THE SPECTRA OF NONSTATIONARY RANDOM PROCESSES

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ABSTRACT

The use of spectral techniques for the computation of the expected power output of linear time invariant filters subjected to a nonstationary noise is studied. The two-dimensional power spectrum is defined, and its use for computing the time varying expected power output is illustrated. The derivation of the one-dimensional energy spectrum from the two-dimensional power spectrum is shown. The derivation of the instantaneous power spectrum as the derivative of the truncated energy spectrum is shown. It is concluded that the instantaneous power spectrum is not a useful engineering technique since there are no expressions relating the instantaneous power spectrum at the filter output to that at the input. For the special case in which the nonstationary noise is the product of a modulation function and a stationary noise, it is shown that the problem can often be reduced to an equivalent stationary problem and solved in a well known way.

The two-dimensional power spectrum is used to compute the optimum bandwidth of an RC filter for detection of a rectangular pulse in exponentially decaying white noise. Curves showing that the optimum bandwidth as a function of time and the product of pulse length and decay time constant are developed. The results are compared with the results obtained for a stationary noise.

Finally, the problem of computing the response of an RLC band-pass filter to volume reverberation is studied. An asymptotic expansion is derived which provides sufficient accuracy for most engineering work. Bounds on the error are obtained.

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CHAPTER I

INTRODUCTION

1.1 Statement of the Problem

This thesis is an investigation into the applications of spectral representations of nonstationary random processes. The investigation determines how the spectra are useful for computing output statistics of a linear time invariant filter whose input is a nonstationary process. Some of the nonstationary processes which the engineer encounters are radar clutter, noise in large auditoriums and back-scattered sound in the sea, known as reverberation, which will be frequently cited as an example. Many engineers have studied enough communication theory to be able to efficiently solve problems of the detection of signals in stationary noise background, but, when the background is a nonstationary noise, most are confused. One first wonders how to solve for the output at all. Are there techniques for solving for the total noise power over all time as well as for the expected noise power as a function of time? If the transform of the autotranslation function is computed as is usually done in spectrum computations, how should the results be interpreted? These questions are discussed in this thesis.

First, the equations used to compute the output of a linear filter to a stationary random noise are reviewed. There results, not justified here, are taken from Lee.[1] A random process is weakly or wide sense stationary if its autocorrelation function, defined by

$$R_{xx}(t+\tau,t) \equiv E[x(t+\tau)x(t)] \tag{1.1}$$

and its expected value, or mean, $E[x(t)]$, are independent of t for any given τ. In this thesis, for simplicity, the term stationary is used instead of weakly or wide sense stationary.

The power spectrum of a stationary random process is defined by

$$S_{xx}(\omega) \equiv \int_{-\infty}^{\infty} R_{xx}(\tau)e^{-j\omega\tau}d\tau \quad , \tag{1.2}$$

where $R_{xx}(\tau) = E[x(t+\tau)x(t)]$. Observe that two symbols, $R_{xx}(t+\tau,t)$ and $R_{xx}(\tau)$, have been used to represent $E[x(t+\tau)x(t)]$. The distinction between these two symbols is that $R_{xx}(t+\tau,t)$ is used when $E[x(t+\tau)x(t)]$ is a function of t and τ, and $R_{xx}(\tau)$ is used when $E[x(t+\tau)x(t)]$ is independent of t and is a function of τ only.

Taking the inverse Fourier transform of $S_{xx}(\omega)$ gives

$$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)e^{j\omega\tau}d\omega \quad , \tag{1.3}$$

and setting $\tau = 0$ gives

$$R_{xx}(0) = E[x^2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)d\omega \quad . \tag{1.4}$$

Thus, for a stationary random process, the integral of the spectrum,

$S_{xx}(\omega)$, over ω is proportional to the expected power.

One of the most obvious characteristics of reverberation is that the expected power decreases with time. It is clear, then, that reverberation cannot be a stationary process. For this reason, the conventional definition of the spectrum of $S_{xx}(\omega)$ cannot be used.

It is possible to treat reverberation as a transient with finite energy. In such cases, Lee defines the spectrum as the transform of the function,

$$\int_{-\infty}^{\infty} x(t+\tau)x(t)dt \quad .$$

The principal difference between transients as considered by Lee and reverberation is the random nature of reverberation. This brings to mind the possibility of treating reverberation as a random transient and defining its spectrum as the Fourier transform of the auto-translation function,

$$\begin{aligned} \overline{R}_{xx}(\tau) &= E\left[\int_{-\infty}^{\infty} x(t+\tau)x(t)dt\right] \\ &= \int_{-\infty}^{\infty} R_{xx}(t+\tau,t)dt \quad . \end{aligned} \tag{1.5}$$

This can be done and is derived in Section 2.3. It provides a satisfactory answer as long as the problem does not require a time-varying spectrum or a time-varying statistical description. This requires a transformation of the autocorrelation function $R_{xx}(t+\tau,t)$ as will be seen in Section 2.2.

1.2 Reverberation Model

The purpose of the research is not to derive a justifiable model of reverberation, but it is to study spectral techniques for describing reverberation in order to determine useful mathematical tools for the engineer. It is, therefore, helpful to review a little of what has been written about reverberation and its spectrum.

A sound signal projected into water will encounter various impurities, inhomogeneities and a changing index of refraction which are all lumped together under the term scatterers. Each of the scatterers produces an echo and the sum of these echoes is called reverberation. The random location, motion and strength of the scatterers makes reverberation a random process.

Returns from scatterers not located at a boundary are called volume reverberation. For volume reverberation, the propagation loss is proportional to

$$g_v(t) = \begin{cases} \dfrac{e^{-\alpha t}}{t} & t \geq T \\ 0 & t < T \end{cases} , \qquad (1.6)$$

where α is a constant dependent on the frequency and water chemistry (and is proportional to the factor usually called absorption coefficient), and T is the transmitted pulse length. In this equation, $g(t)$ describes the variation of sound pressure level with time. The variation of sound power with time is proportional to

$$g_v^{\,2}(t) = \begin{cases} \dfrac{e^{-2\alpha t}}{t^2} & t \geq T \\ 0 & t < T \end{cases} \qquad (1.7)$$

Similarly, for boundary reverberation, the variation of sound pressure level with time is proportional to

$$g_b(t) = \begin{cases} \dfrac{e^{-\alpha t}}{t^{3/2}} & t \leq T \\ 0 & t < T \end{cases} , \qquad (1.8)$$

so that

$$g_b^{\,2}(t) = \begin{cases} \dfrac{e^{-2\alpha t}}{t^3} & t \geq T \\ 0 & t < T \end{cases} \qquad (1.9)$$

In observing records of actual reverberation, the author has observed that sound pressure level varies greatly from ping to ping but, when a great number of pings are averaged, the sound pressure level does vary according to the above equations. There is, of course, no way of knowing if the variation is from variation in propagation loss or in the scatterers.

Faure[19] derives some interesting results by assuming that the scatterers are distributed in the medium according to the Poisson probability distribution. From this assumption, Faure is able to derive the well-known laws of reverberation:[18]

1) reverberation power is proportional to the transmitted energy,

2) the expected power of volume reverberation decreases as $1/t^2$ and the expected power of boundary reverberation decreases as $1/t^3$ if absorption is neglected.

Faure then shows that, if the transmitted pulse is short, the auto-translation function of reverberation is approximately proportional to the propagation loss $g^2(t)$ times the autotranslation function of the received echoes which is a function of the probability distribution of the Doppler shift of the echoes. From this, it is argued that reverberation is approximately stationary and the spectrum of reverberation can be defined as the product of $g^2(t)$ times the Fourier transform of the autotranslation function of the received echoes. This finally leads to the interesting conclusion that the spectrum so defined appears as a time function $g^2(t)$ multiplied by the convolution product of the spectrum of the transmitted signal (the Fourier transform of its autotranslation function) and the probability density of the Doppler shift; i.e.,

$$\overline{S}_{rr}(\omega) = E[\sigma^2]\rho(t)g^2(t)\int_{-\infty}^{\infty} |\overline{S}_{ss}(\omega + \phi)|^2 p(\phi)d\phi \quad , \qquad (1.10)$$

where $\overline{S}_{rr}(\omega)$ is the Fourier transform of the autotranslation function of reverberation, σ is a random factor proportional to the scatterer cross-section, $\rho(t)$ is the probability density of the arrival of echoes at time t , $g^2(t)$ is the expected variation of the propagation loss, $\bar{S}_{ss}(\omega)$ is the energy spectrum of the transmitted signal, ϕ is the approximate Doppler shift of the echo, and $p(\phi)$ is the probability density of ϕ. Although the study will not be restricted to models based on assumptions of approximate stationarity, it will be useful to represent reverberation as suggested by Faure's conclusion that it is approximately a stationary noise times a time function. Thus, nonstationary processes of the form

$$y(t) = g(t)x(t) \tag{1.11}$$

where $g(t)$ is a deterministic function and $x(t)$ is a stationary random process with spectrum $S_{xx}(\omega)$ will often be considered to illustrate spectral techniques.

Other authors, for example, Ol'shevskii,[20] state flatly that reverberation can be represented by Equation (1.11). In addition, all laboratory simulators seen by the author model reverberation as in Equation (1.11), except for those which contain tape recordings of actual reverberation. It should be mentioned that Faure concluded that the bandwidth of the random process $x(t)$ should actually decrease with time and that this has been experimentally verified.

CHAPTER II

FOURIER TRANSFORMS OF THE AUTOCORRELATION AND AUTOTRANSLATION FUNCTIONS

2.1 Introduction

In this chapter, the Fourier transforms of the autocorrelation and the autotranslation functions which lead to different spectral descriptions of real valued nonstationary random processes are considered. This is done in order to discover useful techniques for designing time-invariant linear filters and to learn if there is a physical interpretation of the resulting spectra. It is assumed that a spectral description of a random process is useful if it results in an equation relating the filter transfer function and the spectrum to the expected output of the linear filter. If it does not, the spectrum is rejected as being impractical for our purposes.

The autocorrelation function of $y(t)$ was defined as

$$R_{yy}(t+\tau,t) = E[y(t+\tau)y(t)] \quad .$$

Letting $t_1 = t+\tau$ and $t_2 = t$ gives an equivalent definition,

$$R_{yy}(t_1,t_2) = E[y(t_1)y(t_2)] \quad .$$

Then, if $t_1 = t_2$, $R_{yy}(t_1,t_2)$ is the expected instantaneous power of the random process at time t_1 , $E[y^2(t_1)]$.

2.2 Two-Dimensional Power Spectrum

Let the spectrum $S_{yy}(\omega,\nu)$ be defined as the following double transform of the autocorrelation function:

$$S_{yy}(\omega,\nu) \equiv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{yy}(t_1,t_2)e^{-j\omega t_1}e^{j\nu t_2}dt_1dt_2 \quad . \tag{2.1}$$

It follows from an iterative use of the inversion theorem for Fourier transforms that

$$R_{yy}(t_1,t_2) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{yy}(\omega,\nu)e^{j\omega t_1}e^{-j\nu t_2}d\omega d\nu \quad . \tag{2.2}$$

The advantage of the two-dimensional spectrum $S_{yy}(\omega,\nu)$ is that it readily leads to an equation for the autocorrelation function of the output of a linear filter. If $z(t)$ is the output of a linear time-invariant filter having weighting function $h(t)$ for a sample input $y(t)$, then the convolution gives

$$E[z(t_1)z(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)E[y(t_1-\tau_1)y(t_2-\tau_2)]d\tau_1 d\tau_2 \quad . \tag{2.3a}$$

Since $E[y(t_1-\tau_1)y(t_2-\tau_2)] = R_{yy}(t_1-\tau_1,t_2-\tau_2)$, Equation (2.3a) can also be written as

$$E[z(t_1)z(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_{yy}(t_1-\tau_1,t_2-\tau_2)d\tau_1 d\tau_2 \quad . \tag{2.3b}$$

Substituting Equation (2.2) in Equation (2.3b) gives

$$E[z(t_1)z(t_2)] = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)\cdot$$

$$\cdot\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{yy}(\omega,\nu)e^{j\omega(t_1-\tau_1)}e^{-j\nu(t_2-\tau_2)}\,d\omega d\nu d\tau_1 d\tau_2$$

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)e^{-j\omega\tau_1}d\tau_1 \cdot$$

$$\cdot\int_{-\infty}^{\infty} h(\tau_2)e^{j\nu\tau_2}d\tau_2 S_{yy}(\omega,\nu)\cdot e^{j\omega t_1}e^{-j\nu t_2}d\omega d\nu$$

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(\omega)H^*(\nu)S_{yy}(\omega,\nu)e^{j(\omega t_1-\nu t_2)}d\omega d\nu \;. \tag{2.4}$$

Equation (2.4) is the result desired, an equation involving only a transform of the input autocorrelation function and the filter transfer function which gives the output autocorrelation function. By letting $t_1 = t_2$ in Equation (2.4), the expected value of the filter output power as a function of time is obtained; that is,

$$E[z^2(t)] = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(\omega)H^*(\nu)S_{yy}(\omega,\nu)e^{j(\omega-\nu)t} d\omega d\nu \quad . \tag{2.5}$$

The definition stated by Equation (2.1) is justified because it provides useful information through Equations (2.4) and (2.5).

It is interesting to observe that the integral of the spectrum $S_{yy}(\omega,\nu)$ over ω and over ν is proportional to the expected instantaneous power at $t = 0$. To obtain this result, let the filter transfer function be $H(\omega) = 1$ for all ω . Then,

$$E[z^2(t)] = E[y^2(t)] = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_{yy}(\omega,\nu)e^{j(\omega-\nu)t} d\omega d\nu \quad .$$

Now, if $t = 0$,

$$E[y^2(0)] = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_{yy}(\omega,\nu) d\omega d\nu \quad ,$$

which is the result desired. If the process is stationary, the expected instantaneous power is independent of time and

$$E[y^2(t)] = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_{yy}(\omega,\nu) d\omega d\nu$$

for all t .

2.2.1 Two-Dimensional Power Spectrum of a Stationary Random Process. If $y(t)$ is a sample of a stationary random process, than it is interesting to study the relation between the two-dimensional power spectrum $S_{yy}(\omega,\nu)$ and the one-dimensional power spectrum defined by

$$S_{yy}(\omega) \equiv \int_{-\infty}^{\infty} R_{yy}(\tau)e^{-j\omega\tau}d\tau \quad , \tag{2.6a}$$

where

$$R_{yy}(\tau) \equiv E[y(t+\tau)y(t)] \quad . \tag{2.6b}$$

To find this relation, evaluate $S_{yy}(\omega,\nu)$ from Equation (2.1) for y, a sample of stationary random process. Letting $\tau = (t_1 - t_2)$ in Equation (2.1), there results

$$S_{yy}(\omega,\nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{yy}[(t_2+\tau),t_2]e^{-j\omega(t_2+\tau)}e^{j\nu t_2}dt_2 d\tau$$

$$= \int_{-\infty}^{\infty} e^{-j(\omega-\nu)t}dt \int_{-\infty}^{\infty} R_{yy}(t+\tau,t)e^{-j\omega\tau}d\tau \quad . \tag{2.7}$$

It has been assumed that $y(t)$ belongs to a stationary process. Then, comparing the definition of $R_{yy}(t_1,t_2)$ with the definition of $R_{yy}(t)$ in Equation (2.6b) shows that

$$R_{yy}(t+\tau,t) = R_{yy}(\tau) \quad ,$$

and the τ integration in Equation (2.7) is given by Equation (2.6a). Substituting Equation (2.6a) in Equation (2.7),

$$S_{yy}(\omega,\nu) = \int_{-\infty}^{\infty} e^{-j(\omega - \nu)t} dt S_{yy}(\omega) \quad . \tag{2.8}$$

Finally, recognizing the t integral in Equation (2.8) as $2\pi\delta(t)$, where $\delta(t)$ is the Dirac delta function, results in

$$\begin{aligned} S_{yy}(\omega,\nu) &= 2\pi S_{yy}(\nu)\delta(\omega - \nu) \\ &= 2\pi S_{yy}(\omega)\delta(\omega - \nu) \quad . \end{aligned} \tag{2.9}$$

Equation (2.9) is the desired relation between the two-dimensional spectrum and the one-dimensional spectrum. For a stationary process, the two-dimensional spectrum is equal to the one-dimensional spectrum multiplied by the Dirac delta function $\delta(\omega - \nu)$.

By substituting Equation (2.9) in Equation (2.5), it can be seen that Equation (2.5) reduces to the well-known relation for the average output of a filter subjected to a stationary random input. Carrying this out, there results

$$E[z^2(t)] = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(\omega)H^*(\nu)S_{yy}(\omega)2\pi\delta(\omega - \nu)d\omega d\nu$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega)H^*(\omega)S_{yy}(\omega)d\omega \quad , \tag{2.10a}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 S_{yy}(\omega)d\omega \tag{2.10b}$$

for a stationary random input.

2.2.2 Two-Dimensional Spectrum of the Product of a Deterministic Function and Band-Limited White Noise. Suppose that we generate a nonstationary random process

$$y(t) = g(t)x(t) \quad ,$$

where $g(t)$ is a deterministic function and $x(t)$ is a stationary random process with spectrum

$$S_{xx}(\omega) = \begin{cases} \pi/2 & \omega_o - \beta \le |\omega| \le \omega_o + \beta \\ 0 & \text{otherwise} \end{cases} \quad ,$$

where $\omega_o > \beta$. $S_{xx}(\omega)$ represents the output spectrum of an ideal band-pass filter with white noise input. The autocorrelation function of $x(t)$ can be determined from the Wiener-Khintchine theorem:

$$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)e^{j\omega\tau}d\omega \tag{2.11}$$

$$= \frac{1}{2\pi}\int_{-(\omega_o+\beta)}^{-(\omega_o-\beta)} \frac{\pi}{2}\, e^{j\omega\tau}d\omega + \frac{1}{2\pi}\int_{\omega_o-\beta}^{\omega_o+\beta} \frac{\pi}{2}\, e^{j\omega\tau}d\omega$$

$$= \frac{1}{4}\, \frac{e^{j\omega\tau}}{j\tau}\Bigg|_{-(\omega_o+\beta)}^{-(\omega_o-\beta)} + \frac{1}{4}\, \frac{e^{j\omega\tau}}{j\tau}\Bigg|_{\omega_o-\beta}^{\omega_o+\beta}$$

$$= \frac{\sin\beta\tau}{\tau} \cos \omega_o\tau \quad . \tag{2.12}$$

The expected instantaneous power of the stationary random process $E[x^2(t)]$ can be determined by setting $\tau = 0$ in Equation (2.12), which gives

$$E[x^2(t)] = R_{xx}(0) = \frac{\sin \beta\tau}{\tau} \cos \omega_o \tau \Big|_{\tau=0} = \beta \quad .$$

The two-dimensional spectral density of $y(t)$ is given by Equation (2.1),

$$S_{yy}(\omega,\nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{yy}(t_1,t_2)e^{-j\omega t_1}e^{j\nu t_2}dt_1 dt_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(t_1)g(t_2)R_{xx}(t_1,t_2)e^{-j\omega t_1}e^{j\nu t_2}dt_1 dt_2 \quad .$$

Since $x(t)$ is stationary,

$$R_{xx}(t_1,t_2) = R_{xx}(\tau) \quad ,$$

where $\tau = t_1 - t_2$. Substituting $\tau = t_1 - t_2$ in Equation (2.12) gives

$$R_{xx}(t_1,t_2) = \frac{\sin \beta(t_1 - t_2)}{(t_1 - t_2)} \cos \omega_o(t_1 - t_2) \quad ;$$

and $S_{yy}(\omega,\nu)$ becomes

$$S_{yy}(\omega,\nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(t_1)g(t_2)\,\frac{\sin\beta(t_1 - t_2)}{(t_1 - t_2)}\,\cdot$$

$$\cdot\ \cos\omega_o(t_1 - t_2)e^{-j\omega t_1}e^{j\nu t_2}dt_1dt_2$$

$$= \int_{-\infty}^{\infty} g(t_2)e^{j\nu t_2}\int_{-\infty}^{\infty} g(t_1)\,\frac{\sin\beta(t_1 - t_2)}{(t_1 - t_2)}\,\cdot$$

$$\cdot\ \cos\omega_o(t_1 - t_2)e^{-j\omega t_1}dt_1dt_2 \quad . \qquad (2.13)$$

The t_1 integration in Equation (2.13) can be obtained from Parseval's theorem,

$$\int_{-\infty}^{\infty} f_1(t)f_2^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\eta)F_2^*(\eta)d\eta \quad ,$$

where $F_1(\eta)$ and $F_2(\eta)$ are the Fourier transforms of $f_1(t)$ and $f_2(t)$, respectively. To apply Parseval's theorem here, let

$$f_1(t_1) = g(t_1)e^{-j\omega t_1}$$

and

$$f_2(t_1) = \frac{\sin\beta(t_1 - t_2)}{(t_1 - t_2)}\cos\omega_o(t_1 - t_2) \quad .$$

Then,

$$F_1(\eta) = G(\eta + \omega) \quad ,$$

where $G(\eta)$ is the Fourier transform of $g(t)$. Comparing $f_2(t)$ with the relations derived between $S_{xx}(\omega)$ and $R_{xx}(\tau)$, it is obvious that

$$F_2(\eta) = \begin{cases} \frac{\pi}{2} e^{-j\eta t_2} & \omega_o - \beta \le |\eta| \le \omega_o + \beta \\ 0 & \text{otherwise} \end{cases} ,$$

and, therefore, that the conjugate of $F_2(\eta)$ is

$$F_2^*(\eta) = \begin{cases} \frac{\pi}{2} e^{j\eta t_2} & \omega_o - \beta \le |\eta| \le \omega_o + \beta \\ 0 & \text{otherwise} \end{cases} .$$

Thus, the t_1 integration results in

$$\int_{-\infty}^{\infty} g(t_1)e^{-j\omega t_1} \frac{\sin \beta(t_2 - t_1)}{(t_1 - t_2)} \cos \omega_o(t_1 - t_2)dt_1 =$$

$$= \frac{1}{4}\int_{-(\omega_o+\beta)}^{-(\omega_o-\beta)} G(\eta + \omega)e^{j\eta t_2}d\eta + \frac{1}{4}\int_{\omega_o-\beta}^{\omega_o+\beta} G(\eta + \omega)e^{j\eta t_2}d\eta$$

and substituting in Equation (2.13), there results

$$S_{yy}(\omega,\nu) = \frac{1}{4}\int_{-\infty}^{\infty} g(t_2)e^{j\nu t_2}\left\{\int_{-(\omega_o+\beta)}^{-(\omega_o-\beta)} G(\eta+\omega)e^{j\eta t_2}d\eta\right.$$

$$\left.+\int_{\omega_o-\beta}^{\omega_o+\beta} G(\eta+\omega)e^{j\eta t_2}d\eta\right\} dt_2$$

$$= \frac{1}{4}\int_{-(\omega_o+\beta)}^{-(\omega_o-\beta)} G(\eta+\omega)\int_{-\infty}^{\infty} g(t_2)e^{j(\eta+\nu)t_2}dt_2 d\eta$$

$$+\int_{\omega_o-\beta}^{\omega_o+\beta} G(\eta+\omega)\int_{-\infty}^{\infty} g(t_2)e^{j(\eta+\nu)t_2}dt_2 d\eta$$

$$= \frac{1}{4}\int_{-(\omega_o+\beta)}^{-(\omega_o-\beta)} G(\eta+\omega)G^*(\eta+\nu)d\eta$$

$$+\frac{1}{4}\int_{\omega_o-\beta}^{\omega_o+\beta} G(\eta+\omega)G^*(\eta+\nu)d\eta \quad . \qquad (2.14)$$

Equation (2.14) is a useful form for the computation of the two-dimensional spectrum of a nonstationary random process which is the product of a modulation function and band-limited white noise. In the laboratory, reverberation is very often simulated in this way, although the band-limited white noise $x(t)$ is only approximated by a good band-pass filter.

In the event that it is desired to solve for the two-dimensional spectrum when $x(t)$ is generated by a low-pass filter, let $\omega = \beta$ in Equation (2.14) and there results

$$S_{yy}(\omega,\nu) = \frac{1}{4}\int_{-2\beta}^{2\beta} G(\eta + \omega)G^*(\eta + \nu)d\eta \quad . \tag{2.15}$$

2.2.3 Example 1. Two-Dimensional Power Spectrum of Exponentially Decaying Band-Limited White Noise. Consider the nonstationary random process $y(t)$ obtained by multiplying band-limited white noise $x(t)$ by the exponential modulation function

$$g(t) = \begin{cases} e^{-\alpha t} & t \geq 0 \ (\alpha \geq 0) \\ 0 & t < 0 \end{cases} \quad .$$

If the spectrum of $x(t)$ is given by

$$S_{xx}(\omega) = \begin{cases} \pi & |\omega| \leq \beta \\ 0 & \text{otherwise} \end{cases} \quad ,$$

then, from Equation (2.15), it follows immediately that

$$S_{yy}(\omega,\nu) = \frac{1}{4}\int_{-2\beta}^{2\beta} \frac{1}{(j\eta + j\omega + \alpha)(-j\eta - j\nu + \alpha)}\, d\eta \quad . \tag{2.16}$$

Multiplying the numerator and denominator by (j^2), there results

$$S_{yy}(\omega,\nu) = -\frac{1}{4}\int_{-2\beta}^{2\beta} \frac{1}{(-\eta - \omega + j\alpha)(\eta + \nu + j\alpha)}\, d\eta \quad . \tag{2.17}$$

From Pierce's Tables,[16] integral number 42,

$$S_{yy}(\omega,\nu) = \frac{1}{4(\omega - \nu - 2j\alpha)} \ln \left(\frac{\nu + \eta + j\alpha}{-\omega - \eta + j\alpha}\right) \Bigg|_{\eta = -2\beta}^{\eta = 2\beta}$$

$$= \frac{1}{4(\omega - \nu - 2j\alpha)} \ln\left[\frac{(\nu + 2\beta + j\alpha)}{(\omega + 2\beta - j\alpha)}\,\frac{(\omega - 2\beta - j\alpha)}{(\nu - 2\beta + j\alpha)}\right] . \tag{2.18}$$

2.3 One-Dimensional Energy Spectrum

In this section, it will be shown that, if $\omega = \nu$ in Equation (2.1), the resulting spectrum $S_{yy}(\omega,\omega)$ is the energy spectrum and that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{yy}(\omega,\omega)\, d\omega = \int_{-\infty}^{\infty} E[y^2(t)]\, dt \quad . \tag{2.19}$$

Also, $S_{yy}(\omega,\omega)$ is equal to $\overline{S}_{yy}(\omega)$ defined by

$$\overline{S}_{yy}(\omega) = \int_{-\infty}^{\infty} \overline{R}_{yy}(\tau) e^{-j\omega\tau} d\tau \quad , \tag{2.20}$$

where $\overline{R}_{yy}(\tau)$ is the autotranslation function defined by

$$\overline{R}_{yy}(\tau) = E\left[\int_{-\infty}^{\infty} y(t+\tau)y(t)dt\right] \tag{2.21a}$$

$$= \int_{-\infty}^{\infty} R_{yy}(t+\tau,t)dt \quad . \tag{2.21b}$$

The spectrum $\overline{S}_{yy}(\omega)$ defined by Equations (2.20) and (2.21) is, except for the expected value, identical to the energy spectrum of a transient function as described by Y. W. Lee.

This suggests considering nonstationary processes with finite energy as simply an ensemble of random transients which are treated in the frequency domain in the same way as deterministic transients, except that an ensemble average of the spectra must be computed. This technique is suitable if one is interested in the expected output power of a filter integrated over time as when the ratio of signal-to-noise energy is being computed. This technique is not suitable if one is interested in the expected output as a function of time as when the ratio of signal-to-noise power at a particular time is being computed. In this case, the two-dimensional spectrum is needed.

To prove that

$$S_{yy}(\omega,\omega) = \overline{S}_{yy}(\omega) \quad , \tag{2.22}$$

take Equation (2.2) which follows directly from the definition of $S_{yy}(\omega,\nu)$, Equation (2.1), and let $\tau = t_1 - t_2$. This gives

$$R_{yy}(t_2 + \tau, t_2) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{yy}(\omega,\nu)e^{j\omega(t_2+\tau)}e^{-j\nu t_2}d\omega d\nu \quad . \tag{2.23}$$

Now, integrating both sides of Equation (2.23) with respect to t_2 gives

$$\int_{-\infty}^{\infty} R_{yy}(t_2 + \tau, t_2)dt_2 = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{yy}(\omega,\nu)e^{j\nu t} \cdot \int_{-\infty}^{\infty} e^{j(\omega-\nu)t_2}dt_2 d\omega d\nu \quad . \tag{2.24}$$

Comparing the left side of Equation (2.24) with the definition of the autotranslation function, Equation (2.21), and recognizing that

$$\delta(\omega - \nu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j(\omega-\nu)t_2}dt_2 \quad ,$$

where $\delta(\omega - \nu)$ is the Dirac delta function, there results

$$\overline{R}_{yy}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{yy}(\omega,\omega)e^{j\omega\tau}d\omega \quad . \tag{2.25}$$

Finally, recognizing the right side of Equation (2.25) as the transform of $S_{yy}(\omega,\omega)$ shows that taking the transform of both sides of Equation (2.25 gives

$$\int_{-\infty}^{\infty} \overline{R}_{yy}(\tau)e^{-j\omega\tau}d\tau = S_{yy}(\omega,\omega) \quad . \tag{2.26}$$

Since the left side of Equation (2.26) defines $\overline{S}_{yy}(\omega)$, Equation (2.22) is proven.

Taking the equation for the expected output power of a filter, Equation (2.5), and integrating both sides with respect to t , there results

$$\int_{-\infty}^{\infty} E[z^2(t)]dt = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(\omega)H^*(\nu)S_{yy}(\omega,\nu)\int_{-\infty}^{\infty} e^{j(\omega-\nu)t}dt\,d\omega\,d\nu$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(\omega)H^*(\nu)S_{yy}(\omega,\nu)\delta(\omega - \nu)d\omega\,d\nu$$

and

$$E\left[\int_{-\infty}^{\infty} z^2(t)dt\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)H^*(\omega)\overline{S}_{yy}(\omega)d\omega \quad . \tag{2.27}$$

Equation (2.27) shows that $\overline{S}_{yy}(\omega)$ is an energy spectrum. If $H(\omega)$ is the transfer function of an ideal band-pass filter,

$$H(\omega) = \begin{cases} 1 & \omega_\ell \le |\omega| \le \omega_H \\ 0 & \text{otherwise} \end{cases} ,$$

then Equation (2.27) states that

$$E\left[\int_{-\infty}^{\infty} z^2(t)dt\right] = \frac{1}{2\pi}\int_{\omega_\ell}^{\omega_h} \bar{S}_{yy}(\omega)d\omega \quad .$$

In other words, $\bar{S}_{yy}(\omega)$ gives the energy associated with the frequency ω . Letting $H(\omega)$ be an all-pass filter, that is, letting $H(\omega) = 1$ for all ω , immediately proves Equation (2.19).

2.3.1 Example 2. One-Dimensional Energy Spectrum of Exponentially Decaying Band-Limited White Noise. The one-dimensional energy spectrum of the nonstationary random process described in Example 1 can be immediately obtained from Equation (2.16) by setting $\omega = \nu$.

$$\bar{S}_{yy}(\omega) = \frac{j}{8\alpha} \ln \frac{(\omega + 2\beta + j\alpha)}{(\omega + 2\beta - j\alpha)} \frac{(\omega - 2\beta - j\alpha)}{(\omega - 2\beta + j\alpha)}$$

$$= \frac{j}{8\alpha} \ln \frac{\omega^2 - 4\beta^2 + \alpha^2 - 4j\alpha\beta}{\omega^2 - 4\beta^2 + \alpha^2 + 4j\alpha\beta} ,$$

let

$$r = [(\omega^2 - 4\beta^2 + \alpha^2)^2 + (4\alpha\beta)^2]^{1/2} ,$$

and

$$\phi = \tan^{-1} \frac{4\alpha\beta}{\omega^2 - 4\beta^2 + \alpha^2} \quad .$$

Then,

$$\overline{S}_{yy}(\omega) = \frac{j}{8\alpha} \ln \frac{re^{-j\phi}}{re^{j\phi}}$$

$$= \frac{j}{8\alpha} \ln e^{-2j\phi}$$

$$= \frac{j}{8\alpha} (-2j\phi)$$

$$= \frac{1}{4\alpha} \tan^{-1} \frac{4\alpha\beta}{\omega^2 - 4\beta^2 + \alpha^2} \quad .$$

2.4 Instantaneous Power Spectrum

Another well-known spectral representation of a nonstationary random process is the instantaneous power spectrum, first proposed by Page.[7] Page did not propose it as a spectrum for a nonstationary process, but as a qualitative method of determining when a suddenly applied noise had lasted long enough to be considered stationary. Lampard[8] independently developed the same concept as a generalization of the Wiener-Khintchine Theorem for nonstationary process. Turner[9] and Levin[10] developed further the concept.

If the instantaneous power spectrum of a stationary random process is computed, the same answer is obtained as from the Wiener-Khintchine Theorem,

$$S_{yy}(\omega) = 2 \int_0^\infty R_{yy}(\tau) \cos \omega\tau d\tau \quad .$$

However, for nonstationary process, the instantaneous power spectrum can be negative at some frequencies, although the integral of the spectrum over all frequencies is always positive and equal to the instantaneous power.

A more serious objection to the use of the instantaneous power spectrum is that no published technique exists for obtaining information about the output of a filter from its transfer function and the instantaneous power spectra.

To develop the mathematical representation of the instantaneous power spectrum, suppose that all the samples $y(t)$ of a random process are truncated at some time $t = T$. If the time function of a truncated sample is represented by

$$y^T(t) \equiv \begin{cases} y(t) & t \leq T \\ 0 & t > T \end{cases} , \qquad (2\text{-}28)$$

then the autocorrelation function of the nonstationary process consisting of all the truncated samples would be

$$R_{y_T y_T}(t_1,t_2) \equiv E[y^T(t_1)y(t_2)]$$

$$= \begin{cases} R_{yy}(t_1,t_2) & t_1 \leq T,\ t_2 \leq T \\ 0 & \text{otherwise} \end{cases} .$$

The two-dimensional power spectrum of the ensemble of $y_T(t)$ is

$$S_{y_T y_T}(\omega,\nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{y_T y_T}(t_1,t_2)e^{-j\omega t_1}e^{j\nu t_2}dt_1 dt_2$$

$$= \int_{-\infty}^{T}\int_{-\infty}^{T} R_{yy}(t_1,t_2)e^{-j\omega t_1}e^{j\nu t_2}dt_1 dt_2 \quad .$$

In Section 2.3, it was shown that, if $\omega = \nu$ in the two-dimensional spectrum, the resultant spectrum is the energy spectrum. That is,

$$S_{y_T y_T}(\omega,\nu) = \overline{S}_{y_T y_T}(\omega)$$

$$= \int_{-\infty}^{T}\int_{-\infty}^{T} R_{yy}(t_1,t_2)e^{-j(t_1-t_2)}dt_1 dt_2$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{T} S_{y_T y_T}(\omega)d\omega = \int_{-\infty}^{T} E[y^2(t)]dt \quad . \qquad (2.29)$$

The instantaneous power spectrum is defined as the derivative of $\overline{S}_{y_T y_T}(\omega)$ with respect to T . Letting $\rho(T,\omega)$ represent the instantaneous power spectrum

$$\rho(T,\omega) \equiv \frac{\partial}{\partial T}\overline{S}_{y_T y_T}(\omega) \quad . \qquad (2.30)$$

Differentiating both sides of Equation (2.29) with respect to T ,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\partial}{\partial T}\,\overline{S}_{y_T y_T}(\omega)\,d\omega = E[y^2(T)] \quad ,$$

and making use of the definition, Equation (2.30),

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\rho(T,\omega)\,d\omega = E[y^2(T)] \quad . \tag{2.31}$$

Equation (2.31) justifies the name, instantaneous power spectrum, since its integral over ω is the expected instantaneous power at time $t = T$.

To derive the relation between the instantaneous power spectrum and the autocorrelation function of the ensemble of truncated samples, apply Equation (2.21b) to $y^T(t)$,

$$\overline{R}_{y_T y_T}(\tau) = \int_{-\infty}^{T} R_{yy}(t+\tau,t)\,dt \qquad \tau \leq 0 \quad . \tag{2.32}$$

Since $\overline{R}_{y_T y_T}(\tau)$ is an even function, it is sufficient to write the integral for negative τ only, and the energy spectrum can be written from Equation (2.20) as

$$\overline{S}_{y_T y_T}(\omega) = \int_{-\infty}^{\infty}\overline{R}_{y_T y_T}(\tau)e^{-j\omega\tau}\,d\tau$$

$$= 2\int_{-\infty}^{\infty} \overline{R}_{y_T y_T}(\tau) \cos \omega\tau d\tau \quad . \tag{2.33}$$

Substituting Equation (2.32) in Equation (2.33),

$$\overline{S}_{y_T y_T}(\omega) = 2\int_{-\infty}^{0} \left[\int_{-\infty}^{T} R_{yy}(t + \tau, t) dt \right] \cos \omega\tau d\tau$$

$$= 2\int_{-\infty}^{T} dt \int_{-\infty}^{0} R_{yy}(t + \tau, t) \cos \omega\tau d\tau \quad . \tag{2.34}$$

Differentiating both sides of Equation (2.34), there results

$$\frac{\partial}{\partial T} \overline{S}_{y_T y_T}(\omega) = 2\int_{-\infty}^{0} R_{yy}(T + \tau, T) \cos \omega\tau d\tau \quad , \tag{2.35}$$

or

$$\rho(T,\omega) = 2\int_{-\infty}^{0} R_{yy}(T + \tau, T) \cos \omega\tau d\tau \quad .$$

For $y(t)$, a stationary random process $R_{yy}(T + \tau, T)$ is independent of T and can be replaced by $R_{yy}(\tau)$, giving

$$\rho(T,\omega) = 2\int_{-\infty}^{0} R_{yy}(\tau) \cos \omega\tau d\tau = S_{yy}(\omega) \quad .$$

2.4.1 <u>Instantaneous Power Spectrum of an Exponentially Decaying Low-Pass Noise</u>. As an example of the instantaneous power spectrum, consider the following signal:

$$y(t) = \begin{cases} e^{-\alpha t} & t \geq 0 \\ 0 & t < 0 \end{cases} ,$$

where $x(t)$ is a stationary random process with autocorrelation function $e^{-\beta|\tau|}$. The spectrum has been changed from that used in Example 1 to avoid difficulties arising from the $\sin(\beta\tau)/\tau$ which would appear in such a way as to make the required integration impossible. For the autocorrelation function given here, the spectrum is

$$S_{xx}(\omega) = \frac{2\beta}{\omega^2 + \beta^2} .$$

For $t \geq 0$, the autocorrelation function of $y(t)$ is

$$R_{yy}(t + \tau, t) = \begin{cases} E[e^{-\alpha(t+\tau)} x(t+\tau) e^{-\alpha t} x(t)] & \tau \geq -t,\ t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} e^{-2\alpha t} e^{-\alpha\tau} R_{xx}(\tau) & \tau \geq -t,\ t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} e^{-2\alpha t} e^{-\alpha\tau} e^{-\beta|\tau|} & \tau \geq -t,\ t \geq 0 \\ 0 & \text{otherwise} \end{cases} .$$

From Equation (2.35), the instantaneous power spectrum is, for $T \geq 0$,

$$\rho(T,\omega) = 2\int_{-T}^{0} R_{yy}(T+\tau,T)\cos\omega\tau\, d\tau$$

$$= 2e^{-2\alpha T}\int_{-T}^{0} e^{-\alpha\tau}e^{\beta\tau}\cos\omega\tau d\tau$$

$$= 2e^{-2\alpha T}\left\{ \frac{e^{(\beta-\alpha)\tau}(\beta-\alpha)\cos\omega\tau + \omega\sin\omega\tau}{(\beta-\alpha)^2+\omega^2} \Bigg|_{-T}^{0} \right\}$$

$$= 2e^{-2\alpha T}\left\{ \frac{(\beta-\alpha) - e^{-(\beta-\alpha)T}[(\beta-\alpha)\cos\omega T - \omega\sin\omega T]}{(\beta-\alpha)^2+\omega^2} \right\}$$

Observe that for $\beta >> \alpha$, this reduces to

$$\rho(T,\omega) \cong e^{-2\alpha T}\frac{2\beta}{\beta^2+\omega^2} \qquad T \geq 0 \quad ,$$

which is the product of the square of the modulation function and the spectrum of $x(t)$. This would be intuitively expected for a power spectrum of the process.

CHAPTER III

REDUCTION TO AN EQUIVALENT STATIONARY PROBLEM

3.1 Summary of the Result

In some cases, the nonstationary random process is written as the product of a stationary random process $x(t)$ and a modulation function $g(t)$. Brown[11] showed that the problem of computing some output statistics of a filter for the input $g(t)x(t)$ can often be reduced to the problem of finding the corresponding output statistics of a different filter for the input $x(t)$, a stationary random process. Miller[12] generalized Brown's result to a random modulation function $g(t)$. Here, only a deterministic modulation function is considered. Specifically, if $h(t)$ is the filter weighting function, the transfer function of a new filter is computed:

$$B(\omega,t) = \int_{-\infty}^{\infty} h(t - \xi)g(\xi)e^{-j\omega\xi}d\xi \quad .$$

The mean square value of the output of the new filter in response to the stationary input $x(t)$ is the same as the output of the actual filter to the nonstationary input $g(t)x(t)$. If $z(t)$ is the filter output, then

$$E[z^2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} |B(\omega,t)|^2 \, S_{xx}(\omega)d\omega$$

as in stationary problems.

3.2 Proof of the Result

The steps in Brown's proof are repeated here. If $y(t) = g(t)x(t)$ is the input to the filter having the weighting function $h(t)$, the filter output is

$$z(t) = \int_{-\infty}^{\infty} h(t - \xi)y(\xi)d\xi \quad . \tag{3.1}$$

$z(t)$ can be considered as a sample of the output or as a random variable. Assume that $h(t)$ and $g(t)$ are real values with $g(t)$ uniformly bounded on $(-\infty,\infty)$ and that $h(t)$ is absolutely integrable and square integrable. In addition, assume $x(t)$ is wide-sense stationary with square integrable autocorrelation function $R_{xx}(\tau)$ and spectrum $S_{xx}(\omega)$.

The autocorrelation function of the filter output is given by

$$R_{zz}(t_1,t_2) = E\left[\int_{-\infty}^{\infty} h(t_1 - \xi)g(\xi)x(\xi)d\xi \int_{-\infty}^{\infty} h(t_2 - \eta)g(\eta)x(\xi)d\xi\right], \tag{3.2a}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t_1 - \xi)h(t_2 - \eta)g(\eta)g(\xi)E[x(\xi)x(\eta)]d\xi d\eta \quad , \tag{3.2b}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t_1 - \xi)h(t_2 - \eta)g(\eta)g(\xi)R_{xx}(\xi - \eta)d\xi d\eta \quad . \tag{3.2c}$$

Now, we define $B(\omega,t)$ as follows:

$$B(\omega,t) = \int_{-\infty}^{\infty} h(t - \xi)g(\xi)e^{-j\omega\xi}d\xi \quad . \tag{3.3}$$

Then, $B(\omega,t)$ is the Fourier transform of $h(t - \xi)g(\xi)$, and $S_{xx}(\omega)e^{j\eta\omega}$ the Fourier transform of $R_{xx}(\xi - \eta)$. Then, by Parseval's theorem, the ξ integration in Equation (3.2c) becomes

$$\int_{-\infty}^{\infty} h(t_1 - \xi)g(\xi)R_{xx}(\xi - \eta)d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} B(\omega,t_1)S_{xx}(\omega)e^{j\eta\omega}d\omega \quad . \tag{3.4}$$

Then, substituting Equation (3.4) in Equation (3.2c):

$$R_{zz}(t_1,t_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B(\omega,t_1)S_{xx}(\omega)h(t_2 - \eta)g(\eta)e^{j\eta\omega}d\omega d\eta \tag{3.5a}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} B(\omega,t_1)B^*(\omega,t_2)S_{xx}(\omega)d\omega \quad . \tag{3.5b}$$

The expected value of the output power is obtained by letting $t_1 = t_2$,

$$E[z^2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} |B(\omega,t)|^2 S_{xx}(\omega)d\omega \quad . \tag{3.6}$$

This is the desired result which will be used extensively in Chapter V.

An alternate form for computation of $B(\omega,t)$ can be obtained by applying Parseval's theorem to the definition of $B(\omega, t)$, if $g(t)$ is also square integrable. Using the time and frequency translation theorems for Fourier transforms [Campbell and Foster,[15] pair (206) and (207)], the Fourier transform of the terms $h(t - \xi)$ and $g(\xi)e^{-j\omega\xi}$ in the integrand of Equation (3.3) are

$$\int_{-\infty}^{\infty} h(t - \xi)e^{-j\eta\xi}d\xi = e^{j\eta t}H(\eta) \quad . \tag{3.7a}$$

$$\int_{-\infty}^{\infty} [g(\xi)e^{-j\omega\xi}]e^{-j\eta\xi}d\xi = G(\eta + \omega) \quad , \tag{3.7b}$$

and, using Parseval's theorem, Equation (3.3) can be written in the alternate form:

$$B(\omega,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\eta)G(\eta + \omega)e^{j\eta t}d\eta \quad . \tag{3.8}$$

3.3 Interchange of the Modulation Function with the Filter Weighting Function

It can also be shown that the same results are obtained if the modulation function and filter weighting function are interchanged. First, write the convolution integral in the alternate form:

$$z(t) = \int_{-\infty}^{\infty} h(\xi)y(t - \xi)d\xi \quad ,$$

then,

$$R_{zz}(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\xi)h(\eta)g(t_1 - \xi)g(t_2 - \eta) \cdot$$

$$\cdot \quad E[x(t - \xi)x(t - \eta)]d\xi d\eta \tag{3.9a}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\xi)h(\eta)g(t_1 - \xi)g(t_2 - \eta)R_{xx}(\xi - \eta)d\xi d\eta \quad . \tag{3.9b}$$

Define

$$B'(\omega,t_1) = \int_{-\infty}^{\infty} h(\xi)g(t_1 - \xi)e^{-j\omega\xi}d\xi \tag{3.10}$$

similar to the definition of $B(\omega,t)$.

Then,

$$R_{zz}(t_1,t_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B'(\omega,t_1)S_{xx}(\omega)h(\eta)g(t_2-\eta)e^{j\eta\omega}d\omega d\eta \tag{3.11a}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} B'(\omega,t_1)B^*(\omega,t_2)S_{xx}(\omega)d\omega \quad . \tag{3.11b}$$

Comparison of Equation (3.11b) with Equation (3.5b) shows that the output autocorrelation function is unchanged if we interchange the modulation function and the filter weighting function.

3.4 Practical Application

The advantage of this technique, when it can be applied, is that the problem can immediately be reduced to a stationary problem whenever the Fourier transform of $h(t)$ and $g(t)$ can be determined. Then, after computing the transfer function of the imaginary filter $B(\omega,t)$, we can solve for the power output as a function of time. Although, in general, $B(\omega,t)$ is a complicated filter transfer function so that the integral $\int|B|^2 S_{xx}d\omega$ will often be difficult, this integration can usually be done graphically or numerically. It is also worthwhile to observe that the time variation of the filter power output can be determined from $B(\omega,t)$ without doing the integration over frequency.

Chapter V uses Brown's result to solve for the output power of an RLC filter subjected to a reverberation input. It will be seen that the calculations are quite complicated and require a computer for all but an asymptotic solution.

CHAPTER IV

OPTIMUM BANDWIDTH OF AN RC FILTER FOR DETECTION OF A PULSE IN EXPONENTIALLY DECAYING WHITE NOISE

4.1 Introduction

Schwartz,[2] in discussing matched filters for detection of pulses in stationary white noise, also gives some comparative results for an RC low-pass filter. Schwartz shows that the optimum bandwidth of the RC filter is 0.2 divided by the pulse length and that the output signal-to-noise ratio of such a filter is only 1 dB less than the output signal-to-noise ratio of the matched filter. The purpose of this chapter is to perform some similar calculations for white noise in which the spectrum level varies exponentially with time, according to the modulation function,

$$[g(t)]^2 = \begin{cases} e^{-2\alpha t} & t \geq 0 \\ 0 & t < 0 \end{cases} \quad . \tag{4.1}$$

It will be found that the optimum bandwidth is a function of time for small t and a constant for large t and that it is different from the optimum bandwidth for a stationary random noise. In the case when $\alpha = 0$, the noise becomes stationary for very large t and, as expected, the optimum bandwidth for large t is the same as that computed by Schwartz.

4.2 Practical Significance of the Problem

From the point of view of the designer, the practical significance of this problem is not the detection of a rectangular pulse in white noise. It is fact that the solution is related to the practical problem of detecting a pulsed sine wave in white noise using a simple RLC filter. Specifically, if the frequency of the pulsed sine wave is the same as the resonant frequency of the RLC filter, the optimum bandwidth of the RLC filter is exactly twice the optimum bandwidth computed for the RC filter and rectangular pulse.

4.3 Optimum Bandwidth for Stationary White Noise

Figure 1 shows the response of an RC filter to a rectangular pulse, the analogous response of an RLC filter to a pulsed sine wave and the white noise background. The definitions of signal power and noise power are also illustrated in this sketch. It will be useful to first compute the output signal-to-noise ratio in a general form. This will provide an upper bound on the output signal-to-noise ratio which is attained by the matched filter. Then, the output signal-to-noise ratio will be computed for an RC filter. The purpose of these computations is to provide a basis for comparison with the exponentially varying noise.

If the Fourier transform of the signal is $V_1(\omega)$, then the Fourier transform of the filter output due to the signal is

$$V_o(\omega) = V_1(\omega)H(\omega) \quad .$$

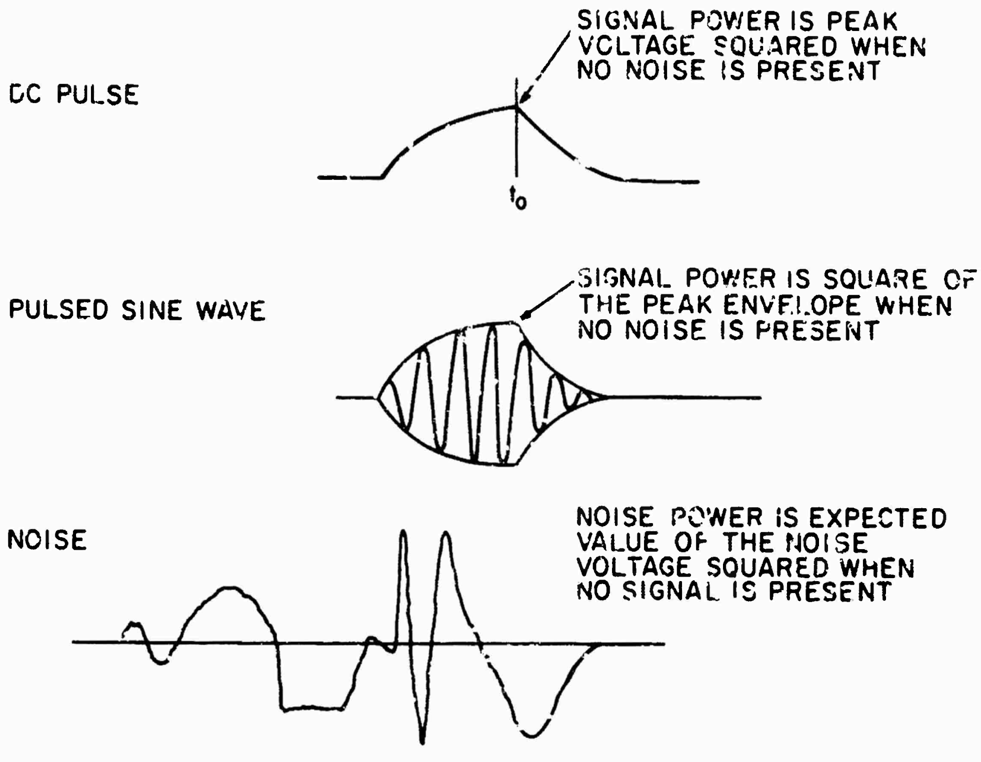

Figure 1. Definition of Signal and Noise Power

The filter output as a function of time is the inverse transform of $V_o(\omega)$,

$$S_o(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} V_i(\omega)H(\omega)e^{j\omega t}d\omega \quad .$$

If t_o is the time at which the filter output due to signal alone is maximum, then the peak signal output power is

$$S_o^2(t_o) = \frac{1}{(2\pi)^2}\left|\int_{-\infty}^{\infty} V_i(\omega)H(\omega)e^{j\omega t_o}d\omega\right|^2 \quad .$$

For white noise with a spectrum level of r watts per Hz over positive frequencies only, or of $r/2$ watts per Hz over both positive and negative frequencies, the expected noise output power is

$$E[n_o^2] = \frac{r}{4\pi}\int_{-\infty}^{\infty} |H(\omega)|^2 d\omega \quad ,$$

giving an output signal-to-noise ratio of

$$\frac{S_o^2}{E[n_o^2]} = \frac{1}{\pi}\,\frac{\left|\int_{-\infty}^{\infty} V_i(\omega)H(\omega)e^{j\omega t_o}d\omega\right|^2}{r\int_{-\infty}^{\infty} |H(\omega)|^2 d\omega} \quad .$$

From Parseval's theorem, the signal energy is

$$E = \int_{-\infty}^{\infty} S_i^{\,2}(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |V_i(\omega)|^2 d\delta \quad .$$

Multiplying both signal and noise by the signal energy, the output signal-to-noise ratio can be expressed as

$$\frac{S_o^{\,2}}{E[n_o^{\,2}]} = \frac{2E}{r}\,\mu \quad , \tag{4.2}$$

where

$$\mu \equiv \frac{\left| \int_{-\infty}^{\infty} V_i(\omega)H(\omega)e^{j\omega t_o}d\omega \right|^2}{\int_{-\infty}^{\infty} |V_i(\omega)|^2 d\omega \int_{-\infty}^{\infty} |H(\omega)|^2 d\omega} \quad .$$

From Schwarz inequality, the maximum value of $\mu = 1$, so the maximum possible output signal-to-noise ratio is $2E/r$. The filter transfer function, such that $\mu = 1$ is, by definition, a matched filter. For the rectangular pulse in white noise background,

$$\left(\frac{S_o^{\,2}}{E[n_o^{\,2}]} \right)_{max} = \frac{2V^2T}{r} \quad .$$

Having derived an upper bound on the output signal-to-noise ratio which is obtainable with a matched filter, what is the maximum output signal-to-noise ratio obtainable with a simple RC filter? Applying a pulse of amplitude V and duration T to an RC filter gives an output

$$S_o(t) = \begin{cases} V(1 - e^{-t/RC}) \\ V\{1 - e^{-t/RC} - [1 - e^{-(t-T)/RC}]\} \end{cases} \qquad (4.3)$$

which is maximum at $t = T$. The maximum value of $S_o^2(t)$ is, therefore,

$$S_o^2(T) = V^2(1 - e^{-WT})^2 \quad . \qquad (4.4)$$

where $W = 1/RC$. The output noise power is

$$E[n_o^2] = \frac{\pi}{2}\frac{1}{2\pi RC} r = \frac{r}{4RC} = \frac{rW}{4} \quad ,$$

since the equivalent noise bandwidth of an RC filter is $(\pi/2)$ times the half-power bandwidth. The output signal-to-noise ratio is, therefore,

$$\frac{S_o^2}{E[n_o^2]} = \frac{2V^2T}{r}\frac{(1 - e^{-WT})^2}{WT/2} \quad . \qquad (4.5)$$

Comparing this equation with Equation (4.2), it is seen that

$$\mu = \frac{(1 - e^{-WT})^2}{WT/2} \quad . \qquad (4.6)$$

The maximum value of μ occurs at

$$WT = 2\pi \cdot 0.2 = 0.4\pi \quad ,$$

which gives

$$\mu = \frac{(1 - e^{-0.4\pi})^2}{0.2\pi} = 0.81 \quad ,$$

only 1 dB below 1.0 provided by the matched filter.

4.4 Optimum Bandwidth for Exponentially Decaying White Noise

In order to perform analogous calculations for exponentially decaying white noise, let the noise be equal to the modulation function $g(t)$ times stationary white noise having spectrum level r . Since the signal will be exactly the same, it will be desirable to express the output signal-to-noise ratio in the form

$$\frac{S_o^2(t)}{E[n_o^2(t)]} = \frac{2V^2T}{re^{-2\alpha t}} \mu(T) \quad ,$$

where $\mu(t)$ is a function of time to be determined by the following calculations. It will be assumed that the signal pulse has always arrived T seconds before t so that the signal reaches its maximum value at the filter output at time t .

4.4.1 Time-Varying Expected Noise Power.

To compute the expected noise power as a function of time, the two-dimensional power spectrum derived in Example 1 can be used. Since the assumed spectral density of the stationary random process is $r/2$ watts per Hz for a spectrum containing positive and negative frequencies, the spectrum assumed in Example 1 must be multiplied by $r/2\pi$. Then, from Equation (2.16),

$$S_{yy}(\omega,\nu) = \frac{r}{4\pi}\int_{-\infty}^{\infty} \frac{1}{(j\eta + j\omega + \alpha)(-j\eta - j\nu + \alpha)}\, d\eta \quad . \tag{4.7}$$

The transfer function of an RC low-pass filter is

$$H(\omega) = \frac{1/RC}{j\omega + 1/RC} \quad . \tag{4.8}$$

Substituting Equations (4.7) and (4.8) in Equation (2.5) gives the expression for the expected output noise:

$$E[n_o^2(t)] = \frac{r}{2(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{1/RC\ e^{j\omega t}}{j\omega + 1/RC}\right)\left(\frac{1/RC\ e^{-j\nu t}}{-j\nu + 1/RC}\right) \cdot$$

$$\cdot \int_{-\infty}^{\infty} \frac{d\eta}{(\alpha + j\eta + j\omega)(\alpha - j\eta - j\nu)}\, d\omega d\nu \quad .$$

Reversing the order of integration gives

$$E[n_o^2(t)] = \frac{r}{2(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1/RC\ e^{j\omega t}}{(j\omega + 1/RC)(j\omega + j\eta + \alpha)}\, d\omega \cdot$$

$$\cdot \int_{-\infty}^{\infty} \frac{1/RC\ e^{-j\nu t}}{(-j\nu + 1/RC)(-j\nu - j\eta + \alpha)}\, d\nu d\eta \quad .$$

Using Campbell and Foster,[15] transform pair 448,

$$\int_{-\infty}^{\infty} \frac{1/RC \; e^{j\omega t}}{(j\omega + 1/RC)(j\omega + j\eta + \alpha)} \, d\omega = \frac{2\pi}{RC} \; \frac{e^{-(\alpha + j\eta)t} - e^{-t/RC}}{-j\eta - (\alpha - 1/RC)} \quad ,$$

and, therefore,

$$\int_{-\infty}^{\infty} \frac{1/RC \; e^{-j\nu t}}{(-j\nu + 1/RC)(-j\nu - j\eta + \alpha)} \, d\nu = - \int_{\infty}^{-\infty} \frac{1/RC \; e^{j\nu t}}{(j\nu + 1/RC)(j\nu - j\eta + \alpha)} \, d\nu$$

$$= \int_{-\infty}^{\infty} \frac{1/RC \; e^{j\nu t}}{(j\nu + 1/RC)(j\nu - j\eta + \alpha)} \, d\nu$$

$$= \frac{2\pi}{RC} \; \frac{e^{-(\alpha - j\eta)t} - e^{-t/RC}}{j\eta - (\alpha - 1/RC)} \quad .$$

Substituting these transforms into the previous expression for $E[n_o^2(t)]$,

$$E[n_o^2(t)] = \frac{r}{4\pi (RC)^2} \left\{ \left(e^{-2\alpha t} + e^{-2t/RC} \right) \cdot \right.$$

$$\cdot \int_{-\infty}^{\infty} \frac{d\eta}{\eta^2 + (\alpha - 1/RC)^2} - 2e^{-(\alpha + 1/RC)t} \cdot$$

$$\left. \cdot \int_{-\infty}^{\infty} \frac{\cos \eta t}{\eta^2 + (\alpha - 1/RC)^2} \, \eta \right\} \quad .$$

The first integral on the left is

$$\lim_{\beta\to\infty} \int_{-\beta}^{\beta} \frac{1}{\eta^2 + (W - \alpha)^2}\, d\eta = \frac{2}{(W - \alpha)} \left\{ \lim_{\beta\to\infty} \tan^{-1} \frac{\eta}{W - \alpha} \Bigg|_{\eta=\beta}^{\eta=\beta} \right\}$$

$$= \frac{\pi}{|W - \alpha|} \quad . \tag{4.10}$$

To obtain the integral on the right, let

$$u = \frac{\eta}{|W - \alpha|} \quad .$$

Then,

$$\int_{-\infty}^{\infty} \frac{\cos \eta t}{\eta^2 + (W - \alpha)^2}\, d\eta = \frac{|W - \alpha|}{(W - \alpha)^2} \int_{-\infty}^{\infty} \frac{\cos[(W - \alpha)ut]}{1 + u^2}\, du \quad ,$$

which is a tabulated definite integral [Peirce,[16] number (505)],

$$\int_{0}^{\infty} \frac{\cos ax}{1 + x^2}\, dx = \frac{\pi}{2} e^{-a} \quad . \tag{4.11}$$

Finally, substituting the two integrals in Equation (4.9),

$$E[n_o^2(t)] = \frac{W^2 r}{4|W - \alpha|} \left\{ e^{-2\alpha t} + e^{-2Wt} - 2e^{-(W+\alpha)t} e^{-|W-\alpha|t} \right\} \quad .$$

If $(W - \alpha) \geq 0$,

$$e^{-(W + \alpha)t} e^{-|W - \alpha|t} = e^{-2Wt} \quad ,$$

and, if $(W - \alpha) \leq 0$,

$$e^{-(W+\alpha)t}e^{-|W-\alpha|t} = e^{-2\alpha t} \quad .$$

This gives the expression for the output noise:

$$E[n_o^2(t)] = \frac{W^2 r}{4|W-\alpha|} \left| e^{-2\alpha t} - e^{-2Wt} \right| \quad . \tag{4.12}$$

The expected output noise power is plotted in Figure 2 as a function of time for $\alpha = 1$ and $W = 10\pi$, a 5 Hz low-pass filter.

4.4.2 Time-Varying Optimum Bandwidth. The output signal-to-noise ratio obtained from Equations (4.4) and (4.12) is

$$\frac{s_o^2(T)}{E[n_o^2(t)]} = \frac{4|W-\alpha|}{W^2 r} \frac{V^2(1-e^{-WT})^2}{|e^{-2\alpha t} - e^{-2WT}|} \quad . \tag{4.13}$$

To compare this with Equation (4.5), the corresponding equation for a stationary noise background, note that r is analogous to $re^{-2\alpha t}$,

$$\frac{s_o^2(T)}{E[n_o^2(t)]} = \frac{2E}{re^{-2\alpha t}} \frac{(1-e^{-WT})^2}{\left(\frac{W^2}{2|W-\alpha|}\right) \cdot T \cdot \left|1 - e^{-2(\omega-\alpha)T}\right|} \quad . \tag{4.14}$$

The function $\mu(t)$ is, therefore, given by

$$\mu(t) = \frac{(1-e^{-WT})^2}{\left(\frac{W^2}{2|W-\alpha|}\right) \cdot T \cdot \left|1 - e^{-2(\omega-\alpha)T}\right|} \quad . \tag{4.15}$$

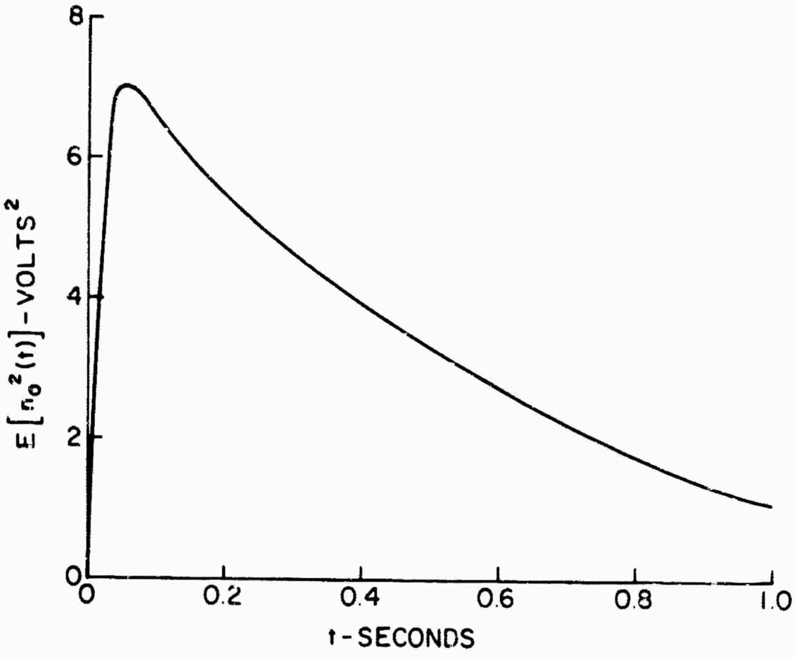

Figure 2. Low-Pass Filter Output Noise

For t sufficiently large so that $e^{-2(W - \alpha)t} << 1$ and for $\alpha << W$, $\mu(t)$ will be nearly equal to μ given by Equation (4.6) and the optimum bandwidth will be the same as for a stationary white noise. In addition, given equivalent input noise spectrum levels at every time, the output signal-to-noise ratios will be nearly equal. If these two conditions are not met, a different optimum bandwidth is expected.

A computer program has been written to solve Equation (4.15) as a function of αT, t/T, and of WT. Basically, an αT and a time t/T are assumed and $\mu(t)$ is computed as a function of the bandwidth pulse length product WT. The program output actually expresses the bandwidth pulse length product as Hz times pulse length instead of radians per second times pulse length. In other words, the bandwidth pulse length product is expressed as

$$BT = \frac{WT}{2\pi} .$$

The program also finds the maximum value of $\mu(t)$, which gives the maximum output signal-to-noise ratio, and the value of BT which gives this maximum output signal-to-noise ratio. This value of BT, of course, is the optimum bandwidth pulse length product which is the object of the calculation.

Figure 3 shows the optimum bandwidth pulse length product BT as a function of the time for $\alpha T = 0.0, 0.5$ and 1.0. Note that, for $\alpha T = 0$, the optimum BT is asymptotic to 0.2, the optimum BT for stationary white noise. This is as expected since, for $\alpha T = 0$, the noise spectrum level does not change after $t = 0$ and, after a while,

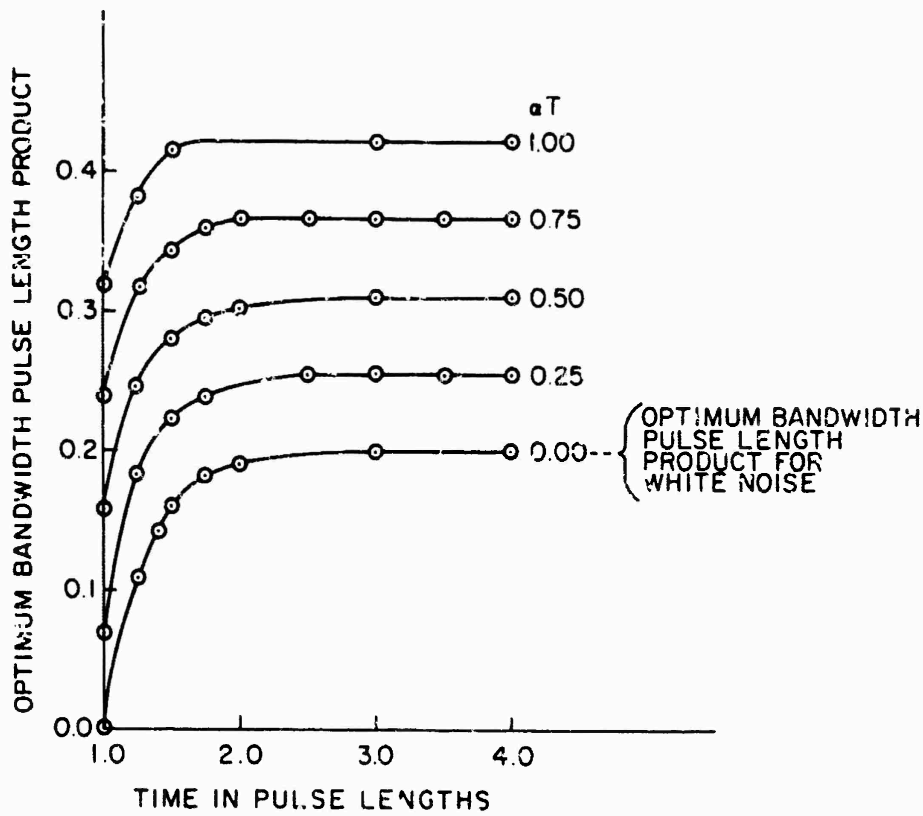

Figure 3. Optimum Bandwidth Pulse Length Product Versus Time

the noise appears stationary to any finite memory filter. For $\alpha T = 1.0$, the bandwidth is asymptotic to 0.41, a value twice that for stationary noise.

Figure 4 shows the maximum value of $\mu(t)$ as a function of time for the same set of αT . Recall that, for a stationary white noise background, the maximum value of $\mu(t)$ is 0.8 for an RC filter and 1.0 for a matched filter.

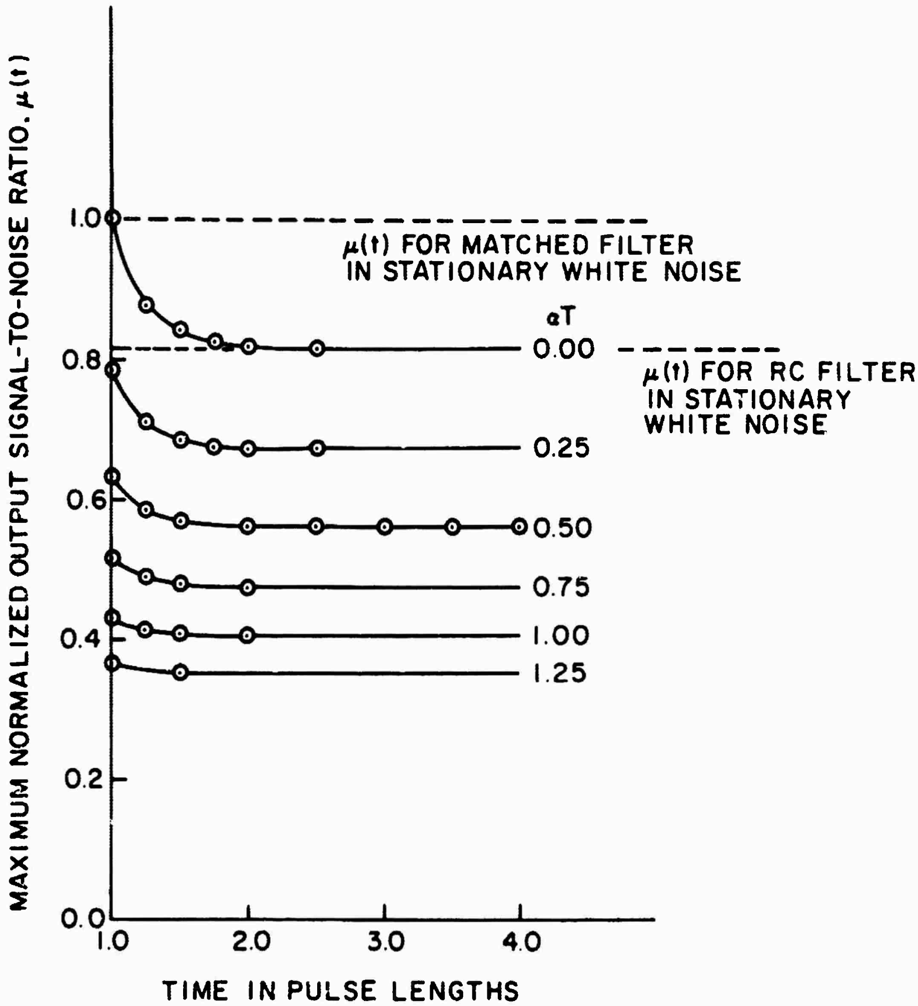

Figure 4. Maximum Output Signal-to-Noise Ratio Versus Time

CHAPTER V

RESPONSE OF A SIMPLE RLC FILTER TO VOLUME REVERBERATION

5.1 Introduction

In Chapters II and III, the required mathematical tools were developed. In Chapter IV, the time variation of the optimum RC filter bandwidth for detection of a signal in exponentially decaying white noise was studied. In this chapter, the practical problem of calculating the response of passive linear circuits to reverberation inputs using spectral techniques will be studied. In particular, the response of the RLC circuit shown in Figure 5 will be computed for a volume reverberation input.

The volume reverberation input $r_i(t)$ is modeled as described in Section 1.2. To review the model, if a pulse of duration T is transmitted at time $t = 0$, the reverberation returns are

$$r_i(t) = \begin{cases} \dfrac{e^{-\alpha t}}{t} x(t) & t \geq T \\ 0 & \text{otherwise} \end{cases} ,$$

where T is the transmitted pulse length, α is a constant proportional to the sound absorption coefficient, and $x(t)$ is a stationary random process with a spectrum $S_{xx}(\omega)$.

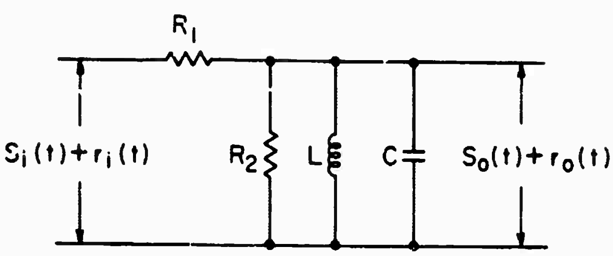

Figure 5. RLC Band-Pass Filter

The equations derived by Brown will be used since they are directly applicable to the reverberation model used and nce the results obtained can be used with any spectrum $S_{xx}(\omega)$. The best spectrum to use for $x(t)$ will depend on the speed of the platform ar l on the directivity pattern of the transducer. Brown's equations allow one to compute a function $B(\omega,t)$ which includes the effect of the modulation function, in this instance,

$$g(t) = \begin{cases} \frac{e^{-\alpha t}}{t} & t \geq T \\ 0 & \text{otherwise} \end{cases} \qquad (5.1)$$

and which contains the effect of the filter. The expected filter output power can then be computed for any $x(t)$ from the equation,

$$E[r_o^2(t)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} |B(\omega,t|^2 \, S_{xx}(\omega)d\omega \quad , \qquad (5.2)$$

just as if $B(\,,t)$ were the transfer function of a filter subjected to the input $x(t)$.

5.2 Filter Weighting Function

To compute $B(\omega,t)$ from Equation (3.3), it will first be necessary to compute the filter weighting function $h(t)$. From Figure 5, the transfer function $H(\omega)$ can be written as

$$H(\omega) = \frac{1}{R_1 C} \frac{j\omega}{(j\omega)^2 + \frac{1}{R_p C} + \frac{1}{LC}} \quad , \qquad (5.3)$$

where R_p is the resistance of R_1 and R_2 in parallel. A partial fraction expansion of Equation (5.3) gives

$$H(\omega) = \frac{1}{\omega_o} \frac{j\omega}{R_1 C} \left(\frac{1}{j\omega + \frac{1}{2R_pC} + j\omega_o} - \frac{1}{j\omega + \frac{1}{2R_pC} - j\omega_o} \right) , \quad (5.4)$$

where $\omega_o \equiv \left(\frac{1}{LC} - \frac{1}{(2R_pC)^2} \right)^{1/2}$.

To find the weighting function $h(t)$, first take the inverse transform of

$$H_1(\omega) \equiv \frac{1}{2} \left(\frac{1}{j\omega + \frac{1}{2R_pC} + j\omega_o} - \frac{1}{j\omega + \frac{1}{2R_pC} - j\omega_o} \right) , \quad (5.5)$$

which gives

$$h_1(t) = \begin{cases} e^{-t/2R_pC} \sin \omega_o t & t \geq 0 \\ 0 & t < 0 \end{cases} . \quad (5.6)$$

Then, the inverse transform of $j\omega H_1(\omega)$ is obtained by differentiation of $h_1(t)$, giving

$$h(t) = \begin{cases} \frac{1}{\omega_o R_1 C} e^{-Wt}(\omega_o \cos \omega_o t - W \sin \omega_o t) & t \geq 0 \\ 0 & t < 0 \end{cases} ,$$

where $W = \frac{1}{2R_pC}$, the half-bandwidth of the RLC filter. It will be easier to work with $h(t)$ if we express $\sin \omega_o t$ and $\cos \omega_o t$ in the exponential form. This gives the following equivalent expression for $h(t)$:

$$h(t) = \frac{1}{\omega_o R_1 C} e^{-Wt} \left\{ \frac{\omega_o}{2} \left[e^{-j\omega_o t} + e^{j\omega_o t} \right] \right.$$

$$\left. - \frac{Wj}{2} \left[e^{-j\omega_o t} - e^{j\omega_o t} \right] \right\} \qquad t \geq 0$$

$$= \frac{1}{\omega_o R_1 C} \left\{ \frac{\omega_o - jW}{2} e^{-Wt} e^{-j\omega_o t} + \frac{\omega_o + jW}{2} e^{-Wt} e^{j\omega_o t} \right\} \quad t \geq 0. \tag{5.8}$$

5.3 Integral Expression for $B(\omega,t)$

$B(\omega,t)$ can now be computed by substituting Equations (4.1) and (5.8) in Equation (3.3):

$$B(\omega,t) = \int_T^t h(t-\xi) g(\xi) e^{-j\omega\xi} d\xi$$

$$= \frac{1}{\omega_o R_1 C} \left\{ \frac{\omega_o - jW}{2} \int_T^t e^{-W(t-\xi) - j\omega_o (t-\xi)} \frac{e^{-\alpha\xi}}{\xi} e^{-j\omega\xi} d\xi \right.$$

$$\left. + \frac{\omega_o + jW}{2} \int_T^t e^{-W(t-\xi) + j\omega_o (t-\xi)} \frac{e^{-\alpha\xi}}{\xi} e^{-j\omega\xi} d\xi \right\}$$

$$= \frac{1}{\omega_o R_1 C} \left\{ \frac{\omega_o - jW}{2} e^{-Wt} e^{-j\omega_o t} \int_T^t \frac{e^{[W-\alpha-j(\omega-\omega_o)]\xi}}{\xi} d\xi \right.$$

$$\left. + \frac{\omega_o + jW}{2} e^{-Wt} e^{j\omega_o t} \int_T^t \frac{e^{[W-\alpha-j(\omega+\omega_o)]\xi}}{\xi} d\xi \right\} . \tag{5.9}$$

Recall that W is one-half the filter bandwidth in radians per second and that α is proportional to the sound absorption coefficient. For frequencies between 1.0 kHz and 100 kHz, the absorption coefficient will vary between 0.003 and 30.0 dB per kiloyard.[18] Since sound travels approximately 1.6 kiloyards per second, the absorption loss at 100 kHz in one second is

$$1.6 \text{ kyd} \cdot 30.0 \, \frac{\text{dB}}{\text{kyd}} = 48.0 \, \frac{\text{dB}}{\text{sec}} \quad .$$

The coefficient α can now be computed from the equation for the absorption loss:

$$10 \log_{10} e^{-2\alpha t} = -48 \, \frac{\text{dB}}{\text{sec}} \quad ,$$

which gives $\alpha \cong 5.5$ at 100 kHz. Similarly, at 1 kHz, $\alpha \cong 5.5 \cdot 10^{-4}$.

It is important to construct a filter such as that shown in Figure 5 which has a bandwidth less than a few Hertz. Assuming a filter bandwidth as low as 5 Hertz, the corresponding value of W would be

$$W = \frac{2\pi \cdot 5}{2} \cong 15.7 \quad ,$$

and $(W - \alpha)$ would be between 10.2 and 15.7. For these reasons, it will be assumed that $(W - \alpha)$ is positive when obtaining an expression for $B(\omega,t)$ in terms of the exponential integral in the next section. Then, since $(W - \alpha) >> 1$, in the following section, an asymptotic expansion of $B(\omega,t)$ is derived.

For simplicity, the following definitions are made:

$$a = W - \alpha - j(\omega - \omega_o) \qquad \text{Re } a \geq 0$$

$$b = W - \alpha - j(\omega + \omega_o) \qquad \text{Re } b \geq 0 \quad .$$

Then, Equation (5.9) can be written as

$$b(\omega,t) = \frac{e^{-Wt}}{\omega_c R_1 C} \left\{ \frac{\omega_o - jW}{2} e^{-j\omega_o t} \int_T^t \frac{e^{a\xi}}{\xi} d\xi + \frac{\omega_o + jW}{2} e^{j\omega_o t} \int_T^t \frac{e^{b\xi}}{\xi} d\xi \right\} \quad . \tag{5.10}$$

5.4 Solution for $B(\omega,t)$ in Terms of the Exponential Integral

An exact solution for $B(\omega,t)$ can be obtained in terms of the exponential integral, defined by

$$Ei(u) \equiv -\fint_{-u}^{\infty} \frac{e^{-v}}{v} dv \quad , \tag{5.11}$$

where $\fint$ indicates the Cauchy Principal Value. Although $Ei(u)$ cannot be integrated in closed form, it has been tabulated for real values of the argument and, in addition, many infinite series and asymptotic series for both positive real, negative real and complex values of the argument have been published. Jahnke and Emde,[17] for example, contains an extensive listing of these series as well as several qualities for $Ei(u)$.

To express $B(\omega,t)$ in terms of $Ei(u)$, let $v = -a$ in Equation (5.10). Then, there results:

$$B(\omega,t) = \frac{e^{-Wt}}{\omega_o R_1 C}\left\{\frac{\omega_0 - jW}{2} e^{-j\omega_o t}\int_{-aT}^{-at}\frac{e^{-v}}{v}\,dv\right.$$

$$\left. + \frac{\omega_o + jW}{2} e^{j\omega_o t}\int_{-bT}^{-bt}\frac{e^{-v}}{v}\,dv\right\} .$$

But, since the real part of $a > 0$, and since T and $t > 0$,

$$\int_{-aT}^{-at}\frac{e^{-v}}{v}\,dv = \int_{-aT}^{\infty}\frac{e^{-v}}{v}\,dv - \int_{-at}^{\infty}\frac{e^{-v}}{v}\,dv = Ei(at) - Ei(aT) .$$

Doing exactly the same thing for the other integral results in

$$B(\omega,t) = \frac{e^{-Wt}}{\omega_o R_1 C}\left\{\frac{\omega_o - jW}{2} e^{-j\omega_o t}[Ei(at) - Ei(aT)]\right.$$

$$\left. + \frac{\omega_o + jW}{2} e^{j\omega_o t}[Ei(bt) - Ei(bT)]\right\} . \tag{5.12}$$

5.5 Asymptotic Expansion for $B(\omega,t)$

The large value of $(W - \alpha)$ mentioned in Section 5.3 will make the real part of (at) and (bt) large for a great many values of t. For this reason, an asymptotic expansion for $B(\omega,t)$ approximates the correct value. Actually, an asymptotic expansion for Ei is tabulated in Reference 17. It is trivial to obtain an expansion for $B(\omega,t)$ from an expansion for Ei. It is simple, however, to derive an asymptotic expansion directly from the integrals in Equation (5.10). This not only avoids integration through the singularity at $\xi = 0$ which is required to express $B(\omega,t)$ in terms of Ei, but it shows that the expension obtained is valid for this problem. This fact is not clear from the tabulations. It will be found that the expansion obtained is identical to that obtained from using the expansion tabulated for Ei.

The resulting asymptotic expansion can be used to simplify the expression for the output of the filter for very large values of t. Furthermore, it will be possible to put an upper bound on the error in the simplified expression, given a value of $(W - \alpha)t$.

5.5.1 Properties of Asymptotic Expansion. This section reviews, without proof, the definition of an asymptotic expansion and some of its useful properties. Goldman[13] points out that asymptotic series have turned out to be particularly practical in transformation calculus. Thus, it appears that resorting to an asymptotic expansion is not a technique peculiar to this study, but a common event in solving practical problems by means of transforms.

Let $S_n(t)$ be the sum of the first $(n + 1)$ terms of the series

$$A_o + \frac{A_1}{t} + \frac{A_2}{t^2} + \ldots + \frac{A_n}{t^n} + \ldots \quad , \tag{5.13}$$

and let $R_n(t) = t^n[f(t) - S_n(t)]$. Then, if

$$\lim_{t\to\infty} R_n(t) = 0 \quad ,$$

for a fixed n , even though

$$\lim_{n\to\infty} |R_n(t)| \to \infty$$

for a fixed t , we say that Equation (5.13) is the asymptotic expansion of $f(t)$. In such a case, it is customary to write

$$f(t) = A_o + \frac{A_1}{t} + \frac{A_2}{t^2} + \ldots + \frac{A_n}{t^n} + \ldots \quad . \tag{5.14}$$

Although an asymptotic series does not converge, it can be used for large values of the variable. If the variable is sufficiently large, the magnitudes of the terms monotonically decrease with increasing n , pass through a minimum, and then monotonically increase. The sum of the first n terms differs from $f(t)$ by less than the $(n + 1)^{th}$ term. Therefore, if the series is carried out to the minimum term, it differs from $f(t)$ by less than the maximum term and a good approximation is obtained. In addition, for a fixed number of terms, the approximation gets better as the value of t increases.

5.5.2 Derivation of the Asymptotic Expansion. The asymptotic expansion is derived from Equation (5.10) by integrating

$$\int_T^t \frac{e^{a\xi}}{\xi}\, d\xi$$

and

$$\int_T^t \frac{e^{b\xi}}{\xi}\, d\xi$$

successively by parts. For simplicity, the expansion of only one of these two integrals will be shown. The expansion of the other is obtained by interchanging a and b . To integrate by parts, let

$$u = 1/\xi$$

$$dv = e^{a\xi}\, d\xi \quad .$$

Then,

$$\int \frac{e^{a\xi}}{\xi}\, d\xi = uv - \int v\,du = \frac{e^{a\xi}}{\xi} + \frac{1}{a}\int \frac{e^{a\xi}}{\xi^2}\, d\xi \quad .$$

Similarly, to integrate

$$\frac{1}{a}\int \frac{e^{a\xi}}{\xi^2} d\xi \quad ,$$

let $\mu = 1/\xi^2$ and let dv remain as before. This gives

$$\frac{1}{a}\int \frac{e^{a\xi}}{\xi^2}\,d\xi = \frac{e^{a\xi}}{(a\xi)^2} + \frac{2}{a^2}\int \frac{e^{a\xi}}{\xi^3}\,d\xi \quad .$$

Continuing this method of integration and summing all the terms gives:

$$\int_T^t \frac{e^{a\xi}}{\xi}\,d\xi = e^{a\xi}\left\{\frac{1}{a\xi} + \frac{1}{(a\xi)^2} + \frac{2!}{(a\xi)^3} + \ldots + \frac{(n-1)!}{(a\xi)^n}\right\}\Bigg|_T^t +$$

$$+ \frac{n!}{a^n}\int_T^t \frac{e^{a\xi}}{\xi^{n+1}}\,d\xi$$

$$= e^{at}\left\{\frac{1}{at} + \frac{1}{(at)^2} + \frac{2!}{(at)^3} + \ldots + \frac{(n-1)!}{(at)^n} +\right.$$

$$\left. + \frac{n!}{a^n}\,e^{-at}\int_{-\infty}^t \frac{e^{a\xi}}{\xi^{n+1}}\,d\xi\right\} -$$

$$- e^{at}\left\{\frac{1}{aT} + \frac{1}{(aT)^2} + \frac{2!}{(aT)^3} + \ldots + \frac{(n-1)!}{(aT)^n} +\right.$$

$$\left. + \frac{n!}{a^n}\,e^{-aT}\int_{-\infty}^T \frac{e^{a\xi}}{\xi^{n+1}}\,d\xi\right\} \quad .$$

From the definition in Section 5.5.1,

$$\left\{ \frac{1}{at} + \frac{1}{(at)^2} + \frac{2!}{(at)^3} + \ldots + \frac{(n-1)!}{(at)^n} \right\}$$

is an asymptotic expansion if

$$\lim_{t\to\infty} \left\{ \frac{n!}{a^n} e^{-at} t^n \int_{-\infty}^{t} \frac{e^{a\xi}}{\xi^{n+1}} d\xi \right\} = 0 \quad .$$

From L'Hospital's rule,

$$\lim_{t\to\infty} \frac{n!}{a^n} e^{-at} t^n \int_{-\infty}^{t} \frac{e^{a\xi}}{\xi^{n+1}} d\,\xi = \lim_{t\to\infty} \frac{\frac{n!}{a^n} \frac{d}{dt} \int_{-\infty}^{t} \frac{e^{a\xi}}{\xi^{n+1}} d\xi}{\frac{d}{dt} e^{at} t^{-n}}$$

$$= \lim_{t\to\infty} \frac{n!}{a^n} \frac{e^{at}}{t^{n+1}} \frac{1}{e^{at}(-nt^{-n-1}) - \frac{t^{-n} e^{at}}{a}}$$

$$= \lim_{t\to\infty} \frac{n!}{a^n} \frac{1}{-n + t/a} = 0 \quad ,$$

and the expansion is asymptotic. This allows us to write

$$\int_{T}^{t} \frac{e^{a\xi}}{\xi} d\xi \simeq e^{at} \left\{ \frac{1}{at} + \frac{1}{(at)^2} + \frac{2!}{(at)^3} + \ldots \frac{(n-1)!}{(at)^n} \right\} - $$

$$-e^{aT} \left\{ \frac{1}{aT} + \frac{1}{(aT)^2} + \frac{2!}{(aT)^3} + \ldots \frac{(n-1)!}{(aT)^n} \right\} \quad , \tag{5.15}$$

and the error will be less than the minimum term of the series.

Finally, substituting the asymptotic expansion of the integral into the expression for $B(\omega,t)$, Equation (5.10), there results:

$$
\begin{aligned}
B(\omega,t) \cong \frac{1}{\omega_o R_1 C} \Bigg\{ \frac{\omega_o - jW}{2} \Bigg[& e^{-\alpha t} e^{-j\omega t} \left(\frac{1}{at} + \frac{1}{(at)^2} + \frac{2!}{(at)^3} + \ldots \right) \\
& -e^{-\alpha T} e^{-j\omega T} e^{-W(t-T)} e^{-j\omega_o (t-T)} \left(\frac{1}{aT} + \frac{1}{(aT)^2} + \frac{2!}{(aT)^3} + \ldots \right) \Bigg] \\
& + \frac{\omega_o + jW}{2} \Bigg[e^{-\alpha t} e^{-j\omega t} \left(\frac{1}{bt} + \frac{1}{(bt)^2} + \frac{2!}{(bt)^3} + \ldots \right) \\
& -e^{-\alpha T} e^{-j\omega T} e^{-W\,(t-T)} e^{j\omega_o (t-T)} \left(\frac{1}{bT} + \frac{1}{(bT)^2} + \frac{2!}{(bT)^3} + \ldots \right) \Bigg] \Bigg\} \\
\cong \frac{1}{\omega_o R_1 C} \Bigg\{ & e^{-\alpha t} e^{-j\omega t} \Bigg[\left(\frac{\omega_o + jW}{2} \right) \left(\frac{1}{at} + \frac{1}{(at)^2} + \frac{2!}{(at)^3} + \ldots \right) \Bigg] \\
& + e^{-\alpha t} e^{-j\omega t} \Bigg[\left(\frac{\omega_o + jW}{2} \right) \left(\frac{1}{bt} + \frac{1}{(bt)^2} + \frac{2!}{(bt)^3} + \ldots \right) \\
& - e^{-W(t-T)} e^{-\alpha T} e^{-j\omega T} \Bigg[e^{-j\omega_o (t-T)} \left(\frac{{}_o - jW}{2} \right) \cdot \\
& \cdot \left(\frac{1}{aT} + \frac{1}{(aT)^2} + \frac{2!}{(aT)^3} + \ldots \right) \\
& + e^{j\omega_o (t-T)} \left(\frac{\omega_o + jW}{2} \right) \left(\frac{1}{bT} + \frac{1}{(bT)^2} + \frac{2!}{(bT)^3} + \ldots \right) \Bigg] \Bigg] \Bigg\} .
\end{aligned}
\tag{5.16}
$$

5.5.3 Accuracy of the Asymptotic Expansion. Recalling that $a = W - \alpha - j(\omega - \omega_o)$, it is clear that at $\omega = \omega_o$, a is real and has its minimum magnitude for fixed $(W - \alpha)$. Thus, if the real part of a is sufficiently large so that, at $\omega = \omega_o$,

$$e^{at}\left\{\frac{1}{at} + \frac{1}{(at)^2} + \frac{2!}{(at)^3} + \ldots\right\} = \frac{e^{at}}{at} , \tag{5.17}$$

then, for $\omega \neq \omega_o$, the magnitudes of the terms

$$\left\{\frac{1}{(at)^2}, \frac{2!}{(at)^3}, \frac{3!}{(at)^4}, \ldots\right\}$$

are even a smaller percentage of $1/at$ than for $\omega = \omega_o$. Thus, the approximation represented by Equation (5.17) is better at $\omega \neq \omega_o$ than at $\omega = \omega_c$. It is found by trial and error that

$$Ei(21) = 0.6613 \cdot 10^8$$

and

$$\frac{e^{21}}{21} = 0.629 \cdot 10^8 .$$

The error in using $e^{21}/21$ in place of $Ei(21)$ is 4.8 percent. For arguments with real parts greater than 21, $Ei(at)$ may be computed from e^{at}/at with less than 5 percent error in magnitude.

Similarly, if $Ei(a) << Ei(at)$ for real a, than $Ei(aT)$ will be even more negligible for complex a. To see this, form the ratio $Ei(aT)/Ei(at)$ and assume this fraction is sufficiently small at

$\omega = \omega_o$, where a is real. In forming this ratio, remember that it is being assumed that (at) is sufficiently large so that $Ei(at) = e^{at}/at$.

$$\frac{Ei(aT)}{Ei(at)} \cong \frac{ate^{aT}}{e^{at}} \left\{ \frac{1}{aT} + \frac{1}{(aT)^2} + \frac{2}{(aT)^3} + \dots \frac{(n-1)!}{(aT)^n} \right\}$$

$$= te^{-a(t-T)} \left\{ \frac{1}{T} + \frac{1}{aT^2} + \frac{2}{a^2T^3} + \dots \frac{(n-1)!}{a^{n-1}T^n} \right\} . \qquad (5.18)$$

Since this ratio is assumed sufficiently small at $\omega = \omega_o$, it is even smaller for $\omega \neq \omega_o$, where a is complex, since the magnitude of $e^{-a(t-T)}$ is unchanged for complex a but the magnitudes of all the terms containing a in the denominator decrease.

It is found that $Ei(13) = 0.3720 \cdot 10^5$ so that if the real part of aT is less than 13, the magntidue of $Ei(aT)$ is less than 0.1 percent of the magnitude of $Ei(at)$ and can be neglected.

5.5.4 Simplified Expression for $E[r_o^2(t)]$ for Large t . Assuming that $(W - \alpha)t > 21$ and that $(W - \alpha)T < 13$, then

$$Ei(at) - Ei(aT) \cong \frac{e^{at}}{at} \qquad (5.19)$$

with less than 5 percent error. Since this statement also applied to (bt) , then substituting in Equation (5.12) gives

$$B(\omega,t) \cong \frac{e^{-\alpha t}e^{-j\omega t}}{2\omega_o R_1 Ct} \left\{ \frac{\omega_o - jW}{a} + \frac{\omega_o + jW}{b} \right\} \qquad (5.20)$$

and the magnitude will be correct within 5 percent. Substituting for a and b in the above expression gives

$$B(\omega,t) \cong \frac{e^{-\alpha t} e^{-j\omega t}}{2\omega_o R_1 C t} \left\{ \frac{(\omega_o - jW)b + (\omega_o + jW)a}{ab} \right\}$$

$$\cong \frac{e^{-\alpha t}}{t} \frac{e^{-j\omega t}}{R_1 C} \left\{ \frac{-(\alpha + j\omega)}{(j\omega)^2 - j\omega 2(W - \alpha) + \omega_o^2 + (W - \alpha)^2} \right\} . \tag{5.21}$$

Since $|j\omega| >> |\alpha|$ over any realistic range of reverberation frequencies, this expression can be further simplified to

$$B(\omega,t) \cong \frac{e^{-\alpha t}}{t} e^{-j\omega t} \frac{1}{R_1 C} \left\{ \frac{-j\omega}{(j\omega)^2 - j\omega 2(W - \alpha) + \omega_o^2 + (W-\alpha)^2} \right\} . \tag{5.22}$$

Recalling that

$$\omega_o^2 \equiv \frac{1}{LC} - W^2 ,$$

and comparing Equation (5.22) with Equation (5.3), it appears that $B(\omega,t)$ is the product of $e^{-j\omega t}$, the modulation function $e^{-\alpha t}/t$, and the complex conjugate of the transfer function of an RLC filter. The RLC filter is the same as shown in Figure 5 except that the bandwidth is equal to $(W - \alpha)$ instead of W. If $H_e(\omega)$ represents the transfer function of this RLC filter, that is, if

$$H_e(\omega) \equiv \frac{1}{R_1C}\left\{\frac{j\omega}{(j\omega)^2 + j\omega 2(W-\alpha) + \omega_o^2 + (W-\alpha)^2}\right\} , \quad (5.23)$$

then,

$$B(\omega,t) \cong \frac{e^{-\alpha t}}{t^2} e^{-j\omega t} H_e{}^*(\omega) \quad . \quad (5.24)$$

From Equation (3.6), the expected filter output power is

$$E[r_o{}^2(t)] \cong \frac{e^{-2\alpha t}}{t^2} \frac{1}{2\pi} \int_{-\infty}^{\infty} H_e(\omega) H_e{}^*(\omega) S_{xx}(\omega) d\omega \quad . \quad (5.25)$$

Equation (5.25) states that the filter output power is nearly equal to the square of the modulation function multiplied by the expected output of the filter, with bandwidth decreased by α radians per second, for a stationary random input having a spectrum $S_{xx}(\omega)$. This equation is correct to within a factor of $(1.05)^2$ or about 10 percent since $B(\omega,t)$ is correct to within 5 percent provided $(W - \alpha)t > 21$ and $(W - \alpha)T < 13$.

The interesting thing about this result is that, in Chapter IV, where the modulation function was $e^{-\alpha t}$ instead of $e^{-\alpha t}/t$, the filter output power for large t was also equal to the square of the modulation function times the output of the same RLC filter $H_e(\omega)$ for an input having a spectrum $S_{xx}(\omega)$.

CHAPTER VI

SUGGESTIONS FOR FURTHER RESEARCH

This thesis is an investigation of the interpretations and the advantages of several definitions of the spectrum of a nonstationary random process. In particular, it was desired to learn how to describe such processes with a mathematical model leading to a spectrum so that the output of linear time-invariant filters can be computed. Although the nonstationary process discussed most frequently is backscattered sound in the sea, the emphasis was more on studying the mathematical tools available than on actually designing filters. Since there is no data available which gives spectral descriptions of reverberation based on actual observation, before obtaining such data, it is necessary to decide on a method for giving a spectral description which appears practical. Then, it is necessary to examine actual reverberation to determine the constants fitting the spectral description. This task alone is material enough for a doctoral dissertation in acoustics and is the next logical step.

Certain things are clear already. The time function describing the decay of the expected volume or boundary reverberation is well established. If the reverberation can be well described by multiplying this time function by a stationary random process, much work can be done in advance of having a complete spectral description using the equations of Chapter III. Thus, Chapter V provides a partial solution to the output of a linear filter for volume reverberation. This solution is useful no matter what is the best stationary random process $x(t)$ for simulating true reverberation. It would be useful to continue the work in Chapter V so that a computer program for $B(\omega,t)$ is obtained. This will lead easily to a computer program for the expected filter output power for any $x(t)$ and, finally, to a third program which will compute the optimum bandwidth of the filter for any $x(t)$ just as was done in Chapter IV. On the basis of the results obtained in Chapter IV, one would expect the optimum bandwidth to be a function of time. It is hoped that the author can continue this work and at least obtain some results for various assumed $x(t)$'s if not those based on actual data. Having done this, of course, there would remain the problem of doing the entire study over for boundary reverberation.

Finally, there is the interesting problem of optimum filter configuration which was not considered at all. Even though time invariant filters were assumed, it was found in Chapter IV that the optimum bandwidth for detection of the signal was a function of time. This suggests that time varying filters are called for in such problems. It was also found that the maximum output signal-to-noise ratio for the RC filter in Chapter IV never exceeded that of the classical matched filter for stationary noise backgrounds. This makes one wonder whether or not the matched filter gives an upper bound on the output signal-to-noise ratio which is more generally applicable than stated in Chapter IV.

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