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STOCHASTIC INTEGRALS FOR PROCESSES WITH COVARIANCE

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ABSTRACT

For a stochastic process $x(t,\omega)$, the covariance $\chi(\Delta, \Delta^*)$ of the t-intervals Δ, Δ^* means the integral in ω of the product of differences $\Delta x\, \Delta^* \bar{x}$. The processes considered here are those for which (i) $|\chi(\Delta,\Delta^*)| \leq \varphi(|\Delta|)\varphi(|\Delta^*|)$ when Δ, Δ^* are non-overlapping, and (ii) $\chi(\Delta,\Delta) \leq \chi^2(|\Delta|)$ where χ is much larger than φ. Such processes are termed discriminatory. If $\varphi(u)/u$ is square integrable they turn out to be rather similar to the limiting case where $\varphi = 0$, which is that of a process with orthogonal increments; for instance (ii) is then a consequence of (i) if we take $\chi(u) = \sqrt{u}$; moreover we can define a stochastic integral with respect to $x(t,\omega)$ for functions $y(t)$ of certain integrated Lipschitz classes, where the graph of $y(t)$ may very well fill a square. If $\varphi(u)/u$ is not square integrable, we can still define our stochastic integral, but the class of functions $y(t)$ is then correspondingly smaller. Besides defining our stochastic integral and proving its existence in the above cases, we establish inequalities which can be used, for instance, for passage to the limit under the integral sign. The stochastic integrals studied here are partly suggested by the MRC Technical Report #677, to which this is a sequel. However, they are otherwise quite different from any previously considered in the theory of stochastic processes.

STOCHASTIC INTEGRALS FOR PROCESSES WITH COVARIANCE

L. C. Young

§1. Introduction. This is a sequel to three notes on derivatives and integrals.* We wish to define stochastic integrals, more general than in the third note, which are not included in those treated in Doob's book, nor in any other definitions in the literature. The covariance between non-overlapping time-intervals need vanish no longer, instead it can be majorized by a suitable product of estimates. In the case we are most interested in, which we term that of self-pairability, this leads to majorizing also the covariance of coincident intervals, in terms of a measure on the t-axis, exactly as in the special case of a process with orthogonal increments. In the case of non-self-pairability, any majorisation of the covariance of coincident intervals becomes a new hypothesis, and our results for this case will be somewhat less complete.

We again understand a stochastic process as defined by a function $x(t, \omega)$, which is complex-valued and square integrable in ω for each t, the measure $d\omega$ being some fixed probability measure. Further, given any pair of intervals Δ, Δ^* of the form $(t, t+h)$, $(t^*, t^* + k)$, we write Δx, $\Delta^* x$ for the differences, or increments

$$x(t + h, \omega) - x(t, \omega),\ x(t^* + k, \omega) - x(t^*, \omega),$$

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and $\varkappa(\Delta, \Delta^*)$ for the scalar product

$$\int \Delta x \Delta^* \bar{x} d\omega .$$

We term $|\varkappa|$ the covariance, for the process, of the intervals Δ, Δ^*, or alternatively, of the increments Δx, $\Delta^* x$; we speak of self covariance if Δ, Δ^* coincide, and of cross-covariance if they are non-overlapping. Incidentally, intervals in this paper will generally be neutral intervals, i.e. figures rather than sets. (See Saks Theory of the integral Chapter III.)

For the processes considered here, we shall suppose that there exists on the relevant part of the t-axis, a monotone increasing function $\tau(t)$, such that

$$(1.1) \quad \begin{cases} \text{(i)} & |\varkappa(\Delta, \Delta)| \leq (\chi(\Delta\tau))^2 , \\ \text{(ii)} & |\varkappa(\Delta, \Delta^*)| \leq \varphi(\Delta\tau)\varphi(\Delta^*\tau), \end{cases}$$

whenever the time-intervals Δ, Δ^* are non-overlapping. Here $\Delta\tau$, $\Delta^*\tau$ mean $\tau(t+h) - \tau(t)$, $\tau(t^*+k) - \tau(t^*)$, while φ, χ denote increasing functions of u for $u \geq 0$. The process is "non-discriminatory" if $\chi = \varphi$. We shall be concerned rather with the case of a heavily "discriminatory" process, where φ is small compared with χ as $u \to 0$, i.e. with the case where there is a genuine difference of order between (i) and (ii), and where (ii) is not simply a consequence of (i).

For such a process, minute successive time-intervals Δ, Δ^* will have, at best, a very "poor" covariance $|\chi|$, compared with coincident ones. In a natural process, this may seem, at first, paradoxical: for if there is a memory at all, one would expect this memory to be "good" for a short enough time. However, the paradox is only superficial, since instantaneous memory is a myth, except for quite simple events, whereas in many natural stochastic processes the events taking place in the smallest distinguishable time-intervals are as complex as in millennia. One has only to open a newspaper to realize that it takes time and effort to recall even a comparatively simple chain of events, such as an assassination, in all its details.

From the mathematical point of view, the existence of discriminatory processes, other than those of the third note, in which we set $\varphi = 0$, follows from the principle of superposition. This seems to be a physical requirement also. Stochastic processes must form a linear space. In particular, by superposing a process with its translations, we pass from those of the third note to more or less non-discriminatory processes, and by superposing one of these with a process with orthogonal increments, we can clearly obtain a discriminatory process. It is, for instance, sufficient to superpose in this way two such real processes on orthogonal axes in the plane, the function $\chi(u)$ being $\sqrt{u}$, and the function φ small compared with $\sqrt{u}$. If the superposition is not on orthogonal axes, we end up with a process obeying formally more general conditions than (1.1), in which the right-hand side of (ii) has been augmented by terms such as $\varphi(\Delta\tau)\chi(\Delta^{*}\tau)$; we shall ignore this minor refinement, although, actually, our arguments would apply with φ replaced by $\sqrt{(\varphi\chi)}$. At any rate, we see not only that discriminatory stochastic processes must exist, but also that they include the result of perturbing a process with orthogonal increments by superposing a suitable non-discriminatory process.

It is convenient to give to (1.1) an equivalent, but in appearance more general, form. We shall stipulate that there must exist a pair of monotone increasing functions $\tau_1(t)$, $\tau_2(t)$, such that, under the same conditions,

$$
(1.2) \qquad \begin{cases} \text{(i)} \quad |K(\Delta,\Delta)| \leq (\chi(\Delta\tau_1))^2, \\ \text{(ii)} \quad |K(\Delta,\Delta^{*})| \leq \varphi(\Delta\tau_2)\varphi(\Delta^{*}\tau_2). \end{cases}
$$

To derive (1.1) from this form, it is sufficient to set $\tau(t) = \tau_1(t) + \tau_2(t)$. In these relations, the function φ will always be taken to be the first of a pair of estimate-functions φ, ψ as defined in the first note; the second function ψ of such a pair will play an important part also.

We shall distinguish two kinds of discriminatory processes: those for which φ is self-pairable, i.e. for which the repetition φ, φ constitutes a pair of estimate functions, or what comes to the same, for which

$$\int \frac{\varphi(u)}{u}\, d\varphi(u) \text{ converges at } 0;$$

and those for which this is not assumed. In the former case, our basic hypothesis is only the cross-covariance inequality (1.2) (ii), and the process is termed of restricted cross-covariance, with self-pairable φ. Otherwise, we assume (1.1) and we term the process of biased covariance; the additional self-covariance inequality then makes it necessary to introduce, besides φ, ψ, a second pair ρ, σ of estimate functions, where $\rho(u) = \chi(u)\sqrt{u}$. It should be remarked that, in the case of restricted cross-covariance, although φ is self-pairable, we still pair it with some ψ, which may be quite different from φ; moreover, that the self-covariance inequality (1.2) (i) will still play an important part, without being a hypothesis, the relevant function χ being then $\sqrt{u}$.

The general discussion of stochastic integrals for our processes, under the above assumptions, is complicated by troublesome side-issues. These we prefer to ignore in this paper, by aiming rather at cases in which formal simplifications occur, but which are sufficiently typical to indicate the scope of our methods. Thus we set $\tau(t) = t$ in (1.1), or $\tau_2(t) = t$ in (1.2) (ii), since this virtually amounts to making a smoothing substitution in which τ_1 or τ_2, becomes the new variable. Again, for a process of biased covariance, we suppose each of the pairs φ, ψ and ρ, σ to be reversible, i.e. ψ, φ and σ, ρ are also to be pairs of estimate functions; we do so partly because, if we suppose φ not self-pairable, and if φ, ψ are orders of magnitude in the scale of powers and logarithms and their combinations, then the pair of estimate functions φ, ψ is reversible; and partly because of the symmetry of the roles of ρ, σ and φ, ψ. On the other hand, we make no reversibility assumption in the case of restricted cross-covariance, wtih self-pairable φ; in this case, however, our hypotheses, described above, are shown to imply a self-covariance inequality $|\varkappa(\Delta, \Delta)| \leq \Delta Q$, where $Q(t)$ is a monotone increasing function, the "quadratic variation" from 0 to t, which takes the place of $\tau_1(t)$ in (1.2) (i); and we here make again a simplifying assumption which amounts to a virtual change of variable.

For the purposes of this paper, a process of biased covariance will thus satisfy the conditions

$$(1.3) \qquad \begin{cases} \text{(i)} & |\varkappa(\Delta, \Delta)| \leq (\chi(|\Delta|))^2, \\ \text{(ii)} & |\varkappa(\Delta, \Delta^*)| \leq \varphi(|\Delta|)\varphi(|\Delta^*|), \end{cases}$$

where Δ, Δ^* are non-overlapping, and $|\Delta|$, $|\Delta^*|$ are their lengths. A process of restricted cross-covariance will be subject to

$$(1.4) \qquad |\varkappa(\Delta, \Delta^*)| \leq \varphi(|\Delta|)\varphi(|\Delta^*|)$$

where Δ, Δ^* are non-overlapping, and φ is now self-pairable; and it will also satisfy a further condition according to which a certain "quadratic measure" dq does not exceed dt. Because of the special role of the variable t and the measure dt, we shall refer to the process as canonical.

Our main object is to define for our processes on a finite time-interval T, a stochastic integral

$$(1.5) \qquad \int_T y(t) d_t x(t, \omega)$$

for a large enough class of "deterministic" integrands $y(t)$, i.e. of functions independent of ω. (In the theory of stochastic processes, classes of functions really are large, and need to be.) Besides defining the integral (1.5) and establishing its existence, we shall seek to obtain for it an inequality, which can be used for passage to the limit under the integral sign. The relevant class of integrands $y(t)$ will consist of those which are square integrable, and which satisfy, for small $h > 0$,

$$(1.6) \qquad \begin{cases} \text{(i)} & \int |y(t) - y(t-h)| dt \leq \psi(h), \\ \text{(ii)} & \int |y(t) - y(t-h)|^2 dt \leq (\sigma(h))^2 . \end{cases}$$

The second condition can be omitted in the case of restricted cross-covariance, with self-pairable φ. Generally, by varying ψ, or ψ, σ

subject to the conditions connecting them to φ or χ, we see easily that the space of integrands becomes a linear one, in which our integral has all the usual elementary properties. It is a particularly large space in the case of restricted cross-covariance with self-pairable φ: it then includes, as a matter of course, all functions $y(t)$ of bounded variation, and also many other functions, whose generality fully matches, to say the least, that of the functions of t obtained, for almost all $\omega = \text{const}$, from $\chi(t,\omega)$. For instance, when $\varphi(u) = u^{3/4}$, the continuous $y(t)$ include some which may fill a square, which is almost never the case of a curve $x(t,\omega)$ for constant ω.

§2. The stochastic integral for a canonical process with a biased covariance. We tackle this first, as it is easiest, and the method is clearest. By hypothesis, we have (1.3). Moveover, to avoid even minor side-issues, as far as possible, we shall make also some minor assumptions. We suppose $x(t,\omega)$ extended, if necessary, or modified, outside T, which we take to be the unit interval $0 \leq t \leq 1$; and for this purpose we set, in an interval T^* slightly larger than T,

$$x(t, \omega) = x(0, \omega) \quad \text{for } t < 0,$$

$$x(t, \omega) = x(1, \omega) \quad \text{for } t > 1.$$

We arrange too for $x(t,\omega)$ to be square integrable in (t,ω), and hence in t for almost every ω; this can be done by subtracting $x(0,\omega)$. In fact, this subtraction from $x(t,\omega)$ does not affect (1.5), and by (1.1)(i), with $\tau(t) = t$, the difference is square integrable in (t, ω), and so in t for almost every ω, by Fubini's theorem. As regards $y(t)$, we also suppose its definition extended outside T, and we strengthen (1.6) (i) and (ii) by changing the interval of integration, from T to T_+.

We shall also modify slightly the definition of stochastic integral of the third note. We intend to define (1.5), not as the derivative (in the mean) of the convolution

$$\int_T y(t)x(t+k, \omega)\, dt.$$

taken with respect to k, but as the corresponding limit of the expression

$$(2.1) \qquad \int_{-k}^{1} y(t)\hat{x}(t,\omega)dt, \text{ where } \hat{x}(t, w) = \frac{x(t+k, \omega) - x(t, \omega)}{k}$$

and where $k \to +0$. The difference arises from the limit of the quantity

$$q = \frac{1}{k}\int_{-k}^{0} y(t)\{x(t+k, \omega) - x(0, \omega)\}dt;$$

of course, this depends on how we extend $y(t)$ for $t < 0$, but, in any case, by Schwarz's inequality, we have

$$\int |q|^2 d\omega \leq k^{-1}(\chi(k))^2 \int_{-k}^{0} |y(t)|^2 dt .$$

This last quantity certainly tends to 0 with k in the case in which $\chi(u) = \sqrt{u}$, and therefore under the hypotheses of the third note. Thus the definition there used accords with our present one. What is important here is that, with the above minor change, the definition will be expressed in a more convenient form below, which we shall then use throughout the rest of this section. In order to obtain this alternative expression, we need a few preliminary remarks.

For this purpose, we recall, in our present context, the elementary definition of Stieltjes integral, for a scalar function $\hat{y}(t)$, with respect to a vector-valued function $X(t)$, where $\hat{y}(t)$ is of bounded variation, and $X(t)$ is continuous; the integral then exists, as the limit of the Riemann sums, and integration by parts is valid. Here we shall choose for $X(t)$ the vector whose value, for any fixed t, is the function of ω, defined by $x(t,\omega)$. In the space of such vectors f, i.e. of such functions $f(\omega)$, we take as norm $|f|$ the quantity $(\int |f(\omega)|^2 d\omega)^{1/2}$; the space is thus a Hilbert space. In this way, for a function $\hat{y}(t)$ of bounded variation, we define

$$(2.2) \qquad \int_T \hat{y}(t) d_t x(t,\omega) = \int_T \hat{y}(t) dX(t),$$

where the right-hand side is the elementary Stieltjes integral.* In particular, in the rest of this section, we choose

*Such elementary integrals will actually be found to be special cases of our definition of (1.5). In the meantime, there is no conflict of notation, since the wider definition is not used until after the proof of Theorem (2.9).

$$\hat{y}(t) = \frac{1}{k}\int_0^k y(t-u)du ;$$

the integral (2.2) may then be written, after integrating by parts, in the form $(A - B)/k$, where

$$\begin{aligned} A &= kx(1,\omega)\hat{y}(1) + \int_0^1 x(t,\omega)y(t-k)dt \\ &= x(1,\omega)\int_{1-k}^1 y(t)dt + \int_{-k}^{1-k} x(t+k,\omega)y(t)dt \\ &= \int_{-k}^1 x(t+k,\omega)y(t)dt, \end{aligned}$$

$$\begin{aligned} B &= kx(0,\omega)\hat{y}(0) + \int_0^1 x(t,\omega)y(t)dt \\ &= \int_{-k}^1 x(t,\omega)y(t)dt. \end{aligned}$$

Thus (2.2) coincides with the value of the expression (2.1), and our task will be to prove that it has a limit as $k \to 0$ and to define (1.5) as the value of this limit; naturally, here, the limits are in the topology of our vector space of functions of ω, just like the definition of the elementary Stieltjes integral (2.2), itself.

In other terms, if we denote by $\hat{I}$ the vector, or the function of ω, defined by (2.2), and therefore dependent on k, we have to establish the existence of a vector I, such that $|\hat{I} - I| \to 0$ as $k \to 0$.

We shall need several simple lemmas, of which the most important is the first. In referring to (1.1), it will always be understood, in this section, that $\tau(t) = t$.

(2.3) <u>Lemma. Let</u> $z(t)$ $t \in T$ <u>take constant values</u> z_i <u>in the interiors of the partial intervals</u> Δ_i <u>of a subdivision of</u> T <u>into equal parts of length</u> h, <u>and let</u>

$$R_h(z) = |I(z)|^2 - \sum_i |z_i|^2 \chi(\Delta_i, \Delta_i),$$

<u>where</u> $|I(z)|$ <u>denotes the Hilbert norm of the vector</u>

$$I(z) = \sum_i z_i \Delta_i x = \int_T z(t) d_t x(t, \omega).$$

<u>Then</u>

$$|I(z)|^2 \leq R_h(z) + h^{-1}(\chi(h))^2 \int_T |z|^2 dt, \quad |R_h(z)| \leq (h^{-1}\varphi(h) \int_T |z| dt)^2 .$$

<u>Proof.</u> We have

$$|I(z)|^2 = \int d\omega (\sum_i z_i \Delta_i x)(\sum_j \bar{z}_j \Delta_j \bar{x}) = \sum_j \sum_j z_i \bar{z}_j \chi(\Delta_i, \Delta_j).$$

Hence by (1.3) (ii),

$$|R_h(z)| = |\sum_{i \neq j}\sum z_i \bar{z}_j \chi(\Delta_i, \Delta_j)| \leq (h^{-1}\varphi(h))^2 \sum_i \sum_j h^2 |z_i||z_j| ,$$

while by (1.3) (i)

$$|I(z)|^2 - R_h(z) = \sum_i |z_i|^2 \chi(\Delta_i, \Delta_i) \leq h^{-1}(\chi(h))^2 \sum_i h|z_i|^2.$$

Here $\sum_i h|z_i|^2$ is the integral of $|z|^2$, while $\sum_i \sum_j h^2 |z_i||z_j|$ has been increased by adding the terms for which $i = j$, so that it becomes the square of $\sum_i h|z_i|$, i.e. of the integral of $|z|$.

We denote by T_+ some fixed interval $(-k_0, 1)$ where $k_0 > 0$, and, for $1 \leq p \leq 2$, by $\psi_p(u)$, $\psi_p^+(u)$ the suprema in c, for $0 < c < u$, of the expressions $\int_T |y(t) - y(t-c)|^p dt$, $\int_{T_+} |y(t) - y(t-c)|^p dt$. The number k in the definition of $\hat{y}(t)$ will be supposed $< k_0$. Further, for a subdivision of T into equal parts Δ_i of length h, let $z(t)$ be a step-function equal, in the interior of each Δ_i to the corresponding mean value of $y(t)$, and let $\hat{z}(t)$ be there similarly the corresponding mean value of $\hat{y}(t)$; finally, let $z^*(t)$ be a step-function similarly constructed from $y(t)$, but for a subdivision of T into equal parts of length $h^* = h/N$, where N is a positive integer.

(2.4) <u>Lemma. With these notations, we have</u>

(i) $$\int_T |\hat{y}(t) - \hat{y}(t-h)|^p dt \leq \int_{T_+} |y(t) - y(t-h)|^p dt \leq \psi_p^+(h),$$

(ii) $$\int_T |z(t) - \hat{z}(t)|^p dt \leq \int_T |y(t) - \hat{y}(t)|^p dt \leq \psi_p(k),$$

(iii) $$\int_T |z(t) - z^*(t)|^p dt \leq 2\psi_p(h).$$

<u>Proof</u>. It will mainly be a matter of applying, in each case, Holder's inequality. Thus

$$\int_T |\hat{y}(t) - \hat{y}(t-h)|^p dt = \int_T |\frac{1}{k}\int_0^k \{y(t-u) - y(t-h-u)\}du|^p dt$$

$$\leq \int_T \frac{1}{k}\int_0^k |y(t-u) - y(t-u-h)|^p du dt$$

$$= \frac{1}{k}\int_0^k du \int_T dt |y(t-u) - y(t-u-h)|^p$$

$$\leq \frac{1}{k}\int_0^k du \int_{T_+} |y(t) - y(t-h)|^p dt,$$

which proves (i). Similarly

$$\int_T |y(t) - \hat{y}(t)|^p dt = \int_T |\frac{1}{k}\int_0^k \{y(t) - y(t-u)\}du|^p dt$$

$$\leq \int_T \frac{1}{k}\int_0^k |y(t) - y(t-u)|^p du\, dt$$

$$= \frac{1}{k}\int_0^k du \int_T |y(t) - y(t-u)|^p dt \leq \psi_p(k),$$

which is the second part of (ii). Again, if t_i is the initial point of Δ_i, and z_i, $\bar{z}_i$ are the constant values of $z(t)$, $\bar{z}(t)$ in the interior of Δ_i, we see that

$$\int_{\Delta_i} |z(t) - \hat{z}(t)|^p dt = h|z_i - \hat{z}_i|^p = h|\frac{1}{h}\int_{\Delta_i} \{y(t) - \hat{y}(t)\}dt|^p$$

$$\leq h \cdot \frac{1}{h}\int_{\Delta_i} |y(t) - \hat{y}(t)|^p dt,$$

do that, by addition, we obtain the first part of (ii).

Finally, to get (iii), it will suffice to verify, for each i, that

$$(2.5) \qquad \int_{t_i}^{t_i+h} |z(t) - z^*(t)|^p dt \le \frac{2}{N} \sum_{\lambda=1}^{N-1} \int_{t_i+\lambda h^*}^{t_i+h} |y(t)-y(t-\lambda h^*)|^p dt.$$

Keeping i fixed, we consider, for this purpose, the subintervals Q_μ of Δ_i in which t has the form $t_i + \mu h^* + u$ $(0 < u < h^*)$; here $\mu = 0, 1, \ldots, N-1$. In Q_μ we have

$$|z(t) - z^*(t)|^p = \left|\frac{1}{N} \sum_{\nu=0}^{N-1} \frac{1}{h} \int_0^h \{y(t_i+\nu h^*+u) - y(t_i+\mu h^*+u)\}du\right|^p$$

$$\le \frac{1}{N} \sum_{\nu=0}^{N-1} \frac{1}{h} \int_0^h |y(t_i+\nu h^*+u) - y(t_i+\mu h^*+u)|^p du.$$

By integrating in t over Q_μ, which means multiplying by h, and by then summing in μ, we see that the left-hand side of (2.5) cannot exceed the quantity

$$M = \frac{1}{N} \sum_\mu \sum_\nu \int_0^h |y(t_i+\nu h^*+u) - y(t_i+\mu h^*+u)|^p du .$$

However, M can be expressed as twice the corresponding sum for $\mu > \nu$; and if we then set $\lambda = \mu-\nu$, change the order of summation, and revert from u to $t = t_i + \mu h^* + u$ as our variable of integration, we find that M is indeed the right-hand side of (2.5). This proves (iii).

We shall make use also of the following elementary remark: let ρ, σ be a reversible pair of estimate functions, then

$$(2.6) \qquad \sum_{\nu=N+1}^{\infty} 2^{\nu}\rho(2^{-\nu})\sigma(2^{-(\nu-1)}) \leq 16 \int_0^{2^{-N}} u^{-2}\rho(u)\sigma(u)du < \infty .$$

In fact

$$\int_0^1 \frac{\rho(u)\sigma(u)}{u^2}\, du \leq \rho(1)\sigma(1) + \int_0^1 \frac{\rho(u)\sigma(u)}{u^2}\, du$$

$$= \int_0^1 \frac{\rho(u)d\sigma(u)}{u} + \int_0^1 \frac{\sigma(u)d\rho(u)}{u} < \infty;$$

and moreover, if we write $h = 2^{-\nu}$ for short and denote by $\tilde{u}$ a suitable u in $h \leq u \leq 2h$,

$$\int_h^{2h} \frac{\rho(u)\sigma(u)}{u^2}\, du = h\frac{\rho(\tilde{u})\sigma(\tilde{u})}{(\tilde{u})^2} \geq (\frac{h}{4}) \frac{\rho(2h)\sigma(2h)}{4h^2} \geq \frac{\rho(h)\sigma(2h)}{16h} ,$$

whence the first half of (2.6) follows by summation. Besides (2.6), we shall have, of course, the corresponding result for the reversible pair of estimate functions φ, ψ.

In what follows, the step functions $z(t)$, $\hat{z}(t)$, determined as earlier, but now for $h = 2^{-\nu}$, will be denoted by $y_{\nu}(t)$, $\hat{y}_{\nu}(t)$. Further we write

$$I_{\nu} = \int_T y_{\nu}(t)d_t x(t,\omega), \quad \hat{I}_{\nu} = \int_T \hat{y}_{\nu}(t)d_t x(t,\omega),$$

so that these are functions $I_\nu(\omega)$, $\hat{I}_\nu(\omega)$, or what comes to the same, they are vectors in our Hilbert space. We shall understand the norms to be in that space, whenever ω is omitted from our notation.

(2.7) Lemma. Both sums $\sum_\nu |I_{\nu-1} - I_\nu|$, $\sum_\nu |\hat{I}_{\nu-1} - \hat{I}_\nu|$ are dominated, term by term, by twice the series

$$\sum_\nu 2^\nu \{\rho(2^{-\nu})\sigma(2^{-(\nu-1)}) + \varphi(2^{-\nu})\psi(2^{-(\nu-1)})\} .$$

Moreover, $|I_\nu - \hat{I}_\nu|$ cannot exceed the expression

$$2^\nu \{\rho(2^{-\nu})\sigma(k) + \varphi(2^{-\nu})\psi(k)\}.$$

Proof. All this follows readily from lemma (2.3), if we replace in it, successively, $z(t)$ by the three step-functions $y_{\nu-1} - y_\nu$, $\hat{y}_{\nu-1} - \hat{y}_\nu$, $y_\nu - \hat{y}_\nu$, and if we then replace further the integrals of $|z(t)|^p$ $(p = 1, 2)$ by the larger quantities given in lemma (2.4).

(2.8) Lemma. Let

$$S(h) = \int_0^h u^{-2}\{\rho(u)\sigma(u) + \varphi(u)\psi(u)\}du.$$

Then, in our Hilbert space, the limits $I = \lim_\nu I_\nu$, $\hat{I} = \lim_\nu \hat{I}_\nu$ exist and we have

$$|I_\nu - I| \leq 16\, S(2^{-(\nu-1)}), \quad |\hat{I}_\nu - \hat{I}| \leq 16\, S(2^{-(\nu-1)}) .$$

Proof. This follows at once from the preceding lemma together with (2.6).

(2.9) Theorem. *Let* $x(t,\omega)$ *be subject to* (1.3), *where* χ *is increasing, and let* $\rho(u) = u^{1/2}\chi(u)$. *Further let* $y(t)$ *satisfy* (1.6), *where integration is over an interval containing in its interior the closed finite interval* T, *and suppose that* (ρ, σ), (φ, ψ) *are reversible pairs of estimate-functions. Finally, let* $\hat{I}$ *now denote the elementary integral*

$$\int_T \hat{y}(t)d_t x(t,\omega), \quad \textit{where} \quad \hat{y}(t) = \frac{1}{k}\int_0^k y(t-u)du.$$

Then, in our Hilbert space, the limit $I = \lim_{k\to 0}\hat{I}$ *exists, and we have* $|\hat{I} - I| \leq 400\, \mathcal{S}(k)$, *where the function* $\mathcal{S}$ *is defined as in the preceding lemma. We term* I *the stochastic integral in* T, *of* $y(t)$ *with respect to* $x(t,\omega)$.

Proof. We define I, $\hat{I}$ provisionally, not as in the assertion, but as in the preceding lemma. However, $\hat{I}$ then coincides with the elementary integral in question, which is the unique limit of the approximating Riemann sums, and in particular of the $\hat{I}_\nu$. It then only remains to verify the inequality asserted for $|\hat{I} - I|$, which clearly implies that $I = \lim_{k\to 0}\hat{I}$. Further, we see by changing u to $2u$ and using the relations of the type (2.1) of the first note, that $\mathcal{S}(2k) \leq 8\,\mathcal{S}(k)$. Therefore it will suffice to prove that $|\hat{I} - I| \leq 50\,\mathcal{S}(2k)$.

For this purpose, we choose ν so that $1 \leq k2^{-\nu} < 2$ and we observe that

$$|\hat{I} - I| \leq |\hat{I} - \hat{I}_\nu| + |I - I_\nu| + |\hat{I}_\nu - I_\nu| .$$

By lemma (2.8), each of the first two terms on the right is $\leq 16\delta(2k)$; the same is true of the third term, by the last part of lemma (2.7) and the remarks about (2.6). This completes the proof, with a little to spare.

An important, and typical, special case of our theorem is that in which X is a constant, and φ a power $\frac{1}{2} - \alpha$ $(0 \leq \alpha < \frac{1}{2})$ of u; we can then choose σ, ψ of power-orders $\frac{1}{2} + \epsilon$, $\frac{1}{2} + \alpha + \epsilon$, so that $y(t)$ lies in the intersection of two integrated Lipschitz classes of Hardy and Littlewood [5].

Our inequality for $|\hat{I} - I|$ is reminiscent of ideas used to refine Schwarz's inequality, by separating non-diagonal terms; see, for instance, van der Corput [4]. This is useful for estimates of trigonometric sums in diophantine approximation.

§3. <u>The more elaborate form of the basic lemma, and the notion of quadratic measure</u>. We pass on to the case of a process of restricted cross-covariance, with φ self-pairable. We assume, for the present, only (1.4), so that this is a generalisation of the state of affairs of the third note. Our arguments will be more complicated than in the preceding section, and we shall not wholly avoid side-issues.

In the first place, we need to modify the notion of step-function, and this is one of the side-issues. For this purpose, intervals are, for the moment, to be regarded as neither open, nor closed, but as neutral, as in the algebraization of integration. Given a subdivision $\mathfrak{J}$ of T into such neutral intervals Δ, we term (step)-function associated with $\mathfrak{J}$, the map which attaches to each such $\Delta \epsilon \mathfrak{J}$ a constant complex value $z(\Delta)$. A (step)-function z is thus a function, not of $t \epsilon T$, but of $\Delta \epsilon \mathfrak{J}$. However, if z has the same value in two adjacent Δ, we identify it with the (step)-function with this value in their union and with the original values outside. Similarly, if we superpose on $\mathfrak{J}$ a further subdivision, we shall define z on the new parts so that it retains the original value in any subinterval of an interval of $\mathfrak{J}$, and the new (step)-function, thus obtained, will be identified with the old. Any two (step)-functions can thus be associated with a same subdivision, and hence (step)-functions form an algebra. If z is any (step)-function, we shall write $|z|$ for the (step)-function with the values $|z(\Delta)|$; moreover, any integral of z denotes the appropriate finite sum, for instance we interpret $\int_T z\,dt$ and the stochastic integral

$$I(z) = \int_T z\, d_t x(t, \omega)$$

to mean the finite sums $\sum z(\Delta)\Delta t$, $\sum z(\Delta)\Delta x$, where Δt, Δx stand for the difference in t, at the ends of Δ, applied to the functions t, $x(t, \omega)$, and where the sums extend to the partial intervals Δ of the associated subdivision of T. These integrals do not change when we pass from one subdivision associated with z to another, i.e. from z to a (step)-function identified with z.

We term (step h)-function, a step-function associated with the subdivision $\mathfrak{J}$ of T into equal parts of length h, and we speak of h as the "length of a step". This length will always be a power of $1/2$, and it may depend on a suffix ν, in which case it is understood to decrease as ν increases. As previously in lemma (2.3), we write

$$R_h(z) = \int d\omega \left| \int_T z d_t x(t,\omega) \right|^2 - \sum_{\Delta \in \mathfrak{J}} |z(\Delta)|^2 \varkappa(\Delta,\Delta).$$

(3.1) Lemma. For $\nu = 0, 1, \ldots, N$, let z_ν denote a (step h_ν)-function, and let s_ν denote, for $\nu < N$, the sum of the block of terms, given by $h_\nu > 2^{-r} \geq h_{\nu+1}$, in the series $\sum 2^r \varphi^2(2^{-r})$, while s_N denotes the remainder, the sum for $h_N > 2^{-r}$. Further let $\zeta_0 = z_0$, $\zeta_\nu = z_\nu - z_{\nu-1}$ ($\nu > 0$). Then if z is a (step h)-function identical with z_N, where $h \leq h_N$, we have

$$|R_h(z)| \leq \sum_{\nu=0}^{N} s_\nu \int_T |z_\nu|^2 dt + 2\left(\sum_{\nu=0}^{N} h_\nu^{-1} \varphi(h_\nu) \int_T |\zeta_\nu| dt\right)^2.$$

Proof. By setting $\zeta_\nu = 0$ when $h_0 2^{-\nu}$ is not one of the h_ν, we may suppose that $h_\nu = h_0 2^{-\nu}$, $h_N = h$, and remove the term $\nu = N$ from the first sum on the right. The case $N = 0$ follows from lemma (2.3).

We can proceed by induction. We have to show that (3.1) implies the corresponding statement with N replaced by N+1. We write $h^* = h/2$, $z^* = z_{N+1} = z+\zeta = z+\zeta_{N+1}$, so that z^* is a (step h^*)-function, we denote by $\mathfrak{J}^\nu$ the subdivision of T into equal parts Δ^ν of length $h_0 2^{-\nu}$, and, for $\nu = N+1$, we write $\mathfrak{J}^*$, Δ^* in place of $\mathfrak{J}^\nu$, Δ^ν.

We first observe that the difference

$$P = \sum_{\Delta\epsilon\mathfrak{J}} |z(\Delta)|^2 K(\Delta,\Delta) - \sum_{\Delta^*\epsilon\mathfrak{J}^*} |z(\Delta^*)|^2 K(\Delta^*,\Delta^*)$$

may be written as the sum, for $\Delta\epsilon\mathfrak{J}$, of the real part of

$$2|z(\Delta)|^2 K(\Delta', \Delta''),$$

where Δ', Δ'' are the two halves of Δ; the absolute value of this difference is thus at most

$$\sum_{\Delta\epsilon\mathfrak{J}} h|z(\Delta)|^2(h^*)^{-1}\varphi^2(h^*) = (h^*)^{-1}\varphi^2(h^*)\int_T |z|^2 dt.$$

Hence it will suffice to show that

$$R(z^*) - R(z) - P$$

cannot exceed the quantity

$$\Lambda = 2\int_T |\zeta| dt (h^*)^{-1}\varphi(h^*) \left(\sum_{\nu=0}^{N} + \sum_{\nu=0}^{N+1}\right) h_\nu^{-1}\varphi(h_\nu)\int_T |\zeta_\nu| dt.$$

However, we have

$$R(z^*) - R(z) - P = |I(z^*)|^2 - |I(z)|^2 - M,$$

where

$$M = \sum_{\Delta^* \epsilon \mathfrak{J}^*} (|z^*(\Delta^*)|^2 - |z(\Delta^*)|^2) K(\Delta^*, \Delta^*)$$

$$= \sum_{\Delta^* \epsilon \mathfrak{J}^*} \text{Real part} \left\{ \left(\sum_{\nu=0}^{N} + \sum_{\nu=0}^{N+1} \right) \zeta_\nu(\Delta^*) \zeta(\Delta^*) K(\Delta^*, \Delta^*) \right\}.$$

Moreover $|I(z^*)|^2 - |I(z)|^2$ is the real part of

$$\int d\omega \{ \bar{I}(\zeta) I(z + z^*) \} =$$

$$= \sum_{\Delta^* \epsilon \mathfrak{J}^*} \bar{\zeta}(\Delta^*) \int d\omega \{ \Delta^* \bar{x} I(z + z^*) \},$$

and here

$$\int d\omega \{ \Delta^* \bar{x} I(z + z^*) \} = \left(\sum_{\nu=0}^{N} + \sum_{\nu=0}^{N+1} \right) \int d\omega \{ \Delta^* \bar{x} I(\zeta_\nu) \}.$$

Consequently

$$R(z^*) - R(z) - P = \sum_{\Delta^* \epsilon \mathfrak{J}} \text{Real part} \left(\sum_{\nu=0}^{N} + \sum_{\nu=0}^{N+1} \right) \bar{\zeta}(\Delta^*) f_\nu(\Delta^*),$$

where

$$f_\nu(\Delta^*) = \int d\omega \{ \Delta^* \bar{x} I(\zeta_\nu) \} - \zeta_\nu(\Delta^*) K(\Delta^*, \Delta^*).$$

By comparing the expression found for $R(z^*) - R(z) - P$ with that for the quantity Λ, remembering that

$$\sum_{\Delta^* \in \mathfrak{J}^*} |\zeta(\Delta^*)| h^* = \int_T |\zeta| dt,$$

we see that it will now suffice to prove that $|f_\nu(\Delta^*)|$ has the bound, independent of Δ^*, given by the expression

$$2\varphi(h^*)\varphi(h_\nu) h_\nu^{-1} \int_T |\zeta_\nu| dt.$$

To this effect, we note that

$$f_\nu(\Delta^*) = \left\{ \sum_{\Delta^\nu \in \mathfrak{J}^\nu} \zeta_\nu(\Delta^\nu) \varkappa(\Delta^\nu, \Delta^*) \right\} - \zeta_\nu(\Delta^*) \varkappa(\Delta^*, \Delta^*).$$

Now only one Δ^ν intersects Δ^*, and for this Δ^ν we have $\zeta_\nu(\Delta^\nu) = \zeta_\nu(\Delta^*)$ and moreover, as is easy to see,

$$|\varkappa(\Delta^\nu, \Delta^*) - \varkappa(\Delta^*, \Delta^*)| \leq 2\varphi(h^*)\varphi(h_\nu),$$

while for all the other $\Delta^\nu \in \mathfrak{J}^\nu$, we have $\varkappa(\Delta^\nu, \Delta^*) \leq \varphi(h^*)\varphi(h_\nu)$. Hence, remembering that

$$\sum_{\Delta^\nu \in \mathfrak{J}^\nu} |\zeta_\nu(\Delta^\nu)| h_\nu = \int_T |\zeta_\nu| dt,$$

we see at once that $|f_\nu(\Delta^*)|$ has the desired bound.

The lemma just proved is already of importance in a very simple special case. We choose $h_\nu = 2^{-\nu}$ and we denote by $\tilde{\Delta}$ an interval

whose extremities are integer multiples of $h = 2^{-N}$. We define the (step h_ν)-function z_ν so that $z_\nu(\Delta^\nu) = 1$ if $\Delta^\nu \subset \tilde{\Delta}$, and $z_\nu(\Delta^\nu) = 0$ otherwise. Thus the final (step h)-function z can be identified with the (step)-function which is 1 on $\tilde{\Delta}$, and 0 on the set of (at most two) complementary intervals; moreover, for each ν, the (step)-function ζ_ν will be 0 except on at most two of the Δ^ν, on which it is 1.
In this case $|I(z)|^2 = \mathcal{K}(\tilde{\Delta}, \tilde{\Delta})$, so that lemma (3.1) yields the inequality

$$|\mathcal{K}(\tilde{\Delta}, \tilde{\Delta}) - \sum_{\substack{\Delta \subset \tilde{\Delta} \\ \Delta \in \mathfrak{J}}} \mathcal{K}(\Delta, \Delta)| \leq A + 2B^2,$$

where

$$A = |\tilde{\Delta}| \sum_\nu h_\nu^{-1} \varphi^2(h_\nu),$$

$$B = \sum_\nu h_\nu^{-1} \varphi(h_\nu) \int_T |\zeta_\nu| dt = \sum_\nu (h_\nu^{-\frac{1}{2}} \varphi(h_\nu))(h_\nu^{-\frac{1}{2}} \int_T |\zeta_\nu| dt).$$

If we apply Schwarz's inequality to this last sum, noting that

$$\int_T |\zeta_\nu| dt \leq 2h_\nu, \quad \sum_\nu \int_T |\zeta_\nu| dt = |\tilde{\Delta}|,$$

and therefore that

$$\sum_\nu h_\nu^{-1} (\int_T |\zeta_\nu| dt)^2 \leq 2|\tilde{\Delta}|,$$

we find that

$$B^2 \leq 2|\tilde{\Delta}| \sum_\nu h_\nu^{-1} \varphi^2(h_\nu).$$

Thus

$$A + 2B^2 \leq 5\mathcal{S},$$

where

$$\mathcal{S} = \sum_{\nu} 2^{\nu} \varphi^2(2^{-\nu}),$$

and here we can now sum in ν from 0 to ∞. Of course, we could have discarded originally the values of ν for which $2^{-\nu} > |\tilde{\Delta}|$, since, for these values, no interval of the form Δ^{ν} is contained in $\tilde{\Delta}$. We may therefore, if we prefer, discard these small values of ν in the definition of $\mathcal{S}$, and so sharpen slightly our bound.

We shall write, for brevity, and only temporarily, $\sum(\tilde{\mathfrak{I}})$ for the function of $\tilde{\mathfrak{I}}$ given by the sum

$$\sum_{\tilde{\Delta} \in \tilde{\mathfrak{I}}} \mathcal{H}(\tilde{\Delta}, \tilde{\Delta})$$

where $\tilde{\mathfrak{I}}$ denotes any subdivision of T. The inequality, derived above from lemma (3.1), leads at once, by addition, to the following one:

$$\left|\sum(\tilde{\mathfrak{I}}) - \sum(\mathfrak{I})\right| \leq 5\mathcal{S},$$

provided that $\mathfrak{I}$ is as before, and that $\tilde{\mathfrak{I}}$ has all its points of division among those of $\mathfrak{I}$. Hence, further, (3.2) $$|\sum(\tilde{\mathfrak{J}}) - \sum(\mathfrak{J})| \leq 10\delta,$$

if $\mathfrak{J}$ and $\tilde{\mathfrak{J}}$ now denote any two subdivisions of T, whose points of division are integral multiples of 2^{-N}, for some N. We shall extend this last result further:

(3.3) Lemma. The inequality (3.2) remains valid for an arbitrary pair $\mathfrak{J}$, $\tilde{\mathfrak{J}}$ of subdivisions of T.

Proof. It is clearly sufficient to show that any subdivision $\tilde{\mathfrak{J}}$ of T can be associated with another $\tilde{\mathfrak{J}}_h$, where the points of division of $\tilde{\mathfrak{J}}_h$ are integer multiples of $h = 2^{-N}$ for some N, in such a manner that

$$\sum(\tilde{\mathfrak{J}}_h) \to \sum(\tilde{\mathfrak{J}}) \quad \text{as } N \to \infty.$$

To this effect, let t_i $(i = 1, 2, \ldots, n-1)$ be the points of division of $\tilde{\mathfrak{J}}$. We mark off, for each t_i, the nearest integer multiples of h on the two sides of t_i; they have the form $t_i - \epsilon_i$, $t_i + \epsilon_i'$ where $0 \leq \epsilon_i \leq h$, $0 \leq \epsilon_i' \leq h$. We shall suppose N large enough for the intervals determined by these pairs of points to be disjoint. We denote by $\tilde{\mathfrak{J}}_h$ the subdivision of T whose points of division are the $t_i - \epsilon_i$ and the $t_i + \epsilon_i'$, and by $\tilde{\mathfrak{J}}_0 \tilde{\mathfrak{J}}_h$ the subdivision whose points of division are those of $\tilde{\mathfrak{J}}$ together with those of $\tilde{\mathfrak{J}}_h$. From the bilinear character of $\mathcal{K}$ and the relation (1.2) (ii) with $\tau_2(t) = t$, we easily estimate the differences

$$|\sum(\tilde{\mathfrak{J}}) - \sum(\tilde{\mathfrak{J}}_0 \tilde{\mathfrak{J}}_h)|, \; |\sum(\tilde{\mathfrak{J}}_h) - \sum(\tilde{\mathfrak{J}}_0 \tilde{\mathfrak{J}}_h)|;$$

the former will not exceed a fixed number of products of the form $\varphi(h)\varphi(|\tilde{\Delta}|)$ $\tilde{\Delta}\epsilon\tilde{\mathfrak{J}}$, and the latter will not exceed a fixed multiple of $\varphi^2(h)$. Hence $|\sum(\tilde{\mathfrak{J}}_h) - \sum(\tilde{\mathfrak{J}})|$ tends to 0 with h, and this completes the proof.

In (3.2), the quantity $\mathcal{S}$ is finite, since we suppose φ self-pairable. (See the remarks after (2.6) of the preceding section). It is convenient further to denote by $\mathcal{S}(\epsilon)$ the sum derived from that defining $\mathcal{S}$ by discarding the initial terms, for which $2^{-\nu} > \epsilon$. If the lengths $|\Delta|$, $|\tilde{\Delta}|$ of the intervals of $\mathfrak{J}$, $\tilde{\mathfrak{J}}$ are all $< \epsilon$, we can, as already noted, discard these initial terms, so that in (3.2), $\mathcal{S}$ can then be replaced by $\mathcal{S}(\epsilon)$, which is small with ϵ.

We remark also that (3.2) implies, for every subdivision $\tilde{\mathfrak{J}}$ of T,

$$\sum(\tilde{\mathfrak{J}}) \leq K(T, T) + 10\,\mathcal{S}.$$

This has an important interpretation: if we again write X(t) for the vector in Hilbert space, defined by $x(t,\omega)$, we have

$$\sum(\tilde{\mathfrak{J}}) = \sum_{\tilde{\Delta}\epsilon\tilde{\mathfrak{J}}} |\tilde{\Delta}X|^2,$$

and the supremum in $\tilde{\mathfrak{J}}$ of this quantity is termed the quadratic variation of X(t). Thus we see that X(t) is a function of bounded quadratic variation. We shall denote more generally by Q(t) the supremum in $\mathfrak{J}$ of the sum

$$(3.4) \qquad \sum_{\Delta \in \mathfrak{J}} \mathcal{K}(\Delta, \Delta) = \sum_{\Delta \in \mathfrak{J}} |\Delta X|^2 ,$$

for subdivisions $\mathfrak{J}$, not of the whole original interval T, but only of the part between 0 and t. We term $Q(t)$ the quadratic variation from 0 to t of the vector-valued function $X(t)$, or of the process $x(t, \omega)$. It is covenient also to denote by $Q_\epsilon(t)$ the corresponding supremum when $\mathfrak{J}$ is further restricted by the condition that $|\Delta| < \epsilon$ for every $\Delta \epsilon \mathfrak{J}$.

Evidently $Q(t)$ is increasing and $\Delta Q \geq \mathcal{K}(\Delta, \Delta)$ for every interval $\Delta \subset T$. In other words the relation (1.2) (i) must hold for $\chi(u) = \sqrt{u}$ and $\tau_1(t) = Q(t)$, so that the diagonal distortion, resulting from the fact that $\tau_1(t)$ need not be linear in t, is wholly determined by the quadratic variation $Q(t)$. At the same time, $Q(t)$ replaces, in this respect, the function $\mu(t)$ of the third note. We have thus:

(3.5) <u>Theorem</u>. <u>Suppose that</u> (1.2) (ii) <u>holds with</u> $\tau_2(t) = t$ <u>and</u> φ <u>self-pairable</u>. <u>Then</u> (1.2) (i) <u>holds with</u> $X(u) = \sqrt{u}$ <u>and</u> $\tau_1(t) = Q(t)$, <u>where</u> $Q(t)$ <u>is the quadratic variation from</u> 0 <u>to</u> t <u>for the process</u>.

Nevertheless, the role of the measure $d\mu$ of the third note, will mainly be taken over, not by $dQ(t)$, but by a quadratic measure $dq(t)$ derived from the function $q(t) = \mathrm{Lim}_{\epsilon \to 0} Q_\epsilon(t)$. This limit exists, since $Q_\epsilon(t)$ decreases with ϵ; and $q(t)$ is clearly monotone increasing. We shall speak of this quadratic measure, as defined, not only for dq-measurable sets, but also for neutral intervals Δ, when it becomes the corresponding difference Δq of $q(t)$.

From the earlier remarks about discarding initial values of ν in the quantity $\mathcal{S}$ in (3.2), it follows that:

(3.6) <u>Lemma</u>. <u>The quadratic measure</u> $\tilde{\Delta} q$ <u>of any neutral interval</u> $\tilde{\Delta}$ <u>is the unique limit, as</u> $h \to 0$, <u>of the sum</u> (3.4), <u>for subdivisions</u> $\mathfrak{J}$ <u>of</u> $\tilde{\Delta}$ <u>such that each</u> $\Delta \in \mathfrak{J}$ <u>has length</u> $\leq h$.

A slight variant of this lemma will be needed also. For a given $\tilde{\Delta}$, we denote by $\sum_+(\mathfrak{J})$ the sum

$$\sum \mathcal{K}(\Delta, \Delta)$$

extended to those $\Delta \in \mathfrak{J}$ for which Δ intersects $\tilde{\Delta}$; and by $\sum_-(\mathfrak{J})$ the same sum extended to those $\Delta \in \mathfrak{J}$ for which $\Delta \subset \tilde{\Delta}$. Here $\mathfrak{J}$ now denotes an arbitrary subdivision of T.

From the preceding lemma, we deduce easily that, if $\tilde{\Delta}$ is an interval at whose ends $q(t)$ is continuous, or else an interval, one of whose ends is 0 or 1, and the other is a point of continuity of $q(t)$, then $\tilde{\Delta} q$ is the unique limit of $\sum_+(\mathfrak{J})$, and also of $\sum_-(\mathfrak{J})$, for subdivisions $\mathfrak{J}$ of T such that each $\Delta \in \mathfrak{J}$ has length $\leq h$.

Lemma (3.6), together with this variant, is itself a special case of the following result, easily deduced by addition:

(3.7) *Lemma. Let* $\tilde{\mathfrak{J}}$ *be a subdivision of* T, *and let* $\tilde{z}$ *be a (step)-function associated with* $\tilde{\mathfrak{J}}$, *i.e. one for which the constant values* $\tilde{z}(\tilde{\Delta})$ *are defined when* $\tilde{\Delta} \in \tilde{\mathfrak{J}}$. *Then the integral*

$$J = \int_T \tilde{z}\, dq$$

is the unique limit, as $h \to 0$ *of the sum*

$$\sum_{\Delta \in \mathfrak{J}} \tilde{z}(\Delta)\mathfrak{K}(\Delta, \Delta)$$

for subdivisions $\mathfrak{J}$ *of* T, *such that each* $\Delta \in \mathfrak{J}$ *is of length* $\leq h$, *and that the points of division of* $\mathfrak{J}$ *include all those of* $\tilde{\mathfrak{J}}$. *Further, if* $\tilde{z}$ *is real-valued, and we denote, for any interval* Δ, *by* $\tilde{z}^*(\Delta)$ *any value between the greatest and least values of* $\tilde{z}$ *for intervals* $\tilde{\Delta} \in \tilde{\mathfrak{J}}$ *which intersect* Δ, *then the sum*

$$\sum_{\Delta \in \mathfrak{J}} \tilde{z}^*(\Delta)\mathfrak{K}(\Delta, \Delta)$$

for subdivisions $\mathfrak{J}$ *of* T, *such that each* $\Delta \in \mathfrak{J}$ *is of length* $\leq h$, *also has the unique limit* J *as* $h \to 0$, *provided that the points of division of* $\tilde{\mathfrak{J}}$ *are points of continuity of* $q(t)$.

We shall need the following analogue of lemma (3.7) for ordinary functions:

(3.8) *Lemma.* *Let* $f(t)$ *be a bounded real-valued function, Riemann-Stieltjes integrable in* T *with respect to the quadratic measure* dq, *and for each interval* $\Delta \subset T$ *let* $g(\Delta)$ *denote a real number between the supremum and the infimum of* $f(t)$ *in* Δ. *Then, for subdivisions* $\mathfrak{J}$ *of* T *such that each* $\Delta \epsilon \mathfrak{J}$ *has length* $\leq h$, *the integral* $\int_T f(t)dq$ *is the unique limit, as* $h \to 0$, *of*

$$\sum_{\Delta \epsilon \mathfrak{J}} g(\Delta)\mathcal{K}(\Delta, \Delta).$$

Proof. We write $\gamma(f)$ for the sum in question, in so far as it depends on f, and we note that, if $f_1 \leq f \leq f_2$, we can always determine corresponding sums $\gamma(f_2)$, $\gamma(f_2)$ so that

$$\gamma(f_1) \leq \gamma(f) \leq \gamma(f_2).$$

Now by Riemann-Stieltjes integrability of f with respect to dq, given $\epsilon > 0$, there exist step-functions f_1, f_2, which have no common discontinuities with $q(t)$, and which satisfy

$$f_1 \leq f \leq f_2, \quad \int_T \{f_2(t) - f_1(t)\}dq < \epsilon.$$

By the preceding lemma, the integrals of f_1, f_2 are the limits of all the corresponding sums $\gamma(f_1)$, $\gamma(f_2)$. Hence the upper and lower limits of $\gamma(f)$ must lie between these two integrals, which differ by $< \epsilon$ from one another, and *a fortiori* from the integral of f with respect to dq. This last integral is therefore the desired limit, as asserted.

We return to lemma (3.1), and we put it into a convenient form for our main application. We write, when z denotes a (step)-function, and more generally

$$R(z) = \int d\omega \left| \int_T z d_t x(t,\omega) \right|^2 - \int_T |z|^2 dq .$$

(3.9) <u>Lemma</u>. <u>With the notation and hypotheses of lemma</u> (3.1), <u>we have</u>

$$R(z) \leq \sum_{\nu=0}^{N} s_\nu \int_T |z_\nu|^2 dt + 2\Big(\sum_{\nu=0}^{N} h_\nu^{-1} \varphi(h_\nu) \int_T |\zeta_\nu| dt \Big)^2 .$$

<u>Proof</u>. We need only make $h \to 0$ in lemma (3.1), and use lemma (3.7).

§4. <u>The stochastic integral for a canonical process with restricted cross-covariance and self-pairable</u> φ. We now limit ourselves to the case in which $dq \leq dt$; this is the case that we term canonical, in dealing with a process subject to (1.4). In studying it, we need to modify slightly the line of argument of section 2, by using our more elaborate lemma.

We first fix ψ so that φ, ψ is a pair of estimate functions according to the first note, and we determine an infinite decreasing sequence h_ν $\nu = 0, 1, \ldots,$ consisting of powers of $1/2$ and with limit 0, such that

(4.1) $$2\psi(h_\nu) \leq \psi(h_{\nu-1}) \leq 8\psi(h_\nu),$$

and therefore, by (2.1) of the first note, such that

$$h_\nu^{-1}\varphi(h_\nu)\psi(h_{\nu-1}) \le 16(h_{\nu+1})^{-1}\varphi(h_{\nu+1})\psi(h_\nu)\ . \tag{4.2}$$

This subsequence h_ν of the binary sequence $2^{-\nu}$ was introduced in the first note to ensure the behavior of a geometric series for $\sum\psi(h_\nu)$, and (more to the point) for the series of its differences, so that, in the latter, the remainder $\psi(h_\nu)$ be of the same order as the ν-th term $\psi(h_\nu) - \psi(h_{\nu+1})$. We keep the sequence h_ν fixed except for discarding initial terms, so that h_0 may be arbitrarily small.

We denote by $y(t)$ $t \in T_+$ a square integrable function subject to (1.6) (i), and consequently also to (1.6) (ii), for some $\sigma(u)$ which is continuous and increasing for $u \ge 0$ with the initial value $\sigma(0) = 0$. We write again

$$\hat{y}(t) = \frac{1}{k}\int_0^k y(t-u)du;$$

moreover, if $\mathfrak{J}^\nu$ denotes the subdivision of T into equal parts Δ^ν of length h_ν, we define (step h_ν)-functions y_ν, $\hat{y}_\nu$ by stipulating that their values, for each $\Delta^\nu \in \mathfrak{J}^\nu$, are given by

$$y_\nu(\Delta^\nu) = \frac{1}{h_\nu}\int_{\Delta^\nu} y(t)dt, \quad \hat{y}_\nu(\Delta^\nu) = \frac{1}{h_\nu}\int_{\Delta^\nu}\hat{y}(t)dt.$$

Alternatively, we can regard these as the values taken in the interior of Δ^ν, and define y_ν, $\hat{y}_\nu$ as ordinary step-functions; the distinction will be immaterial in this section, as it was in section 2. The integrals of y_ν, $\hat{y}_\nu$ with respect to $d_t x(t,\omega)$ on T will be denoted for short by I_ν, $\hat{I}_\nu$; they exist as integrals of ordinary step-functions, and the same applies to corresponding integrals in dt or dq. Further, from lemma (2.4), we draw the following consequences for $\nu = 1, 2, \ldots$: (i) the integrals on T of $|y_1 - y_\nu|^2$ and $|\hat{y}_1 - \hat{y}_\nu|^2$ in dt, and also in dq (which is $\leq$ dt), are each $< 2\sigma^2(h_1)$; (ii) those of $|y_\nu - y_{\nu+1}|$ and $|\hat{y}_\nu - \hat{y}_{\nu+1}|$ are each $< 2\psi(h_\nu)$; and (iii) those of $|y_1 - \hat{y}_1|$ and $|y_1 - \hat{y}_1|^2$ are, respectively, less than $\psi(k)$ and $\sigma^2(k)$. Finally, from (4.2) above and from (2.5) of the first note, we have

$$\sum_{\nu=1}^{N} h_\nu^{-1}\varphi(h_\nu)\psi(h_{\nu-1}) < 2^4 \sum_{\nu=2}^{N+1} h_\nu^{-1}\varphi(h_\nu)\psi(h_{\nu-1}) < 2^{11} \int_0^{h_1} \frac{\varphi(u)}{u}\, d\psi(u),$$

while in lemma (3.1), the sum of the s_ν, other than s_0, is $\mathfrak{s}(h_1)$, where

$$\mathfrak{s}(k) = \sum_{2^{-\nu} \leq k} 2^\nu \varphi^2(2^{-\nu}) \leq 16 \int_0^k \frac{\varphi^2(u)}{u^2}\, du \leq 32 \int_0^k \frac{\varphi(u)}{u}\, d\varphi(u),$$

by the remarks following (2.6) in section 2 above.

If we now apply lemma (3.9) with $z_0 = \zeta_0 = 0$, first for $\zeta_\nu = y_\nu - y_{\nu-1}$ $\nu \geq 1$, then for $\zeta_\nu = \hat{y}_\nu - \hat{y}_{\nu-1}$ $\nu \geq 1$, and finally, with $N = 1$, $\zeta_1 = 2 = y_1 - \hat{y}_1$, we find that $|I_1 - I_N|^2$ and $|\hat{I}_1 - \hat{I}_N|^2$ are each at most

#932

$$(4.3) \qquad 2\sigma^2(h_1)\{1+\delta(h_1)\} + 2(2^{11}\int_0^{h_1} \frac{\varphi(u)}{u}\, d\psi(u))^2,$$

and that $|I_1 - \hat{I}_1|^2$ is at most

$$(4.4) \qquad 2\sigma^2(k)\{1+\delta(h_1)\} + 2(h_1^{-1}\varphi(h_1)\psi(k))^2.$$

Hence, as in section 2, by making $h_1 \to 0$, it follows that the Hilbert-space limits $I = \lim I_1 = \lim I_\nu$, and $\hat{I} = \lim \hat{I}_\nu$, exist, and that $|I_1 - I|^2$, $|\hat{I}_1 - \hat{I}|^2$ are each at most the quantity (4.3). Further, if we choose h_0 in our binary subsequence, so that

$$h_0 > k \geq h_1,$$

and remark that the second expression in (4.4) is then at most twice the square of the quantity

$$h_1^{-1}\varphi(h_1)\psi(h_0) \leq \sum_{\nu=1}^{N} h_\nu^{-1}\varphi(h_\nu)\psi(h_{\nu-1}) < 2^{11}\int_0^{h_1} \frac{\varphi(u)}{u}\, d\psi(u),$$

we find that each of the expressions $|I_1 - I|^2$, $|\hat{I}_1 - \hat{I}|^2$, $|I_1 - \hat{I}_1|^2$ is at most

$$(4.5) \qquad 2\sigma^2(k)\{1 + \delta(k)\} + 2(2^{11}\int_0^{k} \frac{\varphi(u)}{u}\, d\psi(u))^2.$$

Finally, since, as in section 2, $\hat{I}$ is identical with the elementary Stieltjes integral, we obtain the following result:

(4.6) Theorem. Let $x(t,\omega)$ be subject to (1.4), where φ is self-pairable, and let $y(t)$ satisfy (1.6) in T_+. Further suppose the quadratic measure dq, defined in section 3, satisfies $dq \le dt$, and that the function ψ in (1.6) is such that φ, ψ is a pair of estimate functions according to the first note. Finally, let $\hat{I}$ denote the elementary integral

$$\int_T \hat{y}(t)\,d_t x(t,\omega), \quad \text{where} \quad \hat{y}(t) = \frac{1}{k}\int_0^k y(t-u)\,du.$$

Then, in our Hilbert space, the limit $I = \lim_{k\to 0} \hat{I}$ exists and we have

$$|\hat{I} - I| \le (5+\epsilon)\sigma(k) + \eta, \tag{4.7}$$

where

$$\epsilon = 50\int_0^k \frac{\varphi(u)}{u}\,d\varphi(u), \quad \eta = 10000\int_0^k \frac{\varphi(u)}{u}\,d\psi(u).$$

§5. Additional comments. The non-discriminatory case may be considered to have been treated in [1], and the reader will find there also some indication of the manner in which a reduction from (1.1) to the canonical case (1.3) can be effected. We intend to deal with this more in detail at a later date, and also to discuss the corresponding extension of theorem (4.6). In this connection, we draw attention to an oversight in the third note, where no special assumption is made in regard to dq: it should have been assumed that y is almost everywhere in dq the derivative of its indefinite integral in dt. This is the case for almost every (in dt) translation of a function which satisfies the other conditions stated, as follows from a theorem of Wiener and R. C. Young [3].

Finally there is the further problem of extending our stochastic integral to non-deterministic integrands $y(t,\omega)$. This also we plan to come to at a later date.

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13. ABSTRACT

this report extends

This is a sequel to MRC Technical Report #677 and it is devoted to extending further the notion of stochastic integral in conformity with the ideas and methods of that report. The integrand is taken to be a deterministic function belonging to one or more integrated Lipschitz classes; the stochastic process, with respect to which we integrate is one for which the covariance between non-overlapping fine intervals, and that between an interval and itself, are subject to certain inequalities.

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