AN ODD THEOREM

BY

B. CURTIS EAVES

TECHNICAL REPORT NO. 69-10
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Operations Research House
Stanford University
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Abstract

Let C be a bounded convex polyhedral set and let $f:C \to C$ be continuous and piecewise linear. Using notions from complementary pivot theory, it is shown that if each fixed point of f lies interior to some piece of linearity, then f has an odd number of fixed points. In addition, an algorithm is given for computing a fixed point of f.

1. Introduction

Using the main ideas of complementary pivot theory (see [1] - [8]), we prove the following theorem.

Theorem: Let C be a bounded convex polyhedral set and let $f:C \to C$ be continuous and piecewise linear. If each fixed point of f lies interior to some piece of linearity, then f has an odd number of fixed points.

An algorithm for computing (finitely quick) a fixed point of f (whether or not the interior condition is met) is a by-product of the proof of the theorem.

The essential difference between our attitude and that of [1], [4], [8], and hence Sperner's Lemma is that we label vertices of a triangulation with vectors instead of integers. For a simplex to be "completely labeled," there must be a convex combination of the vector labels which generate zero.

2. Graph Principle

Our proof will rest on a simple graph principle; the same principle used in [1] - [8]. By a graph (N,A), we mean a finite set N together with a symmetric anti-reflexive relation A on N. If aAb (hence bAa and $a \neq b$), we say a and b are adjacent. We call an element a of N odd or even if it is adjacent to an odd or even number of elements of N, respectively. Recall that a graph has an even number of odd elements. In the next section, we construct a particular graph and use this device to prove our theorem. In this graph each element will be adjacent to exactly one or two elements; in this case, the odd elements have a natural pairing.

3. The Theorem and Proof

Let C be a finite dimensional bounded convex polyhedral set. We can assume that C lies in n-dimensional Eucledian space and that it has an interior. Let T be a triangulation of C (i.e., T is a complex and $|T| = C$, see [9]), and let $f: C \to C$ be a continuous function which is linear (i.e., affine) on each simplex of T. Let (C,T,f) denote such a triple. If each fixed point of f is interior to an n-simplex of T, then we say that (C,T,f) is <u>nondegenerate</u>. To prove our result, it is sufficient to prove the following theorem.

<u>Theorem</u>: If (C,T,f) is nondegenerate, then f has an odd number of fixed points.

Given (C,T,f) we notice that f is completely determined by its action on the vertices of T. Indeed, if $r \varepsilon S \varepsilon T$, then $f(r) = \Sigma_s f(s)x_s$ where $r = \Sigma_s sx_s$, $\Sigma_s x_s = 1$, and $x_s \geq 0$ (where s ranges over the vertices of S).

A simplex S of T contains a fixed point if and only if the system

$$\Sigma_s (f(s)-s)x_s = 0$$

$$\Sigma_s x_s = 1$$

has a nonnegative solution in the x_s (where s ranges over the vertices of S). In this case $\Sigma_s sx_s$ is the fixed point. If (C,T,f) is nondegenerate, then it follows that a simplex will contain at most one fixed point; if S contains a fixed point, then S is an n-simplex and the solution x_s of the system above is unique and positive.

Assume (C,T,f) is nondegenerate. Let C' be an n-simplex which contains C in its interior. We shall extend both T and f to C' to generate (C',T',f'). Let $v_o,\ldots,v_n$ be the vertices of C'.

Extend T to a triangulation T' of C' without introducing new vertices; that is, a vertex of T' is either a vertex of T or a vertex of C'. Hence each (n-1)-face of C' is an element of T'.

Temporarily let r be any point of C. Define f' on C' by setting $f'(t) = f(t)$ for vertices of T and $f'(v_i) = r$ for vertices of C' and then by extending f' linearly on the simplexes of T'. Now we further specify r. Select $r \varepsilon C$ such that for any (n-1)-simplex S of T' the system

$$(f'(v_o)-v_o)x_S + \Sigma_s (f'(s)-s)x_s = 0$$

$$x_S + \Sigma_s x_s = 1$$

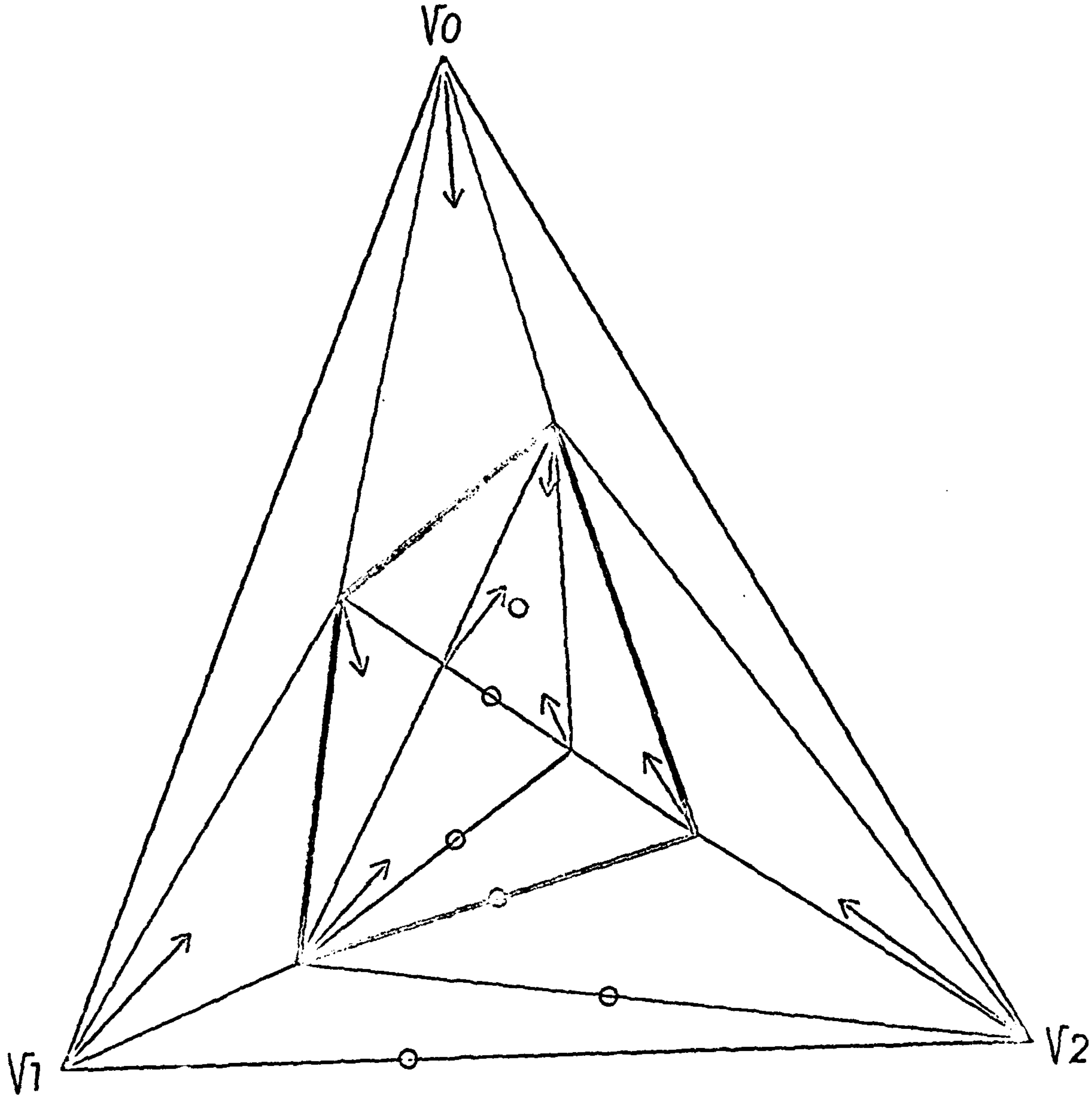
either has a unique positive solution in x_S and the x_s or else has no nonnegative solution (where s ranges over the vertices of S). Such r's are very available; in fact, almost every element of C will suffice.

We can now define a particular graph. Let (C',T',f') be generated as just described. Let N_1 *be the set of simplexes of* T' *which contain* fixed points; these simplexes will be n-simplexes of T. Let N_2 be the set of (n-1)-simplexes S in T' for which the system

$$(f'(v_o)-v_o)x_S + \Sigma_s(f'(s)-s)x_s = 0$$

$$x_S + \Sigma_s x_s = 1$$

has a nonnegative solution in x_S and the x_s (where s ranges over the vertices of S); these solutions will be positive and unique. Let $N = N_1 \cup N_2$. We define two distinct simplexes of N to be adjacent if they lie in a common simplex of T'.


V0
V1
V2

Let S_o be the (n-1)-simplex with vertices $\{v_1,\ldots,v_n\}$. One can now establish that $S_o \varepsilon N_2$, that each element of $N_1 \cup \{S_o\}$ is adjacent to exactly one element of N, and that each element of $N_2 \sim \{S_o\}$ is adjacent to exactly two elements of N. From the graph principle, we see that N_1 contains an odd number of elements; this establishes the theorem.

The figure illustrates the structure for a 2-dimensional C. The arrow at a vertex t denotes the direction of f'(t)-t (further specification is unnecessary), the heavy lines denote the boundary of C, and the small circles denote the simplexes which are in N.

4. The Algorithm

The preceding development gives a procedure for calculating finitely quick a fixed point of (C,T,f). After constructing (C',T',f'), one begins at S_o and proceeds to an adjacent simplex, etc. This step from simplex to simplex is essentially a "pivot" as known in linear programming. One eventually terminates with a simplex containing a fixed point, and hence, one has the fixed point.

The next section shows that if (C,T,f) is degenerate, the algorithm may still be applied to find a fixed point ((C,T,f) is altered slightly to make it nondegenerate; however, from solely computational considerations, there are far more efficient methods of dealing with degeneracy).

The section on Brouwer's Theorem demonstrates that if $g:C\to C$ is a continuous function and if $\varepsilon>0$, then the algorithm can be used to compute a point $t \varepsilon C$ such that $|g(t)-t| \leq \varepsilon$. Scarf's procedure [8] has this capability.

Kuhn [4] describes an extremely efficient data handling procedure which can be adapted to our algorithm.

5. Perturbation and Stability

Here we show that nondegeneracy is stable and that nondegeneracy can be achieved via a perturbation.

Suppose that (C,T,f) is nondegenerate. Then there is an $\varepsilon>0$ such that (S,T,g) is nondegenerate and such that a simplex of T will contain a fixed point of f if and only if it contains a fixed point of g, if $|f-g|\leq\varepsilon$.

Consider (C,T,f) and (C,T,g). Suppose that $g(C) = r$ and that r is interior to an n-simplex of T. Then there is an $\varepsilon_o>0$ such that for $0<\varepsilon\leq\varepsilon_o$, $(C,T,(1-\varepsilon)f+\varepsilon g)$ is nondegenerate. Further, if a simplex of T contains a fixed point of $(1-\varepsilon)f+\varepsilon g$, then it contains a fixed point of f for $0<\varepsilon\leq\varepsilon_o$.

6. Brouwer's Theorem and Extensions

From Sections 4 and 6 we see that for any (C,T,f) there is a fixed point. We can now prove Brouwer's fixed point theorem.

Let $g:C\to C$ be a continuous function. Choose (C,T_n,f_n) such that $|f_n-g|\leq\frac{1}{n}$ for $n=1,2,\ldots$. Let s_n be a fixed point of f_n. We have $|g(s_n)-s_n|\leq\frac{1}{n}$. If s is a custer point of the s_n sequence, then clearly s is a fixed point of g.

For the general case $g:C\to C$ where g is continuous and C is compact and convex, our theorem has implications regarding the parity of the number of fixed points. These results will be reported on in another paper.

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13. ABSTRACT

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