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THERMAL SHOCK FOLLOWING RAPID UNIFORM HEATING OF SPHERES AND LONG CYLINDRICAL RODS

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Aerospace Corporation
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Prepared for
SPACE AND MISSILE SYSTEMS ORGANIZATION
AIR FORCE SYSTEMS COMMAND
Air Force Unit Post Office
Los Angeles, California 90045

THE AEROSPACE CORPORATION
Technology Division
San Bernardino Operations

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FOREWORD

This report for The Aerospace Corporation, San Bernardino Operations, has been prepared under Contract No. F04701-69-C-0066 as TR-0066(S5816-71)-1. The Air Force program monitor is Lt. Col. James C. Scheuer, USAF (SMTAN).

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This technical report has been reviewed and is approved.

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UNCLASSIFIED ABSTRACT

THERMAL SHOCK FOLLOWING RAPID UNIFORM HEATING OF SPHERES AND LONG CYLINDRICAL RODS, by P. Mann-Nachbar and W. Nachbar

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Stress waves, that develop following rapid uniform internal heating of linear-elastic spheres and long cylindrical rods, display a focusing effect as they proceed radially towards the center in these geometries. This effect can cause peak interior dynamic stresses in both tension and compression that are much larger in magnitude than the magnitude of the uniform compressive stress which is initially induced by constrained thermal expansion. Two fundamental problems are treated by the Heaviside wave method, one for the heating of a long cylindrical core in an infinite medium, and the other for heating of a complete sphere with zero surface traction. A simple closed-form formula for the stress at the center of the cylinder in the first problem allows determination of the effect of a finite heating time (a ramp function) on reducing the amplitude of the peak tensile stress at the center. The peak tensile stress at the center becomes infinite for the cylinder problem in the limit as heating time goes to zero. For the sphere problem, the stress at the center is calculated for a duration of several wave-reflection times, and the center stress is also found to be infinite for the limit of zero heating time. The infinity of center stress in the sphere is a tensile Dirac delta function, while it is of finite slope for the cylinder. For both the cylinder and the sphere, the tensile stress impulse associated with the center stress infinity is finite for compressible materials. In the limit of the incompressible material (Poisson's ratio equal to 0.5) the tensile stress impulse tends to infinity for both cylinder and sphere. (Unclassified Report)

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CONTENTS

INTRODUCTION 1

PART A - THE CYLINDRICAL GEOMETRY 4

1. Conventional Thermoelastic Equations 5
2. Solution by Laplace Transform 8
3. The Heated Cylindrical Core Problem 10
4. Inversion of the Transform 17
5. Stress-Free Boundary 19

PART B - THE SPHERICAL GEOMETRY: STRESS-FREE BOUNDARY 23

1. Conventional Thermoelastic Equations 23
2. Solution by Laplace Transform 25
3. Inversion of the Transform 28
4. Nature of the Singularity at the Origin 33

CONCLUSION 47

REFERENCES 49

FIGURES

1. Cylindrical Geometry 2

2. Nondimensional Center Stress s_{r_1} ① $(0+, \tau)$ versus Nondimensional Time τ for $\nu = \frac{1}{3}$ 15

3. Nondimensional Radial Stress at the Origin at $\tau = 1^+$ 32

4. Nondimensional Stress versus Nondimensional Time at a Point Close to the Origin 36

5. Nondimensional Radial Stress at the Origin versus Nondimensional Time τ for $\nu = \frac{1}{3}$ 38

6. Nondimensional Displacement versus Nondimensional Time at a Point Close to the Origin 43

7. Nondimensional Displacement Profile at $\tau = 1-\epsilon$ 45

TABLES

1 Values of Nondimensional Stress at the Center of the Cylinder 13

INTRODUCTION

In this paper, we are concerned in the first part with axially-symmetric elastic stress waves that develop when a long cylindrical rod of circular cross section with radius "a" is suddenly heated internally in one of two configurations (Figure 1). In the first configuration, the lateral surface of the rod, $r = a$, is attached to an infinite, unheated medium having the same elastic properties as the rod. In the second configuration, the lateral surface of the rod is free of surface traction. Because of the length of the rod, the effects of the ends of the rod are neglected, and it is assumed that the center portion of the rod is in a condition of plane strain (i.e., zero axial strain).

If the effective depth of the internal heating in from the lateral surface is small compared to "a," then it is expected that the initial behavior of the stress waves in an elastic rod is similar to the behavior that has been found for plane waves in a thermally-shocked, semi-infinite elastic medium (Ref. 8). If, however, the effective depth of heating is "a," as, for example, if the rod is "instantaneously" brought to a uniform temperature rise T_0 above ambient, then the curvature of the surface is likely to affect the problem significantly. We expect that if the rod is heated uniformly throughout to temperature T_0, then the initial compressive state of stress caused by thermal strains of magnitude αT_0 will be altered, first by an unloading wave proceeding inwards from the surface, and then by the effects of convergence of the cylindrical waves toward the center.

The study of the cylindrical geometry in Part A is followed by a study in Part B of the spherical geometry for a similar problem. Spherically-symmetric stress waves are studied for an elastic sphere in which the surface is free of traction. The sphere is suddenly heated to a uniform temperature throughout. This problem for the sphere is here solved in terms of elementary functions by application of the Heaviside wave method (Ref. 5, p. 101). However, the solution is not nearly so accessible for the cylindrical geometry with traction-free surface. (Section 5 of Part A below), the second of the two configurations mentioned above in the first paragraph. The cylindrical problem leads to an expression for the Laplace Transform

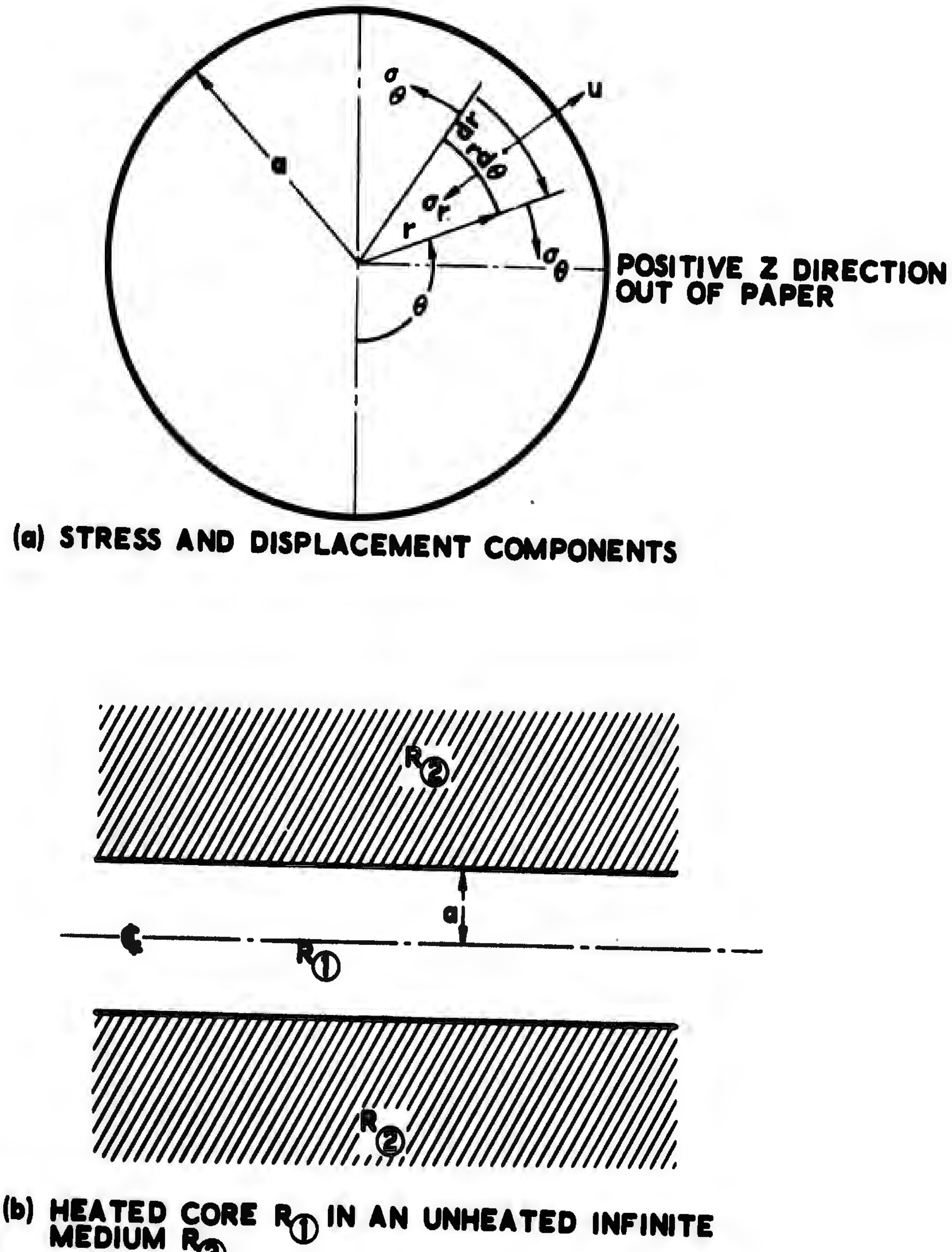

Figure 1. Cylindrical Geometry

(Part A, Eq [34]) that cannot be inverted by use of standard tables of inverses. Consequently, in the cylindrical case, we restrict ourselves here only to a "small time" solution.

On the other hand, the first configuration for the cylinder is solved here (Section 4 of Part A) exactly and in finite form. A complete numerical solution of this problem requires computer tabulation of the functions defined by definite integrals (Eqs [24] and [32] of Part A); however, simple expressions valid for all time are found for the radial stress component at the center, and these are listed below in Table 1. The solution obtained here corrects a previous but erroneous solution for this problem that was given in Ref. 3.

The cylindrical solutions obtained in this paper are all valid for a finite time of heating t_o, assuming that the temperature rise during heating is linear in time. Further, solutions are obtained for the limiting case $t_o \longrightarrow 0$, which is called instantaneous heating. In the cylindrical geometry under instantaneous heating, the stress at the center becomes <u>singular</u> and jumps to an infinite tensile value at a characteristic time given by a/c_1, where c_1 is the dilatational speed. This singularity disappears if t_o is positive, however, and the stress at the center will not become tensile at all if t_o exceeds a value of the order of a/c_1. The spherical geometry solution is similar to the cylindrical in that there is a singular behavior at the origin at time a/c_1 in the limit $t_o \longrightarrow 0$, but the nature of the singularity is different for the two geometries (compare Figure 5 with Figure 2).

The investigation of Nelson (Ref 9) was seen after completion of the present work. The problem considered in Ref 9 is the transient motion following sudden removal of uniform radial pressure on the surface of a long cylinder at ambient temperature. This problem is mathematically equivalent to the problem of uniform instantaneous heating of the cylinder with a stress-free surface. In the present report, only a "short-time" solution is presented for this problem (see Part A, Sect 5, below). The solution for all time that is derived in Ref 9 is obtained by inversion of a finite Hankel transform and corresponds to zero heating time in our problem. The numerical results shown for the stress at various radii as functions of time exhibit propagating discontinuities of infinite amplitude in stress.

PART A - THE CYLINDRICAL GEOMETRY

Nomenclature

σ_r, σ_z, σ_θ	radial, axial and circumferential stress components
u	radial displacement component
r, θ	polar coordinates
$T(r, \theta)$	temperature rise above ambient
α	linear coefficient of thermal expansion
E, ν	Young's modulus and Poisson's ratio
ρ	density (mass per unit volume)
t	time

1. CONVENTIONAL THERMOELASTIC EQUATIONS (Ref 1, Pg 170, Eq VIII-24)

$$\frac{\partial u}{\partial r} = \frac{1}{E}\left[\sigma_r - \nu\,(\sigma_\theta + \sigma_z)\right] + \alpha T \tag{1a}$$

$$\frac{u}{r} = \frac{1}{E}\left[\sigma_\theta - \nu\,(\sigma_r + \sigma_z)\right] + \alpha T \tag{1b}$$

$$0 = \frac{1}{E}\left[\sigma_z - \nu\,(\sigma_r + \sigma_\theta)\right] + \alpha T \tag{1c}$$

From Eq (1c)

$$\sigma_z = \nu\,(\sigma_r + \sigma_\theta) - E\alpha T$$

$$\frac{\nu\sigma_z}{E} = \frac{\nu^2}{E}\,(\sigma_r + \sigma_\theta) - \nu\alpha T$$

Therefore:

$$\frac{\partial u}{\partial r} = \frac{1}{E}\left[(1 - \nu^2)\,\sigma_r - \nu\,(1 + \nu)\,\sigma_\epsilon\right] + (1 + \nu)\,\alpha T \tag{2a}$$

$$\frac{u}{r} = \frac{1}{E}\left[(1 - \nu^2)\,\sigma_\theta - \nu\,(1 + \nu)\,\sigma_r\right] + (1 + \nu)\,\alpha T \tag{2b}$$

Equilibrium

Refer to Figure (1a) for sign conventions. We have

$$\frac{\partial}{\partial r}\,(\sigma_r r d\theta)\,dr - \sigma_\theta dr d\theta = \rho\frac{\partial^2 u}{\partial t^2}\,.\,r d\theta dr$$

for a unit length in the z-direction. Hence

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = \rho\,\frac{\partial^2 u}{\partial t^2} \tag{2c}$$

Nondimensionalization

Let L be a characteristic length, c a characteristic speed, Σ a characteristic stress and T_o a characteristic temperature. We define dimensionless variables as follows:

$$\xi = r/a$$

$$w = u/L$$

$$\tau = tc/L$$

$$s = \sigma/\Sigma$$

$$\Theta = \frac{T}{T_o}$$

Equations (2a, b, c) become, upon substitution from the above,

$$\frac{L}{a}\frac{\partial w}{\partial \xi} = \frac{\Sigma}{E}\left[(1-\nu^2)\, s_r - \nu(1+\nu)\, s_\theta\right] + (1+\nu)\,\alpha T_o \Theta \tag{3a}$$

$$\frac{L}{a}\frac{w}{\xi} = \frac{\Sigma}{E}\left[(1-\nu^2)\, s_\theta - \nu(1+\nu)\, s_r\right] + (1+\nu)\,\alpha T_o \Theta \tag{3b}$$

$$\frac{\Sigma}{a}\left[\frac{\partial s_r}{\partial \xi} + \frac{s_r - s_\theta}{\xi}\right] = \frac{\rho L}{(L^2/c^2)}\frac{\partial^2 w}{\partial \tau^2} \tag{3c}$$

It is convenient to define the parameter κ as

$$\kappa \equiv \frac{\nu}{1-\nu} \tag{4a}$$

and to set the nondimensional parameter groups in Eq (3a, b, c) equal to numbers as follows

$$\frac{\Sigma a}{LE}(1-\nu^2) = 1 \tag{4b}$$

$$\frac{(1+\nu)\,\alpha T_o a}{L} = 1 \tag{4c}$$

$$\frac{\Sigma L}{a\rho c^2} = 1 - \kappa^2 \tag{4d}$$

These equations give three relations between the four parameters L, c, Σ, T_o:

$$\Sigma \equiv \frac{E}{1-\nu^2}\left(\frac{L}{a}\right) \tag{5a}$$

$$(1+\nu)\,\alpha T_o \equiv \frac{L}{a} \tag{5b}$$

$$c^2 \equiv \frac{L}{a}\;\frac{\Sigma}{(1-\kappa^2)\rho} \equiv \frac{1-\nu}{(1-2\nu)(1+\nu)}\;\frac{E}{\rho}\left(\frac{L}{a}\right)^2 \equiv c_1^2\left(\frac{L}{a}\right)^2 \tag{5c}$$

where c_1, given by,

$$c_1 \equiv \left[\frac{1-\nu}{(1-2\nu)(1+\nu)}\;\frac{E}{\rho}\right]^{1/2} \tag{5d}$$

is the dilatational wave speed.

In consequence of Eq (5a, b, c), the dimensional quantities will be given in terms of their nondimensional counterparts by the following relations:

$$\sigma = \frac{E}{1-\nu}\,\alpha T_o \cdot s \quad \text{for } \sigma_r,\ \sigma_s,\ \sigma_z \tag{5e}$$

$$u = (1+\nu)\,\alpha T_o \cdot aw \tag{5f}$$

$$t = \frac{a}{c_1}\,\tau \tag{5g}$$

Substitution from Eq (4a, b, c, d) into Eq (3a, b, c) gives

$$\frac{\partial w}{\partial \xi} = s_r - \kappa s_\theta + \Theta \tag{6a}$$

$$\frac{w}{\xi} = s_\theta - \kappa s_r + \Theta \tag{6b}$$

$$\frac{\partial s_r}{\partial \xi} + \frac{s_r - s_\theta}{\xi} = (1 - \kappa^2)^{-1} \frac{\partial^2 w}{\partial \tau^2} \tag{6c}$$

Equation (6b) can be eliminated by writing it as

$$s_\theta = \frac{w}{\xi} + \kappa s_r - \Theta$$

Upon substitution for s_θ in Eqs (6a) and (6c),

$$\frac{\partial w}{\partial \xi} = s_r - \kappa\left(\frac{w}{\xi} + \kappa s_r - \Theta\right) + \Theta$$

$$\frac{\partial s_r}{\partial \xi} + \frac{s_r}{\xi} - \frac{1}{\xi}\left(\frac{w}{\xi} + \kappa s_r - \Theta\right) = (1 - \kappa^2)^{-1} \frac{\partial^2 w}{\partial \tau^2}$$

After collection of terms, these two equations take the form

$$\frac{\partial w}{\partial \xi} + \kappa \frac{w}{\xi} - (1 - \kappa^2)\, s_r = (1 + \kappa)\Theta \tag{7a}$$

$$\frac{\partial s_r}{\partial \xi} + (1 - \kappa) \frac{s_r}{\xi} - \frac{w}{\xi^2} - (1 - \kappa^2)^{-1} \frac{\partial^2 w}{\partial \tau^2} = -\frac{1}{\xi}\Theta \tag{7b}$$

2. SOLUTION BY THE LAPLACE TRANSFORM

Let the Laplace transform of a function $f(\xi, \tau)$ be indicated by $\overline{f}(\xi, p)$, viz.

$$\mathcal{L}(f) = \overline{f}(\xi, p) = \int_0^\infty f(\xi, \tau)\, e^{-p\tau}\, d\tau.$$

If the initial conditions

$$w(\xi, 0) = \frac{\partial w}{\partial \tau}(\xi, 0) = 0$$

are assumed, then application of the Laplace transform to Eq (7a, b) gives $\left(\overline{f}\,' = \frac{\partial \overline{f}}{\partial \xi}\right)$

$$\overline{w}\,' + \kappa\frac{\overline{w}}{\xi} - (1 - \kappa^2)\, \overline{s}_r = (1 + \kappa)\, \overline{\Theta} \tag{8a}$$

$$\overline{s}_r' + (1 - \kappa) \frac{\overline{s}_r}{\xi} - (1 - \kappa^2)^{-1} \left(\frac{1 - \kappa^2}{\xi^2} + p^2\right) \overline{w} = -\frac{1}{\xi}\, \overline{\Theta} \tag{8b}$$

Upon differentiation of both sides of Eq (8a), and the combining of the resulting equation with (8a) itself, the equation

$$\bar{w}'' + \frac{\bar{w}'}{\xi} - \kappa^2 \frac{\bar{w}}{\xi^2} - (1 + \kappa)\left(\bar{\Theta}' + \frac{1 - \kappa}{\xi}\bar{\Theta}\right) = (1 - \kappa^2)\left[\bar{s}_r' + (1 - \kappa)\frac{\bar{s}_r}{\xi}\right]$$

is obtained. Substitution into this equation from Eq (8b) then gives

$$\bar{w}'' + \frac{\bar{w}'}{\xi} - \left(\frac{1}{\xi^2} + p^2\right)\bar{w} = (1 + \kappa)\,\bar{\Theta}' . \tag{9}$$

The general solution to the homogeneous ($\bar{\Theta}' \equiv 0$) equation (9) is (Ref 2, Sect 9.6), expressible in terms of the modified Bessel functions I_1 and K_1 as follows:

$$\bar{w}(\xi) = AI_1(p\xi) + BK_1(p\xi) \tag{10}$$

Here A and B are functions of p only. Equation (10) is also the complete solution for a temperature distribution that is independent of ξ. If thermal conduction is entirely neglected for the short time intervals to be considered, and if t_o is a heating time, $\tau_o = \frac{c_1}{a} t_o$ the corresponding nondimensional heating time, then the nondimensional temperature is independent of ξ and is assumed here to have the following form for $0 < \xi < 1$:

$$\begin{aligned} \Theta &= \frac{\tau}{\tau_o} & 0 \le \tau < \tau_o \\ \Theta &= 1 & \tau \ge \tau_o \end{aligned} \tag{11}$$

The limiting case $\tau_o = 0$ corresponds to instantaneous heating. Consequently,

$$\bar{\Theta} = \frac{1}{p^2 \tau_o}\left(1 - e^{-p\tau_o}\right) \tag{12a}$$

and in the limiting case $\tau_o = 0$,

$$\bar{\Theta} = \frac{1}{p} \tag{12b}$$

The symbol $\bar{\Theta}$ when written hereafter will refer specifically to Eqs (12a, b).

Since, by Ref 2, Eq (9.6.26),

$$\xi \frac{d}{d\xi} I_1 (p\xi) = p\xi I_2 (p\xi) + I_1 (p\xi)$$

$$\xi \frac{d}{d\xi} K_1 (p\xi) = - p\xi K_2 (p\xi) + K_1 (p\xi)$$

then the stress component transform $\bar{s}_r$ can be written from Eqs (8a) and (10) as

$$(1 - \kappa) \bar{s}_r (\xi) = \frac{1}{1 + \kappa} \left\{ A \left[pI_2 (p\xi) + \frac{1 + \kappa}{\xi} I_1 (p\xi) \right] + B \left[- pK_2 (p\xi) + \frac{1 + \kappa}{\xi} K_1 (p\xi) \right] \right\} - \bar{\Theta} \quad (13)$$

3. THE HEATED CYLINDRICAL CORE PROBLEM

We consider next the problem of a suddenly-heated long cylindrical region imbedded in an unheated, infinite, elastic medium of the same material. Denote by $R_{①}$ the region corresponding to the cylindrical core and by $R_{②}$ the surrounding medium (Figure 1b). Similarly, let $w_{①}$ be the displacement in $R_{①}$ etc. The requirement of bounded displacements applied to Eq (10) implies that

$$B_{①} = A_{②} = 0 \quad (14)$$

Therefore, for $0 \leq \xi \leq 1$,

$$\bar{w}_{①} (\xi) = A_{①} I_1 (p\xi) \quad (15a)$$

$$(1 - \kappa) \bar{s}_{r①} (\xi) = A_{①} \left[\frac{p}{1 + \kappa} I_2 (p\xi) + \frac{1}{\xi} I_1 (p\xi) \right] - \bar{\Theta} \quad (15b)$$

and for $\xi \geq 1$,

$$\bar{w}_{②} (\xi) = B_{②} K_1 (p\xi) \quad (15c)$$

$$(1 - \kappa) \bar{s}_{r②} (\xi) = B_{②} \left[- \frac{p}{1 + \kappa} K_2 (p\xi) + \frac{1}{\xi} K_1 (p\xi) \right] \quad (15d)$$

Continuity of $\bar{w}$ (ξ) and $\bar{s}_r$ (ξ) at $\xi = 1$ requires

$$A_{①} I_1(p) = B_{②} K_1(p) \tag{16a}$$

$$A_{①}\left[\frac{p}{1+\kappa} I_2(p) + \frac{1}{\xi} I_1(p)\right] - \bar{Ⓟ} = B_{②}\left[-\frac{p}{1+\kappa} K_2(p) + \frac{1}{\xi} K_1(p)\right] \tag{16b}$$

Substitution from Eq (16a) into Eq (16b) gives

$$A_{①}\left[\frac{p}{1+\kappa} I_2(p) + \frac{1}{\xi} I_1(p)\right] - \bar{Ⓟ} = A_{①} \frac{I_1(p)}{K_1(p)} \cdot$$

$$\cdot \left[-\frac{p}{1+\kappa} K_2(p) + \frac{1}{\xi} K_1(p)\right]$$

Making use of the Wronskian (Ref 2, Pg 375)

$$K_1(p) I_2(p) + K_2(p) I_1(p) = \frac{1}{p} \tag{17}$$

we then obtain

$$A_{①} = (1+\kappa)\, \bar{Ⓟ} K_1(p) \tag{18a}$$

$$B_{②} = (1+\kappa)\, \bar{Ⓟ} I_1(p) \tag{18b}$$

The solution in R ① will be carried out in detail because it is of primary interest. The solution in R ② is directly analogous. In R ① , i.e., $0 \le \xi \le 1$, Eqs (15a, b) and (18a) give

$$\bar{w}_{①}(\xi) = (1+\kappa)\, \bar{Ⓟ} K_1(p) I_1(p\xi) \tag{19a}$$

$$(1-\kappa)\, \bar{s}_{r_{①}}(\xi) = \bar{Ⓟ}\left[p K_1(p) I_2(p\xi) + \frac{1+\kappa}{\xi} K_1(p) I_1(p\xi) - 1\right] \tag{19b}$$

The inversion of Eq (19a, b) is considered in the following section. We will first consider only the stress at the center $\xi = 0$. For fixed p (Ref 2, Eq (9.6.7)),

$$\lim_{\xi \to 0} I_2 (p\xi) = 0$$

$$\lim_{\xi \to 0} \frac{I_1 (p\xi)}{\xi} = \frac{1}{\xi} \cdot \frac{1}{2} p\xi = \frac{1}{2} p.$$

Hence, Eq (19b) becomes

$$(1 - \kappa) \bar{s}_{r\,①} (0+) = \frac{1 + \kappa}{2 \tau_o} \left(1 - e^{-p\tau_o}\right) \frac{1}{p} K_1 (p) - \frac{1}{p^2 \tau_o} \left(1 - e^{-p\tau_o}\right) \tag{20}$$

From Ref 4, pg 277,

$$\mathcal{L}^{-1} \left[\frac{1}{p} K_1 (ap)\right] = \frac{1}{a} (\tau^2 - a^2)^{1/2} H (\tau - a) \tag{21}$$

where H(x) is the Heaviside step function that is zero for $x < 0$ and one for $x > 0$. Furthermore, if

$$\mathcal{L}^{-1} \left[f (p)\right] = F (\tau),$$

then

$$\mathcal{L}^{-1} \left[\left(1 - e^{-p\tau_o}\right) f (p)\right] = F (\tau) - H (\tau - \tau_o) F (\tau - \tau_o) \tag{22}$$

Equations (21) and (22) are used to invert $\bar{s}_{r\,\textcircled{1}}(0+)$ from Eq (20) to give the value of $s_{r\,\textcircled{1}}(\xi, \tau)$ at $\xi = 0+$ for $\tau \geq 0$:

$$(1-\kappa)\, s_{r\,\textcircled{1}}(0+, \tau) = \frac{1+\kappa}{2\tau_o} \left\{ (\tau^2 - 1)^{1/2} H(\tau - 1) - \left[(\tau - \tau_o)^2 - 1 \right]^{1/2} H(\tau - \tau_o - 1) \right\} - \frac{1}{\tau_o} \left[\tau - (\tau - \tau_o) H(\tau - \tau_o) \right] \tag{23}$$

Table 1

VALUES OF NONDIMENSIONAL STRESS AT THE CENTER OF THE CYLINDER, FROM EQ (23)

$\tau_o < 1$	$\tau_o > 1$	$(1-\kappa)\, s_{r\,\textcircled{1}}(0+, \tau) =$
$0 < \tau < \tau_o$	$0 < \tau < 1$	$-\frac{\tau}{\tau_o}$
$\tau_o < \tau < 1$		-1
	$1 < \tau < \tau_o$	$-\frac{\tau}{\tau_o} + \frac{1+\kappa}{2\tau_o}(\tau^2 - 1)^{1/2}$
$1 < \tau < \tau_o + 1$	$\tau_o < \tau < \tau_o + 1$	$-1 + \frac{1+\kappa}{2\tau_o}(\tau^2 - 1)^{1/2}$
$\tau > 1 + \tau_o$	$\tau > 1 + \tau_o$	$-1 + \frac{1+\kappa}{2\tau_o}\left\{(\tau^2 - 1)^{1/2} - \left[(\tau - \tau_o)^2 - 1\right]^{1/2}\right\}$

Examination of Eq (23) is facilitated by Table 1 and Figure 2, from which the following conclusions can be drawn about s_r at $\xi = 0+$:

a. A uniform compressive state of stress, which is a particular integral of Eq (7a, b), persists at the center until the relief wave from the surface $\xi = 1$ arrives at $\tau = 1$. This state is

$$s_r = -\frac{\Theta}{1-\kappa} = s_\theta,$$

$$w = 0.$$

b. For sufficiently small τ_o, the maximum <u>tensile</u> stress occurs at $\tau = \tau_o + 1$, and it has the value $s_{r,\,max}$ in nondimensional form:

$$s_{r,\,max} = \frac{1}{2}\,\frac{1+\kappa}{1-\kappa}\left(1+\frac{2}{\tau_o}\right)^{1/2} - \frac{1}{1-\kappa} \tag{24}$$

This peak value becomes infinite in the limit $\tau_o = 0$. Tensile stresses disappear entirely for

$$\tau_o > 2\left[\frac{4}{(1+\kappa)^2} - 1\right]^{-1} \tag{25}$$

The spherical geometry solution is similar to the cylindrical in that there is a singular behavior at the origin at time a/c_1 in the limit $t_o \rightarrow 0$, but the nature of the singularity is different for the two geometries (compare Figure 5 with Figure 2).

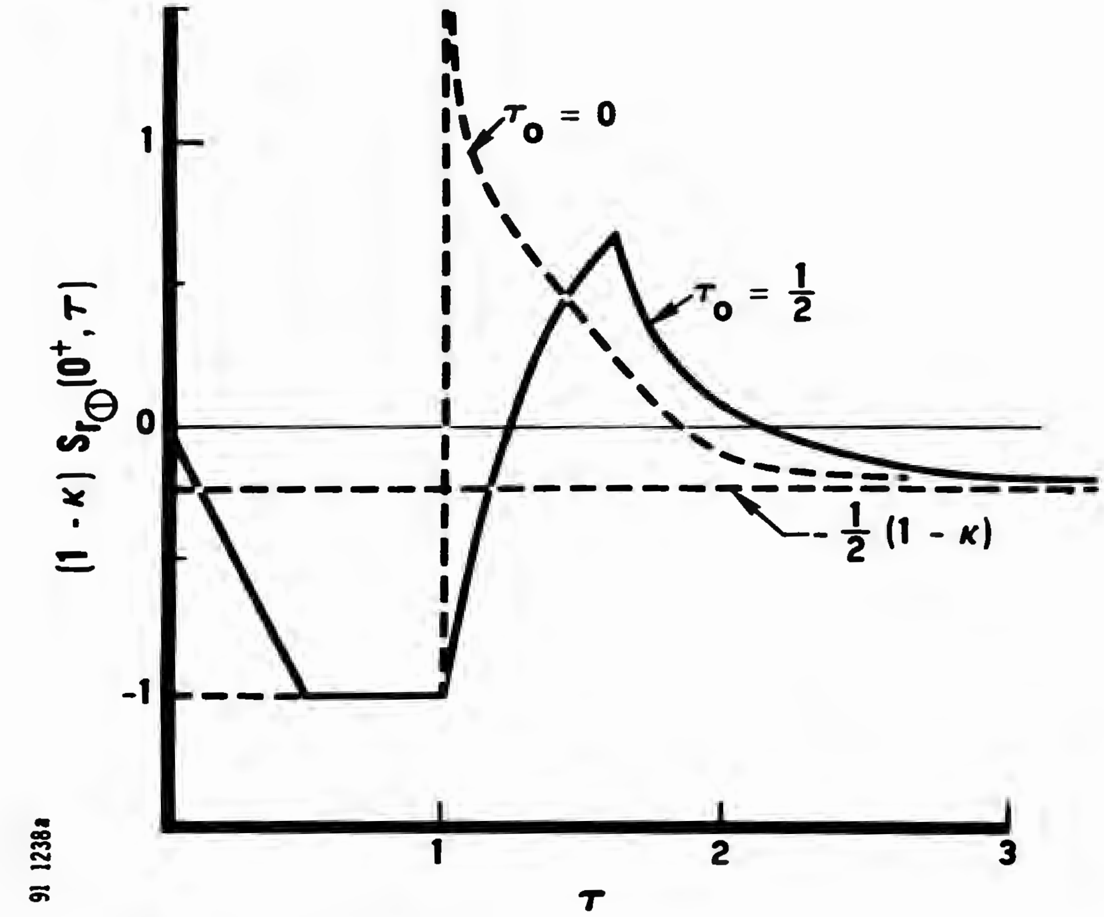

Figure 2. Nondimensional Center Stress s_r ① $(0+, \tau)$ versus Nondimensional Time for $\nu = \frac{1}{3}$

The right-hand side of inequality (25) depends only on ν. It has its smallest value for $\nu = \kappa = 0$, and this value is $\frac{2}{3}$; for $\nu = \frac{1}{3}$, $\kappa = \frac{1}{2}$, the value is 18/7. The right hand side is unity for $\kappa = \frac{2}{\sqrt{3}} - 1 = 0.155$ or $\nu = 0.135$.

c. For $\tau_0 = 1$, in wl :h case the heating time t_0 is equal to the wave transit time a/c_1, the peak tensile stress is

$$(1 - \kappa)\, s_{r,\ max} = \frac{\sqrt{3}}{2} (1 + \kappa) - 1$$

Since $0.155 < \kappa < 1$, this peak tensile stress is less than the magnitude of the peak compressive stress. Hence, tensile stresses due to the focussing effects will be relatively large compared to peak compressive stresses only for $\tau_0 < 1$.

d. There is only one wave reflection in this problem. For all τ_0, $s_{r\,①}(0+, \tau)$ tends monotonically to the value $-\frac{1}{2}$. This limit can be shown generally from the bottom line of Table 1 and, in a specific case, from Eq (26) below. It can also be shown that for all finite τ_0,

$$\lim_{\tau \to \infty} s_{r\,①}(\xi, \tau) = -\frac{1}{2}, \quad 0 \leq \xi \leq 1,$$

by use of Eq (19b) and the relation (Ref 5, Sect 114)

$$\lim_{\tau \to \infty} s_{r\,①}(\xi, \tau) = \lim_{p \to 0+} p\, \bar{s}_{r\,①}(\rho, \xi).$$

e. In the limiting case of instantaneous heating, a singularity appears in $(1 - \kappa)\, s_{r\,①}(0+, \tau)$ at $\tau = 1$, at which time this nondimensional stress jumps from (-1) at $\tau = 1-$ to $(+\infty)$ at $\tau = 1+$.

$$(1 - \kappa) \lim_{\tau_0 \to 0} s_{r\,①}(0+, \tau) = \frac{1 + \kappa}{2} \frac{\tau}{(\tau^2 - 1)^{1/2}} H(\tau - 1) - 1 \qquad (26)$$

4. INVERSION OF THE TRANSFORM

Equations (19a) and (12a) can be written as

$$\bar{w}_{①}(\xi) = \frac{(1+\kappa)}{\tau_o}\left(1 - e^{-p\tau_o}\right)\left[\frac{1}{p^2} e^p K_1(p)\right]$$

$$\cdot \left[e^{-p\xi} I_1 (p\xi)\right] \cdot e^{-p(1-\xi)} \qquad (27)$$

Let

$$\bar{G}_1 \equiv \frac{1}{p^2} e^p K_1 (p)$$

$$\bar{G}_2 \equiv e^{-p\xi} I_1 (p\xi)$$

From Ref 4, pp 276 and 278, and Ref 5, p 5, for $\tau > 0$ and $0 < \xi < 1$:

$$\mathcal{L}^{-1}(\bar{G}_1) = \int_0^\tau (\lambda^2 + 2)^{-1/2} \, d\lambda = \sinh^{-1}\left(\frac{1}{\sqrt{2}} \tau\right) \qquad (28a)$$

$$\mathcal{L}^{-1}(\bar{G}_2) = \frac{1}{\pi\xi} (\xi - \tau)(2\xi\tau - \tau^2)^{-1/2} \, H(2\xi - \tau) \qquad (28b)$$

Consequently (Ref 5, p 7), if

$$\mathcal{L}^{-1}(\bar{G}_1 \bar{G}_2) = G(\tau, \xi)$$

then by convolution

$$G(\tau, \xi) = \frac{1}{\pi\xi} \int_0^\tau \frac{\sinh^{-1}\left[\frac{1}{\sqrt{2}} (\tau - \lambda)\right] (\xi - \lambda)}{(2\xi\lambda - \lambda^2)^{1/2}} H(2\xi - \lambda) \, d\lambda \qquad (29)$$

Using the shift theorem (Ref 5, p 7, Theorem V) $w_{①}(\xi)$ can be inverted to give

$$w_{①}(\tau, \xi) = \frac{1+\kappa}{\tau_0}\Big[G(\tau - 1 + \xi, \xi)\, H(\tau - 1 + \xi)$$

$$- G(\tau - 1 - \tau_0 + \xi, \xi)\, H(\tau - 1 - \tau_0 + \xi)\Big] \qquad (30)$$

It can be shown from Eq (29) that for all $\tau > 0$

$$\lim_{\xi \to 0} G(\tau, \xi) = 0,$$

and so for all $\tau > 0$, it is verified that Eq (30) gives

$$w_{①}(\tau, 0+) = 0.$$

Equation (30) for w can be evaluated numerically by a numerical integration to obtain $G(\tau, \xi)$ from Eq (29). By similar means, one can obtain $s_{r①}(\xi)$ from inversion of Eq (19b).

For the case of instantaneous heating, the limit $\tau_0 \to 0$, one can make use of a relation that follows from Eq (22), under the condition $F(0+) = 0$.

$$\lim_{\tau_0 \to 0} \mathcal{L}^{-1}\left[\frac{1}{\tau_0}\left(1 - e^{-p\tau_0}\right) f(p)\right] = F'(\tau) \qquad (31)$$

If we take

$$f(p) = (1 + \kappa)\, \overline{G}_1\, \overline{G}_2\, e^{-p(1-\xi)}$$

then

$$F(\tau) = (1 + \kappa)\, G(\tau - 1 + \xi, \xi)\, H(\tau - 1 + \xi)$$

Consequently, Eq (27) can be interpreted for the limit $\tau_o \to 0$ as in the form of Eq (31), and so

$$w_{①}(\tau, \xi) = F'(\tau) = (1+\kappa)\, G^*(\tau - 1 + \xi, \xi)\, H(\tau - 1 + \xi)$$

where, with use of Eq (27a) and (27b),

$$G^*(\tau, \xi) = \frac{\partial}{\partial \tau} G(\tau, \xi) = \frac{1}{\pi \xi} \int_0^{\tau} \frac{(\xi - \lambda)\, H(2\xi - \lambda)}{\left\{\left[(\tau - \lambda)^2 + 2\right]\left[2\xi\lambda - \lambda^2\right]\right\}^{1/2}}\, d\lambda \tag{32}$$

The displacement is everywhere continuous and vanishes, as required, at the origin. The heated cylindrical core problem for $\tau_o = 0$ was formulated incorrectly in Ref 3. The result was that an infinite discontinuity of radial displacement was obtained (see Figure 2 of Ref 3) which travelled out from $\xi = 0$ at $\tau > 1$.

5. STRESS-FREE BOUNDARY

If the surface $\xi = 1$ is stress-free, then $\bar{s}_{r①}(1) = 0$, and we determine from Eq (15b) that

$$A_{①} = \frac{\bar{\Theta}}{\frac{p}{1+\kappa} I_2(p) + I_1(p)}$$

and so

$$(1-\kappa)\, \bar{s}_{r①}(\xi) = \bar{\Theta}\left[\frac{\frac{p}{1+\kappa} I_2(p\xi) + \frac{1}{\xi} I_1(p\xi)}{\frac{p}{1+\kappa} I_2(p) + I_1(p)} - 1\right] \tag{33}$$

At $\xi = 0$,

$$(1-\kappa)\, \bar{s}_{r①}(0+) = \bar{\Theta}\left[\frac{\frac{1}{2}p}{\frac{p}{1+\kappa} I_2(p) + I_1(p)} - 1\right] \tag{34}$$

We seek here only to find $s_{r\,①}(0+, \tau)$ for small τ, and hence look for approximations for p large (Ref 5, Chapter XIII). If the numerator and denominator of the fraction in Eq (34) are both multiplied by $K_1(p)$, then by use of Eq (17),

$$\frac{p}{1+\kappa} K_1(p)\, I_2(p) + K_1(p)\, I_1(p) = \frac{1}{1+\kappa} - \frac{p}{1+\kappa} K_2(p)\, I_1(p) + K_1(p)\, I_1(p) \, .$$

For large p, $K_2 I_1 \cong K_1 I_1 \cong \frac{1}{2p}$. Hence, Eq (34) can be written for large p as

$$(1-\kappa)\,\bar{s}_{r\,①}(0+) = \bar{Ⓗ}\left[\frac{\frac{1}{2}\, p\, K_1(p)}{\frac{1}{2(1+\kappa)} + \frac{1}{2p}} - 1\right]$$

$$= (1+\kappa)\,\bar{Ⓗ}\left[\frac{p\, K_1(p)}{1 + \frac{1+\kappa}{p}}\right] - \bar{Ⓗ}$$

$$= \frac{1+\kappa}{\tau_o}\left(1 - e^{-p\tau_o}\right)\left[\frac{\frac{1}{p}\, K_1(p)}{1 + \frac{1+\kappa}{p}}\right] - \frac{1}{p^2 \tau_o}\left(1 - e^{-p\tau_o}\right) \qquad (35)$$

If the added term $\left(\frac{1+\kappa}{p}\right)$ in the denominator of Eq (35) is neglected altogether in comparison to unity, then the solution obtained for $(1-\kappa)\,\bar{s}_{r\,①}(0+)$ for the stress free boundary in Eq (35) with this approximation can be found from the solution previously obtained for the cylindrical core problem. It is only necessary to observe that the approximate Eq (35) can be obtained from Eq (20) by replacing $\frac{1}{2}(1+\kappa)$ on the right-hand side of Eq (20) by $(1+\kappa)$. Hence, $(1-\kappa)\, s_{r\,①}(0+, \tau)$ is immediately obtained from Table 1 by replacing the factor $\frac{1}{2}(1+\kappa)$ by $(1+\kappa)$ in the expressions in the right-hand column of this table. These expressions are then only asymptotic representations of $(1 - \kappa)\, s_{r\,①}\,(0+, \tau)$ for $\tau \to o$. A better approximation could be found by use of the following:

$$\mathcal{L}^{-1}\left[\frac{\frac{1}{p} K_1 (p)}{1 + \frac{1+\kappa}{p}}\right] \cong \mathcal{L}^{-1}\left[\frac{1}{p} K_1 (p) - \frac{1+\kappa}{p^2} K_1 (p)\right]$$

$$= (\tau^2 - 1)^{1/2} H(\tau - 1) - (1 + \kappa) \int_o^{\tau} (\lambda^2 - 1)^{1/2} H(\lambda - 1)\, d\lambda$$

$$\int_o^{\tau} (\lambda^2 - 1)^{1/2} H(\lambda - 1)\, d\lambda = H(\tau - 1) \int_1^{\tau} (\lambda^2 - 1)^{1/2} d\lambda$$

$$= \frac{1}{2} H(\tau - 1)\left[\tau (\tau^2 - 1)^{1/2} - \cosh^{-1} \tau\right]$$

In the limiting case, $\tau_o = 0$, Eq (35) becomes

$$(1 - \kappa)\, \bar{s}_{r\,①}\,(0+) \cong (1 + \kappa) K_1 (p)\left[1 - \frac{(1+\kappa)}{p}\right] - \frac{1}{p} \tag{36}$$

Eq (36) is valid for large p; under the assumption that it is valid for τ of order 1, the inversion of Eq (36) gives

$$\lim_{\tau_o \to 0} (1 - \kappa)\, s_{r\,①}\,(0+, \tau) = (1 + \kappa) \frac{\tau}{(\tau^2 - 1)^{1/2}} H(\tau - 1) - 1$$

$$- (1 + \kappa^2)(\tau^2 - 1)^{1/2} H[\tau - 1] \tag{37}$$

This singular solution too can be obtained from the corresponding stress for the imbedded core cylinder, Eq (26), by changing the factor $\left(\frac{1+\kappa}{2}\right)$ to $(1 + \kappa)$ and adding the correction term

$$- (1 + \kappa^2)(\tau^2 - 1)^{1/2} H[\tau - 1]$$

In Ref 7, the case of a cylinder with a free surface is treated, and there is obtained the (singular) asymptotic solution for the stress at the origin in a form similar to that given by Eq (37) above. The development shown in Ref 7 is restricted, however, to the condition of instantaneous heating, $\tau_0 \to 0$, and it does not consider non-vanishing rise times τ_0. We have now shown that the representation of s_r $(0+, \tau)$ for this problem for small times τ can be obtained from Table 1 for the case $\tau_0 > 0$, with the limit $\tau_0 \to 0$ being adequately represented by the first two terms on the right-hand side of Eq (37) near $\tau = 1$.

PART B - THE SPHERICAL GEOMETRY: STRESS-FREE BOUNDARY

1. CONVENTIONAL THERMOELASTIC EQUATIONS

Introduce the usual spherical coordinate system (r, θ, ϕ) with origin at the center of a sphere of radius a.

The principal stress components are

σ_r: radial

σ_θ $(= \sigma_\phi)$: tangential

All other quantities are as defined in Part A.

The governing equations are the stress-displacement relations (Ref 1, p. 170 Eq VIII-24)

$$\frac{\partial u}{\partial r} = \frac{1}{E}\left[\sigma_r - 2\nu\sigma_\theta\right] + \alpha T \tag{1a}$$

$$\frac{u}{r} = \frac{1}{E}\left[\sigma_\theta - \nu(\sigma_r + \sigma_\theta)\right] + \alpha T \tag{1b}$$

which can also be written:

$$\sigma_\theta = \frac{E}{(1+\nu)(1-2\nu)}\left[\frac{u}{r} + \nu\frac{\partial u}{\partial r}\right] - \frac{E}{1-2\nu}\alpha T \tag{2a}$$

$$\sigma_r = \frac{E}{(1+\nu)(1-2\nu)}\left[2\nu\frac{u}{r} + (1-\nu)\frac{\partial u}{\partial r}\right] - \frac{E}{1-2\nu}\alpha T \tag{2b}$$

and the equation of equilibrium:

$$\frac{\partial \sigma_r}{\partial r} + \frac{2}{r}(\sigma_r - \sigma_\theta) = \rho\frac{\partial^2 u}{\partial t^2} \tag{3}$$

In terms of the displacement, u, Eq (3) becomes:

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} - \frac{2u}{r^2} = \frac{(1+\nu)(1-2\nu)}{(1-\nu)E}\,\rho\,\frac{\partial^2 u}{\partial t^2} \tag{4}$$

where use has been made of the fact that the temperature field is uniform.

Introduce dimensionless forms for the radial coordinate, time, temperature, stress and displacement:

$$\xi \equiv \frac{r}{a}$$

$$w \equiv \frac{1-\nu}{(1+\nu)\alpha T_o a}\, u$$

$$s_r \equiv \frac{1-\nu}{\alpha T_o E}\,\sigma_r$$

$$s_\theta \equiv \frac{1-\nu}{\alpha T_o E}\,\sigma_\theta$$

$$\Theta \equiv \frac{T}{T_o}$$

$$\tau = \frac{tc}{a}$$

where c is the "dilatational" wave speed:

$$c^2 = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)\rho}$$

Equations (2a), (2b), and (4) now become:

$$s_\theta = \frac{1}{1-2\nu}\left[\frac{w}{\xi} + \nu\frac{\partial w}{\partial \xi}\right] - \frac{(1-\nu)}{1-2\nu}\,\Theta \tag{5a}$$

$$s_r = \frac{1}{1 - 2\nu} \left[2\nu \frac{w}{\xi} + (1 - \nu) \frac{\partial w}{\partial \xi} \right] - \frac{(1 - \nu)}{1 - 2\nu} \Theta \tag{5b}$$

and

$$\frac{\partial^2 w}{\partial \xi^2} + \frac{2}{\xi} \frac{\partial w}{\partial \xi} - \frac{2w}{\xi^2} = \frac{\partial^2 w}{\partial \tau^2} \tag{6}$$

Solution of the problem consists of finding a function w which satisfies Eq (6), satisfies the initial conditions

$$w(\xi, 0) = 0 \, ; \; \left. \frac{\partial w}{\partial \tau} \right]_{(\xi, 0)} = 0 \tag{7}$$

and is such that the stress, s_r, satisfies the boundary condition:

$$s_r(1, \tau) = 0. \tag{8}$$

2. SOLUTION BY MEANS OF THE LAPLACE TRANSFORM

As before, let the Laplace transform of a function $\bar{f}(\xi, \tau)$ be designated by $\bar{f}(\xi, p)$. Application of the Laplace transform to Eq (6), under the conditions of Eq (7) gives

$$\frac{\partial^2 \bar{w}}{\partial \xi^2} + \frac{2}{\xi} \frac{\partial \bar{w}}{\partial \xi} - \frac{2w}{\xi^2} = p^2 \bar{w} \tag{9}$$

Because the displacement must remain finite at the origin, the only acceptable solution of Eq (9) has the form

$$\bar{w} = A(p) \left[\frac{\pi/2}{p \xi} \right]^{1/2} I_{3/2}(p\xi) \tag{10}$$

where

$$\left(\frac{\pi/2}{p \xi} \right)^{1/2} I_{3/2}(p\xi) \equiv f_1(p\xi)$$

is a modified spherical Bessel Function of the first kind (Ref 2, p. 443) and A(p) is to be determined from the boundary condition.

At $\xi = 1$, we have, from Eqs. (5b) and (8)

$$2\nu \frac{\overline{w}}{\xi} + (1 - \nu) \frac{\partial \overline{w}}{\partial \xi} - (1 - \nu) \overline{\Theta} = 0 \tag{11}$$

and, therefore

$$A(p) \left[2\nu \; f_1(p) + (1 - \nu) \; pf_1'(p) \right] = (1 - \nu) \; \overline{\Theta} \tag{12}$$

The derivative, $f_1'(p)$, can be written in terms of $f_0(p)$ and $f_1(p)$ (Ref 2, p. 444). The factor A(p) then becomes

$$A(p) = \frac{\overline{\Theta}}{p \left[f_0(p) - \frac{2}{p} f_1(p) \right] + \frac{2\nu}{1 - \nu} f_1(p)}$$

$$= \frac{\overline{\Theta}}{p \, f_0(p) - 2 \frac{(1 - 2\nu)}{1 - \nu} f_1(p)} \tag{13}$$

If the heating is instantaneous, then

$$\Theta = 1 \text{ and } \overline{\Theta} = \frac{1}{p} \tag{14}$$

If a gradual rise is permitted over a time, τ_o, say

$$\Theta = \frac{\tau}{\tau_o} \qquad \tau \le \tau_o$$

$$\Theta = 1 \qquad \tau \ge \tau_o$$

then

$$\overline{\Theta} = \frac{1}{p \cdot p \tau_o} \left[1 - e^{-p \tau_o} \right] \tag{15}$$

Consider now the heating pattern of Eq. (14), (instantaneous) with

$$\overline{w} = \frac{1}{p\left[pf_0(p) - \frac{2(1-2\nu)}{1-\nu} f_1(p)\right]} f_1(p\xi) \tag{16}$$

The functions f_0 and f_1 can be expressed in terms of hyperbolic sines and cosines, or alternatively, negative and positive exponentials (Ref 2, p. 443). Set

$$\frac{1-2\nu}{1-\nu} \equiv \kappa^* \tag{17}$$

for simplicity. Then making use of the relationships (for any argument, Z),

$$f_0(Z) = \frac{\sinh Z}{Z} = \frac{1}{Z}\frac{(e^Z - e^{-Z})}{2} ,$$

$$f_1(Z) = -\frac{\sinh Z}{Z^2} + \frac{\cosh Z}{Z} = -\frac{1}{Z^2}\frac{(e^Z - e^{-Z})}{2} + \frac{1}{Z}\frac{(e^Z + e^{-Z})}{2} ,$$

equation (16) becomes

$$\overline{w} = \frac{e^{-p(1-\xi)}}{p \cdot p^2 \xi^2}\left[\frac{1}{\left(1 - \frac{2\kappa^*}{p} + \frac{2\kappa^*}{p^2}\right)\left(1 - \frac{Q_2}{Q_1} e^{-2p}\right)}\right]$$

$$\cdot \left[(p\xi - 1) + (p\xi + 1) e^{-2p\xi}\right] \tag{18}$$

where

$$Q_2 \equiv 1 + \frac{2\kappa^*}{p} + \frac{2\kappa^*}{p^2}$$

$$Q_1 \equiv 1 - \frac{2\kappa^*}{p} + \frac{2\kappa^*}{p^2}$$

The factor

$$\left(1 - \frac{Q_2}{Q_1} e^{-2p}\right)^{-1}$$

can be written as a series in ascending powers of

$$\frac{Q_2}{Q_1} e^{-2p}.$$

The function $\overline{w}$ then has the series expansion:

$$\overline{w} = \frac{1}{\xi^2 p} \left\{ \frac{(p\xi - 1)}{p^2 - 2\kappa^* p + 2\kappa^*} e^{-p(1-\xi)} + \frac{p\xi + 1}{p^2 - 2\kappa^* p + 2\kappa^*} e^{-p(1+\xi)} \right.$$

$$\left. + \frac{Q_2}{p^2 Q_1^2} (p\xi - 1) e^{-1(3-\xi)} + \frac{Q_2}{p^2 Q_1^2} (p\xi + 1) e^{-p(3+\xi)} + - - - - \right\} \qquad (19)$$

the terms of which represent an inward moving wave and its successive reflections. The inversion of $\overline{w}$ can now be carried out term by term. It should be noted, however, that in order to compute the displacement, or stress, at the origin, the terms must be taken in pairs; i. e., the first two or the first four, etc. The first term of each pair gives the contribution of the wave arriving at $\xi = 0$. The wave is then immediately reflected and the second term of the pair gives the contribution of the reflected wave.

3. INVERSION OF THE TRANSFORM

Denote the first term of Eq. (19) by $\overline{w}_1$, the first two by $\overline{w}_{1,2}$ etc. Consider

$$\overline{w}_{1,2} = \frac{1}{\xi^2 p} \left\{ \frac{p\xi - 1}{p^2 - 2\kappa^* p + 2\kappa^*} e^{-p(1-\xi)} \right.$$

$$\left. + \frac{p\xi + 1}{p^2 - 2\kappa^* p + 2\kappa^*} e^{-p(1+\xi)} \right\} \qquad (20)$$

The roots of the polynomial $p^2 - 2\kappa^* p + 2\kappa^*$ are $p = r_1$ and $p = r_2$:

$$r_1 = \kappa^* + i\sqrt{2\kappa^* - \kappa^{*2}}\,;\; r_2 = \kappa^* - i\sqrt{2\kappa^* - \kappa^{*2}}\,,$$

Since $0 \le \nu < 1/2$, then $1 \geqq \kappa^* > 0$, and, therefore, $2\kappa^* - \kappa^{*2} > 0$. The inverse of $\bar{w}_{1,2}$ from Eq. (20) is readily found to be (Ref 4, Vol. 1, p. 230; Ref 5, p. 7)

$$w_{1,2} = \frac{1}{\xi^2}\left\{\frac{1 - r_1\xi}{(r_2 - r_1)r_1}\, e^{r_1[\tau - (1-\xi)]} + \frac{1 - r_2\xi}{(r_1 - r_2)r_2}\, e^{r_2[\tau - (1-\xi)]} - \frac{1}{r_1 r_2}\right\} \cdot H[\tau - (1-\xi)]$$

$$- \frac{1}{\xi^2}\left\{\frac{1 + r_1\xi}{(r_2 - r_1)r_1}\, e^{r_1[\tau - (1+\xi)]} + \frac{1 + r_2\xi}{(r_1 - r_2)r_2}\, e^{r_2[\tau - (1+\xi)]} - \frac{1}{r_1 r_2}\right\} \cdot H[\tau - (1+\xi)] \tag{21}$$

The function $w_{1,2}$ is identical with w for the time interval $\tau < 3 - \xi$. The third term of Eq. (19) contributes only for $3 - \xi < \tau$ and the fourth for $3 + \xi < \tau$ etc. Equations (5a), (5b), (14), and (17) determine the stress; thus:

$$\kappa^* s_r = \frac{\partial w}{\partial \xi} + 2(1 - \kappa^*)\frac{w}{\xi} - 1 \tag{22a}$$

$$\kappa^* s_\theta = (1 - \kappa^*)\frac{\partial w}{\partial \xi} + (2 - \kappa^*)\frac{w}{\xi} - 1 \tag{22b}$$

Stress wave fronts at $\tau = n - \xi$, $n = 1, 3, 5 \ldots\ldots$, travel in to the center, the first arriving there at $\tau = 1$. Behavior at the origin at $\tau = 1$, which is singular, is discussed in Section 4.

Consider here first the stress and displacement at the center in the open interval $1 < \tau < 3$. The displacement may be disposed of very simply. It is easily shown that

$$\lim_{\xi \to 0} w_{1,2}(\xi, \tau) = 0 \qquad 1 < \tau < 3$$

as required.

To compute the stress, write $w_{1,2}$ as

$$w_{1,2} = \frac{1}{\xi^2} G_1 \left[\xi, \tau - (1-\xi)\right] H\left[\tau - (1-\xi)\right]$$

$$+ \frac{1}{\xi^2} G_2 \left[\xi, \tau - (1+\xi)\right] H\left[\tau - (1+\xi)\right]$$

Then for $\tau > 1 + \xi$

$$\frac{\partial w_{1,2}}{\partial \xi} = \frac{1}{\xi^2} \frac{\partial (G_1 + G_2)}{\partial \xi} - \frac{2}{\xi^3} (G_1 + G_2)$$

and

$$\kappa^* s_{r_{1,2}} = \frac{1}{\xi^2} \frac{\partial (G_1 + G_2)}{\partial \xi} - \frac{2\kappa^*}{\xi^3} (G_1 + G_2) \tag{23a}$$

$$\kappa^* s_{\theta_{1,2}} = \frac{(1-\kappa^*)}{\xi^2} \frac{\partial (G_1 + G_2)}{\partial \xi} + \frac{\kappa^*}{\xi^3} (G_1 + G_2) \tag{23b}$$

where $\kappa^* s_{r_{1,2}}$ and $\kappa^* s_{\theta_{1,2}}$ are stresses associated with the first incoming wave and its reflection from the origin.

By repeated use of L'Hospital's rule it can be shown that

$$\lim_{\xi \to 0} \frac{1}{\xi^2} \frac{\partial}{\partial \xi} (G_1 + G_2) = 3 \left\{ \lim_{\xi \to 0} \frac{G_1 + G_2}{\xi^3} \right\}$$

$$= 3 \left\{ \frac{1}{(r_2 - r_1)} \left[\frac{-2r_1^{\,2} e^{r_1 (\tau - 1)}}{3} \right] + \frac{1}{(r_1 - r_2)} \left[\frac{-2r_2^{\,2} e^{r_2 (\tau - 1)}}{3} \right] \right\} \tag{24}$$

At the origin, therefore,

$$\kappa^* s_r (0, \tau) = \kappa^* s_\theta (0, \tau) = -1 \qquad \text{for } 0 < \tau < 1 \tag{25a}$$

$$\kappa^* s_r(0,\tau) = \kappa^* s_\theta(0,\tau) = \lim_{\xi \to 0} \left[(3 - 2\kappa^*) \frac{G_1 + G_2)}{\xi^3} \right] - 1$$

$$= -\frac{2}{3}(3 - 2\kappa^*) \left[\frac{r_1^2}{r_2 - r_1} e^{r_1(\tau - 1)} + \frac{r_2^2}{r_1 - r_2} e^{r_2(\tau - 1)} \right] - 1 \qquad (25b)$$

for $1 < \tau < 3$

At $\tau = 1^+$

$$s_r(0, 1^+) = + \frac{2}{3\kappa^*}(3 - 2\kappa^*) \left[r_2 + r_1 \right] - \frac{1}{\kappa^*}$$

$$= \frac{4}{3}(3 - 2\kappa^*) - \frac{1}{\kappa^*} \qquad (26)$$

The function $s_r(0, 1^+)$ is plotted in Fig. 3 as a function of ν

More generally, for $\xi = 0$, $1 < \tau < 3$:

$$\kappa^* s_r(0,\tau) = \frac{4}{3}(3 - 2\kappa^*)\kappa^* e^{\kappa^*(\tau - 1)} \left\{ \cos\left[\sqrt{2\kappa^* - \kappa^{*2}}(\tau - 1) \right] + \frac{(\kappa^* - 1)}{\sqrt{2\kappa^* - \kappa^{*2}}} \sin\left[\sqrt{2\kappa^* - \kappa^{*2}}(\tau - 1) \right] \right\} - 1 \qquad (27)$$

Equations (25a) and (27) show that in the case of a compressible material the stress state at $\xi = 0$ goes from homogeneous compression, $\kappa^* s_r = -1$, in the open interval $0 < \tau < 1$ to finite values in the open interval $1 < \tau < 3$. The limit from the right, denoted by $s_r(0, 1^+)$, is also finite for all ν in the interval $0 < \nu < \frac{1}{2}$; this limit is tensile for ν up to about 0.4 as shown in Figure 3. Equations (25a) and (27) also show that the stress approaches $-\infty$ as ν tends to 1/2, or as κ^* tends to 0 [cf. Eq. (17)] . The case $\nu = \frac{1}{2}$ is for an ideally incompressible material in which the dilatational speed would also be infinite.

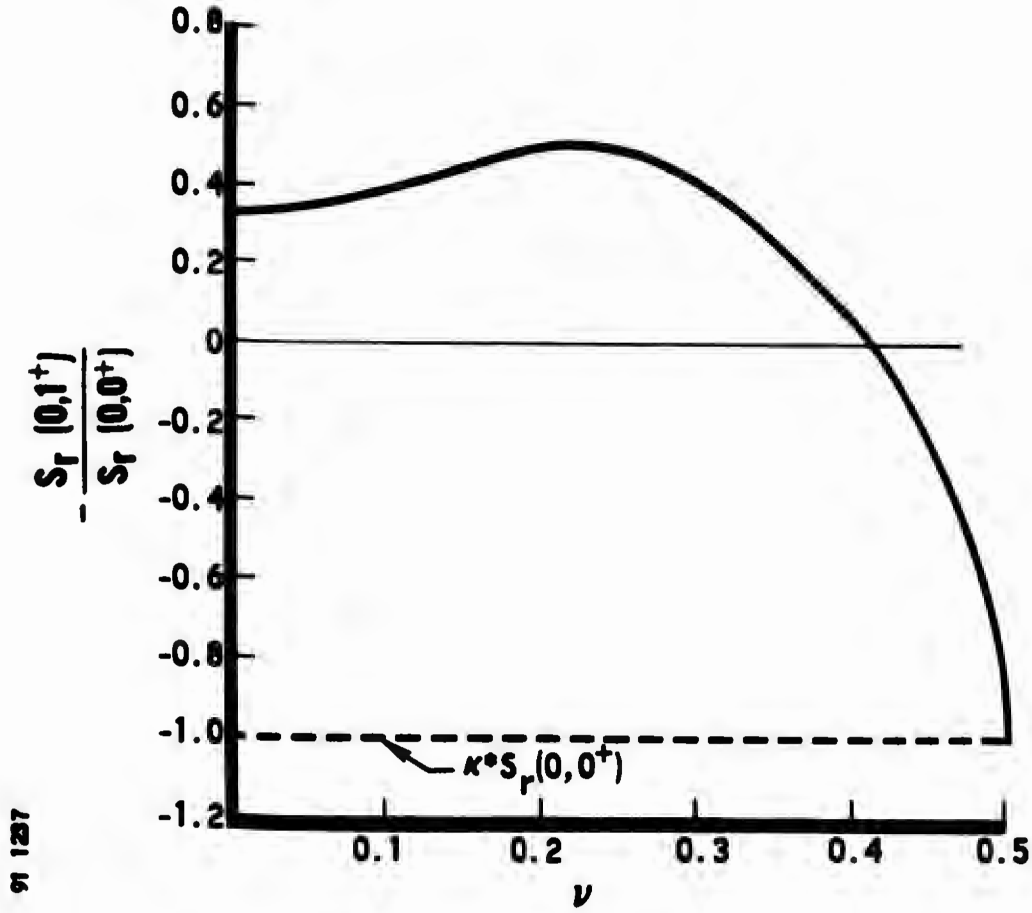

Figure 3. Radial Stress at the Origin at $\tau = 1^+$

Further investigation of the singular behavior of both stress and displacement at the origin for $\tau = 1$ is carried out in the following section where it is shown that a Dirac delta function at $\tau = 1$ must be added to the right-hand side of Eq. (27) in order that the interval be extended to include $\tau = 1$. Equations (25a) and (27) are plotted in Figure 4.

4. NATURE OF THE SINGULARITY AT THE ORIGIN

To determine the stress at the origin for $\tau = 1$, consider the stress at a neighboring point $\xi = \epsilon$ in the time interval $1 - \epsilon \leq \tau < 1 + \epsilon$. Let $s_{r,1}$ be the nondimensional radial stress associated with the first incoming (tension) wave and $s_{r,2}$ the nondimensional radial stress associated with its reflection at the origin. $s_{r,1}$ arrives at $\xi = \epsilon$ at time $\tau = 1 - \epsilon$, and $s_{r,2}$ does not arrive until $\tau = 1 + \epsilon$. Therefore, for $\tau < 1 + \epsilon$,

$$\kappa^* s_r = \kappa^* s_{r,1} - 1,$$

where $s_{r,1}$ is given by Eq. (22a) with the substitution on the right-hand side of

$$w = w_1 = \frac{1}{\xi^2} \left\{ \frac{1 - r_1 \xi}{(r_2 - r_2) r_1} e^{r_1 [\tau - (1 - \xi)]} + \frac{1 - r_2}{(r_1 - r_2) r_2} e^{r_2 [\tau - (1 - \xi)]} - \frac{1}{r_1 r_2} \right\} H[\tau - (1 - \xi)] \tag{28}$$

Let

$$\tau = 1 - \epsilon + 2\epsilon\eta, \qquad \eta \geqq 0 .$$

Substituting Eq. (29) into Eq. (28) and using a Taylor representation, we obtain, to third order in ϵ,

$$w_1 = \frac{1}{\epsilon^2} \left\{ 2(\eta - \eta^2)\epsilon^2 + \frac{4}{3}\kappa^* (3\eta^2 - 2\eta^3)\epsilon^3 + O(\epsilon^4) \right\} H[\eta] \tag{29}$$

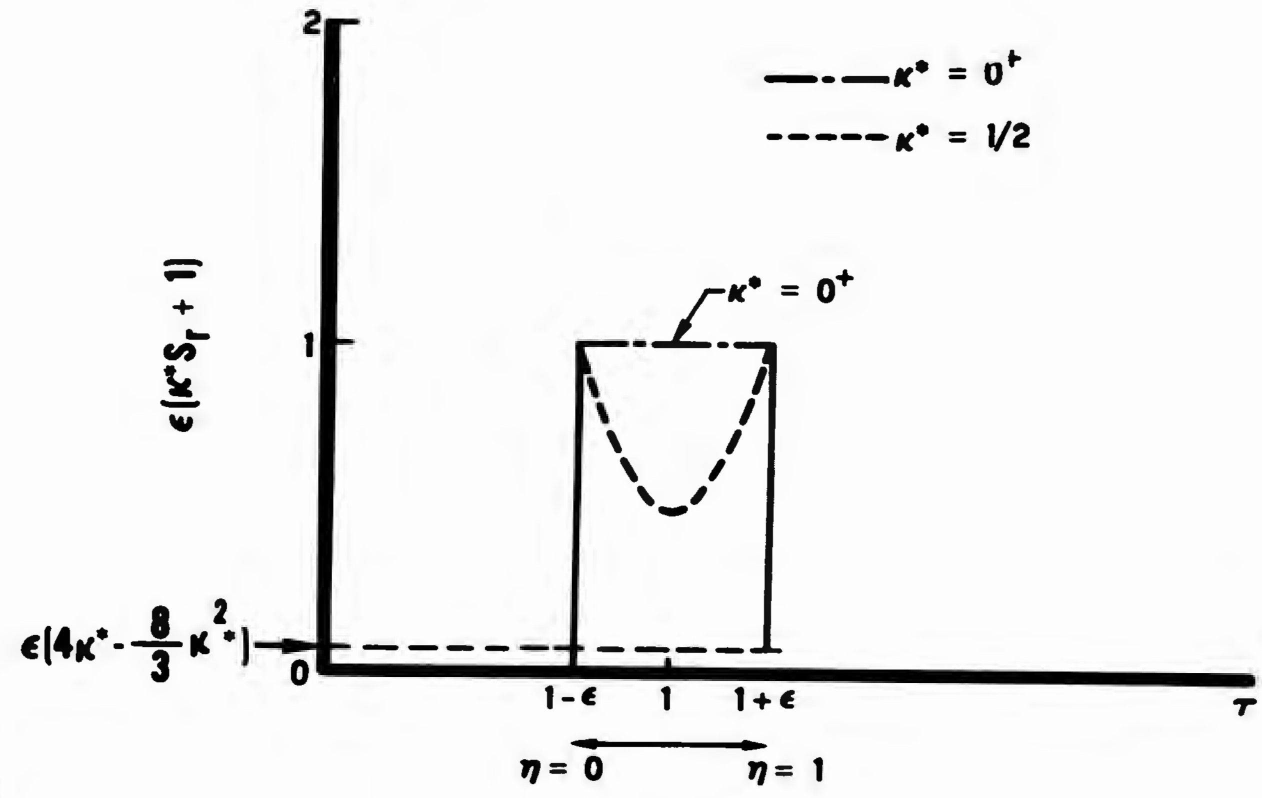

Figure 4. Nondimensional Stress versus Nondimensional Time at a Point Close to the Origin

Similarly, from Eq. (28),

$$\frac{\partial w_1}{\partial \xi} = \frac{1}{\xi^2} \left\{ - \frac{r_1 \xi}{r_2 - r_1} e^{r_1 [\tau - (1 - \xi)]} - \frac{r_2 \xi}{r_1 - r_2} e^{r_2 [\tau - (1 - \xi)]} \right\} H[\tau - (1 - \xi)] - \frac{2}{\xi} w_1 \tag{30}$$

Again, after substitution and expansion,

$$\frac{\partial w_1}{\partial \xi} = \frac{1}{\epsilon^2} \left\{ \epsilon + 4\kappa^* \eta \epsilon^2 + 4\kappa^* (2\kappa^* - 1)\eta^2 \epsilon^3 + O(\epsilon^4) \right\} H[\eta] - \frac{2}{\epsilon} w_1 \tag{31}$$

Therefore

$$\kappa^* s_{r,1} = \left\{ \frac{1}{\epsilon} - \frac{4\kappa^* (\eta - \eta^2)}{\epsilon} + 4\kappa^* - \frac{8}{3} \kappa^{*2} (3\eta^2 - 2\eta^3) + O(\epsilon) \right\} H[\eta] \tag{32}$$

At $\eta = 1$, i.e., at $\tau = 1 + \epsilon$, the reflected wave arrives at $\xi = \epsilon$, and therefore

$$\kappa^* s_r = \kappa^* s_{r,1} + \kappa^* s_{r,2} - 1, \qquad \tau = (1 + \epsilon)^+ \tag{33}$$

To obtain $s_{r,2}$ return to Eq. (21) to obtain w_2 and $\frac{\partial w_2}{\partial \xi}$ as follows:

$$w_2 = - \frac{1}{\xi^2} \left\{ \frac{1 + r_1 \xi}{(r_2 - r_1) r_1} e^{r_1 [\tau - (1 + \xi)]} + \frac{1 + r_2 \xi}{(r_1 - r_2) r_2} e^{r_2 [\tau - (1 + \xi)]} - \frac{1}{r_1 r_2} \right\} H[\tau - (1 + \xi)] \tag{34a}$$

$$\frac{\partial w_2}{\partial \xi} = \frac{1}{\xi^2}\left\{\frac{r_1\xi}{r_2 - r_1}\, e^{r_1[\tau - (1+\xi)]} + \frac{r_2\xi}{r_1 - r_2}\, e^{r_2[\tau - (1+\xi)]}\right\} H[\tau - (1+\xi)] - \frac{2w_2}{\xi} \tag{34b}$$

For $\xi = \epsilon$ and $\tau = (1+\epsilon)^+$, directly behind the reflected wave,

$$w_2 = 0$$

$$\frac{\partial w_2}{\partial \xi} = -\frac{1}{\epsilon^2} \cdot |\epsilon|$$

and therefore

$$\kappa^* s_{r,2} = -\frac{1}{\epsilon} \qquad \tau = (1+\epsilon)^+ \tag{35}$$

Combining Eqs. (32), (33), and (35), we obtain at $\tau = (1+\epsilon)^+$ and $\xi = \epsilon$:

$$\kappa^* s_r = 4\kappa^* - \frac{8}{3}\kappa^{*2} - 1 + O(\epsilon) \tag{36}$$

The quantity $\epsilon(\kappa^* s_r + 1)$ is shown in Fig. 4 over the range $0 \leq \tau \leq 1 + \epsilon$. In the limit, as $\epsilon \to 0$, the stress behaves as a Dirac delta function; its value at $\tau = 1^+$ is easily seen to be

$$\lim_{\epsilon \to 0} \kappa^* s_r(\epsilon, (1+\epsilon)^+) = 4\kappa^* - \frac{8}{3}\kappa^{*2} - 1 \tag{37}$$

Comparison with Eq. (26) shows this limit to be identical with the expression previously obtained for $s_r(0, 1^+)$.

The results of this section can be combined with Eqs. (25a) and (25b) to give the stress over the entire range $0 \leq \tau < 3$; thus,

$$\kappa^* s_r(0, \tau) = -\frac{2}{3}(3 - 2\kappa^*)\left[\frac{r_1^{\,2}}{r_2 - r_1}\, e^{r_1(\tau - 1)} + \frac{r_2^{\,2}}{r_1 - r_2}\, e^{r_2(\tau - 1)}\right] H(\tau - 1) + \delta(\tau - 1) - 1 \tag{38a}$$

$$= \frac{4}{3}(3 - 2\kappa^*)\kappa^* e^{\kappa^*(\tau - 1)}\left\{\cos\left[\sqrt{2\kappa^* - \kappa^{*2}}\,(\tau - 1)\right] + \frac{k^* - 1}{\sqrt{2\kappa^* - \kappa^{*2}}}\sin\left[\sqrt{2\kappa^* - \kappa^{*2}}\,(\tau - 1)\right]\right\} H(\tau - 1)$$

$$+\, \delta(\tau - 1) - 1 \tag{38b}$$

Equation (38b) for $\nu = 1/3$ ($\kappa^* = 1/2$) is plotted in Fig. 5. The stress state at $\xi = 0$ goes from homogeneous compression for $0 \leq \tau < 1$ to an infinite tensile spike at $\tau = 1$, then immediately falls to a tensile stress about 1/3 the initial compressive stress at $\tau = 1^+$. At $\tau = 3^-$ the stress is again compressive and measures three times the initial value.

Further information requires evaluation of w_3 and w_4. It is expected that a second infinite tensile spike will occur at $\tau = 3$.

Using the general expression for w_1 and w_2, Eq. (21), to obtain the solution at the center, while instructive, is tedious. A more satisfactory approach is to return to Eq. (16) and interchange the operations of inversion and $\lim_{\xi \to 0}$, i.e., to compute

$$\mathcal{L}^{-1}\left[\lim_{\xi \to 0} \bar{w}\right]$$

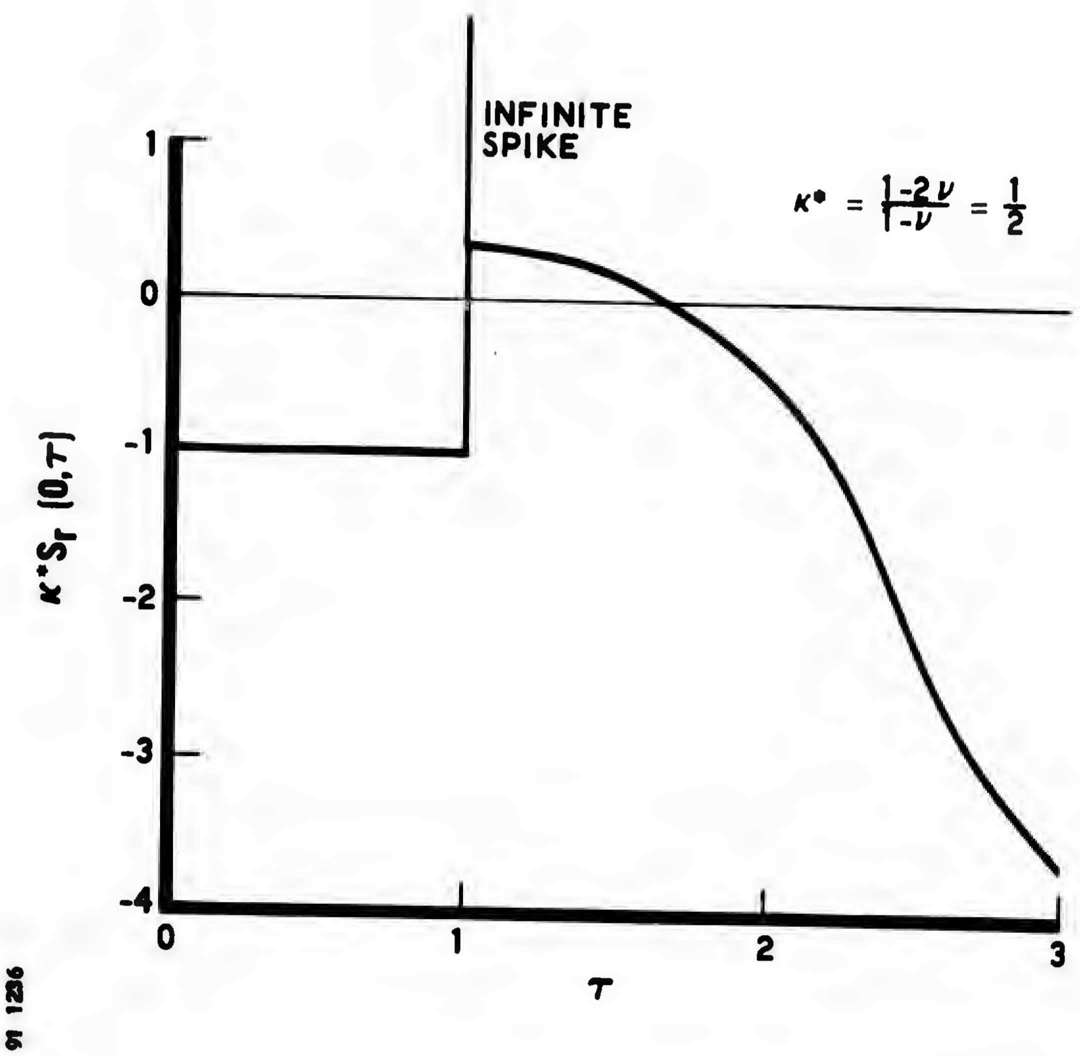

Figure 5. Radial Stress at the Origin vs Nondimensional Time τ for $\nu = 1/3$

This gives directly the complete time history at the origin. From Eq. (16) and the form of f_0 and f_1 it follows directly that

$$\lim_{\xi \to 0} \frac{\overline{w}}{\xi} = \frac{1}{3[\, p f_0(p) - 2\kappa^* f_1(p)]} = \lim_{\xi \to 0} \frac{\partial \overline{w}}{\partial \xi} \tag{39}$$

Therefore,

$$\lim_{\xi \to 0} \kappa^* \overline{s}_r = \frac{3 - 2\kappa^*}{3} \left[\frac{1}{p f_0(p) - 2\kappa^* f_1(p)} \right] - \frac{1}{p} \tag{40}$$

$$= \frac{2}{3} (3 - 2\kappa^*)\, e^{-p} \left[\frac{1}{1 - \frac{2\kappa^*}{p} + \frac{2\kappa^*}{p^2}} \right] \cdot \left[1 + \frac{Q_2}{Q_1} e^{-2p} + \left(\frac{Q_2}{Q_1} \right)^2 e^{-4p} + \ldots \right] - \frac{1}{p} \tag{41}$$

where Q_2 and Q_1 are as defined on Page 27.

The first term in the series as shown gives the stress associated with the first incoming wave and its reflection, the second with the second incoming wave and its reflection, etc., thus

$$\lim_{\xi \to 0} \kappa^* (\overline{s}_{r_{1,2}}) = \frac{2}{3} e^{-p} \left[\frac{3 - 2\kappa^*}{1 - \frac{2\kappa^*}{p} + \frac{2\kappa^*}{p^2}} \right] \tag{42}$$

To carry out the inversion, write this as

$$\lim_{\xi \to 0} \kappa^* (\overline{s}_{r_{1,2}}) = \frac{2}{3} (3 - 2\kappa^*) \left\{ e^{-p} \left[p\overline{x}(p) - 1 \right] + e^{-p} \right\} \tag{43}$$

where

$$\bar{x}(p) \equiv \frac{p}{p^2 - 2\kappa^* p + 2\kappa^*} \equiv \frac{p}{(p - r_1)(p - r_2)}$$

and r_1, r_2 are given on Page 29.

Let

$$\mathcal{L}^{-1}\left[\bar{x}(p)\right] \equiv x(\tau)$$

Then

$$x(\tau) = \frac{-r_1}{r_2 - r_1} e^{r_1 \tau} - \frac{r_2}{r_1 - r_2} e^{r_2 \tau}$$

Since

$$\lim_{\tau \to 0} x(\tau) \to 1 \qquad \text{and} \qquad \lim_{\tau \to \infty} e^{-p\tau} x(\tau) \to 0$$

Theorem II, p. 5 of Ref. 5 may be applied to give

$$\mathcal{L}^{-1}\left[p\bar{x}(p) - 1\right] = \frac{-r_1^2}{r_2 - r_1} e^{r_1 \tau} - \frac{r_2^2}{r_1 - r_2} e^{r_2 \tau}$$

Returning now to Eq. (43), applying Theorem V, p. 7 of Ref. 5, and recalling Eq. (41) for the total transform we obtain for $\tau < 3$

$$\kappa^* s_r(0, \tau) = -\frac{2}{3}(3 - 2\kappa^*)\left\{\frac{r_1^2 e^{r_1(\tau - 1)}}{r_2 - r_1} + \frac{r_2^2 e^{r_2(\tau - 1)}}{r_1 - r_2}\right\} \cdot H\left[\tau - 1\right] + \delta(\tau - 1) - 1 \tag{44}$$

which is identical with Eq. (38a).

This derivation lends itself well to an evaluation of the effect of a finite heating time, i.e., a ramp load, on the singularity at the center.

In the transform domain, solutions for a gradual temperature rise Eq. (15), are obtained from the solutions for instantaneous heating by introduction of the factor $\frac{[1 - e^{-p\tau_o}]}{p\tau_o}$. The stress transform at the center then is (Equation 41)

$$\lim_{\xi \to 0} \kappa^* \bar{s}_r = \lim_{\xi \to 0} \left[\kappa^* \bar{s}_{r_{1,2}} + \kappa^* \bar{s}_{r_{3,4}} + \dots \right] - \frac{(1 - e^{-p\tau_o})}{p^2 \tau_o} \tag{45a}$$

with

$$\kappa^* \bar{s}_{r_{1,2}} = \frac{2}{3} \frac{(3 - 2\kappa^*)}{\tau_o} \left[\frac{p}{p^2 - 2\kappa^* p + 2\kappa^*} \right] \left[e^{-p} - e^{-p(1 + \tau_o)} \right] \tag{45b}$$

For $\tau < 3$, standard inversion techniques, as used previously in this paper give

$$\kappa^* s_r(0, \tau) = -\frac{2}{3} \frac{(3 - 2\kappa^*)}{\tau_0} \left\{ \left[\frac{r_1 e^{r_1(\tau - 1)}}{r_2 - r_1} + \frac{r_2}{r_1 - r_2} e^{r_2(\tau - 1)} \right] H(\tau - 1) \right.$$

$$\left. - \left[\frac{r_1 e^{r_1(\tau - \tau_0 - 1)}}{r_2 - r_1} + \frac{r_2}{r_1 - r_2} e^{r_2(\tau - \tau_0 - 1)} \right] H(\tau - \tau_0 - 1) \right\}$$

$$- \left\{ \frac{\tau}{\tau_0} - \frac{(\tau - \tau_0)}{\tau_0} H(\tau - \tau_0) \right\} \tag{46}$$

For $\tau_o > 0$, this function remains finite and well-behaved throughout the time interval $0 \leq \tau < 3$. In the limit, as $\tau_o \to 0$, Eq. (46) reduces to Eq. (44).

The displacement also exhibits singular behavior at the center for instantaneous heating, the singularities disappearing if the heating time is finite.

Referring to Equation (29), for $\xi = \epsilon$, $\tau = 1 - \epsilon + 2\epsilon\eta$

$$w_1 = \left[2(\eta - \eta^2) + \frac{4}{3}\kappa^* (3\eta^2 - 2\eta^3) \cdot \epsilon + O(\epsilon^2) \right] H(\eta) \tag{47}$$

Similarly, it can be shown that w_2, as given by Equation (34a), can be written

$$w_2 = \left[- 2(\eta - \eta^2) + \frac{4}{3}\kappa^* (\eta - 1)^2 (1 + 2\eta) \cdot \epsilon + O(\epsilon^2) \right] H(\eta - 1) \tag{48}$$

with $\tau = 1 - \epsilon + 2\epsilon\eta$ as before.

Figure 6 shows $w_{1,2} = w_1 + w_2$ at $\xi = \epsilon$ as a function of τ. The displacement starts out at zero, rises to a maximum which is independent of ϵ and then has fallen to a value of order ϵ just as the first reflected wave arrives from the center at $\tau = 1 + \epsilon$ (i.e., $\eta = 1$). As $\epsilon \to 0$, the interval in time represented by the values $0 \leq \eta \leq 1$ shrinks and in the limit, at the center, a spike of height 1/2 appears for the displacement.

Alternatively, we can examine the displacement profile in space as the first tensile wave travels in toward the center. Instead of considering w_1 at a given point, $\xi = \epsilon$, take a given time, $\tau = 1 - \epsilon$ and write

$$\xi = \epsilon + 2\epsilon\lambda \qquad \lambda \geq 0$$

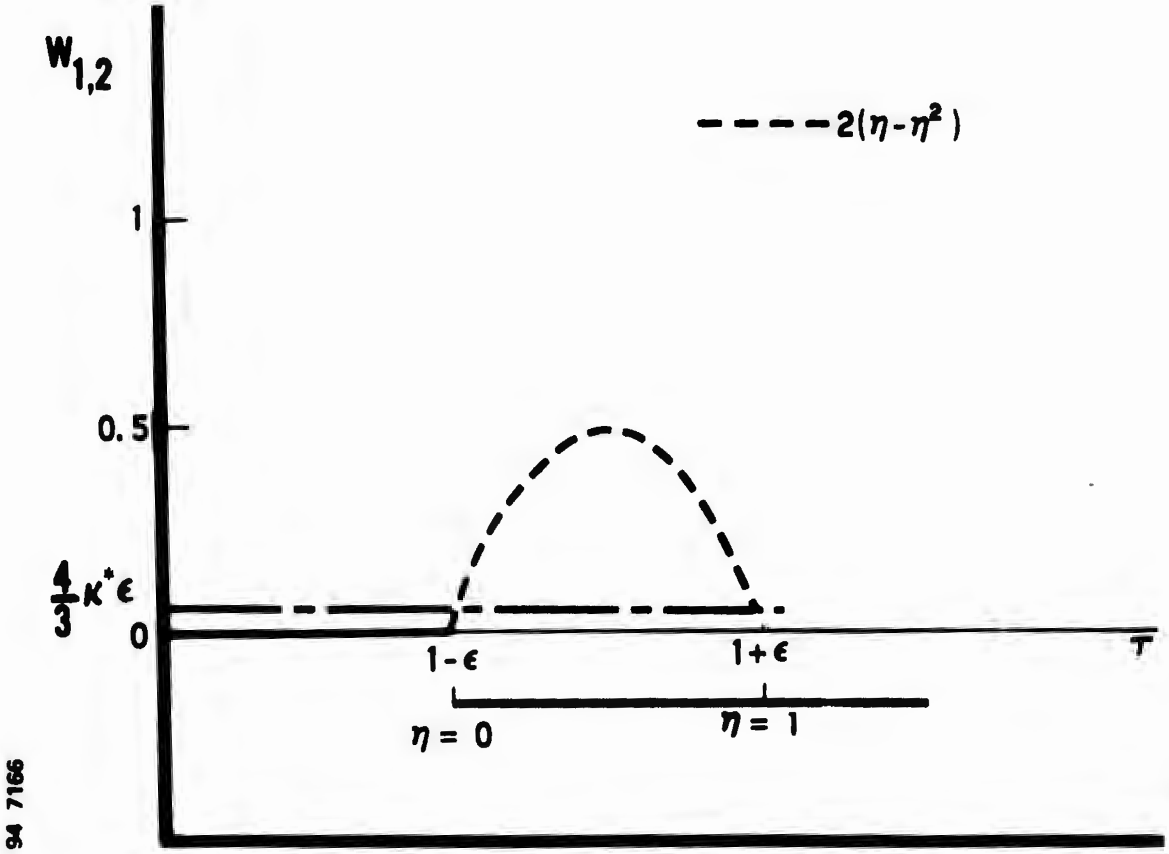

Figure 6. Nondimensional Displacement versus Nondimensional Time at a Point Close to the Origin

so that $2\epsilon\lambda$ represents distance behind the incoming wave. Substitution into Eq. (28) and expansion in ϵ gives

$$w_1 = \frac{1}{(1+2\lambda)^2} \left\{ 2(\lambda + \lambda^2) + 4\kappa^{*}\lambda^2 \left(1 + \frac{8}{3}\lambda\right)\epsilon + O(\epsilon^2) \right\} H[\lambda] \tag{49}$$

Equation (49) is shown in Fig. 7. At $\lambda = 0$, the slope of the displacement curve, as given by the derivative with respect to λ, is 2. As $\epsilon \to 0$, a given change in λ corresponds to smaller and smaller changes in ξ ($\frac{\partial}{\partial\lambda} = 2\epsilon\frac{\partial}{\partial\xi}$). Therefore, the wave front grows gradually steeper as it travels in to the center confirming the results shown in Fig. 6. These results for the displacement are consistent with the appearance of a delta function at the center for the stress.

Finally, it will be shown that when the heating is gradual ($\tau_o \neq 0$), displacements of order one do not appear near the center and the slope of the wave front remains finite as the front moves in to the center.

To obtain w_1 for a ramp load, multiply the first term in Eq. (19) by $\left[\frac{1 - e^{-p\tau_o}}{p\tau_o}\right]$ and invert, thus

$$\begin{aligned} w_1 = \frac{1}{\xi^2\tau_o} & \left\{ \frac{1 - r_1\xi}{(r_2 - r_1)r_1^2} e^{r_1[\tau - (1-\xi)]} + \frac{1 - r_2\xi}{(r_1 - r_2)r_2^2} e^{r_2[\tau - (1-\xi)]} \right. \\ & \left. - \frac{\tau}{r_1 r_2} \right\} \cdot H[\tau - (1-\xi)] - \frac{1}{\xi^2\tau_o} \left\{ \frac{1 - r_1\xi}{(r_2 - r_1)r_1^2} e^{r_1[\tau - \tau_o - (1-\xi)]} \right. \\ & \left. + \frac{1 - r_2\xi}{(r_1 - r_2)r_2^2} e^{r_2[\tau - \tau_o - (1-\xi)]} - \frac{\tau - \tau_o}{r_1 r_2} \right\} H[\tau - \tau_o - (1-\xi)] \end{aligned} \tag{50}$$

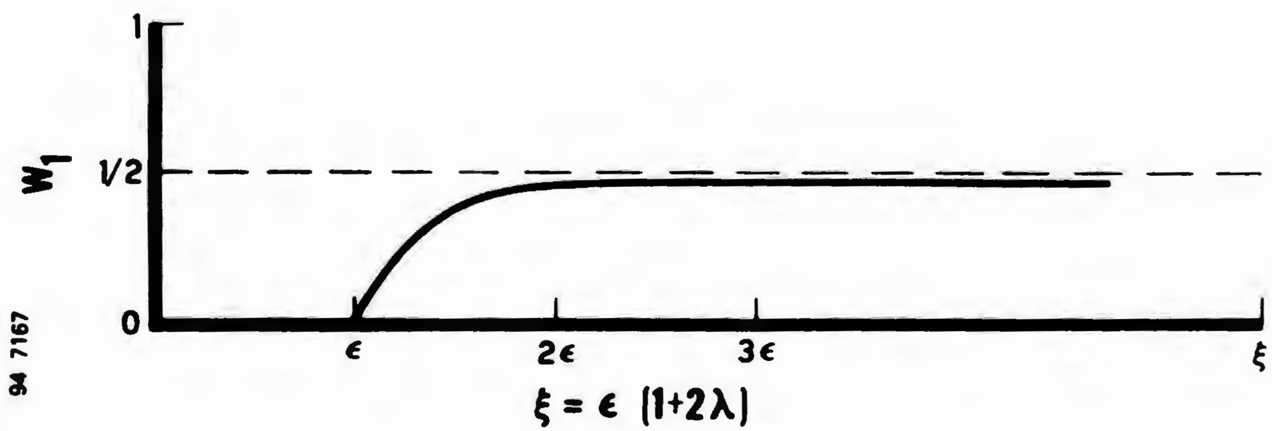

Figure 7. The Nondimensional Displacement Profile at $\tau = 1-\epsilon$

Consider again a given time, $\tau = 1 - \epsilon$, and let $\xi = \epsilon + 2\epsilon\lambda$ with $\lambda \geq 0$. Since τ_0 is fixed and non-zero and since we will wish to take ϵ arbitrarily small, take also $\tau - \tau_0 < 1 - \xi$. Substituting into Eq. (50) and expanding in ϵ we obtain

$$w_1 = \frac{1}{(1 + 2\lambda)^2 \tau_0} \left\{ \left[2\lambda^2 + \frac{8}{3}\lambda^3 \right] . \epsilon + O(\epsilon^2) \right\} H(\lambda) \qquad (51)$$

The leading term in the displacement is now proportional to ϵ. Further, the derivative $\frac{\partial w_1}{\partial \lambda}$ is also proportional to ϵ so that the slope of the wave front, given by $\frac{\partial w_1}{\partial \xi}$, remains finite as $\epsilon \to 0$.

CONCLUSION

Consider the impulse associated with the tensile stress at the origin for the first wave reflection; in nondimensional form, this impulse is called m,

$$m = \int_{1-}^{\tau_t} s_r(0^+, \tau)\, d\tau , \tag{52}$$

where τ_t is the value of τ at which s_r becomes zero (cf. Fig. 2 and Fig. 5) where $\tau_t \sim 2$ for $t_o = 0$. Substitution from Table 1 into Eq. (52) and taking the limit gives for the cylinder problem

$$(1 - \kappa)\, m_o = \lim_{\tau_o \to 0} \left\{ \int_1^{\tau_o + 1} \left[-1 + \frac{1+\kappa}{2\tau_o} (\tau^2 - 1)^{1/2} \right] d\tau \right.$$

$$\left. + \int_{\tau_o + 1}^{\tau_t} \left[-1 + \frac{1+\kappa}{2\tau_o} \left\{ (\tau^2 - 1)^{1/2} - [(\tau - \tau_o)^2 - 1]^{1/2} \right\} \right] d\tau \right\} \tag{53}$$

Let $\frac{1+\kappa}{2} (\tau^2 - 1)^{1/2} = f(\tau)$. The first integral in Eq. (53) becomes

$\lim_{\tau_o \to 0} \frac{1}{\tau_o} \int_1^{1+\tau_o} f(\tau) d\tau = 0$ and non-vanishing part of the right-hand side of

Eq. (53) thus gives for m_o:

$$(1 - \kappa)^{-1} \left\{ (1 - \tau_t) + \lim_{\tau_o \to 0} \frac{1}{\tau_o} \int_{1+\tau_o}^{\tau_t} \left[f(\tau) - f(\tau - \tau_o) \right] d\tau \right\} = m_o$$

The limit of the integral above is just

$$\int_1^{\tau_t} f'(\tau)\, d\tau = f(\tau_t)$$

and so m_o for the cylinder problem is given by

$$m_o = (1 - \kappa)^{-1} \left[1 - \tau_t + \frac{1+\kappa}{2} (\tau_t^2 - 1)^{1/2} \right] \tag{54}$$

For the spherical case, it is seen from Figure 4 that the tensile stress impulse associated with the delta function is not greater than

$$m_o = \lim_{\epsilon \to 0} \int_{1-\epsilon}^{1+\epsilon} s_r(\epsilon\, \tau) d\tau = \lim_{\epsilon \to 0} \int_{1-\epsilon}^{1+\epsilon} \left(\frac{1}{\epsilon} - 1 \right) \frac{1}{\kappa^*} d\tau$$

$$= 2/\kappa^* \tag{55}$$

There is also a finite contribution from the stresses given by Eq. (27) (cf. Fig. 5) for the time interval $1 < \tau < \tau_t$.

Hence, the stress-impulse, which is a crude measure of the dynamic fracture capability, is finite in the limit $\tau_o \to 0$ for both the cylindrical and the spherical geometry. For the limit of $\nu \to \frac{1}{2}$, which is the incompressible material, κ for the cylinder tends to $+1$ while κ^* for the sphere tends to 0. Equations (54) and (55) then show that the nondimensional stress impulse becomes infinite for both the cylinder and the sphere for $\nu = \frac{1}{2}$.

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13. ABSTRACT Stress waves, that develop following rapid uniform internal heating of linear-elastic spheres and long cylindrical rods, display a focussing effect as they proceed radially towards the center in these geometries. This effect can cause peak interior dynamic stresses in both tension and compression that are much larger in magnitude than the magnitude of the uniform compressive stress which is initially induced by constrained thermal expansion. Two fundamental problems are treated by the Heaviside wave method, one for the heating of a long cylindrical core in an infinite medium, and the other for heating of a complete sphere with zero surface traction. A simple closed-form formula for the stress at the center of the cylinder in the first problem allows determination of the effect of a finite heating time (a ramp function) on reducing the amplitude of the peak tensile stress at the center. The peak tensile stress at the center becomes infinite for the cylinder problem in the limit as heating time goes to zero. For the sphere problem, however, the stress at the center is calculated for a duration of several wave-reflection times, and the center stress is found to be finite even for the limit of zero heating time. (Unclassified Report)

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