

Division of Engineering
BROWN UNIVERSITY
PROVIDENCE, R. I.

2017

CONSTITUTIVE RELATIONS FOR
AN INELASTIC GRANULAR MEDIUM

J. B. WEIDLER, Jr. and P. R. PASLAY

ADLON 780

TECHNICAL LIBRARY
BLDG 813
ABERDEEN PROVING GROUND MD.
STEAP-TL

Advanced Research Projects Agency
Department of Defense
Contract SD-86
Materials Research Program

*BU
ARPA-E-67*

ARPA E67

June 1969

*copy
10/11/69*

4

CONSTITUTIVE RELATIONS FOR AN INELASTIC GRANULAR MEDIUM

by

II

*Brown and
Paslay Inc.*

J. B. Weidler, Jr. and P. R. Paslay

Brown University
Division of Engineering
Providence, Rhode Island

- 1. Soil mechanics
- 2. Granular materials - stress-strain equation.
- 3. Strain and stresses.

ARPA E67

June 1969

TECHNICAL LIBRARY
BLDG. #18
ABERDEEN PROVING GROUND, MD.

20060223356



CONSTITUTIVE RELATIONS FOR AN INELASTIC GRANULAR MEDIUM¹

By J. B. Weidler, Jr.² and P. R. Paslay³

ABSTRACT

The results of an experimental and analytical program to develop stress-strain relations for a granular medium are presented. Explicit three-dimensional relations are given which represent the inelastic deformation of the medium. A simple version of the formulation is used to represent experimental results for a locally quarried quartz, and an evaluation of the relations is presented. The simplified relations are seen to be a compromise between mathematical tractability on the one hand and accuracy of prediction on the other.

¹ Work presented in this paper was sponsored by ARPA Contract SD-86.

² Brown and Root Inc., Houston, Texas, formerly Brown University.

³ Brown University, Providence, Rhode Island.

CONSTITUTIVE RELATIONS FOR AN INELASTIC GRANULAR MEDIUM¹

By J. B. Weidler, Jr.² and P. R. Paslay³

INTRODUCTION

Technological advances in soil mechanics have followed a different development pattern from strength of materials and elasticity theory for structural and machine design applications. While elasticity theory is capable of predicting most structural and machine design behavior within design accuracy, there is no corresponding theory for soil mechanics.

Mathematical description of soil behavior is extremely difficult. In addition to having elastic characteristics under certain loadings, its behavior can be plastic, viscoelastic or thixotropic under other situations. The development of a mathematical theory which encompasses all these characteristics, even as an engineering approximation, has not been accomplished. The acceptable mathematical complexity of any material description is limited by the ability to solve practical boundary value problems with the description. Although numerical methods and digital computers have recently extended the acceptable mathematical complexity, there are considerations in soil mechanics, such as material stability, that are still beyond these extended capabilities.

Most efforts to deduce useful mathematical results for practical soil mechanics problems fall into two categories. The first of these is elementary, three dimensional, theories which describe a limited type of behavior. The idealizations of Elasticity, Plasticity and Coulomb theory represent examples

¹ Work presented in this paper was sponsored by ARPA Contract SD-86.

² Brown and Root Inc., Houston, Texas, formerly Brown University.

³ Brown University, Providence, Rhode Island.

of this approach. The results obtained using these methods in soil mechanics have been more helpful in establishing trends rather than providing quantitative answers.

The second category is equations which have been deduced empirically to predict behavior under certain circumstances. Most design formulas fall into this category. The results in this category often have a theoretical basis but, in the long run, rely on empiricism. This approach in soil mechanics has provided the technical basis for practically all design and will probably continue to play this role for many years.

The work presented here falls into the first category. The idealization under consideration is, broadly speaking, a granular medium. This means the behavior is time independent (the stress-strain equations do not have a characteristic time). A pore pressure is not included in this formulation although it is thought to be straightforward to include this phenomenon in the theory.

Even in this restricted area adequate mathematical representations of material behavior is lacking. The most common representation for granular materials is the Coulomb theory. The basis for this theory is that whenever on any plane the ratio of resolved shearing stress to the sum of the normal stress and a constant called the "cohesive strength" reaches a critical value, the material flows. The description of how the material flows is the part of this theory that is inadequately resolved. One approach which is used, for example, in Terzaghi's text¹ is to assume the flow is incompressible. It is well known that granular media can expand or compact during flow. Another objection to the incompressibility assumption is that Drucker² has shown that normality of the

¹ Terzaghi, K., Theoretical Soil Mechanics, Wiley, New York, 1943.

² Drucker, D. C., "A More Fundamental Approach to Plastic Stress-Strain Relations," Proc. First U. S. Nat. Congress of App. Mech., ASME, 1951, pp. 487-491.

deformation rate vector to the loading surface is required if material stability is to be insured. The condition of normality necessitates a volume change and this is the basis for a second approach to the description of the material flow. A normality requirement leads to a plasticity type formulation for soils³. Hansen⁴ has conducted experiments to try to evaluate the normality assumption for soils. He observed volume changes which were between the zero volume change required by an incompressibility assumption and the change required for normality. The significance of these results are that 1) there are appreciable volume changes during flow and 2) unstable material behavior should be anticipated for certain flow conditions.

Earlier work^{5,6} by the authors on granular media was directed to trying to develop a plasticity type theory with a modified "stability" requirement so that normality was not a consequence of the requirement. The reason such an approach is attractive is that the stability postulate due to Drucker has a physical basis which can be deduced from energy considerations. In the authors' earlier work it was postulated that energy can be stored and released in the material. The energy is taken to be a function of the specific volume. The result of this study was that the amount of energy storage and release would be greater, in order to explain Haythornthwaite's results, than one would expect on physical grounds.

³ Drucker, D. C., Henkel, D. J. and Gibson, R. E., "Soil Mechanics and Work-Hardening Theories of Plasticity," ASCE Trans., Vol. 122, 1957, pp. 338-346.

⁴ Hansen, J. B., Earth Pressure Calculation, Danish Technical Press, Copenhagen, Denmark, 1953.

⁵ Weidler, J. B. and Paslay, P. R., "Analytical Description of Behavior of Granular Media," Jour. of Engr. Mech. Div., Proc. ASCE, April 1969, Vol. 95, EM2, pp. 379-395.

⁶ Paslay, P. R. and Weidler, J. B., "An Analysis of the Triaxial Test for Cohesionless Soils," Jour. of Engr. Mech. Div., Proc. ASCE, June 1969, Vol. 95, EM3.

The work presented here is a different attempt to formulate stress-strain equations for granular materials which are predictive and yet tractable. The notion of a limit surface has been modified to make the presentation of this theory straightforward. A relationship is introduced between the mean normal stress, the density and the rate of change of volume. The result is a specific three-dimensional theory that predicts several phenomena which earlier theories could not. Experimental results from shear tests are used to give a critical evaluation of the theory which suggests it is an engineering compromise between accuracy and complication which should be adequate for many practical situations.

THEORY; GENERAL CONSIDERATIONS

This development depends on several physical assumptions which are stated here at the onset so the mathematical development can proceed without interruption.

When an element of a granular medium, large enough to be treated as a continuum, is loaded monotonically, it responds elastically and it is assumed that one of two phenomena occur when the load becomes sufficiently high. The first is that the individual grains crush. This will occur, for example, if the loading is hydrostatic compression. The second is that the element distorts due to shearing motion as in a shear test. Undoubtedly some particles fracture during shearing, but this is neglected. The only response which this theory considers is the shearing response.

The magnitude of the shear stress required to initiate flow is assumed to be a function of the mechanical state of the material and mean normal stress. This mechanical state is assumed to be measured by the specific volume of the material.

In conventional plasticity theory the yield surfaces are all cylinders in principal stress space and are identified by their cross section on the π plane (the Mises condition is a circle while the Tresca condition is a hexagon). The Coulomb condition is a six sided pyramid in principal stress space. On the π plane a six-sided figure with threefold symmetry results. In the theory developed here, a surface of revolution will represent the limit surface at any time. The intersection of a π plane with the surface of revolution will be a circle whose radius depends on the mechanical state. This choice of limit surface makes possible an approximation to Coulomb theory and is mathematically easier to work with than the Coulomb theory for many problems.

Experiments, such as shear box tests, indicate that when flow is induced, there is a volume change. As shearing proceeds, the volume reaches an equilibrium value which is related to the normal load applied during shearing. If the specific volume is above (below) its equilibrium value when shearing commences, then the specific volume decreases (increases) during the shearing flow and approaches the equilibrium value.

In order to incorporate the above ideas into a theory, it is convenient to work in a strain space rather than a stress space owing to the mechanical state being described in terms of specific volume. The reduced elastic components of strain are deduced from the difference of the total reduced strain \underline{e} and the reduced inelastic strain \underline{a} . The stresses and elastic strains are to be related through a linear Hooke's law so that the Mises condition may be written as

$$(\underline{e} - \underline{a}) \cdot (\underline{e} - \underline{a}) \leq \rho^2(\sigma, \alpha) \dots \dots \dots (1)$$

where $\rho(\sigma, \alpha)$ abbreviated as ρ is the elastic shear strain required to initiate inelastic flow, σ is the mean normal stress (tensile stresses measured as

positive) and α is the inelastic volumetric strain. The function $\rho(\sigma, \alpha)$ will be assumed to have the following properties

$$\rho(0, \alpha) = 0 \dots\dots\dots (2)$$

$$\rho(\sigma, 0) = 0 \dots\dots\dots (3)$$

The first property simply implies that when there is no mean normal compression, there is no resistance to flow. The second property adjusts the reference state from which the plastic volume change is measured. Fig. 1 depicts the loading surface graphically.

A second function is introduced to determine the equilibrium volume for a given mean normal stress. The function is denoted by $\alpha^0(\sigma)$ or abbreviated as α^0 .

The requirements for inelastic flow to occur are that the Mises condition equality

$$(\underline{e} - \underline{a}) \cdot (\underline{e} - \underline{a}) = \rho^2(\sigma, \alpha) \dots\dots\dots (4)$$

be satisfied and that a loading condition, deduced by differentiation of the Mises condition, inequality,

$$(\underline{e} - \underline{a}) \cdot \dot{\underline{e}} - \frac{1}{2} (\rho^2)_{,\sigma} \dot{\sigma} > 0 \dots\dots\dots (5)$$

be satisfied. If Eqs. 4 and 5 are not simultaneously satisfied, then only elastic deformation occurs.

When inelastic flow does occur, a flow law is required to describe the deformation. To describe the flow law introduced here, it is convenient to refer to a principal strain space where the Mises condition describes a cylinder. If the loading conditions, Eqs. 4 and 5, are satisfied, then the direction of the deformation rate vector in the strain space is such that its

projection on the π plane going through point A is normal to the cylinder. The component of the deformation rate vector perpendicular to the π plane is determined by the mean normal stress and the separation between the current inelastic volume strain, α , and the equilibrium volume strain $\alpha^0(\sigma)$. Analytically this may be expressed as follows

$$\dot{\underline{a}} = \lambda(\underline{e} - \underline{a}) \quad \text{if yield and loading} \quad \dot{\underline{a}} = 0 \quad \dots \dots \dots (6)$$

$$\dot{\alpha} = \lambda h(\sigma, \alpha^0 - \alpha) \quad \text{conditions are satisfied,} \quad \dot{\alpha} = 0 \quad \dots \dots \dots (7)$$

otherwise

In Eqs. 6 and 7, λ is an arbitrary parameter and $h(\sigma, \alpha^0 - \alpha)$ abbreviated as h is a function which has the following property

$$h(\sigma, 0) = 0 \quad \dots \dots \dots (8)$$

since the inelastic volume change and the equilibrium volume change coincide in this circumstance. λ may be eliminated from Eqs. 6 and 7 by differentiating the Mises condition equality (which must remain valid during inelastic flow) and substituting Eqs. 6 and 7. This procedure leads to

$$\lambda = \frac{(\underline{e} - \underline{a}) \cdot \dot{\underline{e}} - \frac{1}{2} (\rho^2)_{,\sigma} \dot{\sigma}}{\rho^2 + \frac{1}{2} (\rho^2)_{,\alpha} h} \quad \dots \dots \dots (9)$$

so that, by noting Eq. 5, λ is positive when $\rho^2 + \frac{1}{2} (\rho^2)_{,\alpha} h$ is positive. When λ is positive, it simply means that the projection of the deformation rate vector on the π plane at point A is directed radially outward from the Mises condition surface. Substitution of Eq. 9 into 6 and 7 gives

$$\dot{\underline{a}} = \frac{(\underline{\epsilon} - \underline{a}) \cdot \dot{\underline{\epsilon}} - \frac{1}{2} (\rho^2)_{,\sigma} \dot{\sigma}}{\rho^2 + \frac{1}{2} (\rho^2)_{,\alpha} h} (\underline{\epsilon} - \underline{a}) \quad \text{if yield and loading conditions are satisfied,} \quad \dot{\underline{a}} = 0 \quad \dots \dots \dots (10)$$

$$\dot{\underline{a}} = \frac{(\underline{\epsilon} - \underline{a}) \cdot \dot{\underline{\epsilon}} - \frac{1}{2} (\rho^2)_{,\sigma} \dot{\sigma}}{\rho^2 + \frac{1}{2} (\rho^2)_{,\alpha} h} h \quad \text{otherwise} \quad \dot{\underline{a}} = 0 \quad \dots \dots \dots (11)$$

Finally, the stress and elastic strain states must be related through Hooke's law. The reduced elastic strain is $(\underline{\epsilon} - \underline{a})$ and the elastic volume change per unit volume is $(\epsilon - \alpha)$ so that

$$\underline{s} = 2G(\underline{\epsilon} - \underline{a}) \quad \dots \dots \dots (12)$$

$$\sigma = K(\epsilon - \alpha) \quad \dots \dots \dots (13)$$

where K and G are functions of α and σ .

Eqs. 4, 5, 10, 11, 12 and 13 complete the constitutive formulation for the material. For the solution of boundary value problems, the equilibrium equations and strain compatibility equations must be included to form an appropriate set of field equations.

THEORY; SIMPLEST CASE

In order to make a practical evaluation of the formulation presented in the preceding section, a specific choice has been made for each of the functions $\rho(\sigma, \alpha)$, $\alpha^0(\sigma)$ and $h(\sigma, \alpha^0 - \alpha)$. The simplest case which yields qualitatively correct predictions for elementary deformations is as follows

$$G(\sigma, \alpha) \rho(\sigma, \alpha) = +\alpha \sigma G_0 \rho_{11} \quad \dots \dots \dots (14)$$

$$\alpha^0(\sigma) = \alpha_0^0 + \alpha_1^0 \sigma \quad \dots \dots \dots (15)$$

$$h(\sigma, \alpha^0 - \alpha) = \sqrt{-\sigma} (\alpha^0 - \alpha) h_1 \dots \dots \dots (16)$$

where $G_o \rho_{11}$, α_o^0 , α_1^0 and h_1 are constants dependent on the particular granular medium being represented. The Eqs. 14, 15, 16 satisfy the Conditions 2, 3, 8.

With the simplified definition loading occurs whenever

$$G^2(\sigma, \alpha)(\underline{e} - \underline{a})(\underline{e} - \underline{a}) = G_o^2 \rho_{11}^2 \alpha^2 \sigma^2 \dots \dots \dots (17)$$

and

$$G^2(\sigma, \alpha)(\underline{e} - \underline{a}) \cdot \dot{\underline{e}} - G_o^2 \rho_{11}^2 \alpha^2 \sigma^2 (1 - \frac{\sigma G_{,\sigma}}{G}) \frac{\dot{\sigma}}{\sigma} > 0 \dots \dots \dots (18)$$

otherwise there is no loading. The stress-deformation relations then become

$$\dot{\underline{a}} = \frac{(G^2(\sigma, \alpha)(\underline{e} - \underline{a}) \cdot \dot{\underline{e}} / G_o^2 \rho_{11}^2 \alpha^2 \sigma^2) + (\frac{\sigma G_{,\sigma}}{G} - 1) \frac{\dot{\sigma}}{\sigma}}{1 + \sqrt{-\sigma} h_1 (\alpha_o^0 + \alpha_1^0 \sigma - \alpha) \frac{1}{\alpha} (1 - \frac{\sigma G_{,\sigma}}{G})} (\underline{e} - \underline{a}) \dots \dots \dots (19)$$

} loading

$$\dot{\underline{a}} = \frac{(G^2(\sigma, \alpha)(\underline{e} - \underline{a}) \cdot \dot{\underline{e}} / G_o^2 \rho_{11}^2 \alpha^2 \sigma^2) - (1 - \frac{\sigma G_{,\sigma}}{G}) \frac{\dot{\sigma}}{\sigma}}{1 + \sqrt{-\sigma} h_1 (\alpha_o^0 + \alpha_1^0 \sigma - \alpha) \frac{1}{\alpha} (1 - \frac{\sigma G_{,\sigma}}{G})} h_1 (\alpha_o^0 + \alpha_1^0 \sigma - \alpha) \dots \dots \dots (20)$$

$$\dot{\underline{a}} = 0 \dots \dots \dots (21)$$

} no loading

$$\dot{\alpha} = 0 \dots \dots \dots (22)$$

In the following sections these stress-deformation relations are evaluated against experimental results. The form chosen for $G(\sigma, \alpha)$ is one which has been determined experimentally, reported in the literature⁷ and can be approximated by

⁷ Hardin, Bobby O., "Dynamic Versus Static Shear Modulus for Dry Sand," ASTM Materials Research and Standards, Vol. 5, No. 5, pp. 232-235.

$$G(\sigma, \alpha) = 10,850 \sqrt{-\sigma} (.0833 - \alpha) , \quad \sigma < 0 \quad \dots \dots (23)$$

where σ and $G(\sigma, \alpha)$ are in psi.

COMPARISON OF SIMPLEST THEORY WITH EXPERIMENTAL RESULTS

The comparison with experiment to be presented here is based on 77 shear tests which were run on locally quarried quartz sand. The shear test apparatus is shown graphically in Fig. 2. The tests used in this evaluation are part of a large set of tests run as part of another investigation and have been reported in detail⁸. The test consists of shearing the top half of a shear box at a slow enough rate to make inertia effects negligible. The shearing force, T , required to accomplish this motion is measured. The normal load N is applied through weights so that it remains constant during the test. The vertical and horizontal components of the shear box motion are measured.

There are many drawbacks to using such tests for the evaluation of constitutive relations. The most obvious is that the state of deformation in the sample is not uniform. It will be observed in the following that a number of assumptions is required in order to reach a critical evaluation of the theory from these tests.

A typical set of data for a compacted sand is sketched in Fig. 3a. If the data beyond about two grain diameters is observed, it is seen that the average specific volume decreases while the shear load also decreases. As the shearing progresses, both T and δ_v appear to asymptotically approach constant values. The authors have ignored δ_H displacements of less than two grain diameters because extraneous settling and orientation effects are occurring which would

⁸ Weidler, J. B., "The Flow Behavior of Sand at Failure," Brown University, Division of Engineering Report ARPA E59, August 1968.

possibly not be present in a natural, undisturbed foundation. When a loose sand is tested, the results follow the pattern shown in Fig. 3b. For each test the following data were recorded:

initial specific volume, v ($\text{in}^3/\#$)

nominal normal stress, $N \div A$ ($\#/\text{in}^2$)

nominal initial shearing stress, $T_{\text{max}} \div A$ or $T_2 \text{ gr. dia} \div A$ ($\#/\text{in}^2$)

$$\left. \frac{d\delta_v}{d\delta_H} \right|_{T_{\text{max}}} \quad \text{or} \quad \left. \frac{d\delta_v}{d\delta_H} \right|_{2 \text{ gr. dia}}$$

The experiments reported here were run for 5 values of $N \div A$ which were 46.1, 37.3, 28.5, 19.7 and 10.9 psi and for assorted values of v from 17.5 to 20.5 in^3/lb . It should be noted that out of the 77 tests, only one had the character of Fig. 3b owing to the difficulty of obtaining sufficiently loose initial states.

Fig. 4 shows the result of plotting the nominal initial shearing stress as a function of initial specific volume for the different values of nominal normal stress. An approximation to the value of $G(\alpha)_p_{11}$ can be achieved from this curve if it is assumed that the nominal initial shearing stress defines the initiation of inelastic flow for a mean normal stress equal to the nominal normal stress and a volumetric inelastic strain given by

$$\alpha = \frac{v-24}{24} \text{ (v in units of } \text{in}^3/\# \text{) } \dots \dots \dots (24)$$

Eq. 24 is deduced from the figure by 1) extrapolating the curves for different values of $N \div A$ in Fig. 4 to a nominal initial shearing stress of zero, 2) finding that the average v determined this way for the different values

of $N \div A$ is $24 \text{ in}^3/\text{lb}$ and 3) measuring the volumetric strain from the reference value of $24 \text{ in}^3/\text{lb}$. The above scheme then leads to the following equation:

$$\text{nominal initial shearing stress} = G_o \rho_{11} (N \div A)(-\alpha) \dots \dots (25)$$

The straight lines shown in Fig. 4 are deduced by setting

$$G_o \rho_{11} = 4.00 \dots \dots \dots (26)$$

The evaluation of α_o^o and α_1^o , in Eq. 15 can be accomplished by plotting $\frac{d\delta_v}{d\delta_H}$ as a function of α for each different nominal normal stress. In this case five curves are obtained. By extrapolating these curves to the point where $\frac{d\delta_v}{d\delta_H}$ is zero, an approximation to the equilibrium volume for each nominal normal stress may be obtained. The necessary data for accomplishing this procedure are plotted in Fig. 5. A plot of the equilibrium volumes as a function of nominal normal stress which results from this procedure is given in Fig. 6. If $N \div A$ is interpreted as $-\sigma$, the straight line approximation shown in Fig. 6 is represented by

$$\alpha_o^o = - .124 \dots \dots \dots (27)$$

$$\alpha_1^o = +.0009 \text{ 1/psi} \dots \dots \dots (28)$$

To make use of measured values of $\frac{d\delta_v}{d\delta_H}$ for the determination of $G(\sigma, \alpha)h_1$, a brief derivation is required. For the first increment of flow from some specified state, the stress-deformation Relations 19 and 20 predict the rates of change of strain and volume. If the ratio of the xy component of \dot{a} from Eq. 19 to \dot{a} from Eq. 20 is formed, the following equation results

$$\frac{\dot{a}_{xy}}{\dot{\alpha}} = \frac{(e_{xy} - a_{xy})}{\sqrt{-\sigma} h_1 (\alpha_0^o + \alpha_1^o \sigma - \alpha)} \dots \dots \dots (29)$$

Combining this with the yield condition from Eq. 17 for this case yields

$$G(\sigma, \alpha) \frac{\dot{a}_{xy}}{\dot{\alpha}} = \frac{G_o \rho_{11} \alpha \sqrt{-\sigma}}{h_1 (\alpha_0^o + \alpha_1^o \sigma - \alpha)} \dots \dots \dots (30)$$

Now an approximation to $\frac{\dot{a}_{xy}}{\dot{\alpha}}$ from the experiments can be obtained as follows

$$\frac{\dot{a}_{xy}}{\dot{\alpha}} \approx - \frac{1}{2} \frac{d\delta_H}{d\delta_V} \dots \dots \dots (31)$$

so that

$$G(\sigma, \alpha) h_1 = \frac{2G_o \rho_{11} \alpha \sqrt{-\sigma}}{(\alpha_0^o + \alpha_1^o \sigma - \alpha)} \left(- \frac{d\delta_V}{d\delta_H} \right) \dots \dots \dots (32)$$

This equation then determines $G(\sigma, \alpha) h_1$ for each shear test in terms of the constants $G_o \rho_{11}$, α_0^o , α_1^o which have already been determined and the measured values α , σ and $d\delta_V/d\delta_H$. As before σ is approximated by $-N \div A$ and the procedure is straightforward. The value given for $G(\sigma, \alpha)$ from Eq. 23 then makes possible a determination of h_1 for each experiment. As h_1 was supposed constant in the theory, this calculation procedure furnishes a crucial test for the theory. Fig. 7 shows the results of the calculations for h_1 .

Whether or not Fig. 7 is regarded as a partial verification of the theory or not is a matter of viewpoint. In classical plasticity theory the predicted and measured values of strain increment ratios⁹ vary by as much as 2:1 and yet

⁹ Naghdi, P. M., Rowley, J. C., and Beadle, C. W., "Experiments Concerning the Yield Surface and the Assumption of Linearity in the Plastic Stress-Strain Relations," Journal of Applied Mechanics, Vol. 22, No. 3, 1955, pp. 416-420.

the theory is widely accepted. It appears that deviations of the same order are present here for a much more complicated material behavior so that the authors view Fig. 7 as encouraging.

APPENDIX - NOTATION

A	= cross-sectional area of shear test apparatus; see Fig. 2
\underline{a}	= reduced inelastic strain tensor
$\underline{b} \cdot \underline{b}$	= quadratic invariant of \underline{b} , $\underline{b} \cdot \underline{b} = \sum_i \sum_j b_{ij} b_{ij}$ where b_{ij} are components of \underline{b} referred to Cartesian coordinates
\underline{e}	= reduced total strain tensor
$G = G(\alpha, \sigma)$	= shear modulus which depends on inelastic volume change and mean normal stress; see Eq. 12
$G_0, \rho_{II}, \alpha_0^o, \alpha_1^o, h_1$	= a set of constants which describe the simplified theory. See Eqs. 1, 2, 3 and 10
$h = h(\sigma, \alpha^o - \alpha)$	= a function first appearing in Eq. 7
$K = K(\alpha, \sigma)$	= bulk modulus which depends on inelastic volume change and mean normal stress; see Eq. 13
N	= normal force on shear test apparatus; see Fig. 2
\underline{s}	= reduced stress tensor, $s_{ij} = \sigma_{ij} - (\frac{1}{3} \sum_k \sigma_{kk}) \delta_{ij}$ where $\underline{\sigma}$ is the stress
T	= tangential force on shear test apparatus; see Fig. 2
v	= specific volume
α	= inelastic volume strain measured from loose state
$\alpha^o = \alpha^o(\sigma)$	= a function first appearing in Eq. 7
δ_V	= displacement downward of shear test apparatus cover; see Fig. 2
δ_H	= horizontal displacement of shear test apparatus cover; see Fig. 2
ϵ	= total volume change measured from loose state
λ	= parameter first appearing in Eqs. 6 and 7

$\rho = \rho(\sigma, \alpha)$ = function of mean normal stress and inelastic
volume change first appearing in Eq. 1

σ = mean normal stress

CAPTIONS FOR FIGURES

- Fig. 1 - Sketch showing relation of strain point to limit surface in strain space.
- Fig. 2 - Shear box testing apparatus.
- Fig. 3 - Sketches of Typical Results from Shear Box Tests. a) for a Compacted Sand and b) for a Loose Sand.
- Fig. 4 - Nominal initial shearing stress versus initial specific volume for indicated values of the nominal normal stress.
- Fig. 5 - Rate of change of the vertical displacement with respect to the horizontal displacement as a function of the initial inelastic volume strain measured from loose state for indicated values of the nominal normal stress.
- Fig. 6 - Equilibrium inelastic volume strain as a function of the nominal normal stress.
- Fig. 7 - Variation of h_1 with inelastic volume strain and nominal normal stress. For a perfect match between theory and experiment h_1 is constant.

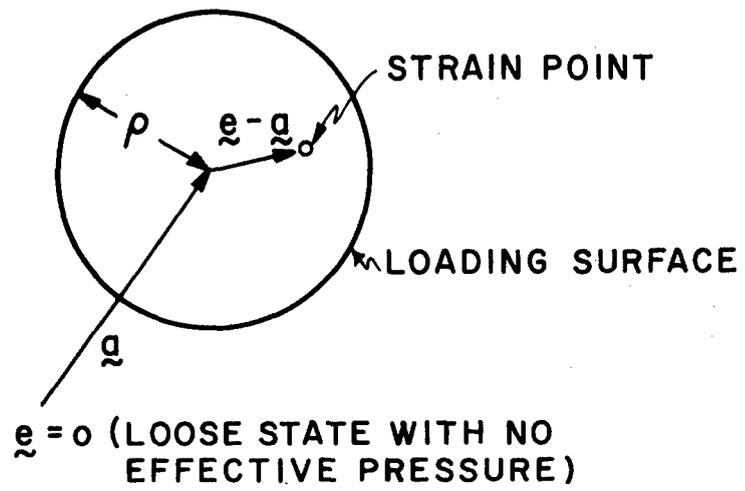
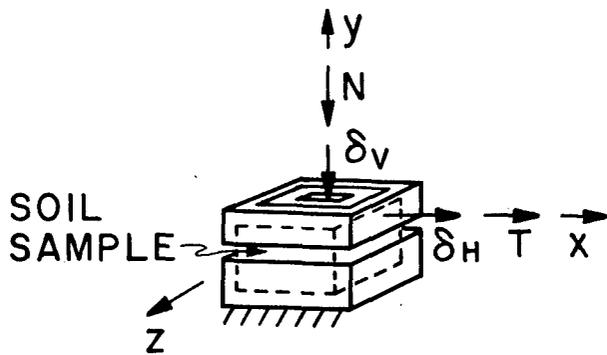


FIGURE 1



N IS CONSTANT

δ_H IS INCREASED MONOTONICALLY

T AND δ_V ARE OBSERVED AS
FUNCTIONS OF δ_H

A IS AREA OF SAMPLE WHICH
IS SHEARED

FIGURE 2

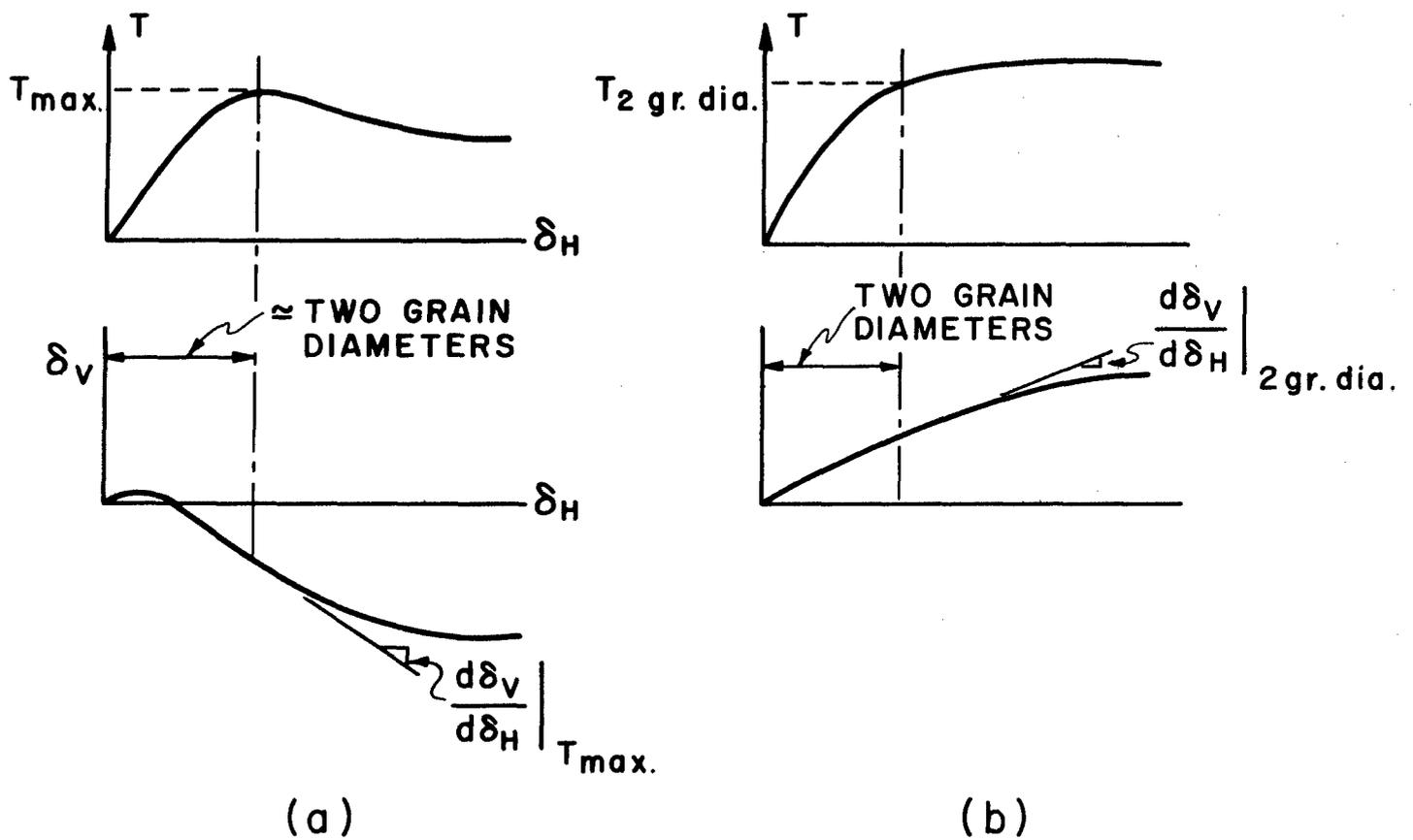


FIGURE 3

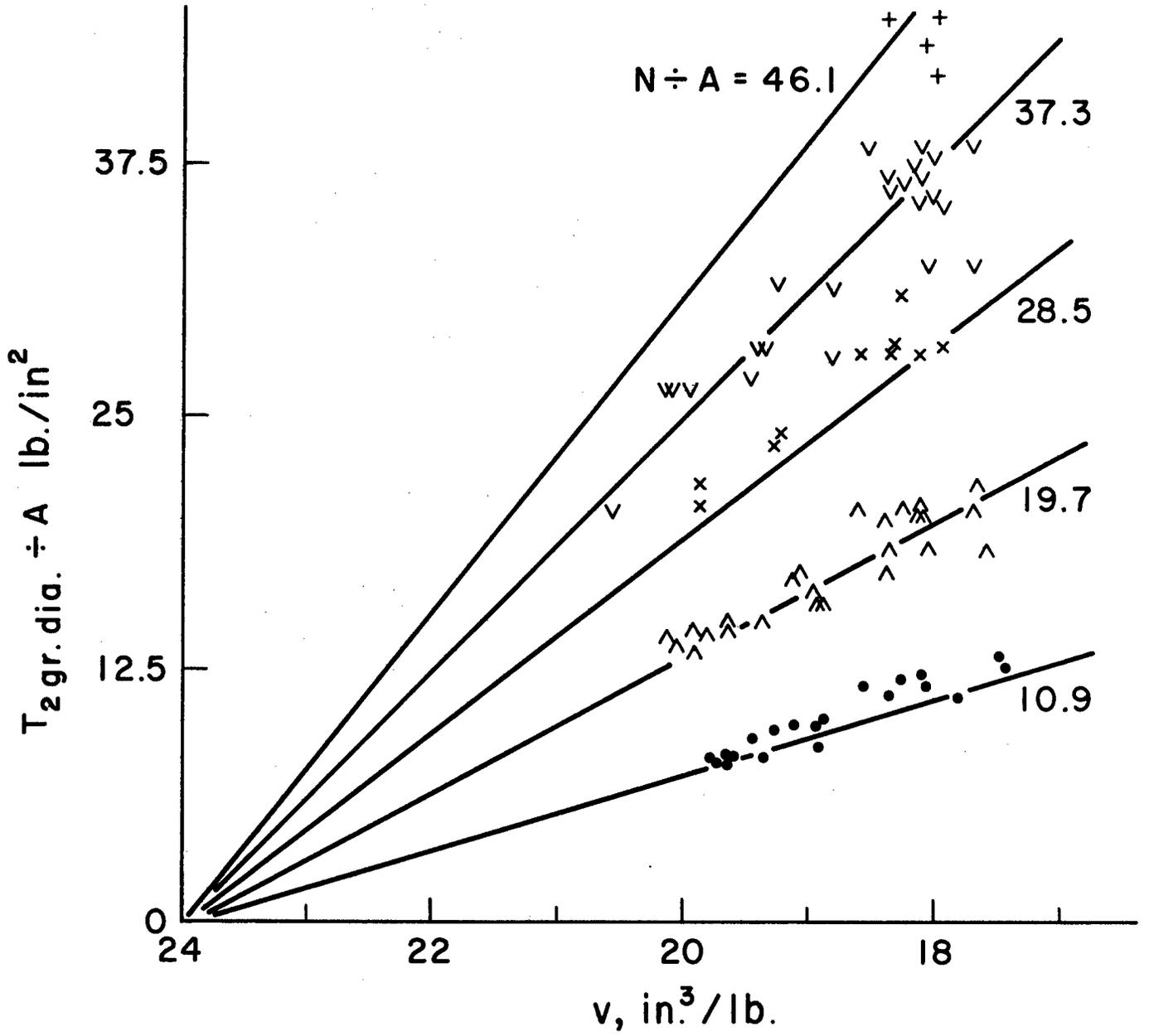


FIGURE 4

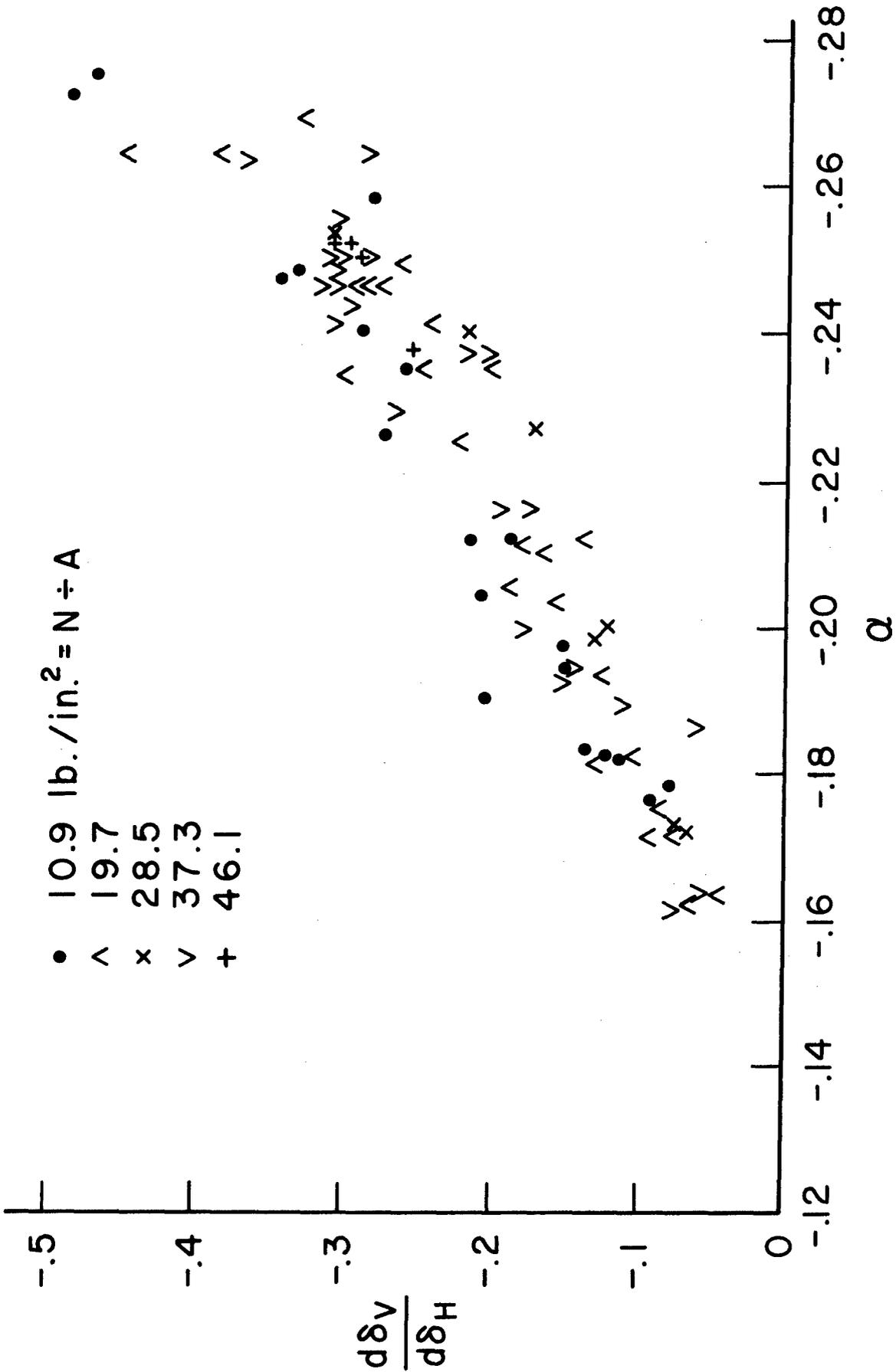


FIGURE 5

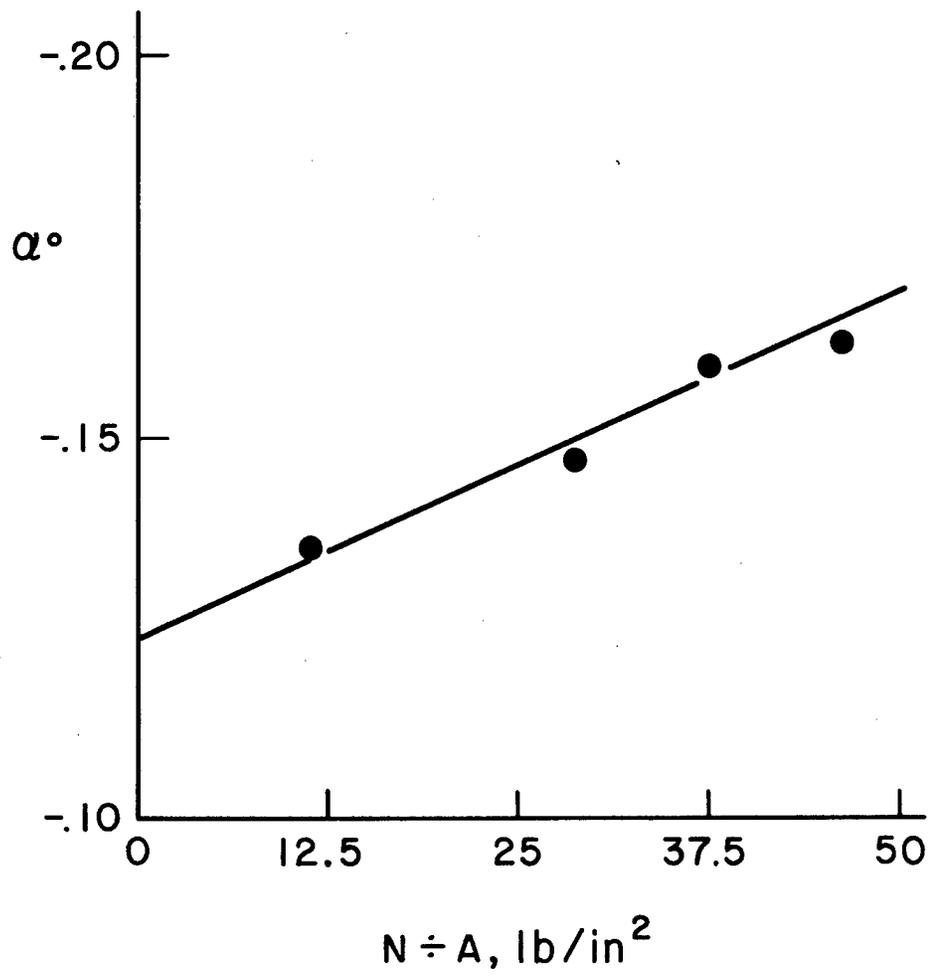


FIGURE 6

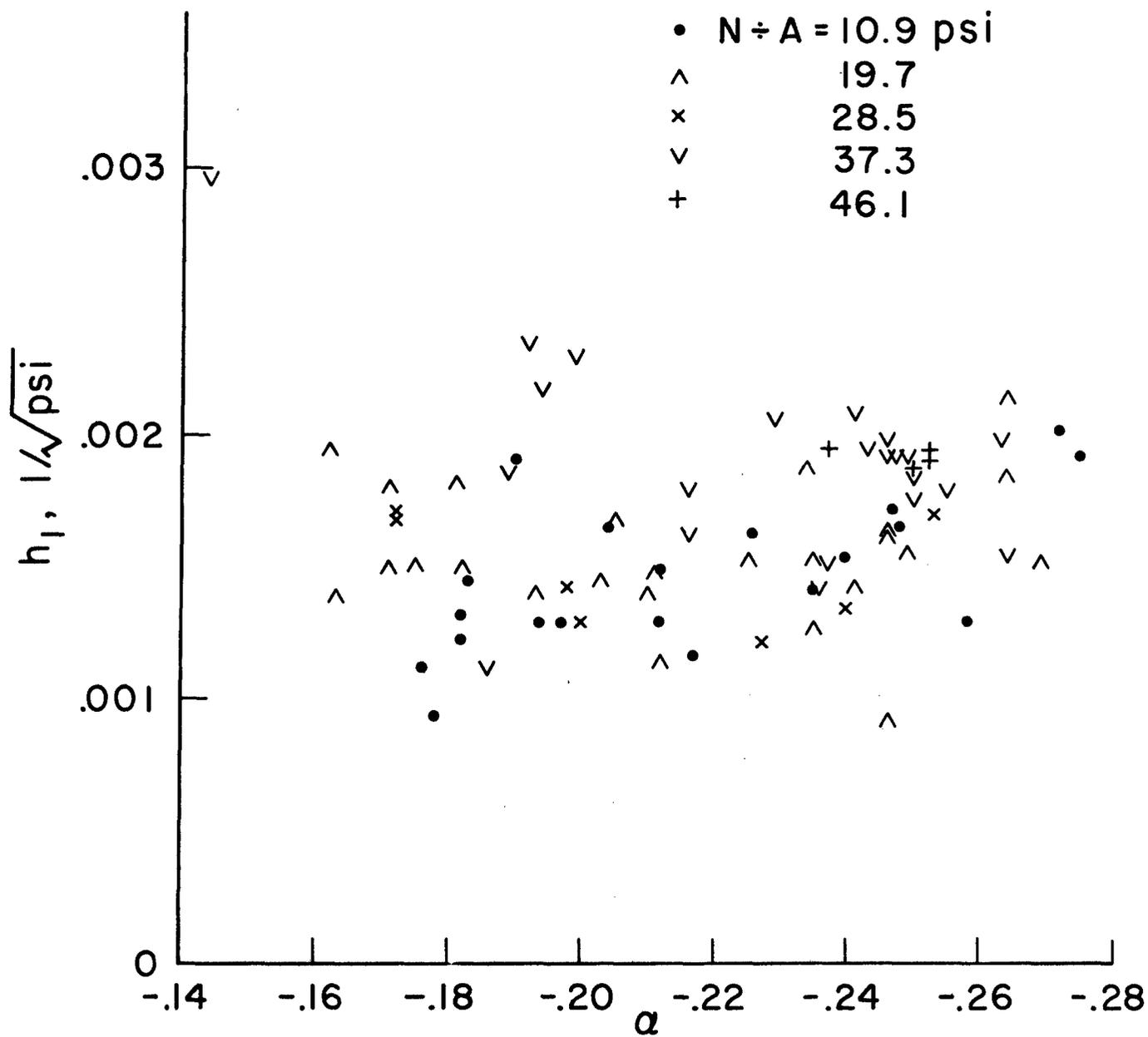


FIGURE 7