

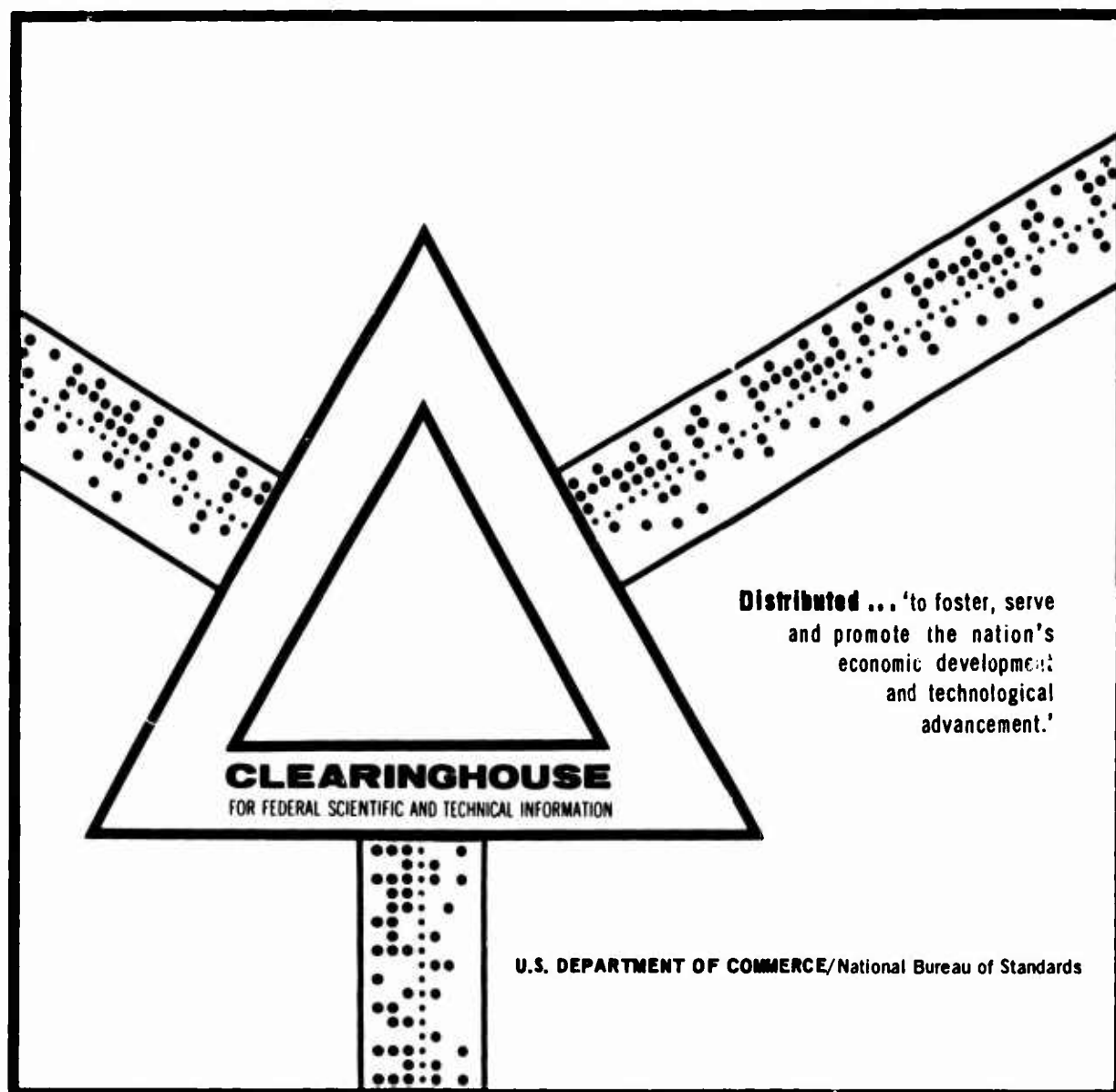
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RENEWAL REWARD PROCESSES

Mark Brown, et al

California University
Berkeley, California

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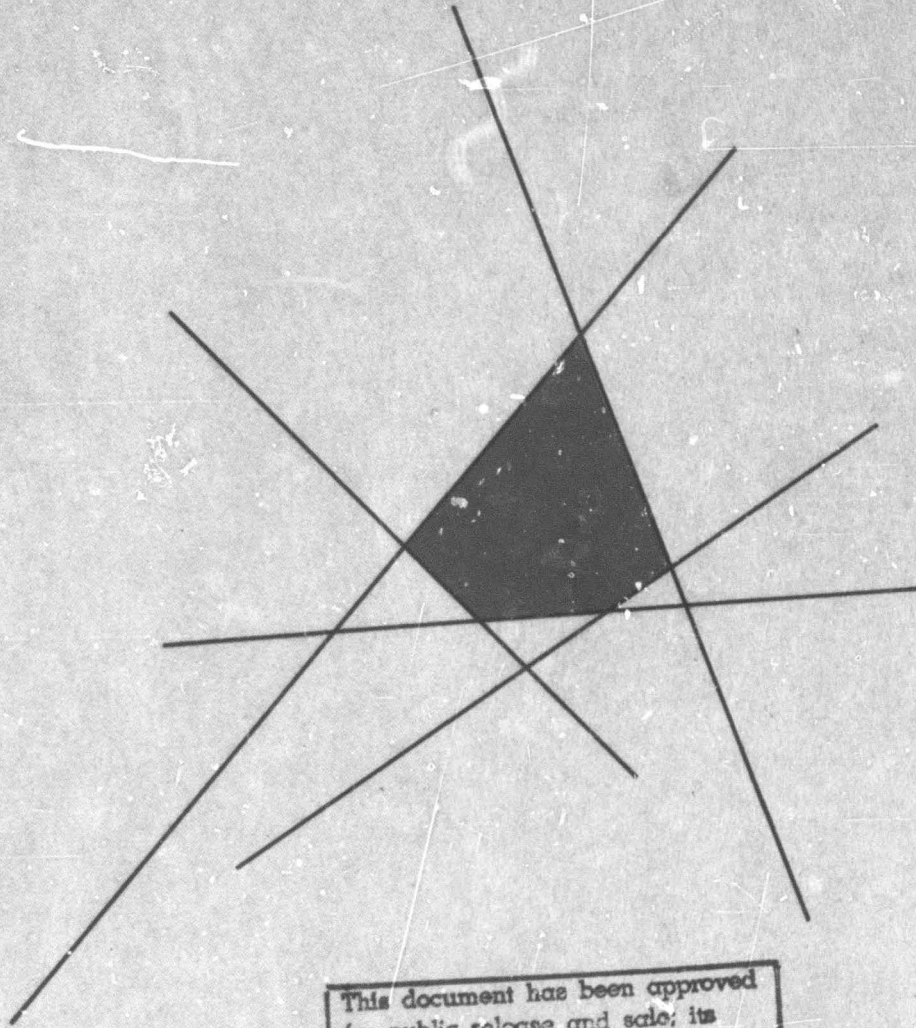


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MARK BROWN
and
SHELDON M. ROSS

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RENEWAL REWARD PROCESSES

by

Mark Brown
Department of Operations Research
Cornell University
Ithaca, New York

and

Sheldon M. Ross[†]
Department of Industrial Engineering
and Operations Research
University of California, Berkeley

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ABSTRACT

We consider a process in which rewards are being earned and for which there exist time points at which the process begins anew. That is, we ~~suppose~~ that there exists an embedded renewal process. An expression for the asymptotic mean reward earned during any time interval is then obtained. In the final section we ~~consider~~ the special case of a regenerative reward process, and we ~~present~~ a simple expression for the long run average reward earned per unit time.

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0. INTRODUCTION

Let $X_1, X_2, \dots$, be the interarrival times for a renewal process with interarrival distribution F . Suppose that at the time of the i th renewal we receive a reward Y_i . Y_i may depend on X_i , but it is assumed that the pairs (X_i, Y_i) , $i = 1, 2, \dots$, are independent and identically distributed. If we let

$$Y(t) = \sum_{i=1}^{N(t)} Y_i,$$

where $N(t)$ is the number of renewals by time t , then $Y(t)$ represents the total reward earned by time t . The stochastic process $\{Y(t), t \geq 0\}$ is called a renewal reward process. In the first section of this paper, we will prove the analogue of Blackwell's theorem for a renewal reward process. In Section 3 we will consider processes of the form $Y(t) = \int_0^t V(s)ds$, where V is a real-valued regenerative process. An important result of Smith [5], p. 262, asserts under mild conditions that $\frac{Y(t)}{t}$ converges a.s. and in expectation to κ_1/μ_1 , where κ_1 is the expected value of the integral of V over a regenerative cycle and μ_1 is the expected length of the regeneration cycle. Our result is that $\kappa_1/\mu_1 = EV(\infty)$, where $V(\infty)$ is the limiting distribution of $V(t)$.

1. BLACKWELL'S THEOREM FOR RENEWAL REWARD PROCESSES

The following proposition is well known:

Proposition 1:

If either EY_1 or EX_1 is finite, then

$$(i) \quad \lim_{t \rightarrow \infty} \frac{Y(t)}{t} = \frac{EY_1}{EX_1} \text{ with probability 1, and}$$

$$(ii) \quad \lim_{t \rightarrow \infty} \frac{EY(t)}{t} = \frac{EY_1}{EX_1}.$$

A proof, based on a Tauberian theorem, is given by Johns and Miller [2] and credited to Bell. The proposition also follows from a more general result of Smith [5]. Part (i) of the above is clearly the analogue of the elementary renewal theorem. We shall now prove the analogue of Blackwell's theorem.

Theorem 1:

If $EY_1 < \infty$, F is not lattice, and $EX_1Y_1 < \infty$, then

$$\lim_{t \rightarrow \infty} E[Y(t+h) - Y(t)] = h \frac{EY_1}{EX_1} \text{ for all } h \geq 0.$$

Proof:

Let $m(t) = E[N(t)]$. Now,

$$\begin{aligned} (1) \quad E[Y(t)] &= E\left[\sum_{i=1}^{N(t)+1} Y_i\right] - E[Y_{N(t)+1}] \\ &= (m(t) + 1)EY_1 - E[Y_{N(t)+1}] \end{aligned}$$

where the last identity follows from Wald's equation. Hence,

$$E[Y(t+h) - Y(t)] = (m(t+h) - m(t))EY_1 - E[Y_{N(t+h)+1} - Y_{N(t)+1}] ,$$

and the result would follow from Blackwell's theorem if we can show that

$\lim_{t \rightarrow \infty} E[Y_{N(t)+1}]$ exists and is finite. Toward this end, let $g(t) = E[Y_{N(t)+1}]$.

Then

$$g(t) = \int_0^{\infty} E[Y_{N(t)+1} | X_1 = x] dF(x) \\ \int_t^{\infty} E[Y_1 | X_1 = x] dF(x) + \int_0^t g(t-x) dF(x) .$$

This renewal type equation has the solution

$$(2) \quad g(t) = h(t) + \int_0^t h(t-x) dm(x) ,$$

where

$$h(t) = \int_t^{\infty} E[Y_1 | X_1 = x] dF(x) .$$

Suppose now that all rewards are nonnegative; then by the key renewal theorem, it follows that

$$\lim_{t \rightarrow \infty} g(t) = \frac{\int_0^{\infty} h(t) dt}{EX_1} = \frac{\int_0^{\infty} \int_t^{\infty} E[Y_1 | X_1 = x] dF(x) dt}{EX_1} \\ = \frac{\int_0^{\infty} x E[Y_1 | X_1 = x] dF(x)}{EX_1} = \frac{EX_1 Y_1}{EX_1}$$

where the interchange of integrals is justified by the nonnegativity of rewards. In the general case, the result may be proven by breaking up the rewards into their positive and negative parts and applying the above argument separately to each.

Remark 1:

The proof of Theorem 1 may also be used to prove (ii) of Proposition 1. This is done in the following manner: Assume first that $EY_1 < \infty$. Then, from Equation (1) and the elementary renewal theorem, it follows that (ii) holds if $\frac{g(t)}{t} \rightarrow 0$. This, however, easily follows from (2), the assumption that $EY_1 < \infty$, and the elementary renewal theorem. If $EY_1 = \infty$ (but $EX_1 < \infty$), then the result follows by truncation.

In the above, we have assumed that the rewards are earned at the end of the renewal intervals. However, in many applications the rewards (or costs) are earned gradually during the renewal intervals. For instance, in an inventory model for which an (s, S) policy is employed, the costs are gradually incurred during the renewal cycle. In order to generalize Theorem 1 to include this possibility, let $W(s)$ denote the expected reward earned during the first s time units of a renewal interval of length greater than s . Then, the expected reward earned by t , $EY(t)$ will be given by

$$E[Y(t)] = E\left[\sum_{i=1}^{N(t)} Y_i\right] + E[W(Z(t))]$$

where

$$Z(t) = t - \sum_{i=1}^{N(t)} X_i$$

is the age of the renewal process at time t . Let

$$F_t(a) = \begin{cases} F(a) - \int_0^{t-a} (1 - F(t-y)) d\mu(y), & a \leq t \\ 1, & a > t. \end{cases}$$

It is well known that F_t is the distribution of Z_t .

Following Smith [6], p. 11, define G to be the class of all distributions F on $[0, \infty)$ having the property that for some K , the K th iterated convolution of F with itself has an absolutely continuous component.

Theorem 2:

- (i) If F is nonlattice, $EY_1 < \infty$, $EX_1 < \infty$, $EX_1 Y_1 < \infty$, then if W is continuous and uniformly integrable with respect to the family $\{F_t, t > 0\}$, then

$$(3) \quad \lim_{t \rightarrow \infty} E[Y(t+h) - Y(t)] = h \frac{EY_1}{EX_1}$$

- (ii) Under the conditions of (i) but with $F \in G$, (3) holds iff W is uniformly integrable with respect to $\{F_t, t > 0\}$ (W need not be continuous).

Proof:

- (i) It follows from Smith [5], p. 259 condition B, that $Z(t)$ converges in distribution to a random variable with c.d.f.

$$F_e(a) = \frac{\int_0^a (1 - F(x)) dx}{EX_1}.$$

Thus $E(W(Z(t)))$ converges to $\int_0^\infty W(x) dF_e(x)$ by the Helly-Bray theory

(Loève [3], p. 183). Hence, the result follows from Theorem 1.

(11) It follows from Smith [5], p. 259 condition C, that $P(Z_t \in A)$ converges for all Borel sets A to $F_e(A) = \int_A (1 - F(x)) dx / EX_1$.

$$\text{Let } W^a(x) = \begin{cases} W(x), & |W(x)| \leq a \\ 0 & |W(x)| > a. \end{cases}$$

Let $W^{a,\delta}$ be a simple function having the property that $\sup_x |W^a(x) - W^{a,\delta}(x)| < \delta$.

$W^{a,\delta}$ can be chosen by choosing $\frac{1}{n} < \delta$ and letting $W^{a,\delta}(x) = \frac{1}{n}$ when

$\frac{1}{n} \leq W(x) < \frac{1+\delta}{n}$, $1 = -\frac{a}{n} \dots \frac{a}{n}$. Note that the strong convergence in distribution

implies that $E_{F_t} W^{a,\delta} \rightarrow E_{F_e} W^{a,\delta}$ for all a, δ . The result now follows by:

$$\begin{aligned} |E_{F_e} W - E_{F_t} W| &\leq |E_{F_e} W - E_{F_e} W^a| + |E_{F_e} W^a - E_{F_e} W^{a,\delta}| + |E_{F_e} W^{a,\delta} - E_{F_t} W^{a,\delta}| \\ &\quad + |E_{F_t} W^{a,\delta} - E_{F_t} W^a| + |E_{F_t} W^a - E_{F_t} W|. \end{aligned}$$

Now the 1st and 5th terms on the right go to 0 uniformly in t as $a \rightarrow \infty$, by assumption. The 2nd and 4th terms on the right go to 0 uniformly in t , for fixed a , as $\delta \rightarrow 0$. The 3rd term goes to 0 as $t \rightarrow \infty$ for fixed a, δ . Thus, by first choosing a sufficiently large, then fixing a and choosing δ sufficiently small, and then fixing a, δ and choosing t sufficiently large, we can make the right-side smaller than any preassigned $\epsilon > 0$.

The necessity of uniform integrability follows from an argument in Loève [3], p. 183.

2. REGENERATIVE REWARD PROCESSES

Let $\{V(t), t \geq 0\}$ be a regenerative process [5], p. 256, with imbedded generalized renewal sequence $\{X_i, i \geq 0\}$. By generalized renewal sequence we mean that X_0 is independent of the i.i.d. sequence $\{X_i, i > 0\}$ but may have a different distribution. The random elements $V(t)$ take values in an abstract measurable space $(F, \mathcal{A})$. If F , the distribution of X_1 , belongs to G and if $\mu_1 = EX_1 < \infty$, then it follows from Smith [5], p. 259, that:

$$(4) \quad \Pr(X_t \in A) = \frac{1}{\mu_1} \int_0^t \Pr(V_s \in A, X_0 > s \mid \text{renewal at } 0) ds \\ = \mu_-(A) \quad \text{for all } A \in \mathcal{A}.$$

It follows from Fubini's theorem that μ_- is a probability measure on $(F, \mathcal{A})$.

If in addition V is a real valued process with a measurable modification then it follows from Smith [5], p. 262, that $\frac{1}{t} \int_0^t V(s) ds$ converges a.s. and in

expectation to κ_1/μ_1 , where $\kappa_1 = E \int_{X_0}^{X_0+X_1} V(s) ds$ (assuming κ_1 exists). A natural question to pose is whether or not $\kappa_1/\mu_1 = E(V(-)) = \int_{-\infty}^{\infty} x d\mu_-(x)$.

We will show that this is the case.

It will be convenient to convert the imbedded renewal process $\{X_i, i \geq 0\}$ into a stationary renewal process. This can be done by inserting a renewal to the left of 0, its distance from 0 having the same distribution as the limiting distribution of $Z(t)$, the age of the renewal process at time t , discussed in Section 1. Formally, we let $\{X'_i, i = 0, \pm 1, \dots\}$ be a doubly infinite sequence of i.i.d. random variables distributed as X_1 . Let $T_n = X'_0 + \sum_{i=1}^n X'_i$ for $n \geq 0$, $T_n = X'_0 - \sum_{i=1}^{-n} X'_{-i}$ for $n < 0$. Then $\{T_n, n = 0, \pm 1, \dots\}$ generates a strictly stationary renewal process on $(-\infty, \infty)$ (see [1], p. 162).

Start the regenerative process V at the first T_1 point to the left of 0. Call this point T^0 and call the resulting regenerative process V' . Now

$$(5) \quad \begin{aligned} \Pr(V'_0 \in A) &= E[\Pr(V'_0 \in A \mid T^0)] \\ &= \frac{1}{\mu_1} \int_{-\infty}^0 \Pr(V'_0 \in A \mid T^0 = -t) (1 - F(t)) dt. \end{aligned}$$

But

$$(6) \quad \Pr(V_t \in A, X_0 > t \mid \text{renewal at } 0) = \Pr(V_t \in A \mid X_0 > t, \text{renewal at } 0).$$

$$\begin{aligned} \Pr(X_0 > t \mid \text{renewal at } 0) &= (1 - F(t)) \Pr(V_t \in A \mid Z(t) = t) \\ &= (1 - F(t)) \Pr(V'_0 \in A \mid T^0 = -t). \end{aligned}$$

Thus, from (4), (5), (6)

$$(7) \quad \Pr(V'_0 \in A) = \Pr(V(\infty) \in A) = \mu_\infty(A).$$

Assume that V has a measurable modification and that $E|V'_0| < \infty$. This implies that $Y(t) = \int_0^t V'(s) ds$ exists a.s. for all t . Start V with a renewal at time 0 (thus X_0 has same distribution as X_1) and call the resulting process V'' . Define:

$$W(t) = \begin{cases} V''(t), & t < X_0 \\ 0, & t \geq X_0. \end{cases}$$

Then $\int_0^{X_0} |V''(s)| ds = \int_0^\infty |W(s)| ds$, possibly infinite. Now

$$\begin{aligned}
\int_0^{\infty} E|W(s)|ds &= \int_0^{\infty} E(|V''(s)| \mid X_0 > s)(1 - F(s))ds \\
&= \int_0^{\infty} E(|V(s)| \mid Z(s) = s)(1 - F(s))ds \\
&= \int_0^{\infty} E(|V'(0)| \mid T^0 = -s)(1 - F(s))ds = \mu_1 E|V'(0)| < \infty.
\end{aligned}$$

Thus, we have proved:

Theorem 3:

Let $\{V(t), t \geq 0\}$ be a regenerative process with a measurable modification and such that $F \in G$, $\mu_1 < \infty$. Then EV'_0 exists iff κ_1 exists and

$$\frac{\kappa_1}{\mu_1} = EV'_0 = EV_{\infty}.$$

Comments:

1. Regenerative reward processes (real-valued regenerative processes) arise frequently in queuing theory. They are often of the form $V(t) = W(S(t))$, where W is a real-valued function, and $S(t)$ an abstract valued regenerative process. For example in an $M/G/s$ queue with $\frac{\mu_G}{\lambda} < s$, the imbedded renewal sequence consists of epochs at which busy periods begin (the interarrival times satisfy $\mu_1 < \infty$, $F \in G$) $S(t)$ consists of the number of customers in service at time t with their arrival times, and the number of customers in the queue, and $W(S(t))$ may be the number of customers in service, or the number in the queue, or the unit cost of the service system for handling the number of customers present, or an indicator variable

$$W(S(t)) = \begin{cases} 1 & \text{if number in queue} = k \\ 0 & \text{otherwise} \end{cases}, \text{ etc.}$$

Assume that S is a regenerative process with arbitrary state space $(F, \mathcal{A})$ and jointly measurable as a map from $(\Omega, \mathcal{C}) \times (R, \mathcal{B})$ to $(F, \mathcal{A})$. Here $(\Omega, \mathcal{C}, P)$ is the probability space on which each $S(t)$ is defined, R the real line and $\mathcal{B}$ the Borel sets. If W is a Borel measurable real-valued function then $\{V(t) = W(S(t)), t \geq 0\}$ is a real-valued measurable regenerative process. If $\mu_1 < \infty$, $F \in \mathcal{G}$, then since $\Pr(S(t) \in A) \rightarrow \Pr(S(\infty) \in A)$ for all $A \in \mathcal{A}$, it follows that $\Pr(W(S(t)) \in B) \rightarrow (\mu_\infty W^{-1})(B) = \Pr(W(S(\infty)) \in B)$, for all Borel sets. Thus, if $E|W(S(\infty))| < \infty$ then it follows from Theorem 3 that:

$$(8) \quad \frac{1}{t} \int_0^t W(S(x)) dx \rightarrow E(W(S(\infty)))$$

a.s. and in expectation. Note also that if $E|\frac{1}{t} \int_0^t W(S(x)) dx|^{p_0} \rightarrow E|W(S(\infty))|^{p_0}$ for some $p_0 > 1$, then $\frac{1}{t} \int_0^t W(S(x)) dx \rightarrow E(W(S(\infty)))$ in L^p , for $0 \leq p \leq p_0$.

2. If $EV(t)$ converges then $EV(\infty)$ must be its limit, since $EV(\infty) = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t E(V(s)) ds$. In this case Theorem 4 is trivial. However, $E(V(t))$ may not converge and Theorem 4 may still hold. For example, start with a renewal at time 0 and let the interarrival time c.d.f. $F \in \mathcal{G}$, have an atom at 1. Choose a regenerative process V so that $E(V(t) | Z(t) = 1/2) = \infty$, $E(V(t) | Z(t) > 0) = 0$. Then clearly $EV(n + \frac{1}{2}) = \infty$ for all integers n , but $EV(\infty) = 0$. A necessary and sufficient condition for convergence of $EV(t)$ to $EV(\infty)$ is uniform integrability of $g(s) = E(V(t) | Z(t) = s)$ with respect to the family $\{F_t, t \geq 0\}$, discussed in Section 1.

3. Also note that if $F \neq G$ but Smith's alternative conditions [5], p. 259 hold, so that $\Pr(V_t \in B) \rightarrow \Pr(V_\infty \in B)$ for all Borel sets, then Theorem 5 still applies. If $V(t)$ does not have a limiting distribution, it still holds that if $\mu_1 < \infty$, V has a measurable modification and $E|V'(0)| < \infty$, then

$$\frac{1}{t} \int_0^t V(s) ds \rightarrow \kappa_1 / \mu_1 = EV'(0), \text{ a.s. and in expectation.}$$

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