertical fire: $\alpha = 90°$.

Consequently: $\delta^{\circ}_1 = 0$.

Since the initial yaw vanishes, the gyroscopic motion consists of oscillations from the origin to the summit, and even beyond that (see my previous reports). So we certainly have, at the origin, $\delta_1 < 0$, and since the sense of orientation is counterclockwise, we must write:

$$- u \quad \eta' = \quad \ell(k - 1)\delta_1. \tag{13}$$

On the other hand, we would have to take the + sign if the sense of orientation were clockwise, i.e., if GZ were directed forward (Fig. 2).

Since $u_o > 0$ and $\delta_1^o = 0$, whatever the value of k may be, at the origin,

$$\psi'_o = 0.$$

On the other hand, with θ and τ as defined,

$$v\theta' = -g\cos\theta + R(k-1)\delta_2, \tag{14}$$

with

$$v\tau' = -g\cos\tau,$$

and, if the departure is perfectly regular, $\delta_2^o = 0$.

So, at the origin:

$$\theta_o = \tau_o = \alpha \quad \text{and } \theta'_o = \tau'_o,$$

while, in general,

$$\theta \approx \tau \quad \text{and } \theta' \approx \tau'.$$

Since

$$p_o = \psi_o = \eta'_o = 0,$$

therefore

$$\delta'_o + \tau'_o = 0.$$

The term $\eta'\cos\theta\sin\psi$ is extremely small at the origin, for it contains two factors which vanish. It might be said that this term is of the "second order"; but it is sometimes dangerous to extend to the "very small", notions which are clear only for the "infinitesimal".

Let us retain the formula:

$$\delta'_o + \tau'_o = 0, \text{ or } \delta'_o = |\tau'_o,| \tag{15}$$

relative to the case of perfectly regular departure. Notice, besides, that, in this case:

$$\omega_3^o = \omega_4^o = \omega_6^o = \omega_7^o = M^o = N^o = 0,$$

and consequently, $q_o = 0$.

In order to have $r_o = 0$, we must set $\psi'_o = 0$.

Then, for a perfectly regular departure:

$$\delta_o = \psi_o = \psi'_o = p_o = q_o = r_o = 0.$$

Now, let us find out whether M/δ and ω_6/δ have limits for $t = 0$.

If M/δ has a limit, ω_6/δ will have a limit, and it will be the same limit. The expression for M/δ is:

$$\frac{M}{\delta} = -\theta' \frac{\sin\psi}{\delta} + \frac{\eta'}{\delta} \cos\theta \cos\psi - \eta' \sin\theta.$$

The last term vanishes at the origin, as η'_o does.

The first term vanishes at the origin, as ψ'_o does. To see this, it is sufficient to apply L'Hospital's rule to the ratio $\frac{\sin\psi}{\delta}$ and utilize formula (15).

Now, let us study the second term. By (13),

$$-u \frac{\eta'}{\delta} = R (k - 1) \frac{\delta_1}{\delta},$$

while

$$\delta_1 = -\delta \sin\psi .$$

As $\sin\psi$ vanishes at the origin, so does the second term.

So, in the case of perfectly regular departure,

$$\lim_{t=o} \frac{M}{\delta} = \lim_{t=o} \frac{\omega_6}{\delta} = 0. \tag{16}$$

Now, we must calculate the derivative $\frac{dp}{dt} = p'$.

Equation (9) gives:

$$p' = \delta'' + \theta'' \cos\psi + \eta'' \cos\theta \sin\psi - \eta' \theta' \sin\theta \sin\psi + \omega_4 \psi'.$$

Let:

$$\Phi \equiv \theta'' \cos\psi + \eta'' \cos\theta \sin\psi - \eta'\theta' \sin\theta \sin\psi , \tag{17}$$

$$\Theta \equiv \theta'' \cos\psi + \eta'' \cos\theta \sin\psi . \tag{18}$$

The derivative of p will then have the form:

$$p' = \delta'' + \Phi + \omega_4 \psi' . \tag{19}$$

At the origin:

$$\lim_{t=0} \frac{\omega}{\delta}4 = \lim_{t=0} \psi' = \lim_{t=0} \frac{\eta'}{\delta} = \lim_{t=0} \sin \psi = 0.$$

So, the terms $\frac{\omega}{\delta}4 \ \psi'$ and $\frac{\eta'}{\delta} \ \theta' \sin\theta \sin\psi$ each contain two factors which vanish at the origin. Therefore, if t is extremely small,

$$\frac{p'}{\delta} \approx \frac{\delta'' + \theta}{\delta}, \text{ with } t \approx 0, \tag{20}$$

and the closer t is to zero, the more satisfactory the approximation will be.

6. USE OF THE MOMENT OF MOMENTUM THEOREM. Let OX, OY, OZ be absolute axes, and GX', GY', GZ' be axes respectively parallel to the former. Here, the absolute axes are related to Earth; this is a sufficient approximation. The motion with respect to the trihedron of reference GX'Y'Z' is, by definition, the "motion with respect to the center of gravity". With this new trihedron of reference, we can use the fundamental moment of momentum theorem, without modifying the external forces: The derivative of the vectorial moment of momentum (with respect to t) is equipollent to the vectorial sum of the moments (with respect to G) of the external forces*.

We do not have to bother ourselves with fixing the absolute trihedron OXYZ; we are going to take the projections on the axes of the free trihedron, which has been defined.

The components of the moment of momentum on Gx, Gy, Gz are:

$$Bp, \quad Bq, \quad A\rho$$

The components of the relative rate of change (with respect to Gxyz) are:

$$Bp', \quad Bq', \quad A\rho'.$$

* See e.g., Appell's Traité de Mécanique, or else Painlevé's Cours de Mécanique, Gauthier-Villars, 1930.

The components of the translational rate of change are the minors of the matrix:

$$\begin{vmatrix} p & q & r \\ Bp & Bq & A\rho \end{vmatrix},$$

i.e.,

$$Aq\rho - Bqr, \; -(A\,p\rho - Bpr), \; Bpq - Bpq.$$

We have, then, the three equations:

$$B\frac{dp}{dt} + Aq\rho - Bqr = M_x,$$

$$B\frac{dq}{dt} - A\,p\rho + B\,pr = M_y, \qquad (21)$$

$$A\frac{d\rho}{dt} = M_z,$$

M. Esclangon, who utilized the moment of momentum theorem in another form, fortunately discovered that an algebraic equation of the second degree in ψ' can be rigorously deduced from these equations. The <u>condition of reality</u> will give a lower limit of Ω.

Before seeking this equation of the second degree, I shall make a remark. The spin of the projectile has a negligible variation along the trajectory. So, the role of M_z is almost negligible. To simplify the explanation, I assume that

$$M_z = 0, \text{ whence } \rho = \text{const.}$$

In the case of perfectly regular departure,

$$\rho_o = \varphi'_o = \Omega. \qquad (22)$$

In the contrary case,

$$\rho_o = \Omega_1 \approx \Omega. \qquad (23)$$

In fact, the initial values δ_o, ψ'_o, η'_o are extremely small if the projectile is well designed and if the tube is not eroded.

Note that, if $\mathcal{N}_z$ is not completely negligible, it is sufficient to add to the expression for ρ a function $D(t)$, which vanishes at the origin, decreases, and is extremely small if the projectile is normal.

The first of equations (21):

$$\mathcal{B}\,p' + \mathcal{A}\,q\rho - \mathcal{B}qr = \mathcal{N}_x, \tag{24}$$

would be an absolutely rigorous equation if the parameters k and l, which figure in $\mathcal{N}_x$, were known. These parameters certainly vary along the trajectory; but they vary little if the stability is suitable, so that the drift is regular and feeble. If we assume that k increases between the origin and the dangerous zone, and that it can take values between 1 and 3 or 4, approximately, this is an hypothesis which appears acceptable. Furthermore, it is possible that the variations of l are in the sense opposite to those of k.

In the case of flat fire, it is extremely probable that the variations of kl are quite small. It is difficult to give general indications from this point of view in the case of very high angle fire.

We could study equation (24) with great precision by keeping $\sin\delta$ and $\cos\delta$, by not assuming any approximation. However, that would be difficult. So, I shall make certain approximations which will not alter the character of the equation at all.

Let us consider, e.g., the derivatives θ' and τ'.

At the origin, θ' or τ' will be very small if V_o is very large.

Later, θ' or τ' will remain very small if V_m remains quite large. I shall always suppose that V_m remains quite large, so I exclude not only the case: $\alpha = 90°$, but also the case: $\alpha \approx 90°$. I do not study almost vertical fire because, near the summit, the minimum velocity would be quite close to zero. The phenomena then become entirely different from those which we are examining.

Now, let us consider the derivative η'.

If $\delta° = 0$, or $\delta_1° = 0$, we shall have: $\eta_0' = 0$.

In the case of perfectly regular departure, $\eta_o' = 0$.

In the contrary case, η' will be very small at the origin, as δ_1 is. Later, η' will remain small if V_m remains quite large.

The variations of k and l play a role here. Let us repeat that this role is a minimum in the case of flat fire if the angle of projection is large.

As long as V_m remains large, our ignorance on the subject of k and l probably gives us no great embarassment.

When I say that the minimum of V_m must be quite large, this statement is not precise enough. Suppose V_o very large and α quite near 90°. The summit S will have a high altitude and, consequently, at the point of minimum velocity, the ballistic coefficient c will be very much diminished with respect to its initial value c_o. There results a great diminution in the value of the stability function Q. So, the numerical value of the mean yaw might become too large, with regard to the general conditions of stability. I shall say, then: V_m must always remain quite large, and all the larger, the higher the corresponding point of the trajectory is.

So, I always suppose that the initial velocity V_o is very large, that the minimum velocity is not small, and that the angle of projection α is different from 90°. Otherwise, the stability is doubtful a priori; absolutely nothing would be known about the parameters k and l; certain hypotheses such as:

$$\theta \approx \tau, \; \theta' \approx \tau', \; \theta'' \approx \tau'',$$

might have no foundation. Besides, there would be no sense in writing:

$$\sin \delta \approx \delta, \text{ and } \cos \delta \approx 1.$$

Without actually insisting on these questions of approximation, I must underline their importance.

Let us analyze all the terms of equation (24).

Neglecting the product $\ddot{\eta}'\theta'$, we have:

$$p' \approx \delta'' + \theta + \omega_4 \psi'.$$

Neglecting products such as: $\theta'\delta^2$, $\eta'\delta^2$, $\theta'^2\delta$, $\eta'^2\delta$, $\eta'\theta'\delta$, etc., we have:

$$qr \approx \delta \psi'^2 + \psi' (2\omega_3\delta + \omega_4).$$

Therefore,

$$p' - qr \approx \delta'' + \theta - \delta \psi'^2 - 2\omega_3\delta\psi . \tag{25}$$

On the other hand, if we suppose $M_z = 0$,

$$\rho = \rho_0 = \Omega_1.$$

The very approximate expression for q is:

$$q \approx \delta \psi' + M.$$

So:

$$q\rho \approx \Omega_1(\delta \psi' + \omega_3\delta + \omega_4). \tag{26}$$

The first member of equation (24) will be, with a good approximation if we remain within the limits previously indicated:

$$A\Omega_1(\delta \psi' + \omega_3\delta + \omega_4) + B(\delta'' + \Theta - \delta \psi'^2 - 2\omega_3\delta \psi').$$

However, we know, a priori, that Ω_1 must be a large number and, on the other hand, B/A is certainly less than 8 or 10, for example.

Besides, $|\omega_3|$ remains very small.

So, we can certainly neglect the last term, for it is always extremely small with respect to the first. So, for the primitive equation (24), we can substitute the following approximate equation:

$$-B\delta\psi'^2 + A\Omega_1\delta \psi' + B(\delta'' + \Theta) + A\Omega_1 M - Rk l \delta = 0. \tag{27}$$

Roughly, the approximation is good if δ remains small, and the better, the closer δ is to zero.

The condition of reality of ψ' will be:

$$(A\Omega_1\delta)^2 + 4B\delta[B(\delta'' + \Theta) + A\Omega_1 M - Rk l \delta] \geq 0. \tag{28}$$

I recall that:

$$R = m\mathcal{R} = mcF(v),$$

$$\theta = \theta'' \cos\psi + \eta'' \cos\theta \sin\psi ,$$

$$M \approx \omega_6 \approx \omega_4,$$

and

$$\omega_4 = -\theta' \sin\psi + \eta' \cos\theta \cos\psi .$$

This is M. Esclangon's criterion, mutatis mutandis (loc. cit., p. 780).

I shall put it in the following form:

$$S\delta^2 + 4B^2\delta(\delta'' + \dot{\theta}) + 4AB\Omega_1\delta M \geqq 0, \tag{29}$$

setting

$$S \equiv (A\Omega_1)^2 - 4B\,Rkl .$$

Since the yaw δ is positive, the criterion (29) can be put in the form:

$$S\delta + 4B^2(\delta'' + \theta) + 4AB\Omega_1 M \geqq 0. \tag{30}$$

M. de Sparre, in his aforementioned Report of 1904 (pp. 293 and 295), gave two successive criteria:

$$(A\Omega)^2 > 4B(Rkl)_o, \tag{31}$$

or

$$S_o > 0, \quad \text{with} \quad \Omega_1 = \Omega ,$$

and

$$(A\Omega)^2 > \frac{16}{3} B(Rkl)_o. \tag{32}$$

The first criterion is deduced from the equations of gyroscopic motion, derived in the Report of 1904, and the second criterion is deduced from very ingenious considerations of the physical character of the air resistance.

I shall study criterion (30) at the origin, i.e., for $t \approx 0$. As a result of this, if the departure is perfectly regular, the second criterion is applicable again, for $t \approx 0$.

As a result of this, if the departure is perfectly regular, the second criterion is applicable again, for $t \approx 0$. However, if the departure has the least irregularity, it must be foreseen that the second criterion will be insufficient for $t \approx 0$. M. de Sparre always implicitly supposed that the departure was perfectly regular.

After the criterion has been studied at the origin, it must be studied for any values of t whatever; but actually, I put this question aside.

M. Esclangon made a penetrating study of this question by distinguishing zones of stability and zones of instability on a trajectory.

7. INTEGRATION OF A DIFFERENTIAL SYSTEM. I suppose that the departure is perfectly regular and that $\mathcal{M}_z = 0$. Under these conditions, equations (21) become:

$$\begin{aligned} \mathcal{B}\,p' + q\,(\mathcal{A}\,\Omega - \mathcal{B}\,r) &= \mathcal{M}_x, \\ \mathcal{B}\,q' - p\,(\mathcal{A}\,\Omega - \mathcal{B}\,r) &= \mathcal{M}_y, \end{aligned} \tag{33}$$

with

$$\rho = \rho_0 = \Omega .$$

Besides,

$$r_0 = 0.$$

So, the term $\mathcal{B}\,r$ is negligible with respect to $\mathcal{A}\,\Omega$ at the origin, i.e., when $t \approx 0$, and the above equations can be replaced by the following with an excellent approximation:

$$\begin{aligned} \mathcal{B}\,p' + q\mathcal{A}\,\Omega &= \mathcal{M}_x, \\ \mathcal{B}\,q' - p\,\mathcal{A}\Omega &= \mathcal{M}_y. \end{aligned} \tag{34}$$

These are the differential equations which M. Sugot very fortunately used to derive the approximate equations of gyroscopic motion. His equations are similar to those of M. Esclangon and the approximations are the same. I am going to study equations (34), so as to be able to discuss criterion (30).

Let $\omega = \mathcal{A}\,\Omega/\mathcal{B}$; ω will be a large number, like Ω . I suppose, in fact, that the ratio $\mathcal{B}/\mathcal{A}$ is less than 7 or 8, for example.

We shall write:

$$\frac{dp}{dt} + \omega q = \omega P\delta \equiv f_1,$$

$$\frac{dq}{dt} - \omega p = \omega\Lambda\,\delta \equiv f_2. \tag{35}$$

Although it appears a bit paradoxical at first glance, I am going to integrate equations (35) as if f_1 and f_2 were known functions of t. These functions are continuous and differentiable, since we suppose essentially that the derivatives δ' and δ'' exist, in all that precedes.

Note that the ratio $f_2/f_1 = \Lambda/P$ is supposed constant and very small for $t \approx 0$. We assumed that this ratio can have a value such as 2% or 3% at the origin.

The classic method of "variation of constants" immediately gives the solution:

$$p = \lambda\cos\omega t + \mu\sin\omega t,$$

$$q = \lambda\sin\omega t - \mu\cos\omega t. \tag{36}$$

Whence:

$$\lambda' = f_1\cos\omega t + f_2\sin\omega t,$$

$$\mu' = f_1\sin\omega t - f_2\cos\omega t. \tag{37}$$

To have $p_o = q_o = 0$, we must take $\lambda_o = \mu_o = 0$. It will be convenient to introduce the following symbols:

$$f_1/\omega \equiv F_1,\ f_2/\omega \equiv F_2.$$

It should be remarked that, always,

$$p^2 + q^2 = \lambda^2 + \mu^2.$$

I shall have to be occupied with the case where F_1 and F_2 are linear functions of t:

$$F_1(t) = H_1 t,\ F_2(t) = H_2 t,\ H_2/H_1 = \Lambda/P. \tag{38}$$

It will be convenient to introduce the auxiliary variable:

$$T = \omega t.$$

Immediately:

$$\omega \lambda = H_1 \int_0^T T \cos T \, dT + H_2 \int_0^T T \sin T \, dT,$$

$$\omega \lambda = H_1(T \sin T + \cos T - 1) + H_2 (-T \cos T + \sin T), \qquad (39)$$

and, similarly,

$$\omega \mu = H_1(- T \cos T + \sin T) - H_2(T \sin T + \cos T - 1). \qquad (40)$$

The curves of λ and μ can be constructed according to the formulas, remarking that the extremes depend uniquely on the ratio Λ/P, from the point of view of their position.

On the other hand, the terms in H_2 will generally be negligible with respect to the terms in H_1, because the constant H_1 will be a very small number, and because the ratio Λ/P is small with respect to unity.

I adopt the following notation:

$$x = x_0, x_1, x_2, x_3, x_4,$$

for

$$T = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi,$$

or

$$t = 0, \frac{\pi}{2\omega}, \frac{\pi}{\omega}, \frac{3\pi}{2\omega}, \frac{2\pi}{\omega}.$$

Hence:

$$\omega \lambda_1 = H_1 \left(\frac{\pi}{2} - 1\right) + H_2, \quad \omega\mu_1 = H_1 - \left(\frac{\pi}{2} - 1\right)H_2,$$

$$\omega\lambda_2 = - 2H_1 + \pi H_2, \quad \omega\mu_2 = \pi H_1 + 2H_2,$$

$$\omega \lambda_3 = - \left(3\frac{\pi}{2} + 1\right)H_1 - H_2, \quad \omega \mu_3 = - H_1 + \left(3\frac{\pi}{2} + 1\right)H_2,$$

$$\omega \lambda_4 = - 2\pi H_2, \qquad \omega \mu_4 = - 2\pi H_1.$$

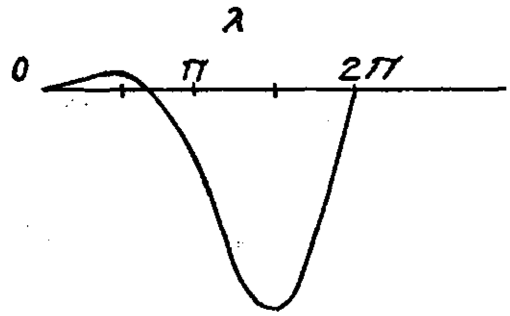

FIG. 6a

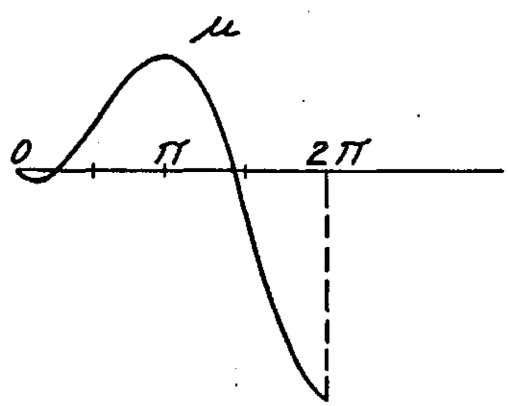

FIG. 6b

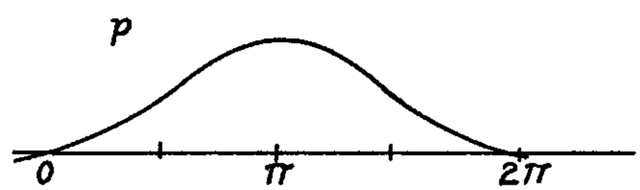

FIG 7a.

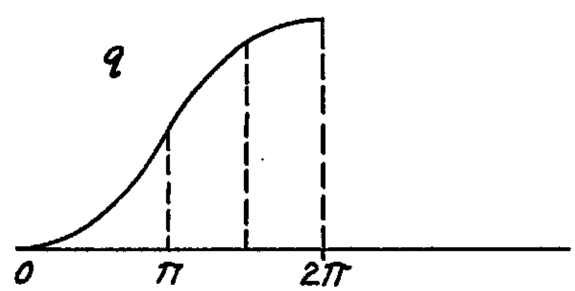

FIG. 7b.

Consequently:

$$p_1 = \mu_1 \approx \frac{H_1}{\omega}, \qquad p_2 = -\lambda_2 \approx \frac{2H_1}{\omega},$$
$$p_3 = -\mu_3 \approx \frac{H_1}{\omega}, \qquad p_4 = \lambda_4 = 2\pi\frac{H_2}{\omega}, \tag{41}$$

$$q_1 = \lambda_1 \approx \left(\frac{\pi}{2} - 1\right)\frac{H_1}{\omega}, \qquad q_2 = \mu_2 \approx \pi\frac{H_1}{\omega},$$
$$q_3 = -\lambda_3 \approx \left(3\frac{\pi}{2} + 1\right)\frac{H_1}{\omega}, \qquad q_4 = -\mu_4 = 2\pi\frac{H_1}{\omega}. \tag{42}$$

The curves of p and q can be constructed, remarking that the extremes of p are given by the equation:

$$q(t) = F_1(t), \tag{43}$$

and the extremes of q by the equation:

$$p(t) = -F_2(t), \tag{44}$$

so that the extremes of p correspond to the following values:

$T = 0$, $T \approx \pi$, and $T = 2\pi$.

The extremes of q correspond to the values:

$T = 0$, $T = 2\pi$.

Figures 6 and 7 represent the curves of λ, μ, p, and q in the interval from $T = 0$ to $T = 2\pi$.

Besides, we must note this: whatever the functions P(t) and $\delta(t)$ are, if they are finite and continuous in the neighborhood of the point $t = 0$,

$$\lim_{t=o} \frac{p}{\delta} = \lim_{t=o} \frac{q}{\delta} = 0.$$

In fact, let us return to equations (36) and (37), and suppose:

$$\lambda_o = \mu_o = p_o = q_o = 0.$$

The functions f_1 and f_2 contain δ as a factor. It will be sufficient to apply the theorem of the mean in the interval from $t = 0$ to $t = \varepsilon$, and to make ε approach zero as a limit.

Remark. I must make a remark. In this theory, there exists an appearance of a vicious circle, for I seek the value of Ω and, at the same time, I sometimes rest on the fact that Ω will be a large number. However, this is only an appearance of a vicious circle. In fact, Ω is not a parameter that is susceptible to vary from zero to infinity. With a given caliber and initial velocity, we are certain that Ω will vary between two rather close limits Ω' and Ω''. That is why we can regard Ω, *a priori*, as a large number with respect to other elements which come into play.

Finally, when Ω will have been determined according to the criterion, we shall see well whether the method followed is good, mediocre, or bad. In the last case, the method will have to be perfected.

For example, the major calibers might require particular studies; but I am not actually making a complete discussion of allthe cases.

8. STUDY OF THE YAW AT THE ORIGIN. I suppose that the departure is perfectly regular; I am going to make the approximate calculation of δ at the origin, i.e., when t varies from 0 to $2\pi/\omega$.

I shall make a first approximation by starting from formulas (9) and (15), and setting

$$\delta \approx |\tau'|_{0}\; t. \tag{45}$$

That will permit the knowledge of p, with a great approximation, according to the preceding study of p and q.

Knowing p, I shall make a second approximation of δ by setting

$$\delta' \approx |\tau'|_{0} + p. \tag{46}$$

That will give the form of the curve $\delta(t)$ at the origin.

Note that t varies here in an extremely small interval if ω is a large number. As a result of this, we know, we can regard $P(t)$ and $\tau'(t)$ as constants and equal respectively to P_0 and τ'_0 in the interval from $T = 0$ to $T = 2\pi$.

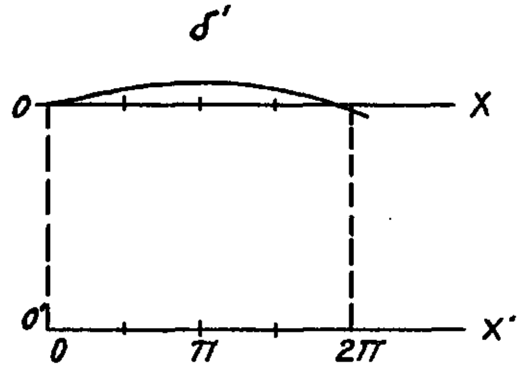

FIG. 8a

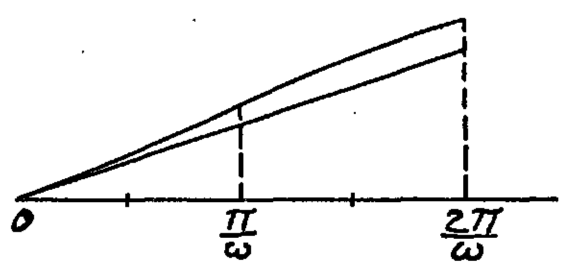

FIG. 8b

So, the first approximation will give:

$$P\delta \approx P_o | \tau_o' | \quad t. \tag{47}$$

If we refer to the previous notations, we must set, here:

$$H_1 = P_o | \tau_o' | \quad , \text{ or } \frac{H_1}{\omega} = \frac{P_o}{\omega} | \tau_o' |. \tag{48}$$

I suppose that the ratio P_o/ω is very small. (Besides, τ_o' is very small if V_o is very large). Consequently, the ordinates of the curve of p (Fig. 7) are always very small with respect to τ_o'. The curve which represents $\delta'(t)$ will be, with a very great approximation, the curve of p, but referred to an axis O'X', parallel to OX, the distance OO' being equal to $|\tau_o'|$(Fig. 8).

The function δ' will have a maximum for $T = \pi + h$, where $h \approx 0$, and a minimum for $T = 2\pi$.

So, the curve of δ will have a point of inflection for $T = \pi + h$, and another for $T = 2\pi$.

Here, then, is an essential fact: in the interval from $T = 0$ to $T = \pi + h$, $\delta'' > 0$, and, in the interval from $\pi + h$ to 2π, $\delta'' < 0$.

When t varies from 0 to about π/ω, the curve $\delta(t)$ turns its <u>concavity</u> upwards, and when t varies from about π/ω to $2\pi/\omega$, the curve $\delta(t)$ turns its <u>concavity</u> downwards; this is an <u>important fact</u>.

9. STUDY OF THE CRITERION AT THE ORIGIN. DEPARTURE PERFECTLY REGULAR. I suppose that the departure is perfectly regular. So, it will be necessary to substitute Ω for Ω_1 in the inequalities (29) and (30).

Besides, I actually study the criterion only at the origin, i.e., for $t \approx 0$.

By the relation (16),

$$\lim_{t=o} \frac{M}{\delta} = 0,$$

and M/δ is negligible for $t \approx 0$ in the inequality (30).

If there is, for $t \approx 0$, a double inequality of the type:

$$0 < \frac{\delta'' + \theta}{\delta} < A, \tag{49}$$

where A is positive, then, at the origin, we shall be able to adopt the criterion (31). Moreover, it must be remarked that this criterion is obtained immediately if it is assumed that, at the origin, the projectile is similar to the top -- a rather fragile intuition, when it concerns so delicate a discussion*.

However, I have just shown that δ'' changes sign in the immediate neighborhood of the origin. As a result of this, the possibility of this relation (49) is not probable, and I am going to study carefully the values of $\frac{\delta'' + \theta}{\delta}$ for $t \approx 0$.

With reference to formula (20):

$$\frac{p'}{\delta} \approx \frac{\delta'' + \theta}{\delta}, \text{ for } t \approx 0;$$

it will be sufficient to study the values of p'/δ for $t \approx 0$.

With a sufficient approximation,

$$p' = \omega(P_o\delta - q), \tag{50}$$

$$\frac{p'}{\delta} = \omega(P_o - \frac{q}{\delta}). \tag{51}$$

So, it is necessary to study the variations of q/δ for $t \approx 0$.

In the interval from $T = 0$ to $T = \pi + h$, $p' > 0$. However, in the interval from $\omega + h$ to 2π, $p' < 0$.

So, it is necessary to know the maximum of the modulus of $(P_o\delta - q)$ in this interval; it is this maximum which intervenes in the criterion.

The following values are very approximate:

$$\text{for } T = \pi, \quad \delta = \frac{\pi}{\omega}\,|\tau_o'|\;, \quad q = \frac{\pi}{\omega}\,|\tau_o'|\;P_o;$$

$$\text{for } T = 2\pi, \quad \delta = \frac{2\pi}{\omega}\,|\tau_o'|\;, \quad q = \frac{2\pi}{\omega}\,|\tau_o'|\;P_o.$$

* On the subject of the elementary theory of tho top, see e.g., Béghin, Statics and Dynamics, Vol. II, collection Armand Colin, p. 168.

So, at these two points, the value of q/δ is the same, with a great approximation.

The following values are also <u>very approximate</u>:

$$\text{for } T = \frac{3\pi}{2},\ \delta = \frac{3\pi}{2\omega}\,|\,\tau_o'\,|\ ,\ q = \frac{1}{\omega}\left(\frac{3\pi}{2} + 1\right)|\,\tau_o'\,|\ \ P_o.$$

So, at this point, the value of q/δ is greater than at the points π and 2π. If the number P_o/ω is very small (which can be verified when the value of Ω is fixed), all these approximations are <u>satisfactory</u>; but they are approximations; so it is useless to seek the exact maximum of q/δ. We have immediately a <u>superior limit of this maximum</u>, by taking the <u>largest</u> value of q in the interval, and the <u>smallest</u> value of δ. That gives:

$$\sup\lim q/\delta = 2P_o, \tag{52}$$

$$\sup\lim (q/\delta - P_o) = P_o \tag{53}$$

in the interval from $T = \pi + h$ to $T = 2\pi$, or, finally,

$$\sup\lim |\,p'/\delta\,| = \omega P_o, \tag{54}$$

$$\sup\lim \left|\frac{\delta'' + \theta}{\delta}\right| = \omega P_o \tag{55}$$

The inequality (30) can then be put in the following form for $t \approx 0$:

$$S \geqq 4\ B^2\ \ \omega P_o, \tag{56}$$

where

$$S \equiv (\,A\,\Omega)^2 - 4\ \ B\,(Rkl\ )_o\ ;$$

i.e.,

$$(A\,\Omega)^2 \geqq 8\ B\ (Rkl\ )_o, \tag{57}$$

for

$$P \equiv \frac{Rkl}{A\,\Omega}\ .$$

The coefficient 8 in (57), which I obtain thus, is rather high. M. de Sparre's coefficient in (32) is 16/3. However, I immediately find M. de Sparre's coefficient again intuitively. According to the preceding, the maximum of q/δ will occur approximately at $T = 3\pi/2$. At this point, with a great approximation,

$$q_3 = \left(\frac{3\pi}{2} + 1\right) \frac{P}{\omega} |\tau'_o| \ , \ \delta_3 = \frac{3\pi}{2\omega} | \tau'_o | ,$$

$$\frac{q_3}{\delta_3} = P_o \frac{3\pi/2 + 1}{3\pi/2}, \quad \frac{q_3}{\delta_3} - P_o = P_o \frac{2}{3\pi} .$$

If I substitute 1/3 for $2/3\pi$, I obtain M. de Sparre's coefficient. As this theory has no point of contact with M. de Sparre's, I consider the concordance of figures as being very satisfactory. However, this accord exists only in the exceptional case of perfectly regular departure.

10. INTEGRATION OF A DIFFERENTIAL SYSTEM NEAR THE PRECEDING SYSTEM. We shall now have to consider a non-vanishing initial yaw. As a result of this, it will be opportune to integrate the system (35) by taking f_1 and f_2 as constants. I obtain the expression of these constants by taking the values of P and δ at the instant $t = \pi/\omega$, for my study will be limited to the interval of time from $t = 0$ to $t = 2\pi/\omega$. By thus taking constant approximate values, I introduce an artificial periodicity; but that offers no inconvenience at all, since this study is actually limited to the interval of time indicated above.

Let

$$f_1 = \omega A_1 \quad \text{and} \quad f_2 = \omega A_2 .$$

Hence,

$$A_2/A_1 = \Lambda/P,$$

and we know that this ratio is very small. For example, at the origin, its value might be 1% or 2% or 3%.

First, suppose:

$$\lambda_o = \mu_o = p_o = q_o = 0.$$

Integration of equations (37) gives immediately:

$$\lambda = A_1 \sin \omega t + A_2(1 - \cos \omega t),$$
$$\mu = A_1(1 - \cos \omega t) - A_2 \sin \omega t. \tag{58}$$

It is convenient to introduce the variable $T = \omega t$.

It is seen immediately that λ has a maximun near A_1 for $T \approx \pi/2$, and a minimum near $-A_1$ for $T \approx 3\pi/2$, since A_2 is negligible with respect to A_1.

Similarly, μ has a maximum near $2A_1$ for $T \approx \pi$, and a very small minimum for $T \approx 0$. Also, it is evident that the curves of λ and p are very close to each other, and that the curves of μ and q are very close to each other. That originates, particularly, in the following fact:

$$p^2 + q^2 = \lambda^2 + \mu^2 = 4(A_1^2 + A_2^2) \sin^2 (\omega t/2), \tag{59}$$

so that the points (λ, μ) on the one hand, and (p,q) on the other, describe two circles of the same diameter, passing thru the origin (Fig. 10). If we consider the tangents at the origin, these two tangents are symmetric with respect to Ox, and very slightly inclined to Ox. So, the centers B_1 and B_2 of the two circles are symmetric with respect to Oy, and are very close to it; they would be confused if the ratio A_2/A_1 were zero. The symmetry of the tangents with respect to Ox results immediately from the following expressions:

$$p'_o = \omega A_1, \; q'_o = \omega A_2, \; \lambda'_o = \omega A_1, \; \mu'_o = - \omega A_2,$$

$$\mu'_o / \lambda'_o = - q'_o / p'_o .$$

It must be noted that the forms of the four curves (Fig. 9) are determined by the value of the ratio A_2/A_1, and this is what makes this approximation interesting.

Now, suppose:

$$p_o \neq 0, \; q \neq 0.$$

Since

$$p_o = \lambda_o \text{ and } q_o = - \mu_o,$$

we will have:

$$\lambda_o \neq 0 \text{ and } \mu_o \neq 0.$$

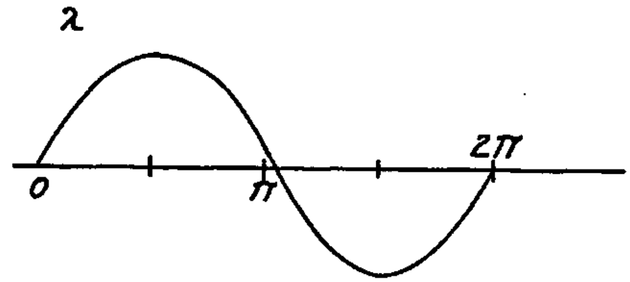

FIG. 9a

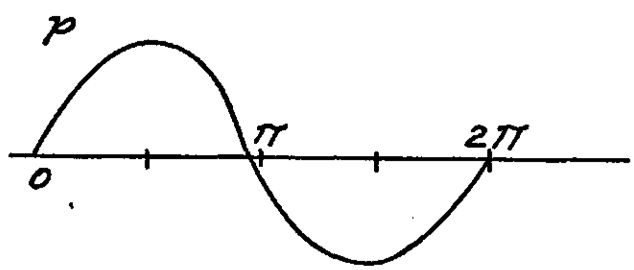

FIG. 9b

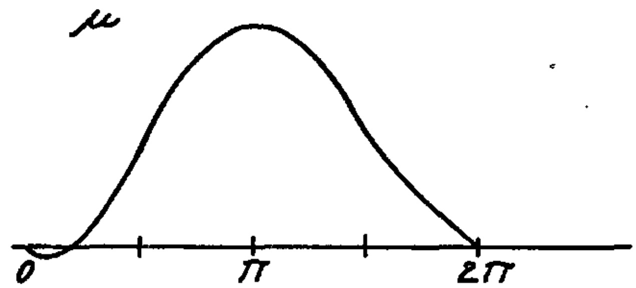

FIG. 9c

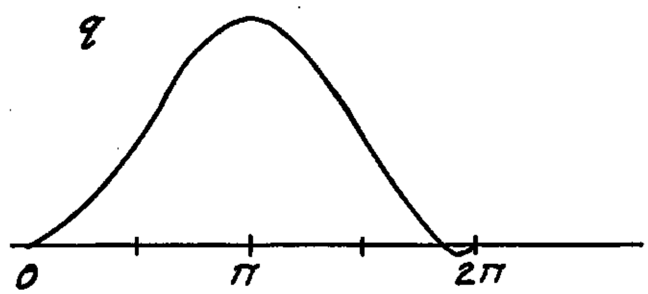

FIG. 9d

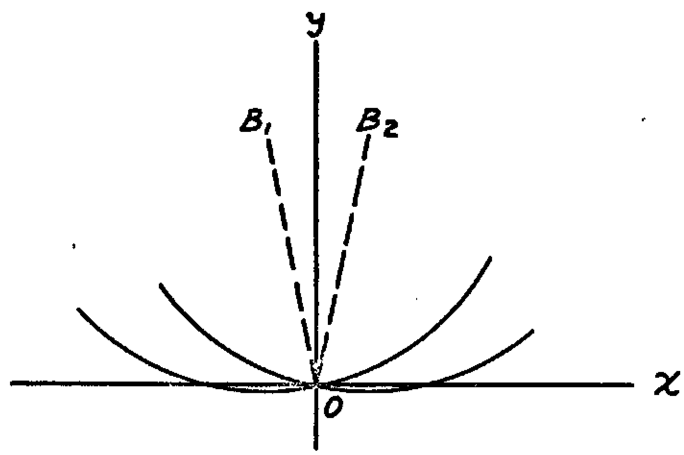

FIG. 10

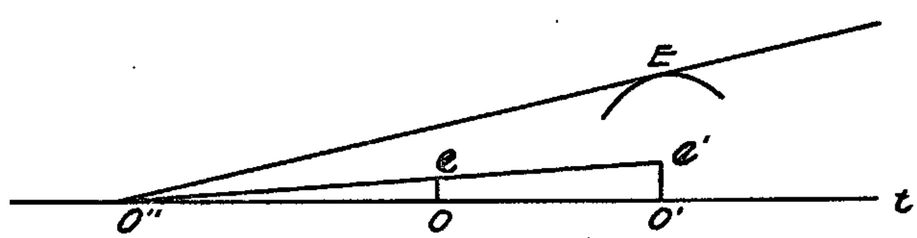

FIG. 11

Let us keep the notation λ, μ, p, q for the expressions which have been calculated and which vanish at the origin. We must now replace λ and μ by $\lambda + \lambda_o$ and $\mu + \mu_o$.

The new expression of p will be A:

$$A = p + \lambda_o \cos \omega t + \mu_o \sin \omega t.$$

The new expression of q will be B:

$$B = q + \lambda_o \sin \omega t - \mu_o \cos \omega t.$$

What has been done permits us to integrate the system (35) in somewhat more general conditions.

Let (p_1, q_1) be the solution, which vanishes at the origin, for the case:

$$f_1 = \omega A_1, \; f_2 = \omega A_2.$$

Let (p_2, q_2) be the solution, which vanishes at the origin, for the case:

$$f_1 = \omega H_1 t, \; f_2 = \omega H_2 t;$$

$(p_1 + p_2, q_1 + q_2)$ will be the solution, which vanishes at the origin, for the following case:

$$f_1 = \omega(A_1 + H_1 t), \; f_2 = \omega(A_2 + H_2 t).$$

Furthermore, to have a solution that passes thru the point (p_o, q_o) at the origin, it is sufficient to add the periodic functions:

$$W_1 = \lambda_o \cos \omega t + \mu_o \sin \omega t, \tag{60}$$

$$W_2 = \lambda_o \sin \omega t - \mu_o \cos \omega t. \tag{61}$$

To avoid all confusion, we could designate the solution of (35), which <u>vanishes at the origin</u>, by p and q, and the general solution, which assumes the values p_o and q_o at the origin, by A and B.

11. COMMENTARY on IRREGULAR DEPARTURES. It must be foreseen that the departure could not be perfectly regular. It certainly depends on us to utilize only well inspected shells and uneroded tubes, but other facts do not depend on us. Violent reactions, which make the trunnions of the gun vibrate, can be

produced at the contact of the earth and the carriage. Furthermore, at the muzzle, the expansion of the gases and their inflammation can generate certain vortices and cavitations which act on the projectile as deviating forces. From the analytic point of view, we say that an irregular departure corresponds to a non-vanishing initial value of the yaw, and an initial value of $\frac{d\delta}{dt}$ different from the regular initial value $|\tau'_0|$.

However, a distinction must be made.

If very violent perturbations existed at the departure, we would then have to study the primitive equations (21), and not the equations such as (27), deduced from the primitive equations by simplifications which are permitted by reason of a certain regularity of the physical phenomenon, of which we have a well founded intuition. The study of very violent perturbations is not a question which properly concerns ballistics: I do not speak of it.

I am going to consider exclusively the following conditions, which seem to me normal if the initial perturbation is small.

I assume the existence of an extremely small initial value δ_0, say of the order of a thousandth of a degree, this figure being only an indication.

Furthermore, I assume the existence of a modification of the regular value of δ'_0, which is $|\tau'_0|$, a small modification compared to the regular value $|\tau'_0|$.

I shall remain almost in the preceding frame of study, and shall say that the departure is <u>almost perfect</u>.

To simplify the facts, I suppose $\psi_0 = 0$, i.e., $\mathring{\delta}_1 = 0$. Whence, $\delta_0 = \delta_2^0$, while $\delta_0 = 0$ in the case of perfectly regular departure.

As a result of this, $\eta'_0 = 0$.

Consequently, as in the case of perfectly regular departure:

$$\mathring{\omega}_2 = \mathring{\omega}_3 = \mathring{\omega}_4 = \mathring{\omega}_6 = \mathring{\omega}_7 = 0,$$

$$M^0 = N^0 = 0.$$

Let us consider the expressions for p, q, ρ. We have:

$$p_0 = \delta'_0 + \theta'_0 = \delta'_0 + \tau'_0 . \tag{62}$$

I assume again that $\theta'_0 = \tau'_0$, because δ_0 is extremely small.

Then:

$$q_\circ = \psi_\circ' \sin \delta_\circ \approx \psi_\circ' \delta_\circ, \tag{63}$$

$$r_\circ = \psi_\circ' \cos \delta_\circ \approx \psi_\circ', \tag{64}$$

$$\rho_\circ = \Omega + r_\circ. \tag{65}$$

Let us return to the fundamental system (21), supposing $N_z = 0$, for greater simplicity.

The third equation will give:

$$\rho = \rho_\circ = \Omega + r_\circ, \tag{66}$$

$$\rho \approx \Omega + \psi_\circ'. \tag{67}$$

I shall suppose essentially that the ratio $|r_\circ|/\omega$ is small; consequently, the ratio $|r_\circ|/\Omega$ will be a very small number. We shall then have a sufficient approximation if we substitute Ω for ρ in equations (21), and $A\rho$ for $A\rho - Br$, for $t \approx 0$. So, we again find equations (34), but with not quite so good an approximation as in the case of perfectly regular departure; for, in the latter case, $r_\circ = 0$.

I have thus substituted the following for the system (21):

$$\begin{aligned} \frac{dp}{dt} + \omega q &= N_x, \\ \frac{dq}{dt} = \omega p &= N_y, \\ \rho &= \rho_\circ. \end{aligned} \tag{68}$$

The system (21) is not an ordinary system defining p, q, ρ, because δ, which figures in the second members, is also an unknown function of t. However, I am going to place myself in particular conditions. I consider only the very small interval of time from $t = 0$ to $t = 2\pi/\omega$, corresponding to a very small zone $Z_\circ$ on the trajectory. In this zone $Z_\circ$, I shall be able to obtain a very satisfactory approximation of δ, as I have already done, and that will permit me to regard the system (21) as a differential system defining p, q, and ρ. Of course, I suppose the parameters k and l to be well known at the origin.

Furthermore, considering (21) or (68) from this point of view, I shall be able to choose $p_\circ$, $q_\circ$, $\rho_\circ$ arbitrarily, taking account of the condition which I have imposed on myself concerning the limitation of $|r_\circ|/\omega$.

To give p_o, q_o, ρ_o, or p_o, q_o, r_o, is to give the <u>initial perturbation</u> of the free trihedron.

I essentially suppose that this perturbation is a minimum. For example, the ratio $|p_o/\tau'_o|$ will be less than 1/20 or 1/30.

The value of δ_o, e.g., could be less than 1/10,000 or 1/100,000, which corresponds to about $\frac{57.3°}{10,000}$ or $\frac{57.3°}{100,000}$.

I shall set $\psi'_o = \sigma P_o$, where $|\sigma|$ can be comprised between zero and a number quite close to 1, for example.

If

$$|\tau'_o| = 1/100,$$

e.g., we will have

$$|p_o| < 1/2000.$$

Then take:

$$\delta_o = 1/60,000, \text{ about } 1°/1000; \text{ and } \psi'_o = 5;$$

this will give:

$$q_o = 1/12,000, \text{ and } r_o \approx 5.$$

In general, the figures which I give are only slightly probable numbers. The last ones, in particular, constitute only a vague indication. In each case, it will be necessary to examine what can be a normal initial perturbation. Furthermore, from the point of view of approximations, the caliber plays an important role, according to the fundamental formula:

$$\Omega = 2V_o j/a.$$

The design of the shell also plays its role, since

$$\omega = A\,\Omega/B.$$

We have previously seen the role which the ratio P_o/ω plays, and I supposed that this ratio is small. It is

$$\frac{P_o}{\omega} = \frac{B\,P_o}{A\,\Omega},$$

and, if a designates the diameter,

$$P_o/\Omega \quad = nac;$$

so that P_o/ω is generally small, because the ballistic coefficient c is a very small number. However, it will be necessary to examine all the possible cases, in order to evaluate the coefficient n.

Since the departure is almost perfect, my method of approximation will be modified a little. Formula (62) gives

$$\delta'_o = |\tau'_o| + p_o.$$

The first approximation of δ' will be $\delta' = \delta'_o$, which gives

$$\delta = \delta_o + (|\tau'_o| + p_o)t. \tag{69}$$

The function p, or A(t), will be calculated, taking the value p_o at the origin; and the second approximation of δ' will be

$$\delta' = |\tau'_o| + A(t). \tag{70}$$

p, or A(t), and q, or B(t), should be calculated according to the system (35), in which the second members comprise a constant and a term in t, according to the approximate expression for δ, given by formula (69).

We shall be able to construct the curve of q, or B(t), relative to the zone Z_o. In this zone, the first approximation of the yaw is represented by the straight line O"ee', whose slope is δ'_o; and OE = δ_o.

Draw the tangent O"E to the curve q; the ratio of the slopes of the straight lines O"E and O"e gives the maximum of the ratio q/δ (Fig. 11).

The constants p_o and q_o can again effect a raising of the curve of q and, consequently, increase the maximum of q/δ.

It will not be useless actually to consider the nature of the approximations adopted in this study. Equation (9) is an exact relation. The last term in η' is negligible in the zone Z_o, for it contains two factors which vanish at the origin. We can substitute $\cos\psi_o$ for $\cos\psi$, if ψ'_o is quite small. Besides, here, $\cos\psi_o = 1$.

So, I shall replace $\cos\psi$ by 1, assuming that ψ'_o is quite small. I assume intuitively that, in the zone Z_o, $\theta = \tau$ and $\theta' = \tau'$; in particular, that $\theta'_o = \tau'_o$.

If the variation of τ' is very feeble in this zone -- which occurs, in general -- we can substitute τ'_o for θ', which gives, with a great approximation, in the zone Z_o,

$$p = \delta' + \tau'_o.$$

Furthermore, by substituting the approximate expression $A(t)$ for p, we obtain formula (70). This formula puts in relief the points of <u>inflection</u> of the curve $\delta(t)$ -- an important fact.

Then, I again take the approximate expression (69) for δ. This approximation will be good if $|p/\tau_0'|$ remains small in the zone Z_0. In particular, if $p_0 \neq 0$, the ratio $|p_0/\tau_0'|$ will have to be small for the approximation to be valid.

In general, the smaller $|p/\tau_0'|$ is, the better the approximation will be.

In these questions, we must always try to see the conditions which concern a very good approximation. If great difficulties sometimes keep us at a little distance from the conditions required for a good approximation, we shall then know that the method is less sure, and shall have to introduce some prudence in the terms of the conclusion.

For example, when I say that Ω or ω is a large number with respect to certain parameters, this fact must be discussed and verified. A complete discussion of the approximation would involve a classification of projectiles according to their diameter, their ballistic coefficient, etc., and it would be seen that such an approximation, which is excellent for certain calibers, is weak for others. I content myself here with quite general indications.

In my study of the system (35), I always supposed P constant, and substituted P_0 for P. That requires some enlightenment.

According to the preceding,

$$(P'/P_0) = -n/10, \tag{71}$$

where n is quite close to 1, but I am not more precise.

Now, suppose V_0 comprised between 500 and 1500 m/s. To give an indication that is probable but not precise, I shall say that Ω could be comprised between 1000 and 3000. Moreover, taking probable values for the ratio A/B, we could say that ω is comprised among such numbers as $70\varkappa$, $100\varkappa$, and $150\varkappa$.

In these conditions, the zone Z_0 would correspond to a duration of 2/70, 2/100, and 2/150 sec.

Now, let us study the ratio δ'/δ in the zone Z_0.

Let us first suppose $\delta_0 = 0$. We have:

$$\begin{gathered} \delta' \approx \delta_0', \ \delta \approx \delta_0' t, \\ \delta'/\delta \approx 1/t. \end{gathered} \tag{72}$$

In the zone Z_o, this derivative is extremely large.

Now, let us suppose $\delta_o > 0$, and extremely small. This concerns almost perfect departure, and I suppose essentially that the ratio $\delta_o'/\delta_o = |\tau_o'|/\delta_o$ is a <u>large number</u>.

In the zone Z_o,

$$\delta' \approx \delta_o', \ \delta \approx \delta_o + \delta_o' t.$$

Set

$$\delta_o' t = L\delta_o.$$

For $t = 0$,

$$L = 0.$$

At the end of the zone Z_o,

$$t = 2/70 \text{ or } 2/100 \text{ or } 2/150,$$

approximately, and, e.g.,

$$L = 2, 3, 4, \ldots..$$

Write

$$\frac{\delta'}{\delta} \approx \frac{\delta_o'}{\delta_o(1+L)}. \tag{73}$$

Evidently, this ratio is quite a large number, a little less than δ_o'/δ_o, in the zone Z_o, if ω is a large number.

In all cases, the ratio δ'/δ is much larger than $|P'/P|$ in the zone Z_o, if ω is a large number.

In the zone Z_o, the product $P\delta$ is increasing, altho P may be decreasing; the variations of P are insignificant with respect to those of δ, in general.

To avoid all equivocation, I must repeat that I do not examine <u>all possible cases</u>, when I refer to the numerical value of Ω , ω, or L. That would require a much more detailed discussion.

12. SUMMARY STUDY OF AN ALMOST PERFECT DEPARTURE. I have defined the almost perfect departure. In particular, $(M/\delta)_o = 0$, so that, in the zone Z_o, we shall put the criterion (30) in the

form:

$$S + 4B^2 \frac{\delta'' + \theta}{\delta} > 0, \tag{74}$$

or else:

$$S + 4B^2 \frac{p'}{\delta} > 0. \tag{75}$$

When p' is negative, we must seek the maximum of $|p'/\delta|$ and the criterion takes the form:

$$S > 4B^2 \left|\frac{p'}{\delta}\right| \max. \tag{76}$$

I recall that

$$S = (A\Omega_1)^2 - 4B\,Rk\ell.$$

Furthermore,

$$\left|\frac{p'}{\delta}\right| \max \approx \omega\left[\left(\frac{q}{\delta}\right) \max - P_o\right], \tag{77}$$

so that we ought to seek the maximum of q/δ.

Set

$$\left(\frac{q}{\delta}\right)_{max} = (s + 1)\ P_o, \tag{78}$$

so that

$$\left|\frac{p'}{\delta}\right|_{max} \approx \omega s P_o, \tag{79}$$

and the criterion will take the form:

$$S > 4\ Bs(Rk\ell)_o, \tag{80}$$

for

$$\omega P = \frac{Rk\ell}{B}.$$

A general study of the curve of q, or B(t), would have to be made in the case of almost perfect departure. To abridge the discussion, which would require quite long developments, I shall only make a sketch, by examining two particular cases.

First case. I suppose:

$$\delta_\circ = \psi_\circ = \psi_\circ' = 0, \; p_\circ = \lambda_\circ < 0.$$

Then:

$$q_\circ = r_\circ = 0, \; \mu_\circ = 0.$$

Moreover, the ratio $|p_\circ/\tau_\circ'|$ is very small, which will permit me to keep the method of approximation previously adopted. Here, as in the case of perfect departure,

$$\lim_{t=0} \frac{M}{\delta} = 0,$$

but with $p_\circ \neq 0$; so, this is an almost perfect departure.

Calculate the solutions p and q of the system (35), which vanish at the origin; they are represented by the curves already constructed (Fig. 7). To pass from p to A(t) and from q to B(t), it is sufficient to add respectively the periodic functions W_1 and W_2. These functions, defined by formulas (60) and (61), have here the simpler form:

$$U_1 = \lambda_\circ \cos \omega t, \tag{81}$$

$$U_2 = \lambda_\circ \sin \omega t. \tag{82}$$

Consequently, the curve of q will be raised in the interval from $T = \pi$ to $T = 2\pi$, and the curve of p will be raised in the interval from $T = \pi/2$ to $T = 3\pi/2$.

The solid line (Fig. 12) represents the curve of q, and the dotted line represents the curve B(t). We ought to draw the tangent OE to the latter curve; it is evident that the tangent is raised. At the same time, the straight line Oe', which represents the first approximation of δ, will be lowered a little. So, the maximum of q/δ is augmented by the presence of the non-vanishing parameter $p_\circ$.

Therefore, if we take M. de Sparre's coefficient $s = 1/3$ in the case of perfect departure, it will be possible to have $s > 1/3$ in the case of almost perfect departure.

Second case. Take:

$$V_\circ = 1200, \; g \cos \alpha = 1, \alpha \approx 84^\circ$$

Then

$$|\tau_\circ'| = 1/1200.$$

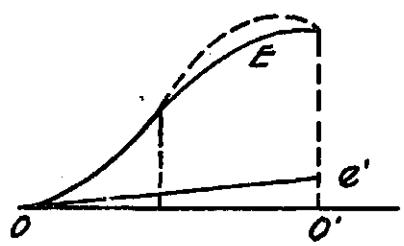

FIG. 12

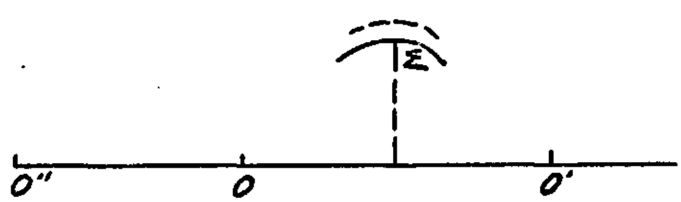

FIG. 13

We shall take

$$\delta_\circ = 1/120,000.$$

which gives

$$\delta_\circ / |\tau_\circ^!| = 1/100.$$

The first approximation of δ will be

$$\delta = \frac{1}{1200}\left(\frac{1}{100} + t\right).$$

Altho ω is unknown a priori, we know that its numerical value will be quite large if the diameter of the projectile is not very large. So, suppose the caliber medium, e.g.: 80 mm, 100 mm, 140 mm, ... Represent the value of δ for $T = \varkappa$ by δ_m, and the value of δ for $T = 2\varkappa$, which corresponds to the end of the zone $Z_\circ$, by δ_n. For example, if $\varkappa/\omega$ were near 1/200, we would have:

$$\delta_m \approx \frac{3\delta_\circ}{2}, \ \delta_n \approx 2\delta_\circ.$$

If $\varkappa/\omega$ were near 1/100, we would have:

$$\delta_m \approx 2\delta_\circ, \ \delta_n \approx 3\delta_\circ.$$

Since the variation of δ is not large in the zone $Z_\circ$, I shall be able to make an approximation which is somewhat rougher, but rapid and convenient. I shall take a mean value of $P\delta$ in $Z_\circ$, the value at the instant $T = \varkappa$. The fact that ω is not known is unimportant, as we shall see. So, the mean value of $P\delta$ will be $P_\circ\delta_m$, very approximately.

So, I integrate the system (35), by setting:

$$f_1 = \omega P_\circ \delta_m, \ f_2 = \omega \frac{\Lambda}{P} P_\circ \delta_m.$$

We know the solutions p and q, which vanish at the origin, and we know that:

$$|p|_{max} \approx P_\circ\delta_m, \ q_{max} \approx 2P_\circ\delta_m . \qquad (83)$$

To obtain the maximum of q/δ, I ought to draw a tangent to the curve thru the point O". I can assume, as a new approximation, that the point of contact E is confused with the summit Σ of the curve. By the hypothesis which was made about the numerical value of Λ/P, the maximum of q occurs for $T \approx \varkappa$.

So:

$$\left(\frac{q}{\delta}\right)_{max} \approx 2P_o. \qquad (84)$$

This approximate result is independent of the numerical value of δ_m, and that is what makes the approximation easy. In the actual case, with this approximation, $s \approx 1$.

Now, suppose the departure almost perfect, i.e., p_o, q_o, r_o non-vanishing and small enough for the adopted approximation to be valid.

From the point of view of the approximation, we must examine the numerical value of P_o. The approximation will be good if P_o is quite small, i.e., if the ballistic coefficient is very small. That results from the examination of the formulas (83).

Suppose the initial values p_o and q_o negative. The curve of q being represented by the solid line (Fig. 13), the curve B(t) will be represented by the dotted line; the curve will be raised in the region of the summit Σ, and the maximum of B will be able to have the same abscissa as the maximum of q. Then:

$$\left(\frac{q}{\delta}\right)_{max} \approx (2 + h)P_o, \quad h > 0,$$

i.e., $s > 1$.

Since this method of approximation is not very precise, the value of s is not very well determined; but I have shown intuitively that a minimum initial perturbation can make the numerical value of s greater.

Since the criterion relative to Ω is represented by formula (80) at the origin, we see quite simply that minimum irregularities at the departure might increase the coefficient s.

Moreover, if the artillery adopts a coefficient s greater than M. de Sparre's, that signifies that it foresees quite strong initial perturbations*.

I have examined one of the numerous aspects of the problem, and I shall complete these researches later.

Translated by

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*G. Sugot, Theoretic Exterior Ballistics, Mém. de l'art. fr., Vol. VI, 1927.