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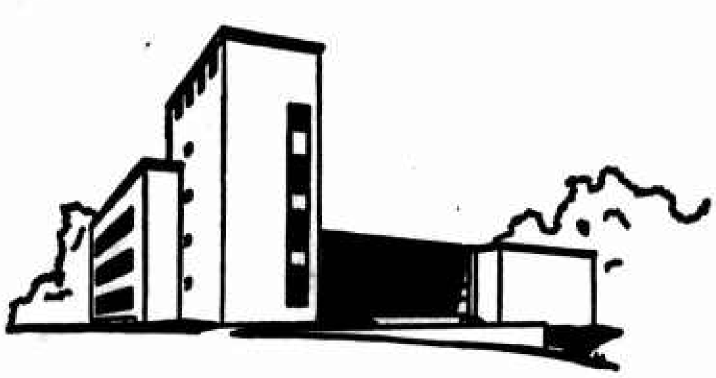

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SOME BAYES ESTIMATES OF LONG-RUN AVAILABILITY IN A TWO-STATE SYSTEM

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Abstract

The method of Bayesian statistical inference is used to derive estimates of operational parameters in a simple system that can be in one of two states: failed, or operative. The sampling plan consists of occasional observations of the system for finite periods: snapshop, plus patch sampling. Numerical examples are given.

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I. Introduction

In investigations in the area of systems reliability it is often appropriate to consider systems which alternate between two capacity states, up or down, according to some random process. One measure of effectiveness that is of importance in such systems is the long-run probability that the system will be up or productive when needed; the latter is equivalent to the long-run fraction of the time during which the system is productive. This parameter is also known in the literature as the long-run availability or the operational readiness of the system, but these terms are not standardized. The problem of estimation of the above availability parameter received some consideration in a recent paper [4], where the authors pointed out that operator log-book data on field operations of systems may be untrustworthy and suggested some sampling procedures for obtaining supplementary estimates of the system parameters. These sampling plans yield the following two types of observations on the system: a) those which reveal only the state of the system at isolated time-points, and b) those which continuously record the duration of the up and down times of the system throughout intervals of fixed or random length. Cox [2] has considered a similar sampling plan for a problem in computer acceptance testing.

The papers [2] and [4] use the method of maximum likelihood to obtain estimates of the system parameters from the two combined sources of data. Frequently, however, prior information about the system parameters is available and is based on experience with previous or current systems of a similar kind. A strong impetus to use this information often exists. Frequently it is incorporated informally by engineers who are familiar with particular systems. But there may exist merit in carrying out a more formal analysis by expressing at least some of the additional information in the form of prior distributions for system parameters, and applying the technique of Bayesian inference. In this paper such analysis is provided for a particular class of sampling plans, with the objective of estimating the long-run system availability.

Since the maximum likelihood estimates are not guaranteed to possess any desirable optimal properties in the non-asymptotic case, we believe that it will be useful to investigate other estimates which have known properties in the usual small sample situation. The Bayes estimates are, formally speaking, optimum with respect to the assumed prior distributions and loss structure; in addition they offer one plausible way for combining the above two sources of data, provided the underlying model remains approximately valid. The fact that some of these estimates appear to improve upon the maximum likelihood estimates in particular cases investigated (see Section V) offers heuristic justification for proposing these estimates.

In Section II a description of our sampling procedure and of the system model is given. In Section III mathematical expressions for the Bayes estimates are derived using the assumptions stated in Section II. It turns out that these estimates can be represented in explicit forms for a special case of the sampling plan; these are obtained in Section IV. Finally, in Section V some illustrative numerical comparisons are given for different estimates obtained in this paper.

II. Assumptions

A. Description of the System Model

We assume that the times spent in the up and down states are mutually statistically independent; both up and down times have the exponential distribution. Specifically, if U_i and D_i are the i-th up and down times, then their densities are given by

$$(1) \qquad f_U(x) = \lambda e^{-\lambda x} \qquad [x \geq 0] \, ,$$

$$(2) \qquad f_D(y) = \mu e^{-\mu y} \qquad [y \geq 0] \, ,$$

where λ and μ are unknown positive parameters. Thus the system is described by a two-state renewal process (see Cox [1]). The assumption of exponential density for the up and down times is made here because it has been found to represent, at least approximately, a large variety of empirical failure and repair time distributions and because of its attractive mathematical properties.

Using (1) and (2) it can be shown that the probability of finding the system in the up state a long time after it has been put into operation is given by

$$(3) \qquad p = P\{\text{up in the long run}\} = \frac{E[U]}{E[U]+E[D]} = \frac{\mu}{\lambda+\mu}$$

so that

$$(4) \qquad q = 1-p = P\{\text{down in the long run}\} = \frac{\lambda}{\lambda+\mu} \, .$$

Suppose that observations are made at widely spaced instants, meaning that prior knowledge is utilized to assure that the sampling interval is greater than, say, $5(\lambda+\mu)^{-1}$. Then the states observed are effectively independent, and the probability that the system is observed to be up is p.

B. Sampling Procedure

Following [2] and [4], we consider sampling procedures which yield the following two types of information on the system:

(a) A sequence of "snapshot" observations, which are available only at widely dispersed instants. These snapshots merely reveal whether the system is up or down at the instant when the observation is made.

(b) A sequence of continuous observations on the up and down times of the system throughout intervals of fixed or random duration. Following the terminology of [4], these observations will be called "patches". Some examples of patch observations are given below.

When the exponential distribution holds and snapshots are made at random instants, it is the time remaining in a given state after the snapshot has been made (rather than the total time in the state, before and after the observation) that follows the exponential law with the parameter appropriate to the state observed; intuitively this is because long times in state tend to be observed, but renewal theory mathematics provides a proof. Thus under the exponential assumption we can consider sampling plans, e.g., Case II below, where observations are not made for the complete lengths of the up and down times.

We now define the quantities which arise in the above sampling procedure and develop notations for them.

a: total number of up intervals observed during patches.

x_+: total up time observed; $x_+ = \Sigma_{i=1}^{a} x_i$, x_i being an individual up interval (or forward recurrence time).

b; total number of down intervals observed during patches; b may be fixed or random depending on the sampling plan.

y_+: total down time observed; $y_+ = \Sigma_{i=1}^{b} y_i$, y_i being an individual down interval (or forward recurrence time).

α: total number of snapshots showing the system to be up

β: total number of snapshots showing the system to be down

It follows from the above description that the likelihood function of the observations following the above sampling procedure will be given by

$$(5) \qquad L(\lambda, \mu) = e^{-\lambda x_+} \lambda^a e^{-\mu y_+} \mu^b \left(\frac{\mu}{\lambda+\mu}\right)^{\alpha} \left(\frac{\lambda}{\lambda+\mu}\right)^{\beta} .$$

A number of alternative sampling plans are special cases of the general procedure described above. Two examples appear below.

Case I

A system's up and down history is continuously recorded through k cycles, each consisting of one up period and one down period, (possibly during the developmental phase of the system). Thereafter m rare snapshots are taken, r of which show the system to be up. In this case

$$a = b = k$$

$$\alpha = r; \ \beta = m - r.$$

Case II

A system is observed m times at rare intervals, and each time the system state and the remaining time in that state are recorded. If r is the number of snapshots showing the system to be up, then

$$a = \alpha = r$$

$$b = \beta = m - r$$

C. A Priori Distribution

We assume next that λ and μ have independent priors belonging to the gamma family:

$$f_1(\lambda) = \gamma(c,\xi;\lambda) = \frac{1}{\Gamma(c)} \xi^c \lambda^{c-1} e^{-\xi\lambda} \qquad (\lambda \geq 0) \tag{6}$$

and

$$f_2(\mu) = \gamma(d,\eta;\mu) = \frac{1}{\Gamma(d)} \eta^d \mu^{d-1} e^{-\eta\mu} \qquad (\mu \geq 0) \tag{7}$$

Containing (5), (6) and (7), we obtain the posterior density

$$(8) \qquad L_p(\lambda,\mu) = L(\lambda,\mu)f_1(\mu)f_2(\lambda)$$

$$= e^{-\lambda(x_+ +\xi)}\lambda^{a+c-1}\, e^{-\mu(y_+ +\eta)}\mu^{b+d-1}\left(\frac{\mu}{\lambda+\mu}\right)^{\alpha}\left(\frac{\lambda}{\lambda+\mu}\right)^{\beta} .$$

In the above, c, ξ, d and β are assumed to be known positive constants which will perhaps be estimated from the prior data available about the distributions of λ and μ. In the case when there is no prior information we shall put $c = d = 1$, $\xi = \eta = 0$. It is seen from (8) that for these values of the parameters, the modified likelihood function $L_p(\lambda,\mu)$ reduces to $L(\lambda,\mu)$ as defined in (5). The prior distributions have been assumed to belong to the gamma family since the latter is conjugate to the exponential distribution.

D. Loss Functions

Let δ denote an estimate of long-run system availability, p. The loss $L(\delta, p)$ from estimating δ when, in fact, p prevails may be specified in various ways. In general the loss should increase as the difference between δ and p increases, but the loss associated with a negative error may not be the same as that associated with a positive error of equal magnitude. The loss functions to be used here, however, will be symmetrical; they are all related to the familiar mean squared error

$$(9)\qquad L(\delta, p) = C(p)(\delta - p)^2 .$$

We shall first consider the loss function which is the unweighted mean-squared error

$$(10)\qquad L_1(\delta, p) = C(\delta - p)^2 .$$

where C is a positive constant.

It seems clear, however, that when estimating a probability the importance of an error of the magnitude $e = |\delta - p|$ may not be independent of the probability level; an error of 0.1 may be of more importance at $p = 0.05$ or 0.95 than at 0.5. If so, the loss function (9) with $C(p) = C[p(1-p)]^{-1}$ reflects this qualitative condition and leads to the following loss function:

$$(11)\qquad L_2(\delta, p) = \frac{C}{p(1-p)} (\delta - p)^2,$$

C again being a positive constant.

Still another criterion for estimating a probability, superficially plausible when p is large, is that the fractional (per cent) error of the estimate be minimized. This leads to $C(p) = p^{-2}$ and the loss function

$$(13)\qquad L_3(\delta, p) = C(\frac{\delta}{p} - 1)^2 .$$

This paper is devoted to obtaining Bayes estimates with respect to the above loss functions, which are all variations on the mean-squared error theme. In specific instances other loss functions may well occur, and the estimates implied must be constructed anew. It is perhaps worth mentioning that the point estimation problem that concerns us here is by no means the only relevant, nor even the most important, statistical question likely to arise in practice. For instance, one may wish to provide confidence limits for the unknown readiness parameter on the basis of sample information, or, alternatively, to test the hypothesis that one system is an improvement upon another. Such problems are not considered here.

III. Bayes Estimates for Long-Run Availability under Patch-Snapshot Sampling

In this section we derive formal expressions for the Bayes estimates of the measure of effectiveness under patch-snapshot sampling for the system model we have described above.

Let δ_1^*, δ_2^* and δ_3^* be the Bayes estimates with respect to the loss functions L_1, L_2, L_3 when the prior distributions of λ and μ are given by (6) and (7) respectively. In terms of the posterior distribution of p the estimates corresponding to L_1, L_2, and L_3 are respectively

(14) $$\delta_1^* = E(p) ,$$

(15) $$\delta_2^* = \frac{E[(1-p)^{-1}]}{E[p^{-1}(1-p)^{-1}]} ,$$

and

(16) $$\delta_3^* = \frac{E[p^{-1}]}{E[p^{-2}]} ,$$

Notice first that if the sampling plan is simply to take m snapshots, r of which find the system up, and if the prior distribution of p is uniform, then $\delta_1^* = (r+1)(m+2)^{-1}$ while $\delta_2^* = r/m$, which equals the maximum likelihood estimate. On the other hand, under similar circumstances,

$$\delta_3^* = \begin{cases} 0 & \text{for } r = 0, 1 \\ \frac{r-1}{m} & \text{for } r = 2, 3, \ldots, m. \end{cases}$$

If the prior is proportional to p, however, $\delta_3^* = r/m+1$.

Since the Bayes estimate does not depend on the order in which the patch and snapshot observations have been taken because of the exponential assumption, we assume without any loss of generality that the patch observations are taken first and then followed by the snapshots. Thus under the above assumption, x_+ and y_+ denote respectively the total up and down times observed in a sequence of (a+b) successive observations. It now follows from the familiar reproducibility property of the gamma prior distributions with respect to exponential distributions that the posterior distribution of λ and μ after such observations have been made will be given by $\gamma(a+c, \xi+x_+;\lambda)$ and $\gamma(b+d, \eta+y_+; \mu)$ respectively; [see (6)]. Further, the posterior distribution of the ratio $p = \frac{\mu}{\lambda+\mu}$ after $X_1, X_2, \ldots, X_a, Y_1, Y_2, \ldots, Y_b$ have been observed can be shown to be of the form

$$g(p \mid X_i = x_i;\ Y_j = y_j,\ i = 1,2,\ldots,a;\ j = 1,2,\ldots,b)$$

(17)

$$= \frac{\Gamma(a+b+c+d)}{\Gamma(a+c)\Gamma(b+d)} \; \frac{(\xi+x_+)^{a+c}(\eta+y_+)^{b+d} p^{b+d-1}(1-p)^{a+c-1}}{[x_+ + \xi + (\eta + y_+ - \xi - x_+)p]^{a+b+c+d}} ; \quad 0 \le p \le 1.$$

Moreover, after $(\alpha+\beta)$ snapshots have been taken which result in α successes, the posterior distribution will be

$$(18) \qquad g(p \mid x_1, x_2, \ldots, x_a;\ y_1, y_2, \ldots, y_b; \alpha, \beta) = \frac{h(p)}{\int_0^1 h(p)\, dp} ,$$

where

$$(19) \qquad h(p) = p^{b+d+\alpha-1}(1-p)^{a+c+\beta-1}[x_+ + \xi + (\eta + y_+ - \xi - x_+)p]^{-(a+b+c+d)}; \quad 0 \le p \le 1.$$

Now using (14), (15), (16) and (18) we finally get (see [3]) that

(20)

$$\delta_1^* = \frac{B(a+\alpha+c+1,\ b+\beta+d)}{B(a+\alpha+c,\ b+\beta+d)} \cdot \frac{F(a+b+c+d,\ a+\alpha+c+1;\ a+b+\alpha+\beta+c+d+1; z)}{F(a+b+c+d,\ a+\alpha+c;\ a+b+\alpha+\beta+c+d; z)}$$

$$= \frac{b+\alpha+d}{a+b+\alpha+\beta+c+d} \cdot \frac{F(a+b+c+d,\ b+d+\alpha+1;\ a+b+\alpha+\beta+c+d+1; z)}{F(a+b+c+d,\ b+\alpha+d;\ a+b+\alpha+\beta+c+d; z)} .$$

where $F(\cdot,\cdot;\cdot;\cdot)$ is a hypergeometric function (See [3]) representable as

$$F(\alpha,\beta;\gamma;z) = 1 + \frac{\alpha\cdot\beta}{1\cdot\gamma} z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)} z^2 + \ldots$$

and

$$z = \frac{x_+ + \xi - \eta - y_+}{x_+ + \xi} .$$

Similarly,

$$(21) \qquad \delta_2^* = \frac{b+d+\alpha-1}{a+b+c+d+\alpha+\beta-2} \cdot \frac{F(a+b+c+d,\ b+\alpha+d;\ a+b+\alpha+\beta+c+d-1; z)}{F(a+b+c+d,\ b+\alpha+d-1;\ a+b+\alpha+\beta+c+d-2; z)}$$

and

(22) $$\delta_3^* = \frac{b+\alpha+d-2}{a+b+c+d+\alpha+\beta-2} \cdot \frac{F(a+b+c+d,\ b+\alpha+d-1;\ a+b+c+d+\alpha+\beta-1;z)}{F(a+b+c+d,\ b+\alpha+d-2;\ a+b+c+d+\alpha+\beta-2;z)},$$

However, to ensure convergence of the hypergeometric series occurring in (20), (21) and (22), we require that

(23) $$|z| < 1,$$

which implies that

(24) $$0 < y_+ + \eta < 2(x_+ + \xi).$$

This does not seem too severe a restriction in view of the fact that for most of the systems in practice, p will be near 1 and therefore with a high probability the total up time in a given situation will be larger than the total down time. When (24) does not hold, the form of solution given in (20)-(22) will not be usable.

We thus see that the three Bayes estimates we have considered can all be expressed as products of the quotient of two hypergoemetric functions and a simple fraction. In general it will not be possible to simplify the above expressions further and reduce them to a more convenient form. However an equivalent representation in terms of continued fractions can be given. We notice that the two hypergoemetric series associated with the Bayes estimate are of the form $F(\ell, m+1; n+1; z)$ and $F(\ell, m, n; z)$. From a result given in [3], we have

(25) $$\frac{F(\ell, m+1; n+1; z)}{F(\ell, m; n; z)}$$

$$= \cfrac{1}{1 - \cfrac{u_1 z}{1 - \cfrac{v_1 z}{1 - \cfrac{u_2 z}{1 - \cfrac{v_2 z}{\cdots}}}}},$$

where

(26) $$u_s = \frac{(\ell+s-1)(n-m+s-1)}{(n+2s-2)(n+2s-1)}$$

$$v_s = \frac{(m+s)(n-\ell+s)}{(n+2s-1)(n+2s)}, \qquad s = 1, 2, \ldots .$$

Thus the ratio of the two hypergeometric functions appearing in the expressions for the Bayes estimates can be replaced by a continued fraction; there seems to be a possibility of extracting useful approximations from this representation. When $\alpha = \beta = 0$(i.e. no shapshot information is used), it can be shown that $v_2 = 0$ in the continued fraction expansions for the Bayes estimates δ_2^* and δ_3^* so that they may be represented in simple explicit forms in this special case. In this case they can, however, be obtained more directly and because of the relative case with which they can be evaluated, the direct derivation of these estimates under patch-sampling alone is given in the following section.

IV. Bayes Estimates for Long-Run System Availability under Patch Sampling

The preceding developments have yielded the formulae (20), (21) and (22) for the Bayes estimates of the system availability, p, when patch and snapshot information is available. The purpose of this section is to show that when patch infromation alone is utilized, the Bayes estimates δ_2^* and δ^*_3 resulting from loss functions L_2 and L_3 may be presented in simple explicit forms.

If a up intervals have been observed having total uptime x_+, then the posterior density of λ is gamma

$$\gamma_1(c+a;\xi+x_+;\lambda) = e^{-(\xi+x_+)\lambda}\frac{[(\xi+x_+)\lambda]^{c+a-1}}{\Gamma(c+a)}(\xi+x_+) \tag{27}$$

and similarly the posterior density of μ is the independent gamma

$$\gamma_2(b+d,\eta+y_+;\mu) = e^{-(\eta+y_+)\mu}\frac{[(\eta+y_+)\mu]^{d+b-1}}{\Gamma(b+d)}(\eta+y_+) \;. \tag{28}$$

The Estimate δ_2^* under Patch Sampling

If posterior information concerning $p = \frac{\mu}{\lambda+\mu}$ is contained in (27) and (28) then we may rewrite (15) as follows:

$$\delta_2^* = \frac{\int_0^\infty\int_0^\infty (1+\frac{\mu}{\lambda})\gamma_1\gamma_2 \, d\lambda d\mu}{\int_0^\infty\int_0^\infty (1+\frac{\lambda}{\mu})(1+\frac{\mu}{\lambda})\gamma_1\gamma_2 \, d\lambda d\mu} \;. \tag{29}$$

The gamma integrals are easily evaluated, and we obtain

(30) $$\delta_2^* = \frac{1 + \left(\frac{b+d}{y_+ + \eta}\right)\left(\frac{x_+ + \xi}{a+c-1}\right)}{2 + \left(\frac{b+d}{y_+ + \eta}\right)\left(\frac{x_+ + \xi}{a+c-1}\right) + \left(\frac{a+c}{x_+ + \xi}\right)\left(\frac{y_+ + \eta}{b+d-1}\right)} .$$

For the special case in which $\xi = \eta = 0$, $c = d = 1$ ("flat" priors)

(31) $$\delta_2^* = \frac{\bar{x}\bar{y} + \frac{b+1}{b}\bar{x}^2}{\frac{b+1}{b}\bar{x}^2 + 2\bar{x}\bar{y} + \frac{a+1}{a}\bar{y}^2},$$ where

$\bar{x} = x_+/a$ and $\bar{y} = y_+/b$. As a and b become large (31) approaches the maximum likelihood estimator $\frac{\bar{x}}{\bar{x}+\bar{y}}$; for finite a and b, the right hand side of (31) is somewhat larger than the maximum likelihood estimate when the latter exceeds $\frac{1}{2}$.

The Estimate δ_3^* under Patch Sampling

The form of the estimate δ_3^* may be found in exactly the same manner as was used to obtain δ_2^*. From (16) we similarly obtain

(32) $$\delta_3^* = \frac{1 + \left(\frac{a+c}{x_+ + \xi}\right)\left(\frac{y_+ + \eta}{b+d-1}\right)}{1 + 2\left(\frac{a+c}{x_+ + \xi}\right)\left(\frac{y_+ + \eta}{b+d-1}\right) + \frac{(a+c)(a+c+1)}{(x_+ + \xi)^2} \cdot \frac{(y_+ + \eta)^2}{(b+d-2)(b+d-1)}} .$$

V. Numerical Results

The estimates we have obtained above are optimum when it is assumed that the system parameters have prior distributions given by (6) and (7) and loss functions are as given by (10), (11) and (12). It is however of considerable importance to investigate their performance in samples of realistic sizes when the actual system samples is given a fixed set of parameter values. Since any analytical results appear to be difficult to obtain for the general case, we carry out the investigation by means of experimental sampling for a specific system involving the following set of parameter values:

λ = 0.2 (expected up-time of 5 units)

μ = 1 (expected down-time of 1 unit)

The particular sampling plan used is that of Case I described in Section IIB. The following parameters are chosen for the sampling plan:

k = 5 (number of initial periods observed)

m = 10 (number of later snapshots).

A synthetic system realization is observed continuously through five consecutive up-and-down times, after which merely the state -- up or down -- is noted; snapshots are taken at intervals of approximately 15 time units. Altogether five hundred such sample realizations are examined. From the data for each realization the estimates δ_1^*, δ_2^* and δ_3^* are computed using (20), (21) and (22) and substituting c = d = 1; $\xi = \eta = 0$ (the parameters corresponding to "flat" prior distribution) in the above expressions. The data used for the purpose of these investigations are the same as reported in [4] and it is therefore possible to compare the maximum likelihood estimate obtained there with the above Bayes estimates. In addition, the estimates are computed using merely the outcomes of the $k = 5$ initial period observations, omitting the snapshots. A summary of the results (computed from 500 independent realizations) showing the averages of the estimates and their mean squares about the true value are given in Table I. The last row in this table gives the value of the Cramér-Rao lower bound for the variance of the unbiased estimate of p, which has been computed from the formula given in Lehmann [5]. For Case 1 of the sampling plan, this lower bound is given by the right hand side of the following inequality:

$$\sigma_T^2 \geq \frac{2kp^2(1-p)^2}{2k^2+2k(m-2k)p(1-p)-(1-2p)^2k^2} ,$$

where T is any unbiased estimate of p.

TABLE 1
Estimates of Long Run Availability.
$\lambda = 1$, $\mu = 0.2$, $p = 0.833$, $k = 5$, $m = 10$
(The Bayes estimates have been computed with respect to "flat" prior distributions)

Estimate	Averages		Mean Squares	
	Patch-Snapshot	Patch only	Patch-Snapshot	Patch only
δ_1^*	0.8155	0.8017	.005725	.009735
δ_2^*	0.8353	0.8288	.005486	.009497
δ_3^*	0.8007	0.7710	.007778	.01789
Maximum Likelihood	0.8223	0.8154	.005774	.009745
Cramér-Rao Lower Bound	----	----	.00495	.00774

The above table indicates that even in the absence of *a priori* information the estimates δ_1^* and δ_2^* are perhaps a little better than the maximum likelihood estimate for the particular set of parameter values, λ and μ, assumed. The estimate δ_2^* seems to have a slight edge over both δ_1^* and the maximum likelihood estimate. On the other hand, the estimate δ_3^* appears to be definitely inferior to the other estimates. In view of the above fact, a more detailed investigation of the estimates δ_1^* and δ_2^* seems to be warranted. For all these estimates, patch-snapshot sampling considerably reduces the mean square errors below those obtained from patch sampling alone.

If there exists accurate prior information about the distribution of system parameters, we expect that the mean square should be reduced considerably by incorporating the available prior information in our estimates. To investigate the effect of this reduction, the above estimates were computed with respect to three sets of gamma distributions each having $E(\mu) = 1.0$ and $E(\lambda) = 0.2$ for one hundred sample realizations of the system described above. The mean squares of these estimates are displayed in Table 2. Columns (1) and (5) of the table give the results coresponding to the generalized maximum likelihood estimates obtained by maximizing (8) and using the formulae (2.8) and (2.9) of [4]. This table shows that reduction of the mean square error is indeed effected by using the above set of priors, as is to be anticipated. This table indicates that δ_3^* is definitely inferior but as to the relative efficacy of the other estimates, Table 2 is somewhat uncertain. Also it must be emphasized that only one parameter value, p, and a few prior distributions were considered.

Table 2
Mean Squares of Estimates of Long-Run Availability Using Prior Information
$\mu = 1$, $\lambda = 0.2$, $p = 0.833$, $k = 5$, $m = 10$

Parameters of the gamma prior distributions	Patch-Snapshot				Patch			
	GML	δ_1^*	δ_2^*	δ_3^*	GML	δ_1^*	δ_2^*	δ_3^*
	(10	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1. $c=0.2; d=2.0$ $\xi=1.0; \eta=2.0$	.004214	.004309	.004340	.005763	.006206	.006835	.006420	.01182
2. $c=0.2; d=10.0$ $\xi=1.0; \eta=10.0$	.002973	.003236	.003194	.004198	.003785	.004878	.004076	.006675
3. $c=10.0; d=2.0$ $\xi=50.0; \eta=2.0$	.002730	.002540	.002541	.003195	.002994	.002922	.002808	.004253

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13. ABSTRACT

The method of Bayesian statistical inference is used to derive estimates of operational parameters in a simple system that can be in one of two states: failed, or operative. The sampling plan consists of occasional observations of the system for finite periods: snapshop, plus patch sampling. Numerical examples are given.

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