LARGE-SCALE LINEAR PROGRAMMING

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OPERATIONS RESEARCH HOUSE

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LARGE-SCALE LINEAR PROGRAMMING

by George B. Dantzig*

Large-Scale Systems and the Computer Revolution:

From its very inception, it was envisioned that linear programming would be applied to very large, detailed models of economic and military systems. Kantorovitch's 1939 proposals, which were before the advent of the electronic computer, mentioned such possibilities, [78]. Linear programming evolved out of the U.S. Air Force interest in 1947 in finding optimal time-staged deployment plans in case of war, [126]; a problem whose mathematical structure is similar to that of finding an optimal growth pattern of a developing economy and similar to other control problems, [41], [58], [123]. Structurally the dynamic problems are characterized in discrete form by staircase matrices representing the inputs and outputs from one time period to the next. Treated as an ordinary linear program, the number of rows and columns grows in proportion to the number of time periods T and the computational effort grows by T^3 and possibly higher. This fact has limited the use of linear programming as a tool for planning over many time periods.

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At the present 1967 stage of the computer revolution, there is growing interest on the part of practical users of linear programming models to solve larger and larger systems [40]. Such applications imply that eventually automated systems will obtain information from counters and sensing devices, process data into the proper form for optimization and finally implement the results by control devices. There has been steady progress in this mechanization of flow to and from the computer. Hitherto, this has been one of the obstacles encountered in setting-up and solving large-scale systems, [113]. The second obstacle has been the cost and the time required to successfully solve large problems, [74].

It is difficult to measure the potential of large-scale planning. Certain developing countries appear, according to optimal calculations on simplified models to be able to grow at the rate of 15% per year implying a doubling of their industrial base in 5 years. But administrators apparently ignore plans and make decisions based on political expediency which restrict growth to 2 or 3% and sometimes -2%. It is the belief of the author that the mechanization of data flow (at least in advanced countries) in the next decade will provide pathways for constructing large models and the effective implementation of the results of optimization. This application of mathematics to decision processes will eventually become as important as the classical applications to physics and will, in time, change the emphasis in pure mathematics.

Three Approaches to Solving Large-Scale Systems:

There have been a great number of papers on this subject as evidenced from the list of references attached. I have broadly classified them into:

I	Decomposition Principle (Sub-optimization using interior path)
II	Compact Inverse (Using a simplex variant)
III	Parametric Variation (Sub-optimization using simplex variant)

The aim is to say a little about each, citing some references and some structures to which they are applicable. We shall begin with

I: The Decomposition Principle, [47]:

Consider the non-linear programming problem: Find $x = (x_1,\dots,x_n)$ such that

$$(1)\qquad \begin{array}{ll} g(x) = \text{Min} & \\ f_1(x) \le 0 & :\lambda_1 \\ f_2(x) \le 0 & :\lambda_2 \\ f_3(x) \le 0 & \\ \text{-------------} & \\ f_m(x) \le 0 & \end{array}$$

We assume $g(x)$ and $f_i(x)$ are convex functions of x. Assigning Lagrange Multipliers λ_i to a subset of the constraints, say the first two, we obtain the SUBPROBLEM: Find x and Min $\phi(x)$ satisfying

$$(2)\qquad \phi(x) = g(x) + \lambda_1 f_1(x) + \lambda_2 f_2(x)$$

$$f_3(x) \le 0,\dots,f_m(x) \le 0$$

Theorem: <u>If for given</u> λ_i , $x = \hat{x}$ <u>solves the subproblem</u> (2) <u>and if</u> $f_i(\hat{x}) \leq 0$ <u>for all</u> i <u>and</u> $\lambda_i f_i(\hat{x}) = 0$ <u>for</u> $i = (1,2)$ <u>then</u> $x = \hat{x}$ <u>solves</u> (1).

We shall discuss a method where we assign values to λ_i and if the resulting $x = \hat{x}$ does not satisfy the conditions in the theorem, this fact can be used to improve the values of λ_i .

(Ia) <u>Example</u>:

FIND $x \geq 0$, Min $f_o(x)$:

$$
(3) \quad \begin{array}{llllll}
c_1 x_1 + c_2 x_2 & + c_3 x_3 + c_4 x_4 & + c_5 x_5 & = f_o(x) & & \\
a_{11}x_1 + a_{12}x_2 & + a_{13}x_3 + a_{14}x_4 & + a_{15}x_5 & = b_1 & : \lambda = \lambda_1 & \boxed{\text{GUESS}} \\
a_{21}x_1 + a_{22}x_2 & & & \leq b_2 & & \\
a_{31}x_1 + a_{32}x_2 & & & \leq b_3 & & \\
& a_{43}x_3 + a_{44}x_4 & & \leq b_4 & & \text{SUBPROBLEM} \\
& a_{53}x_3 + a_{54}x_4 & & \leq b_5 & & \\
& & a_{65}x_5 & \leq b_6 & & \\
\hline
\phi_1(x_1,x_2) & + \phi_2(x_3,x_4) & + \phi_3(x_5) & = \phi(x) \text{ Min} & &
\end{array}
$$

where $\phi_1 = (c_1 + \lambda a_{11})x_1 + (c_2 + \lambda a_{12})x_2$; $\phi_2 = (c_3 + \lambda a_{13})x_3 + (c_4 + \lambda a_{14})x_4$;

$\phi_3 = (c_5 + \lambda a_{15})x_5$

Note that the subproblem decomposes into three separate problems; hence the term: "Decomposition Principle".

(Ib) <u>Equivalent Generalized Linear Program</u>:

Returning to problem (1) we now restate it in the form of

Wolfe's Generalized Linear Program, [38, Chapter 22]. This differs from an ordinary linear program in that the coefficients in each column P_j instead of being fixed are freely drawn from a convex set C_j . It can be shown that the following problem is equivalent to (1).

FIND Min z , $w_i \geq 0$ such that

$$(4)\quad \underbrace{\begin{bmatrix}0\\0\\0\\1\end{bmatrix} \geq \begin{bmatrix}-1\\0\\0\\0\end{bmatrix} z + \begin{bmatrix}g(x^1)\\f_1(x^1)\\f_2(x^1)\\1\end{bmatrix} w_1 + \dots + \begin{bmatrix}g(x^t)\\f_1(x^t)\\f_2(x^t)\\1\end{bmatrix} w_t}_{\text{RESTRICTED MASTER}} + \begin{bmatrix}g(\bar{x})\\f_1(\bar{x})\\f_2(\bar{x})\\1\end{bmatrix} w \qquad \begin{matrix}:\ 1\\:\ \lambda_1\\:\ \lambda_2\\:\ \mu\end{matrix}$$

where x^i and $\bar{x}$ satisfy $f_3(x) \leq 0, \dots, f_m(x) \leq 0$ and the solution to (1) is

$$(5)\qquad \hat{x} = \sum w_i x^i + w\bar{x}.$$

<u>(Ic) Iterative Process</u>:

At the start of iteration t , $x^1, \dots, x^t$ are known. An improved guess of (λ_1 , λ_2) and a new $x = x^{t+1}$ is obtained by solving the "restricted master" linear programming indicated in (4). Let the optimal dual variables be $(1, \lambda_1^t, \lambda_2^t, \mu^t)$ and let $w_i = w_i^t$ be the optimal primal variables. Then

$$(6)\qquad \hat{x}^t = \sum_{i=1}^{t} w_i^t x^i$$

is an optimal solution to (1) if

$$(7)\quad \text{Min } [g(\bar{x}) + \lambda_1^t f_1(\bar{x}) + \lambda_2^t f_2(\bar{x}) + \mu] \geq 0 ,$$

$$f_1(\bar{x}) \leq 0 , \qquad f_2(\bar{x}) \leq 0 ;$$

i.e. if the last column "prices out" non-negative for all admissable $\bar{x}$. But (7) is the same as solving the subproblem (2) using $(\lambda_1, \lambda_2) = (\lambda_1^t, \lambda_2^t)$. If in (7), $x = x^{t+1}$ yields a Min < 0 , this x is used to generate a new column of (4).

The successive $\hat{x}^t$ satisfy $f_i(x) \leq 0$ for all i and $g(\hat{x}^t) \rightarrow \text{Min } g(x)$. The iterative process is finite when applied to a linear program like the preceeding example (3).

This completes our discussion of the decomposition approach. To be useful, the generated subproblems must be <u>easy</u> to solve, [38,Chapter 24].

<u>(II) Compact Inverse</u>:

The second approach accepts the standard simplex or any of the numerous variants and tries to arrange the arithmetic to take advantage of structure. It is clear that if the number of iterations is fixed, the only savings can come from doing each iteration efficiently: i.e. doing the pricing and those operations involving the inverse of the basis efficiently.

<u>(IIa) Sparse Matrices</u>:

The larger problems become the lower, in practice, become the density of non-zero coefficients. For problems of 200 equations a density of 5% is typical; for larger problems the density drops to .5% or less. It is possible, however, that the inverses of bases drawn from such matrices to be 100% dense, for example:

$$(8)\quad B = \begin{bmatrix} 4 & 1 & 1 & 1 \\ 1 & 1 & & \\ 1 & & 1 & \\ 1 & & & 1 \end{bmatrix} ; \; B^{-1} = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 2 & 1 & 1 \\ -1 & 1 & 2 & 1 \\ -1 & 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ -1 & & 1 & \\ -1 & & & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 & -1 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

However, if the inverse is expressed as products of elementary matrices of either the row or column type or both in any order, the number of off-diagonal non-zeros in this representation can often be quite low.

Unsolved problem: Given a basis express the inverse as the product of elementary matrices such that the count of off-diagonal non-zero elements is minimal.

Markowitz [90] proposed that the elementary matrices correspond to upper and lower diagonalization operations using as pivot element the one that locally creates as few additional non-zeros as possible. Variations of this idea have been incorporated in commercial codes in the early 1960's, see [43]. The inverse of a 5% dense basis often running not more than 7% dense and the running time often is cut by a factor of 5. In example (8), the inverse in product form has the same number of non-zeros as the originating basis.

(IIb) Dynamic Structures:

As noted earlier these have important applications [95]. One such is to linear control processes, see [114], [128]. As early as 1954, the author published a paper on how to compact the inverse representation of the basis with a staircase structure, (9); see [32].

Again, in 1963, I discussed another method which also permitted one to find a compact inverse and efficiently maintain the compactness in moving from one iteration to the next, [37]. There have been other proposals, all excellent, that seek to apply the simplex method to the full system by compacting the inverse. As far as I know, none of these <u>direct</u> proposals have been realized in computer codes. See [5], [56], [71].

(9)

A_1 1st period input			
$-\bar{A}_1$ 1st period output	A_2 2nd period input		
	$-\bar{A}_2$ 2nd period output	A_3 3rd period input	
		$-\bar{A}_3$ 3rd period output	A_4 4th period input
			$-\bar{A}_4$ 4th period output

An important special case is the <u>Dynamic Leontief Economic Model with Substitution</u> [33]. Another Special Case is a <u>Markov Process with Alternative Policies</u> [125], [76]. These cases are known to be mathematically equivalent and to have a remarkably simple solution. A Leontief System is defined by: (1) a non-negative right hand side,

(2) exactly one positive coefficient in each column, and (3) the existence of a feasible solution for some positive right hand side. In the dynamic case, we further assume that the positive coefficient always appears in the input block along the diagonal.

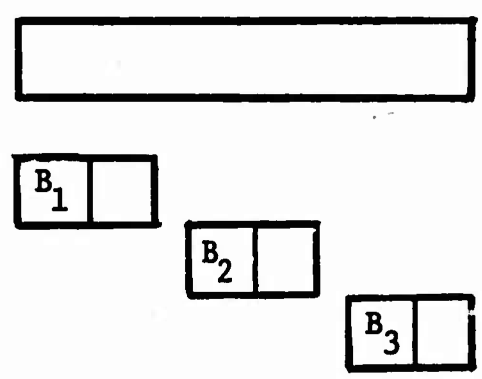
Theorem: The optimal choice of basic columns associated with the last period is independent of the choice in prior periods.

This permits the determining of optimal basis and Lagrange multipliers for the last block of equations. Weighting these equations by their multipliers, the last period equations are subtracted from the cost equations to produce a modified-cost equation. Dropping these equations, the optimal choice of columns for the next-to-last period and prices are next determined using the modified-cost equation; etc. backwards in time until the first period is reached. When the basic columns of the first period become known, the value of its basic variables can be calculated, these in turn can be used to determine those of the second period, etc. forward in time. [118].

The essential characteristic of the basis in the dynamic Leontief case and in the Markov Process case is that the blocks of non-zero coefficients are square and non-singular and the entire basis is block triangular. Hence only the inverses of blocks along the diagonal are needed; the rest of the calculations can be done by substitution below the diagonal. An ideal block-triangular structure! Unfortunately, the general staircase problem does not have this property. It would be very worthwhile to see if one can find a meaningful economic extension of the Leontief model (like the introduction of activities that generate capital) that is tractable.

(IIc) Block Angular Systems: These consist of M general linear equations and L sets of equations which have no variable in common. The blocks of non-zero coefficients are depicted below.

(10)



Several proposals have been made to compact the inverse, see particularly Bennett [21]; also [79], [106]. Essentially they all chose square non-singular submatrices B_i from the basis along the block diagonal which are used as block pivots to initiate the elimination. After the elimination, a square $m \times m$ submatrix is left. Many practical problems satisfy this structure. One important subclass are the multi-commodity network problems, [54], [77], sometimes referred to as the traffic assignment problem [24]:

(11) Find $x_{ijk} \geq 0$, Min z :

$$\sum_k x_{ijk} \leq c_{ij} \qquad (i,j,k) = 1,\ldots,n.$$

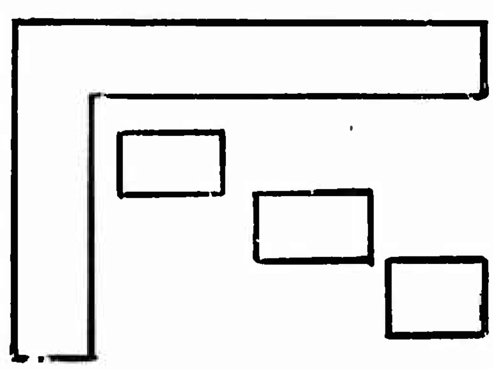
$$\sum_i x_{ipk} - \sum_j x_{pjk} = a_{pk} \qquad (p,k) = 1,\ldots,n$$

$$\sum_i \sum_j \sum_k t_{ijk} x_{ijk} = z$$

In another type of application involving the allocation of many orders to several plants, the diagonal blocks consist of one equation each. Such a system is referred to as a <u>generalized upper-bound structure</u>, [46]. In one application $L = 4000$ and $M = 20$. An important property of such systems is that when L is large relative to M most (in fact $L-M$ or more) of the diagonal equations have exactly one basic variable among the set of its variables. The fact that most basic variables are at their upper-bound value can be used to advantage. The first code along these lines was developed by M. Kasatkin and J. Bigelow for a problem of Crown Zellerbach paper corporation. Running time on an example was about 1/10 the time that was required by a general purpose code. See also [65].

<u>(IId) Bordered Angular Systems</u>: This consists of blocks along the diagonal of non-zero coefficients and a border of non-zeros along the top and left.

(12)



This structure is sufficiently general yet specialized to usefully cover a majority of current applications except the staircase type. Generalization (of the procedures just discussed) have been made by Heesterman, [72]. Ritter [99] has generalized Rosen's parametric scheme, [103].

III. Parametric Variation:

The third and last approach depends on the system being weakly linked i.e. on the existence of a few rows and columns which, if removed, makes the solution of the remaining system trivial. For example, a network-flow problem with an extra budget constraint. By assigning a Lagrange Multiplier to the latter, the constraint could be removed and the objective equation modified by adding to it the multiple of the removed constraint. The resulting pure network could then be easily solved. If the solution does not satisfy the constraint and complementary slackness conditions, then the Lagrange Multiplier is varied until it does. This is also the idea behind the decomposition principle but the proposed methods of variation (such as those below) are more direct:

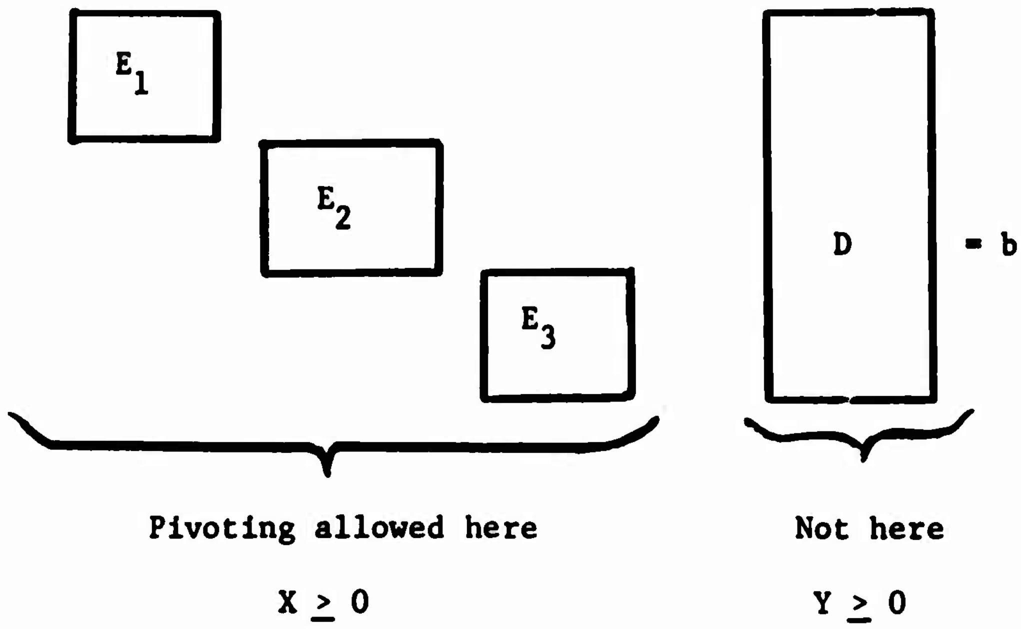
Rosen: "Partition Programming" [103], Ritter [99].
Kron: "Diakoptics" [83].
Balas: "Infeasibility Pricing" [10].
Beale: "Pseudo Basic Variables" [17].
Abadie & Williams: [3].
Gass: "Dualplex Method" [59].

(IIIa) Dualplex Method:

As representative of the parametric approaches I have selected

Gass' "Dualplex Method" which is related to Rosen's "Partition Programming" in dual form. It is clear if we had a transposed block-angular structure

(13)



that pivoting in the right hand interconnecting part would destroy the angular structure but pivoting anywhere in E_1, E_2, E_3 would not. We assume that for a given $Y = Y^o \geq 0$ (variables associated with D) a feasible solution $X = X^o \geq 0$ exists and is optimal. Let the system be reduced to optimal canonical form restricting pivots to only columns of E_i :

(14)
$$\begin{aligned} IX_B + \bar{A}X_N + \bar{D}Y &= \bar{b} \\ \bar{c}X_N + \bar{d}Y &= z - Z_o \ \text{(Max)} \end{aligned}$$

where X_B are basic variables and X_N, Y non-basic. Holding $X_N = 0$ for the moment, we solve the subproblem

(15)
$$\bar{D}Y \leq \bar{b} , \quad Y \geq 0 , \quad \text{Max } \bar{d}Y .$$

The dual of this subproblem is

$$\text{(16)} \qquad \Pi\bar{D} \geq \bar{d} \; , \quad \Pi \geq 0 \; , \quad \text{Min } \Pi\bar{b} \; .$$

Since $\bar{D}^T$ is presumed to consist of few rows and many columns, it is suitable for solution by the standard simplex method. Let $\Pi = \Pi^1$ be an optimal solution and $Y = Y^1 \geq 0$ be optimal to its dual. Denote by $\bar{D}_i$ the i-th row of $\bar{D}$ and by $\bar{A}_j$ the j-th column of A. Let the basic X_B be partitioned into $X_I = 0$ and $X_{II} > 0$ according as components $x_i = 0$ or $x_i > 0$ where $x_i + \bar{D}_i Y' = \bar{b}_i$; Let the non-basic X_N be partitioned into X_{III} and X_{IV} according as $\delta_j = \bar{c}_j - \Pi\bar{A}_j > 0$ or ≤ 0 .

$$\text{(17)} \qquad \begin{array}{cc|cc|c|c|c|l}
 & & & & \delta_{III} > 0 & \delta_{IV} \leq 0 & & \\
X_I = 0 & & X_{II} > 0 & & X_{III} = 0 & X_{IV} = 0 & Y' \geq 0 & \\
\hline
1 & & & & \boxed{\begin{array}{l}\text{Block}\\ \text{Pivot}\end{array}} & & & = \bar{b} \\
 & 1 & & & & & & \\
\hdashline
 & & 1 & & & & & \\
 & & & & & \bar{A} & \bar{D} & \\
 & & & 1 & & & & \\
 & & & & \bar{c} & & & = z - z_o \text{ (Max)}
\end{array}$$

The block pivot:

The next step is to find the block pivot of highest rank that switches the role of as many basic and nonbasic variables in X_I and X_{III} as possible. Since both sets are at zero value this does not effect the current feasible solution. If there is a choice of block pivot its columns are selected from those with highest δ_j values.

After the pivot the new dual subproblem is solved using as starting basis, the one corresponding to the final basis of the previous subproblem. $Y' \geq 0$ is still a feasible price vector of the dual subproblem but Π' no longer satisfies it. However,

Theorem (Gass):

<u>If after the block pivot those components</u> Π'_j <u>of</u> Π' <u>corresponding to</u> $\delta_i > 0$ <u>are replaced by the value</u> $-\delta_i$, <u>the new</u> Π <u>constitutes an infeasible basic solution to the new subproblem;</u> $Y' \geq 0$ <u>remains as a feasible vector of dual simplex multipliers.</u>

Because of infeasibility, the new subproblem can be improved (using the dual simplex method). This is repeated iteratively until all $\delta_j \leq 0$ or $z \to +\infty$. Associated with each iteration is a basic feasible to the full problem so that usual proof of a finite-number-of-iterations applies.

The parametric methods should be regarded as important variants of the standard simplex process.

<u>Concluding Remarks</u>:

This completes the survey of the three types of approaches to solving large-scale systems: Decomposition, Compact Inverse, and Parametric Variation, and of the type of matrix structures that each are best suited. Little has been said about how different proposals compare on test problems. At present, there does not appear to be a practical way to do this. The program of instructions for the computer are often an order of magnitude more complex than a good commercial linear program system and the latter can cost two to five hundred thousand dollars to develop. The author feels that better computer languages have to be developed to facilitate the experimental coding and comparision of large-scale system proposals, [74].

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13. ABSTRACT

From its inception Linear Programming was envisioned as being applied to large detailed dynamic models of economic and industrial systems. Difficulties of obtaining input data, making use of detailed output data, and the cost of computation have in the past limited applications. Three types of approaches have been proposed for efficient computation. These are reviewed in terms of typical matrix structures to which they are applicable. A list of 128 references is appended.

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