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Linear and Nonlinear Theory of Grid Excitation of Low-Frequency Waves in a Plasma

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ABSTRACT

The steady-state excitation of longitudinal waves by a pair of idealized grids immersed in a collisionless plasma and driven at a frequency small compared with the ion plasma frequency is investigated theoretically. In linear theory the Fourier-inversion integral which determines the spatial behavior of the potential in the plasma is expressed as a sum of two integrals which embody the interactions of phase-velocity components of the wave with ions and electrons. An appropriate choice of the deformed contour of integration permits evaluation of the response as the sum of the residue of the dominant "ion-acoustic" pole and of the two

branch-cut integrals. A perturbation-series expansion of the potential and the species distribution functions in the (nonlinear) Vlasov equation yields a hierarchy of equations. In each order the equations are linear in the perturbation quantities of that order and have driving terms composed of quadratic combinations of lower-order quantities. For sufficiently small amplitude of excitation the principal contributions to the response come from the first-order (linearized Vlasov) equation and the second-order equation. In second order the steady-state response consists of zero-frequency and double-frequency components. The second-order equations are Laplace-Fourier transformed and resulting velocity integrals are expressed in terms of plasma dispersion functions. By approximating the driving terms by their dominant-pole component, one can express the steady-state double-frequency response as a single Fourier-inversion integral. As in the linear problem, the integral can be evaluated as the sum of a residue and of "ion like" and "electron-like" branch-cut integrals. Numerical results are presented for the linear and nonlinear cases.

I. INTRODUCTION

The calculation by Gould[1] of the steady-state response in linear theory of a collisionless plasma to grid excitation of longitudinal waves at frequencies below the ion plasma frequency exhibits good agreement with the experiment of Wong, D'Angelo, and Motley,[2] and supports the interpretation that spatial damping of the wave is due to Landau damping. The importance of understanding the nature of the damping and of studying nonlinear processes in plasmas suggest the desirability of extending the theory into the nonlinear regime. As the amplitude of excitation is increased, one expects the steady-state response to include harmonics of the applied frequency. If the amplitude is sufficiently small, only zero-frequency and double-frequency harmonics are significantly excited, and a perturbation procedure should yield this nonlinear response.

In Section II the response in linear theory is determined by a method which possesses certain advantages over that of Gould. The response is expressed as the sum of two branch-cut integrals which contain the velocity derivatives of the ion and the electron distribution functions and give insight into the role of the two species in the Landau damping of the wave. The response may also be expressed as the sum of the residue of the ion-acoustic pole and of the two integrals evaluated along deformed contours which are well-suited for obtaining high accuracy. Calculations are performed both for the case of negligible grid spacing (dipole limit) considered by Gould and for the case of finite spacing between the grids; the comparison illuminates the role of the model of the grid in determining the character of the response. In Section III perturbation-series expansions of

electric field and species distribution functions are introduced into the Vlasov equation; the equations for lowest-order nonlinear response are obtained. Performing the appropriate Laplace-Fourier transforms, analytically continuing (in the manner of Landau[3]) functions of two complex variables defined by velocity integrals which occur in the formulation, and relating these functions to the plasma dispersion function, a double Fourier-inversion integral for the lowest-order nonlinear response in the plasma is obtained. This integral contains a quadratic combination of the electric field in linear theory as a driving term. In Section IV reduction of the double integral for the double-frequency component to a single Fourier-inversion integral is achieved by approximating the electric field in linear theory by its dominant-pole component. The nonlinear response is expressed as the sum of a residue and of ion-like and electron-like integrals, as in the linear problem. In Section V the branch-cut integrals and the residue are considerably simplified by utilizing the square root of the mass ratio as a smallness parameter. Calculations are performed for a range of values of the electron-to-ion temperature ratio. In Section VI the zero-frequency component of the nonlinear response is shown to consist of a polarization of the plasma, with no species current densities.

II. LINEAR PROBLEM

The steady-state oscillatory potential produced in a uniform and infinite collisionless plasma by excitation at a frequency ω_0 of a pair of closely-spaced, idealized grids, which produces an external oscillating charge density but intercepts no particles, is given by $\phi(x,t) = \phi(x)\exp(-i\omega_0 t)$ + complex conjugate, in which

$$\phi(x) = -\frac{i\sigma_0 x_0}{2\pi\epsilon_0}\left[\int_{-\infty}^{0}\frac{\exp(ikx)\,dk}{k K_-(\omega_0,k)} + \int_{0}^{\infty}\frac{\exp(ikx)\,dk}{k K_+(\omega_0,k)}\right]. \qquad (1)$$

Here σ_0 is the amplitude of the surface-charge density on either grid and x_0 is the separation between the grids; the dipole limit, $x_0 \to 0$, $\sigma_0 x_0 = \text{const}$, is considered. (The effect of finite spacing between the grids is examined below.) The "plus" and "minus" dielectric functions are given by

$$K_\pm(\omega,k) = 1 - \frac{\omega_{pi}^2}{k^2 a_i^2} Z_\pm'\left(\frac{\omega}{k a_i}\right) - \frac{\omega_{pe}^2}{k^2 a_e^2} Z_\pm'\left(\frac{\omega}{k a_e}\right) \qquad (2)$$

in which the corresponding plasma dispersion functions are defined by

$$Z(\zeta) = \int_{-\infty}^{\infty}\frac{\exp(-t^2)\,dt}{\pi^{1/2}(t-\zeta)} \qquad (3)$$

when $\mathrm{Im}\{\zeta\}$ is positive and negative, respectively, and by the analytic continuation of these integrals elsewhere. The functions $Z_\pm(\zeta)$ have a branch-cut along the real axis; the situation may be understood by considering the integral to be the limit of an integral over the finite range $-c$ to c, as $c \to \infty$. Branch points occur at $\zeta = \pm c$. The mapping of the branch-cut and branch points onto the k plane for $\omega = \omega_0 + i\varepsilon$ is shown in Fig. 1a, along with the primitive Fourier inversion contour. The first step in evaluating the integral is to fold the left half of the primitive contour in the upper half-plane (for x positive) over onto the right half of the primitive contour, as shown in Fig. 1b . (A residue contribution at $k = ik_D$, where $k_D = 2^{1/2}\,[\omega_{pe}^2\, a_e^{-2} + \omega_{pi}^2\, a_i^2]^{1/2}$, may be neglected except for very small values of x . There is a pole at $k=0$ which gives a contribution to the potential

which is proportional to sign $\{x\}$ and therefore has no physical significance.) This is the contour used by Gould[1] for numerical evaluation of the Fourier inversion integral. It has the disadvantages that accuracy is reduced as x increases, because of the rapid oscillatory behavior of the factor $\exp(ikx)$, and that it does not provide for separate determination, as a residue, of the contribution to the response arising from the ion-acoustic pole at k_1, which is dominant within a considerable range of x.

The method used here avoids these disadvantages. It provides some additional physical insight into the damping of the wave and leads to a method of dealing with the branch-cut integrals which is useful in the vastly more complicated nonlinear case.

The physical understanding of Landau damping of a weakly damped wave involves an interaction between the wave and particles with velocities very near the phase velocity of the wave.[4] Presumably the case of strong damping considered here involves interactions between the wave, which consists of a superposition of phase-velocity components, and particles in the corresponding band of phase velocities. Therefore a transformation of variable of integration from k to the dimensionless complex phase velocity $\zeta = \omega_0/ka_i$, which is considered a more "natural" variable, is introduced. The usual dimensionless variables $z = \omega_0 x/a_i$, $f = \omega_0/\omega_{pi}$, $\Phi = -\epsilon_0\phi/\sigma_0 x_0$, and $\tilde{T} = T_i/T_e$ are introduced. The mass ratio is $\mu = m_e/m_i$. Making use of the relation

$$Z'(\zeta) - Z'_{-}(\zeta) = 4\pi^{\frac{1}{2}} i\, \zeta \exp(-\zeta^2), \tag{4}$$

the following separation into an "ion-like" and an "electron-like" integral is achieved:

$$\frac{\Phi}{f^2} = \frac{2}{\pi^{\frac{1}{2}}} \int_0^\infty \frac{\exp(iz\zeta^{-1} - \mu\zeta^2)\, d\zeta}{\zeta^{-2}\left[f^2 K_+(\zeta, f^2)\right]\left[f^2 K_-(\zeta, f^2)\right]}$$

$$+ \mu^{\frac{1}{2}} \tilde{T}^{\frac{1}{2}} \frac{2}{\pi^{\frac{1}{2}}} \int_0^\infty \frac{\exp(iz\zeta^{-1} - \mu\tilde{T}\zeta)\, d\zeta}{\zeta^{-2}\left[f^2 K_+(\zeta, f^2)\right]\left[f^2 K_-(\zeta, f^2)\right]} . \qquad (5)$$

(Since the factor $\exp(iz\zeta^{-1})$ has an essential singularity at $\zeta=0$, giving z a small positive imaginary part assures the existence of the integral.) The dielectric functions are here and henceforth indicated as functions of the alternative arguments ζ and f^2. The form of Eq. (5) displays clearly the role of ion and electron distribution functions in determining the character of the response. The usual factor K_+^{-1} is modulated in each integral by the velocity derivative of the distribution function, which indicates the range of particle velocities which interact with each phase-velocity component of the wave. The ion-like integral, which is strongly affected by the ion-acoustic root of the dispersion relation at $\zeta_1 = \omega_0/ka_i$, principally determines the response from the grid to the beginning of the interference region. See the dashed curve of Fig. 2. In this region electrons easily follow the low-frequency ion motion and, as is well known, neutralize the ion charge density quite effectively. In the region where the ion-like and electron-like integrals are of comparable magnitude the interference is obtained. At large distances the response is principally determined by the electron-like integral; the phase-mixing factor $\exp(iz\zeta^{-1})$ limits the phase velocities involved to values greater than those of most ions so that only an electron charge density arises. This part of the response corresponds

to electron shielding of the disturbance produced by the grid.

By deforming the contours so that they proceed from the origin in the direction of negative imaginary values, pass below the pole at ζ_1, and that pole only, and approach infinity well within the range $-\pi/4 < \arg\{\zeta\} < \pi/4$, one expresses the response as the sum of the residue of the pole at ζ_1 and of the two integrals.

The ion-like integral may be evaluated along the path of steepest descents for the function $\exp(iz\zeta^{-1} - \zeta^2)$, shown in Fig. 1c, or along the simpler contour shown in Fig. 1d, which is independent of z. The path of the steepest descents is the archetype of desirable contours in that it transforms the dominant exponential behavior of the integrand into a gaussian; for the simpler contour considered, the exponential becomes predominantly damped (as opposed to oscillatory) as ζ approaches the origin and infinity. Thus the oscillatory behavior which plagues an integration along the real axis for large x is avoided here.

The evaluation of the electron-like integral requires a somewhat different treatment. Since its integrand decreases slowly as $|\zeta|$ increases, the dimensionless wave number is an appropriate variable and the contour of Fig. 1e is a suitable contour. The practical difference between using one variable and the other is that in the numerical integration one treats the variable chosen as having a distribution of discrete values which is not radically different from a uniform distribution. The path of steepest descents for the function $\exp(iz\zeta^{-1} - \mu\zeta^2)$ or a contour which bears the same relation to it as Fig. 1d bears to Fig. 1c is unsatisfactory because a large number of roots of the dispersion relation are swept past in going to such a contour from the positive real axis.

Computations were performed both for the dipole grid model and for the model of the double grid with finite spacing. The latter case was considered in order to gain an appreciation of the importance of the assumed theoretical model of the grid excitation in determining the character of the response. The factor $[\sin(kx_0/2)/(kx_0/2)]$ introduced into the integrand of Eq. (1) gives the integral for the case of spacing between the grids equal to x_0. The potential for points $x > x_0/2$ is $\phi(x) = x_0^{-1}[\phi_d(x + \frac{1}{2}x_0) - \phi_d(x - \frac{1}{2}x_0)]$, in which

$$\phi_d(x) = -\frac{\sigma_0 x_0}{2\pi\varepsilon_0}\int_0^\infty \left[\frac{1}{K_+(\omega_0,k)} - \frac{1}{K_-(\omega_0,k)}\right]\frac{\exp(ikx)dk}{k^2}. \qquad (6)$$

(The pole at $k=0$ gives unimportant contributions to $\phi(x)$ as in the dipole limit.)

Computations were performed on the Culler On-Line Computer of TRW Systems, Inc., Redondo Beach, California. Figure 2 shows the results for the dipole limit and the finite-spacing case (with $z_0 = \omega_0 x_0/2a_i = 4$) when $\tilde{T}=1$ and $f^2 \ll 1$. (In the limit $f^2 \ll 1$ one has $f^2K_\pm(\zeta,f^2) \to f^2K_\pm(\zeta) = -\zeta^2[Z_\pm'(\zeta) + \tilde{T}Z_\pm'(\mu^{1/2}\,T^{1/2}\,\zeta)]$). Comparison of the curves suggests that the general character of the response is unaffected by a choice of reasonable models of the grid excitation, but that fine details such as the precise character of the interference between ion and electron waves should not be taken very seriously.

III. FORMULATION OF THE NONLINEAR PROBLEM

Examination of the (nonlinear) Vlasov equation indicates that the steady-state response to time-harmonic grid excitation at frequency ω_0 involves frequency components $\omega=0$, ω_0, $2\omega_0$,.... Strong nonlinearity involves substantial contributions from a large number of harmonics and a coupled system which is probably difficult, if not impossible, to solve. Accordingly a solution is sought for the case of weak nonlinearity by introducing a perturbation-series expansion of the potential and species distribution functions. (This is equivalent to the procedure used by Montgomery and Gorman[5] to study the initial-value problem.) Introducing for convenience the parmeter λ, which is finally to be set to unity, one obtains the following expansions to order λ^2 of the potential in the plasma and the species distribution functions:

$$\Phi(x,t) = \lambda\,\phi(x,t) + \lambda^2\delta\phi(x,t) + \cdots \tag{7}$$

$$F_\alpha(x,v,t) = f_{0\alpha}(v) + \lambda\, f_\alpha(x,v,t) + \lambda^2\,\delta f_\alpha(x,v,t) + \cdots \tag{8}$$

Substituting these expressions into the Vlasov equation and collecting like powers of λ one obtains in first order the linearized Vlasov equation and in second order the set

$$\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial x}\right)\delta f_\alpha-\frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial x}\delta\phi\frac{\partial}{\partial v}f_{0\alpha}=-\frac{q_\alpha}{m_\alpha}E\frac{\partial}{\partial v}f_\alpha \qquad (9)$$

$$-\frac{\partial^2}{\partial x^2}\delta\phi=\sum_\alpha\frac{q_\alpha n_{0\alpha}}{\epsilon_0}\int_{-\infty}^{\infty}\delta f_\alpha\,dv \qquad (10)$$

in which $E = -\partial\phi/\partial x$. By proceeding to higher orders one obtains a hierarchy of pairs of equations for higher-order quantities. In each order the equation corresponding to Eq. (9) differs in having a different sum of quadratic combinations of lower-order terms on the right-hand side; the structure of the other equation is the same as that of Eq. (10). For sufficiently small amplitude of excitation only the first-order and second-order components of the response need be considered.

The second-order equations are Laplace-Fourier transformed; the transform operator is

$$T=\int_0^\infty dt\,\exp(i\omega t)\int_{-\infty}^{\infty}dx\,\exp(-ikx)\,, \qquad (11)$$

The transform-inversion operator is denoted by T^{-1}. (Primed and double-primed transform variables, ω', ω'', k', and k'', are used below; the corresponding operators are denoted by T', T'^{-1}, T'', and T''^{-1}.) The distribution functions $\delta f_\alpha(\omega,k,v)$ are eliminated from Eqs. (9) and (10) and the resulting equation is solved for $\delta\phi(\omega,k)$. The electric field and species

distribution functions in linear theory are expressed as inversions of the transforms $E(\omega'',k'')$ and

$$f_\alpha(\omega',k',v) = \frac{i q_\alpha}{m_\alpha} \frac{\partial f_{0\alpha}/\partial v}{(k'v-\omega')} E(\omega',k') \tag{12}$$

in which $E(\omega',k')$ is the transform of the external field of the grid divided by the dielectric function. Here ω' initially has a positive imaginary part. The velocity integrals which arise (see Eq. (10)) are analytically continued into the entire ω/k and ω'/k' planes. The deformation of the contour of integration in the velocity plane around the pole at $v = \omega/k$ is the same as that in $K(\omega,k)$; the deformation around the pole at $v = \omega'/k'$ is the same as that in $K(\omega',k')$. The Laplace-Fourier inversion of $\delta\phi(\omega,k)$ is performed. The result of these operations is

$$\delta\phi(x,t) = T^{-1}\left\{\frac{i}{k^3 K(\omega,k)} \sum_\alpha \omega_{p\alpha}^2 \int_\Lambda \frac{T[T''^{-1}(E)T'^{-1}(\partial f_\alpha/\partial v)]}{(v-\omega k^{-1})} dv\right\} \tag{13}$$

in which E and f_α are Laplace-Fourier transforms of the field and distribution-function perturbations in linear theory, and Λ denotes the contour in the velocity plane deformed as described above. Making use of the notation $U_\alpha = q_\alpha \omega_{p\alpha}^2 / m_\alpha a_\alpha^4$ and

$$V(\zeta_\alpha,\zeta_\alpha') = a_\alpha^{-1} \int_\Lambda \frac{1}{(v-\omega k^{-1})} \frac{\partial}{\partial v}\left[\frac{\partial f_{0\alpha}/\partial v}{(v-\omega' k'^{-1})}\right] dv \tag{14}$$

in which $\zeta_\alpha = \omega/ka_\alpha$ and $\zeta_\alpha' = \omega'/k'a_\alpha$ one may express the response as

$$\delta\phi = T^{-1}\left[\frac{-1}{k^3 K(\omega,k)}\sum_\alpha U_\alpha T\left\{T''^{-1}\left[E(\omega'',k'')\right]T'^{-1}\left[E(\omega',k')k'^{-1}V(\zeta_\alpha,\zeta_\alpha')\right]\right\}\right] \quad (15)$$

For a plasma composed of electrons and of one species of singly-charged ions, $U_e/U_i = -\hat{T}^2$. Performing an integration by parts and making a partial fraction expansion in Eq. (14), one obtains

$$V(\zeta,\zeta') = \frac{[Z'(\zeta')-Z'(\zeta)]}{(\zeta-\zeta')^2} + \frac{Z''(\zeta)}{(\zeta-\zeta')}. \quad (16)$$

IV. DOMINANT-POLE APPROXIMATION

The practical impossibility of evaluating $\delta\phi(x,t)$ without introducing a further simplification is now apparent. Application of the convolution theorem to $T\{\ \}$ in Eq. (15) results in a single integration with respect to k' which contains a quadratic combination of a function of k' and a function of $(k-k')$. The inversion $k \to x$ leads to an integration with respect to k. In addition to the requirement of evaluating a double Fourier integral is the further complication that the functions of k' and $(k-k')$ within the double integral have branch-point pairs in the k and k' planes when $k'=0$ and $(k-k')=0$, just as there is a branch-point pair at $k=0$ in the linear problem. An escape from this impasse is provided by the dominance

of the exponentially damped contribution of the residue of the ion-acoustic pole over a wide range of x in the linear problem. The deviation from this response near the grid is so strongly damped that its contribution to the nonlinear response is probably highly localized; the deviation due to the electron wave at large distances is at such a low level that it may be neglected.

The dominant-pole approximation is obtained by representing the steady-state electric field in linear theory by the residue contribution

$$E(x,t) = f^2 A \exp[i(k_1|x| - \omega_0 t)] + \text{c.c.} \tag{17}$$

in which

$$A = \frac{i\omega_0 \sigma_c x_0}{2\epsilon_0 v_i} \frac{1}{\zeta_1^2 [f^2 K_+'(\zeta_1)]} \tag{18}$$

and $k_1 = \omega_0/\zeta_1 a_i$.

To obtain the double-frequency response, it is sufficient to consider the partial linear response given by the first term of Eq. (17). The corresponding nonlinear response plus its complex conjugate is the real double-frequency response. The Laplace-Fourier transform of the partial excitation is

$$E^{+}(\omega,0) = f^2 A \frac{i}{\omega - \omega_c} \left[\frac{1}{i(\zeta - \zeta_1)} - \frac{1}{i(\zeta + \zeta_1)} \right], \tag{19}$$

Examining Eqs. (1) and (5) one notes that the first term of this expression comes from $K_+(\omega_0,k)$ and that the second term comes from $K_-(\omega_0,k)$. Because of the replacement of the dielectric function in the transform of the response in linear theory by its dominant-pole component it is necessary to indicate whether the plasma dispersion functions in $V(\zeta_\alpha,\zeta'_\alpha)$ (see Eqs. (15) and (16)) are plus or minus functions. This is accomplished by the notation $V_{s\,s'}(\zeta_\alpha,\zeta'_\alpha)$, in which s and s' are plus or minus; the first subscript is associated with the first argument and the second subscript with the second argument.

As the first step in determining the explicit form of Eq. (15) in the dominant-pole approximation with the partial excitation described above, one obtains the inversion

$$T'^{-1}[E^+(\omega',k')k'^{-1}V(\zeta_\alpha,\zeta'_\alpha)] = f^2 A a_1 \omega_0^{-1}\zeta_1 \exp[i(k_2|x'|-\omega_0 t')]$$
$$\times[\theta(x')V_{(-)+}(\zeta_{1\alpha},\zeta_{2\alpha})-\theta(x')V_{(-)-}(\zeta_{1\alpha},-\zeta_{2\alpha})]\,. \tag{20}$$

Here $\theta(x')$ is the unit step function; $\zeta_{1\alpha}$ equals ζ_1 for $\alpha=i$ and $\mu^{1/2}\tau^{1/2}\zeta_1$ for $\alpha=e$. The subscripted parentheses indicate that plasma dispersion functions of argument ζ_α may be plus or minus functions. The further steps indicated in Eq. (15) are performed; one obtains $\delta\phi^+(x,t) = \exp(-2i\omega_0 t)\delta\phi^+(x)$, in which

$$f^{-2}\delta\phi^+(x) = iA f^2 a_1 \omega_0^{-2}\zeta_1 \sum_\alpha U_\alpha$$
$$\times\int_{-\infty}^{\infty}\frac{\exp(ikx)}{k^2[f^2K(k,\omega_0)]}\left[\frac{V_{(-)+}(\zeta_{1\alpha},\zeta_{2\alpha})}{(k-k_2)}-\frac{V_{(-)-}(\zeta_{1\alpha},-\zeta_{2\alpha})}{(k+k_2)}\right]\frac{dk}{2\pi}\,. \tag{21}$$

Here ω has been set equal to $2\omega_0$ so that $\zeta_\alpha = 2\omega_0/ka_\alpha$. The definition

of $\zeta_{1\alpha}$ is unchanged.

The folding of the primitive contour in the k plane and the transformation of the variable of integration from k to ζ, which is now defined to be equal to $2\omega_0/ka_i$, are performed. The contour in the ζ plane is now deformed so that it proceeds from the origin in the direction of negative imaginary values, passes below the pole at ζ_1, and approaches infinity well within the range $-\pi/4<\arg\{\zeta\}<\pi/4$. Such a contour is denoted by C. As in the linear problem there is a simple pole at $k=0$, which has no physical significance. For the case $f^2<<1$ ($f = \omega_0/\omega_{pi}$, as before), to which consideration is now limited, $f^2K_+(2\omega_0,k) \to \frac{1}{4} f^2K_\pm(\zeta)$. For purpose of comparison with integrals in the linearized problem it is convenient to consider $[\delta\phi^+(x)/f^6\ (-2\pi iC_i)]$, in which $C_i = C_\alpha|_{\alpha=i}$ and

$$C_\alpha = \frac{i}{16\pi}\left(\frac{\sigma_0 X_0}{\epsilon_0}\right)^2 \left\{\frac{1}{\zeta_1[f^2K_+(\zeta_1)]}\right\}^2 \left(\frac{v_i}{\omega_0}\right)^2 U_\alpha. \qquad (22)$$

One obtains

$$f^{-6}(-2\pi i\, C_i)^{-1}\,\delta\phi^+(x) = I^{(i)} + I^{(e)} + R \qquad (23)$$

in which

$$I^{(i)} = \int_C \frac{d\zeta \exp(2iz\zeta^{-1})}{(-2\pi i)\zeta^{-2}[f^2K_+(\zeta)][f^2K_-(\zeta)]}$$

$$\times\left\{\frac{[f^2K_-(\zeta)V_{++}(\zeta,\zeta_1) - f^2K_+(\zeta)V_{-+}(\zeta,\zeta_1)]}{(\zeta-\zeta_1)}\right.$$

$$\left. - \frac{[f^2K_-(\zeta)V_{+-}(\zeta,-\zeta_1) - f^2K_+(\zeta)V_{--}(\zeta,-\zeta_1)]}{(\zeta+\zeta_1)}\right\} \qquad (24)$$

$$I^{(e)} = -\tau^2 \int_C \frac{d\zeta \exp(2iz\zeta^{-1})}{(-2\pi i)\zeta^{-2}[f^2K_+(\zeta)][f^2K_-(\zeta)]}$$

$$\times\left\{\frac{[f^2K_-(\zeta)V_{++}(\xi,\xi_1) - f^2K_+(\zeta)V_{-+}(\xi,\xi_1)]}{(\zeta-\zeta_1)}\right.$$

$$\left. - \frac{[f^2K_-(\zeta)V_{+-}(\xi,-\xi_1) - f^2K_+(\zeta)V_{--}(\xi,-\xi_1)]}{(\zeta+\zeta_1)}\right\} \qquad (25)$$

and R is the residue contribution, which will be discussed and evaluated below. For compactness the notation $\xi = \mu^{1/2}\tau^{1/2}\zeta$ and $\xi_1 = \mu^{1/2}\tau^{1/2}\zeta_1$ has been introduced. The definition of z remains unchanged, $z = \omega_0 x/a_i$.

Expression of $I^{(i)}$ and $I^{(e)}$ as sums of ion-like and electron-like integrals, that is integrals whose integrands have the exponential behavior $\exp(iz\zeta^{-1} - \zeta^2)$ and $\exp(iz\zeta^{-1} - \mu\tau\zeta^2)$, respectively, is achieved by manipulations which are here described briefly. By expressing $Z''(\zeta)$ in terms of $Z'(\zeta)$ in Eq. (16), one obtains

$$V(\zeta,\zeta') = \left\{(\zeta-\zeta')^{-2}Z'(\zeta') + 2[\zeta(\zeta-\zeta')]^{-1}\right\}$$

$$+\left\{-(\zeta-\zeta')^{-2} + (1-2\zeta^2)[\zeta(\zeta-\zeta')]^{-1}\right\}Z'(\zeta). \qquad (26)$$

In $I^{(i)}$ and $I^{(e)}$ only the ζ_α dependence of the V's remains. To indicate the structure of the manipulations one may denote the V's by $W_+(\zeta_\alpha) = V_{+\pm}(\zeta_\alpha, \pm\zeta_{1\alpha})$, $W_-(\zeta_\alpha) = V_{-\pm}(\zeta_\alpha, \pm\zeta_{1\alpha})$, in which

$$W_\alpha(\zeta_\alpha) = P(\zeta_\alpha) + Q(\zeta_\alpha)\, Z'_\pm(\zeta_\alpha). \tag{27}$$

The P's and Q's depend on the species α and on the suppressed $\pm$ index associated with the arguments $\pm\zeta_{1\alpha}$. But they do not contain $Z'_\pm(\zeta_\alpha)$ and hence are independent of the $\pm$ index associated with ζ_α . Appropriate elements of the integrands of $I^{(i)}$ and $I^{(e)}$ are expressed as

$$\begin{aligned}&\left[f^2 K_-(\zeta) W_+(\zeta_\alpha) - f^2 K_+(\zeta) W_-(\zeta_\alpha)\right] \\ &= W_+(\zeta_\alpha)\left[f^2 K_-(\zeta) - f^2 K_+(\zeta)\right] + Q(\zeta_\alpha)\left[Z'_-(\zeta_\alpha) - Z'_+(\zeta_\alpha)\right].\end{aligned} \tag{28}$$

The desired expression of $I^{(i)}$ and $I^{(e)}$ as sums of ion-like and electron-like integrals is achieved by making use of Eq. (4). Defining, for $n = 2,3$, the functions

$$D_n(\zeta_\alpha) = \left[(\zeta_\alpha - \zeta_{1\alpha})^{-n} - (\zeta_\alpha + \zeta_{1\alpha})^{-n}\right] \tag{29}$$

and making use of the relation $Z'_\pm(-\zeta) = Z'_\mp(\zeta)$, one obtains finally for $\alpha = i,e$

$$I^{(\alpha)} = I^{(\alpha i i)} + I^{(\alpha i e)} + I^{(\alpha e i)} + I^{(\alpha e e)} \tag{30}$$

in which

$$I^{(\alpha\beta n)} = \frac{2}{\pi^{1/2}} \int_C \frac{\exp(2iz\zeta^{-1}) G^{(\alpha\beta n)} \, d\zeta}{\zeta^{-2} [f^2 K_+(\zeta)][f^2 K_-(\zeta)]} \tag{31}$$

The functions $G^{(\alpha\beta n)}$ are

$$G^{(iii)} = \exp(-\zeta^2) \zeta^3 \left[Z_+'(\zeta_1) + \tilde{T} Z_+'(\xi) \right] D_3(\zeta) \tag{32}$$

$$G^{(iia)} = \exp(-\zeta^2) \zeta^2 \left[2 - \tilde{T}(1 - 2\zeta^2) Z_+'(\xi) \right] D_2(\zeta) \tag{33}$$

$$G^{(iea)} = \mu^{\frac{1}{2}} \tilde{T}^{\frac{3}{2}} \exp(-\xi^2) \zeta^3 \left[Z_+'(\zeta_1) - Z_+'(\xi) \right] D_3(\zeta) \tag{34}$$

$$G^{(iia)} = \mu^{\frac{1}{2}} \tilde{T}^{\frac{3}{2}} \exp(-\xi^2) \zeta^2 \left[2 + (1 - 2\zeta^2) Z_+'(\xi) \right] D_2(\zeta) \tag{35}$$

$$G^{(eia)} = -\mu^{-1} \tilde{T} \exp(-\zeta^2) \zeta^3 \left[Z_+'(\xi) - Z_+'(\xi) \right] D_3(\zeta) \tag{36}$$

$$G^{(e12)} = -\mu^{-\frac{1}{2}} \tilde{T}^{\frac{3}{2}} \exp(-\zeta^2)\, \zeta^3\, Z_+''(\xi)\, D_2(\zeta) \tag{37}$$

$$G^{(ee3)} = -\mu^{-\frac{1}{2}} \tilde{T}^{\frac{3}{2}} \exp(-\xi^2)\, \zeta^3 \left[\tilde{T}\, Z_+'(\xi) + Z_+'(\zeta)\right] D_3(\zeta) \tag{38}$$

$$G^{(ee2)} = -\mu^{-\frac{1}{2}} \tilde{T}^{\frac{3}{2}} \exp(-\xi^2)\, \zeta^2 \left[2\tilde{T} - (1-2\xi^2) Z_+'(\zeta)\right] D_2(\zeta). \tag{39}$$

In Eqs. (36) and (38) the relation $D_3(\xi) = \mu^{-3/2} \tilde{T}^{-3/2} D_3(\zeta)$ has been used; in Eqs. (37) and (39) the relation $D_2(\xi) = \mu^{-1} \tilde{T}^{-1} D_2(\zeta)$ has been used.

V. SIMPLIFICATION AND NUMERICAL RESULTS

The numerical evaluation of the branch-cut integrals may be simplified considerably by the introduction of approximations and an ordering of terms based on the smallness of $\mu^{1/2}$. In lowest order $I^{(i)}$ may be neglected. In next order one element of $I^{(i)}$, namely $I^{(ii2)}$, makes a contribution to the result which, although small, is enhanced by the character of its integrand above that indicated by the ordering.

The relation $[Z'_+(\zeta_1) + \tilde{T}Z'_+(\xi_1)] = 0$ implied by the dispersion relation facilitates the comparison of the relative magnitudes of the corresponding elements of $I^{(i)}$ and $I^{(e)}$ which contain D_3. Using this relation one obtains $[G^{(ii3)}/G^{(ei3)}] = \mu$ and $[G^{(ie3)}/G^{(ee3)}] = \mu\tilde{T}^{-1}$.

The corresponding elements of $I^{(i)}$ and $I^{(e)}$ which contain D_2 are now compared. Consider first $I^{(ie2)}$ and $I^{(ee2)}$. In the small-argument region of the contour C, $|\zeta| \lesssim 4$, the factor $[2 + (1-2\zeta^2)Z'_+(\zeta)]$, which appears in $G^{(ie2)}$, and the factor $[2\tilde{T} - (1-2\xi^2)Z'_+(\zeta)]$, which appears in $G^{(ee2)}$, are both of order unity. Thus the ratio of the absolute value of the small-argument part of $I^{(ie2)}$ to that of $I^{(ee2)}$ is approximately μ. In the asymptotic region $|\zeta| \gtrsim 4$ the two factors have the behavior $[2 + (1-2\zeta^2)Z'_+(\zeta)]\ [= \zeta Z'_+(\cdot)] \sim -2\zeta^{-2}$ and $[2\tilde{T} - (1-2\xi^2)Z'_+(\zeta)] \sim 2\tilde{T}$. Thus the ratio of the absolute value of the asymptotic part of $I^{(ie2)}$ to that of $I^{(ee2)}$ is less than μ.

The last comparison to be made is that of $I^{(ii2)}$ and $I^{(ei2)}$. In the range $|\zeta| \lesssim 4$ over which numerical evaluation of ion-like integrals is performed, the corresponding factors $[2 - \tilde{T}(1-2\zeta^2)Z'_+(\xi)]$, which appears in $G^{(ii2)}$, and $\zeta Z''_+(\xi)$, which appears in $G^{(ei2)}$, are both of

order unity. Therefore the ratio of the absolute value of $I^{(ii2)}$ to that of $I^{(ei2)}$ is $O(\mu^{1/2})$. It will be seen that, due to the absence of inverse powers of $(\zeta-\zeta_1)$ in the lowest order of an expansion of the integrand of the sum $[I^{(ei3)} + I^{(ei2)}]$, $I^{(ii2)}$ contributes more importantly to the total response than the ordering indicates. Therefore it is retained.

The integrands which have been retained are now simplified. Consider first the sum of ion-like integrals, $[I^{(ei3)} + I^{(ei2)} + I^{(ii2)}]$. Expanding the factor $[Z'_+(\xi_1) - Z'_+(\xi)]$ which appears in $G^{(ei3)}$ in powers of $(\xi-\xi_1)$ and approximating $Z''_+(\xi)$ in $G^{(ei2)}$ by $Z''_+(\xi_1)$, one obtains in lowest order

$$[G^{(ei3)}+G^{(ei2)}]=\mu^{-\frac{1}{2}}\tau^{\frac{3}{2}}\exp(-\zeta^2)\zeta^3\left[\tfrac{1}{2}\zeta_1 Z''_+(\xi_1)\right](\zeta-\zeta_1)^{-3}. \qquad (40)$$

The reason for retaining $I^{(ii2)}$ should now be apparent. Although $G^{(ii2)}$ is formally smaller than $[G^{(ei3)} + G^{(ei2)}]$ by the factor $\mu^{1/2}$, the presence of a factor $(\zeta-\zeta_1)^{-2}$ in the former and the absence of negative powers of $(\zeta-\zeta_1)$ in the latter result in a contribution of $I^{(ii2)}$ substantially larger than that indicated by the ordering. (The next order of the expansion of $[G^{(ei3)} + G^{(ei2)}]$ makes a negligible contribution to the result, despite the fact that one of two terms contains a factor $(\zeta-\zeta_1)^{-1}$.) Since only lowest order terms in $\mu^{1/2}$ are retained (with the exception of the anomalously large contribution $I^{(ii2)}$) the approximations $Z''_+(\xi_1) \simeq -2\pi^{1/2}i$ and $Z''_+(\xi) \simeq -2$ may be made in Eqs. (33) and (40), respectively.

Consider now the sum of electron-like integrals, $[I^{(ee3)} + I^{(ee2)}]$.

They are evaluated as the sum of two parts: a small-argument component evaluated along the same contour as the ion-like integrals and a large-argument component evaluated along a contour in the complex phase velocity plane of the same general character as that of Fig. 1e but with an upper limit which is the reciprocal of the upper limit of integration in the ζ plane. In the small-argument region an expansion in $\mu^{1/2}$ gives

$$\left[G^{(ee3)}+G^{(ee2)}\right]=-\mu^{-\frac{1}{2}}\tilde{T}^{\frac{3}{2}}\exp(-\xi^2)\,\zeta^3(2\zeta_1)(\zeta^2-\zeta_1^2)^{-3}$$

$$\times\left\{\left[Z_+'(\zeta)-2\tilde{T}\right](\zeta^2+3\zeta_1^2)+\mu^{\frac{1}{2}}\tilde{T}^{\frac{3}{2}}\left[Z_+''(0)\zeta_1\right](3\zeta^2+\zeta_1^2)+\cdots\right\} \tag{41}$$

The term of order $\mu^{1/2}$ makes an anomalously large contribution to the integral because $[Z'_+(\zeta) - 2\tilde{T}]$ vanishes at a point with a distance of order $\mu^{1/2}$ from ζ_1 and is therefore retained. In the large-argument region $Z'_+(\zeta)$ is represented by a suitable number of terms of its asymptotic expansion.

The residue contribution to Eq. (23) is

$$R=R_{es}\left\{\frac{\zeta^2\exp(\beta\zeta^2)}{f^2K_+(\zeta)}\left[\frac{V_{++}(\zeta,\zeta_1)-V_{++}(\zeta_1,\zeta_1)}{(\zeta-\zeta_1)}\;\frac{V_{+-}(\zeta,-\zeta_1)-V_{+-}(\zeta_1,\zeta_1)}{(\zeta+\zeta_1)}\right]\right\} \tag{42}$$

One expands the factors of Eq. (42) in powers of $(\zeta-\zeta_1)$ to obtain an explicit expression for the residue. To develop an approximate expression which is convenient for computational purposes one expands the residue in powers of $\mu^{1/2}$. The final result is

$$R = \mu^{-\frac{1}{2}} \tilde{T}^{\frac{3}{2}} [F^2 K_+'(\zeta_1)]^{-1} \exp(2iz\zeta^4)[s_0 + s_1(2z)] \qquad (43)$$

in which $s_0 = -(\pi^{1/2}/2)i + O(\mu^{1/2})$ and $s_1 = \mu^{1/2}\tilde{T}^{-3/2}(-i/2)[Z'''_+(\zeta_1) + 2\pi^{1/2}\tilde{T}^2 i]$. The term proportional to z is retained even though it is formally of order $\mu^{1/2}$ because it can make a substantial contribution to the result at large values of z which are within the probable range of validity of the dominant-pole approximation for large values of the ratio T_e/T_i.

The results of numerical calculations are shown in Figs. 3 and 4 for the cases $\tilde{T} = 1$, .5, and .25. It is difficult to estimate the range of z over which the dominant-pole approximation is valid. If the dominant-pole component of the linear response, Eq. (17), is considered to be mixed with itself in the absence of further dispersion or damping, the double-frequency harmonic is attenuated as $\exp[-2\mathrm{Im}(\zeta_1)x]$. The change in x dependence $x \to 2x$ is adopted as the basis for choosing a maximum value of z. For the equal temperature case the deviation of the linear response from the dominant-pole approximation is small (except very near the grid) until $z \approx 22$. Thus, $z = 11$ is chosen as the maximum value for the calculation of the double-frequency response. For the case $T_e/T_i = 2$, the corresponding maximum value for the linear response is $z = 38$. (See the results of Gould, Fig. 6 of Ref. 1.) Accordingly, $z = 19$ is selected as the maximum value for the calculation of the double-frequency response.

For this and higher values of the temperature ratio the double-frequency response has a characteristic repetitive pattern; accordingly, in the interest of conserving computer time the corresponding increases in the maximum values of z are not made beyond $T_e/T_i = 2$.

The principal features of the numerical results are three: a repetitive modulation of the spatially damped behavior of the logarithm of the absolute value of the potential perturbation; a reduction in spatial damping as the electron-to-ion temperature ratio increases; and a transition in the overall spatial rate of change of phase with temperature ratio increase, from a small rate characteristic of a non-wave-like disturbance similar to the "electron wave" at large distances in the linear problem, to a large linear rate characteristic of the mixing of the dominant-pole component with itself and similar to the behavior of the phase of the ion-acoustic wave in the linear problem. The modulation of the amplitude of the response and the non-wave-like character of the response for values of the electron-to-ion temperature ratio close to unity are consequences of the fact that the electron-like integral in the double-frequency response is not diminished relative to the ion-like integral by the presence of a factor $\mu^{1/2}$, as is the case in the linear response.

VI. ZERO-FREQUENCY RESPONSE

The nonlinear response at zero frequency is now shown to be a polarization of the plasma with no associated current density. This result is established in the context of the assumptions of small-amplitude excitation, uniform and stationary distribution function in the absence

of excitation, and one-dimensionality. There are a number of conditions in the experiments which are inconsistent with these assumptions; the inclusion of these complcations would make the analysis vastly more difficult.

The zero-frequency component of potential in the plasma must be an even function of x because there is no preferred direction in space when an average over one cycle of the excitation is considered. The only possible source of a zero-frequency component of potential in this theory is the inequality of the charge-to-mass ratio for the two species, which would give a potential which is an even function of x.

The Vlasov equation implies species continuity equations

$$\frac{\partial}{\partial t} n_\alpha + \frac{\partial}{\partial \underline{x}} \cdot \underline{\Gamma}_\alpha = 0 \tag{44}$$

in which $\underline{\Gamma}_\alpha$ is the particle current density of species α. For the zero-frequency component this equation states that the species particle current densities are divergenceless. The spatial symmetry of the zero-frequency component of the potential implies that species particle current densities are zero at $x = 0$; hence they must be zero everywhere.

The zero-frequency component of the lowest order nonlinear contribution to the potential was determined in Ref. 6. Since the zero-frequency component is probably difficult to detect in an experiment, that development is not included here.

VII. CONCLUSIONS

The double-frequency response produced by the nonlinear interaction of a grid-excited ion-acoustic wave with itself exhibits complicated behavior for values of the electron-to-ion temperature ratio near unity. The electron-wave component of the response is not diminished by the presence of a factor $\mu^{1/2}$, as occurs in the linear case. This indicates the operation of an electron shielding mechanism in the double frequency response. As the temperature ratio approaches a value of four, the response comes closer to displaying exponentially damped behavior.

The complications involved in determining nonlinear response to grid excitation are reduced considerably by expressing the Fourier inversion integrals which result as sums of "ion-like" and "electron-like" integrals. Physical interpretation of the results of a nonlinear grid excitation problem, is rendered somewhat less difficult by such a representation.

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Figure Captions

Fig. 1. Integration contours for Fourier inversion integrals.

(a) Primitive Fourier inversion contour in k plane and mapping of branch-cut and branch points of $Z_{\pm}(\zeta)$ for $\omega = \omega_0 + i\varepsilon$.

(b) Folding of left half of primitive contour onto right half of primitive contour in k plane. First few members of infinite set of roots of dispersion relation $K_+ = 0$ are indicated.

(c) Path of steepest descents for function $\exp(iz\zeta^{-1} - \zeta^2)$ and mapping of roots of dispersion relation shown in (b) in ζ plane.

(d) Simple contour for evaluation of ion-like integrals.

(e) Contour in k plane for evaluation of electron-like integrals which reduces oscillatory behavior of integrand.

Fig. 2. Numerical results in linear theory (cesium plasma).

Fig. 3. Numerical results in nonlinear theory (cesium plasma): natural logarithm of absolute value of $[I^{(i)} + I^{(e)} + R]$ as a function of z for temperature ratios $\tilde{T} = 1, .5, .25$.

Fig. 4. Numerical results in nonlinear theory (cesium plasma): argument of $[I^{(i)} + I^{(e)} + R]$ as a function of z for temperature ratios $\tilde{T} = 1, .5, .25$.

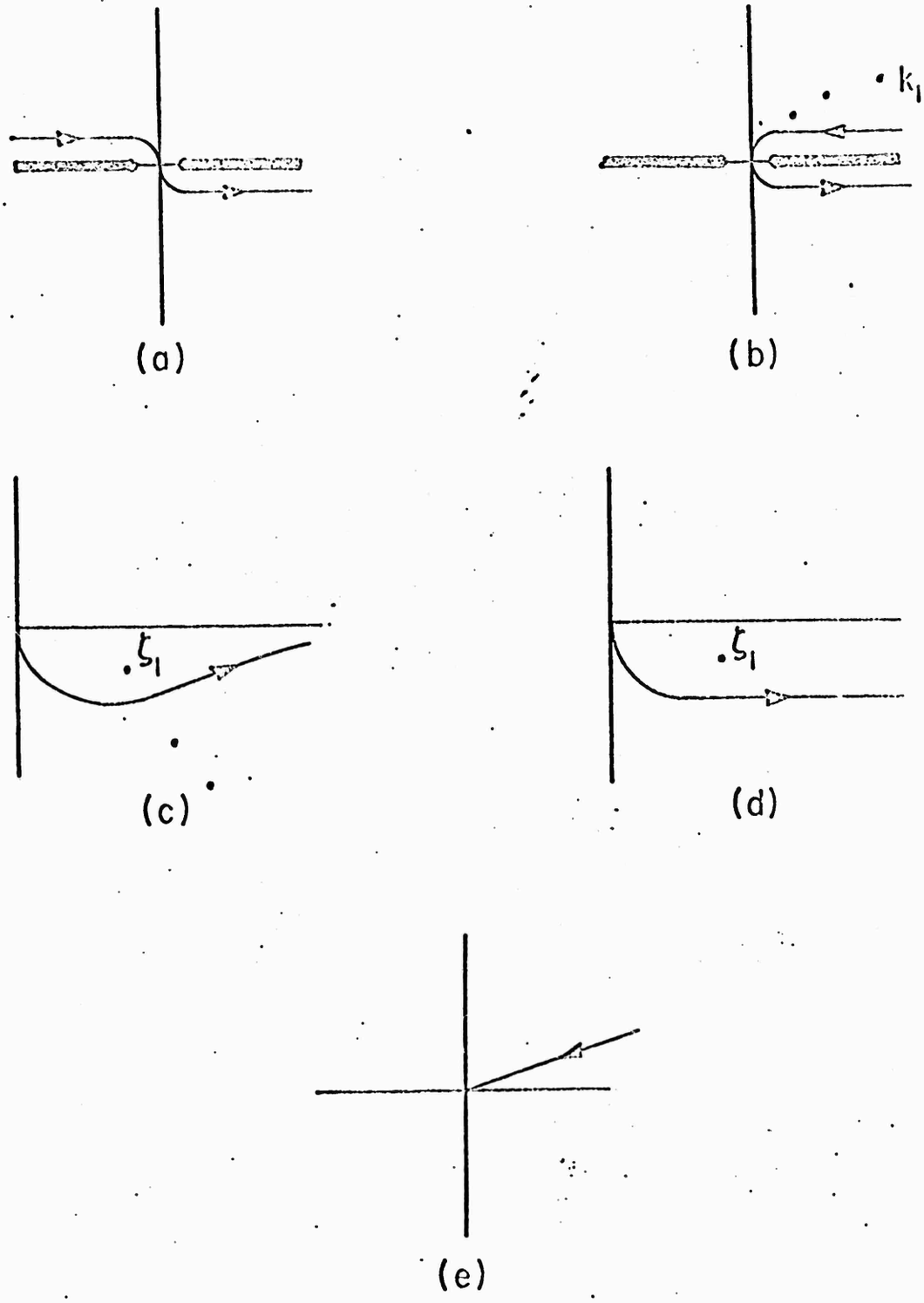

Figure 1.

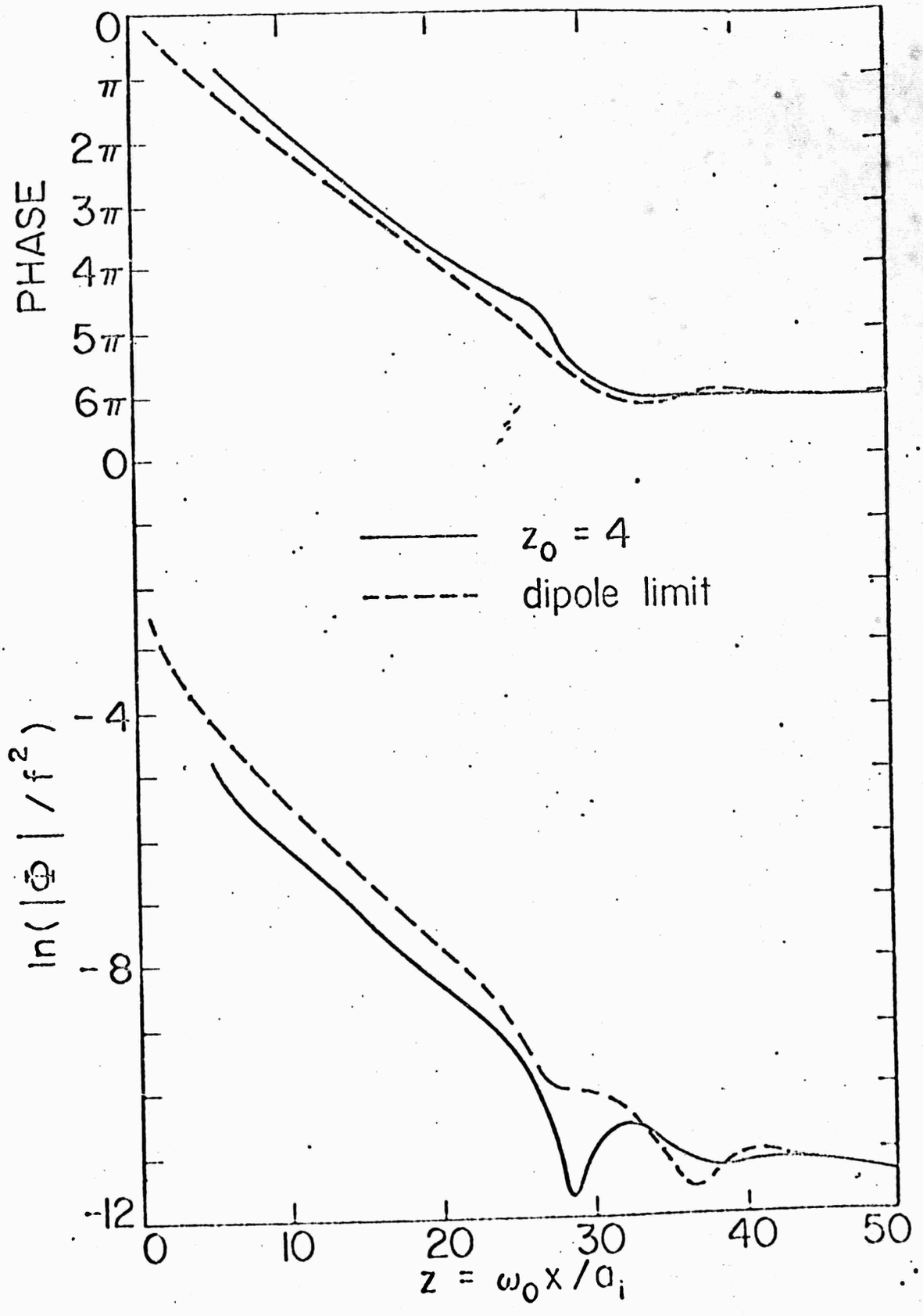
PHASE
0
π
2π
3π
4π
5π
6π
0
$z_0 = 4$
dipole limit
$\ln(|\Phi|/f^2)$
-4
-8
-12
0
10
20
30
40
50
$z = \omega_0 x / a_i$

Figure 2.

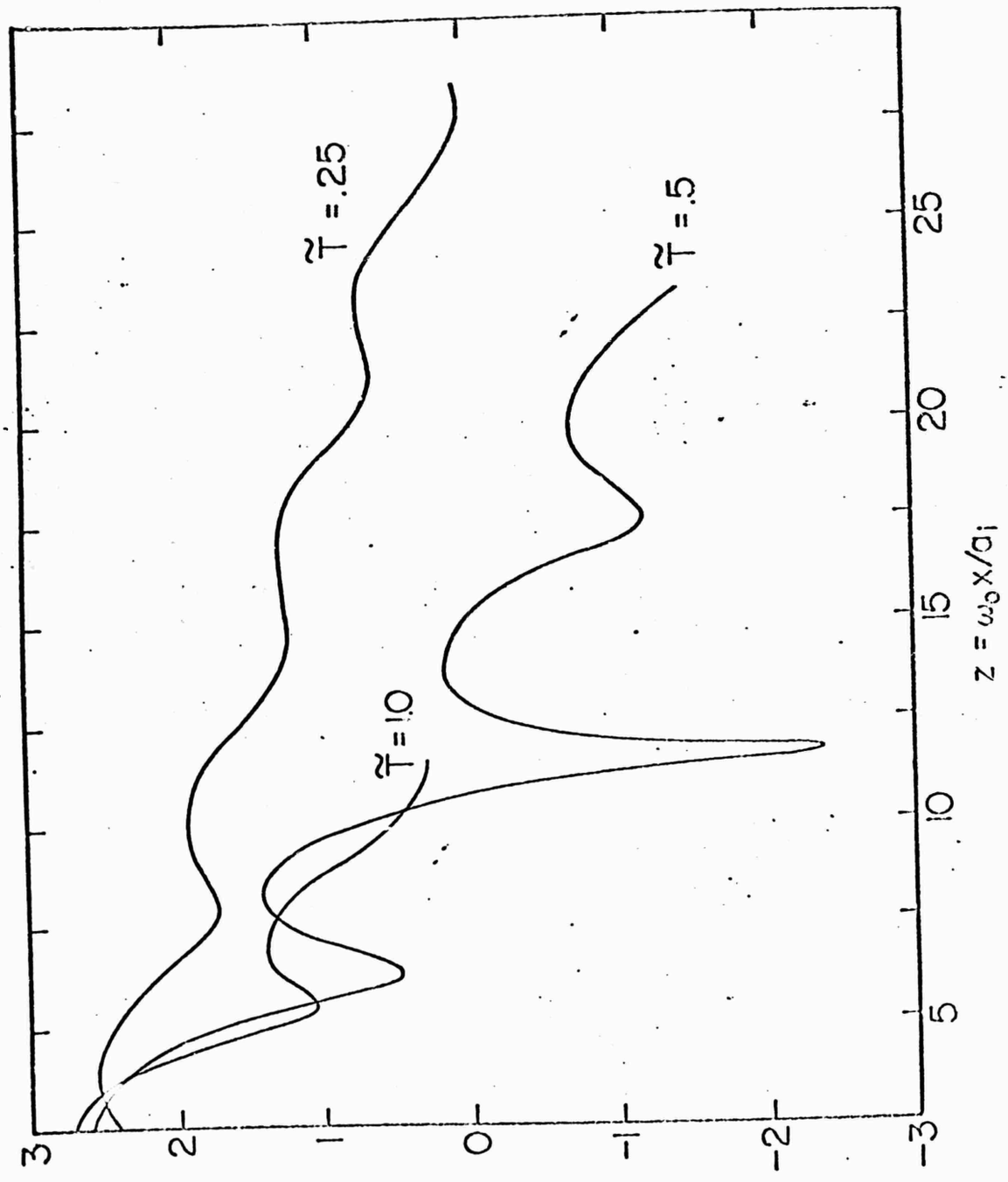

Figure 3.

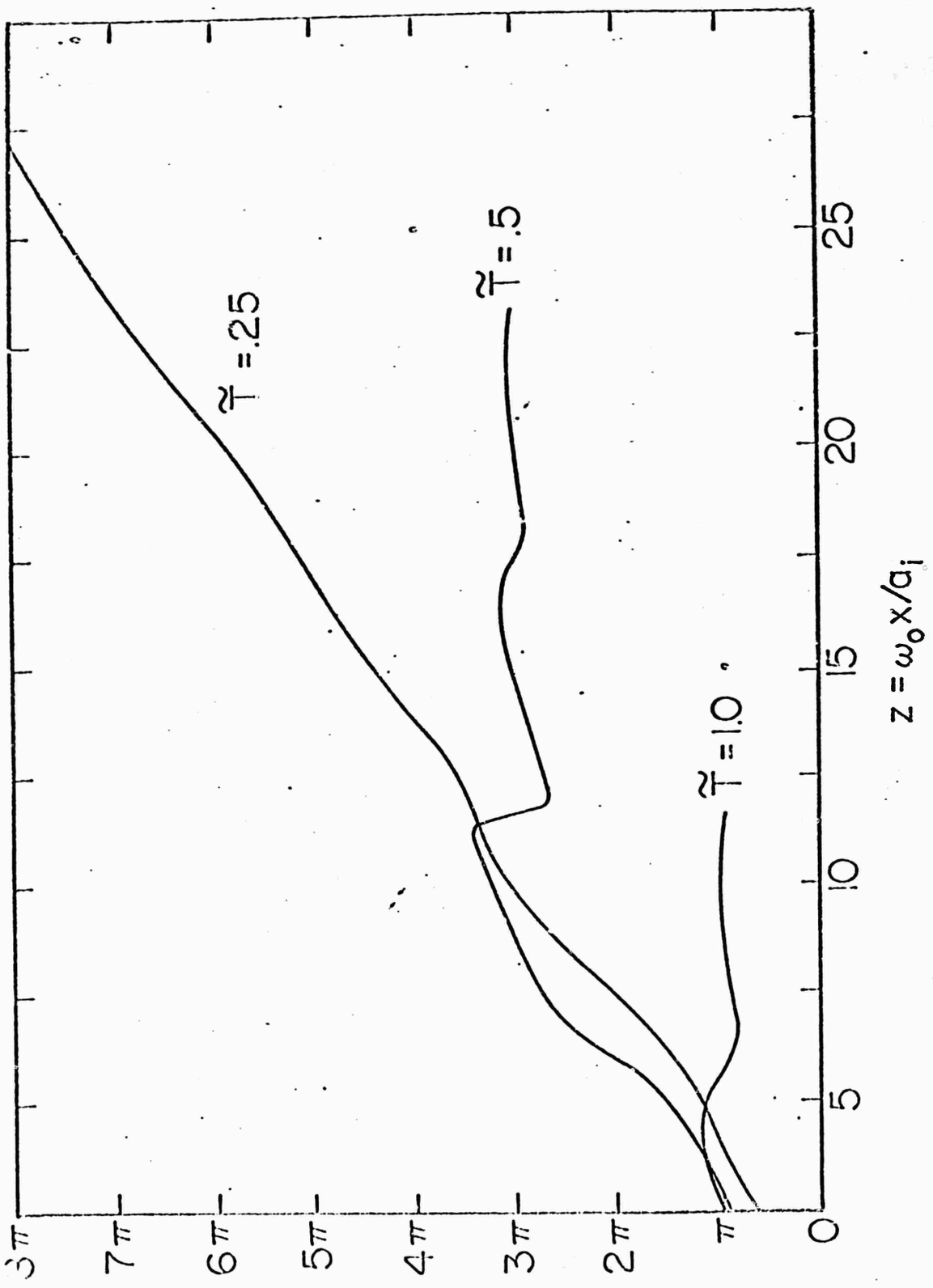
$\tilde{T}$ = .25
$\tilde{T}$ = .5
$\tilde{T}$ = 1.0
$z = \omega_0 x/a_i$
0
5
10
15
20
25
π
2π
3π
4π
5π
6π
7π

Figure 4.