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Technical Report No. 13

OPTIMIZATION OF ARMAMENT DESIGN

by

H. O. Hartley, R. B. Hocking  
L. R. LaMotte and H. H. Oxspring

Texas A&M Research Foundation  
Office of Naval Research  
Contract N00014-68-A-0140  
Project NRO47-700

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ATTACHMENT I

OPTIMIZATION OF ARMAMENT DESIGN

THEMIS OPTIMIZATION RESEARCH PROGRAM  
Technical Report No. 13  
July 1969

INSTITUTE OF STATISTICS  
Texas A&M University

Research conducted through the  
Texas A&M Research Foundation  
and sponsored by the  
Office of Naval Research  
Contract N00014-68-A-0140  
Project NRO47-700

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ATTACHMENT II

# OPTIMIZATION OF ARMAMENT DESIGN

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## 1. Introduction

In this report, we describe a mathematical programming solution to the problem of optimizing the design of the warhead of a missile for the purpose of destroying enemy aircraft. The warhead consists of fragments and explosive charges. Upon detonation of the explosive, the fragments are projected outward. By suitably packing the fragments and the explosive it is possible to control the fragment pattern, thus the problem is to determine that fragment pattern which will be most effective. The velocity and flight path of the target relative to the intercept missile will vary, but it is assumed that they vary according to known probability laws. In view of this, the problem to be solved is that of determining the fragment pattern which will maximize the probability of destroying the target.

In Section 2 we develop the basic notation and assumptions for the special case in which the flight path of the target and the intercept missile are in the same plane. In Section 3 the optimization problem is described and in Section 4 the mathematical programming problem is formulated for this coplanar case. The formulas necessary for implementing the solution are developed in Section 5 and an example is given in Section 6. The generalization to the non-coplanar case is developed in Section 7, and other extensions developed in Section 8.

## 2. Notation

The notation for the problem will be developed in terms of Figure 1 which illustrates a typical intercept situation. It is assumed that at time  $t = 0$ ,

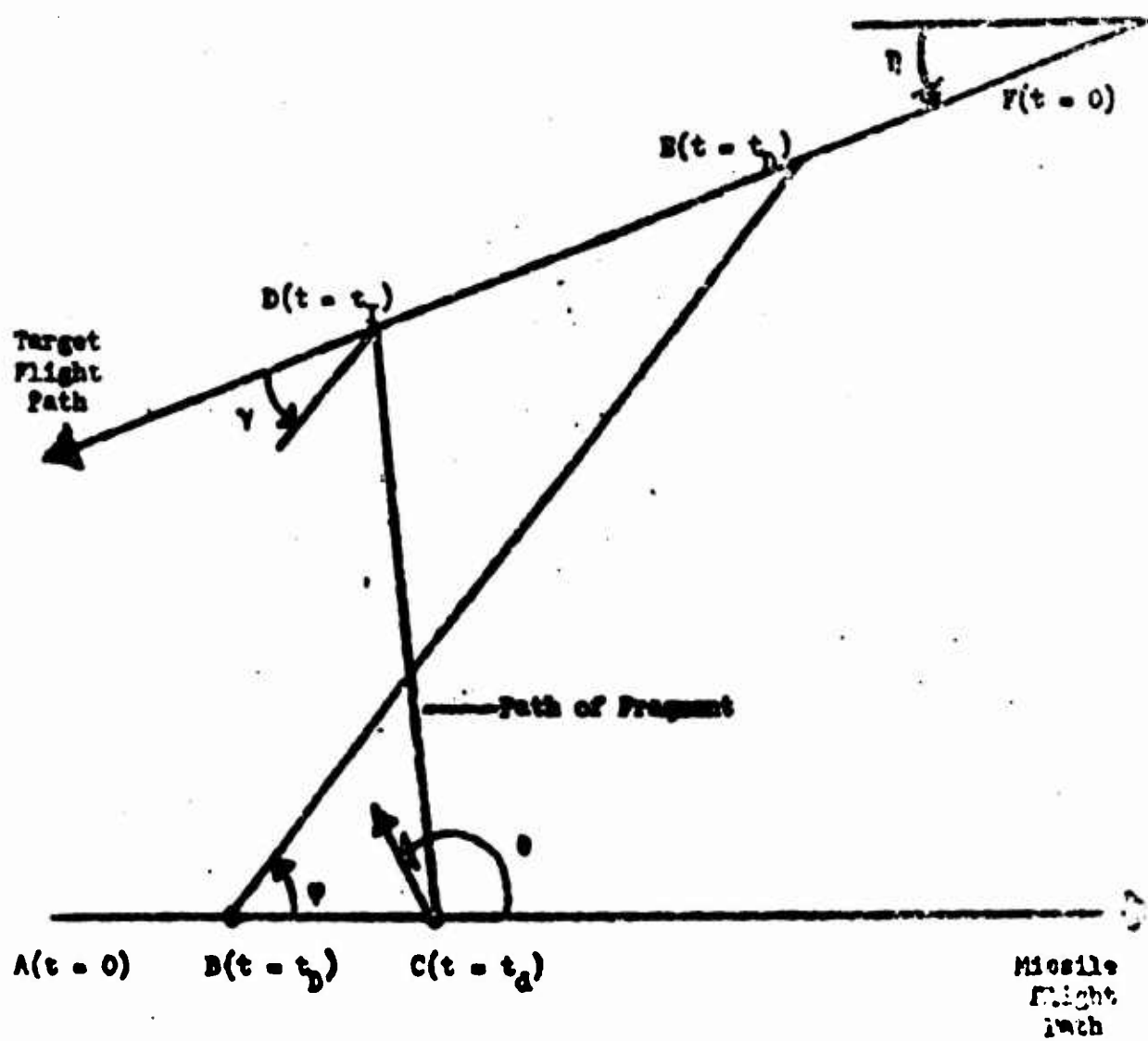


Figure 1. Typical Interception

the missile is at point A on the indicated flight path traveling with velocity  $V_M$  and that the target is at point F on the indicated flight path and has velocity  $V_T$ . At time  $t = t_D$ , the missile is at point B and the target is detected at point E. The angle of detection  $\phi$  is known. At time  $t = t_d$ , the missile is at point C and the explosives are detonated yielding fragments whose velocity is  $V_F$ . Finally, a fragment, projected at an angle  $\theta$ , is assumed to intercept the target at point D with  $t = t_I$ . The angle  $\gamma$  denotes the angle at which the fragments strike the missile at a velocity  $V_S$ . The vulnerability of the target will be denoted by  $A(\gamma, V_S)$  which is assumed to be a known function of  $\gamma$  and  $V_S$ .

The following quantities are assumed known:

- (1)  $V_M$ , the missile velocity
- (2)  $t_d - t_D$ , the delay time after detection
- (3)  $V_F$ , the fragment velocity
- (4)  $\phi$ , the detection angle

The velocity of the target  $V_T$ , the angle  $\eta$  between the two flight paths and the coordinates of the point E are not assumed known. However, it is assumed that they follow a known probability law. This probability law is described in terms of  $V_T$ ,  $\eta$  and  $R_m$  where  $R_m$  denotes the minimum distance between the missile and the target if they follow the present flight path. We shall denote the joint distribution of  $V_T$ ,  $\eta$ ,  $R_m$  by  $f(V_T, \eta, R_m)$ .

### 3. Formulation of the Optimization Problem in the Coplanar Case

The probability of destroying the target may be expressed in terms of  $R$ , the distance the fragment travels,  $\theta$ , the angle at which the fragment is projected,  $\gamma$ , the angle at which the fragment strikes the target and  $V_S$  the velocity at interception. For given values of  $R$ ,  $\theta$ ,  $\gamma$  and  $V_S$ , this probability is assumed to be given by

$$P(R, \theta, \gamma, V_s) = 1 - \exp\left\{-A(\gamma, V_s) \rho(\theta)/R^2\right\}. \quad (1)$$

The function  $\rho(\theta)$  represents the fragment density function in terms of the angle  $\theta$ . (It is assumed that the fragment pattern is symmetric about the axis of the missile.) The 'intercept geometry',  $(R, \theta, \gamma, V_s)$  depends on the missile constants  $V_M, t_d - t_D, V_F$  and  $\phi$  and on the target variables  $V_T, \eta$ , and  $R_m$  which are distributed according to  $f(V_T, \eta, R_m)$ . In Section 5, the formulas for determining the intercept geometry  $(R, \theta, \gamma, V_s)$  in terms of the missile constants and the target variables are developed. Thus a probability distribution is induced on the intercept geometry. Denote the joint distribution of the intercept geometry variables by  $g(R, \theta, \gamma, V_s)$ . The average probability of destroying the target is then given by

$$\int g(R, \theta, \gamma, V_s) \left\{1 - \exp(-A(\gamma, V_s) \rho(\theta)/R^2)\right\} \quad (2)$$

where the integration is over the range on the variables  $(R, \theta, \gamma, V_s)$ .

The objective is to determine the density function  $\rho(\theta)$  so as to maximize the 'kill' probability given by (2). Since

$$\int g(R, \theta, \gamma, V_s) = 1 \quad (3)$$

it is sufficient to consider the minimization of

$$\int g(R, \theta, \gamma, V_s) \exp(-A(\gamma, V_s) \rho(\theta)/R^2). \quad (4)$$

The function  $\rho(\theta)$  is constrained to satisfy the requirements

$$\int \rho(\theta) d\theta = M \quad (5)$$

and

$$\rho(\theta) \geq 0 \quad \text{for all } \theta. \quad (6)$$

The constraint (5) is the requirement that a given number of particles will be packed into the warhead and (6) is the obvious requirement that the density function be non-negative.

The minimization of (4) with respect to  $\rho(\theta)$  where  $\rho(\theta)$  must satisfy (5) and (6) is a problem in control theory, that is, the calculus of variations further complicated by the inequality constraint (6). Only special cases of this general problem have been solved and the complexity of the integrand in (4) forces us to use an approximate method of solution.

#### 4. The Mathematical Programming Solution

In order to solve the problem formulated in Section 3 it is necessary to assume that

- (i) the distribution  $f(V_T, \eta, R_m)$  is discrete and,
- (ii) it is sufficient to determine  $\rho(\theta)$  for only a finite set of values and then approximate  $\rho(\theta)$  by a LaGrange interpolating polynomial.

With these assumptions the problem described by (4) (5) and (6) is now written as

$$\text{minimize } \sum_i g_i \exp\left\{ -A(v_i, v_{s_i}) \rho_i / R_i^2 \right\} \quad (7)$$

$$\text{subject to } \sum_i a_i \rho_i = M \quad (8)$$

$$\rho_i \geq 0 \quad (9)$$

The sum in (7) and (8) is over all points  $(R, \theta, \gamma, v_s)$  which have positive probability in the discrete distribution  $g(R, \theta, \gamma, v_s)$ . The variables  $\rho_i$  represent the value of the fragment density function at the corresponding values of  $\theta$ , that is,  $\rho_i = \rho(\theta_i)$ . The constraint (8) is just a suitable approximation



to the constraint (5) for the relevant values of  $\theta$ .

Rather than determine the density at the points  $\theta_i$  as determined by the distribution  $g(R, \theta, \gamma, V_s)$  it has been decided to specify in advance a set of variables  $\theta_k^*$ ,  $k = 1, \dots, r$  at which  $\rho(\theta)$  will be determined. The approximation to  $\rho(\theta)$  will then be given by a LaGrange interpolation over these points, that is

$$\rho(\theta) \doteq \sum_{k=1}^r L_k(\theta) \rho_k, \quad \rho_k = \rho(\theta_k^*) \quad (10)$$

where

$$L_k(\theta) = \frac{\prod_{\substack{q=1 \\ q \neq k}}^r (\theta - \theta_q^*)}{\prod_{\substack{q=1 \\ q \neq k}}^r (\theta_k^* - \theta_q^*)} \quad (11)$$

The functional (7) is then written as

$$\sum_i g_i \exp \left\{ - A(\gamma_i, V_{s_i}) \sum_{k=1}^r L_k(\theta_i) \rho_k / R_i^2 \right\} \quad (12)$$

and the minimization is over the  $\rho_k$ .

It now remains to determine the distribution  $g(R, \theta, \gamma, V_s)$  induced by the distribution on  $V_T, \eta, R_m$ . The appropriate formulas will be developed in Section 5. In fact, it is not necessary to explicitly develop this distribution but rather, the functional (12) is expressed directly in terms of the variables  $V_T, \eta, R_m$  as follows:

$$\sum_{j=1}^N f_j \exp \left\{ - A(\gamma_j, V_{s_j}) \sum_{k=1}^r L_k(\theta_j) \rho_k / R_j^2 \right\} \quad (13)$$

In (13), it is assumed that the distribution  $f(V_T, \eta, R_m)$  assigns probability

to  $N$  points,  $(V_T, \eta, R_m)$  of  $f_j$ ,  $j = 1, \dots, N$ . Further,  $R_j, \theta_j, \gamma_j, V_{s_j}$  denote the values of these variables for a given point  $(V_T, \eta, R_m)_j$  as computed from the formulas in the next section.

The function (13) is seen to be convex in the variables  $\rho_k$  and hence its minimization subject to the constraints (8) and (9) is just a problem in convex programming and can be solved by a number of algorithms. In Section 6 we describe an example using the Hartley-Hocking (1963) convex programming algorithm.

### 5. Development of Formulas for the Coplanar Case

The solution of the programming problem developed in Section 4 required that we develop expressions for  $R, \theta, \gamma$  and  $V_s$  in terms of  $V_T, \eta, R_m$  as well as the constants  $V_M, t_d - t_D, V_F$  and  $\phi$ .

For convenience, we shall assume that  $t_D = 0$ , hence the point  $(X_0, Y_0)$  represents the coordinates at which detection is made. In general, the point on the target which we wish to hit, say the target center, will not be the same as the point which is first detected. Let  $(X_0^*, Y_0^*)$  be the coordinates of the target center at  $t = 0$  where  $X_0^* = X_0 + c_1, Y_0^* = Y_0 + c_2$ .

Let the coordinates of the target center at time  $t$  be denoted by  $(X_t, Y_t)$  where

$$\begin{aligned} X_t &= X_0^* - t V_T \cos \eta \\ Y_t &= Y_0^* - t V_T \sin \eta . \end{aligned} \quad (14)$$

Similarly the coordinates of the missile  $(X_t^M, Y_t^M)$  at any time  $t$  are given by

$$\begin{aligned} X_t^M &= t V_M \\ Y_t^M &= 0 . \end{aligned} \quad (15)$$

The coordinates of a fragment projected at angle  $\theta$  for any time  $t \geq t_d$  are given by  $(X_t^F, Y_t^F)$  where

$$\begin{aligned} X_t^F &= t V_M + (t - t_d) V_F \cos \theta \\ Y_t^F &= (t - t_d) V_F \sin \theta \\ 0 &\leq \theta \leq 2\pi . \end{aligned} \quad (16)$$

The first task in describing the intercept geometry is to determine the time of intercept  $t_I$ . This is achieved by solving the equations

$$\begin{aligned} X_t &= X_t^F \\ Y_t &= Y_t^F \end{aligned} \quad (17)$$

Using (14) and (16) the equations (17) become

$$\begin{aligned} X_O^* - t V_T \cos \eta - t V_M &= (t - t_d) V_F \cos \theta \\ Y_O^* - t V_T \sin \eta &= (t - t_d) V_F \sin \theta \end{aligned} \quad (18)$$

Squaring both sides, adding and simplifying yields the following quadratic in  $t$ :

$$\begin{aligned} &t^2(V_T^2 + V_M^2 + 2 V_T V_M \cos \eta - V_F^2) \\ &- 2t(X_O^* V_T \cos \eta + X_O^* V_M + Y_O^* V_T \sin \eta - V_F^2 t_d) \\ &+ (X_O^{*2} + Y_O^{*2} - V_F^2 t_d^2) = 0 \end{aligned} \quad (19)$$

For certain values of the input variables and missile constants, this quadratic may have either zero, one or two real roots which exceed  $t_d$ . The usual case will be that only one root exceeds  $t_d$ , and this will be  $t_I$ . If no roots exceed  $t_d$  for the  $j^{\text{th}}$  case that is  $(V_T, \eta, R_m)_j$  then it is impossible to intercept with this choice of input variables. In this case, the  $j^{\text{th}}$  term is deleted from (13) and the solution obtained as usual. However the kill probability (2) will now be decreased by an amount  $f_j$ . The case of two roots greater than  $t_d$  indicates that two 'hits' are possible. At the present time we only consider one and hence let  $t_I$  be the smallest root greater than  $t_d$ .

The angle  $\theta_I$  yielding the interception is now determined from (18) by setting  $t = t_I$  and solving for  $\theta$ . The distance the fragment must travel,  $R_I$ , is then given by

$$R_I^2 = (X_{t_I} - V_M t_d)^2 + Y_{t_I}^2. \quad (20)$$

Finally, the angle at which the fragment strikes the target is given by

$$\gamma = \gamma^* - \eta$$

where  $\tan \gamma^* = \frac{V_{sy}}{V_{sx}}$  (21)

and  $V_{sy} = V_F \sin \theta + V_T \sin \eta$

$$V_{sx} = V_M + V_F \cos \theta + V_T \cos \eta.$$

The striking velocity is given by

$$V_s^2 = V_{sx}^2 + V_{sy}^2 \quad (22)$$

Inspection of these results shows that we have expressed the intercept geometry  $(R, \theta, \gamma, V_s)$  in terms of  $X_o^*, Y_o^*$  the coordinates of the target center at  $t = 0$  rather than the miss distance  $R_m$  as desired. The modification of the formulas is easily achieved. For a specified detection angle  $\phi$ , the detection point satisfies the equation

$$Y_o = X_o \tan \phi. \quad (23)$$

The squared distance between the missile and the detection point at any time  $t$  is given by

$$R^2 = (X_o - t(V_T \cos \eta + V_M))^2 + (Y_o - t V_T \sin \eta)^2. \quad (24)$$

This distance is seen to be minimum for  $t = t_m$  where

$$t_m = \frac{X_o(V_T \cos \eta + V_M) + Y_o V_T \sin \eta}{V_t^2 + V_M^2 + 2 V_T V_M \cos \eta} \quad (25)$$

In view of (23) we have

$$t_m = X_o \frac{V_T \cos \eta + V_M + V_T \sin \eta \tan \phi}{V_T^2 + V_M^2 + 2 V_T V_M \cos \eta} \quad (26)$$

$$= X_o H(V_T, V_M, \eta, \phi) .$$

The square of the miss distance is thus given by substituting (23) and (26) into (24). Thus,

$$R_m^2 = X_o^2 \left\{ [1 - (V_T \cos \eta + V_M)H]^2 + [\tan \phi - V_T \sin \eta H]^2 \right\}$$

$$= X_o^2 G(V_T, V_M, \eta, \phi) \quad (27)$$

Thus, for a specified value of  $R_m$  we can determine  $X_o$  and hence  $X_o^*$  and thus specify the intercept geometry. The procedure is summarized as follows:

- (i) From (27), determine

$$X_o = R_m G^{-\frac{1}{2}} \quad \text{and} \quad X_o^* = X_o + c_1 \quad (28)$$

(Note: It is assumed that  $G$  is different from zero. The case  $G = 0$

arises if  $\tan \phi = \frac{V_T \sin \eta}{V_T \cos \eta + V_M}$  and corresponds to zero miss distance.)

- (ii) From (23) determine  $Y_o$  and  $Y_o^* = Y_o + c_2$  and then determine the time of intercept  $t_I$  from (19).
- (iii) The intercept geometry ( $R, \theta, \gamma, V_s$ ) is then determined from (20), (18), (21) and (22).

Thus each term in the functional (13) may be evaluated. Using the same information the derivatives of (13) with respect to  $\rho_k$  may be evaluated for any particular point  $\rho_k = \rho_k^*$ . These two evaluations are all that are necessary in

the Hartley-Hocking convex programming algorithm and hence the optimal solution is readily obtained.

## 6. Illustration of Computations in Coplanar Case

The optimization problem as formulated in Section 4 required the minimization with respect to  $\rho_k$ ,  $k = 1, \dots, r$  of

$$\sum_{j=1}^N f_j \exp\left\{-A(v_j, v_{s_j}) \sum_{k=1}^r L_k(\theta_j) \rho_k / R_j^2\right\} \quad (29)$$

subject to the constraints

$$\sum_{k=1}^r a_k \rho_k = M \quad (30)$$

$$\rho_k \geq 0 \quad k = 1, \dots, r. \quad (31)$$

The  $\rho_k$ ,  $k = 1, \dots, r$  represent the value of  $\rho(\theta)$  for specified angles  $\theta_k^*$ ,  $k = 1, \dots, r$  and the approximation to  $\rho(\theta)$  is then given by

$$\rho(\theta) = \sum_{k=1}^r L_k(\theta) \rho_k \quad (32)$$

where  $L_k(\theta)$  are the LaGrange interpolation coefficients as given by (11). It is clear that the constraint (31) is not sufficient to guarantee that the approximate fragment density function (32) will be non-negative for all  $\theta$ . To partially rectify this situation we adjoin to the problem the constraints

$$\sum_{k=1}^r L_k(\theta_j) \rho_k \geq 0 \quad j = 1, \dots, N. \quad (33)$$

Since these constraints are linear in the variables  $\rho_k$  the problem may still be

solved using the convex programming algorithm. In addition to giving added assurance that the approximation to  $\rho(\theta)$  will be non-negative, the constraints (33) insure that the optimal solution of (29) is between zero and one which is essential.

As observed earlier, the minimization of the functional (29) subject to the constraints (30), (31) and (32) by the Hartley-Hocking algorithm requires that we be able to evaluate (29) and its derivatives with respect to  $\rho_k$  for specified values of  $\rho_k$ ,  $k = 1, \dots, r$ . In this section we shall illustrate the procedure used to prepare the problem for solution. In particular, consider the evaluation of one term of the functional (29), that is

$$f_j \exp \left\{ - A(v_j, v_{s_j}) \sum_{k=1}^r L_k(\theta_j) \rho_k / R_j^2 \right\}. \quad (34)$$

Recall that the missile constants  $V_M$ ,  $t_d - t_D$ ,  $V_F$  and  $\phi$  are fixed but that  $N$  triples  $(V_T, \eta, R_m)_j$ ,  $j = 1, \dots, N$  are specified.

The example considered has,

$$V_M = 5000 \text{ ft./sec.}$$

$$V_F = 11,000 \text{ ft./sec.}$$

$$A(v_j, v_{s_j}) = 1000$$

$$t_d - t_D = .002 \text{ sec.}$$

$$\phi = \pi/4$$

the density,  $\rho(\theta)$ , will be determined for the points

$$\theta_k^* = \frac{\pi}{4} + \frac{(k-1)\pi}{16}, \quad k = 1, \dots, r = 9.$$

The distributions on  $V_T$ ,  $\eta$ , and  $R_m$  are assumed to be

$$f(V_T) = \begin{cases} \frac{1}{4} & V_T = 4000, 4250, 4500, 5000 \text{ ft./sec.} \\ 0 & \text{otherwise} \end{cases}$$

$$f(\eta) = \begin{cases} \frac{1}{3} & \eta = \frac{\pi}{8}, 0, -\frac{\pi}{8} \\ 0 & \text{otherwise} \end{cases}$$

$$f(R_m) = \begin{cases} \frac{1}{3} & R_m = 50, 100, 150 \text{ ft.} \\ 0 & \text{otherwise.} \end{cases}$$

The numbers used for this test problem are not based on real data but are merely designed to provide a simple example.

For simplicity, it is assumed that the variables  $V_T$ ,  $\eta$  and  $R_m$  are independent, thus

$$f(V_T, \eta, R_m) = f(V_T) f(\eta) f(R_m) .$$

The first step in the evaluation of (34) is to determine which triple  $(V_T, \eta, R_m)$  is being considered. Let  $V_T = 4500$ ,  $\eta = 0$ ,  $R_m = 100$ . We see that there are  $N = 36$  triples and in this simple example,

$$f_j = \frac{1}{36} \quad j = 1, \dots, N . \quad (35)$$

We then determine  $X_0$  according to (28) and  $Y_0$  according to (23) as

$$X_0 = 100$$

$$Y_0 = 100 .$$

We assume  $X_0 = X_0^*$  and  $Y_0 = Y_0^*$  for this example. It should be observed that, in general, detection could occur for either  $\angle = \pi/4$  or  $\angle = -\pi/4$ . We adopt the convention that  $\tan \angle \geq 0$ , that is,  $Y_0 \geq 0$ .



Substituting into the quadratic (19) and solving yields

$$t_I \doteq 0.0111$$

We now solve (18) for  $\theta$ . Let

$$\sin \theta = \frac{Y_O - V_T t_I \sin \eta}{(t_I - t_d)V_F} = v$$

$$\cos \theta = \frac{X_O - (V_T \cos \eta + V_M)t_I}{(t_I - t_d)V_F}.$$

The value of  $\theta$  is obtained according to the following rules:

- (a) If  $\arcsin(v) \leq 0$  then
  - (1)  $\theta = 2\pi + \arcsin(v)$  if  $\cos \theta > 0$
  - (2)  $\theta = \pi - \arcsin(v)$  if  $\cos \theta < 0$
- (b) If  $\arcsin(v) \geq 0$ 
  - (1)  $\theta = \arcsin(v)$  if  $\cos \theta > 0$
  - (2)  $\theta = \pi - \arcsin(v)$  if  $\cos \theta < 0$ .

In this case we obtain the intercept angle corresponding to the current choice

$(V_T, \eta, R_m)_j$  as

$$\theta_j = 1.626 \text{ RAD}.$$

From (14) the position of the target at intercept is given by

$$\begin{aligned} X_{t_I} &= X_O^* - t_I V_T \cos \eta \\ &= 100 - (4500)(0.0111) = 50.5 \end{aligned}$$

$$Y_{t_I} = Y_O^* - t_I V_T \sin \eta = 100$$

and hence from (20) we obtain

$$R_j^2 = 1.1602 \times 10^4 .$$

Next, we must evaluate the Lagrange interpolation coefficients  $L_k(\theta)$  for the current value of  $\theta_j = 1.6257$ . For example,

$$L_1(\theta_j) = \frac{\prod_{q=2}^9 (\theta_j - \theta_q^*)}{\prod_{q=2}^9 (\theta_1^* - \theta_q^*)} = 0.00083$$

Repeating this for  $k = 2, 3, \dots, r = 9$  would yield the terms  $L_k(\theta_j)$ ,  $k = 1, \dots, 9$ .

Finally, from (21) and (22) we could obtain  $\gamma_j$  and  $V_{s_j}$  and evaluate  $A(\gamma, V_s)$ . In the current example the function  $A(\gamma, V_s)$  was assumed constant so this is not necessary. At this point we have available the necessary information to evaluate (34) for any choice of  $\rho_k$  and also its derivative with respect to any  $\rho_k$ . The procedure described above is then repeated for all possible choices  $(V_T, \eta, R_m)_j$  to generate the  $N$  terms in the sum (29). It should be emphasized that all of this computation is done prior to the application of the convex programming algorithm. This algorithm will not be described at this time.

The approximation (30) to the integral constraint (5) was trapezoidal.

Thus

$$\begin{aligned} a_1 &= a_9 = \frac{1}{2} \\ a_k &= 1 \quad k = 2, \dots, 8 . \end{aligned}$$

The value of  $M$  was assumed to be  $2000 \frac{16}{\pi}$ .

For this data the convex programming algorithm gave the solution shown in Table 1.

TABLE 1. Solution to Example Problem

k	$\theta_k^*$	$\rho_k$
1	0.250 $\pi$	0
2	0.3125 $\pi$	0
3	0.3750 $\pi$	0
4	0.4375 $\pi$	1359.2
5	0.5000 $\pi$	433.9
6	0.5625 $\pi$	17.8
7	0.6250 $\pi$	109.9
8	0.6875 $\pi$	79.2
9	0.7500 $\pi$	0.0

The approximation to  $\rho(\theta)$  is then given by (10) and (11). The kill probability (2) is given by

$$P = 1 - 0.1535 = 0.8465$$

Execution time on the IBM 360-65 for this example was 0.04 minutes.

## 7. Development of the Three-Dimensional Case

In this section we consider the extension of the problem to the more general, three-dimensional case. A typical intercept situation is illustrated in Figure 2. The X-axis is again the flight path of the intercept missile. At time  $t = 0$  the target is detected at position  $(X_0, Y_0, Z_0)$  traveling with velocity  $V_T$  on a flight path described by the angles  $\eta$  and  $\omega$ . At time  $t = t_D = 0$  the target is detected with detection angle  $\phi$ . At time  $t = t_d$  the explosives are detonated and finally at time  $t = t_I$  a fragment projected in a direction described by the angles  $\alpha$  and  $\beta$  is assumed to intercept the target. The striking angle  $\gamma$  and velocity  $V_s$  are defined as in the planar case.

Following the development in Section 5 we shall now develop the expressions for the variables describing the intercept geometry in terms of the missile constants and the target variables. Let the coordinates of the target at time  $t$  be

$$\begin{aligned} X_t &= X_0^* - t V_T \cos \alpha \cos \eta \\ Y_t &= Y_0^* - t V_T \cos \alpha \sin \eta \\ Z_t &= Z_0^* - t V_T \sin \alpha \end{aligned} \quad (35)$$

where

$$X_0^* = X_0 + c_1, Y_0^* = Y_0 + c_2, Z_0^* = Z_0 + c_3.$$

The coordinates of the missile are

$$\begin{aligned} X_t^M &= t V_M \\ Y_t^M &= 0 \\ Z_t^M &= 0 \end{aligned} \quad (36)$$

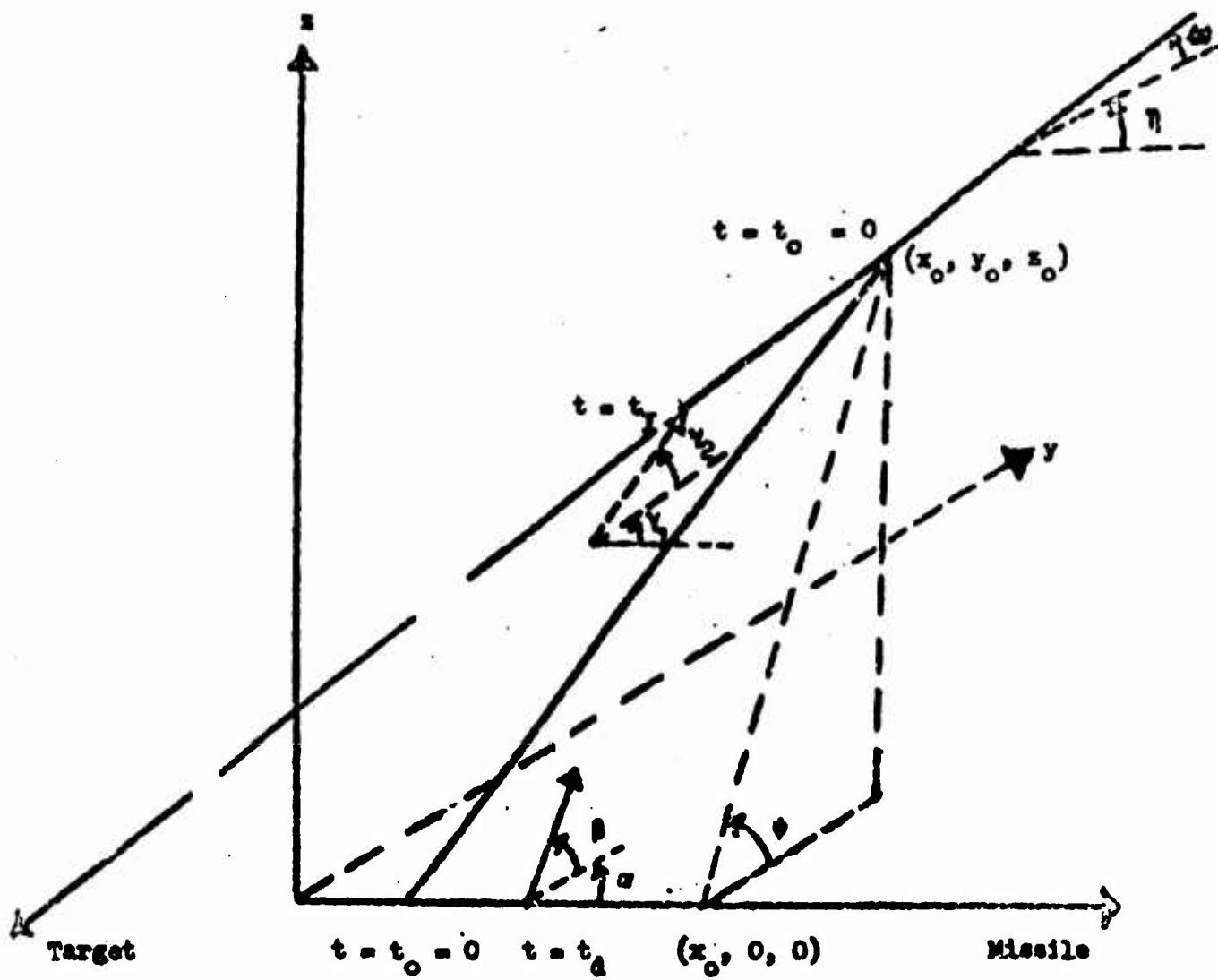


Figure 2. Interception in Three-Dimensions

The coordinates of a typical fragment are given by ( $t > t_d$ ).

$$\begin{aligned} X_t^F &= t V_M + (t - t_d) V_F \cos \beta \cos \alpha \\ Y_t^F &= (t - t_d) V_F \cos \beta \sin \alpha \\ Z_t^F &= (t - t_d) V_F \sin \beta \\ 0 \leq \alpha \leq 2\pi \quad - \quad \frac{\pi}{2} \leq \beta \leq \frac{\pi}{2} \end{aligned} \quad (37)$$

As before, the time of intercept  $t_I$  is determined by setting  $X_t = X_t^F$ ,  $Y_t = Y_t^F$ ,  $Z_t = Z_t^F$ , and solving for  $t$ .

Using (35) and (37) we obtain

$$\begin{aligned} X_O^* - t V_T \cos \omega \cos \eta &= t V_M + (t - t_d) V_F \cos \beta \cos \alpha \\ Y_O^* - t V_T \cos \omega \sin \eta &= (t - t_d) V_F \cos \beta \sin \alpha \\ Z_O^* - t V_T \sin \omega &= (t - t_d) V_F \sin \beta \end{aligned} \quad (38)$$

Eliminating  $\alpha$  and  $\beta$  yields the following quadratic in  $t$

$$\begin{aligned} &t^2 (V_T^2 + V_M^2 + 2 V_M V_T \cos \omega \cos \eta - V_F^2) \\ &- 2t (X_O^* V_T \cos \omega \cos \eta + X_O^* V_M + Y_O^* V_T \cos \omega \sin \eta + Z_O^* V_T \sin \omega - V_F^2 t_d) \\ &+ X_O^{*2} + Y_O^{*2} + Z_O^{*2} - V_F^2 t_d^2 = 0 \end{aligned} \quad (39)$$

The last real root of (39) which exceeds  $t_d$  is the time of intercept  $t_I$ . We then obtain  $\theta_I$  and  $\beta_I$  from (38) and finally  $R_I^2$  is obtained from

$$R_I^2 = (X_{t_I} - V_M t_d)^2 + Y_{t_I}^2 + Z_{t_I}^2. \quad (40)$$

Inspection of these results shows that we have described the intercept geometry in terms of  $X_O^*$ ,  $Y_O^*$ ,  $Z_O^*$  and the two angles  $\omega$  and  $\eta$ . Assuming that the

fragment pattern is symmetric with respect to the axis of the missile, it is not necessary to specify  $\alpha$  and  $\beta$  but only the angle that the fragment direction makes with the missile axis. Denoting this angle by  $\theta$  as in the planar case we see that  $\theta$  is given by

$$\cos \theta = \cos \alpha \cos \beta. \quad (41)$$

The direction of the fragment and its velocity relative to the target are determined by the quantities

$$\begin{aligned} V_{sx} &= V_M + V_F \cos \beta \cos \alpha + V_T \cos \omega \cos \eta \\ V_{sy} &= V_F \cos \beta \sin \alpha + V_T \cos \omega \sin \eta \\ V_{sz} &= V_F \sin \beta + V_T \sin \omega. \end{aligned} \quad (42)$$

The angles  $\gamma_1$  and  $\gamma_2$  as indicated in Figure 2 are given by

$$\gamma_1 = \gamma_1^* - \eta, \quad \gamma_2 = \gamma_2^* - \omega \quad (43)$$

where

$$\tan \gamma_1^* = \frac{V_{sy}}{V_{sx}} \quad \text{and} \quad \tan \gamma_2^* = \frac{V_{sz}}{\left(V_{sx}^2 + V_{sy}^2\right)^{\frac{1}{2}}} \quad (44)$$

The squared velocity is given by

$$V_s^2 = V_{sx}^2 + V_{sy}^2 + V_{sz}^2 \quad (45)$$

We now consider the problem of expressing the intercept geometry in terms of the minimum miss distance.

For a specified detection angle  $\phi$  we see that

$$Y_o^2 + Z_o^2 = \tan^2 \phi X_o^2 \quad (46)$$

where  $(X_o, Y_o, Z_o)$  is the detection point. The squared distance between the detection point and the missile at any time  $t$  is given by

$$\begin{aligned}
R^2 = & (X_o - t(V_M + V_T \cos \omega \cos \eta))^2 \\
& + (Y_o - V_T t \cos \omega \sin \eta)^2 \\
& + (Z_o - t V_T \sin \omega)^2
\end{aligned} \tag{47}$$

This distance is seen to be minimum for  $t = t_m$  where

$$t_m = \frac{X_o(V_M + V_T \cos \omega \cos \eta) + Y_o(V_T \cos \omega \sin \eta) + Z_o V_T \sin \omega}{V_M^2 + V_T^2 + 2 V_M V_T \cos \omega \cos \eta} \tag{48}$$

The square of the minimum miss distance is thus given by substituting (48) into (47). To use the formulas developed thus far we now attempt to determine  $X_o, Y_o, Z_o$  in terms of  $R_m$ . It is seen that an additional angle must be specified namely the angle which describes deviation from the coplanar case. Accordingly, we define the angle  $\psi$  by

$$\tan \psi = \frac{Z_o}{Y_o} . \tag{49}$$

Using (49) we obtain

$$\begin{aligned}
Y_o &= X_o \tan \phi \cos \psi \\
Z_o &= X_o \tan \phi \sin \psi
\end{aligned} \tag{50}$$

Substituting (50) into (48) yields

$$t_m = X_o H(V_T, V_M, \eta, \omega, \phi) \tag{51}$$

and hence (47) and (51) yields



$$\begin{aligned}
 R_m^2 = X_o^2 \{ & (1 - H(V_M + V_T \cos \omega \cos \eta))^2 \\
 & + (\tan \phi \cos \psi - V_T H \cos \omega \sin \eta)^2 \\
 & + (\tan \phi \sin \psi - H V_T \sin \omega)^2 \} .
 \end{aligned} \tag{52}$$

It should be observed that there is no loss of generality in assuming that the angle  $\psi$  is zero or equivalently that  $Z_o$  is zero. This is equivalent to assuming that the X-Y plane in Figure 2 is determined by the line of flight of the intercept missile and the point at which the target is detected. Care must be taken so that the angles  $\eta$ , and  $\omega$  defining the target flight path are described relative to this convention.

In view of the above comments, we see that either  $X_o$ ,  $R_m$  or for that matter the variable  $R_o$  defined as the distance between missile and target at the time of detection may be used to determine the intercept geometry keeping in mind that all of these variables refer to the point of detection of the target. The choice of which of these variables to use will depend on the data available.

From this point, the procedure is just as in the planar case. That is, specification of the missile constants and the variables  $V_T$ ,  $\eta$ ,  $\omega$ , and either  $R_m$ ,  $X_o$  or  $R_o$  enables us to determine the intercept geometry ( $R$ ,  $\theta$ ,  $\gamma$ ,  $V_s$ ) and hence the mathematical programming algorithm may be used to determine the optimum fragment density.

### 8. Extension to Multiple Targets

In Sections 5 and 7, the formulas for computing the intercept geometry for the special case of a single point target were developed. These formulas were then used to develop the mathematical programming problem for the determination of the optimum fragment pattern. Frequently, the incoming aircraft will have more than one vulnerable point or target, for example the engine and the pilot are normally thought of as distinct point targets. Usually, destruction of some combination of these point targets is sufficient to destroy the aircraft. Such combinations as (i) two out of three (ii) at least one and (iii) either targets 1 and 2 or targets 3 and 4 are encountered. The development of the appropriate mathematical program for solving this extended problem usually causes no difficulty but, in some cases, the resulting functional may no longer possess the desired convexity.

To indicate the procedure, consider the special case of an aircraft with two point targets such that the destruction of either will result in destruction of the aircraft. Assuming that the location of the targets (1 and 2) relative to the point of detection is known, then using the formulas of Section 5 (or 7), two intercept geometries  $(R, \theta, \gamma, V_s)_1$  and  $(R, \theta, \gamma, V_s)_2$  may be computed. The associated kill probabilities, say  $P_1$  and  $P_2$  are given by (1). The probability of destroying at least one of the targets for this intercept geometry is given by

$$1 - (1 - P_1)(1 - P_2) \quad (53)$$

Assuming that  $A_1(\gamma, V_s)$  and  $A_2(\gamma, V_s)$  represent the vulnerability functions for the two targets then the function to be minimized, that is, one minus the average kill probability (analogous to (13)) is

$$\begin{aligned} & \sum_{j=1}^N f_j \exp \left\{ -A_1(\gamma_{1j}, v_{s_{1j}}) \sum_{k=1}^r L_k(\theta_{1j}) \rho_k / R_{1j}^2 \right\} \\ & \cdot \exp \left\{ -A_2(\gamma_{2j}, v_{s_{2j}}) \sum_{k=1}^r L_k(\theta_{1j}) \rho_k / R_{2j}^2 \right\} \end{aligned} \quad (54)$$

This function is seen to be convex in the variables  $\rho_k$  and hence the minimization of it subject to the constraints (8) and (9) as well as (33) is a convex programming problem.

More general requirements on the combination of point targets which must be destroyed to insure the destruction of the aircraft cause no difficulty in formulation but in general lead to a lack of uniqueness and optimality of the solution. For example, suppose that the destruction of both targets is required. The probability of destroying the aircraft for a given intercept geometry is given by  $P_1 P_2$  where  $P_1$  and  $P_2$  are given by (1) in terms of  $(R, \theta, \gamma, v_s)_1$  and  $(R, \theta, \gamma, v_s)_2$ . In this case, after reformulation as a mathematical program we see that the function to be minimized is no longer convex. In this case, more than one relative minimum may exist. The algorithm of Fiacco and McCormick (1964) may be used to determine relative minima and by using several initial points more than one relative minimum may be obtained. That is, designer may have several fragment patterns suggested each of which are locally optimal. Assuming that they are all physically and economically practical he would then select the one which yields the highest average kill probability. There is no guarantee, however, that there does not exist a superior solution.

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Security Classification		
DOCUMENT CONTROL DATA - R & D		
(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified.)		
1. ORIGINATING ACTIVITY (Corporate author)		20. REPORT SECURITY CLASSIFICATION
Texas A&M University		Unclassified
		20. GROUP
		Unclassified
2. REPORT TITLE		
OPTIMIZATION OF ARMAMENT DESIGN		
3. DESCRIPTIVE NOTES (Type of report and inclusive dates)		
Final Report		
4. AUTHOR(S) (First name, middle initial, last name)		
H. O. Hartley, R. R. Hocking, L. R. LaMotte and H. H. Oxspring		
5. REPORT DATE	70. TOTAL NO. OF PAGES	70. NO. OF REFS
July 1969	25	2
60. CONTRACT OR GRANT NO.	90. ORIGINATOR'S REPORT NUMBER(S)	
N00014-68-A-0140	Number 13	
7. PROJECT NO.		
NRO47-700		
8.	90. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)	
9.		
10. DISTRIBUTION STATEMENT		
This document has been approved for public release and sale; its distribution is unlimited.		
11. SUPPLEMENTARY NOTES		12. SPONSORING MILITARY ACTIVITY
		Office of Naval Research
13. ABSTRACT		
<p>In this report a method is developed for determining the optimal warhead design for an intercept missile. The problem posed is that of determining the fragment density pattern which will maximize the average probability of destroying an incoming aircraft when the flight parameters of the target are not assumed known but are assumed to follow a known probability law. The problem is formulated as a mathematical programming problem requiring as preliminary computation the determination of an 'intercept geometry' for each possible combination of target parameters. The formulas for these computations are developed, and an example is given.</p>		

ATTACHMENT III

DD FORM 1473 (PAGE 1)

/N 0101-907-6801

Security Classification

**Security Classification**

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Optimal Design						
Convex Programming						
Design of Intercept Missile						

**Security Classification**