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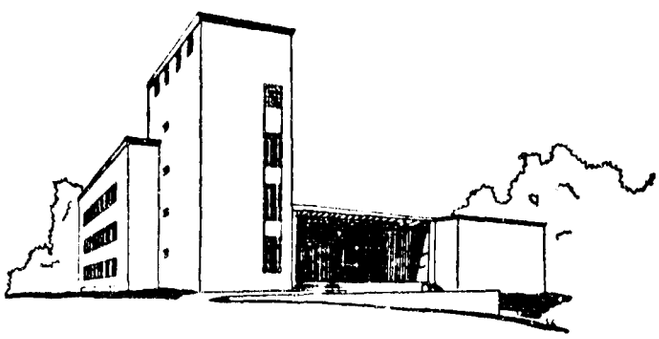

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Sequential Models in Probabilistic Depreciation

by

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Abstract

Probabilistic depreciation is a method of determining the proper depreciation charge in each year of an asset's service life, when the service life is a random variable with known distribution. In this paper, we discuss how the service life distribution is modified as we gain more information about the actual lifetime of the asset. The problem of determining the proper amount to be charged each year to depreciation while at the same time maintaining the proper balance in the accumulated depreciation account is considered. The analysis is done both for a single asset case and for group depreciation. A final section discusses the use of Bayesian analysis for estimating the particular form of the service life distribution while the assets are in service.

1. Introduction

Conventional depreciation methods treat the service life of an asset as a given constant.[1/] When there is some uncertainty as to the actual service life, an average (expected) value is used. The depreciation rates are then calculated based on this given service life. Contrary to this deterministic approach, a probabilistic approach to depreciation has been proposed in [2] where the service life is treated as a random variable which has a given probability distribution. In probabilistic depreciation, the depreciation rates are calculated for each possible service life of the asset and then the weighted average is computed using the service life probabilities as weights. Thus if the asset's life is equally likely to be 1, 2, or 3 years, the depreciation rate for the first year under the probabilistic straight line method is the average of 100%, 50%, and 33 1/3 % or 61 1/9 %. This is in contrast to the conventional straight line method which computes an average service life first, 2 years in this case, and then calculates the depreciation rate as 50% for the first and second year.

It was demonstrated in [2] that for single asset depreciation, conventional depreciation methods result in underdepreciation in the earlier years of an asset's service life. In the case of group depreciation, conventional methods will typically result in underdepreciation throughout the service life of the items in the group. The analysis in [2] computed the depreciation rate for each year based only on a single estimate of the probability distribution of the service life; namely the

one known when the asset was first put into service. As the asset' service life expires, we obtain more information about the actual probability distribution governing the particular asset under study and it becomes reasonable to modify the probability distribution to include the increased knowledge that we have.

In the numerical example above, if the asset survives the first year, we know that the 1 year service life situation did not occur and thus need to consider only the cases of 2 and 3 years of the service life. Therefore, the conditional probability distribution of the service life, given that the asset has survived the first year, is 0.5 for the 2 year life and 0.5 for the 3 year life. Hence, the first year depreciation rate for the asset after it has survived the first year in service is (1/2) (1/2 + 1/3) = 5/12 = 0.42 under the straight line method.

In this paper, we analyze how the depreciation patterns might be affected by such a "sequential probabilistic depreciation" approach as compared with a "static probabilistic" approach discussed in [2]. Section 2 discusses this problem for a single asset case and Section 3 for group depreciation. In Section 4, an application of Bayesian analysis for estimating the service life distribution is considered.

2. Sequential Probabilistic Depreciation: Single Asset Case

Before we proceed in our analysis, we shall define our criteria in selecting depreciation methods. In this paper, we want to set aside the issue of whether or not depreciation rates should be based on the

consumption of service potentials, or on the decline of the market values, or on any other factors. We assume that these economic considerations have been included in the depreciation vector $h_j = (h_{1j}, h_{2j}, \cdots, h_{jj})$ where h_{ij} is the proportion of the depreciable cost (acquisition cost less estimated salvage value) to be depreciated in the i -th year if the service life of the asset is j years. We shall assume that $h_{ij} \geq 0$ for all $i = 1,2,..,j$ and $j = 1,2...$, and that $\sum_{i=1}^{j} h_{ij} = 1$, so that the asset is fully depreciated by the end of its service life. For the straight line method, $h_{ij} = 1/j$; for the sum-of-years-digits method, $h_{ij} = 2(j-i+1)/j(j+1)$; and for the double-declining balance method, $h_{ij} = (1 - 2/j)^{i-1}(2/j)$.

With these assumptions, we consider the situation in which the service life j is not given with certainty but is a random variable with a given probability distribution. The criterion used in selecting a depreciation vector is that the depreciation rate under the selected depreciation vector be, year for year, equal to the expected value of the depreciation rate under the given probability distribution. We refer to this property as the unbiasedness of the depreciation rates. The method of deriving unbiased probabilistic depreciation vectors when the asset is first placed in service is discussed in [2].

In this paper, we shall apply the same criterion of unbiasedness. However, by using the additional information that is available after some number of years, we shall try to minimize the discrepancies between the ex ante depreciation rates based on the prior service life distribution and the ex post depreciation rates based on the actual experience with the asset.

Consider an asset with a service life distribution given by $(p_1, p_2, \ldots,)$ where p_j is the probability that the asset is retired at the end of the j -th year in service. We assume that retirement occurs only at the end of a year.

In order to compute the unbiased estimate of the depreciation rate for the i -th year, we must consider three cases. First, if the asset is retired in the i-1 st year or earlier, we have $d_i^- = 0$ where d_i^- is the depreciation rate for the i -th year when the asset has been retired in an earlier year. If the asset is retired in the i -th year, then $d_i^o = h_{ii}$ where d_i^o is the depreciation rate for the i -th year when the asset has been retired at the end of the i -th year. Note that in both of these cases, the service life was known with certainty so that there is no averaging required to compute the proper depreciation rate.

The third case occurs if the asset survives through the i -th year so that the service life is known to be greater than i. In this situation the probability that the service life is j $(>i)$ years is, by the definition of conditional probability, $p_j / \sum_{k=i+1}^{\infty} p_k = p_j/S_i \quad j = i+1, i+2, \ldots,$ where $S_i = \sum_{k=i+1}^{\infty} p_k$ is the a priori probability that the asset is not retired in the first i years.

If, in fact, the asset is retired in the j -th year, the depreciation rate in the i -th year should be h_{ij}. Since we do not, at this stage, know what the actual service life will be, we compute the expected depreciation rate by averaging over all possible service lives. Thus,

$$d_i^+ = \sum_{j=i+1}^{\infty} h_{ij} \, (p_j/S_i) = (1/S_i) \sum_{j=i+1}^{\infty} h_{ij} p_j$$

where d_i^+ is the depreciation rate for the i -th year when the asset has survived through the i -th year.

Since whether or not the asset is retired before, at the end of, or after the i -th year is not known at the time the asset is first put into service, the actual depreciation rate for the i -th year is a random variable. The expected value of the actual i -th year depreciation rate is given by

$$\sum_{j=1}^{i-1} p_j\, d_i^- + p_i\, d_i^o + \sum_{j=i+1}^{\infty} p_j\, d_i^+$$

$$= p_i h_{ii} + \sum_{j=i+1}^{\infty} p_j\, h_{ij}$$

$$= \sum_{j=i}^{\infty} p_j\, h_{ij}$$

which is equal to the static probabilistic depreciation rate for the i -th year as derived in [2]. Note that the sum of these expected depreciation rates for i = 1,2,.., is unity since

$$\sum_{i=1}^{\infty} \sum_{j=i}^{\infty} p_j\, h_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{j} p_j\, h_{ij} = \sum_{j=1}^{\infty} p_j \sum_{i=1}^{j} h_{ij} = 1 .$$

Suppose that the depreciation rate for the i -th year when the asset has survived the i -th year was chosen to be d_i^* ($\neq d_i^+$) . Then the mean squared deviation between the actual depreciation rate and the proper depreciation rate is given by

$$(1/S_i) \sum_{j=i+1}^{\infty} p_j\, (h_{ij} - d_i^*)^2 .$$

Then,

$$(1/S_i) \sum_{j=i+1}^{\infty} p_j\, [h_{ij} - d_i^*]^2 = (1/S_i) \sum_{j=i+1}^{\infty} p_j [(h_{ij} - d_i^+) + (d_i^+ - d_i^*)]^2$$

$$= (1/S_i) \{ \sum_{j=i+1}^{\infty} p_j\, (h_{ij} - d_i^+)^2 + (d_i^+ - d_i^*)^2 \sum_{j=i+1}^{\infty} p_j + 2(d_i^+ - d_i^-) \sum_{j=i+1}^{\infty} p_j (h_{ij} - d_i^+) \}$$

$$= (1/S_i) \{ \sum_{j=i+1}^{\infty} p_j (h_{ij} - d_i^+)^2 + (d_i^+ - d_i^*)^2\, S_i \},$$

since $\sum_{j=i+1}^{\infty} p_j (h_{ij} - d_i^+) = 0$ by the choice of d_i^+ .

Thus, the mean squared deviation between the actual depreciation rate and the proper depreciation rate is minimized when we choose $d_i^* = d_i^+$.

So far we have concentrated on computing the actual depreciation rate for the i -th year (i = 1,2,..,) to give as accurate a figure as possible for use in the income statement. We now consider what the accumulated depreciation rate should be at the end of the i -th year after the depreciation rate for the i -th year has been charged.

Let a_i be the proportion of the depreciable cost that has already been depreciated by the end of the i -th year. Clearly, if the asset has been retired by the end of the i -th year, a_i should equal 1 . If the asset is still in service at the end of the i -th year, a_i should equal 1 minus the sum of the expected depreciation rates in each of the remaining years in order for a_i to be an unbiased estimate of the proper accumulated depreciation rate when the asset is still in service at the end of the i -th year.

The expected value of the depreciation rate in the k -th year given that the asset is in service at the end of the i -th year is

$$(1/S_i)\left(\sum_{j=i+1}^{k-1} p_j \cdot 0 + p_k\, h_{kk} + \sum_{j=k+1}^{\infty} p_j\, d_k^+\right)$$

$$= (1/S_i) \sum_{j=k}^{\infty} p_j\, h_{kj} \cdot$$

Therefore, the sum of the expected depreciation rates in all years after the i -th is

$$(1/S_i) \sum_{k=i+1}^{\infty} \sum_{j=k}^{\infty} p_j\, h_{kj} = (1/S_i) \sum_{j=i+1}^{\infty} p_j \sum_{k=i+1}^{\infty} h_{kj} \cdot$$

Thus, $a_i^+ = 1 - (1/S_i) \sum_{j=i+1}^{\infty} p_j \sum_{k=i+1}^{j} h_{kj}$

$$= (1/S_i) \sum_{j=i+1}^{\infty} p_j \sum_{k=1}^{i} h_{kj} \quad ,$$

since $\sum_{k=1}^{j} h_{kj} = \sum_{k=1}^{i} h_{kj} + \sum_{k=i+1}^{j} h_{kj} = 1$.

Note that a_i^+ equals the sum of the expected depreciation rates that should have been taken in years 1 to i if it were known that the asset would still be in service at the end of the i -th year.

Knowing a_i^+ , we can compute the expected accumulated depreciation rate after i years to be

$$\sum_{j=1}^{i} p_j \cdot 1 + \left(\sum_{j=i+1}^{\infty} p_j\right) a_i^+$$

$$= \sum_{j=1}^{i} p_j + S_i(1/S_i) \sum_{j=i+1}^{\infty} p_j \sum_{k=1}^{i} h_{kj}$$

$$= \sum_{j=1}^{i} p_j \sum_{k=1}^{j} h_{kj} + \sum_{j=i+1}^{\infty} p_j \sum_{k=1}^{i} h_{kj}$$

$$= \sum_{k=1}^{i} \sum_{j=k}^{i} p_j h_{kj} + \sum_{k=1}^{i} \sum_{j=i+1}^{\infty} p_j h_{kj}$$

$$= \sum_{k=1}^{i} \sum_{j=k}^{\infty} p_j h_{kj}$$

which is exactly equal to the accumulated depreciation rate under the static probabilistic method as derived in [2]. Since this quantity, which is less than unity, is a convex combination of 1 and a_i^+ , we know that the accumulated depreciation for nonretired assets is always less using sequential probabilistic depreciation (a_i^+) than when using static probabilistic depreciation.

Since the accumulated depreciation rate and depreciation rates for each year were derived separately, it is not, in general, true that the accumulated depreciation rate is the sum of depreciation rates in each year. If the asset is still in service at the end of year i-1, its accumulated depreciation rate is given by a_{i-1}^+ . If it survives the i -th year, the depreciation rate is given by d_i^+ . The difference, $a_i^+ - (a_{i-1}^+ + d_i^+)$, may be considered as an entry to an account for

adjusting prior income, and is attributable to the uncertainty in the service life of the asset. Similarly, if the asset is retired in the i -th year [2/] , the depreciation rate is d_i^o and the adjustment is given by $1 - (a_{i-1}^+ + d_i^o)$.

An interesting feature of this method is that a_i^+ does not necessarily increase as i increases. For example, if $p_2 = p_{10} = 0.5$ (all other $p_i = 0$) , and if straight line depreciation is used, the accumulated depreciation rate after one year, a_1^+ , is $.5(1/2) + .5(1/10) = 3/10$. If the asset is still in service at the end of the second year, its accumulated depreciation rate a_2 should be $1\cdot(2/10)$, a decrease of $1/10$ over a_1^+ . Since the asset is still in service at the end of the second year, i.e., the asset's service life is now known to be 10 years with certainty, the first year's depreciation rate should have been $1/10$. This plus the depreciation rate in the second year of $1/10$ equals a_2^+. The difference between $a_1^+ = 3/10$ and what it should have been, $1/10$, is attributable to the uncertainty in the service life. Thus a proper accounting treatment is to depreciate $1/10$ in the second year and make a debit entry of $2/10$ to the accumulated depreciation rate, crediting to an account for adjusting prior income.

In general, the expected value of the adjustment required in the i -th year is

$$
\begin{aligned}
&[a_i^+ - (a_{i-1}^+ + d_i^+)]\, S_i + [1 - (a_{i-1}^+ + d_i^o)]\, p_i \\
&= \sum_{j=i+1}^{\infty} p_j \sum_{k=1}^{i} h_{kj} - \sum_{j=i}^{\infty} h_{ij}\, p_j + p_i - \sum_{j=i}^{\infty} p_j \sum_{k=1}^{i-1} h_{kj} \\
&= \sum_{j=i+1}^{\infty} p_j \sum_{k=1}^{i} h_{kj} - \sum_{j=i}^{\infty} p_j \sum_{k=1}^{i} h_{kj} + p_i \\
&= 0 \ ,
\end{aligned}
$$

so that the adjustment is due solely to the inherent fluctuations caused by the variability in the service life of the asset.

3. Sequential Group Depreciation

The sequential probabilistic depreciation method developed in Section 2 can be easily applied to group depreciation in which assets of a similar nature are grouped together and depreciated in a single account. At the end of any year, the original group of assets can be partitioned into three groups; those that were retired in year i-1 or earlier, those that have just been retired in year i , and those that are still in service. The items in the first group have already been fully depreciated and no further charges are required. Those items which have just been retired are charged at a rate equal to $d_i^o = h_{ii}$ while those items still in service are charged at a rate equal to

$$d_i^+ = (1/S_i) \sum_{j=i+1}^{\infty} h_{ij} P_j .$$

Let N be the number of items originally placed in service at the start of the first year. Let n_i be the number of items that have been retired at the end of the i -th year and let m_i be the number of items that were retired just at the end of the i -th year. Thus,

$$n_i = n_{i-1} + m_i .$$

Therefore the i -th year depreciation rate, based on the total depreciable cost for the N items, is given by

$$\frac{n_{i-1}}{N} d_i^- + \frac{m_i}{N} d_i^o + \frac{(N - n_i)}{N} d_i^+$$

$$= \frac{m_i}{N} h_{ii} + \frac{(N - n_i)}{NS_i} \sum_{j=i+1}^{\infty} h_{ij} P_j ,$$

where d_i^- , d_i^o , and d_i^+ are as defined in Section 2. In order to compute the expected depreciation rate in the i -th year, we need to compute only $E(m_i/N)$ and $E(N - n_i/N)$ since d_i^o and d_i^+ are fixed positive numbers and $d_i^- = 0$. The probability mass functions for the random variables m_i and $N - n_i$ are given by a binomial distribution with parameters $n = N$ (for both variables) and $p = p_i$ and S_i respectively. Since the expected value of a binomial distribution with parameters n and p is np we have that

$$E(m_i/N) = p_i \qquad \text{and}$$

$$E(N - n_i)/N = S_i \quad .$$

Therefore the expected i -th year sequential group depreciation charge is given by

$$[p_i\ h_{ii}] + S_i\ (1/S_i) \sum_{j=i+1}^{\infty} h_{ij}\ p_j$$

$$= \sum_{j=i}^{\infty} h_{ij}\ p_j \quad ,$$

the static group probabilistic rate computed in [2].

A similar analysis shows that the accumulated depreciation rate after i years is

$$\frac{n_i}{N} \cdot 1 + \frac{(N - n_i)}{N}\ a_i^+ = \frac{n_i}{N} + \frac{[(N - n_i)]}{NS_i} \sum_{j=i+1}^{\infty} p_j \sum_{k=1}^{i} h_{kj} \cdot$$

Therefore, the expected accumulated depreciation rate after i years is

$$(1 - S_i) + S_i(1/S_i) \sum_{j=i+1}^{\infty} p_j \sum_{k=1}^{i} h_{kj}$$

$$= \sum_{j=1}^{i} p_j + \sum_{j=i+1}^{\infty} p_j \sum_{k=1}^{i} h_{kj}$$

$$= \sum_{k=1}^{i} \sum_{j=k}^{\infty} p_j\ h_{kj}$$

the accumulated depreciation rate after i years under the static probabilistic method (see [2]).

As in the single asset case, an adjustment is generally needed each year to keep the accounts balanced. The amount of this adjustment is

$$(1 - h_{ii}) \left[\frac{m_i - p_i(N - n_{i-1})}{NS_{i-1}}\right] + \sum_{j=i+1}^{\infty} p_j \sum_{k=1}^{i-1} h_{kj} \left[\frac{N-n_i}{NS_i} - \frac{N-n_{i-1}}{NS_{i-1}}\right]$$

and one can readily check that the expected value of this adjustment is zero.

4. Bayesian Analysis

For the calculations described in the previous sections, we have implicitly assumed that the service life distribution, $\{p_j ; j = 1,2,..,\}$, was precisely known. In [2] we discussed the use of parametric distributions such as the normal, rectangular, Poisson, and geometric to provide convenient representations for the service life distributions. In using these distributions, one makes the assumption that the probabilities of asset retirement follow about the pattern indicated by the distribution so that only one or two parameters need to be specified in order to completely characterize the distribution.

In practice, one may be willing to accept the assumption that a particular distribution is appropriate and yet still be unsure about the parameters that characterize the distribution. In this case, Bayesian analysis might prove useful.[3/]

Assume we have a group of N identical assets each having an independent and identically distributed service life distribution given by $\{p_j ; j=1,2,..,\}$ Then, the probability that exactly n_1 are retired in the first year (assuming $p_1 > 0$) is given by

$$\binom{N}{n_1} p_1^{n_1} (1 - p_1)^{N-n_1} .$$

Assume also that p_1 is not known for certain. The question is how we can use our prior information on p_1 with the actual retirement number, say n_1, in the first year to get a new estimate on p_1 and indirectly on $p_2, p_3, \ldots$. We could then continue in this manner at the end of each year to note the retirement pattern and update our estimate of the parameter(s) of the distribution under study.

Let $f_1(p_1)$ be our prior density function on p_1 where

$f_1(p_1) \geq 0$ for $0 \leq p_1 \leq 1$ and

$\int_0^1 f_1(p_1)\, dp = 1$.

After observing n_1 retirements in the first year, our new (posterior) density function on p_1 (denoted by $f_1(p_1)$) is given by

$$f_1(p_1) = \frac{p_1^{n_1}(1-p_1)^{N-n_1} f(p_1)}{\int_0^1 p_1^{n_1}(1-p_1)^{N-n_1} f(p_1)\, dp} \qquad 0 \leq p_1 \leq 1 \ .$$

In general, however, one would not be solely interested in p_1 but rather in what the information on p_1 implies about the parameter(s) that are specifying the complete service life distribution.

For example, suppose one feels that the geometric distribution is a reasonable approximation to the service life distribution. Therefore

$$p_j = (1 - p)^{j-1} p \ , \quad j = 1,2,\ldots, \ .$$ 4/

One way of estimating p is by noting that the expected lifetime is given by $\sum_{j=1}^{\infty} j\, p_j = 1/p$. However, one may not have had sufficient experience with items similiar to the assets currently being used to place complete faith in a point estimate of p . Therefore one might wish to select a prior density on p and let the experience with the actual items now in service also influence our estimate of p .

For exposition purposes, it is convenient in this situation to choose a beta distribution as the prior density for p[5/] ; i.e.,

$$f(p) = A_0\, p^{r-1}(1-p)^{t-r-1} \quad ; \; 0 \le p \le 1; 0 < r < t \;,$$

where $A_0 = \dfrac{(t-1)!}{(r-1)!\,(t-r-1)!}$.

The parameters r and t may be estimated by using the relations that

$E(p) = r/t$ and

$Var(p) = r/t^2$.

Another interpretation of how r and t may be estimated will become clearer as we proceed.

Therefore, if n_1 out of N items are retired in the first year in service,

$$f_1(p) = \frac{A_0\, p^{n_1}(1-p)^{N-n_1} p^{r-1}(1-p)^{t-r-1}}{A_0 \int p^{n_1}(1-p)^{N-n_1} p^{r-1}(1-p)^{t-r-1}\, dp}$$

$$= \frac{(N+t-1)!}{(r+n_1-1)!\,(N+t-r-n_1-1)!}\; p^{r+n_1-1}(1-p)^{N+t-(r+n_1)-1}$$

which is a β- distribution with parameters $r+n_1$, $N+t$.

At the start of the second period, we have $N - n_1$ items still in service. Suppose that $n_2 - n_1$ (≥ 0) items fail in the second period. The probability of this event, given p , is

$$\binom{N-n_1}{n_2-n_1} p^{n_2-n_1}(1-p)^{N-n_2} \;.$$

Therefore if our prior on p is given by $f_1(p)$ above, the posterior (after the second period) density of p is

$$f_2(p) = A_2\, p^{n_2-n_1}(1-p)^{N-n_2}\, p^{r+n_1-1}(1-p)^{N+t-(r+n_1)-1}$$

$$= A_2\, p^{r+n_2-1}(1-p)^{N+(N-n_1)+t-(n_2+r)-1} ,$$

where $$A_2 = \frac{(N+N-n_1+t-1)!}{(r+n_2-1)!\ (2N+t-n_1-r-n_2-1)!} .$$

Proceeding in this manner, it can be readily shown by induction that if $n_j - n_{j-1}$ items have been retired in the j-th period ($j = 1,2,..,k$; $n_0 = 0$), the posterior density of p after the k-th period is given by

$$f_k(p) = A_k\, p^{r+n_k-1}(1-p)^{t+\sum_{i=0}^{k-1}(N-n_i)-(n_k+r)-1}$$

where

$$A_k = \frac{(t+\sum_{i=0}^{k-1}(n-n_i)-1)!}{(r+n_k-1)!\ (t+\sum_{i=0}^{k-1}(N-n_i)-(r+n_k)-1)!} .$$

Note that $f_k(p)$ dpends on the entire history of retirements (as represented by $n_1,..,n_k$) and not just on n_k, the cumulative number of retirements at the end of the k-th year.

One can see that the initial parameters r and t can be interpreted as if one had a prior experience of observing t of these identical items in use at the start of a period with r of them having been retired by the end of the period. Thus the greater t is, the larger the influence of the prior density on the posterior densities of later periods. In the limit, where we know p with certainty so that a Bayesian analysis is not useful, r and t both go to infinity with the ratio r/t remaining equal to p.

Finally, to compute the probabilistic depreciation rate and accumulated depreciation rate after the k-th period, we need to determine p_j for $j = k+1,k+2,..,$. We know that for a given value of p, $p_j|p = (1-p)^{j-k-1}p$. Therefore, to find the unconditional value of p_j, we multiply by the density

function for p , $f_k(p)$ and integrate over 0 to 1 ; i.e.,

$$p_j = \int_0^1 (p_j|p)\, f_k(p)\, dp = A_k \int_0^1 p^{r+n_k}(1-p)^{[t+\sum_{i=0}^{k-1}(N-n_i)-(r+n_k)+(j-k-1)-1]}\, dp$$

$$= (r+n_k)\, \frac{[t+\sum_{i=0}^{k-1}(N-n_i)-1]!}{[t+\sum_{i=0}^{k-1}(N-n_i)+(j-k)-1]!}\, \frac{[t+\sum_{i=0}^{k-1}(N-n_i)-(n_k+r)+(j-k-1)-1]!}{[t+\sum_{i=0}^{k-1}(N-n_i)-(n_k+r)-1]!}$$

for $j = k+1, k+2, \ldots,$.

Thus, for example,

$$p_{k+1} = \frac{r+n_k}{t+\sum_{i=0}^{k-1}(N-n_i)} \quad \text{and}$$

$$p_{k+2} = \frac{r+n_k}{t+\sum_{i=0}^{k-1}(N-n_i)}\, \frac{t+\sum_{i=0}^{k-1}(N-n_i)-(n_k+r)}{t+\sum_{i=0}^{k-1}(N-n_i)+1}$$

so that if k = 1 (have observed one period with n_1 failures), our revised estimates of p_2 and p_3 are given by

$$p_2 = \frac{r+n_1}{t+N} \quad \text{and} \quad p_3 = \frac{r+n_1}{t+N}\, \frac{t+N-(n_1+r)}{t+N+1} .$$

(One can easily verify that the prior estimates of p_2 and p_3 are given by

$$p_2 = \frac{r}{t} \text{ and } p_3 = \frac{r}{t}\, \frac{t-r}{t+1} .)$$

The values of p_j for $j = k+1, k+2, \ldots,$ are then inserted into the formulas developed in Section 3 to compute the k-th year depreciation rate and accumulated depreciation rate under the sequential Probabilistic method.

In the above example, the posterior was always a β- distribution with the two parameters a simple linear function of the prior estimates r and t , and the observed retirement history. The ease with which we obtained the posterior density of p given $n_1, \ldots, n_k$ was due to the special structure that was assumed: the geometric distribution of the service lives and the β- distribution as the prior density on p . Departures from either one of these assump-

tions will result in considerably more complication in the computation of the posterior density function.

For example, in [2] we postulated the use of a Poisson distribution as a good approximation to actual mortality curves of equipment. With the Poisson distribution, we define $p_i = [\lambda^{i-1}/(i-1)!]\, e^{-\lambda}$ for $i = 1,2,..,$. We would like to be able to treat the case in which λ is a random variable specified initially by a prior density function $f(\lambda)$. The authors are unaware of any prior density for λ that after observing a sequence of retirements will yield a posterior density for λ of the same general form. However, if a prior density on λ is chosen, it is still possible to carry out the required computations to determine, numerically, the posterior density of λ .

Let λ be specified by a prior density, $f(\lambda)$. Then with N identical items, the probability that n_1 are retired in the first period is given by

$$\Pr[n_1 \mid N,\lambda] = \binom{N}{n_1}(e^{-\lambda})^{n_1}(1 - e^{-\lambda})^{N-n_1} .$$

Therefore after observing n_1 retirements, the posterior density of λ is

$f_1(\lambda) = A_1(e^{-\lambda})^{n_1}(1 - e^{-\lambda})^{N-n_1} f(\lambda)$ where A_1 is chosen so that $\int_0^\infty f_1(\lambda)d\lambda = 1$.

In general, let $p_k(\lambda)$ be the probability, given λ , that an item is retired in the k -th period given that it is still in service at the end of the k-1 st period. Therefore,

$$p_k(\lambda) = [\lambda^{k-1}/(k-1)!]/[\sum_{j=k}^{\infty} \lambda^{j-1}/(j-1)!\,] .$$

Also, let $f_{k-1}(\lambda)$ be the posterior density of λ based on the actual retirement experience up through the k-1 st period. Then if $N - n_{k-1}$ items are still in service at the start of the k -th period and $n_k - n_{k-1}$ retire-

ments are observed in the k -th period, the new posterior density of λ is given by

$$f_k(\lambda) = A_k [p_k(\lambda)]^{n_k - n_{k-1}} [1 - p_k(\lambda)]^{N-n_k} f_{k-1}(\lambda) \text{ for } \lambda > 0 \; ; \; k = 1,2,.., ,$$

with A_k chosen so that $\int_0^\infty f_k(\lambda)\, d\lambda = 1$.

Thus to compute the sequential depreciation rate for the k -th period, we need to determine the unconditional value of p_j for $j = k+1,k+2,...,$ as given by

$$p_j = \int_0^\infty \frac{[\lambda^{j-1}/(j-1)!]}{\sum_{i=k+1}^\infty [\lambda^{i-1}/(i-1)!]} f_k(\lambda)\, d\lambda \text{ for } j = k+1,k+2,..., \quad .$$

The above recursive computations are straightforward, albeit tedious, and can be programmed for solution on a digital computer.

FOOTNOTES

1. For a comprehensive treatment of conventional depreciation methods, see Grant and Norton [1] .

2. If the asset is retired in one of the first few years of its possible service life, this adjustment might be quite substantial. However, to offset this effect, the salvage value may be larger than expected because of the few years in service and hence may reduce the amount of this writeoff.

3. See Raiffa and Schlaifer [3] .

4. A more general form of the geometric distribution might allow for a minimum lifetime of m years so that $p_j = 0$ for $j = 1,2,..,m$, and $p_j = (1 - p)^{j-m-1}p$ for $j = m+1,m+2,...,$.

5. The beta distribution is the conjugate prior (see [3]) for the Bernoulli process being described here where the probability of retirement in each year is a constant, p , for those items which are still in service.

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13 ABSTRACT

Probabilistic depreciation is a method of determining the proper depreciation charge in each year of an asset's service life, when the service life is a random variable with known distribution. In this paper, we discuss how the service life distribution is modified as we gain more information about the actual lifetime of the asset. The problem of determining the proper amount to be charged each year to depreciation while at the same time maintaining the proper balance in the accumulated depreciation account is considered. The analysis is done both for a single asset case and for group depreciation. A final section discusses the use of Bayesian analysis for estimating the particular form of the service life distribution while the assets are in service.

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