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"NOTES ON THE METHODOLOGY FOR GENERATION

OF THE REPRESENTATIVE OF A SET"

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Introduction

When a collection of data-points is related to a two-dimensional system of reference and through each point we raise segments perpendicular to the locus of the set, the length of each segment being equal to the corresponding value of some variate, the result is what is called a continuous distribution in space of the third variate over the locus of the set of data-points. Thus, if the data-points represent location, referred to a latitude-longitude system or some other system, and the third variate is population, or income, or potential of either, etc., the distribution is assumed continuous and is represented by a surface of population, or income, or potential, etc. distribution.

Let $(x,y)_i$ represent the coordinate system used for location and p_i the value of some variate, which is said to be the measure of a population at the point $(x,y)_i$. Then, the locus of points $(x,y,p)_i$ is the surface of the continuous distribution of p_i over $(x,y)_i$.

Assume other populations at the same points. Let them be q_i, t_i, u_i, etc. Question: how do we represent, simultaneously, the continuous distribution of p_i, q_i, t_i, etc. over $(x,y)_i$?

This paper proposes alternative methods, suitable for computer programming, and suggests them for that type of continuous spatial distribution.

PRESENT METHODOLOGY

1. Two-dimensional distribution

Let $(x)_i$ represent the coordinate system which locates a set of points. Therefore, the locus of these points is a line. We wish to represent the distribution of a population p_i along this line.

1.1 $(x)_i$ is a straight line

Through each point we simply draw a perpendicular to the line and mark upon it the corresponding value of p_i. The locus of the end-points represent the distribution $(x,p)_i$. Figure 1.
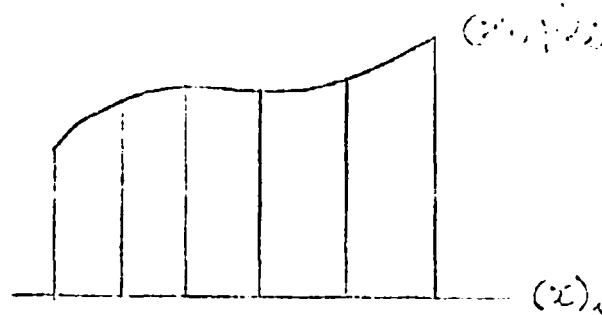

Figure 1.

Assuming now that the curve obtained is to be utilized as another coordinate system for the set of points, along that line, to which it corresponds another set of values of the population p_i, this new distribution, which we will notate as $(x,p)_{1,2}$, can be represented along the normals drawn at each point to the curve. We have then, the case when
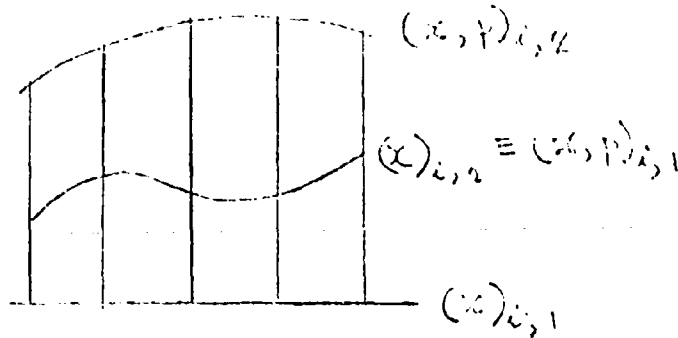
1.2. $(x)_i$ is a curve (plane)

See figure 2.

Figure 2.

This type of representation is the one used by Warntz in the construction of the minimum land acquisition cost routes [1], a method derived from the constructions of the Huygens' diagram for the determination of the path of refracted light [2].

The use of this method of representing a succession of distributions over a line has, as the two examples above mentioned indicated, useful application. In both cases, the relationship among the distributions is utilized for the determination of a minimum path through points in distinct distributions. See figure 3.
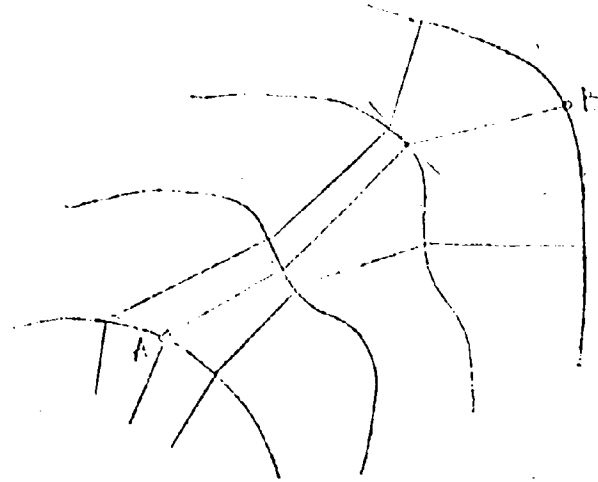
Figure 3.

2. Three-dimensional distribution

Let $(x,y)_i$ represent the coordinate system which locates a set of points. We wish to represent the distribution of a population p_i over the locus of the points.

2.1 $(x,y)_i$ is a plane

Examples of this type of distribution are many. The population p_i need not be a spatial variate, as discussed in several papers of this theoretical series.

Through each point we raise a perpendicular to the plane, mark the corresponding value of p_i and find the locus of the end points. The curved surface obtained represents the continuous spatial distribution $(x,y,p)_i$.

2.2 $(x,y)_i$ is a curved surface

The usual procedure is to determine at each point the normals to the surface, mark on them the corresponding values of p_i in order to obtain another curved surface which represents the distribution $(x,y,p)_i$.

Each of these methods of representing plane and spatial distributions have been successfully used in computer graphics. SYMAP is one of the techniques used in the generation of surfaces representing spatial distributions.

It remains to be seen if there are or not other alternatives, in the representation of these distributions, which may (or may not) facilitate their understanding and study.

NOTES ON THE METHODOLOGY FOR

GENERATION OF THE REPRESENTATIVE

OF A SET

Santos Reis' paper [3], an essay on the generalization of
the representative of a set of points, covers the alternative
approaches which we are seeking. This section of this paper
deals, therefore, only with one of such alternatives, pointing
out some of its advantages over the present methodology. If
the alternative here suggested does not satisfy specific cases
it is always possible to search for another in Santos Reis'
generalized discussion.

But in addition to the generalization indicated above, and
which we will use to derive the suggested methodology, we will
make an attempt in spelling out some classificatory characteristics,
based on geometric transformations and properties, which will
serve to distinguish or to identify the locus of the points
identified by the coordinate systems $(x)_i$ and $(x,y)_i$.

3. On lines and curves $(x)_i$

We shall distinguish between the straight line and the curve
(plane) in the following manner, even though both are one-
dimensional geometric forms:

3.1 A line does not define the plane on which it lies.

3.2 A curve (plane) defines the plane on which it lies.

The observation, naive and simple as it seems, provides,
however, the way by which we shall attain an alternative method
in the derivation of the representative.

Consider a straight line. In this case, it makes no

difference in which direction, in space, we draw the perpendiculars to such a line, in order to obtain the representative $(x,p)_i$. See figure 4.
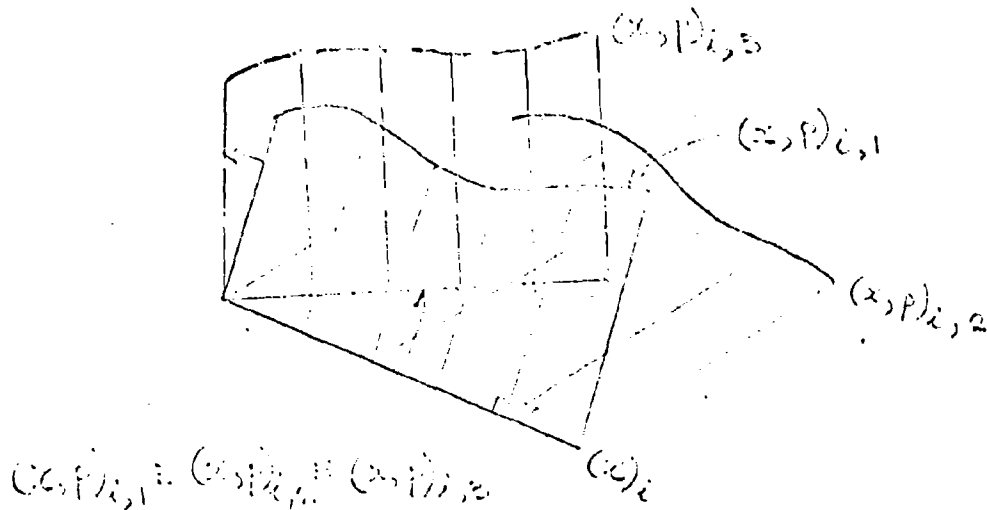
Figure 4.

Consequently, the configuration of the representative is always the same and is independent of the orientation of the plane defined by $(x)_i$ and the representative $(x,p)_i$. We can also say that the same plane is defined by the $(x)_i$ and the direction perpendicular to it.

If we consider this perpendicular direction as a coordinate system for a set of points $(y)_i$, we have the customary (x,y) Cartesian system of coordinates. The representative $(x,p)_i$ is, however, an invariant, being independent of the orientation of the two perpendicular coordinate sets of points $(x,y)_i$.

The above underlining is important because this fact has been overlooked in recent articles discussing the dependency

or independency of the regression line fitted to a set of points coordinated in a (x,y) Cartesian system.

Notice that the representative $(x,p)_1$ can be <u>anything</u>, including the regression line! We have indicated that this representative is an invariant. <u>Therefore, the coefficient of correlation is also an invariant.</u> (Nevertheless, in an article published in the <u>Annals of the Association of American Geographers</u>, by Court [4] commenting on the article by Porter [5, with our comments] Court states that regression lines depend on the axial orientation...The point missed by Court is that (x) and (y) are, actually, the locus of the two sets of points; therefore a rotation of these axes, constitutes a rotation of each set. What Court suggested was a separation of the set of the points in the plane - the representative $(x,p)_1$, according to the usual nomenclature - from the two sets represented on the axis $(x)_1$ and $(y)_1 = (p)_1$. This, of course, is a mistaken interpretation of the one-to-one relationship between image [the representative $(x,p)_1$] and counter-image [the sets $(x)_1$ and $(p)_1 = (y)_1$]. See reference [3].

The methodology derived from the use of $(x)_1$ as a straight line does not, therefore, provide any means of unveiling new properties of the $(x,p)_1$ representative.

The case when $(x)_1$ is a curve (plane) was previously discussed, indicating the "state of the art" in this methodology. However, a different approach can be tried based on the following:

<u>The representative $(x,p)_1$ is obtained on the perpendiculars to the plane of the curve $(x)_1$.</u> More about this later.

4. On planes and curved surfaces

We shall distinguish between the plane and the curved surface in the following manner:

4.1 The plane does not determine the three-dimensional space to which it belongs.

4.2 A curved surface whose curvature, everywhere, is constant and equal to zero does not determine the three-dimensional space to which it belongs.

This is justified by the fact that surfaces enjoying this property have an intrinsic geometry identical to that of the plane (zero curvature everywhere). These surfaces may be made to coincide with a plane and are called, of course, developable surfaces.

4.3 A curved surface whose curvature is different from zero (constant or not) determines the three-dimensional to which it belongs.

We could regroup the three statements above, combining the first two, considering in one group surfaces of zero curvature (which do not determine the 3-D space) and in another group the surfaces whose curvature differs from zero (which determine the 3-D space).

Parallel considerations can now be made, about these two groups of surfaces, to those stated for the case of the straight line and the curve (plane).

Current methodology, as anteriorly discussed, does not make such distinctions when deriving the representative $(x,v,n)_j$

over these surfaces. In other words, the treatment is similar to the one given to the straight line and the curve (plane). It simply recommends the determination of the normals to the surface upon which the representative is obtained.

However, a different approach can be attained by raising perpendiculars to the 3-D space determined by surfaces where the curvature $C \neq 0$. More about this in the next section.

ALTERNATE METHODOLOGY FOR THE DETERMINATION

OF THE REPRESENTATIVE OF A SET OF POINTS

5. The case when $(x)_i$ is a plane curve

Through points of $(x)_i$ raise perpendiculars to its plane and along them mark the corresponding values of $(P)_i$. Figure 5.
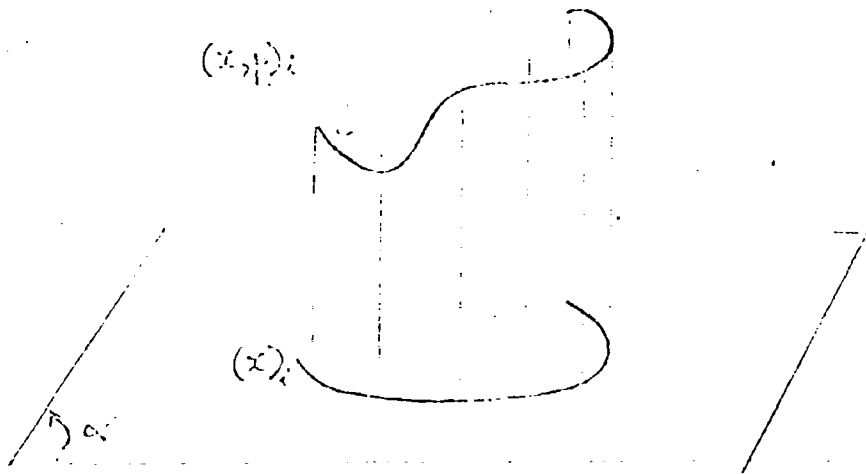

Figure 5.

Immediate advantages can be observed. We take as example the case of the minimum cost route discussed by Warntz in [1]. We pointed out that these routes are obtained orthogonally to a set of distributions $(x,p)_{i,j}$. See figure 3. Let us represent the same distributions in such a manner that each $(x,p)_{i,j}$ distribution is obtained on the perpendiculars to a same plane [the plane of any one of the representatives taken as the curve representative of the set $(x)_i$]. See figure 6.
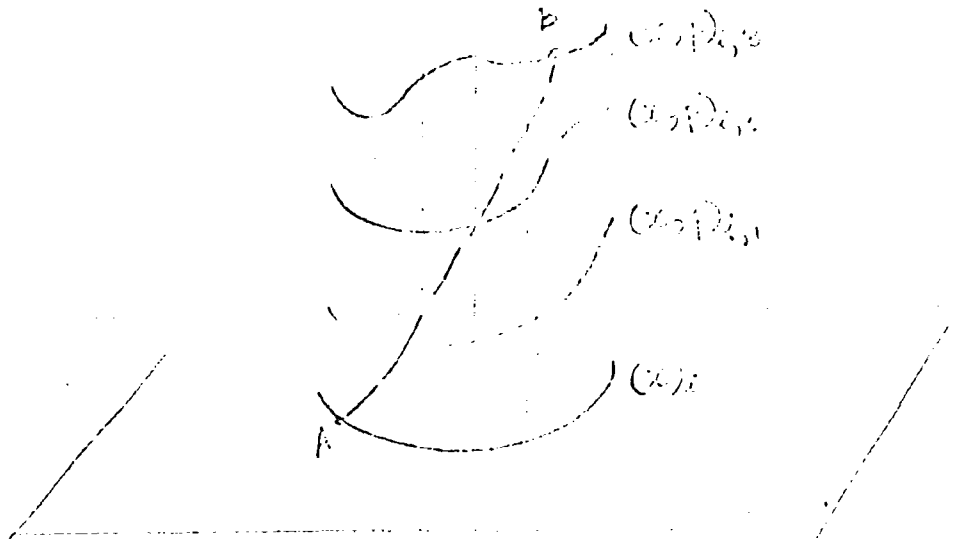
Figure 6.

Obviously, the locus of the representatives $(x,p)_{i,j}$
is a cylindrical surface! If we need to determine the
minimum path from A to B, all that is required is to determine
the geodesic line, through A and B, on that cylindrical surface!
But how can we determine this geodesic line? Simply. Because
the cylindrical surface has constant curvature equal to zero,
it can be developable upon a plane. Thus, the methodology for
.determining the minimum paths (minimum cost routes in Warntz'
problem-case):

 a. Represent the distributions as shown in figure 6.

 b. Develop the cylindrical surface upon a plane.

 c. Locate the points A and B in this development. Simply
draw a <u>straight line</u> from A to B. (The geodesic line, of course,
develops into a straight line.)

The suggested representative indicated in figure 6 gives rise to a new topic of research. For example, the determination of minimum paths, whose discussion is proposed by Lindgren in [6] should be reexamined. This will be done in a forthcoming paper.

6. **The case when $(x,y)_i$ is a nondevelopable curved surface**

A three-dimensional space being determined by the surface, through each point of this surface we raise perpendiculars to the 3-D space (and not to the surface).

That the line perpendicular to this 3-D space does not coincide with the normal to the surface will be demonstrated in the next section. Prior to this demonstration we must introduce some fundamental concepts of four-dimensional geometry. Only after these notions are discussed will we return to the outlining of the methodology for the graphical construction of the perpendicular to the 3-D space of the surface. These constructions, as it will be seen, are suitable for computer programming.

In one additional section we will generalize the problem, discussing the methodology for obtaining the representative of a multi-dimensional spatial distribution.

SOME FUNDAMENTALS IN

FOUR-DIMENSIONAL GEOMETRY

Most of the notions discussed in this section will not
be demonstrated or even presented in detail. References to the
sources where this is properly done are given.

The first notion to be considered is the geometric existence
of a four-dimensional space. Slaby[1] finds it resourceful to
"explain" it in the following manner: a point[2] is the edge view
of a straight line; a line is the edge view of a plane; a plane
is the edge view of a three-dimensional space; a three-dimensional
space is the edge view of a four-dimensional space. He also
proposes the concretization of ideas by relating, in edge views,
a point and a straight line segment; the segment and a square;
the square and a cube; the cube and the hyper-cube in 4-D space.

Manning [7] uses a step procedure, beginning with the line,
plane and three-dimensional space (also referred to as a hyperplane).
Thus his conceptualization: " a space of four dimensions consists
of the points that we get if we take five points not points of
one hyperplane, all points collinear with any two of them, and
all points collinear with any two obtained by this process."

Sommerville [8] and d'Albuquerque [9] simply provide postulates
of existence.

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[1] Steve M. Slaby, Princeton University

[2] Of course point is never defined. It is impossible to construct
a system of geometry without undefined terms.

Lindgren and Slaby [10] employ the postulates of projective geometry to conclude that the only geometric element not postulated as an intersection of two other geometric forms of higher dimensionality is the plane. Only point and line are so postulated:

a. Two lines that belong to the same plane also belong to the same point.

b. Two planes determine a line to which they belong.

With proper utilization of the principle of duality in space and of the concept of belonging, in the geometric sense, one concludes that:

c. Two planes that belong to the same 3-D space also belong to the same line.

This postulate is obtained by replacing point, line, and plane by line, plane, and space, respectively, in postulate a above in order to give a new format to postulate b. Continuing with the application of the duality principle, replace line, plane, and space by plane, 3-D space and 4-D space, to obtain:

d. Two 3-D spaces that belong to the same 4-D space also belong to the same plane.

In conclusion, we can state:

e. The intersection of two 3-D spaces is a plane.

f. Two planes determine a 3-D space. (Just as two points determine a line, and two concurrent lines determine a plane).

g. Two 3-D spaces determine a 4-D space.

The geometric existence of the 4-D space having been

properly analysed, there remains the task of introducing some

notions concerning relationships among the geometric forms in

that space. These are perpendicularity, parallelism, intersections,

etc. A few important ones are:

a. Two planes not belonging to the same 3-D space

intersect at a point.

Here we have the case when we consider a plane α in a

given 3-D space Λ , through a point (a) on it draw a perpen-

dicular to the plane and , through this line, in another 3-D

space Ω , pass a plane β . In this case, planes α and β

intersect at a point (a) and are said to be absolutely per-

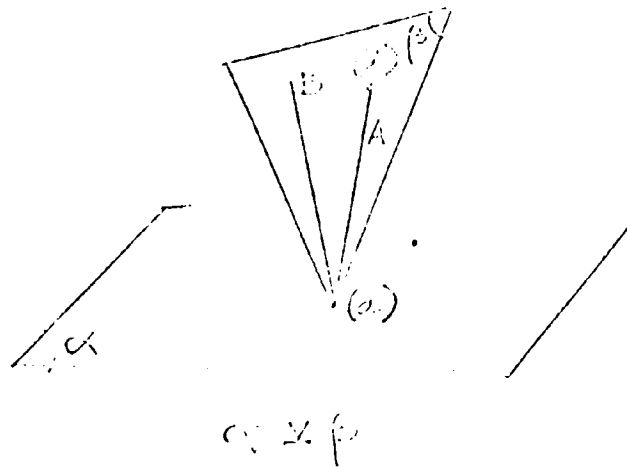pendicular. See figure 7 for a conceptual view of this rela-

tionship.

Figure 7.

The relationship gives rise to the following consequence, very important for our problem in developing the methodology of representing spatial distributions in 4-D space: every line of plane β belonging to a point (a) is perpendicular to plane α.

Consequently, at a point of a plane, in 4-D space, there is more than one perpendicular to the plane through a point on it. Therefore, the reciprocal of this statement IS NOT "through a point, not of the plane, in 4-D space, we can pass one and only one perpendicular to the plane." This is the statement for 3-D geometry. In 4-D geometry one says that "through a point in 4-D space we can pass one and only one absolutely perpendicular plane to the plane."

b. At a point of a 3-D space we can pass one and only one perpendicular to the space.

The reciprocal holds.

c. At a point in 4-D space we can pass one and only one perpendicular to a 3-D space.

d. If a line is perpendicular to a 3-D space at a point, it is perpendicualr to every plane of the 3-D space going through that point.

Evidently, the line is orthogonal[1] to every plane of the

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[1]We differenciate between orthogonality and perpendicularity. These are concepts commonly misinterpreted. Two geometric forms are perpendicular when they intersect. Two geometric forms F_1 and F_2 are orthogonal if, when through a point we pass two other geometric forms F_1' and F_2' parallel to them, F_1' and F_2' turn out to be perpendicular.

3-D space.

    c. As a consequence of the above, if a line is perpendicular to a 3-D space at a point, it is perpendicular to every line of the 3-D space going through that point and orthogonal to all lines of the 3-D space.

    The perpendicualr lines are concurrent and the orthogonal lines are skewed. Thus the necessity of making the differentiation between perpendicualr and orthogonal. See footnote.

    These notions on four-dimensional geometry suffice for the remaining of our presentation in this paper. Since we now must have means of representing these relationships, graphically, in order that they may be mathematically interpreted in a computer programming, we shall introduce, in the next section, some of the fundamentals of the four-dimensional descriptive geometry, one of the many graphical methods suitable for this task.

SOME FUNDAMENTALS IN

FOUR-DIMENSIONAL DESCRIPTIVE GEOMETRY

What is discussed are general notions on the procedure
for representing geometric forms in 4-D space, relating them
to a system of coordinated axes. The justification for using
the system and of the system itself can be found in the author's
Four-Dimensional Descriptive Geometry [10]. The methodology
is similar to that proposed by Monge for the three-dimensional
descriptive geometry [11].

For the development of the methodology to be employed in
the representative of a multi-dimensional spatial distribution
we shall be concerned only with the representation of the point,
the line, the plane and a line perpendicular to a 3-D space.

A brief discussion is made of the Mongean method of
representation followed by the parallel representation in 4-D
space.

The Mongean system

a.   System of reference: three perpendicular axes (Cartesian)
determining three perpendicular planes.

b.   A point is projected upon each plane.

c.   One plane is rotated about the intersection with one
of the other planes until superimposition upon it.

d.   Two points determine a line.

e.   A plane is represented either by three points (or
two concurrent lines, a line and a point) or by its intersections
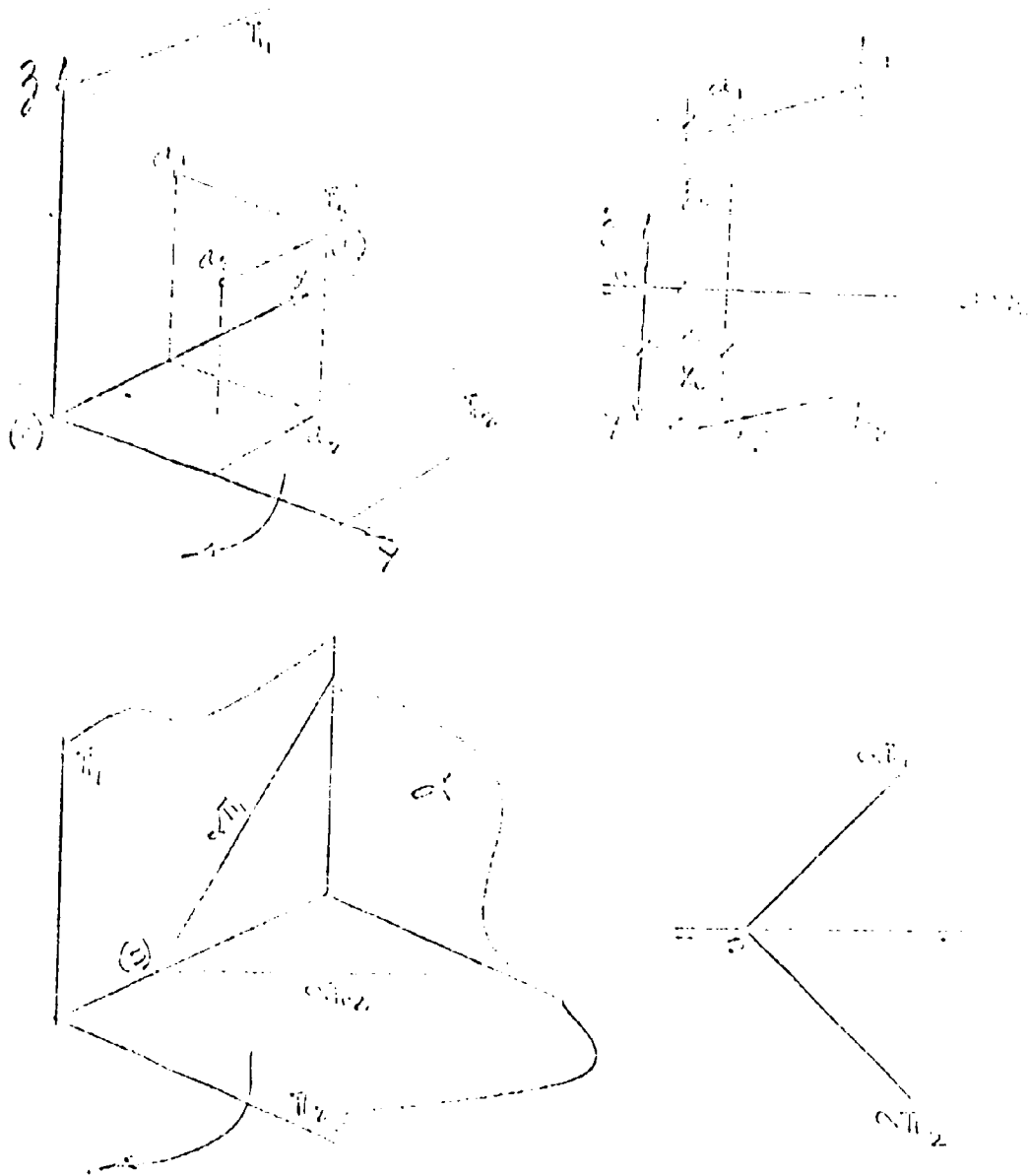with two of the three planes of the system of reference.

Figure 8.

Figure 8.

In 4-D space the system of reference consists of four lines, three-by-three perpendicular determining six planes, three-by-three perpendicular and four 3-D spaces, three-by-three perpendicular. Figure 9.
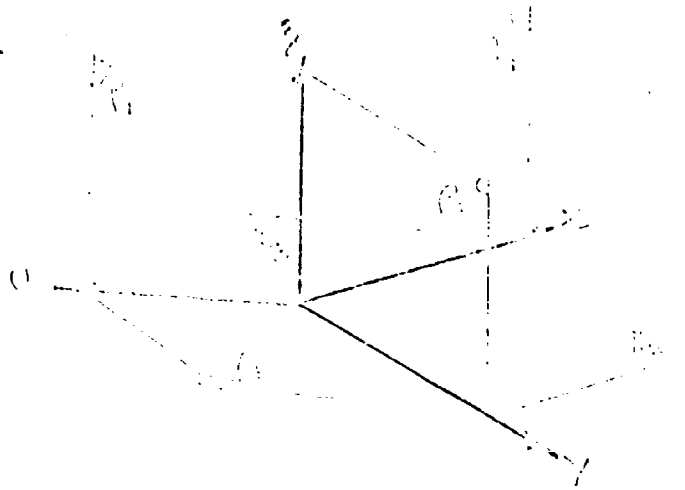

Figure 9.

A point (a) in 4-D space is projected upon each of the 3-D spaces. Figure 10 shows a conceptual view of the transformations where only three of the six planes of the system are indicated. Figure 11 shows the representative after transformed into a plane relationship.
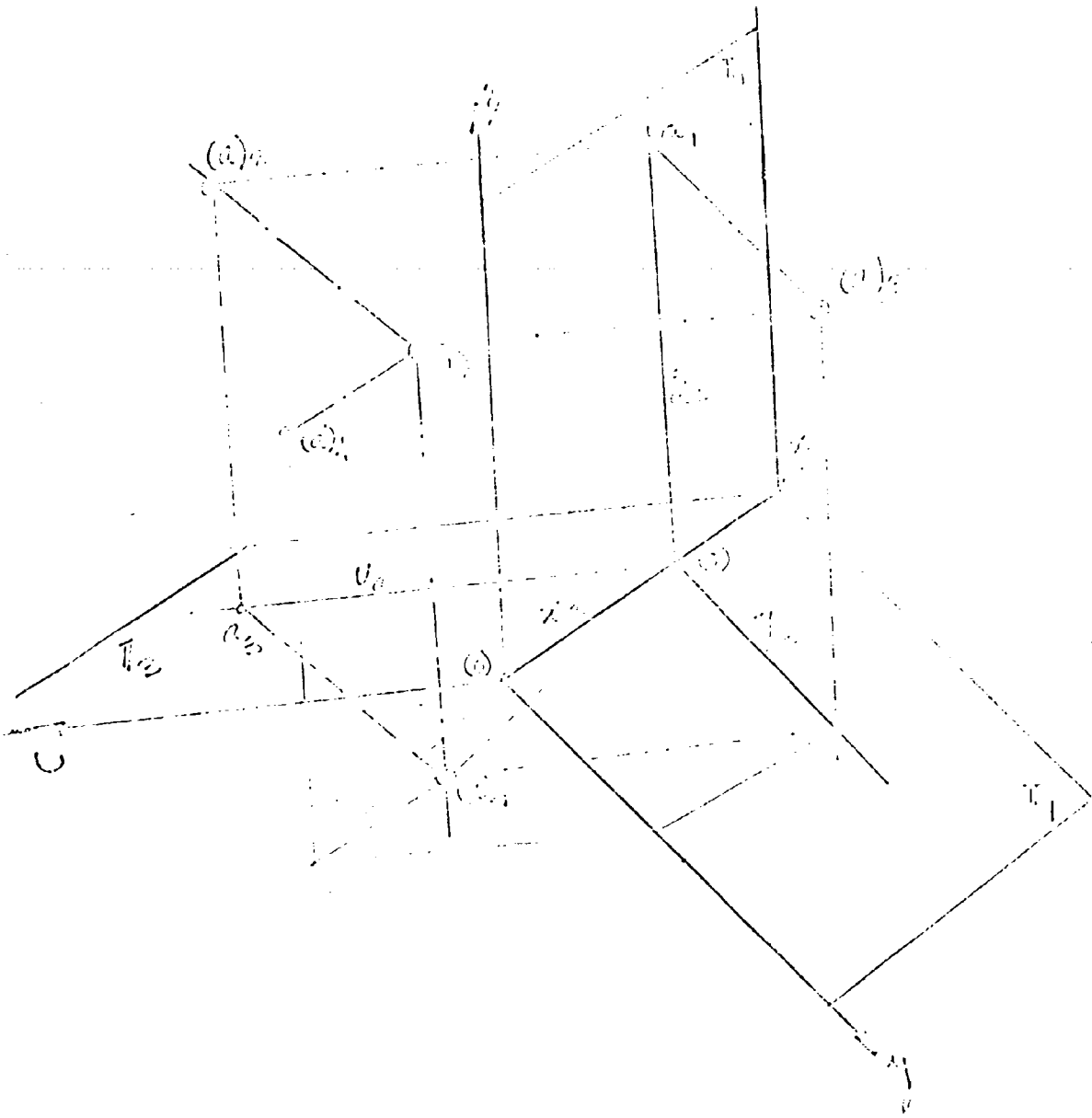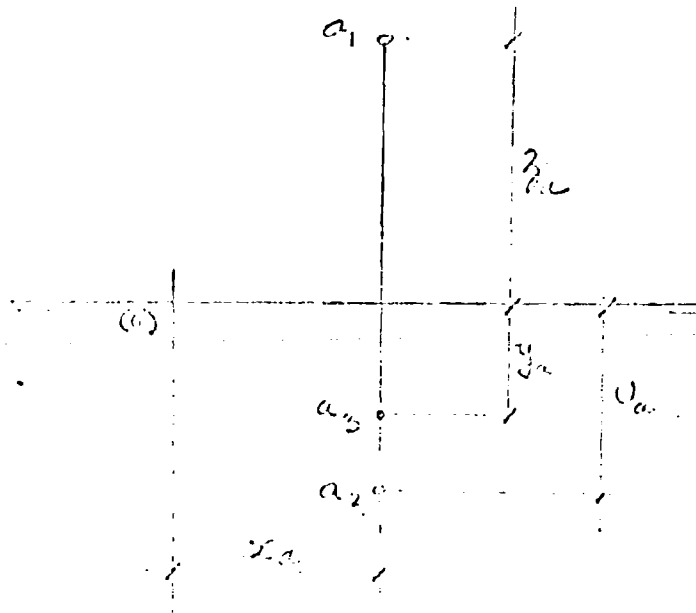
Figure 10.

Figure 11.

The plane is, usually, represented by its points of intersection with planes $\bar{\Pi}_1$, $\bar{\Pi}_2$, and $\bar{\Pi}_3$. (Each plane intersects a plane C_i of the 4-D space at a point, since they do not belong to the same 3-D space.)

A 3-D space Ω is represented by its lines of intersection with the planes $\bar{\Pi}_1$, Π_2, $\bar{\Pi}_3$. (A plane and a 3-D space intersect along a line.)

On this 3-D space we can identify a plane α and on this plane, points and lines. This is shown in figure 12.
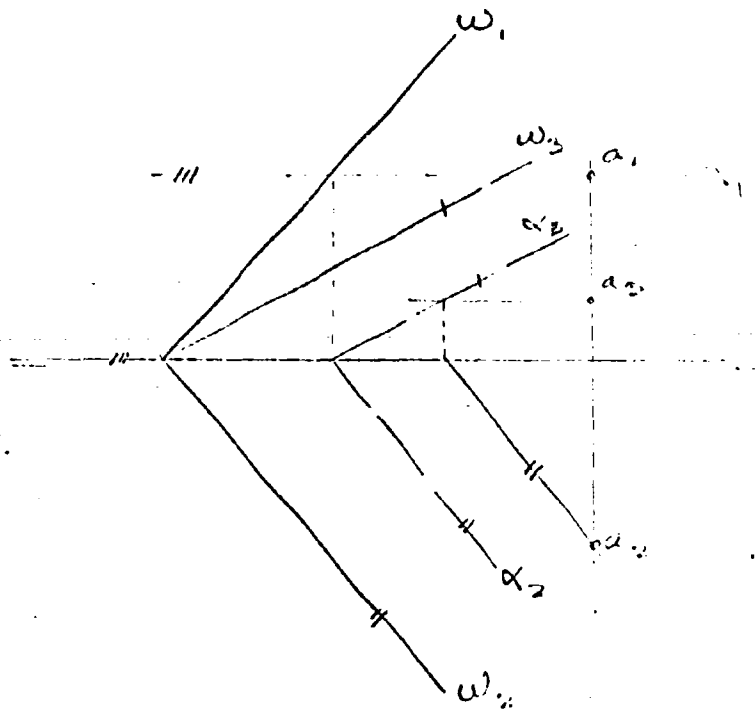
Figure 12.

Recalling that in the methodology proposed for the representative of a distribution involving a non-developed surface it is required to raise a perpendicular to the 3-D space determined by that surface, we will know how to draw that line if we can show how to raise a perpendicular to the 3-D space Ω, shown in figure 12, through a point (a) belonging to it.

The solution is very simple. The demonstration can be found in [10]. All that is required is to draw perpendiculars through a_1, a_2, and a_3 to w_1, w_2, w_3, the lines representing the 3-D space Ω. Figure 13. The line (am) is perpendicular to the 3-D space Ω.

$(a\,m)\perp\Omega.$
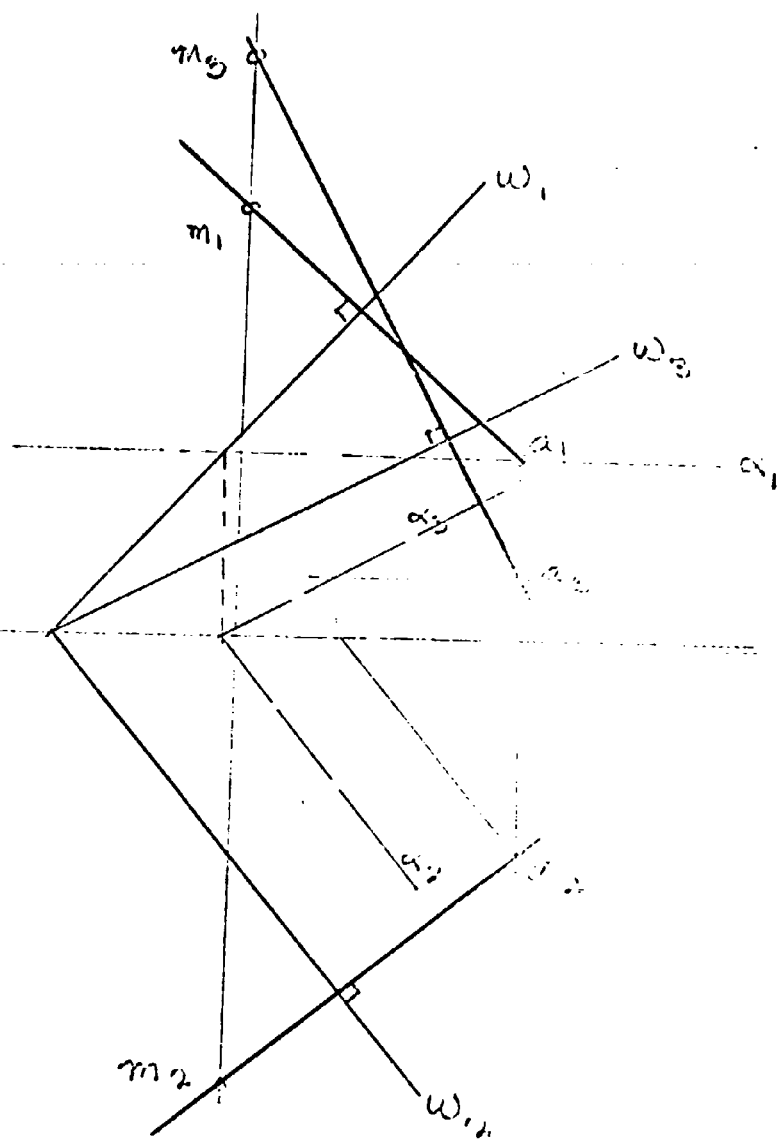
Figure 13.

THE REPRESENTATIVE OF A
DISTRIBUTION IN 4-D SPACE

It has been proposed that the representative of a
distribution over a non-developable surface could be obtained
by considering the three dimensional space determined by
the surface. The representation of this 3-D space required,
in turn, the development of a geometric method. This has
been shown in the preceeding sections. Next, we should find
out how to raise a perpendicular to this 3-D space, since this
is what was proposed originally, replacing the perpendicular
or normal to the surface. The procedure is also indicated
in the preceeding section. It remains, yet, to find ways of
marking on the perpendicular to the 3-D space of the surface,
the value of the population p_i measured at a point. Therefore,
if (a), shown in figure 13, is a point of the surface, and (am)
is the perpendicular to its 3-D space, point (m) will be the
point of the distribution $(x,y,z,p)_i$ _if_ (am) = p_i.

To mark the distance we can make use of the fact that if
a line, in 4-D space, has two of its projections parallel to the
reference line, any segment of this line is projected in true
length in the third projection. Thus the construction shown in
figure 14, is applied when the line does not satisfy the condition.
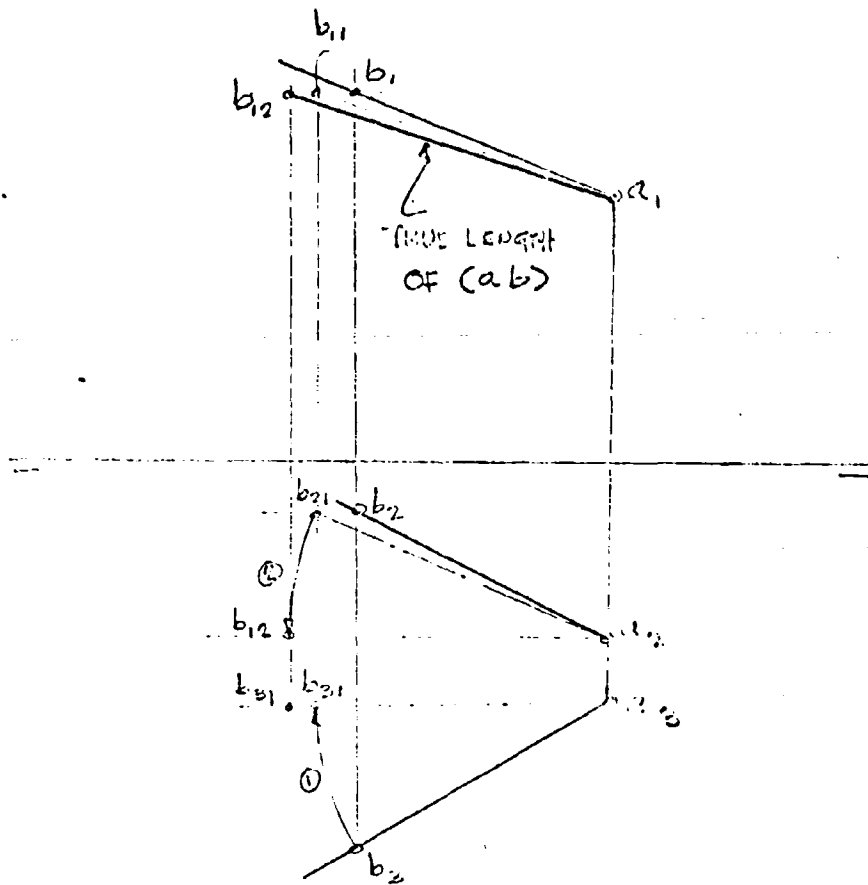
Figure 14.

In figure 15 we show the complete constructions required for the determination of the point (m) corresponding to a point (a) of the surface, so that (am) = p, value of the population at (a) for a distribution p_1.
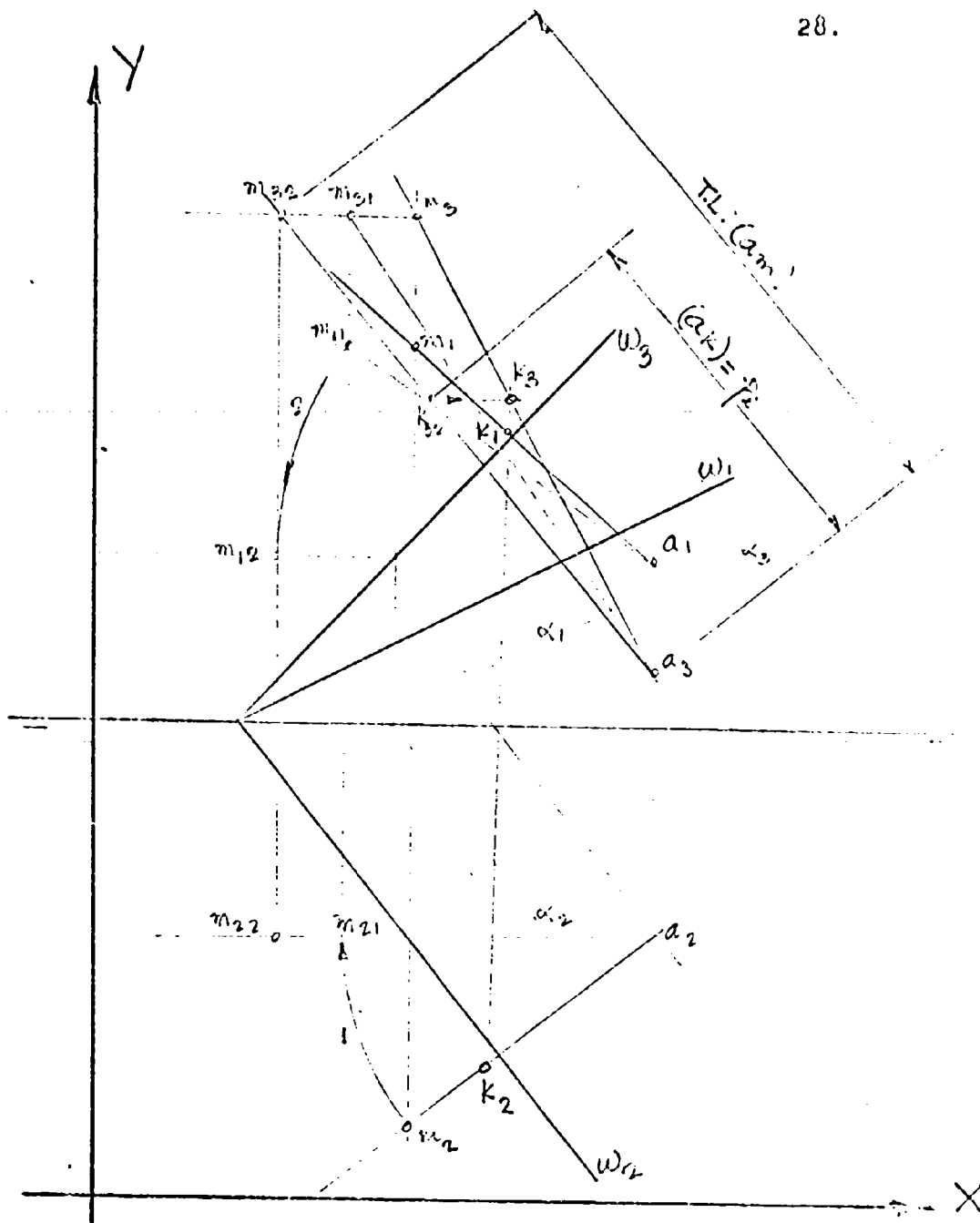
Figure 15.

Indicated in figure 15 is a system of Cartesian axes (x,y)
to indicate that every line involved in the construction can
be referred to that system. It is evident that w_1, w_2, w_3 are
given, since this is the space of the surface. To the set of
points $(a)_1$ of this surface it will correspond a family of
projections $(a_1 m_1)_1$, another of projections $(a_2 m_2)_1$, and a third,
of projections $(a_3 m_3)_1$.

This reference would permit the preparation of a computer
program involving, simply, the writing of the equations of the
geometric elements and their relationships, in the plane.

One final point must be demonstrated and this has to do
with the non-coincidence of the normal to the surface and the
perpendicular to its 3-D space.

This can be verified by checking the relationships among
geometric elements in 4-D space and the concept of absolutely
perpendicular planes.

Assume the surface and the tangent plane at one of its
points. The normal to the surface is perpendicular to that plane.
Consider next the 3-D space of this surface, the perpendicular
to the space and the plane absolutely perpendicular to the
tangent plane. We have seen that the perpendicular to the 3-D
space is unique; however, all lines of the absolutely perpendicular
plane, going through the point of the surface are normals to that
surface, since they are perpendicular to the tangent plane.
Thus, considered in 4-D space, the surface will have an in-
finite number of normals at a point, one of them being the

perpendicular, at that point, to the 3-D space of the surface.

Consequently, in 4-D space, we substitute the concept of line normal to a surface at a point (in three-dimensional geometry) by that of a plane absolutely perpendicular to the plane tangent to the surface at the point. This replacement of concept is similar to the one concerning plane and space curves. A plane curve will have one normal at a point, perpendicular to the tangent. A space curve has an infinite number of normals, all belonging to the normal plane. Of all these, only one is singled out as the principal normal for being perpendicular to the plane tangent to the curve. In 4-D space, of all the normals, only one is also perpendicular to the 3-D space determined by the surface. This is the one selected in this study.

RELATING SEVERAL DISTRIBUTIONS

Suppose that to a set of points identified by a two-dimensional system $(x,y)_i$ - for all purposes let $(x,y)_i$ be a plane - and that we wish to find the representative of the distribution of several populations P_i, Q_i, R_i, etc. over that set.

A distribution $(x,y,P)_i$ may be made, directly, generating, in the general case, a non-developable surface. A distribution $(x,y,z,q)_i$ can then be constructed over the distribution $(x,y,p)_i$ utilizing the methodology proposed in this paper. To obtain a distribution $(x,y,z,u,R)_i$ over the distribution $(x,y,z,\Lambda)_i$ we can have two choices.

1. Consider the distribution related to Q_i as belonging to the same 4-D space as that of the distribution P_i. This is equivalent to the present methodology, which assumes all distributions within the same 3-D space.

2. Consider the distribution related to Q_i as one that determines a new 4-D space. Thus, the distribution for V_i should be obtained by following a methodology parallel to that proposed in this paper. In this case we would make use of the methodology for the five-dimensional geometry. This would not be an impossible task since the basic steps toward it and to higher dimensionalities have already been indicated by Lindgren [12] and Santos Reis [3]. Again, all old graphical constructions are suitable for computer programming.

Obtaining the distribution $(x,y,z,u,R)_i$ we can now search

for the representative of a distribution S_i, selecting similar alternatives as explained above with proper adjustment of the dimensionality.

It is clear that this dimensionality is no longer an obstacle to the possibility of obtaining the distribution. The only drawback, as we see it , is to generate a computer program capable of performing the task. It might take considerable time before it is rendered operational.

If this is the case, perhaps one should limit his methodology for attained spatial distributions to the present approach. One, however, must be resigned to the fact that it will be practically impossible to properly relate several variates. We make reference, then, to the distribution problem discussed in another forth-coming paper of this series.

In that paper we were looking for ways of generating a surface of potential for a function $\underline{A}$ defined as $A = f(a,b,c,...n)$ where a,b,c,d,...,n are the populations of several variates measured at each point of the set over which the distribution is studied. We discussed the problem of evaluating the impact of each variate and we now point out that even this can be accomplished. It still remains the question of determining which form the function $\underline{A}$ takes.

A new path, however, may exist if we use the methodology proposed in this paper. In this case, what we propose is the generation of

1. A surface of potentials involving the first variate $\underline{a}$, obtaining a distribution $(x,y,a)_1$.

2. A surface of potentials involving the second variate **b**, over the previous distribution $(x,y,a)_1$, obtaining a distribution $(x,y,z,b)_1$.

And so on.

But why is this possible? It is possible because, on each surface of potentials we continue to measure <u>distances</u> and, therefore, can procede with the calculation of the potentials U_1 expressed as $\sum \dfrac{p_1}{r}$, where p_1 is the corresponding population $a,b,c,\ldots,n$.

It is necessary to further discuss this point, to clarify matters.

Suppose that we have the set of coordinate points. Then, it is always possible to calculate the distance between any two of them. Let it be $r_{i,j}$.

Let g be a variate. The potential at each point is estimated as $U_{a,i} \cdots$. With these values we generate a surface of potentials.

If on that surface we now determine the geodesic line between any two points, its length - which is a minimum - corresponds to the minimum length, measured on the coordinate surface (a plane, generally) of the points. If the initial unit of measurement is in miles, the length of the geodesic line, on the surface of potentials, continues to be measured in miles. This is due to the fact that the potential constitutes a method of transforming <u>the coordinate surface of the original set of points</u> into another coordinate surface containing the same points now,

however, displaced from their original positions. A graphic view of this transformation is shown in figure 16, where we used a plane curve as an example, the set of points of this curve being transformed into another set, projectively related, however, to the first set.

Figure 16.

If the set of points are coordinated in $(x)_i$ - and only one coordinate is required, due to the one-dimensional character of the line - this set may be transformed into another set $(t)_i$ by means of a factor U_i, estimated at each point of the set $(x)_i$. If the distances between the points in set $(x)_i$ is made in units of length, the distances between the points in set $(t)_i$ is made in the same units. Of course, the unit of measurement of U_i, if it is the potential calculated as $U_i = \sum \frac{p}{r}$, where p is a factor measured at each point of $(x)_i$, is in (units of p) by (unit of length).

Thus, if we generate a potential of population surface, the

length of the geodesic line between two points of the surface, the points corresponding to those in the set $(x,y)_i$, is also measured in unit of length.

So, suppose that we have a population P and that we have estimated the potentials U, generated the surface of potential of population P, determined the geodesic lines and their lengths $r_{i,j}$.

$$U_i = \sum \frac{P_i}{r_{i,j}}$$

On the surface of potential of P we calculate now a new potential, this time taking as variate the values of U. In other words, this will be the surface of potential of the (potential U).

This equals

$$X_i = \sum \frac{U_i}{g_{i,j}}$$

Substituting U by $\frac{P}{r}$ we get:

$$X_i = \sum \frac{P_i}{(gr)_{i,j}}$$

Now, Warntz calls $\frac{P}{r^2}$ the "index of attraction" [13,14]. A surface generated as a function of X_i provides a means of the study of the distribution of this index in a continuous form.

Of course, if we had used as distance between points the same value of r, we would get

$$X_i^1 = \sum \frac{P_i}{r^2_{i,j}}$$

The difference in the values of X_i and X_i^1 indicates the difference between the present methodology and the one proposed

In this paper. The estimation of the distribution of the "index of attraction", according to the present methodology, does not take into account the transformation of the original set of points by the estimation of the potential of P. In other words, after P transforms that set and we want to estimate a new function involving the effect of $\frac{p}{r}$ on the same set, it simply ignores the first transformation. This, however, is obtained by using the proposed methodology. The difference between r and ρ expresses, perhaps, the concept that people (if P is indeed number of people) tends to be as far apart as possible occupying, at the same time, the least area. Perhaps this least area is greater than suggested by r^2, since $\rho > r^2$ for $\rho > r$. These are things for the geographer to interpret.

Let us continue with the discussion related to several populations. We wish to obtain a relationship between them. Let, then, a,b,c,...,n be these populations. With the set of points located by $(x,y)_i$ we calculate distances $r_{i,j}$ and estimate the potential

$$_1U_i = \sum \frac{a_i}{r_{i,j}} \quad .$$

Next, we take the set of points, as they are located in the surface of potentials of a_i, measure the distances $r_{i,j}$ (geodesic lines) and estimate the potential.

$$_2U_i = \sum \frac{b_i}{r_{i,j}}$$

The generation of a surface representing the distribution above is made according to the methodology proposed in this paper.

And so on. After all potentials have been estimated and

each distribution properly made, according to the dimensionality, the point in N-space, corresponding to the point in the original set located by $(x,y)_i$, is characterized in that space by a distance, from the origin of the coordinate system $(x,y)_i$ equal to

$$d_i = \sum_i \, {}_i U_i$$

This is the equivalent to the _multilateral_ representation of a point in N-space as discussed by Santos Reis [3] and expresses the relationship, at each point, among the variates $a, b, c, \ldots, n$ measured at each point of the original set.

BIBLIOGRAPHY

[1] "A Note on Surfaces and Paths and Applications to Geographical Problems", William Warntz, discussion paper No. 6, Michigan Inter-University Community of Mathematical Geographers, Department of Geography, University of Michigan, June 1965.

[2] Breakthrough in Physics , edited by Peter Wolff, The New American Library, 1965.

[3] "The Descriptive Geometry (or Representative) of a Collection of Points Fixed by N-Coordinate Numbers or of N-dimensions", Felippe dos Santos Reis, Harvard Papers in Theoretical Geography, Paper Number Twenty-Four, Laboratory for Computer Graphics, Harvard University, 30 September 1968.

[4] "The Illusive Point of Minimum Travel", Arnold Court, Annals of the Association of American Geographers, Vol. 54, 1964, pages 400-403. This article comments on the article (see reference 5) by P.W. Porter.

[5] "What is the Point of Minimum Aggregate Travel", Philip W. Porter, Annals of the Association of American Geographers, Vol. 53, 1963, pages 220-222; also, "A Comment on the Illusive POint of Minimum Travel", by P.W. Porter, Annals of the Association of American Geographers, Vol. 54, 1964, pages 403-406. In his "Comments" Porter accepts the criticisms and agrees that, indeed, the position of the regression line changes with axial rotation. This renders invalid Porter's method for determining the point of minimum aggregate travel. However, as pointed out by us, the regression line is an invariant. Therefore Porter's method should be reevaluated since Court's criticisms are, now, rendered invalid.

[6] "Proposed Solution for the Minimum Path Problem", C. Ernesto S. Lindgren, Harvard Papers in Theoretical Geography, Paper Number Four, Laboratory for Computer Graphics, Harvard University, 14 October 1967.

[7] Geometry of Four Dimensions , Henry Parker Manning, The MacMillan Company, New York, 1928.

[8] An Introduction to the Geometry of N Dimensions , D.M.Y. Sommerville, Dover Publications, Inc., New York, 1958.

[9] "Contribuição a Geometria Descritiva a Quatro
     Dimensoês", School of Philosophy, Sciences and
     Letters, Guanabara State University, Brazil, 1957.

[10] Four-Dimensional Descriptive Geometry , C. Ernesto
      S. Lindgren and Steve M. Slaby, McGraw-Hill Book
      Co., New York, 1968.

[11] Géométrie Descriptive, Gaspard Monge, Baudouin
     (ed.), Paris, 1798.

[12] N-Dimensional Spatial Analysis and Computer Graphics,
     C. Ernesto S. Lindgren, Harvard Papers in Theoretical
     Geography, Paper Number Sixteen, Laboratory for Computer
     Graphics and Spatial Analysis, Harvard University,
     24 April 1968.

[13] "Geography, Geometry and Graphics", William Warntz,
     Technical Seminar Series, Report No. 7, Department
     of Graphics and Engineering Drawing, Princeton
     University, October 31, 1963.

[14] "Macrogeography and Income Fronts", William Warntz,
     Regional Science Research Institute, Philadelphia,
     Pa., 1965.

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13. ABSTRACT

>The advent of advanced methodology concerning graphical constructions in n-dimensional space provides new means of generation of surfaces. This methodology is summarized and complemented with recently established graphical relationships, suggesting new approaches in the study of spatial distributions. ( )