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AUGUST 1969

SYSTEMS ANALYSIS DIRECTORATE
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FOREWORD

The results of a study conducted under DA Project 1P014501B14A05, AMCMS Code 5011.11.85300.04, are presented in this report.

The design of a weapon system provides a natural setting for an optimization problem. The design requirements stipulate that the system is to perform some task at some index of performance. The optimizer then is to search for the design parameters such that the weapon system not only performs its task, but also maximizes its performance. The objective of this study is to apply a relatively new steepest-descent numerical procedure to an artillery design problem which involves the dynamic behavior of a 105mm howitzer which is fired while resting on rubber tires. The tires act like a spring during the firing cycle which causes the weapon to leave the ground so that the likelihood of it being zeroed in for the next round has been reduced considerably. The purpose, then, will be to minimize the pitch motion of the weapon by obtaining a set of design parameters which are subject to equality as well as inequality constraints prescribed by design requirements.

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SUMMARY

An artillery weapon mounted on tires or tracks has some undesirable features at high angle fire. Unlike the hard mount (weapon rests on a base plate) the flexible mount will have a pitch motion. That is, during the recoil stroke, the tires load up or compress and act like a spring, and just before counterrecoil begins, the tires begin to unload sending the weapon off the ground. Such a phenomenon is known as a secondary recoil effect. The control rod design becomes much more difficult with this secondary recoil effect because an additional acceleration term enters into the recoil equations. Also, it is obvious that when the weapon comes to rest the likelihood of it being zeroed in for the next round has been reduced considerably. The purpose of this study is to reduce the pitch motion of the weapon and at the same time determine the orifice areas for the control rod design.

This study was performed on the XM164, a light weight, 105mm howitzer. The present control rod design for short recoil (75 degrees elevation) yields approximately six inches of "hop". Results from this study show that between 45 and 86 per cent reduction in the pitch motion is possible (depending upon which design option is used) by determining the optimal shape rod force. Once this rod force has been found, the orifice areas can be determined.

A steepest-descent numerical procedure will be used to minimize the pitch motion of the weapon along with satisfying certain design constraints imposed upon the system. This technique starts with an estimated design,

analyzes it, and then improves on the design. It is an iterative process and at each iteration an improvement is made until no significant gains can be achieved.

The results of this study clearly indicate that weapon performance can be improved by using methods of optimal design.

I. Introduction

Weapon systems of today and of the future are becoming more complex and, as a result of this complexity, the engineer's intuition and experience become increasingly more difficult to apply because of the possible trade-offs in the design parameters. Because the task of the engineer becomes more difficult in meeting requested design requirements as weapon systems become more complex, it is important that the design procedure be represented by mathematical modeling, i.e., a translation of the physical description of the problem into mathematical terms. Although a mathematical model may be formulated, the solution may still be difficult to obtain for several reasons The model itself may become very complex and that which is even more difficult to cope with is the fact that some of the parameters may only be engineering estimates based on past experience or perhaps very little is known about the dynamic behavior of a parameter. Also, the solution must be a physical realization of the mathematical design. In short, the conversion of mathematical theory into an engineering accomplishment may not be an easy task.

The design of a weapon system provides a natural setting for an optimization problem assuming a knowledge of all environmental factors which influence the design process. The design requirements specify that the system is to perform some task at some index of performance. To determine the optimum solution, the concept of index of performance is introduced and will be defined as the functional relationship among the system characteristics. The optimizer then is to search for the admissible parameters such that the weapon system not only performs its task, but also maximizes its performance. As design specifications tend to tighten, it becomes increasingly important to design optimum systems relative to some performance criterion, and, in fact, specify that performance be optimized.

It is only natural then that the methods used in the design of optimum systems be of interest for these are the analytical tools which will determine the results for the optimal design problem. Because of the computer, many different disciplines have provided revolutionary aids with respect to analytical tools for the solution to problems that were seemingly hopeless only several years ago. The objective of this study is to apply the relatively new technology to an artillery design problem and to develop a method which will aid the engineer in obtaining design parameters subject to certain constraints and require that the performance of the weapon be optimal in some sense.

II. Statement of the Problem

An artillery weapon mounted on tires or tracks has some undesirable features. Unlike the hard mount (weapon rests on a base plate), the flexible mount will have a pitch motion. During the recoil stroke, when the weapon is fired at 75 degrees elevation, the tires load up or compress; and when counterrecoil begins, the tires act like a spring and unload sending the tires off the ground. It is quite obvious that, when the weapon comes to rest, the likelihood of it being zeroed in for the next round has been reduced considerably, especially for high rate of fire weapons. This phenomenon is known as a secondary recoil effect because an additional acceleration term enters into the recoil equations. Because of this secondary recoil effect, the control rod design becomes much more difficult. For short recoil, the orifice areas in the control rod are designed at maximum elevation (75 degrees); therefore, when elevation is mentioned throughout the remainder of this report, it refers to maximum elevation. The weapon positioned for high-angle fire is shown in Figure 1.

The purpose of this study will be to develop a systematic control rod design procedure characterized by mathematical modeling for the high-speed digital computer. Conceptually, it will be one phase of a study that will give the optimal weapon which meets the given design requirements. To do this, a steepest-descent numerical procedure will be used to minimize the hop or pitch motion of the weapon and, at the same time, to determine the necessary control rod design which will minimize hop.

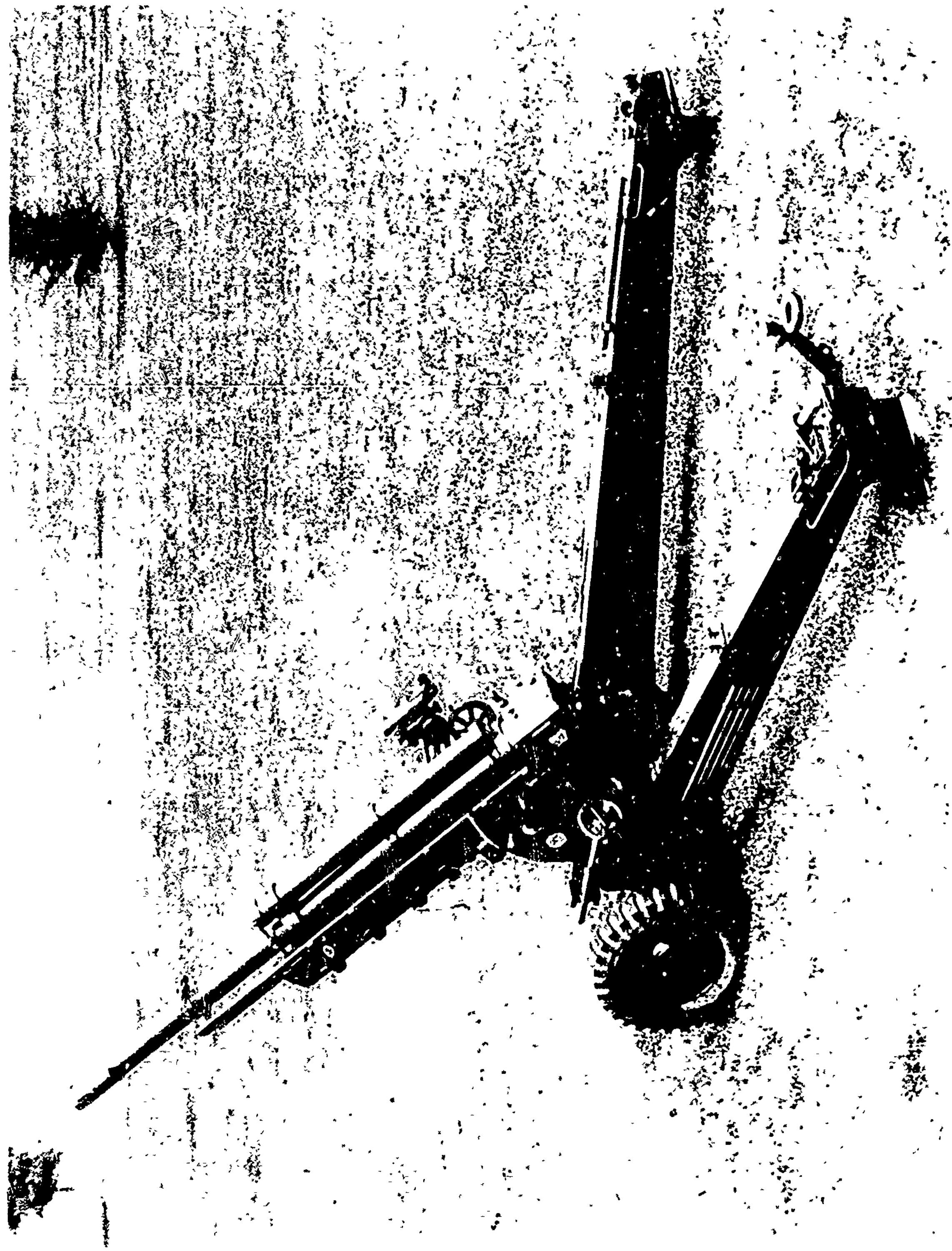


FIGURE 1

A second phase will be to incorporate geometrical effects into the optimal design problem in order to establish optimal geometries for certain configurations.

The recoil equation is of the form of Equation (1)

$$\ddot{x} + f(x)\dot{x}^2 + g(x) = h(t) \tag{1}$$

for a rigid mount. In the second term of this equation, the expression for the effect of the control rod orifice area is defined; however, without any loss of generality, the control rod orifice areas can also be obtained from a predetermined rod-pull force R(t). For the flexible mount, the above equation is coupled with the equation describing the pitch motion of the weapon and thus yielding two second-order nonlinear ordinary differential equations with prescribed initial conditions. The orifice areas are a function of the state of the system. To eliminate state variable inequality constraints, R(t) will be taken as the control variable which is to be determined to minimize hop (the pitch motion of the weapon) subject to other design constraints.

This study was performed on an existing weapon, namely, the XM164. The XM164 is a lightweight, split-trailed towed 105mm howitzer with the XM44 hydropneumatic recoil mechanism. Unlike a rigid mount, the XM164 is a flexible mount and is fired while resting on rubber tires. For a rigid mount weapon, the resisting force R(t) on the recoiling parts is designed with a trapezoidal shape as shown in Figure 2. With the proper design of the control rod orifice area, the flow of oil in the recoil mechanism is controlled and such a force, as shown in Figure 2, can be

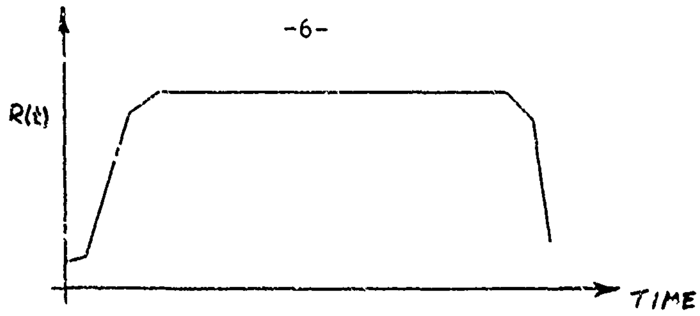


Figure 2. Rod Force During Recoil for a Rigid Mount

obtained. However, when a force (shaped as in Figure 2) is designed for the flexible mount, the question is asked, "Can this force be applied with some other 'best' shape such that it will reduce the pitch of the weapon?" This is the basic question with which this study is concerned.

In this report, the optimum rod force is defined as that curve which, according to some measure (the ncp motion), satisfies all of the requirements imposed upon the system

III. Formulation of the Problem

During the recoil, counterrecoil cycle there are four different times which are of concern. These are shown in Figure 3. At these four times

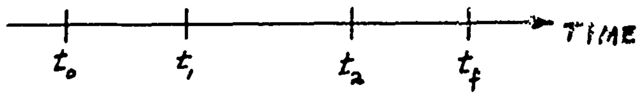


Figure 3

t_0 - initial conditions

t_1 - end of the recoil stroke

t_2 - time at which maximum hop occurs

t_f - end of counterrecoil

certain conditions must be satisfied from the design requirements. At time t_0 the initial conditions for the state of the system are given. At time t_1 the displacement of the recoiling parts is required to be equal to some specified value and the velocity of the recoiling parts must be equal to zero. At time t_2 the velocity of the pitch motion must be zero and the displacement of the pitch motion is to be a minimum. Note that it will be possible for t_2 to vary between t and t_f. Therefore, the hop or pitch motion will be minimized for the entire counterrecoil stroke. At the final time t_f, which is the end of counterrecoil, the recoiling parts must come back to its original position and the velocity of the recoiling parts will be some specified value V_f. This is to insure that the recoiling parts come back to the latch position. It will also be demanded that the total cycle time be equal to c_T seconds.

Formulating the above paragraph into mathematical notation yields

$$\text{Minimize } J = x_4(t_2) \tag{1}$$

subject to the equality constraints

$$\begin{aligned}
\psi_1 &= x_2(t_1) - \eta_0 + \eta_{max} = 0 \\
\psi_2 &= x_2(t_f) - \eta_0 = 0 \\
\psi_3 &= x_1(t_f) - V_f = 0 \\
\Omega^1 &= x_1(t_1) = 0 \\
\Omega^2 &= x_3(t_2) = 0 \\
\Omega^f &= t_f - c_T = 0
\end{aligned} \tag{2}$$

with the full set of initial conditions

$$x_1(0) = x_3(0) = x_4(0) = 0, \quad x_2(0) = \eta_0 \tag{3}$$

where ψ_i, $i = 1,2,3$ are intermediate and terminal constraint functions to be satisfied; Ω^i, Ω^2, and Ω^f define the times at which the intermediate and terminal constraint functions occur; x_1 and x_3 are the velocities of the recoiling parts and pitch motion respectively; x_2 and x_4 are the displacements of the recoiling parts and pitch motion respectively; $\psi_1 = 0$ is the constraint on the displacement of the recoiling parts such that at the end of the recoil stroke the displacement will be exactly equal to η_{max} inches. $\psi_2 = 0$ is the constraint demanding that the recoiling parts return to the latch position at the end of counterrecoil. $\psi_3 = 0$ is the constraint which requires that the velocity of the recoiling parts come into the latch position at a velocity V_f in./sec. $\Omega^1 = 0$ defines the time at which the end of the recoil occurs; $\Omega^2 = 0$ defines the times at which the pitch velocity is zero and the one with the largest displacement is selected, thus defining the time at which maximum hop occurs; $\Omega^f = 0$ defines the total cycle time to be exactly equal to c_T seconds.

It was previously mentioned that in order to eliminate state variable constraints the rod force was taken as the design (control) variable instead of the orifice areas. Using the rod force as the design variable simplifies the problem and it also gives the engineer more insight in the design process since he has an intuitive feel for the force levels the weapon system he is designing can tolerate. Thus, immediately the engineer can specify an admissible upper limit for the rod force say R_{max}, for his design, and this value may be varied by the engineer for any redesign. The following inequality constraint must hold for all time t.

$$\phi = R(t) - R_{max} \leq 0 \qquad 0 \leq t \leq t_f \tag{4}$$

Since the mathematical model must represent a physical realization, to specify one value for R_{max} is not enough. This result was made available from the first set of computer runs and can be seen in Figure 4. Because it was not known how the optimal shape rod force would behave, the design variable R(t) was allowed to take on any shape just as long as it did not exceed R_{max}. It can be seen from Figure 4 that the rod force attained its maximum value at time t_0. The mathematical model says that the best way to reduce the "hop" is to let the recoiling parts move forward first as in the firing-out-of-battery concept. This, of course, is a physical impossibility for the weapon under study since the recoiling parts cannot travel forward beyond the latch position. Additional constraints were subsequently put on the design variable during the first few milliseconds of the recoil stroke.

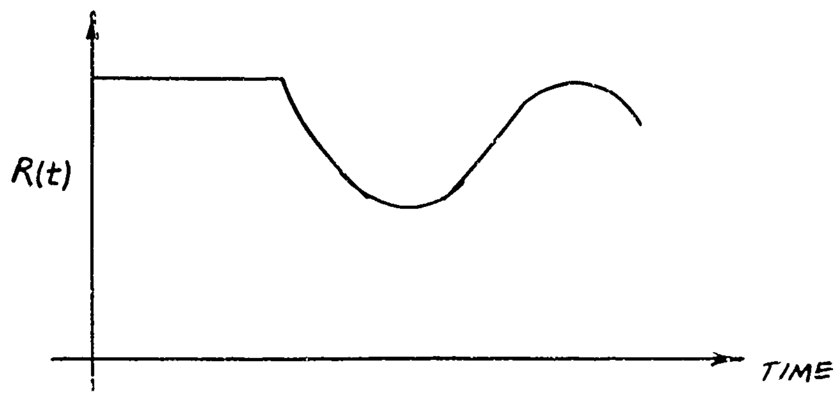


Figure 4. Rod Force With No Rise Constraint For First Few Milliseconds

The optimization problem has now been formulated. The objective function (see Equation III-1) has been defined for the process (see Equations IV-1,2) that is to be optimized subject to the constraints (see Equations III-2,4) that are to be satisfied.

All that must be done now is to put the problem into the steepest-descent formulation. The next section simplifies the equations of motion for the XM164 howitzer.

IV. Translation and Rotational Equations of Motion for the XM164 Howitzer

The differential equations to be solved are given below[Ref. 1]. Equations (1) and (2) are the translational and rotational equations of motion for the XM164 howitzer. Equations (3) and (4) determine the guide friction.

$$M_a[\ddot{\eta} - (\zeta - Y_t \sin\gamma + Z_t \cos\gamma)\ddot{\phi} - (\eta + Y_t \cos\gamma + Z_t \sin\gamma)\dot{\phi}^2]$$
$$= R(t) - B(t) - M_a\, g \sin(\gamma+\phi) - \mu(|S_1| + |S_2|)\,\mathrm{sgn}(\dot{\eta}) \qquad (1)$$

$$\{M_a(\eta + Y_t \cos\gamma + Z_t \sin\gamma)^2 + M_b[(\eta_b \cos\gamma + Y_t - \zeta_b \sin\gamma)^2$$
$$+ (\eta_b \sin\gamma + Z_t + \zeta_b \cos\gamma)^2] + M_d(Y_d^2 + Z_d^2) + I_a + I_b + I_d\}\ddot{\phi}$$
$$+ 2M_a\dot{\eta}\dot{\phi}(\eta + Y_t \cos\gamma + Z_t \sin\gamma) - M_a\dot{\phi}^2(\eta + Y_t \cos\gamma + Z_t \sin\gamma)\cdot$$
$$(\zeta - Y_t \sin\gamma + Z_t \cos\gamma) = B(t)\cdot(\zeta_1 - \zeta) + \qquad (2)$$
$$[R(t) - \mu(|S_1| + |S_2|)\,\mathrm{sgn}(\dot{\eta})]\,(\zeta - Y_t \sin\gamma + Z_t \cos\gamma)$$
$$- g\,\{M_a[\eta + Y_t \cos\gamma + Z_t \sin\gamma] \cos(\gamma+\phi) + M_d(Y_d \cos\phi - Z_d \sin\phi)$$
$$+ M_b[Y_t \cos\phi - Z_t \sin\phi + \eta_b \cos(\gamma+\phi) - \zeta_b \sin(\gamma+\phi)]\} - k(\phi-\phi_{st}) - c\dot{\phi}$$

$$M_a[2\dot{\eta}\dot{\phi} + (\eta + Y_t \cos\gamma + Z_t \sin\gamma)\ddot{\phi} - (\zeta + Z_t \cos\gamma - Y_t \sin\gamma)\dot{\phi}^2]$$
$$= S_1 + S_2 - M_a\, g \cos(\gamma+\phi) \qquad (3)$$

$$I_a\ddot{\phi} = S_1(q_1 - \eta) + S_2\,(q_2 - \eta) - B(t)\cdot(\zeta - \zeta_1)$$
$$+ R(t)\cdot(\zeta - \zeta_2) - \mu[|S_1|(\zeta - \alpha) + |S_2|(\zeta - \beta)]\,\mathrm{sgn}(\dot{\eta}) \qquad (4)$$

For small ϕ the following approximations are made.

$$\sin\phi = \phi$$

$$\cos\phi = 1 - \frac{\phi^2}{2}$$

The $\cos(\gamma+\phi)$ and the $\sin(\gamma+\phi)$ then become

$$\cos(\gamma+\phi) = \cos\gamma - \phi^2 \frac{\cos\gamma}{2} - \phi \sin\gamma$$

$$\sin(\gamma+\phi) = \sin\gamma - \phi^2 \sin\gamma/2 + \phi \cos\gamma .$$

From the above approximations and the following definitions, equations (1) and (2) can be simplified.

$CON1 = M_a$

$CON2 = - M_a(\zeta - Y_t \sin\gamma + Z_t \cos\gamma)$

$CON3 = - \mu(|S_1| + |S_2|) \operatorname{sgn}(\dot{n})$

$CON4 = Y_t \cos\gamma + Z_t \sin\gamma$

$CON5 = M_b[(n_b \cos\gamma + Y_t - \zeta_b \sin\gamma)^2 + (n_b \sin\gamma + Z_t + \zeta_b \cos\gamma)^2]$
$\quad + M_d(Y_d^2 + Z_d^2) + I_a + I_b + I_d$

$CON6 = \zeta - Y_t \sin\gamma + Z_t \cos\gamma$

$CON7 = \zeta_i - \zeta$

$CON8 = SIN\gamma$

$CON9 = COS\gamma$

$CON10 = - M_a \cdot g \cdot CON8$

$CON11 = M_a \cdot g \cdot CON8/2$

$CON12 = M_a \cdot CON4$

$CON13 = - 2M_a$

$CON14 = - 2M_a \cdot CON4$

$$CON15 = -\ k \cdot \phi_{st}$$

$$CON16 = M_a \cdot CON6$$

$$CON17 = M_a \cdot CON6 \cdot CON4$$

$$CON18 = -\ g \cdot M_a \cdot CON9$$

$$CON19 = g \cdot M_a \cdot CON9/2$$

$$CON20 = -\ g \cdot M_a \cdot CON4 \cdot CON9$$

$$CON21 = g \cdot M_a \cdot CON4 \cdot CON9/2$$

$$CON22 = -\ g \cdot M_d \cdot Y_d$$

$$CON23 = g \cdot M_d \cdot Y_d/2$$

$$CON24 = g \cdot M_d \cdot Z_d$$

$$CON25 = -\ g \cdot M_b \cdot Y_t$$

$$CON26 = g \cdot M_b \cdot Y_t/2$$

$$CON27 = g \cdot M_b \cdot Z_t$$

$$CON28 = -\ g \cdot M_b \cdot \eta_b \cdot CON9$$

$$CON29 = g \cdot M_b \cdot \eta_b \cdot CON9/2$$

$$CON30 = g \cdot M_b \cdot \zeta_b \cdot CON8$$

$$CON31 = -\ g \cdot M_b \cdot \zeta_b \cdot CON8/2$$

$$CON32 = g \cdot M_a \cdot CON8$$

$$CON33 = g \cdot M_a \cdot CON4 \cdot CON8$$

$$CON34 = M_b \cdot g \cdot \eta_b \cdot CON8$$

$CON35 = M_b \cdot g \cdot \zeta_p \cdot CON9$

$CON36 = CON20 + CON22 + CON25 + CON28 + CON30$

$CON37 = CON21 + CON23 + CON26 + CON29 + CON31$

$CON38 = CON24 + CON27 + CON33 + CON34 + CON35 - k$

$CON39 = CON15 + CON36$

$CON40 = - M_a \cdot g \cdot CON9$

With the above definitions Equations (1) and (2) may now be written as

$$CON1 \cdot \ddot{\eta} + CON2 \cdot \ddot{\phi} = R(t) - B(t) + CON3 + CON10 + CON11 \cdot \phi^2 + M_a \cdot \eta \cdot \dot{\phi}^2 + CON12 \cdot \dot{\phi}^2 + \phi \cdot CON40. \quad (5)$$

$$\{M_a(\eta + CON4)^2 + CON5\}\ddot{\phi} = CON13 \cdot \dot{\eta} \cdot \dot{\phi} \cdot \eta + CON14 \cdot \dot{\eta}\dot{\phi} + CON38 \cdot \phi + CON32 \cdot \eta\phi + CON39 - c\dot{\phi} + CON16 \cdot \dot{\phi}^2 \eta + CON17 \cdot \dot{\phi}^2 + B(t) \cdot CON7 + [R(t) + CON3] \cdot CON6 + CON18 \cdot \eta + CON19 \cdot \eta \cdot \phi^2 + CON37 \cdot \phi^2 \quad (6)$$

Equations (5) and (6) can be put into the following form

$$v_{11}\ddot{\eta} + v_{12}\ddot{\phi} = v_{13}$$
$$v_{22}\ddot{\phi} = v_{23} \quad (7)$$

where

$v_{11} = CON1$

$v_{12} = CON2$

$$v_{13} = R(t) - B(t) + CON3 + CON10 + CON11\cdot\phi^2 + M_a n\dot{\phi}^2$$

$$+ CON12\cdot\dot{\phi}^2 + CON40\cdot\phi$$

$$v_{21} = 0$$

$$v_{22} = M_a(n + CON4)^2 + CON5$$

$$V_{23} = CON13\cdot\dot{n}\dot{\phi}n + CON14\cdot\dot{n}\dot{\phi} + CON38\cdot\phi + CON32\cdot n\phi$$

$$+ CON39 - c\dot{\phi} + CON16\cdot\dot{\phi}^2 n + CON17\cdot\dot{\phi}^2 + B(t)\cdot CON7$$

$$+ [R(t) + CON3]\cdot CON6 + CON18\cdot n + CON19\cdot n\cdot\phi^2 + CON37\cdot\phi^2$$

Equations (7) can be written as

$$\ddot{n} = [v_{13}\cdot v_{22} - v_{12}v_{23}]/v_{11}\cdot v_{22}$$
$$\ddot{\phi} = v_{23}/v_{22} \qquad (8)$$

By making the following definitions Equations (8) can be put into first order form. The definitions (9) must also be made in the v_{ij}.

$$x_1 = \dot{n}$$
$$x_2 = n$$
$$x_3 = \dot{\phi} \qquad (9)$$
$$x_4 = \phi$$

When this is accomplished, Equations (10) yield the proper formulation which will be used in the steepest-descent scheme.

$$\dot{x}_1 = [v_{13}\cdot v_{22} - v_{12}\cdot v_{23}]/v_{11}\cdot v_{22} \equiv f_1$$
$$\dot{x}_2 = x_1 \equiv f_2$$
$$\dot{x}_3 = v_{23}/v_{22} \equiv f_3 \qquad (10)$$
$$\dot{x}_4 = x_3 \equiv f_4$$

V. Steepest-Descent Formulation

The optimal design problem can be stated as follows: Determine the design (control) variable R(t) in the interval $0 \le t \le t_f$ so as to

$$\text{minimize } J = x_4(t_2) \tag{1}$$

subject to the constraints

$$\begin{aligned}
\psi_1 &= x_2(t_1) - \eta_0 + \eta_{max} = 0 \\
\psi_2 &= x_2(t_f) - \eta_0 = 0 \\
\psi_3 &= x_1(t_f) - V_f = 0 \\
\Omega^1 &= x_1(t_1) = 0 \\
\Omega^2 &= x_3(t_2) = 0 \\
\Omega^f &= t_f - C_t = 0
\end{aligned} \tag{2}$$

$$\phi = R(t) - R_{max} \le 0 \tag{3}$$

and satisfying

$$\dot{x} = f \qquad \text{(Equations IV-10)} \tag{4}$$

with initial conditions

$$x_1(0) = x_3(0) = x_4(0) = 0, \quad x_2(0) = \eta_0 .$$

A. Determination of the Adjoint Equations

The minimization problem stated here starts with an estimated design for R(t), analyzes it, and then improves on the design. This steepest-descent method is an iterative process and at each iteration an improvement is made until no significant gains can be achieved. For a complete development of what is to follow, see [Ref. 2 and 3]. Only the results of those derivations will be used here.

The adjoint equations are

$$\dot{\lambda} = -\left[\frac{\partial f}{\partial x}\right]^T \lambda - \left[\frac{\partial \phi}{\partial x}\right]^T \mu \qquad 0 \leq t \leq t_f$$

where the vectors f and x are defined in Equations IV-10 and $\lambda = [\lambda_1, \lambda_2, \lambda_3, \lambda_4]^T$ where T denotes transpose. It can be seen from Equations (3) that $\frac{\partial \phi}{\partial x} = 0$.

$$\frac{\partial f_1}{\partial x_i} = \left\{ [v_{11}v_{22}][v_{13}\frac{\partial v_{22}}{\partial x_i} + v_{22}\frac{\partial v_{13}}{\partial x_i} - v_{12}\frac{\partial v_{23}}{\partial x_i} - v_{23}\frac{\partial v_{12}}{\partial x_i}] \right.$$

$$\left. - [v_{13}v_{22} - v_{12}v_{23}][v_{11}\frac{\partial v_{22}}{\partial x_i} + v_{22}\frac{\partial v_{11}}{\partial x_i}] \right\} / v_{11}^2 v_{22}^2 \qquad i = 1,2,3,4$$

$$\frac{\partial f_1}{\partial x_1} = - v_{12}x_3[CON13 \cdot x_2 + CON14]/v_{11}v_{22}$$

$$\frac{\partial f_1}{\partial x_2} = \{v_{11}v_{22}[2M_a v_{13}(x_2 + CON4) + M_a v_{22}x_3^2 - v_{12}(CON13 \cdot x_1 x_3$$

$$+ CON32 \cdot x_4 + CON16 \cdot x_3^2 + CON18 + CON19 \cdot x_4^2)]$$

$$- [v_{13}v_{22} - v_{12}v_{23}]\ [2M_a v_{11}(x_2 + CON4)]\}/v_{11}^2 v_{22}^2$$

$$\frac{\partial f_1}{\partial x_3} = \{v_{22}(2M_a x_2 x_3 + 2 \cdot CON12x_3) - v_{12}(CON13 \cdot x_1 x_2 + CON14 \cdot x_1$$

$$- C + 2 \cdot CON16 \cdot x_3 x_2 + 2 \cdot CON17 \cdot x_3)\}/v_{11}v_{22}$$

$$\frac{\partial f_1}{\partial x_4} = \{v_{22}(2 \cdot CON11 \cdot x_4 + CON40) - v_{12}(CON38 + CON32 \cdot x_2$$

$$+ 2 \cdot CON19 \cdot x_2 x_4 + 2 \cdot CON37 \cdot x_4)\}/v_{11}v_{22}$$

$$\frac{\partial f_2}{\partial x_1} = 1, \quad \frac{\partial f_2}{\partial x_2} = 0, \quad \frac{\partial f_2}{\partial x_3} = 0, \quad \frac{\partial f_2}{\partial x_4} = 0$$

$$\frac{\partial f_3}{\partial x_i} = \left\{ v_{22} \frac{\partial v_{23}}{\partial x_i} - v_{23} \frac{\partial v_{22}}{\partial x_i} \right\} / v_{22}^2, \qquad i = 1,2,3,4,$$

$$\frac{\partial f_3}{\partial x_1} \; x_3 (\mathrm{CON13} \cdot x_2 + \mathrm{CON14}) / v_{22}$$

$$\frac{\partial f_3}{\partial x_2} = \{ v_{22} (\mathrm{CON13} \cdot x_1 x_3 + \mathrm{CON32} \cdot x_4 + \mathrm{CON16} \cdot x_3^2$$

$$+ \mathrm{CON18} + \mathrm{CON19} \cdot x_4^2) - 2M_a v_{23} (x_2 + \mathrm{CON4}) \} / v_{22}^2$$

$$\frac{\partial f_3}{\partial x_3} = \{ \mathrm{CON13} \cdot x_1 x_2 + \mathrm{CON14} \cdot x_1 - C + 2 \cdot \mathrm{CON16} \cdot x_3 x_2 + 2 \cdot \mathrm{CON17} \cdot x_3 \} / v_{22}$$

$$\frac{\partial f_3}{\partial x_4} = \{ \mathrm{CON38} + \mathrm{CON32} \cdot x_2 + 2 \cdot \mathrm{CON19} \cdot x_2 x_4 + 2 \cdot \mathrm{CON37} \cdot x_4 \} / v_{22}$$

$$\frac{\partial f_4}{\partial x_1} = 0, \quad \frac{\partial f_4}{\partial x_2} = 0, \quad \frac{\partial f_4}{\partial x_3} = 1, \quad \frac{\partial f_4}{\partial x_4} = 0.$$

The adjoint equations now become

$$\dot{\lambda} = - \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & 1 & \frac{\partial f_3}{\partial x_1} & 0 \\ \frac{\partial f_1}{\partial x_2} & 0 & \frac{\partial f_3}{\partial x_2} & 0 \\ \frac{\partial f_1}{\partial x_3} & 0 & \frac{\partial f_3}{\partial x_3} & 1 \\ \frac{\partial f_1}{\partial x_4} & 0 & \frac{\partial f_3}{\partial x_4} & 0 \end{bmatrix} \lambda \tag{5}$$

where the partial derivatives are defined above.

B. <u>Determination of the Boundary Conditions for the Adjoint Equations</u>

Because of the intermediate constraint functions, we must evaluate λ at t_{2-}, and t_{1-} to allow for any discontinuities which may occur across t_2

and t_1. Since the initial conditions for the adjoint equations are given at t_f, these equations are integrated backwards. Integration is carried out by integrating from t_f to t_{2+}. Using new initial conditions at t_{2-} integration is then performed from t_{2-} to t_{1+}. And finally, using new initial conditions at t_{1-} integration is then performed to t_0.

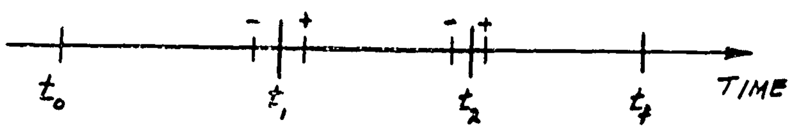


Figure 5

It is the object of this section to determine the initial conditions at t_f, t_{2-}, and t_{1-} for the four different integrations performed on the adjoint equations, that is, for ψ_1, ψ_2, ψ_3 and J.

To get the boundary conditions on the adjoint equations at t_f, we choose

$$\lambda^T_{(t_f)} = \left(\frac{\partial z^f}{\partial x} - \frac{\dot{z}^f}{\dot{\Omega}^f} \frac{\partial \Omega^f}{\partial x} \right)_f$$

where f super and subscripts refer to the time at which the partial derivatives are evaluated.

For an arbitrary function Q, we compute

$$\left(\frac{\partial z^f}{\partial x}\right)_f = \frac{\partial Q}{\partial x_f} , \qquad \left(\frac{\partial z^f}{\partial t}\right)_f = \frac{\partial Q}{\partial t_f}$$

and

$$\dot{z}^f_f = \left(\frac{\partial z^f}{\partial x}\right)_f \dot{x}_f + \left(\frac{\partial z^f}{\partial t}\right)_f$$

$$\dot{\Omega}_f^f = \left(\frac{\partial \Omega^f}{\partial x}\right)_f \dot{x}_f + \left(\frac{\partial \Omega^f}{\partial t}\right)_f$$

or

$$\lambda^{Q^T}_{(t_f)} = \frac{\partial Q}{\partial x_f} - \frac{\left(\frac{\partial Q}{\partial x_f}\right) \dot{x}_f + \left(\frac{\partial Q}{\partial t_f}\right)}{\left(\frac{\partial \Omega^f}{\partial x}\right)_f \dot{x}_f + \left(\frac{\partial \Omega^f}{\partial t}\right)_f} \left(\frac{\partial \Omega^f}{\partial x}\right)_f$$

From Equations (2) of this section it can be seen that Ω^f does not depend upon the state explicitly and therefore, $\left(\frac{\partial \Omega^f}{\partial x}\right)_f = 0$. Thus

$$\lambda^{Q^T}_{(t_f)} = \frac{\partial Q}{\partial x_f}$$

and it follows from Equations (1) and (2) that

$$\begin{aligned} \lambda^{J^T}_{(t_f)} &= [0\ 0\ 0\ 0] \\ \lambda^{\psi_1^T}_{(t_f)} &= [0\ 0\ 0\ 0] \\ \lambda^{\psi_2^T}_{(t_f)} &= [0\ 1\ 0\ 0] \\ \lambda^{\psi_3^T}_{(t_f)} &= [1\ 0\ 0\ 0] \end{aligned} \tag{6}$$

BOUNDARY CONDITIONS AT t_{2-}

We choose

$$\lambda^T_{(t_{2-})} = \left(\frac{\partial z^2}{\partial x} - \frac{\dot{z}^2}{\dot{\Omega}^2} \frac{\partial \Omega^2}{\partial x}\right)_{2-}$$

where the superscript 2 refers to the time t_2.

$$\left(\frac{\partial z^2}{\partial x}\right)_2 = \frac{\partial Q}{\partial x_2} + \lambda^T_{2+} - \left(\frac{\dot{z}^f}{\dot{\Omega}^f} \frac{\partial \Omega^f}{\partial x_2}\right)_f$$

$$\left(\frac{\partial z^2}{\partial t}\right)_2 = \frac{\partial Q}{\partial t_2} - (\lambda^T\dot{x})_{2+} - \left(\frac{\dot{z}^f}{\dot{\Omega}^f}\frac{\partial \Omega^f}{\partial t_2}\right)_f$$

$$\dot{z}^2_{2-} = \left(\frac{\partial z^2}{\partial x}\right)_2 \dot{x}_{2-} + \left(\frac{\partial z^2}{\partial t}\right)_2$$

$$\dot{\Omega}^2_{2-} = \sum_{i=0}^{f-2}\left(\frac{\partial \Omega^{f-1}}{x_i}\dot{x}_i + \frac{\partial \Omega^{f-1}}{\partial t_i}\right)_{f-1} + \left(\frac{\partial \Omega^{f-1}}{\partial x}\dot{x} + \frac{\partial \Omega^{f-1}}{\partial t}\right)_{f-1}$$

$$+ \left(\frac{\partial \Omega^{f-1}}{\partial a}\right)_{f-1}\frac{da}{dt}$$

where a refers to a vector of control parameters and f for this problem is equal to 3, i.e., we have t_0, t_1, t_2, and t_3. In this problem there are no control parameters, however, the additional term is written for completeness of the expression for $\dot{\Omega}^2_{2-}$. The derivatives appearing in the summation are zero since Ω^2 does not depend upon the times t_0 or t_1. The rest of the terms which are zero can be seen immediately by evaluating the derivatives in Equations (2). We how have that

$$\lambda^T_{(t_{2-})} = \frac{\partial Q}{\partial x_2} + \lambda^T_{2+} - \frac{\left[\frac{\partial Q}{\partial x_2} + \lambda^T_{2+}\right]\dot{x}_{2-} - (\lambda^T\dot{x})_{2+}}{\left(\frac{\partial \Omega^2}{\partial x}\right)_2 \dot{x}_{2-}}\left(\frac{\partial \Omega^2}{\partial x}\right)_2$$

Boundary Conditions for $J = x_4(t_2)$ at t_{2-} ($\lambda^T_{2+} = 0$)

$$\lambda^{J^T}_{(t_{2-})} = [0\ 0\ 0\ 1] - \frac{[0\ 0\ 0\ 1]\dot{x}_{2-}}{[0\ 0\ 1\ 0]\dot{x}_{2-}}[0\ 0\ 1\ 0]$$

$$= [0\ 0\ 0\ 1] - \frac{f_4(t_{2-})}{f_3(t_{2-})}[0\ 0\ 1\ 0]$$

$$\lambda^{J^T}_{(t_{2-})} = \left[\ 0\ \ 0\ \ -\frac{f_4(t_{2-})}{f_3(t_{2-})}\ \ 1\ \right] \tag{7}$$

Boundary Conditions for $\psi_1 = x_2(t_1) - \eta_0 + \eta_{max} = 0$ at t_{2-} $(\lambda^T_{2+} = 0)$

$$\lambda^{\psi_1 T}_{(t_{2-})} = [0\ 0\ 0\ 0] \tag{8}$$

Boundary Conditions for $\psi_2 = x_2(t_f) - \eta_0 = 0$ at t_{2-}

$$\lambda^{\psi_2 T}_{(t_{2-})} = \lambda^{\psi_2 T}_{2+} - \frac{\lambda^{\psi_2 T}_{2+}\dot{x}_{2-} - (\lambda^{\psi_2 T}\dot{x})_{2+}}{[0\ 0\ 1\ 0]\,\dot{x}_{2-}}\,[0\ 0\ 1\ 0]$$

$$\lambda^{\psi_2 T}_{(t_{2-})} = \lambda^{\psi_2 T}_{(t_{2+})} - \frac{\lambda^{\psi_2 T}_{2+}\dot{x}_{2-} - (\lambda^{\psi_2 T}\dot{x})_{2+}}{(f_3)_{(t_{2-})}}\,[0\ 0\ 1\ 0] \tag{9}$$

Boundary Conditions for $\psi_3 = x_1(t_f) - V_f = 0$ at t_{2-}

$$\lambda^{\psi_3 T}_{(t_{2-})} = \lambda^{\psi_3 T}_{2+} - \frac{\lambda^{\psi_3 T}_{2+}\dot{x}_{2-} - (\lambda^{\psi_3 T}\dot{x})_{2+}}{[0\ 0\ 1\ 0]\,\dot{x}_{2-}}\,[0\ 0\ 1\ 0]$$

$$\lambda^{\psi_3 T}_{(t_{2-})} = \lambda^{\psi_3 T}_{(t_{2+})} - \frac{\lambda^{\psi_3 T}_{2+}\dot{x}_{2-} - (\lambda^{\psi_3 T}\dot{x})_{2+}}{(f_3)_{(t_{2-})}}\,[0\ 0\ 1\ 0] \tag{10}$$

BOUNDARY CONDITIONS AT t_{1-}

We choose

$$\lambda^T_{(t_{1-})} = \left(\frac{\partial z^1}{\partial x} - \frac{\dot{z}^1}{\dot{\Omega}^1}\frac{\partial \Omega^1}{\partial x}\right)_{1-}$$

$$\left(\frac{\partial z^1}{\partial x}\right)_1 = \frac{\partial Q}{\partial x_1} + \lambda^T_{1+} - \left(\frac{\dot{z}^f}{\dot{\Omega}^f}\frac{\partial \Omega^f}{\partial x_1}\right)_f - \left(\frac{\dot{z}^{f-1}}{\dot{\Omega}^{f-1}}\frac{\partial \Omega^{f-1}}{\partial x_1}\right)_{2-}$$

$$\left(\frac{\partial z^1}{\partial t}\right)_1 = \frac{\partial Q}{\partial t} - (\lambda^T\dot{x})_{1+} - \left(\frac{\dot{z}^f}{\dot{\Omega}^f}\frac{\partial \Omega^f}{\partial t}\right)_f - \left(\frac{\dot{z}^{f-1}}{\dot{\Omega}^{f-1}}\frac{\partial \Omega^{f-1}}{\partial t_1}\right)_{2-}$$

$$\dot{z}^1_{1-} = \left(\frac{\partial z^1}{\partial x}\right)\dot{x}_{1-} + \left(\frac{\partial z^1}{\partial t}\right)_1$$

In general

$$\dot{\Omega}^j_{j-} = \sum_{i=0}^{j-1}\left(\frac{\partial\Omega^j}{\partial x_i}\dot{x}_i + \frac{\partial\Omega^j}{\partial t_i}\right)_{j-} + \left(\frac{\partial\Omega^j}{\partial x}\dot{x} + \frac{\partial\Omega^j}{\partial t}\right)_{j-} + \left(\frac{\partial\Omega^j}{\partial a}\right)_{j-}\frac{da}{dt}$$

For j = 1 we have

$$\dot{\Omega}^1_{1-} = \left(\frac{\partial\Omega^1}{\partial x}\dot{x} + \frac{\partial\Omega^1}{\partial t}\right)_{1-} = \left(\frac{\partial\Omega^1}{\partial x}\dot{x}\right)_{1-}$$

$$\lambda^T_{(t_{1-})} = \frac{\partial Q}{\partial x_1} + \lambda^T_{1+} - \frac{\left[\frac{\partial Q}{\partial x_1} + \lambda^T_{1+}\right]\dot{x}_{1-} - (\lambda^T\dot{x})_{1+}}{\left(\frac{\partial\Omega^1}{\partial x}\dot{x}\right)_{1-}}\left(\frac{\partial\Omega^1}{\partial x}\right)_1$$

Boundary Conditions for $J = x_4(t_2)$ at t_{1-}

$$\lambda^{J^T}_{(t_{1-})} = \lambda^{J^T}_{1+} - \frac{\lambda^{J^T}_{1+}\dot{x}_{1-} - (\lambda^{J^T}\dot{x})_{1+}}{[1\ 0\ 0\ 0]\dot{x}_{1-}}[1\ 0\ 0\ 0]$$

$$\lambda^{J^T}_{(t_{1-})} = \lambda^{J^T}_{1+} - \frac{\lambda^{J^T}_{1+}\dot{x}_{1-} - (\lambda^{J^T}\dot{x})_{1+}}{(f_1)(t_{1-})}[1\ 0\ 0\ 0] \tag{11}$$

Boundary Conditions for $\psi_1 = x_2(t_1) - \eta_0 + \eta_{max} = 0$ at $t_{1-}(\lambda^{\psi_1^T}_{1+} = 0)$

$$\lambda^{\psi_1^T}_{(t_{1-})} = [0\ 1\ 0\ 0] - \frac{[0\ 1\ 0\ 0]\dot{x}_{1-}}{[1\ 0\ 0\ 0]\dot{x}_{1-}}[1\ 0\ 0\ 0]$$

$$= [0\ 1\ 0\ 0] - \frac{(f_2)(t_{1-})}{(f_1)(t_{1-})}[1\ 0\ 0\ 0]$$

$$\lambda^{\psi_1^T}_{(t_{1-})} = \left[\ -\frac{(f_2)(t_{1-})}{(f_1)(t_{1-})}\ 1\ 0\ 0\ \right] \tag{12}$$

Boundary Conditions for $\psi_2 = x_2(t_f) - n_0 = 0$ at t_{1-}

$$\lambda^{\psi_2 T}_{(t_{1-})} = \lambda^T_{1+} - \frac{\lambda^T_{1+}\dot{x}_{1-} - (\lambda^T\dot{x})_{1+}}{[1\ 0\ 0\ 0]\dot{x}_{1-}}[1\ 0\ 0\ 0]$$

$$\lambda^{\psi_2 T}_{(t_{1-})} = \lambda^{\psi_2 T}_{(t_{1+})} - \frac{\lambda^{\psi_2 T}_{(t_{1+})}\dot{x}_{1-} - (\lambda^{\psi_2 T}\dot{x})_{1+}}{(f_1)_{(t_{1-})}}[1\ 0\ 0\ 0] \tag{13}$$

Boundary Conditions for $\psi_3 = x_1(t_f) - V_f = 0$ at t_{1-}

$$\lambda^{\psi_3 T}_{(t_{1-})} = \lambda^{\psi_3 T}_{(t_{1+})} - \frac{\lambda^{\psi_3 T}_{1+}\dot{x}_{1-} - (\lambda^{\psi_3 T}\dot{x})_{+}}{[1\ 0\ 0\ 0]\dot{x}_{1-}}[1\ 0\ 0\ 0]$$

$$\lambda^{\psi_3 T}_{(t_{1-})} = \lambda^{\psi_3 T}_{(t_{1-})} - \frac{\lambda^{\psi_3 T}_{1+}\dot{x}_{1-} - (\lambda^{\psi_3 T}\dot{x})_{1+}}{(f_1)_{(t_{1-})}}[1\ 0\ 0\ 0] \tag{14}$$

C. Determination of the Variation of the Design Variable

The variation or change in the design variable R(t) which makes the greatest reduction in J, the hop, is given by the following expression (see[Ref. 2 and 3]) where the desired change in the constraint function is given by $d\psi$. The ψ constraints of Equations (2) will in general not be satisfied with the nominal choice of R(t). Since the idea is to

$$\delta R(t) = W_u^{-1}(t)[\Lambda^\psi(t)I^{-1}_{\psi\psi}I_{\psi J} - \Lambda^J(t)]\left[\frac{dP^2 - d\psi^T I^{-1}_{\psi\psi}d\psi}{I_{JJ} - I^T_{\psi J}I^{-1}_{\psi\psi}I_{\psi J}}\right]^{1/2} + W_u^{-1}\Lambda^\psi(t)I^{-1}_{\psi\psi}d\psi \tag{15}$$

drive the ψ constraints to be identically equal to zero along with minimizing J, in the selection of perturbations the choice of the desired $d\psi$ will be $- a\psi$. That is

$d\psi = -a\psi \qquad 0 < a \leq 1,$

If a reasonably good estimate is made for R(t), the value of a may be set equal to 1. W_u is a matrix of weighting functions whose elements are functions of time which permits $\delta R(t)$ to be suppressed in sensitive regions or amplified in less sensitive regions. In this problem R(t) was given equal weight throughout the entire recoil and counterrecoil cycle and W_u was set equal to the identity matrix. A few terms and definitions will now be given in order to evaluate the expression of $\delta R(t)$.

$$I_{JJ} = \ell^{J^T} W_\beta^{-1} \ell^J + \int_0^{t_f} \Lambda^{J^T} W_u^{-1} \Lambda^J dt \tag{16}$$

$$I_{\psi\psi} = \ell^{\psi^T} W_\beta^{-1} \ell^\psi + \int_0^{t_f} \Lambda^{\psi^T} W_u^{-1} \Lambda^\psi dt \tag{17}$$

$$I_{\psi J} = \ell^{\psi^T} W_\beta^{-1} \ell^J + \int_0^{t_f} \Lambda^{\psi^T} W_u^{-1} \Lambda^J dt \tag{18}$$

$$\ell^J = \left[\int_0^{t_f} \left(\frac{\partial f}{\partial b} \lambda^J + \frac{\partial \phi}{\partial b} \mu^J \right) dt + \left(\frac{\partial J}{\partial b} \right)^T \right]$$

$$\ell^\psi = \left[\int_0^{t_f} \left[\left(\frac{\partial f}{\partial b} \right)^T \lambda^\psi + \left(\frac{\partial \phi}{\partial b} \right)^T \mu^\psi \right] dt + \left(\frac{\partial \psi}{\partial b} \right)^T \right]$$

$$\Lambda^J(t) = \left(\frac{\partial f}{\partial R} \right)^T \lambda^J + \left(\frac{\partial \phi}{\partial R} \right)^T \mu^J \tag{19}$$

$$\Lambda^\psi(t) = \left(\frac{\partial f}{\partial R} \right)^T \lambda^\psi + \left(\frac{\partial \phi}{\partial R} \right)^T \mu^\psi \tag{20}$$

$$\mu^T \phi(t,x,R,b) = 0 \qquad 0 < t \leq t_f \tag{21}$$

$$\lambda^T \frac{\partial f}{\partial R} + \mu^T \frac{\partial \phi}{\partial R} = 0 \tag{22}$$

W_β is another weighting matrix and will be set equal to the identity matrix. b is a vector of design parameters and since this problem does not contain any, $\ell^J = \ell^\psi = 0$. Taking variations of the last equation yields the linearized version

$$\mu^T \frac{\partial \phi}{\partial x} \delta x + \mu^Y \frac{\partial \phi}{\partial R} \delta R + \mu^T \frac{\partial \phi}{\partial b} \delta b = 0.$$

Since $\frac{\partial \phi}{\partial x} = 0$, $\frac{\partial \phi}{\partial b} = 0$ we obtain the following

$$\mu^T \frac{\partial \phi}{\partial R} \delta R = 0$$

which says that wherever $\delta R \neq 0$, $\mu = 0$ since $\frac{\partial \phi}{\partial R} = 1$. From (21) it is seen that for $\phi < 0$, $\mu(t) = 0$. However, if ϕ is zero over an interval an additional test must be satisfied.

It must be verified that violating a constraint boundary in such an interval would allow an improvement in J. Since ϕ and R(t) are each scalars from (22) we have that

$$\mu = -\frac{\lambda^T \frac{\partial f}{\partial R}}{\frac{\partial \phi}{dR}} \tag{23}$$

and it can be argued that when $\phi = 0$, J will be minimum if μ is a non-negative function. Thus, Equations (21) and (22) provide the equations which determine R(t) and $\mu(t)$. One more vector, $\frac{\partial f}{\partial R}$, must be evaluated now before $\delta R(t)$ is determined.

$$\frac{\partial f_1}{\partial R} = \{v_{11}v_{22}[v_{13}\frac{\partial v_{22}}{\partial R} + v_{22}\frac{\partial v_{13}}{\partial R} - v_{12}\frac{\partial v_{23}}{\partial R} - v_{23}\frac{\partial v_{12}}{\partial R}]$$

$$- [v_{13}v_{22} - v_{12}v_{23}][v_{11}\frac{\partial v_{22}}{\partial R} + v_{22}\frac{\partial v_{11}}{\partial R}]\} / v_{11}^2 v_{22}^2$$

$$\frac{\partial f_1}{\partial R} = [v_{22} - v_{12} \cdot CON6]/v_{11} v_{22}$$

$$\frac{\partial f_2}{\partial R} = 0, \quad \frac{\partial f_4}{\partial R} = 0$$

$$\frac{\partial f_3}{\partial R} = [v_{22} \frac{\partial v_{23}}{\partial R} - v_{23} \frac{\partial v_{22}}{\partial R}]/v_{22}^2$$

$$\frac{\partial f_3}{\partial R} = CON6/v_{22}$$

Λ^J and Λ^{ψ_i} where $i = 1,2,3$ may now be evaluated by replacing Q with J, ψ_1, ψ_2, and ψ_3.

$$\Lambda^Q(t) = \begin{cases} \left(\frac{1}{v_{11}} - \frac{v_{12} \cdot CON6}{v_{11} v_{22}}\right) \lambda_1^Q + \frac{CON6}{v_{22}} \lambda_3^Q & \phi < 0 \\ \left(\frac{1}{v_{11}} - \frac{v_{12} \cdot CON6}{v_{11} v_{22}}\right) \lambda_1^Q + \frac{CON6}{v_{22}} \lambda_3^Q & \phi = 0, \mu > 0 \\ 0 & \phi = 0, \mu \leq 0 \end{cases} \tag{24}$$

$I_{\psi J}$, $I_{\psi\psi}$ and I_{JJ} now become

$$\begin{aligned} I_{\psi J} &= \int_0^{t_f} \Lambda^{\psi^T} \Lambda^J dt \\ I_{\psi\psi} &= \int_0^{t_f} \Lambda^{\psi^T} \Lambda^{\psi} dt \\ I_{JJ} &= \int_0^{t_f} \Lambda^{J^T} \Lambda^J dt \end{aligned} \tag{25}$$

where $I_{\psi J}$ is a (3x1) column vector, $I_{\psi\psi}$ is a (3x3) matrix and I_{JJ} is of order (1x1).

VI. Results and Conclusions

Figure (5) represents the optimal rod force to minimize hop at 75 degrees quadrant elevation with the following constraints

$$\begin{aligned} R(t) &\leq 22000 \text{ lbs.} \\ \text{recoil length} &= 28 \text{ in.} \end{aligned} \tag{1}$$

The resulting hop for the above case is 1.53 inches, i.e., the tires leave the ground 1.53 inches, for a 115 per cent maximum rated pressure breech force. Computer results indicate that the present rod design for short recoil yields 6.26 inches of hop which agrees with firing data. To obtain the 1.53 inches of hop would require a redesign of the orifice areas for short recoil and for counterrecoil. One might question whether the resulting curve in Figure (5) is obtainable with the XM44 recoil mechanism; if it is not, a very simple solution is to alter the curve so that a nearly optimal solution results. If the constraints were such that

$$\begin{aligned} R(t) &\leq 23500 \text{ lbs.} \\ \text{recoil length} &= 29 \text{ in.} \end{aligned} \tag{2}$$

the resulting hop is 0.88 inches.

If one uses the present counterrecoil groove design and requires the constraints in (1) to hold so that it is necessary to redesign the orifice areas for short recoil only, the resulting hop is 3.42 inches or a 45 per cent reduction. For constraint set (1) a 75 per cent reduction is achieved and for constraint set (2) an 86 per cent reduction results.

The acceleration of the recoiling parts during the first portion of counterrecoil is an important factor in reducing the hop. That is, the faster the recoiling parts accelerate during this period, the greater the reduction in hop. As one would expect, an increase in recoil length also reduces hop significantly. An increase in the maximum rod force will also reduce hop, for example, if the rod force is allowed to obtain the value 24160 lbs. in constraint set (1), the hop can be reduced an additional 0.32 inches. Figure (6) shows a possible control rod design for short recoil. The orifice areas were obtained from the rod force in Figure (5). The resulting force levels from the new groove design is indicated by the dotted lines from .110 sec to .13 sec. The rod force is the same as the optimal shaped force curve from 0 to .110 sec. The increase in hop is approximately 0.1 inches. The recoil length changed a very small amount.

An interesting side point is that of the speed of convergence. The nominal design variable, $R(t)$, used for the first iteration was such that at the end of counterrecoil the recoiling arts were 250 inches away from the latch position and the required final velocity of 6 inches/sec. was 96 inches/sec. In approximately 14 iterations, convergence was obtained which seems to be very fast if one considers the complexity of the equations involved.

A computational algorithm is given below.

Step 1. Make an engineering estimate for $R(t)$ and call it $R^0(t)$.

Step 2. Integrate the state Equations (IV-10) with initial conditions (III-3) and determine t_1 and t_2.

Step 3. Integrate the adjoint Equations (V-5) from t_f to t_{2+} with initial conditions (V-6).

Step 4. Evaluate initial conditions (V-7, 8, 9, 10) for adjoint equations at t_{2-} and integrate (V-5) from t_{2-} to t_{1+}

Step 5. Evaluate initial conditions (V-11, 12, 13, 14) for adjoint equations at t_{1-} and integrate (V-5) from t_{1-} to t_0.

Step 6. Evaluate Λ^Q from (V-24) for ψ_1, ψ_2, ψ_3 and J.

Step 7. Perform the definite integration of (V-25) for $I_{\psi J}$, $I_{\psi\psi}$, and I_{JJ}.

Step 8. Choose $d\psi$ and dP in (V-15) where $d\psi$ is the desired change in ψ (V-2).

Step 9. Compute $(dP)^2 - d\psi^T I_{\psi\psi}^{-1} d\psi$. If this quantity is negative, compute $\zeta = [(dP)^2 / d\psi^T I_{\psi\psi}^{-1} d\psi]^{1/2}$ and replace $d\psi$ by $\zeta d\psi$.

Step 10. Evaluate $\delta R(t)$ from (V-15).

Step 11. Compute new estimate $R^{\cdot}(t) = R^{\cdot}(t) + \delta R(t)$.

Step 12. Evaluate gradient squared $(I_{JJ} - I_{\psi J}^T I_{\psi\psi}^{-1} I_{\psi J})$ for convergence. If near zero, stop; if not, go to Step 2.

Results from firing tests show a significant reduction (50% or more) in hop can be achieved simply by increasing the tire pressure. Because tire performance information is not presently available, it was assumed throughout this analysis that the spring rate of the tires was constant. Therefore it is not known what results would be obtained under a dynamic tire response model. Tire manufactures are looking at how they can optimize tire characteristics for the final configuration in the tire itself. In order to obtain optimum weapon performance for flexible mount systems such information as tire performance could be incorporated into the mathematical model and perhaps tire characteristics could also be optimized in the environment for which they are being used.

The technique used in this report has the capability to optimize many design parameters simultaneously. If there exist other sensitive parameters, consideration should be given to optimize them along with the design variable R(t).

This study clearly indicates that weapon performance can be improved by using methods of optimal design.

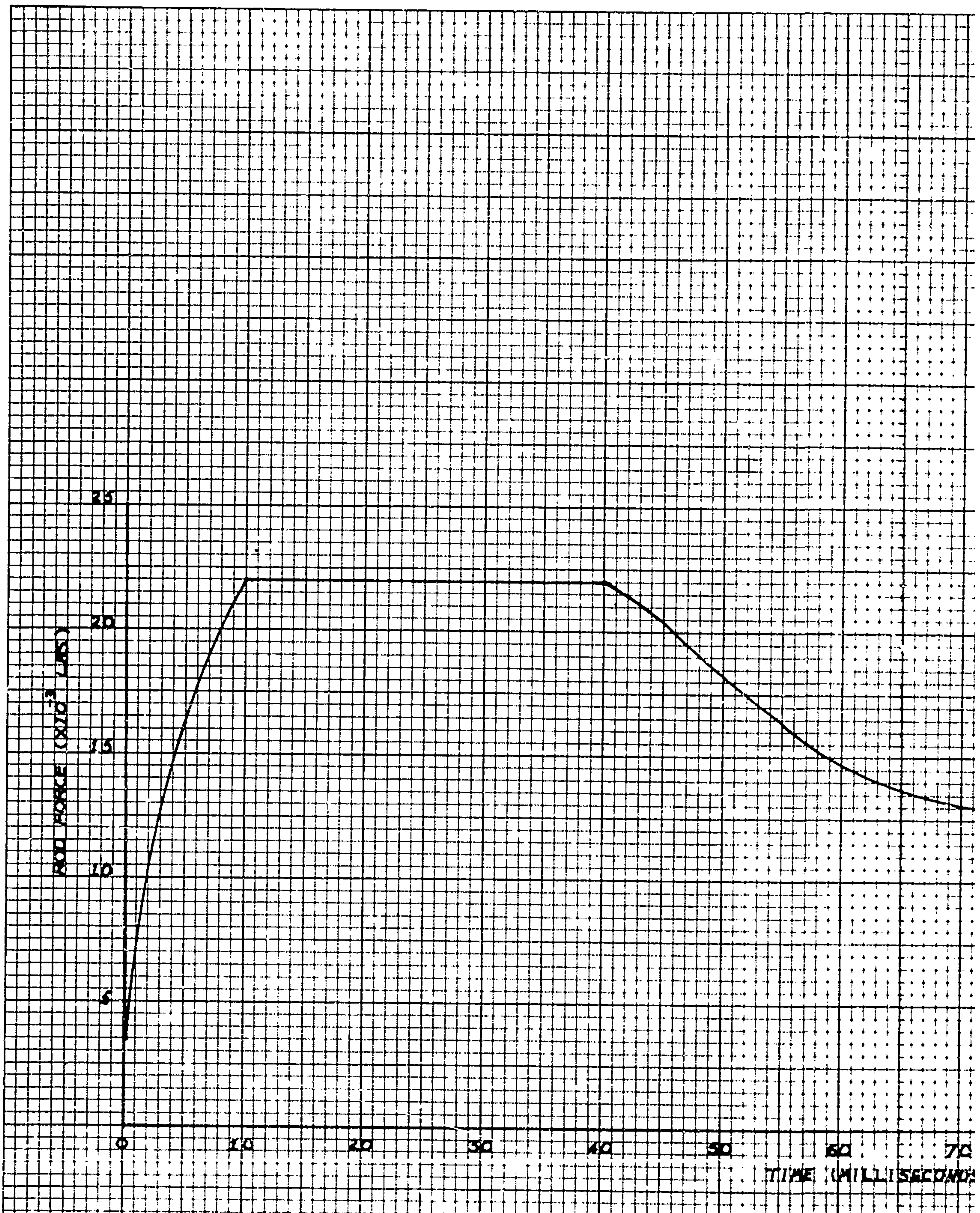

ROD FORCE (X10^-3 LBS)
25
20
15
10
5
0
10
20
30
40
50
60
70
TIME (MILLISECONDS

FIGURE 5

A

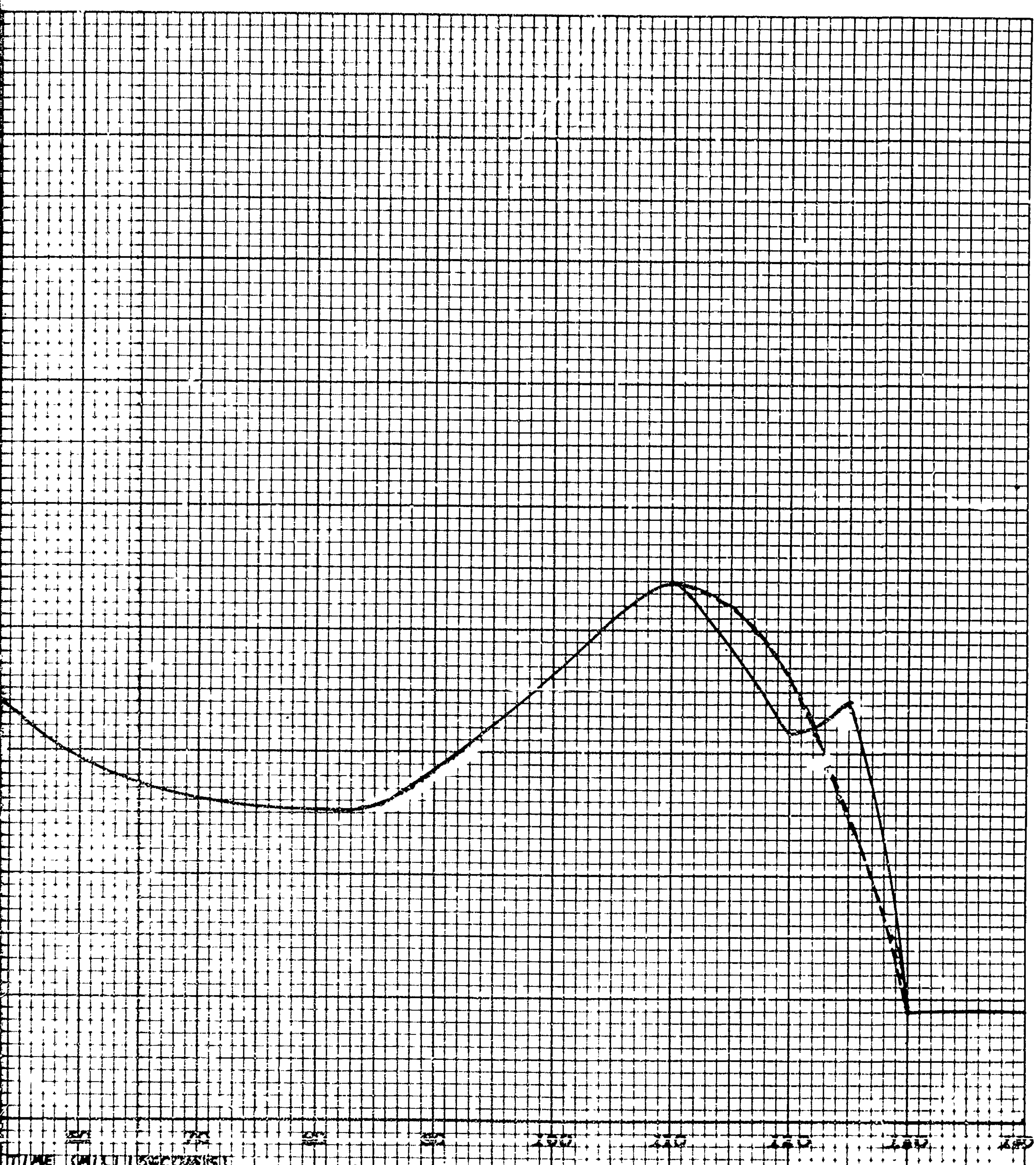
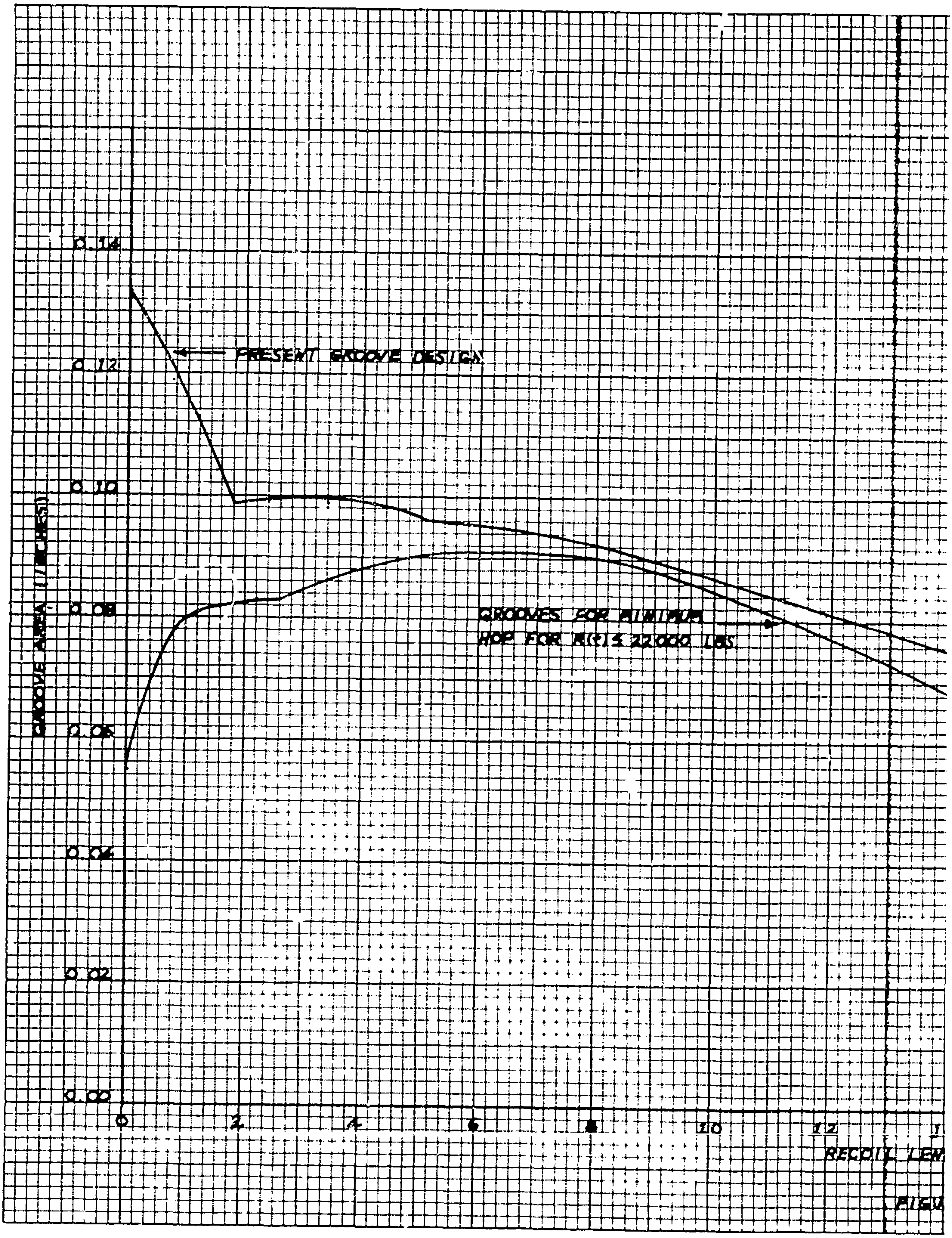


FIGURE 5


0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
GROOVE AREA
PRESENT GROOVE DESIGN
GROOVES FOR MINIMUM
HOP FOR R(t) IS 22,000 LBS
0
2
4
6
10
12
RECOIL LEN
FIGU

A

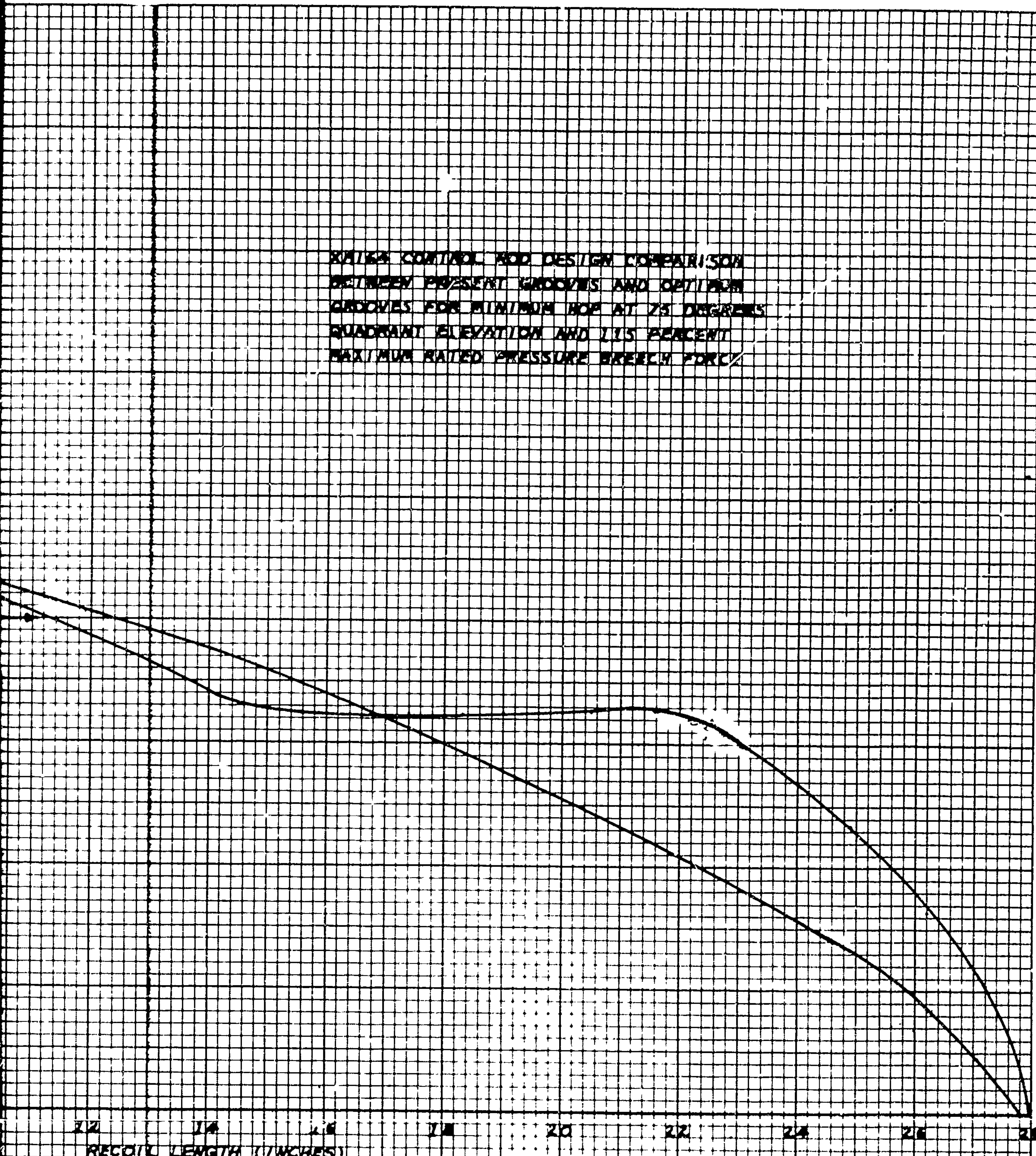

XM164 CONTROL ROD DESIGN COMPARISON
BETWEEN PRESENT GROOVES AND OPTIMUM
GROOVES FOR MINIMUM ROP AT 75 DEGREES
QUADRANT ELEVATION AND 115 PERCENT
MAXIMUM RATED PRESSURE BREECH FORCE
12
14
16
18
20
22
24
26
28
RECOIL LENGTH (INCHES)

FIGURE 6

REFERENCES

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13. ABSTRACT

The problem treated in this report falls into the rapidly developing field of optimal design. The design requirements stipulate that a weapon system is to perform some task at some index of performance. The objective of this study is to apply a relatively new steepest-descent procedure to an artillery design problem which involves the dynamic behavior of a 105mm howitzer which is fired while resting on rubber tires, and determine the design parameters such that the pitch motion of the weapon is minimum at high angle fire. Thus, the weapon will not only perform its task, but will also have maximum performance (in this case, stability).

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