RITTER'S CUTTING PLANE METHOD
FOR NONCONVEX QUADRATIC PROGRAMMING

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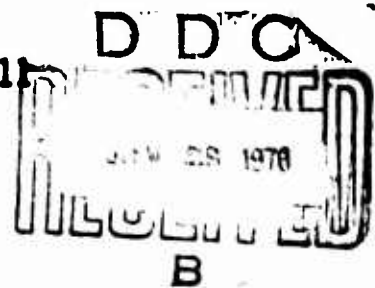R. W. COTTLE
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1. INTRODUCTION

Our purpose here is to review the method of K. RITTER (1964), (1965), (1966) for obtaining a global solution to a linearly-constrained quadratic minimization problem. In writing this primarily tutorial paper, we have made some modifications and introduced an example that we hope will contribute to a better understanding of an important algorithm. As a matter of fact, Ritter's method is the only <u>rigorous</u> procedure we know of for solving the general quadratic programming problem. Other algorithms exist, but they fail to fit this description because of their logical gaps or their less ambitious nature. Examples of the latter type are the frequently advanced proposals for obtaining a local minimum of a concave quadratic function over a polyhedral convex set.

The problem with which we are concerned can be stated without loss of generality as

(1.1)
$$\text{minimize} \quad \varphi(x) = c^T x + \tfrac{1}{2} x^T D x$$

$$\text{subject to} \quad Ax \geq b$$

$$x \geq 0$$

where the matrix A is of order $m \times n$ and $D = D^T$. If D is positive semi-definite, or more precisely, if φ is a convex function on the convex polyhedral constraint set

$$X = \{x : Ax \geq b, \ x \geq 0\} \ ,$$

then (1.1) is called a <u>convex</u> quadratic program. There is an abundant literature on convex quadratic programming, and the interested reader

may consult the volumes by KÜNZI and KRELLE (1962), BOOT (1964), ABADIE (1967), and DANTZIG and VEINOTT (1968) for a selection of algorithms and an ample supply of further references. Ritter's method, described in this paper, is designed to handle problems of the form (1.1) in which the convexity of φ on X is not assumed.

The quadratic programming problem is a sort of bridge between linear and nonlinear programming. It might also be said that the nonconvex quadratic programming problem is a bridge between integer and nonlinear programming. This alone should be sufficient to sustain one's interest in techniques for efficiently solving difficult problems of this kind.

Ritter's method is composed of three distinct phases. Its Phase I is essentially the same as that of the Simplex Method for linear programming (see DANTZIG (1963)). This procedure is used to determine whether there exists a vector satisfying the constraints; indeed, it produces an extreme point of the constraint set X if and only if that set is nonempty. Once this has been settled in the affirmative, another aspect of Phase I deals with expressing the objective function in terms of the independent (i.e., nonbasic) variables. The extreme point at hand then is used as the starting point for Phase II which determines either a local minimum or gives an indication that the objective function is not bounded below on the constraint set. Phase III is a method for constructing a cutting plane that excludes the previously located local minimum without excluding the global minimum if it has not yet been found. After the cutting plane is placed, (i.e., adjoined as a constraint) the Phase I procedure is reapplied to the augmented problem. Termination can occur in Phase I if no feasible points remain after placing the cutting plane, or it can also occur in Phase III after a weak sufficiency condition for

a global minimum is satisfied, or with an indication that φ is not bounded below on the constraint set. The number of calculations preformed in each phase is finite. RITTER (1964), (1966) proved that if X is a bounded set, then the method cycles through the three phases only a finite number of times. The finiteness of this method for unbounded constraint sets is an unsettled question.

It is our intention in this paper to motivate and explain Phases II and III as developed by RITTER. Both of these phases depend on being able to distinguish a local minimum. In Section 2, we set forth the necessary and sufficient conditions characterizing a point as a local minimum of a quadratic programming problem. Section 3 is concerned with the construction of the cutting plane at a given local minimum (i.e., Phase III). We describe Ritter's algorithm for finding a local minimum in Section 4 where we include a small example illustrating how the method works.

2. NECESSARY AND SUFFICIENT CONDITIONS FOR A LOCAL MINIMUM

The quadratic program again is

$$(2.1) \qquad \text{minimize} \quad \varphi(x) = c^T x + \tfrac{1}{2} x^T D x$$

$$\text{subject to} \quad Ax \geq b$$

$$x \geq 0$$

where A is of order m × n and D is symmetric. The problem determines
a polyhedral convex <u>constraint</u> <u>set</u> X given by

$$(2.2) \qquad X = \{ x \in \mathbb{R}^n : Ax \geq b, \ x \geq 0 \}$$

By a <u>local</u> <u>minimum</u> of φ on X we shall mean a vector $\bar{x} \in X$ for which there
exists a positive scalar ϵ and a corresponding ϵ-neighborhood, $N(\bar{x}, \epsilon) =$
$\{ x \in \mathbb{R}^n : \| x - \bar{x} \| < \epsilon \}$, such that $\varphi(\bar{x}) \leq \varphi(x)$ for all $x \in X \cap N(\bar{x}, \epsilon)$. If
the latter inequality holds for all $x \in X$, then $\bar{x}$ is called a <u>global</u>
<u>minimum</u> of φ on X. This is all perfectly standard, and we make this
explicit statement only to emphasize our use of the term local (global)
minimum with reference to the point $\bar{x}$ rather than the functional value
$\varphi(\bar{x})$. By <u>solving</u> (2.1) we mean obtaining a global minimum of φ on X or
showing that none exists.

The Kuhn-Tucker theorem on necessary conditions of optimality is
immediately applicable to the quadratic programming problem because the
constraints are linear and consequently the Kuhn-Tucker constraint qualifi-
cation is satisfied at all relative boundary points $\bar{x} \in X$. As originally
stated, the Kuhn-Tucker theorem pertains to global minima. However, it is
well known (and clear from their proof) that the so-called "Kuhn-Tucker

conditions" must obtain at a local minimum, just as at a global minimum.
It is also well known that these conditions can hold at points which
are not local minima. For this reason, one says that in general they
are necessary but not sufficient. We state them now in a special way
for the problem at hand.

(2.3) THEOREM (KUHN and TUCKER (1951)): If $\bar{x}$ is a local minimum of φ
on X, there exists a vector $\bar{y}$ such that

$$(2.3a) \qquad\qquad c + D\bar{x} - A^T\bar{y} \geq 0$$

$$(2.3b) \qquad\qquad -b + A\bar{x} \geq 0$$

$$(2.3c) \qquad\qquad \bar{x} \geq 0$$

$$(2.3d) \qquad\qquad \bar{y} \geq 0$$

$$(2.3e) \qquad\qquad \bar{x}^T[\ c + D\bar{x} - A^T\bar{y}] = 0$$

$$(2.3f) \qquad\qquad \bar{y}^T[-b + A\bar{x} \quad\ ] = 0$$

We refer to (2.3a) through (2.3f) as the Kuhn-Tucker conditions of (2.1)
and any pair $(\bar{x},\bar{y}) \in R^n \times R^m$ satisfying them will be called a Kuhn-Tucker
point. A stationary point, $\bar{x}$, of φ on X is one for which there exists
a $\bar{y}$ such that $(\bar{x},\bar{y})$ is a Kuhn-Tucker point. If the (slack) vectors u and
v are defined via the equations

$$(2.4a) \qquad\qquad u = c + Dx - A^T y$$

$$(2.4b) \qquad\qquad v = -b + Ax$$

the Kuhn-Tucker conditions can be rendered as

$$(2.5) \qquad u \geq 0,\ x \geq 0,\ y \geq 0,\ v \geq 0,\ x^T u = 0,\ y^T v = 0\ .$$

The variables x_j and u_j are paired and are said to be <u>complementary</u> (or <u>complements</u> of each other). The same terminology applies to the pairs y_i and v_i.

As it turns out, the exposition of this theory is simplified by combining the nonnegativity restrictions $(x \geq 0)$ with the other linear inequalities in one grand system. Thus we can consider a more general system of constraints

$$G_{i\,\cdot} \geq h_i \qquad i = 1, \ldots, p$$

and specialize it by choosing $p = m + n$ and

$$G_{i*} = \begin{cases} A_{i*} & i = 1, \ldots, m \\ I_{i-m,*} & i = m+1, \ldots, m+n \end{cases}$$

$$h_i = \begin{cases} b_i & i = 1, \ldots, m \\ 0 & i = m+1, \ldots, m+n \end{cases}$$

where, for example, G_{i*} denotes the ith row of a matrix G.

Relative to the constraints of the problem and any point $\bar{x} \in R^n$ we may define

$$G(\bar{x}) = \{i : G_{i*}\bar{x} = h_i, \quad i = 1, \ldots, m+n\}$$

The constraints corresponding to indices in $G(\bar{x})$ are said to be <u>binding</u> (or <u>active</u>) at $\bar{x}$. The set

$$\mathcal{L}_i = \{x \in R^n : G_{i*}x = h_i\} \quad i = 1, \ldots, m+n$$

is a linear manifold as is

$$\mathcal{L} = \{x \in R^n : x \in \mathcal{L}_i, \ i \in G(\bar{x})\}$$

which is meant to equal R^n if $G(\bar{x}) = \phi$.

It is clearly important to be able to distinguish local minima from other types of stationary points. This classification problem has been rather thoroughly treated by RITTER. (See RITTER (1964), (1965) as well as the more recent and general work of McCORMICK (1967), or FIAUCO and McCORMICK (1968).) For this discussion, we adopt Ritter's formulation of the problem and consider the program

$$(2.6) \qquad \text{minimize} \quad c^T x + \tfrac{1}{2} x^T D x$$

$$\text{subject to} \quad Gx \geq h$$

which, as we have already noted, includes (2.1). The Kuhn-Tucker conditions of (2.6) are

$$(2.7a) \qquad c + Dx - G^T y = 0$$

$$(2.7b) \qquad -h + Gx \qquad \geq 0$$

$$(2.7c) \qquad \qquad y \geq 0$$

$$(2.7d) \qquad y^T[-h + Gx] \quad = 0$$

For a Kuhn-Tucker point $(\bar{x}, \bar{y})$, i.e., a solution of (2.7), we ᴀine

$$G(\bar{x}, \bar{y}) = \{i \in G(\bar{x}) : \bar{y}_i > 0\} \ .$$

If $i \in G(\bar{x}, \bar{y})$, the ith constraint of (2.6) is said to be strongly binding. We ᴀᴇᴛ

$$\hat{\mathcal{L}} = \{x \in R^n : x \in \mathcal{L}_i, \ i \in G(\bar{x}, \bar{y})\} \ .$$

There is also a corresponding subsystem of (2.7), namely

$$(2.8a) \qquad c + Dx - \hat{G}^T \hat{y} = 0$$

$$(2.8b) \qquad -\hat{h} + \hat{G}x \qquad = 0$$

$$(2.8c) \qquad \qquad \hat{y} > 0$$

in which $\hat{G}$ is the appropriate submatrix of G. In the light of its usage, we will assume the nonsingularity of the matrix

$$(2.9) \qquad \begin{pmatrix} D & -\hat{G}^{\mathsf{T}} \\ \hat{G} & 0 \end{pmatrix} .$$

This implies that the rows of $\hat{G}$ are linearly independent.

Given a Kuhn-Tucker point for (2.6), there is a range over which the vector h may be varied without causing either the strongly binding constraints to become nonbinding or the nonbinding constraints to become strongly binding. Since the matrix specified by (2.9) is nonsingular, equations (2.8a) and (2.8b) can be used to define a Kuhn-Tucker point as a function of $\hat{h}$ when $\hat{h}$ is chosen from the set just described. Over this range of $\hat{h}$, the value of the objective function φ is an implicit function of $\hat{h}$. The following lemma relates the multipliers $\hat{y}$ to the rate of change of φ as $\hat{h}$ is perturbed.

(2.10) <u>LEMMA</u> (RITTER (1965)): For the set of all $\hat{h}$ such that (2.8) has a solution, φ can be regarded as a function of $\hat{h}$ and in this sense,

$$\hat{y}^{\mathsf{T}} = \nabla_{\hat{h}} \varphi$$

Proof. By the chain rule for differentiation

$$\nabla_{\hat{h}} \varphi = \nabla_x \varphi \left[\frac{dx}{d\hat{h}} \right]$$

Rewriting (2.8a) and post multiplying by $\left[\dfrac{dx}{d\hat{h}} \right]$, we obtain

$$[\nabla_x \varphi - \hat{y}^{\mathsf{T}} \hat{G}] \left[\frac{dx}{d\hat{h}} \right] = 0$$

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These two equations imply

$$\nabla_{\hat{h}} \varphi = \hat{y}^T \hat{G} \left[\frac{dx}{d\hat{h}} \right]$$

Since the matrix in (2.9) is nonsingular, $\begin{pmatrix} x \\ \hat{y} \end{pmatrix}$ can be expressed as a function of $\hat{h}$ and (2.8) can be used to obtain

$$\begin{pmatrix} D & -\hat{G}^T \\ \hat{G} & 0 \end{pmatrix} \begin{pmatrix} \left[\frac{dx}{d\hat{h}} \right] \\ \left[\frac{d\hat{y}}{d\hat{h}} \right] \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix}$$

where I is an identity matrix of appropriate order. From this we conclude that

$$\hat{G} \left[\frac{dx}{d\hat{h}} \right] \quad I$$

and the result follows.

We now state Ritter's characterization theorem for stationary points.

(2.11) THEOREM (RITTER (1965)): If $(x,\hat{y})$ is a solution of (2.8) then:

(i)   x is a local minimum of φ on X iff φ is convex on $\hat{\mathcal{L}}$;

(ii) x is a local maximum of φ on X iff $\hat{\mathcal{L}} = R^n$ and φ is concave;

(iii) x is a saddle point of φ on X iff φ is nonconvex on $\hat{\mathcal{L}}$ provided $\hat{\mathcal{L}} \neq R^n$. If $\hat{\mathcal{L}} \quad R^n$, then x is a saddle point iff φ is neither convex nor concave.

Proof. (i) Suppose φ is convex on $\hat{\mathcal{L}}$ and suppose $\bar{x} \in X$. Then $\bar{h} : \hat{G}\bar{x} \sim \hat{h}$. By the assumption (2.9), there will be a unique stationary

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point x' of φ subject to $\hat{G}x = \bar{h}$. Then $\varphi(x') \leq \varphi(\bar{x})$. Now if we choose $\bar{x}$ sufficiently close to x, we may preserve the positivity of the multipliers. The result $\varphi(\bar{x}) \geq \varphi(x') \geq \varphi(x)$ now follows from Lemma (2.10).

Conversely, if x is a local minimum of φ on $\overset{\wedge}{\mathcal{L}}$ then $\varphi(\bar{x}) \geq \varphi(x)$ for all $\bar{x} \in \overset{\wedge}{\mathcal{L}} \cap N(x, \epsilon)$. We have

$$\varphi(\bar{x}) = \varphi(x) + (c^{\mathsf{T}} + x^{\mathsf{T}}D)(\bar{x} - x) + \tfrac{1}{2}(\bar{x} - x)^{\mathsf{T}}D(\bar{x} - x) =$$

$$\varphi(x) + \hat{y}^{\mathsf{T}}\hat{G}(\bar{x} - x) + \tfrac{1}{2}(\bar{x} - x)^{\mathsf{T}}D(\bar{x} - x) \ .$$

Since $\hat{G}(\bar{x} - x) = 0$,

$$\varphi(\bar{x}) - \varphi(x) = \tfrac{1}{2}(\bar{x} - x)^{\mathsf{T}}D(\bar{x} - x) \geq 0 \ .$$

It follows now that the quadratic form $z^{\mathsf{T}}Dz$ is convex on $\overset{\wedge}{\mathcal{L}} \cap N(x, \epsilon)$ and hence on $\overset{\wedge}{\mathcal{L}}$, its carrying plane. Therefore φ is convex on $\overset{\wedge}{\mathcal{L}}$. (See COTTLE (1967).)

(ii) Suppose x is a local maximum and $\overset{\wedge}{\mathcal{L}} \neq R^a$. Then the Lemma (2.10) implies that in any sufficiently small neighborhood of x there exists a point x' such that $\varphi(x') > \varphi(x)$. This contradicts the assumption that x is a local maximum. Hence $\overset{\wedge}{\mathcal{L}} = R^a$ and moreover $c + Dx = 0$. From this we have

$$\varphi(\bar{x}) \leq \varphi(x) = \varphi(x) + \nabla\varphi(x)(\bar{x} - x)$$

for all $\bar{x}$ in an ϵ-neighborhood of x, say $N(x, \epsilon)$. It now follows that φ is concave on $N(x, \epsilon)$ and hence R^a.

If φ is concave on R^a, then of course, the Kuhn-Tucker conditions are sufficient to infer that x is a local—indeed global—maximum.

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(iii) If $\hat{\mathfrak{L}} \neq \mathbb{R}^n$, and x is a saddle point of φ on X then $\dot{\varphi}$ is not

convex on $\hat{\mathfrak{L}}$ by (i). If φ is not convex on $\hat{\mathfrak{L}}$ then x is neither a local

minimum by (i) nor a local maximum since $\hat{\mathfrak{L}} \neq \mathbb{R}^n$, so it must be a saddle

point of φ on X. If $\hat{\mathfrak{L}} = \mathbb{R}^n$, parts (i) and (ii) leave only the conclusion

that x is a saddle point if and only if φ is neither convex nor concave

(i.e. $x^\mathsf{T} D x$ is an indefinite quadratic form).

(2.12) REMARKS. Simple examples show that the exclusion of weakly

binding constraints from the statement of (2.11) is necessary. Another

important fact about this theorem is that if (x^*, y^*) is a Kuhn-Tucker

point for (2.6) and the function φ is convex on the linear manifold $\hat{\mathfrak{L}}$,

then x^* is a local minimum φ on X (i.e., in the problem (2.6)) and not

just in the quadratic program derived from the lata $c, D, \hat{G}, \hat{h}$.

While Theorem (2.11) gives a useful sufficiency condition for a

local minimum, the goal is to determine a global minimum. With this

in mind, we mention here an obvious but useful global sufficiency con-

dition given by RITTER (1964). Let x^* be a feasible vector for (2.6)

such that $G(x^*) \neq \emptyset$. By a suitable permutation of the constraints,

G and h can be partitioned as $\begin{pmatrix} G_0 \\ G_+ \end{pmatrix}$ and $\begin{pmatrix} h_0 \\ h_+ \end{pmatrix}$, respectively, so that

$$G_0 x^* = h_0$$

$$G_+ x^* > h_+$$

(2.13) PROPOSITION: If x^* is a local minimum of (2.6) and

$$\min \{ x^\mathsf{T} D x : G_0 x = 0 \} \geq 0 ,$$

then x^* is a global minimum of φ on X.

Proof. We have the identity

$$\varphi(x) = \varphi(x^*) + \nabla\varphi(x^*)(x - x^*) + \tfrac{1}{2}(x - x^*)^{\mathsf{T}} D(x - x^*).$$

Since x^* is a local minimum, there exists a vector $\bar{y}$ such that $(x^*, \bar{y})$ is a Kuhn-Tucker point. Let $\bar{y} = \begin{pmatrix} \bar{y}_0 \\ \bar{y}_+ \end{pmatrix}$ in conformity with the partitioning of G and h. Then

$$c + Dx^* - G_0^{\mathsf{T}}\bar{y}_0 - G_+^{\mathsf{T}}y_+ = 0$$

$$-h_0 + G_0 x^* \qquad\qquad = 0$$

$$-h_+ + G_+ x^* \qquad\qquad > 0$$

$$\bar{y}_0 \geq 0$$

$$\bar{y}_+ \geq 0$$

$$\bar{y}_0^{\mathsf{T}}[-h_0 + G_0 x^*] = 0 \qquad\qquad \text{(trivially)}$$

$$\bar{y}_+^{\mathsf{T}}[-h_+ + G_+ x^*] = 0$$

The last of these Kuhn-Tucker conditions implies $\bar{y}_+ = 0$, so we arrive at the observation that the linear inequality system

$$G_0^{\mathsf{T}} y_0 = c + Dx \ , \quad y_0 \geq 0$$

has a solution. By the alternative theorem of FARKAS (1902) we see that

$$G_0 x \geq 0, \ (c^{\mathsf{T}} + x^{*\mathsf{T}} D)x < 0$$

does not have a solution. Hence

(2.14) $\qquad G_0 z \geq 0$ implies $(c^{\mathsf{T}} + x^{*\mathsf{T}} D)z \geq 0$.

Suppose $x \in X$. Then $G_0 x \geq h_0$, and it follows that $G_0(x - x^*) \geq 0$. With $z = x - x^*$, (2.14) implies $\nabla\varphi_0(x^*)(x - x^*) = (c^{\mathsf{T}} + x^{*\mathsf{T}} D)(x - x^*) \geq 0$.

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Moreover, by the hypothesis of the Proposition, we have $(x - x^{*T})D(x - x^*) \geq 0$ for all $x \in X$. The result now follows from the identity.

3. CONSTRUCTION OF A CUTTING PLANE

The results of the preceding section enable one to distinguish between stationary points of a quadratic program. Once a local minimum has been reached, it is natural to try to introduce a cutting plane which will exclude the previously determined local minimum without excluding the global minimum if it has not been located so far. In this section we look at the cutting plane problem (Phase III) and postpone until the next section the matter of finding a local minimum (Phase II).

Given a local minimum, it is possible to determine a vertex of the constraint set X that lies in the lowest-dimensional face of X containing the local minimum. Then the primal variables and their complementary dual variables can be relabeled so that the slack variables at that vertex are denoted as v-variables. After this relabeling process, a series of algebraic manipulations can be carried out to put the problem in the form

$$(3.1) \qquad \text{minimize} \quad \varphi(x) = c^T x + \tfrac{1}{2} x^T D x$$

$$\text{subject to} \quad v = -b + Ax$$

$$v \geq 0, \; x \geq 0$$

with $(\bar{x}, \bar{v}, \bar{u}, \bar{y})$ being the Kuhn-Tucker point associated with local minimum $\bar{x}$. We shall assume that this point is a nondegenerate solution so that $\bar{v} > 0$, $\bar{x} \geq 0$. Suppose that x can be partitioned as $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ with $\bar{x}_1 > 0$, $\bar{x}_2 = 0$. The nondegeneracy assumption implies $\bar{u}_1 = 0$, $\bar{u}_2 > 0$. The reader should be warned that here the single subscripts pertain to subvectors and not necessarily to coordinates of the corresponding vectors. This usage should obviate the introduction of more elaborate notation without leading to undue confusion.

As indicated in Section 1, the purpose of a cutting plane is to exclude the previously determined local, but nonglobal, minimum without excluding a global minimum. Geometrically speaking, the previously determined local minimum and the set of global minima are to be separated by the cutting hyperplane. How might one want to proceed in constructing such a hyperplane? Suppose for the moment that the set

$$S = \left\{ s = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \in R^n \ : \ s^T D s < 0, \ s_2 \geq 0 \right\}$$

is nonempty and "known." Then for each $s \in S$ there exists a smallest $\tau_s > 0$ such that

$$\varphi(\bar{x} + \tau s) < \varphi(\bar{x}) \quad \text{for all } \tau > \tau_s \ .$$

If for all $s \in S$, the point $\bar{x} + \tau_s s \notin X$, then $\bar{x}$ is a global minimum. On the other hand, if for some $s \in S$ and some $\tau > \tau_s$ the point $\bar{x} + \tau s \in X$, then an improvement can be made. We define

$$Y = \{ \bar{x} + \tau s \ : \ s \in S, \ \tau \geq \tau_s \} \ .$$

Thus, we are interested in a supporting hyperplane to the set Y.

The procedure described below is a constructive method for choosing a particular supporting hyperplane. Occasionally it is possible to construct a parallel cutting plane that chops off more of X.

Let $e^T x \geq \tau$ denote the constraint to be adjoined. The task is to specify the vector e and the scalar τ. This will be done by considering the problem of finding the global minimum for every value of τ of the quadratic program

$$(3.2) \qquad \text{minimize} \quad c^T x + \tfrac{1}{2} x^T D x$$

$$\text{subject to} \qquad x_2 \geq 0$$

$$e^T x \leq \tau .$$

The approach will be to specify the vector e in a manner which will simplify the solving of (3.2). The inequality constraint $e^T x \leq \tau$ will be called the "capacity constraint" and e will be chosen so that for each value of $\tau \geq 0$, a global minimum for the corresponding program (3.2) exists.

We partition c, D, and e to be compatible with the partitioning of $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Thus

$$c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \quad D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}, \quad \text{and} \quad e = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

Since $(\bar{x}_1, 0)$ is a local minimum for problem (3.1) with $\bar{x}_1 > 0$, it follows that D_{11} is nonsingular. The stationary point at hand is assumed to be a local minimum; thus we know from Theorem (2.11) that φ is convex on the subspace $\{ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in R^n : x_2 = 0 \}$. Hence, we are assured that a global minimum for (3.2) exists when e_1 is set equal to zero and e_2 is choosen to satisfy $e_2 > 0$.

A global minimum in (3.2) must satisfy the Kuhn-Tucker conditions, which can be written as follows:

$$(3.3a) \qquad 0 \quad c_1 + D_{11} x_1 + D_{12} x_2$$

$$(3.3b) \qquad u_2 = c_2 + D_{12}^T x_1 + D_{22} x_2 + e_2 \zeta$$

$$(3.3c) \qquad \omega = \tau \qquad - e_2^T x_2$$

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(3.3d) $\zeta \geq 0, \; \omega \geq 0, \; u_2 \geq 0, \; x_2 \geq 0$

(3.3e) $\zeta \omega = 0, \qquad x_2^T u_2 = 0$

The nonsingularity of D_{11} makes it possible to eliminate x_1 from (3.2) and its Kuhn-Tucker conditions (3.3). In particular,

$$x_1 = -D_{11}^{-1} c_1 - D_{11}^{-1} D_{12} x_2 \; ,$$

and after the substitution, (3.2) becomes

(3.4)   minimize $\bar{\varphi}(x_2) = \bar{c} x_2 + \tfrac{1}{2} x_2^T \bar{D} x_2$

           subject to $x_2 \geq 0$

                               $e_2^T x_2 \leq \tau$

with

$$\bar{c} = c_2 - c_1^T D_{11}^{-1} D_{12}$$

$$\bar{D} = D_{22} - D_{12}^T D_{11}^{-1} D_{12}$$

$$\varphi(\bar{x}) = -\tfrac{1}{2} c_1^T D_{11}^{-1} c_1$$

$$\varphi(x) = \varphi(\bar{x}) + \bar{\varphi}(x_2)$$

The point $x_2 = 0$ is a local minimum with $\bar{\varphi}(0) = 0$ for all $\tau \geq 0$. For τ small enough, $x_2 = 0$ will be the only stationary point in the constraint set of (3.4) and thus it is the global minimum. Consideration of Theorem (2.11) leads one to the observation that at any local minimum of (3.4), other than the point $x_2 = 0$, the capacity constraint $e_2^T x_2 \leq \tau$ must be binding. For this reason, only points satisfying the Kuhn-Tucker conditions with the slack variable of the capacity constraint at zero value (i.e., $\omega = 0$) will be consid   .

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The Kuhn-Tucker conditions for (3.4) are

$(3.5a)$
$$u_2 = \bar{c} + \bar{D}x_2 + e_2\zeta$$

$(3.5b)$
$$\omega = \tau - e_2^I x_2$$

$(3.5c)$
$$u_2 \geq 0, \ x_2 \geq 0, \ \zeta \geq 0, \ \omega \geq 0$$

$(3.5d)$
$$x_2^I u_2 = 0, \ \zeta\omega = 0$$

The problem of determining a global minimum of (3.4) can be carried out by finding a nonnegative basic solution of $(3.5a)$, $(3.5b)$ satisfying the complementary slackness condition $(3.5d)$ and giving the lowest value of $\bar{\varphi}$. The next section should reveal that for a given value of τ, the problem of solving (3.5) in order to produce the smallest value of $\bar{\varphi}$ is simpler than the original problem only in the sense that its size is smaller. The vector e_2 is to be chosen so that having the required solution to (3.5) for a particular value of τ makes it easy to obtain the global solution for (3.4) for all values of $\tau \geq 0$.

Let $e_2 = \bar{c} > 0$. Then for any solution of (3.5) with $\omega = 0$, we have

(3.6)
$$x_2^I \bar{D}x_2 = -(1 + \zeta)\tau$$

From this relationship and the assumption that the capacity constraint is binding, it follows that we are seeking a point satisfying (3.5) that minimizes

(3.7)
$$\bar{\varphi}(x_2) = \tau - \tfrac{1}{2}(1 + \zeta)\tau$$

Putting $\sigma = 1 + \zeta$, we will treat the problem of finding a solution to

(3.8a)
$$u_2 = \bar{D}x_2 + \bar{c}\sigma$$

(3.8b)
$$\omega = \tau - \bar{c}x_2$$

(3.8c)
$$u_2 \geq 0, \; x_2 \geq 0, \; \sigma \geq 0, \; \omega \geq 0$$

(3.8d)
$$x_2^T u_2 = 0, \; \sigma\omega = 0$$

with $\omega = 0$ which makes the value of σ the largest for each value of τ. If $\sigma \geq 1$, the multiplier associated with the capacity constraint is positive, and the corresponding x_2 is a local minimum. Equation (3.7) can be regarded as saying that for $\sigma \geq 2$, the corresponding point x_2 is a global minimum.

The point $x_2 = 0$ is a global minimum of (3.4) when $\tau = 0$. Hence we are only interested in solutions for $\tau > 0$. For such τ we put

$$\tilde{u} := u_2/\tau \qquad \tilde{x} := x_2/\tau$$

$$\tilde{\omega} := \omega/\tau \qquad \tilde{\sigma} := \sigma/\tau$$

and our problem becomes that of finding the solution of

(3.9a)
$$\tilde{u} = \bar{D}\tilde{x} + \tilde{\sigma}\bar{c}$$

(3.9b)
$$\tilde{\omega} = 1 - \bar{c}^T\tilde{x}$$

(3.9c)
$$\tilde{u} \geq 0, \; \tilde{x} \geq 0, \; \tilde{\omega} \geq 0, \; \tilde{\sigma} \geq 0$$

(3.9d)
$$\tilde{x}^T\tilde{u} = 0, \; \tilde{\sigma}\tilde{\omega} = 0$$

with $\tilde{\omega} = 0$ and the largest value of $\tilde{\sigma}$. Let $(\tilde{u}^*, \tilde{x}^*)$, $(0, \tilde{\sigma}^*)$ denote the solution found. The solution of (3.8) can be interpreted according to

the value of τ. In particular:

(i) if $\widetilde{\sigma}^* < 1/\tau$, $x_2 = 0$ is the global minimum, and $x_2 = \tau\widetilde{x}^*$ is no special sort of point;

(ii) if $1/\tau \leq \widetilde{\sigma}^* < 2/\tau$, $x_2 = 0$ is the global minimum and $x_2 = \tau\widetilde{x}^*$ is a local minimum;

(iii) if $2/\tau \leq \widetilde{\sigma}^*$, then $x_2 = \tau\widetilde{x}^*$ is the global minimum of (3.4).

Note that if $\widetilde{\sigma}^* \leq 0$, then the point $\bar{x} = \begin{pmatrix} -D_{11}^{-1}c_1 \\ 0 \end{pmatrix}$, with which we started, is a global minimum for the original quadratic program, for we have satisfied the hypotheses of Proposition (2.13).

The method given by RITTER for finding the required solution of (3.9) requires the examination of all the solutions of (3.9) with $\widetilde{\omega} = 0$. To simplify the operation, he gives some rules making it possible to avoid explicit examination of every solution of (3.9).(See RITTER (1966), pp. 347.)

When D is negative semi-definite the problem is much easier because a solution to problem (3.9) can be found in which $\widetilde{x}^*$ is an extreme point of the constraint region $\{\widetilde{x} : \widetilde{x} \geq 0, \bar{c}'\widetilde{x} \leq 1\}$. Such cases correspond to quadratic programs with concave minimands. The task of setting up problem (3.9) after finding a local minimum will emerge as a by-product of Ritter's method for finding that local minimum.

The cutting plane to be adjoined is $e^\intercal x \geq \tau$, with $e = \begin{pmatrix} 0 \\ \bar{c} \end{pmatrix}$, where only the value τ is yet to be specified. This is done first by computing the largest number τ_1 such that $\varphi(x(\tau_1)) = \varphi^*$ where φ^* is the smallest value of the objective function yet found and $x(\tau) = \begin{pmatrix} -D_{11}^{-1}c_1 - \tau D_{11}^{-1}D_{12}\widetilde{x}^* \\ \tau\widetilde{x}^* \end{pmatrix}$.

Let

$$\tau_2 = \sup \{\tau \,:\, Ax(\tau) \geq b, \; x(\tau) \geq 0\}$$

Thus τ_2 is the largest value of τ such that $x(\tau)$ satisfies the con-
straints ignored in problem (3.2). If $\tau_2 = \infty$, this is another indi-
cation that the objective function φ is not bounded below on the
constraint set X. Let

$$\bar{\tau} = \max \{\tau_1, \tau_2\}$$

The cutting plane that is adjoined is $e^{\mathsf{T}}x \geq \tau$. If $\bar{\tau} = \tau_1$, the cutting
plane is a support to Y, hence it excludes no local minimum yielding
a lower value of the objective function than has yet been determined.
If $\bar{\tau} = \tau_2$, the new lowest value of the objective function is assumed
at the feasible point $x(\bar{\tau})$, and no local minimum giving a lower value
for φ is lost. Now the Phase I procedure must be reapplied.

4. RITTER'S ALGORITHM FOR FINDING A LOCAL MINIMUM

4.1. **Overview.** This section is devoted to the algorithm described by RITTER (1964), (1966) for the calculation of a local minimum of a quadratic program. Our style of presentation differs sharply from that used by RITTER. We regard Ritter's algorithm as a type of principal pivoting method, and our aim here is to interpret it in this context. Examining the algorithm from this point of view simplifies the presentation and results in some simplification in the method itself.

Principal pivoting algorithms are associated with solving <u>linear complementarity problems</u>, that is, systems of the form

(4.1a) $w = q + Mz$

(4.1b) $w \geq 0, \ z \geq 0$

(4.1c) $z^T w = 0$

where $M \in R^{p \times p}$. By suitable identifications, the Kuhn-Tucker conditions of a quadratic program in the form (2.1) become a special case of (4.1). Moreover, a convex quadratic program and its dual can be solved by solving the corresponding linear complementarity problem (4.1). (See COTTLE (1964); for survey references on the solving of linear complementarity problems see LEMKE (1968), COTTLE and DANTZIG (1968) and COTTLE (1968).)

The system expressed in (4.1a) can be recorded in the tabular form

(4.2) $w - \boxed{\begin{array}{c|c} 1 & z \\ \hline q & M \end{array}}$.

A solution of (4.1a) is called <u>nondegenerate</u> if at most p of its 2p unknowns equal zero (or, more generally, at most p of the 2p unknowns equal their "current" lower bound).

The variable w_i is said to be the <u>complement</u> of z_i. Initially, in (4.1a) w_i are <u>basic</u>, the z_i <u>nonbasic</u>. The rules of a particular algorithm may specify that a nonbasic variable be increased, and in this role, it is called the <u>driving</u> <u>variable</u>. If the increase of the driving variable causes a basic variable to reach a specified value (for example zero) then the latter is called a <u>blocking</u> <u>variable</u>. The pair of variables used in the pivot operation (basic exchange) is denoted by

<BLOCKING VARIABLE,   DRIVING VARIABLE> .

The blocking variable specifies a row, and the driving variable specifies a column. Thereby a <u>pivot</u> <u>element</u> (of the tableau) is singled out. The pivot element is the rate of change of the blocking variable with respect to the driving variable.

This type of algorithm can terminate in two ways. The first way termination can occur is with a solution to (4.1). Having a solution to (4.1) indicates that a (possibly vacuous) block pivot could have been performed about a principal submatrix to go directly from (4.1) to a basic solution of (4.1a) satisfying (4.1b) and (4.1c). (See PARSONS (1967).) The second is with an indication that no blocking variable exists. This type of termination must admit of meaningful interpretation

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if the algorithm is to be useful. In our case, the system (4.1) relates to the Kuhn-Tucker conditions of a quadratic program, and the second type of termination must signify that the problem either has an empty constraint set or the objective function is unbounded below on the constraint set. A further requirement of a useful principal pivoting algorithm is termination after a finite number of pivot operations.

4.2. Ritter's Algorithm for Determination of a Local Minimum

The task to be accomplished is to find a local minimum for the problem

(4.3a) minimize $c^T x + \frac{1}{2} x^T D x$

(4.3b) subject to $A x \geq b$

(4.3c) $x \geq 0$

It will be assumed that $x = 0$ is a feasible point; hence $-b \geq 0$. The existence of a feasible point can be checked using the Phase I procedure. Clearly a feasible quadratic program can be written in the form (4.3).

Let us assume that some $c_i < 0$, for otherwise the point $x = 0$ is a local minimum. (The case where $c \geq 0$, but some $c_i = 0$ requires more careful attention and this will be given later.) The algorithm to be described here works by maintaining a solution to the Kuhn-Tucker conditions which yields a local minimum for the problem given by (4.3) augmented with a <u>capacity constraint</u>.

(4.4) $e^T x \leq \tau$ where $e_i \geq 0$ and $e_i > 0$ if $c_i \leq 0$.

The vector e may be any vector satisfying these requirements though it is convenient to use $e_i = 1$ for all i. With the parameter $\tau = 0$, the point $x = 0$ is obviously a local minimum for the augmented problem since no other point is feasible. At the point $x = 0$, the capacity constraint $e^T x \leq \tau = 0$ is binding and has a positive multiplier. Then the capacity constraint is relaxed by an increase of the parameter τ. Next

the Kuhn-Tucker conditions for (4.3), (4.4) are used to compute a local minimum $x = x(\tau)$ for this augmented problem.

However, there are conditions under which it is impossible to increase τ so that $x(\tau)$ remains a local minimum of the augmented problem. Depending on how this situation arises, it will be necessary either to specify that certain nonnegativity constraints $(x_i \geq 0)$ be treated as equality constraints, $(x_i = 0)$, or to change the capacity constraint.

Lemma (2.10) shows that as τ is being increased, the objective function is being decreased as long as the capacity constraint has a positive multiplier. After a finite number of iterations, the algorithm either terminates with an indication that the objective function has no finite lower bound on the feasible region or the multiplier of the capacity constraint reaches the value zero. When this occurs, we have a local minimum to the problem without the capacity constraint, but we may still have the imposed condition that some of the variables be held at the value zero. If there are no such restrictions, the point at hand is a local minimum to (4.1) and Phase III is to be executed. However, if we have a local minimum to the problem with the additional conditions that some of the variables be equal to zero, then a new capacity constraint is to replace the one just dropped, and the procedure is to be continued.

Each iteration of the algorithm corresponds to a different basic solution to the Kuhn-Tucker equations for the augmented problem. If these solutions are nondegenerate, a decrease in the objective function must occur at each iteration. Consequently, no basic solution of the Kuhn-Tucker equations can be repeated and since there are only a finite

number, there can only be a finite number of iterations.

Before stating the details of Ritter's algorithm, it is necessary to mention two pivot-theoretic propositions needed to demonstrate the legitimacy of the algorithm. The proofs of these propositions are straight-forward applications of techniques from numerical linear algebra.

(4.5) <u>PROPOSITION</u>: Consider the system of homogeneous linear equations expressed in tabular form as

$$
\begin{array}{c}
 & \begin{array}{cc} z_1 & z_2 \end{array} \\
\begin{array}{c} w_1 = \\ w_2 = \end{array} &
\begin{array}{|cc|}
\hline
M & m \\
m^T & \mu \\
\hline
\end{array}
\end{array}
$$

where $M \in R^{(p-1) \times (p-1)}$ is symmetric and positive definite, $m \in R^{p-1}$, and $\mu \in R$. After the block pivot making z_1, w_2 basic, the tableau is

$$
\begin{array}{c}
 & \begin{array}{cc} w_1 & z_2 \end{array} \\
\begin{array}{c} z_1 = \\ w_2 = \end{array} &
\begin{array}{|cc|}
\hline
M^{-1} & -M^{-1}m \\
m^T M^{-1} & \mu - m^T M^{-1} m \\
\hline
\end{array}
\end{array}
$$

If $\mu - m^T M m > 0$, then the matrix

$$
M^* = \begin{pmatrix} M & m \\ m^T & \mu \end{pmatrix}
$$

is positive definite.

The other little fact we will need is

(4.6) <u>PROPOSITION</u>: Consider the tableau

$$
\begin{array}{c}
 & \begin{array}{ccc} x_1 & x_2 & y \end{array} \\
\begin{array}{c} u_1 = \\ u_2 = \\ v = \end{array} &
\begin{array}{|ccc|}
\hline
D_{11} & D_{12} & -A^T \\
D_{12}^T & D_{22} & -B^T \\
A & B & 0 \\
\hline
\end{array}
\end{array}
$$

If D_{22} is positive definite, then after the principal block pivot making u_1, x_2, v the basic variables, the entry in the tableau found at the intersection of the row corresponding to v_i and the column corresponding to y_i is nonnegative.

In describing Ritter's algorithm for finding a local minimum, we display the Kuhn-Tucker equations for the quadratic program (4.3) augmented with the capacity constraint (4.4) in the tableau

	1	τ	x	y	ζ
$u =$	c	0	D	$-A^T$	e
$v =$	$-b$	0	A	0	0
$\omega =$	0	1	$-e^T$	0	0

The variables x_j, v_i, ω are <u>primal</u> variables. The variables v_i are slack variables for the inequality constraints, so $Ax - v = b$, and ω is the slack variable for the capacity constraint, so $e^T x + \omega = \tau$. The <u>multipliers</u> are u_j, y_i, and ζ; u is the vector of multipliers for the nonnegativity constraints, y is the vector of multipliers for the inequality constraints $Ax \geq b$, and ζ the multiplier for the capacity constraint.

For any value of τ, say $\bar\tau$, the corresponding value of the vector in a basic solution of of basic variables $\bigwedge$ the Kuhn-Tucker equations is found by adding the first column to $\bar\tau$ times the second column of the tableau.

Now for the details of the algorithm. Specific rules are printed in italics. Comments and justifications are included in each step in ordinary roman type.

STEP 1. Let $c_r/e_r = \min \{c_j/e_j : e_j > 0\}$; it is assumed that $x = 0$, is not a local minimum, and $c_r/e_r < 0$. Perform the pivots $\langle u_r, \zeta \rangle$, $\langle \omega, x_r \rangle$ to complete a 2 × 2 principal block pivot. The values of the variables after the pivot are:

$$(4.7) \quad u_j = \left(c_j - \frac{e_j}{e_r} c_r\right) + \tau\left(\frac{d_{jr}}{e_r} - \frac{e_j}{e_r^2} d_{rr}\right), \quad x_j = 0, \quad (j \neq r; \ j = 1, 2, \ldots, n)$$

$$u_r = 0 \qquad\qquad\qquad x_r = \tau\left(\frac{1}{e_r}\right)$$

$$v_i = -b_i - \tau\left(\frac{a_{ir}}{e_r}\right) \qquad\qquad y_i = 0 \qquad (i = 1, 2, \ldots, m)$$

$$\zeta = \frac{c_r}{e_r} - \tau\left(\frac{d_{rr}}{e_r^2}\right) \qquad\qquad \omega = 0$$

Since $-b \geq 0$, this point satisfies the Kuhn-Tucker conditions when $\tau = 0$. If the problem is nondegenerate, then for some $\tau_1 > 0$, the point given by (4.7) satisfies the Kuhn-Tucker conditions in the range $0 \leq \tau \leq \tau_1$. The specification of τ_1 is given in

STEP 2. If the column of τ is nonnegative, then terminate the procedure. In such a case, τ as a driving variable is not blocked. Lemma (2.10) implies that if τ approaches infinity, the objective function is unbounded below on X. If the τ column contains at least one negative entry, then increase τ until some basic variable becomes zero. Let τ_1 be the value of τ. The assumption of nondegeneracy implies that this basic variable is unique. The basic variable that becomes zero when $\tau = \tau_1$ is the "candidate" to become the blocking variable. If the candidate to become the blocking variable is ζ, the multiplier associated with the capacity constraint, then go to step 6; otherwise go to the next step.

STEP 3. We want to make a principal pivot so as to be able to increase τ a little bit more before any other basic variable becomes negative. The desired pivot is on the blocking variable candidate and its complement, and feasibility can be preserved only if the pivot entry is positive. This can be seen by considering the following tableau where the blocking variable candidate is w_2

	1	τ	z_1	z_2
$w_1 =$	c_1 : f_1		D	d
$w_2 =$	c_2 : f_2		d^T	δ

w_1, z_1, c_1, f, $d \in R^{p-1}$ and w_2, z_2, c_2, f_2, δ are scalars.

For $\tau_1 > \tau < \tau_2$: $-c_2/f_2$,

$$c_2 + \tau f_2 > 0, \; c_1 + \tau f_1 > 0$$

and for $\tau > \tau_2$

$$c_2 + \tau f_2 < 0 .$$

If $\delta > 0$, then performing the pivot $\langle w_2, z_2 \rangle$ results in the tableau

	1	τ	z_1	w_2
w_1	$c_1 - \dfrac{c_2}{\delta} d$: $f_1 - \dfrac{f_2}{\delta} d$		$D - \dfrac{1}{\delta} dd^T$	$\dfrac{1}{\delta} d$
	$-\dfrac{c_2}{\delta}$: $-\dfrac{f_2}{\delta}$		$-\dfrac{1}{\delta} d^T$	$\dfrac{1}{\delta}$

So for $\tau > \tau_2$,

$$-\frac{1}{\delta}(c_2 + \tau f_2) > 0 \; ;$$

for some $\tau_3 > \tau_2$,

$$(c_1 + \tau f_1) - \frac{1}{\delta}(c_2 + \tau f_2) \, d > 0, \quad \text{if } \tau_2 \leq \tau < \tau_3.$$

Hence the rule is: <u>if the desired pivotal entry is positive,</u>
<u>then the blocking variable candidate is the blocking variable and its</u>
<u>complement acts as a driving variable in a pivot operation. Upon</u>
<u>completion of this pivot, repeat step 2. If the desired pivotal entry is</u>
<u>negative or the blocking candidate is a multiplier, go to step 4. If the block-</u>
<u>ing candidate is a primal variable and the pivotal entry is zero, go to step 5.</u>

Since we only make pivots about positive pivotal entries, after
step 1, the principal block pivot entry is positive definite. Repeated
application of Proposition (4.5) indicates that this is so. Hence the
primal variables of the augmented problem that are zero specify a face
of the feasible region containing the current point and restricted to
which the quadratic objective function is convex. The principal block
pivotal entry placing the point on that face is positive definite. Hence
by Theorem (2.11) the current point is a local minimum for the augmented
problem.

If the blocking variable is a multiplier, performing the indicated
principal pivot increases the order of the pivotal block, and the
objective function is to be restricted to a higher-dimensional linear
manifold, since fewer primal variables are at the zero level. If the
blocking variable is a primal variable, then the order of the principal

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pivot block is decreased, and the objective function is to be restricted
to a lower-dimensional linear manifold. Thus, if it is convex on the
higher-dimensional linear manifold, it must be convex on the manifold.

If the blocking candidate is a primal variable, then the pivotal
entry is nonnegative. Proposition (4.6) implies that if the blocking
candidate is a primal variable in the role of a slack variable, then
the pivotal entry is nonnegative. If the blocking variable is a primal
variable not in the role of a primal slack, then the desired coefficient
is on the main diagonal of the current pivot matrix. Since the latter
is positive definite, its diagonal elements are positive.

STEP 4. The blocking variable candidate is a multiplier. Let this
variable become negative and drop it from consideration as a potential
blocking variable as long as it remains negative. Now return to step 2.
The interpretation is that we now have a local minimum for the augmented
problem with the further restriction that the primal variables which are
complements to the multipliers with negative values are restricted to
equal zero and are not merely nonnegative.

STEP 5. The blocking candidate is a primal variable, hence the point
has moved to another face of the convex polytope X. Let v_r denote the
candidate. Make the pivots $\langle v_r , \omega \rangle$, $\langle \zeta , y_r \rangle$ and then drop the capacity
constraint from consideration. All the primal variables must be non-
negative, and if all the basic multipliers are positive the point is a
local minimum. If some of the multipliers are zero but none are negative,
it is necessary to examine the principal submatrix at the intersections of the
rows of the basic multipliers at zero value and their complements. If that matrix

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is positive semi-definite, then the current point is a local minimum. If the current point is a local minimum, terminate Phase II; otherwise go to step 7.

STEP 6. The blocking variable is ζ; make the pivot $\langle\zeta,\omega\rangle$. The pivot entry must be positive since the coefficient in the desired position is the negative of the coefficient of τ in the blocking row. Thus we have a local minimum without the capacity constraint. If all the multipliers are positive, the point is a local minimum to the original problem. Terminate Phase II. If some of the multipliers are negative, then go to step 7.

STEP 7. Let $u_1,\ldots,u_s$ denote the multipliers having negative values. Introduce a new capacity constraint $e^T x = \tau$ with $e_j > 0$, it is customary to set $e_j = 1$, $(j = 1,\ldots,s)$, $e_j = 0(j = s + 1,\ldots,n)$, where x_j, $(j = 1,\ldots,s)$, just denote the complements of the basic multipliers. With $\tau = 0$ the current point is a local minimum to the problem with the new capacity constraint. Return to step 1 and proceed as before.

This completes the description of Ritter's algorithm to determine a local minimum. After finite number of steps, either the objective function is shown to be unbounded below on the feasible region X (and hence we are finished) or we stop with a local minimum and must proceed to Phase III.

When describing his algorithm, RITTER does not employ the tabular form used here. He maintains a distinction between the x-variables and the v-variables which is not necessary.

We now return to the task of identifying the data for the problem to be solved in the construction of a cutting plane. Let $\bar{x}$ be the local

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minimum just found. As stated in Section 3, one could, in principle, determine a vertex of the feasible region which lies in the lowest-dimensional face of X containing $\bar{x}$. The primal variables and their complements could then be relabeled so that the slack variables at that vertex are the v-variables. Two non-principal block pivots could be made in the original tableau to make the v-variables and the u variables basic in the Kuhn-Tucker equations. Let the following tableau represent that situation.

(4.8)

	1	x_1	x_2	y
$u_1 =$	$\widetilde{c}_1$	$\widetilde{D}_{11}$	$\widetilde{D}_{12}$	$-\widetilde{A}'_1$
$u_2 =$	$\widetilde{c}_2$	$\widetilde{D}'_{12}$	$\widetilde{D}_{22}$	$-\widetilde{A}'_2$
$v =$	$-\widetilde{b}$	$\widetilde{A}_1$	$\widetilde{A}_2$	0

where the vector x has been partitioned so that the positive primal variables at the local minimum are x_1 and v. The vector c and u and the matrices D and A are partitioned in a compatible fashion. The tableau corresponding to the local minimum is,

(4.9)

	1	u_1	x_2	y
$x_1 =$	$-\widetilde{D}_{11}^{-1}\widetilde{c}_1$	$\widetilde{D}_{11}^{-1}$	$-\widetilde{D}_{11}^{-1}\widetilde{D}_{12}$	$\widetilde{D}_{11}^{-1}\widetilde{A}'_1$
$u_2 =$	$\widetilde{c}_2 - \widetilde{D}'_{12}\widetilde{D}_{11}^{-1}\widetilde{c}_1$	$\widetilde{D}'_{12}\widetilde{D}_{11}^{-1}$	$\widetilde{D}_{22}-\widetilde{D}'_{12}\widetilde{D}_{11}^{-1}\widetilde{D}_{12}$	$-\widetilde{A}'_2+\widetilde{D}'_{12}\widetilde{D}_{11}^{-1}\widetilde{A}'_1$
$v =$	$-\widetilde{b} - \widetilde{A}_1\widetilde{D}_{11}^{-1}\widetilde{c}_1$	$\widetilde{A}_1\widetilde{D}_{11}^{-1}$	$\widetilde{A}_2 -\widetilde{A}_1\widetilde{D}_{11}^{-1}\widetilde{D}_{12}$	$\widetilde{A}_1\widetilde{D}_{11}^{-1}\widetilde{A}'_1$

So the problem to be solved in Phase III is to find a nonnegative complementary solution for

(4.10)

$$
\begin{array}{c|c|c|c}
 & 1 & \tilde{x} & \tilde{\sigma} \\
\hline
\tilde{u} = & 0 & \tilde{D}_{22} - \tilde{D}_{12}^T \tilde{D}_{11}^{-1} \tilde{D}_{12} & \tilde{c}_2 - \tilde{D}_{12}^T \tilde{D}_{11}^{-1} \tilde{c}_1 \\
\tilde{w} = & 1 & \tilde{c}_2 - \tilde{D}_{12}^T \tilde{D}_{11}^{-1} \tilde{c}_1 & 0 \\
\end{array}
$$

that gives the largest value for $\tilde{\sigma}$.

Tableau (4.10) is formed from data contained in the rows associated with the multipliers that are basic in tableau associated with the local minimum. This indicates that to set up the required Phase II problem, it is only necessary to identify the primal variables that are cumplementary to the multipliers that are basic in the final tableau of Phase II.

$$\varphi(x) = \frac{x_1}{2} - \frac{x_2}{2} - \frac{x_1^2}{2} + \frac{x_2^2}{2} \qquad \text{(Objective Function)}$$
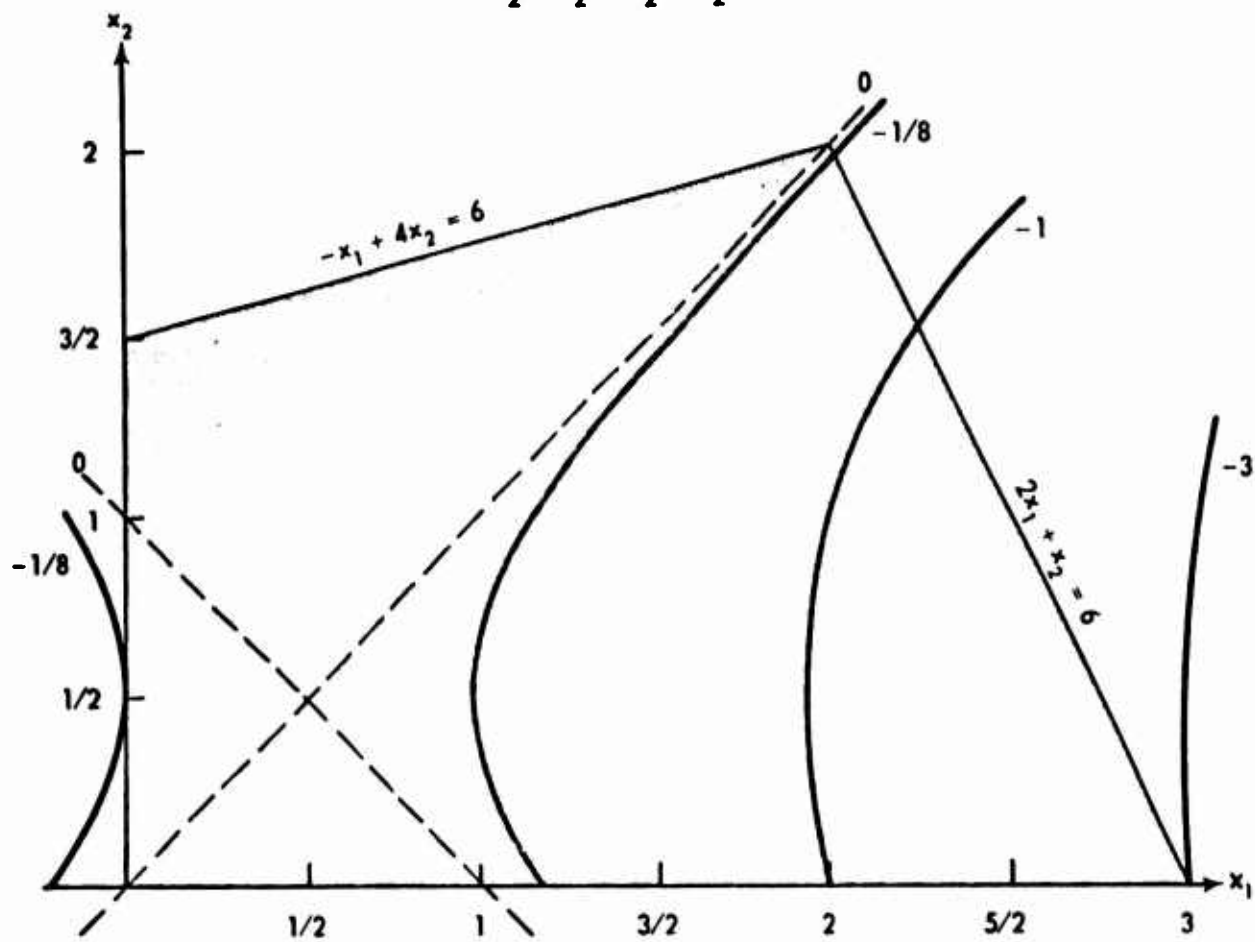
$$X = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : \begin{array}{r} 2x_1 + x_2 \le 6 \\ -x_1 + 4x_2 \le 6 \\ x_1 \ge 0 \\ x_2 \ge 0 \end{array} \right\} \qquad \text{(Constraint Set)}$$

Fig. 1—Example of a Nonconvex Quadratic Program

4.3. An Example Solved Using Ritter's Method

We will illustrate the method on the nonconvex quadratic program:

$$\text{minimize} \quad \varphi(x) := \tfrac{1}{2}x_1 - \tfrac{1}{2}x_2 - \tfrac{1}{2}x_1^2 + \tfrac{1}{2}x_2^2$$

$$2x_1 + x_2 \leq 6$$

$$-x_1 + 4x_2 \leq 6$$

$$x_1 \geq 0, \; x_2 \geq 0$$

Figure 1 depicts the set of feasible points and some of the isovalue contours of φ. As indicated there, the problem has three stationary points; two are local minima, and one is a saddle point.

The point $x = 0$ is a feasible point, so the method can be started in Phase II to determine a local minimum.

Phase II. Determining a local minimum.

The capacity constraint is $x_1 + x_2 \leq \tau$. The first two pivots are for initialization.

	1	τ	x₁	x₂	y₁	y₂	ζ	
u₁ =	1/2	0	-1	0	2	-1	1	Pivot:
u₂ =	-1/2	0	0	1	1	4	1*	⟨u₂,ζ⟩
v₁ =	6	0	-2	-1	0	0	0	
v₂ =	6	0	1	-4	0	0	0	
w =	0	1	-1	-1	0	0	0	

	1	τ	x₁	x₂	y₁	y₂	u₂	
u₁ =	1	0	-1	-1	1	-5	1	Pivot:
ζ =	1/2	0	0	-1	-1	-4	1	⟨w,x₂⟩
v₁ =	6	0	-2	-1	0	0	0	
v₂ =	6	0	1	-4	0	0	0	
w =	0	1	-1	-1*	0	0	0	

	1	τ	x₁	w	y₁	y₂	u₂	
u₁ =	1	-1	0	1	1	-5	1	For τ=1/2
ζ =	1/2	-1	1	1*	-1	-4	1	ζ is the
v₁ =	6	-1	-1	1	0	0	0	blocking
v₂ =	6	-4	5	4	0	0	0	variable.
x₂ =	0	1	-1	-1	0	0	0	Make the principal pivot ⟨ζ,w⟩ .

	1	τ	x₁	ζ	y₁	y₂	u₂	
u₁ =	1/2	0	-1	1	2	-1	0	A Kuhn-Tucker
w =	-1/2	1	-1	1	1	4	-1	point and the
v₁ =	11/2	0	-2	1	1	4	-1	point x̄₁ = 0,
v₂ =	4	0	1	4	4	16	-4	x̄₂ = 1, is a
x₂ =	1/2	0	0	-1	-1	-4	1	local minimum.

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Phase III. Constructing a cutting plane.

To place the cutting plane the complementary solution for

	1	$\widetilde{x}_1$	$\widetilde{\sigma}$
$\widetilde{u}_1 =$	0	-1	1/2
$\widetilde{\omega} =$	1	-1/2	0

with $\widetilde{\omega} = 0$ and the largest value of $\widetilde{\sigma}$ is to be found. Clearly the only possible solution is with $\widetilde{\sigma}$ and $\widetilde{x}_1$ as the basic variables. So the solution tabluau is

	1	$\widetilde{\omega}$	$\widetilde{u}_1$
$\widetilde{\sigma} =$	4	-4	2
$\widetilde{x}_1 =$	2	-2	0

For the problem

$$\text{minimize} \quad \varphi(x) = \tfrac{1}{2}x_1 - \tfrac{1}{2}x_2 - \tfrac{1}{2}x_1^2 + \tfrac{1}{2}x_2^2$$

$$x_1 \geq 0$$

$$\tfrac{1}{2}x_1 \leq \tau$$

and $0 \leq \tau \leq \tfrac{1}{4}$ the point $x_1(\tau) = 2\tau$, $x_2(\tau) = \tfrac{1}{2}$ is not even a stationary point. But for $\tfrac{1}{4} < \tau \leq \tfrac{1}{2}$, $x(\tau)$ is a local minimum and $\bar{x} = \begin{pmatrix} 0 \\ \tfrac{1}{2} \end{pmatrix}$ is the global minimum. Finally for $\tau \geq \tfrac{1}{2}$, $x(\tau)$ is the global minimum.

The final step of constructing the cutting plane requires the determination of τ_1 and τ_2. Solving for the largest τ such that $\varphi(x(\tau)) = \varphi(\bar{x})$ gives $\tau_1 = \tfrac{1}{2}$, and solving for the largest τ such that

$x(\tau) \in X$ gives $\tau_2 = 1/8$. Thus $\bar{\tau} = 11/8$ and the cutting plane to be adjoined is $\frac{1}{2}x_1 \geq 11/8$ (i.e., $x_1 \geq 11/4$).

With the cutting plane adjoined, the problem becomes

$$\text{minimize} \quad \phi(x) = \tfrac{1}{2}x_1 - \tfrac{1}{2}x_2 - \tfrac{1}{2}x_1^2 + \tfrac{1}{2}x_2^2$$

$$\text{subject to} \quad 2x_1 + x_2 \leq 6$$

$$- x_1 + 4x_2 \leq 6$$

$$x_1 \quad\quad \geq 11/4$$

$$x_1 \geq 0, \; x_2 \geq 0 \; .$$

Notice that $x = 0$ is not a feasible point so Phase I must be executed. The result of Phase I is a problem in the form

$$\text{minimize} \quad \varphi(v_3, v_1) = -9/4v_3 + \tfrac{1}{2}(v_3, v_1)\begin{pmatrix} 3 & 2 \\ \sim & 1 \end{pmatrix}\begin{pmatrix} v_3 \\ v_1 \end{pmatrix}$$

$$\text{subject to} \quad -9v_3 - 4v_1 + v_2 \quad\quad = 27/4$$

$$2v_3 + v_1 \quad + x_2 \quad = 1/2$$

$$- v_3 \quad\quad\quad + x_1 = 11/4$$

$$x \geq 0, \; v \geq 0 \; .$$

Phase II. Determining a second local minimum.

Now the capacity constraint $v_1 + v_3 \le \tau$ is introduced. We perform two initialization pivots.

	1	τ	v_3	v_1	y_2	u_2	u_1	ζ	
y_3 =	-9/4 .	0	3	2	-9	2	-1	1*	Pivot:
y_1 =	0 .	0	2	1	-4	1	0	1	⟨y_3,ζ⟩
v_2 =	27/4 .	0	9	4	0	0	0	0	
x_2 =	1/2 .	0	-2	-1	0	0	0	0	
x_1 =	11/4 .	0	1	0	0	0	0	0	
ω =	0 .	1	-1	-1	0	0	0	0.	

	1	τ	v_3	v_1	y_2	u_2	u_1	y_3	
ζ =	9/4 .	0	-3	-2	9	-2	1	1	Pivot:
y_1 =	9/4 .	0	-1	-1	5	-1	1	1	⟨ω,v_3⟩
v_2 =	27/4 .	0	9	4	0	0	0	0	
x_2 =	1/2 .	0	-2	-1	0	0	0	0	
x_1 =	11/4 .	0	1	0	0	0	0	0	
ω =	0 .	1	-1*	-1	0	0	0	0	

In the following tableau when $\tau = 1/4$, x_2 is the blocking candidate. A new vertex of the feasible region has been found. Make the multiplier of the capacity constraint nonbasic by performing ⟨x_2,ω⟩ and ⟨ζ,u_2⟩.

	1	τ	ω	v_1	y_2	u_2	u_1	y_3	
ζ =	9/4 .	-3	3	1	9	-2	1	1	Pivot:
y_1 =	9/4 .	-1	1	0	5	-1	1	1	⟨x_2,ω⟩
v_2 =	27/4 .	9	-9	-5	0	0	0	0	
x_2 =	1/2 .	-2	2*	1	0	0	0	0	
x_1 =	11/4 .	1	-1	-1	0	0	0	0	
v_3 =	0 .	1	-1	-1	0	0	0	0	

	1	τ	x_2	v_1	y_2	u_2	u_1	y_3	
ζ ==	3/2 .	0	3/2	-1/2	9	-2*	1	1	Pivot:
y_1 ==	2 .	0	1/2	-1/2	5	-1	1	1	⟨ζ,u_2⟩
v_2 =	9 .	0	-9/2	-1/2	0	0	0	0	
ω =	-1/4 .	1	1/2	1/2	0	0	0	0	
x_1 =	3 .	0	-1/2	-1/2	0	0	0	0	
v_3 =	1/4 .	0	-1/2	-1/2	0	0	0	0	

The resulting tableau gives another local minimum.

		1	τ	x_2	v_1	y_2	ζ	u_1	y_3
u_2	=	3/4	0	3/4	-1/4	9/2	-1/2	1/2	1/2
y_1	=	5/4	0	-1/4	-1/4	1/2	1/2	1/2	1/2
v_2	=	9	0	-9/2	-1/2	0	0	0	0
ω	=	-1/4	1	1/2	1/2	0	0	0	0
x_1	=	3	0	-1/2	-1/2	0	0	0	0
v_3	=	1/4	0	-1/2	-1/2	0	0	0	0

Placing the next cutting plane results in an empty feasible region.

4.4 Concluding Remarks

In Phase II of Ritter's method, it is not necessary to use the algorithm given here. It can be replaced by any quadratic programming algorithm. Such algorithms have been given by BEALE (1955), (1967) and KELLER (1969). The latter is closely related to Ritter's Phase II but is simpler and makes no use of a capacity constraint. A more detailed analysis of this relationship will be the subject of a future report.

Each phase of Ritter's method has been shown to be finite and in his paper RITTER (1966) has given an argument proving that when X is bounded one will cycle through the three phases only a finite number of times. It is apparent from the description of the method that it may require an inordinate computational effort. For concave problems where there can be a very large number of local minima, one could be so unfortunate as to locate them in order of decreasing value of the objective function, and thereby enumerate them all. Although in this case the computational effort to construct the cutting plane is small, it may be necessary to add a large number of them. This can lead to much work because the effort to locate the local minima is proportional to the number of variables plus the number of constraints. Linear programming can be used to eliminate constraints made superfluous by the cutting plane but this requires additional computational effort. (See RITTER (1965).)

The case where D has a number of both positive and negative eigenvalues can possibly cause this algorithm quite a lot of difficulty. In this case, the finding of a solution to the problem in order to construct

the cutting plane may require an inordinate amount of computational effort.

Since problem (3.9) has such simple structure, it may be fruitful to

devise methods that exploit its special features.

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The cutting plane method of K. Ritter for nonconvex quadratic programming is reviewed and placed in the context of principal pivoting methods. The algorithm is illustrated by an example.

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