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BERNOULLI SAMPLING PLANS WHICH APPROXIMATELY MINIMIZE THE MAXIMUM EXPECTED SAMPLE SIZE SUBJECT TO CERTAIN PROBABILITY REQUIREMENTS

by

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Abstract

The author, in his masters thesis [3], constructed a catalog of 442 Bernoulli sampling plans which approximately minimize the maximum expected sample size among all plans which guarantee certain O.C. probability requirements. Fifty-two of these plans (which would appear to be of greatest practical interest) are presented in this report. A.S.N. curve comparisons are made with plans based on the Wald sequential probability ratio test and the fixed sample size test which guarantee the same O.C. probability requirements.

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Plan 33 --	$\theta_{-1} = .10$,	$\theta_1 = .20$	$\alpha = \beta = .01$
Plan 34 --			.025
Plan 35 --			.05
Plan 36 --			.10
Plan 37 --			.20
Plan 38 --	$\theta_{-1} = .10$,	$\theta_1 = .30$	$\alpha = \beta = .01$
Plan 39 --			.025
Plan 40 --			.05
Plan 41 --			.10
Plan 42 --			.20
Plan 43 --	$\theta_{-1} = .15$,	$\theta_1 = .25$	$\alpha = \beta = .01$
Plan 44 --			.025
Plan 45 --			.05
Plan 46 --			.10
Plan 47 --			.20
Plan 48 --	$\theta_{-1} = .15$,	$\theta_1 = .35$	$\alpha = \beta = .01$
Plan 49 --			.025
Plan 50 --			.05
Plan 51 --			.10
Plan 52 --			.20

4. References

1. Introduction

In a paper by D. Freeman and L. Weiss [1], Weiss proposed using, for a Bernoulli population, a sequential sampling plan with an A.S.N. curve that has the smallest maximum among all sampling plans satisfying

$$\begin{aligned} &P_\theta(\text{plan accepts the population}) \geq 1-\alpha \quad \text{if } \theta \leq \theta_{-1} \\ &P_\theta(\text{plan accepts the population}) \leq \beta \quad \text{if } \theta \geq \theta_1 \end{aligned} \tag{1.1}$$

where θ represents the percent defective in a population and $0<\theta_{-1}<\theta_1<1$, $\alpha>0$, $\beta>0$, $\alpha+\beta<1$. He presented the theory for constructing plans which approximately minimize the maximum expected sample size, and with Freeman developed a few examples of such plans. For all of these Weiss plans, the number of observations required by a plan is a bounded chance variable; the fact that the bound is known for each plan is of obvious practical importance. In Section 2 the theory behind the Weiss test is discussed.

Other important work in the area of sequential sampling has been done by A. Wald [9] and the Statistical Research Group at Columbia University (S.R.G.) [7,8]. Wald introduced sequential sampling plans which minimize the A.S.N. curve at θ_{-1} and θ_1 among all plans satisfying (1.1), and the S.R.G. developed and cataloged sequential plans which in many instances (but not always) possess A.S.N. curves that are lower than those of single- or double-sampling plans which satisfy (1.1).

In a masters thesis [3], the author presented a catalog of 442 sequential-sampling plans as suggested by Weiss. The present paper will display 52 of these plans which are appropriate for small values of θ_{-1}, θ_1 and thus are considered to be of greatest practical interest. The A.S.N. curve characteristics of these plans will be compared with those of corresponding Wald and fixed sample size plans.

2. Theoretical Development

2.1 Introduction

The theory behind sampling plans which approximately minimize the maximum expected sample size was presented by Weiss [1] and will be repeated here, essentially as Weiss presented it, with remarks relating to the application of the theory in constructing plans.

2.2 A Related Problem

The discussion will now center on a problem differing from but related to that described by (1.1). The exact relationship will be described later in Section 2.3. The problem to be discussed is that of constructing a sampling plan which satisfies the conditions

$$\begin{aligned} &P_{\theta_{-1}}(\text{plan accepts the population}) \geq 1-\alpha \\ &P_{\theta_1}(\text{plan accepts the population}) \leq \beta \end{aligned} \qquad (2.1)$$

and which minimizes the A.S.N. curve at a specified value $\theta_0(\theta_{-1}<\theta_0<\theta_1)$ among all plans satisfying (2.1). It is assumed that the observations taken are independent Bernoulli variables. The formulation of the above problem is aimed at controling the A.S.N. curve for values of θ between θ_{-1} and θ_1.

The following notation will be employed. If T is a plan, $P_\theta(A|T)$ denotes the probability that T accepts the population when θ is the proportion of defectives. $P_\theta(A|T;X_1,\cdots,X_m)$ denotes the conditional probability that T accepts the population, given that T has observed $X_1,\cdots,X_m$ and is definitely going to observe X_{m+1} when θ is the proportion of defectives. N denotes the number of observations that will be taken before sampling is terminated. N is a random variable whose distribution depends on θ and on the plan used. $E_\theta(N|T)$ denotes the expected value of N when T is used and θ is the proportion of defectives. $E_\theta(N|T;X_1,\cdots,X_m)$ denotes the conditional expected value of N, given that T has observed $X_1,\cdots,X_m$ and is definitely going to observe X_{m+1}, when θ is the proportion of defectives. Thus $E_\theta(N|T;X_1,\cdots,X_m) \geq m+1$.

If T is a plan which minimizes $E_{\theta_0}(N|T)$ among all plans satisfying (2.1), then there are three positive constants, b_{-1},b_0,b_1, adding to unity, such that

$$\begin{aligned} b_{-1}[1-P_{\theta_{-1}}(A|T)] + b_0E_{\theta_0}(N|T) + b_1P_{\theta_1}(A|T) \leq \\ b_{-1}[1-P_{\theta_{-1}}(A|T')] + b_0E_{\theta_0}(N|T') + b_1(A|T') \end{aligned} \tag{2.2}$$

for each and every plan T' (see [2], Lemma 4.1). A plan T satisfying (2.2) is called a Bayes plan with respect to the a priori distribution b_{-1},b_0,b_1. It should be noted that the Bayes procedures are used here only as devices for constructing plans which possess certain properties. The values of b_{-1},b_0,b_1 depend on α and β, but the exact relationship is unknown. Therefore, rather than preassigning α and β, the values b_{-1},b_0,b_1 will be preassigned, and a Bayes plan T will be constructed with respect to b_{-1},b_0,b_1. This Bayes plan T then has the following property. If T' is any plan satisfying the conditions

$$P_{\theta_{-1}}(A|T') \geq P_{\theta_{-1}}(A|T)$$

$$P_{\theta_1}(A|T') \leq P_{\theta_1}(A|T)$$

then $E_{\theta_0}(N|T') \geq E_{\theta_0}(N|T)$. That is, the values $1-\alpha,\beta$ have been replaced in (2.1) by $P_{\theta_{-1}}(A|T)$, $P_{\theta_1}(A|T)$, respectively.

Define Q(m,X) as equal to unity if m=0, and as equal to

$$\frac{\theta_1^{X_1+\cdots+X_m}(1-\theta_1)^{m-X_1-\cdots-X_m}}{\theta_{-1}^{X_1+\cdots+X_m}(1-\theta_{-1})^{m-X_1-\cdots-X_m}}$$

if m>0. As characterized in Theorem 4 of [4], a Bayes plan T with respect to a given a priori distribution b_{-1},b_0,b_1 has the following properties. There is a finite integer n and two sequences of nonnegative values,

$$a_0 \leq a_1 \leq \cdots \leq a_n = b_{-1}/b_1, \quad r_0 \geq r_1 \geq \cdots \geq r_n = b_{-1}/b_1$$

such that T cannot observe X_{n+1}; and for $m=0,1,\cdots,n-1$, T observes X_{m+1} if $a_m < Q(m,X) < r_m$, T does not observe X_{m+1} and rejects the population if $Q(m,X) > r_m$, T randomizes in any way between observing X_{m+1} or not observing X_{m+1} and rejecting the population if $Q(m,X) = r_m$, T does not observe X_{m+1} and accepts the population if $Q(m,X) < a_m$, T randomizes in any way between observing X_{m+1} or not observing X_{m+1} and accepting the population if $Q(m,X) = a_m$. If $Q(n,X) = b_{-1}/b_1$, T randomizes in any way between accepting or rejecting the population. The values n,a_m,r_m all depend on b_{-1},b_0,b_1.

For the sake of definiteness, the only plans to be considered will be those that definitely observe X_{m+1} whenever $Q(m,X) = a_m$ or r_m for $m<n$ and definitely accept the population whenever $Q(n,X) = b_{-1}/b_1$. The characterization of T just given is presented below.

If m=0 then it is understood that $\sum_{j=1}^{n} X_j$ is equal to zero. If

$$a_m < \frac{\theta_1^{\sum_{j=1}^{m} X_j}(1-\theta_1)^{m-\sum_{j=1}^{m} X_j}}{\theta_{-1}^{\sum_{j=1}^{m} X_j}(1-\theta_{-1})^{m-\sum_{j=1}^{m} X_j}} < r_m$$

for $m=0,1,\cdots,n-1$, then

$$\log a_m < \sum_{j=1}^{m} X_j \log \left[\frac{\theta_1(1-\theta_{-1})}{\theta_{-1}(1-\theta_1)}\right] + m \log \left(\frac{1-\theta_1}{1-\theta_{-1}}\right) < r_m$$

and there is a finite integer n and values $A_0,A_1,\cdots,A_{n-1},R_0,R_1,\cdots,R_{n-1}$, where

$$A_m = \frac{\log a_m - m \log(\frac{1-\theta_1}{1-\theta_{-1}})}{\log[\frac{\theta_1(1-\theta_{-1})}{\theta_{-1}(1-\theta_1)}]} ,$$

$$R_m = \frac{\log r_m - m \log(\frac{1-\theta_1}{1-\theta_{-1}})}{\log[\frac{\theta_1(1-\theta_{-1})}{\theta_{-1}(1-\theta_1)}]} ,$$

and $A_m \leq R_m$ for all $m=0,\cdots,n-1$. For n

$$A_n = R_n = \frac{\log \frac{b_{-1}}{b_1} - n \log(\frac{1-\theta_1}{1-\theta_{-1}})}{\log[\frac{\theta_1(1-\theta_{-1})}{\theta_{-1}(1-\theta_1)}]} . \tag{2.3}$$

Then for $m=0,1,\cdots,n-1$, T observes X_{m+1} if

$$A_m \leq \sum_{j=1}^{m} X_j \leq R_m ,$$

T does not observe X_{m+1} and accepts the population if

$$\sum_{j=1}^{m} X_j < A_m ,$$

T does not observe X_{m+1} and rejects the population if

$$\sum_{j=1}^{m} X_j > R_m ,$$

T accepts the population if

$$\sum_{j=1}^{n} X_j \leq A_n .$$

If A_m is not an integer and $m<n$, A_m can be replaced by the smallest integer greater than A_m, without changing T. Similarly, if R_m is not an integer and $m<n$, R_m can be replaced by the largest integer less than R_m, without changing T. Henceforth, it is assumed that the changes are made, so that A_m and

R_m are integers for all $m<n$.

For a given a priori distribution (b_{-1},b_o,b_1), if $X_1,\cdots,X_m$ have been observed, $c_{j,m}(X_1,\cdots,X_m)$ is defined as

$$\frac{b_j\theta_j^{X_1+\cdots+X_m}(1-\theta_j)^{m-X_1-\cdots-X_m}}{\sum_{i=-1,0,1} b_i\theta_i^{X_1+\cdots+X_m}(1-\theta_i)^{m-X_1-\cdots-X_m}}$$

for $j=-1,0,1$. The set of quantities $(c_{-1,m}(X_1,\cdots,X_m), c_{o,m}(X_1,\cdots,X_m), c_{1,m}(X_1,\cdots,X_m))$ is called the "a posteriori distribution given $X_1,\cdots,X_m$." If $m=0$ the quantity $c_{j,m}(X_1,\cdots,X_m)$ is defined as equal to b_j. If T is a Bayes plan with respect to the a priori distribution b_{-1},b_o,b_1 and T has observed $X_1,\cdots,X_{m-1}$, then T observes X_m if and only if

$$\begin{aligned} &c_{-1,m-1}(X_1,\cdots,X_{m-1})[1-P_{\theta_{-1}}(A|X_1,\cdots,X_{m-1})] \\ &\quad + c_{o,m-1}(X_1,\cdots,X_{m-1})E_{\theta_o}(N|X_1,\cdots,X_{m-1}) \\ &\quad + c_{1,m-1}(X_1,\cdots,X_{m-1})P_{\theta_1}(A|X_1,\cdots,X_{m-1}) \\ &\leq (m-1)c_{o,m-1}(X_1,\cdots,X_{m-1}) \\ &\quad + \min\{c_{-1,m-1}(X_1,\cdots,X_{m-1})c_{1,m-1}(X_1,\cdots,X_{m-1})\} \end{aligned} \tag{2.4}$$

for $m=1,\cdots,n$. Now suppose that the values of A_m and R_m are known along with the values of $P_{\theta_{-1}}(A|X_1,\cdots,X_m)$, $P_{\theta_1}(A|X_1,\cdots,X_m)$, and $E_{\theta_o}(N|X_1,\cdots,X_m)$ for all $X_1,\cdots,X_m$ with $A_m\leq X_1+\cdots+X_m\leq R_m$. Then the following values can be computed: A_{m-1}, R_{m-1}, and $P_{\theta_{-1}}(A|X_1,\cdots,X_{m-1})$, $P_{\theta_1}(A|X_1,\cdots,X_{m-1})$, and $E_{\theta_o}(N|X_1,\cdots,X_{m-1})$ for all $X_1,\cdots,X_{m-1}$ with $A_{m-1}\leq X_1+\cdots+X_{m-1}\leq R_{m-1}$. The method for computing the quantities mentioned above is presented in the following two paragraphs.

It is clear that $A_{m-1}\geq A_m-1$, for if $A_{m-1}<A_m-1$, then if

$$\sum_{i=1}^{m-1} X_i = A_{m-1} ,$$

T would observe X_m and then surely accept the population so that observing X_m would be wasteful. Similarly, $R_{m-1} \leq R_m$. It should be noted that for most sampling plans of a practical interest consecutive values of A_m or R_m will not be farther apart than a value of 1. If

$$\sum_{i=1}^{m-1} X_i = A_m - 1 ,$$

then for all θ, and $m<n$,

$$\begin{aligned} P_\theta(A|X_1,\cdots,X_{m-1}) &= 1-\theta+\theta P_\theta(A|X_1,\cdots,X_{m-1},1) \\ E_\theta(N|X_1,\cdots,X_{m-1}) &= (1-\theta)m+\theta E_\theta(N|X_1,\cdots,X_{m-1},1) . \end{aligned} \tag{2.5}$$

If

$$\sum_{i=1}^{m-1} X_i = R_m ,$$

then for all θ, and $m<n$,

$$\begin{aligned} P_\theta(A|X_1,\cdots,X_{m-1}) &= (1-\theta)P_\theta(A|X_1,\cdots,X_{m-1},o) \\ E_\theta(N|X_1,\cdots,X_{m-1}) &= \theta m+(1-\theta)E_\theta(N|X_1,\cdots,X_{m-1},o) . \end{aligned} \tag{2.6}$$

If

$$A_m - 1 < \sum_{i=1}^{m-1} X_i < R_m ,$$

then for all θ, and $m<n$,

$$\begin{aligned} &P_\theta(A|X_1,\cdots,X_{m-1}) \\ &\quad = (1-\theta)P_\theta(A|X_1,\cdots,X_{m-1},o)+\theta P_\theta(A|X_1,\cdots,X_{m-1},1) \\ &E_\theta(N|X_1,\cdots,X_{m-1}) \\ &\quad = (1-\theta)E_\theta(N|X_1,\cdots,X_{m-1},o)+\theta E_\theta(N|X_1,\cdots,X_{m-1},1) . \end{aligned} \tag{2.7}$$

Now if inequality (2.4) is satisfied when

$$\sum_{i=1}^{m-1} X_i = A_m - 1 ,$$

(using the relations in equation (2.5) to compute the quantities in equation (2.4)) then $A_{m-1} = A_m - 1$. If (2.4) is not satisfied when

$$\sum_{i=1}^{m-1} X_i = A_m - 1 ,$$

then $A_{m-1} > A_m - 1$, and A_{m-1} is the smallest integer greater than $A_m - 1$ for which (2.4) is satisfied (the quantities in (2.4) being computed by using the relations in the appropriate one of (2.6) or (2.7)). Similarly, if inequality (2.4) is satisfied when

$$\sum_{i=1}^{m-1} X_i = R_m ,$$

then $R_{m-1} = R_m$. If (2.4) is not satisfied when

$$\sum_{i=1}^{m-1} X_i = R_m ,$$

then R_{m-1} is the largest integer less than R_m for which (2.4) is satisfied (the quantities in (2.4) being computed by using the relations in the appropriate one of (2.5) or (2.7)).

If the actual value of n is known, as for example in Weiss [10], then $A_n = R_n$ is known and is given by equation (2.3) above. The quantity $A_n = R_n$ is not necessarily an integer and cannot legitimately be replaced by an integer if it is not already one. A_{n-1}, R_{n-1}, $P_\theta(A|X_1,\cdots,X_{n-1})$ and $E_\theta(N|X_1,\cdots,X_{n-1})$ can be computed as described previously except that on the right-hand sides of (2.5), (2.6), and (2.7)

$$E_\theta(N|X_1,\cdots,X_{n-1},k) = n \qquad \text{for } k=0,1; \text{ and}$$

$$P_\theta(A|X_1,\cdots,X_{n-1},k) = 1 \qquad \text{if } X_1+\cdots+X_{n-1}+k \leq A_n,$$

$$P_\theta(A|X_1,\cdots,X_{n-1},k) = 0 \qquad \text{if } X_1+\cdots+X_{n-1}+k > A_n, \text{ for } k=0,1.$$

Thus, if n is known, the entire plan T can be explicitly constructed, since the values of $A_0, A_1, \cdots, A_n, R_0, R_1, \cdots, R_n$ can be found. Also, by repeated use of the recursion formulas (2.5), (2.6), and (2.7), $P_\theta(A|T)$ and $E_\theta(N|T)$ can be computed for any desired values of θ. The above is accomplished when by the recursive construction the point is reached where the quantities P_θ(A|no observations have been taken) and E_θ(N|no observations have been taken) are sought. The proper one of (2.5), (2.6), or (2.7) is then chosen by which of the conditions the value $\sum_{i=1}^{m-1} X_i$ satisfies (here this summation is not precisely defined but is necessarily zero since no observations have been taken).

If the actual value of n is not known before the construction of the plan starts (as is true for all cases except the symmetric ones mentioned in Weiss [10]) but it is known that $n \leq n'$, where n' is a known finite integer, then the Bayes plan T can still be constructed as presented below. The construction technique mentioned earlier is employed, proceeding first as though n' were the correct value for n. If n' is actually greater than n, then for some value $n^* \leq n'$, (2.4) will not be satisfied for any values of $X_1, \cdots, X_{n^*-1}$, otherwise, some sample sequences would not terminate at n, which is a contradiction. This shows that $n \leq n^*-1$, so the same construction is employed again, proceeding as though n^*-1 were the correct value for n. Continuing in this way, the Bayes plan T will eventually be constructed and the value of n will be found in the process. The computation of n' will be discussed in Section 2.4.

2.3 The Relation Between The Two Problems

The Bayes plan T discussed in Section 2.2 is a generalized sequential probability ratio test. Therefore, by Theorem 2 of Lehmann [5] (see also Ghosh [2]), if T satisfies (2.1), it also satisfies (1.1). This shows the relation between the problems discussed in Sections 1 and 2.2, as far as the O.C. curve is concerned.

It is still necessary, however, to investigate the relationship between minimizing the A.S.N. curve at a given value θ_0, and minimizing the maximum value of the A.S.N. curve for a plan which satisfies (2.1) and (1.1). Suppose T is a Bayes plan with respect to the given a priori distribution b_{-1}, b_0, b_1. Let U be any plan satisfying (1.1) with $\alpha = 1 - P_{\theta_{-1}}(A|T)$, $\beta = P_{\theta_1}(A|T)$. Then U satisfies (2.1) with these values of α and β. Define $\Delta(T)$ as

$$\max_\theta E_\theta(N|T) - E_{\theta_0}(N|T),$$

so that $\Delta(T) \geq 0$. Then it will be shown that

$$\max_\theta E_\theta(N|U) \geq \max_\theta E_\theta(N|T) - \Delta(T).$$

That is, T comes within $\Delta(T)$ of minimizing the maximum value of the A.S.N. curve among all plans satisfying (1.1) with $\alpha = 1 - P_{\theta_{-1}}(A|T)$, $\beta = P_{\theta_1}(A|T)$. To show the above, suppose it were not true. Then there would be a plan U satisfying the following conditions:

$$1 - P_{\theta_{-1}}(A|U) \leq 1 - P_{\theta_{-1}}(A|T)$$

$$P_{\theta_1}(A|U) \leq P_{\theta_1}(A|T)$$

$$E_{\theta_0}(N|U) \leq \max_\theta E_\theta(N|U) < \max_\theta E_\theta(N|T) - \Delta(T) = E_{\theta_0}(N|T).$$

But the existence of a plan U satisfying these conditions would mean that T could not satisfy (2.2), which is a contradiction, since T was assumed to be a Bayes plan with respect to b_{-1}, b_0, b_1. This proves that

$$\max_\theta E_\theta(N|U) \geq \max_\theta E_\theta(N|T) - \Delta(T).$$

If $\Delta(T) = 0$, then T actually minimizes the maximum of the A.S.N. curve. In Weiss [10] for a special symmetric case, θ_0 was chosen to make $\Delta(T) = 0$.

In general, however, there is no direct means for choosing θ_0 exactly; so a value of θ_0 is chosen to make $\Delta(T)$ reasonably small. In Section 2.2 it was pointed out that T will continue sampling longest if $Q(m,X)$ is close to b_{-1}/b_1 for all m. Define $\bar{X}(m)$ as

$$\frac{1}{m}\sum_{i=1}^{m} X_i \ .$$

Then $Q(m,X)$ can be written as

$$\left[\frac{\theta_1(1-\theta_{-1})}{\theta_{-1}(1-\theta_1)}\right]^{m\bar{X}(m)} \left[\frac{1-\theta_1}{1-\theta_{-1}}\right]^m \ .$$

If $Q(m,X)$ is close to b_{-1}/b_1, then

$$\left[\frac{\theta_1(1-\theta_{-1})}{\theta_{-1}(1-\theta_1)}\right]^{\bar{X}(m)} \quad \text{is close to} \quad \left[\frac{1-\theta_{-1}}{1-\theta_1}\right] \left[\frac{b_{-1}}{b_1}\right]^{1/m}$$

which for large m is close to $(1-\theta_{-1})/(1-\theta_1)$ if b_{-1}/b_1 is not too far from unity. However, for large m, $\bar{X}(m)$ is with high probability close to θ, the true proportion of defectives in the population. Then it would be expected that the A.S.N. curve is high at a value of θ such that

$$\left[\frac{\theta_1(1-\theta_{-1})}{\theta_{-1}(1-\theta_1)}\right]^{\theta} \quad \text{is close to} \quad \frac{1-\theta_{-1}}{1-\theta_1} \ ,$$

or at a value of θ close to

$$\frac{\log\left[\frac{1-\theta_{-1}}{1-\theta_1}\right]}{\log\left[\frac{\theta_1(1-\theta_{-1})}{\theta_{-1}(1-\theta_1)}\right]} \qquad (2.8)$$

If b_{-1}/b_1 is not close to unity and if there is approximate information available as to the largest number of observations the test could actually take (perhaps by interpolation among the Bayes plans presented later), then the maximizing θ will be approximately, for large $n^{\dagger}$,

$$\frac{\log\left[\frac{1-\theta_{-1}}{1-\theta_1}\right] + \frac{1}{n^{\dagger}} \log\left[\frac{b_{-1}}{b_1}\right]}{\log\left[\frac{\theta_1(1-\theta_{-1})}{\theta_{-1}(1-\theta_1)}\right]}, \tag{2.9}$$

where $n^{\dagger}$ represents the approximate information. If the A.S.N. curve is minimized at the value of θ given by (2.8) or (2.9), it may be expected that the maximum value of the A.S.N. curve will be approximately minimized. Therefore the value of either (2.8) or (2.9) is used in constructing the Bayes plan T, with the expectation that this choice will make $\Delta(T)$ small.

2.4 An Upper Bound on the Largest Possible Sample Size

The computation of the quantity n', an upper bound for n, will be discussed here. The quantity n' is a function of b_{-1}, b_0, b_1, but for typographical simplicity, the notation will not exhibit this dependence.

It would be ideal if the exact value of n were available before starting the construction of the Bayes plan T. However, at present the exact value can be found only in special symmetric cases (as in [10]); so that n' must be used, relying on the actual construction of T to find the value of n. The cruder (greater) the upper bound n', the greater the amount of computer time wasted in working back to n. Below, a value of n' is developed which is used in the construction of T. Wetherill [11] and Ray [6] have studied a different upper bound on n which has not been compared to the bound presented here.

From the discussion immediately following Lemma 4.1 and in Lemmas 4.3 to 4.6 in [4], it is found that there is a continuous nonnegative function K(u) defined for all $u \geq 0$, such that for $m=0,1,2,\cdots$; T observes X_{m+1} if

$$c_{o,m}(X_1,\cdots,X_m) < K\left[\frac{c_{1,m}(X_1,\cdots,X_m)}{c_{-1,m}(X_1,\cdots,X_m)}\right] ;$$

T does not observe X_{m+1} if

$$c_{0,m}(X_1,\cdots,X_m) > K\left[\frac{c_{1,m}(X_1,\cdots,X_m)}{c_{-1,m}(X_1,\cdots,X_m)}\right] ;$$

T can randomize in any way between observing or not observing X_{m+1} if

$$c_{0,m}(X_1,\cdots,X_m) = K\left[\frac{c_{1,m}(X_1,\cdots,X_m)}{c_{-1,m}(X_1,\cdots,X_m)}\right] ;$$

and if T stops sampling immediately after observing X_m, T accepts the population if

$$\frac{c_{1,m}(X_1,\cdots,X_m)}{c_{-1,m}(X_1,\cdots,X_m)} < 1 ,$$

T rejects the population if

$$\frac{c_{1,m}(X_1,\cdots,X_m)}{c_{-1,m}(X_1,\cdots,X_m)} > 1 ,$$

T randomizes in any way between accepting or rejecting if

$$\frac{c_{1,m}(X_1,\cdots,X_m)}{c_{-1,m}(X_1,\cdots,X_m)} = 1 .$$

It should be noted that the function K(u) does not depend on b_{-1},b_0,b_1. Furthermore, by the convexity of the acceptance and rejection regions described by Kiefer and Weiss in [4], K(u) has the following properties:

$$\max_{u\geq 0} K(u) = K(1) < 1 , \quad K(0) = K(\infty) = 0,$$

and if u_1,u_2 are any two values satisfying one of the conditions $0\leq u_1<u_2\leq 1$ or $1\leq u_1<u_2$, then for all d in the closed interval [0,1]

$$K\left[\frac{du_1\{\frac{1-K(u_1)}{1+u_1}\} + (1-d)u_2\{\frac{1-K(u_2)}{1+u_2}\}}{d\{\frac{1-K(u_1)}{1+u_1}\} + (1-d)\{\frac{1-K(u_2)}{1+u_2}\}}\right] \leq dK(u_1)+(1-d)K(u_2) \qquad (2.10)$$

In particular, setting $u_1=0, u_2=0$ in (2.10) and denoting $K(1)$ by k, it is found that

$$K\left[\frac{\frac{1}{2}(1-d)(1-k)}{d+\frac{1}{2}(1-d)(1-k)}\right] \leqq (1-d)k \qquad (2.11)$$

for all d in $[0,1]$. Setting $u_1=1, u_2=\infty$ in (2.10), it is found that

$$K\left[\frac{\frac{1}{2}d(1-k)+(1-d)}{\frac{1}{2}d(1-k)}\right] \leqq dk \qquad (2.12)$$

for all d in $[0,1]$.

If the function $K(u)$ were known, the entire test T could be constructed explicitly, and the exact value of n would be known. However, $K(u)$ is not known. Suppose that a known function $L(u)$ satisfies $L(u) \geqq K(u)$ for all $u \geqq 0$, and that n' is defined as the largest possible sample size of a plan constructed by acting as though $K(u)$ were equal to $L(u)$. Then it is clear that $n' \geqq n$. Furthermore, the closer $L(u)$ is to $K(u)$, the smaller n' will be. The next step is to construct a function $L(u)$ which is known to satisfy $L(u) \geqq K(u)$.

It can be said that a nonnegative value u satisfies the condition S if no Bayes plan with respect to the a priori distribution

$$\frac{1-K(u)}{1+u}\ , \quad K(u)\ , \quad u\left[\frac{1-K(u)}{1+u}\right]$$

can ever observe X_2. From the characterization above, there is a Bayes plan T_0 with respect to the a priori distribution which does not observe X_1, and there is a Bayes plan T which does observe X_1. Since $E_{\theta_0}(N|T_0)=0$ and $P_{\theta_1}(A|T_0)=P_{\theta_{-1}}(A|T_0)$, T_0 makes the left-hand side of (2.2) equal to

$$\min\left\{\frac{1-K(u)}{1+u}\ ,\ u\left[\frac{1-K(u)}{1+u}\right]\right\}.$$

Since T_1 observes X_1 but cannot observe X_2, it is clear that T_1 accepts the population if $X_1=0$ and rejects the population if $X_1=1$, otherwise it would not be necessary for T_1 to observe X_1. Then $E_{\theta_0}(N|T_1)=1$, $P_{\theta_{-1}}(A|T_1)=1-\theta_{-1}$, $P_{\theta_1}(A|T_1)=1-\theta_1$, so that T_1 makes the left-hand side of (2.2) equal to

$$\theta_{-1}\left[\frac{1-K(u)}{1+u}\right] + K(u) + (1-\theta_1)u\left[\frac{1-K(u)}{1+u}\right] .$$

Since both T_o and T_1 minimize the left-hand side of (2.2), then

$$\theta_{-1}\left[\frac{1-K(u)}{1+u}\right]+K(u)+(1-\theta_1)u\left[\frac{1-K(u)}{1+u}\right] = \min\left\{\frac{1-K(u)}{1+u}, u\left[\frac{1-K(u)}{1+u}\right]\right\}.$$

Using this equality, it is found that if u satisfies the condition S, then

$$K(u) = \frac{u\theta_1-\theta_{-1}}{1+u+u\theta_1-\theta_{-1}} \qquad \text{if } u \leq 1$$

$$K(u) = \frac{1-u+u\theta_1-\theta_{-1}}{2+u\theta_1-\theta_{-1}} \qquad \text{if } u \leq 1 . \tag{2.13}$$

Next it will be shown that u=1 satisfies the condition S, so that

$$k = K(1) = \frac{\theta_1-\theta_{-1}}{2+\theta_1-\theta_{-1}} .$$

Using the a priori distribution $\frac{1}{2}(1-k),k,\frac{1}{2}(1-k)$, it is found that if $X_1=0$,

$$c_{o,1}(X_1) = \frac{k(1-\theta_o)}{k(1-\theta_o)+\frac{1}{2}(1-k)(2-\theta_1-\theta_{-1})} , \quad \frac{c_{1,1}(X_1)}{c_{-1,1}(X_1)} = \frac{1-\theta_1}{1-\theta_{-1}} .$$

If

$$d = \frac{\frac{1}{2}(1-k)(\theta_1-\theta_{-1})}{1-\theta_1+\frac{1}{2}(1-k)(\theta_1-\theta_{-1})}$$

in (2.11), it is found that

$$K\left[\frac{1-\theta_1}{1-\theta_{-1}}\right] \leq \frac{k(1-\theta_1)}{1-\theta_1+\frac{1}{2}(1-k)(2-\theta_1-\theta_{-1})} ,$$

and it is easily verified that

$$\frac{k(1-\theta_1)}{1-\theta_1+\frac{1}{2}(1-k)(\theta_1-\theta_{-1})} < \frac{k(1-\theta_o)}{k(1-\theta_o)+\frac{1}{2}(1-k)(2-\theta_1-\theta_{-1})} .$$

Therefore when $X_1=0$,

$$c_{o,1}(X_1) > K\left[\frac{c_{1,1}(X_1)}{c_{-1,1}(X_1)}\right] ,$$

so X_2 cannot be observed. If $X_1=1$,

$$c_{0,1}(X_1) = \frac{k\theta_0}{k\theta_0+\frac{1}{2}(1-k)(\theta_1+\theta_{-1})}, \quad \frac{c_{1,1}(X_1)}{c_{-1,1}(X_1)} = \frac{\theta_1}{\theta_{-1}} .$$

If

$$d = 1 - \frac{\frac{1}{2}(1-k)(\theta_1-\theta_{-1})}{\theta_{-1}+\frac{1}{2}(1-k)(\theta_1-\theta_{-1})}$$

in (2.12), it is found that

$$K\left(\frac{\theta_1}{\theta_{-1}}\right) \leqq \frac{k\theta_{-1}}{\theta_{-1}+\frac{1}{2}(1-k)(\theta_1-\theta_{-1})},$$

and it is easily verified that

$$\frac{k\theta_{-1}}{\theta_{-1}+\frac{1}{2}(1-k)(\theta_1-\theta_{-1})} < \frac{k\theta_0}{k\theta_0+\frac{1}{2}(1-k)(\theta_1+\theta_{-1})} .$$

Therefore when $X_1=1$,

$$c_{0,1}(X_1) > K\left[\frac{c_{1,1}(X_1)}{c_{-1,1}(X_1)}\right],$$

so X_2 cannot be observed. This proves that u=1 satisfies the condition S.

Now, a function $L_1(u)$ is defined as follows:

$$L_1(u) = \frac{k}{1+\frac{1}{2}(1-k)\left(\frac{1-u}{u}\right)} \qquad \text{if } u \leqq 1 ,$$

$$L_1(u) = \frac{k}{1+\frac{1}{2}(1-k)(u-1)} \qquad \text{if } u \geqq 1 .$$

Using (2.11) and (2.12), it can be verified that $L_1(u) \geqq K(u)$ for all $u \geqq 0$. Also $L_1(u)$ is a known function of u, since k=K(1) has been computed explicitly above. Thus $L_1(u)$ could be used as the function L(u) described above. However, more can be done by finding values of u other than unity which satisfy the condition S.

Suppose that values v,w are known, with $v<1<w$, such that all values u in the open interval (v,w) satisfy the condition S. Then it is known that K(u) is given by (2.13) for u in the closed interval [v,w]. Using (2.10) with $u_1=o$, $u_2=v$ yields

$$K(u) \leq \frac{K(v)}{1+(\frac{1}{u}-\frac{1}{v})(\frac{v}{1+v})[1-K(v)]} \qquad \text{for } o \leq u \leq v .$$

Using (2.10) with $u_1=w$, $u_2=\infty$ yields

$$K(u) \leq \frac{K(w)}{1+(u-w)[\frac{1-K(w)}{1+w}]} \qquad \text{for } u \geq w .$$

Therefore, L(u) can be defined as follows:

$$\begin{aligned} L(u) &= \frac{K(v)}{1+(\frac{1}{u}-\frac{1}{v})(\frac{v}{1+v})[1-K(v)]} && \text{for } o \leq u \leq v \\ L(u) &= \frac{u\theta_1-\theta_{-1}}{1+u+u\theta_1-\theta_{-1}} && \text{for } v \leq u \leq 1 \\ L(u) &= \frac{1-u+u\theta_1-\theta_{-1}}{2+u\theta_1-\theta_{-1}} && \text{for } 1 \leq u \leq w \\ L(u) &= \frac{K(w)}{1+(u-w)[\frac{1+K(w)}{1+w}]} && \text{for } u \geq w , \end{aligned} \qquad (2.14)$$

and it is evident that $L(u) \geq K(u)$ for all $u \geq o$. Now the explicit values for v and w can be computed.

First it is noted that K(u) satisfies the following inequalities:

$$\begin{aligned} K(u) &\geq \frac{u\theta_1-\theta_{-1}}{1+u+u\theta_1-\theta_{-1}} && \text{if } u \leq 1 \\ K(u) &\geq \frac{1-u+u\theta_1-\theta_{-1}}{2+u\theta_1-\theta_{-1}} && \text{if } u \geq 1 . \end{aligned} \qquad (2.15)$$

Note that the right-hand sides of these inequalities are identical with the right-hand sides of the equalities (2.13) which hold only for values of u satisfying the condition S, whereas (2.15) holds for all u. The proof of (2.15) proceeds similarly to that of (2.13), noting that T_o is a Bayes plan with respect to the a priori distribution

$$\frac{1-K(u)}{1-u}, \quad K(u), \quad u\left[\frac{1-K(u)}{1+u}\right].$$

But T_1 may not hold for general u, and therefore the left-hand side of (2.2) using T_o is less than or equal to the left-hand side of (2.2) using T_1. The inequalities (2.15) then follow directly.

Define

$$v_1 = \max\left\{\frac{\theta_{-1}(1-\theta_o)}{\theta_1+\theta_{-1}-\theta_o\theta_1-\theta_{-1}\theta_1}, \frac{\theta_{-1}+\frac{\theta_{-1}}{\theta_o}(\theta_1-\theta_{-1})}{\theta_1}\right\}.$$

Then $o<v_1<1$, and a straightforward but somewhat tedious calculation shows that if u is the open interval $(v_1,1)$ and if a Bayes plan is constructed with respect to the a priori distribution

$$\frac{1-K(u)}{1+u}, \quad K(u), \quad u\left[\frac{1-K(u)}{1+u}\right],$$

then

$$c_{o,1}(X_1) > L_1\left[\frac{c_{1,1}(X_1)}{c_{-1,1}(X_1)}\right]$$

For $X_1=0$ or 1, (the first inequality in (2.15) is used in the calculation). Since $L_1(u) \geq K(u)$ for all u, then any u in $(v_1,1)$ satisfies the condition S.

Define

$$w_1 = \min\left\{\frac{(1-\theta_o)(1-\theta_{-1})}{(1-\theta_1)(1+\theta_1-\theta_o-\theta_{-1})}, \frac{1-\theta_{-1}-\left(\frac{\theta_{-1}}{\theta_o}\right)(\theta_1-\theta_{-1})}{1-\theta_1}\right\}.$$

Then $w_1>1$, and calculation shows that if u is in the open interval $(1,w_1)$

and if a Bayes plan is constructed with respect to the a priori distribution

$$\frac{1-K(u)}{1+u}\ , \quad K(u)\ , \quad u\left[\frac{1-K(u)}{1+u}\right] ,$$

then

$$c_{o,1}(X_1) > L_1\left[\frac{c_{1,1}(X_1)}{c_{-1,1}(X_1)}\right] .$$

For $X_1=0$ or 1, the second inequality in (2.15) is used in the calculation. Since $L_1(u) \geq K(u)$ for all u, then any u in $(1,w_1)$ satisfies the condition S. Now define $L_2(u)$ as L(u) was defined in (2.14), using v_1,w_1 in place of v,w. Clearly, $L_2(u) \geq K(u)$ for all u. Noting that v_1,w_1 is found by using the fact that $L_1(u) \geq K(u)$ for all u, and noting that $L_1(u) \geq L_2(u) \geq K(u)$ for all u, it would be possible to find values v_2,w_2, such that all u in (v_2,w_2) satisfy the condition S, where $v_2<v_1, w_2>w_1$. The quantities v_2,w_2 would be computed using $L_2(u)$ just as $L_1(u)$ was used in computing v_1,w_1. Then, using v_2,w_2 in (2.14), there would be a function $L_2(u)$ with $L_2(u) \geq L_2(u) \geq K(u)$. The above process could be continued. Since the establishment of an upper bound n' may be of only relative importance in the wholesale construction of the Bayes plans, the known function $L_1(u)$ defined by using v,w in (2.14) was used here as the final function L(u).

For a given a priori distribution b_{-1},b_0,b_1 define z(m) as the largest integer for which $Q(m,X) \leq b_{-1}/b_1$ when $X_1+\cdots+X_m = z(m)$ for each positive integer m. Then from Section 2.2, it is known that if a Bayes plan cannot observe X_{m+1} when

$$\sum_{i=1}^{m} X_i = z(m)$$

or z(m)+1, the plan cannot observe X_{m+1} under any circumstances. Define c(m) as the value of $c_{o,m}(X_1,\cdots,X_m)$ when

$$\sum_{i=1}^{m} X_i = z(m) ,$$

and define $c'(m)$ as the value of $c_{o,m}(X_1,\cdots,X_m)$ when

$$\sum_{i=1}^{m} X_i = z(m)+1 .$$

Define $f(m)$ as the value of

$$\frac{c_{1,m}(X_1,\cdots,X_m)}{c_{-1,m}(X_1,\cdots,X_m)} \quad \text{when} \quad \sum_{i=1}^{m} X_i = z(m) ,$$

and $f'(m)$ as the value of

$$\frac{c_{1,m}(X_1,\cdots,X_m)}{c_{-1,m}(X_1,\cdots,X_m)} \quad \text{when} \quad \sum_{i=1}^{m} X_i = z(m)+1 .$$

Define n' as the smallest integer satisfying both of the following inequalities:

$$c(n') > L_1(f(n'))$$

$$c'(n') > L_1(f'(n')) .$$

If $L_1(u)$ were equal to $K(u)$, the Bayes plan could not observe $X_{n'+1}$. Since it is known that $L_1(u) \geq K(u)$ for all u, it is also known that the Bayes plan can never observe $X_{n'+1}$. Therefore n' is the upper bound for n described in Section 2.2.

Since n' is a discrete variable and b_o is a continuous variable, in the actual computation it is easier to reverse the above process and search for b_o, rather than for n'. The reverse process is completed by fixing a positive integer m and a value for the ratio b_{-1}/b_1; then $z(m)$, $f(m)$, and $f'(m)$ can be computed without knowing the value of b_o. Define $b_o'(m, b_{-1}/b_1)$ as the smallest value of b_o that makes both of the following inequalities hold:

$$c(m) \geqq L_1(f(m))$$

$$c'(m) \geqq L_1(f'(m)) \; .$$

Then it is clear that there is a Bayes plan with respect to the a priori distribution $b_{-1}, b_0'(m, b_{-1}/b_1), b_1$ which never observes X_{m+1}, where in this a priori distribution b_{-1}/b_1 is equal to the fixed given ratio. Then the entire Bayes plan can be constructed by working back from m as the upper bound on the number of observations.

3. Sequential Sampling Plans Which Approximately Minimize the Maximum Expected Sample Size

The format of each plan (the order of plans is listed in the table of contents by values of $\theta_{-1},\theta_1,\alpha,\beta$) is similar to the following example:

$\theta_{-1} = .001$ $\theta_1 = .011$ $\alpha = \beta = .01$ Plan 1

Comparison of A.S.N. Values

Test	$\theta = .001$	Maximizing θ	$\theta = .011$
Minimax	611.5	735.0	321.0
Wald	568.9	885.0	285.8
Fixed Sample Size		965.0	

n	A_n	R_n	n	A_n	R_n
1	--	--	1346	5	6
3	--	3			
242	--	4			
519	0	4			
664	0	5			
677	1	5			
838	2	5			
1001	3	5			
1046	3	6			
1171	4	6			

The Minimax test refers to the one introduced by Weiss [1]. A.S.N. values for the Wald test [9] were computed using the Wald approximation formulas, and the number of observations required by the fixed sample size test was computed using a normal approximation. The α,β values are approximately those for the given plan. The exact α,β values can be found in [3]. A.S.N. values for the Minimax and Wald tests are compared at θ_{-1},θ_1 and the "Maximizing θ." The value of θ associated with the maximum A.S.N. is usually different for the two sequential tests.

The acceptance and rejection numbers at stage n, are labeled A_n,R_n respectively. The batch of product being sampled is accepted if the cumulative number of defectives after the n^{th} observation is no greater than A_n and is rejected if the cumulative number of defectives is no less than R_n. Otherwise, another observation is taken, and the comparison is repeated. If the A_n column is dashed for the n^{th} observation, the batch cannot yet be accepted, and if the R_n column is dashed for the n^{th} observation, the batch cannot yet be rejected. The values of A_n and R_n are only listed when one or both change value. For example, in Plan 1 presented above no decision to accept or reject can be made until three observations have been taken, for the 3^{rd} through the 241^{st} observations the batch cannot be accepted but can be rejected if a total of 3 or more defectives is observed, etc. The last entries in the A_n and R_n columns differ by one unit and force sampling to terminate if it reaches this point.

θ_{-1} = .001 θ_1 = .001 $\alpha = \beta$ = .01 Plan 1

Comparison of A.S.N Values

Test	θ = .001	Maximizing θ	θ = .011
Minimax	611.5	735.0	321.0
Wald	568.9	885.0	285.8
Fixed Sample Size		965.0	

n	A_n	R_n	n	A_n	R_n
1	--	--	1346	5	6
3	--	3			
242	--	4			
519	0	4			
664	0	5			
677	1	5			
838	2	5			
1001	3	5			
1046	3	6			
1171	4	6			

θ_{-1} = .001 θ_1 = .011 $\alpha = \beta$ = .025 Plan 2

Comparison of A.S.N. Values

Test	θ = .001	Maximizing θ	θ = .011
Minimax	467.3	513.0	233.0
Wald	448.6	546.0	209.9
Fixed Sample Size		670.0	

n	A_n	R_n
1	--	--
2	--	2
131	--	3
401	0	3
539	0	4
571	1	4
744	2	4
915	3	4

$\theta_{-1} = .001$ $\theta_1 = .011$ $\alpha = \beta = .05$ Plan 3

Comparison of A.S.N. Values

Test	$\theta = .001$	Maximizing θ	$\theta = .011$
Minimax	344.5	360.0	179.9
Wald	337.1	>360.0	157.9
Fixed Sample Size		470.0	

n	A_n	R_n
1	--	--
2	--	2
303	0	2
337	0	3
490	1	3
677	2	3

$\theta_{-1} = .001$ $\theta_1 = .011$ $\alpha = \beta = .10$ Plan 4

Comparison of A.S.N. Values

Test	$\theta = .001$	Maximizing θ	$\theta = .011$
Minimax	217.0	217.0	108.1
Wald	214.7	>217.0	100.9
Fixed Sample Size		285.0	

n	A_n	R_n
1	--	1
85	--	2
212	0	2
415	1	2

$\theta_{-1} = .01$ $\theta_1 = .11$ $\alpha = \beta = .01$ Plan 5

Comparison of A.S.N. Values

Test	$\theta = .01$	Maximizing θ	$\theta = .11$
Minimax	59.4	69.2	29.8
Wald	54.8	81.6	26.5
Fixed Sample Size		90.0	

n	A_n	R_n	n	A_n	R_n
1	--	--	131	5	6
3	--	3			
34	--	4			
51	0	4			
66	1	4			
73	1	5			
81	2	5			
97	3	5			
108	3	6			
113	4	6			

$\theta_{-1} = .01$ $\theta_1 = .11$ $\alpha = \beta = .025$ Plan 6

Comparison of A.S.N. Values

Test	$\theta = .01$	Maximizing θ	$\theta = .11$
Maximax	45.0	49.2	22.5
Wald	42.7	52.5	20.6
Fixed Sample Size		65.0	

n	A_n	R_n
1	--	--
2	--	2
16	--	3
39	0	3
54	0	4
55	1	4
71	2	4
87	3	4

$\theta_{-1} = .01$ $\theta_1 = .11$ $\alpha = \beta = .05$ Plan 7

Comparison of A.S.N. Values

Test	$\theta = .01$	Maximizing θ	$\theta = .11$
Minimax	32.6	33.8	17.8
Wald	31.6	>33.8	15.6
Fixed Sample Size		47.0	

n	A_n	R_n
1	--	--
2	--	2
29	0	2
35	0	3
46	1	3
63	2	3

$\theta_{-1} = .01$ $\theta_1 = .11$ $\alpha = \beta = .10$ Plan 8

Comparison of A.S.N. Values

Test	$\theta = .01$	Maximizing θ	$\theta = .11$
Minimax	21.3	21.3	10.6
Wald	21.0	>21.3	9.9
Fixed Sample Size		28.0	

n	A_n	R_n
1	--	1
10	--	2
21	0	2
40	1	2

$\theta_{-1} = .01$ $\theta_1 = .21$ $\alpha = \beta = .01$ Plan 9

Comparison of A.S.N. Values

Test	$\theta = .01$	Maximizing θ	$\theta = .21$
Minimax	24.9	27.4	10.6
Wald	23.3	30.4	9.6
Fixed Sample Size		35.0	

n	A_n	R_n
1	--	--
2	--	2
13	--	3
23	0	3
32	1	3
39	1	4
40	2	4
50	3	4

$\theta_{-1} = .01$ $\theta_1 = .21$ $\alpha = \beta = .025$ Plan 10

Comparison of A.S.N. Values

Test	$\theta = .01$	Maximizing θ	$\theta = .21$
Minimax	18.2	19.1	9.3
Wald	17.5	>19.2	7.6
Fixed Sample Size		25.0	

n	A_n	R_n
1	--	--
2	--	2
17	0	2
26	1	3
37	2	3

$\theta_{-1} = .01$ $\theta_1 = .21$ $\alpha = \beta = .05$ Plan 11

Comparison of A.S.N. Values

Test	θ = .01	Maximizing θ	θ = .21
Minimax	13.4	13.6	6.1
Wald	13.2	>13.6	5.6
Fixed Sample Size		17.0	

n	A_n	R_n
1	--	1
5	--	2
13	0	2
24	1	2

$\theta_{-1} = .01$ $\theta_1 = .21$ $\alpha = \beta = .10$ Plan 12

Comparison of A.S.N. Values

Test	θ = .01	Maximizing θ	θ = .21
Minimax	9.6	9.6	4.3
Wald	9.5	> 9.6	4.0
Fixed Sample Size		11.0	

n	A_n	R_n
1	--	1
10	0	1

$\theta_{-1} = .03$ $\theta_1 = .13$ $\alpha = \beta = .01$ Plan 13

Comparison of A.S.N. Values

Test	$\theta = .03$	Maximizing θ	$\theta = .13$
Minimax	79.9	104.1	52.0
Wald	72.8	131.0	46.6
Fixed Sample Size		136.0	

n	A_n	R_n	n	A_n	R_n	n	A_n	R_n
1	--	--	98	4	9	176	11	13
5	--	5	104	4	10	179	11	14
24	--	6	109	5	10	188	12	14
44	--	7	120	6	10	196	12	15
55	0	7	123	6	11	200	13	15
64	0	8	131	7	11	212	14	15
65	1	8	142	8	12			
76	2	8	153	9	12			
84	2	9	161	9	13			
87	3	9	164	10	13			

$\theta_{-1} = .03$ $\theta_1 = .13$ $\alpha = \beta = .025$ Plan 14

Comparison of A.S.N. Values

Test	$\theta = .03$	Maximizing θ	$\theta = .13$
Minimax	60.9	73.7	39.7
Wald	56.8	84.8	36.4
Fixed Sample Size		99.0	

n	A_n	R_n	n	A_n	R_n	n	A_n	R_n
1	--	--	86	4	8	156	10	12
4	--	4	97	5	8	168	11	12
23	--	5	100	5	9			
42	0	5	109	6	9			
43	0	6	119	6	10			
53	1	6	120	7	10			
62	1	7	132	8	10			
64	2	7	136	8	11			
75	3	7	144	9	11			
81	3	8	154	9	12			

$\theta_{-1} = .03$ $\theta_1 = .13$ $\alpha = \beta = .05$ Plan 15

Comparison of A.S.N. Values

Test	$\theta = .03$	Maximizing θ	$\theta = .13$
Minimax	45.2	51.2	29.9
Wald	43.2	54.8	27.9
Fixed Sample Size		70.0	

n	A_n	R_n	n	A_n	R_n
1	--	--	77	4	7
3	--	3	89	5	7
17	--	4	92	5	8
32	0	4	101	6	8
37	0	5	109	6	9
43	1	5	113	7	9
55	2	5	125	8	9
56	2	6			
66	3	6			
74	3	7			

$\theta_{-1} = .03$ $\theta_1 = .13$ $\alpha = \beta = .10$ Plan 16

Comparison of A.S.N. Values

Test	$\theta = .03$	Maximizing θ	$\theta = .13$
Minimax	30.1	31.8	20.1
Wald	29.3	>31.8	19.1
Fixed Sample Size		42.0	

n	A_n	R_n	n	A_n	R_n
1	--	--	59	5	7
3	--	3	66	6	8
8	--	4	74	7	8
23	--	5			
24	0	5			
31	1	5			
38	2	6			
45	3	6			
51	4	6			
53	4	7			

$\theta_{-1} = .03$ $\theta_1 = .13$ $\alpha = \beta = .20$ Plan 17

Comparison of A.S.N. Values

Test	$\theta = .03$	Maximizing θ	$\theta = .13$
Minimax	14.3	14.3	9.7
Wald	14.1	>14.3	9.2
Fixed Sample Size		18.0	

n	A_n	R_n
1	--	1
5	--	2
13	0	2
21	0	3
27	1	3
38	1	4
40	2	4
53	3	4

$\theta_{-1} = .03$ $\theta_1 = .23$ $\alpha = \beta = .01$ Plan 18

Comparison of A.S.N. Values

Test	$\theta = .03$	Maximizing θ	$\theta = .23$
Minimax	30.0	37.1	17.5
Wald	27.7	45.3	15.6
Fixed Sample Size		48.0	

n	A_n	R_n	n	A_n	R_n
1	--	--	59	5	7
3	--	3	66	6	8
8	--	4	74	7	8
23	--	5			
24	0	5			
31	1	5			
38	2	6			
45	3	6			
51	4	6			
53	4	7			

$\theta_{-1} = .03$ $\theta_1 = .23$ $\alpha = \beta = .025$ Plan 19

Comparison of A.S.N. Values

Test	$\theta = .03$	Maximizing θ	$\theta = .23$
Minimax	22.8	25.7	13.5
Wald	21.2	28.4	12.0
Fixed Sample Size		34.0	

n	A_n	R_n	n	A_n	R_n
1	--	--	54	5	6
2	--	2			
3	--	3			
18	--	4			
19	0	4			
25	1	4			
32	2	4			
33	2	5			
39	3	5			
46	4	6			

$\theta_{-1} = .03$ $\theta_1 = .23$ $\alpha = \beta = .05$ Plan 20

Comparison of A.S.N. Values

Test	$\theta = .03$	Maximizing θ	$\theta = .23$
Minimax	16.6	17.9	9.8
Wald	16.1	18.3	9.1
Fixed Sample Size		24.0	

n	A_n	R_n
1	--	--
2	--	2
10	--	3
14	0	3
21	1	3
24	1	4
29	2	4
36	3	4

$\theta_{-1} = .03$ $\theta_1 = .23$ $\alpha = \beta = .10$ Plan 21

Comparison of A.S.N. Values

Test	$\theta = .03$	Maximizing θ	$\theta = .23$
Minimax	10.7	10.7	6.2
Wald	10.4	>10.7	5.8
Fixed Sample Size		15.0	

n	A_n	R_n
1	--	1
3	--	2
10	0	2
16	0	3
17	1	3
25	2	3

$\theta_{-1} = .03$ $\theta_1 = .23$ $\alpha = \beta = .20$ Plan 22

Comparison of A.S.N. Values

Test	$\theta = .03$	Maximizing θ	$\theta = .23$
Minimax	5.6	5.6	3.4
Wald	5.3	>5.6	3.2
Fixed Sample Size		7.0	

n	A_n	R_n
1	--	1
6	0	1

θ_{-1} = .05 θ_1 = .15 $\alpha = \beta$ = .01 Plan 23

Comparison of A.S.N. Values

Test	θ = .05	Maximizing θ	θ = .15
Minimax	97.8	134.1	71.3
Wald	88.4	172.0	64.0
Fixed Sample Size		175.0	

n	A_n	R_n	n	A_n	R_n
1	--	--	189	15	20
6	--	6	196	16	20
10	--	7	203	16	21
24	--	8	205	17	21
38	--	9	214	18	21
53	--	10	216	18	22
56	0	10	224	19	22
65	1	10	228	19	23
67	1	11	233	20	23
73	2	11	241	20	24
80	2	12	242	21	24
82	3	12	252	22	24
91	4	12	254	22	25
94	4	13	261	23	25
99	5	13	266	23	26
108	6	14	271	24	26
117	7	14	278	24	27
122	7	15	280	25	27
125	8	15	290	26	28
134	9	15	300	27	28
136	9	16			
143	10	16			
149	10	17			
152	11	17			
160	12	17			
163	12	18			
169	13	18			
176	13	19			
178	14	19			
187	15	19			

θ_{-1} = .05 θ_1 = .15 $\alpha = \beta$ = .025 Plan 24

Comparison of A.S.N. Values

Test	θ = .05	Maximizing θ	θ = .15
Minimax	73.6	93.5	53.8
Wald	68.4	109.6	49.6
Fixed Sample Size		127.0	

n	A_n	R_n	n	A_n	R_n	n	A_n	R_n	n	A_n	R_n
1	--	--	78	4	10	131	10	14	186	16	18
5	--	5	84	4	11	137	10	15	188	16	19
15	--	6	86	5	11	140	11	15	196	17	19
29	--	7	95	6	11	149	12	15	201	17	20
43	0	8	97	6	12	150	12	16	206	18	20
51	1	8	104	7	12	158	13	16	213	18	21
57	1	9	111	7	13	163	13	17	215	19	21
60	2	9	113	8	13	168	14	17	225	20	22
69	3	9	122	9	13	176	14	18	235	21	22
70	3	10	124	9	14	177	15	18			

θ_{-1} = .05 θ_1 = .15 $\alpha = \beta$ = .05 Plan 25

Comparison of A.S.N. Values

Test	θ = .05	Maximizing θ	θ = .15
Minimax	55.0	64.8	40.3
Wald	52.1	71.0	37.8
Fixed Sample Size		89.0	

n	A_n	R_n	n	A_n	R_n	n	A_n	R_n
1	--	--	69	4	9	132	11	13
4	--	4	77	5	9	133	11	14
15	--	5	82	5	10	142	12	14
29	--	6	86	6	10	146	12	15
33	0	6	95	7	11	152	13	15
42	1	7	105	8	11	158	13	16
50	2	7	108	8	12	161	14	16
56	2	8	114	9	12	170	14	17
59	3	8	121	9	13	171	15	17
68	4	8	123	10	13	181	16	17

$\theta_{-1} = .05$ $\theta_1 = .15$ $\alpha = \beta = .10$ Plan 26

Comparison of A.S.N. Values

Test	θ = .05	Maximizing θ	θ = .15
Minimax	35.8	38.7	26.3
Wald	34.6	39.5	25.0
Fixed Sample Size		55.0	

n	A_n	R_n	n	A_n	R_n	n	A_n	R_n
1	--	--	67	4	8	126	11	12
3	--	3	68	5	8			
15	--	4	78	6	8			
23	0	4	79	6	9			
28	0	5	87	7	9			
32	1	5	92	7	10			
41	2	6	97	8	10			
50	3	6	104	8	11			
54	3	7	107	9	11			
59	4	7	116	10	12			

$\theta_{-1} = .05$ $\theta_1 = .15$ $\alpha = \beta = .20$ Plan 27

Comparison of A.S.N. Values

Test	θ = .05	Maximizing θ	θ = .15
Minimax	17.2	17.3	13.1
Wald	16.9	>17.3	12.4
Fixed Sample Size		23.0	

n	A_n	R_n	n	A_n	R_n
1	--	--	53	4	6
2	--	2	62	4	7
13	0	2	63	5	7
14	0	3	73	6	7
23	1	3			
26	1	4			
33	2	4			
38	2	5			
43	3	5			
50	3	6			

$\theta_{-1} = .05$ $\theta_1 = .25$ $\alpha = \beta = .01$ Plan 28

Comparison of A.S.N. Values

Test	$\theta = .05$	Maximizing θ	$\theta = .25$
Minimax	34.1	43.6	22.2
Wald	30.9	54.5	19.8
Fixed Sample Size		58.0	

n	A_n	R_n	n	A_n	R_n
1	--	--	47	4	8
4	--	4	53	5	9
9	--	5	59	6	9
20	--	6	64	6	10
25	0	6	65	7	10
30	1	6	71	8	10
32	1	7	74	8	11
36	2	7	77	9	11
42	3	7	84	10	12
43	3	8	90	11	12

$\theta_{-1} = .05$ $\theta_1 = .15$ $\alpha = \beta = .025$ Plan 29

Comparison of A.S.N. Values

Test	$\theta = .05$	Maximizing θ	$\theta = .15$
Minimax	25.8	30.7	16.9
Wald	24.0	35.0	15.4
Fixed Sample Size		41.0	

n	A_n	R_n	n	A_n	R_n
1	--	--	42	4	7
3	--	3	48	5	7
7	--	4	50	5	8
18	--	5	54	6	8
19	0	5	60	6	9
25	1	5	61	7	9
29	1	6	67	8	9
30	2	6			
36	3	6			
40	3	7			

$\theta_{-1} = .05$ $\theta_1 = .25$ $\alpha = \beta = .05$ Plan 30

Comparison of A.S.N. Values

Test	$\theta = .05$	Maximizing θ	$\theta = .25$
Minimax	19.4	21.6	12.9
Wald	18.3	22.9	11.9
Fixed Sample Size		29.0	

n	A_n	R_n
1	--	--
2	--	2
3	--	3
14	--	4
15	0	4
20	1	4
24	1	5
26	2	5
32	3	5
34	3	6
39	4	6
44	4	7
45	5	7
51	6	7

$\theta_{-1} = .05$ $\theta_1 = .25$ $\alpha = \beta = .10$ Plan 31

Comparison of A.S.N. Values

Test	$\theta = .05$	Maximizing θ	$\theta = .25$
Minimax	12.2	12.6	8.3
Wald	11.8	>12.6	7.7
Fixed Sample Size		18.0	

n	A_n	R_n
1	--	--
2	--	2
10	0	3
16	1	3
20	1	4
22	2	4
28	3	4

$\theta_{-1} = .05$ $\theta_1 = .25$ $\alpha = \beta = .20$ Plan 32

Comparison of A.S.N. Values

Test	$\theta = .05$	Maximizing θ	$\theta = .25$
Minimax	5.7	5.7	3.8
Wald	5.6	>5.7	3.6
Fixed Sample Size		8.0	

n	A_n	R_n
1	--	1
5	--	2
6	0	2
12	1	2

$\theta_{-1} = .10$ $\theta_1 = .20$ $\alpha = \beta = .01$ Plan 33

Comparison of A.S.N. Values

Test	$\theta = .10$	Maximizing θ	$\theta = .20$
Minimax	136.0	198.6	112.8
Wald	122.8	258.7	101.4
Fixed Sample Size		259.0	

n	A_n	R_n	n	A_n	R_n	n	A_n	R_n	n	A_n	R_n
1	--	--	143	14	27	259	34	41	370	52	56
10	--	10	145	15	27	262	34	42	376	53	56
14	--	11	151	16	28	265	35	42	377	53	57
22	--	12	157	17	28	270	35	43	382	54	57
30	--	13	159	17	29	271	36	43	385	54	58
38	--	14	163	18	29	277	37	43	389	55	58
46	--	15	167	18	30	278	37	44	392	55	59
54	--	16	169	19	30	283	38	44	395	56	59
56	0	16	175	20	31	286	38	45	400	56	60
62	1	17	181	21	31	289	39	45	402	57	60
68	2	17	183	21	32	293	39	46	407	57	61
71	2	18	187	22	32	295	40	46	408	58	61
74	3	18	191	22	33	301	41	47	414	59	61
79	3	19	192	23	33	308	42	47	415	59	62
80	4	19	198	24	33	309	42	48	421	60	62
85	5	19	199	24	34	314	43	48	422	60	63
87	5	20	204	25	34	317	43	49	427	61	63
91	6	20	207	25	35	320	44	49	429	61	64
95	6	21	210	26	35	324	44	50	434	62	64
97	7	21	215	26	36	326	45	50	437	62	65
103	8	22	216	27	36	332	46	51	440	63	65
109	9	22	223	28	37	339	47	51	444	63	66
111	9	23	229	29	37	340	47	52	447	64	66
115	10	23	231	29	38	345	48	52	451	64	67
119	10	24	235	30	38	347	48	53	453	65	67
121	11	24	239	30	39	351	49	53	458	65	68
127	12	25	241	31	39	355	49	54	460	66	68
133	13	25	247	32	40	357	50	54	466	67	69
135	13	26	253	33	40	362	50	55	473	68	70
139	14	26	254	33	41	364	51	55	479	69	70

$\theta_{-1} = .10$ $\theta_1 = .20$ $\alpha = \beta = .025$ Plan 34

Comparison of A.S.N. Values

Test	$\theta = .10$	Maximizing θ	$\theta = .20$
Minimax	101.9	137.4	84.6
Wald	94.6	163.6	78.2
Fixed Sample Size		188.0	

n	A_n	R_n	n	A_n	R_n	n	A_n	R_n	n	A_n	R_n
1	--	--	126	14	22	242	33	37	350	50	52
8	--	8	127	14	23	243	33	38	355	50	53
15	--	9	132	15	23	248	34	38	357	51	53
23	--	10	135	15	24	251	34	39	362	51	54
31	--	11	138	16	24	255	35	39	363	52	54
39	--	12	142	16	25	259	35	40	369	52	55
42	0	12	144	17	25	261	36	40	370	53	55
47	0	13	150	18	26	266	36	41	376	54	55
48	1	13	156	19	26	267	37	41			
54	2	13	158	19	27	273	38	41			

55	2	14	162	20	27	274	38	42
60	3	14	166	20	28	280	39	42
63	3	15	168	21	28	281	39	43
66	4	15	174	22	29	286	40	43
71	4	16	181	23	29	289	40	44
72	5	16	182	23	30	292	41	44
78	6	16	187	24	30	296	41	45
79	6	17	189	24	31	299	42	45
84	7	17	193	25	31	309	43	46
87	7	18	197	25	32	311	43	47

90	8	18	199	26	32	312	44	47
95	8	19	205	27	33	318	45	48
96	9	19	211	28	33	324	46	48
102	10	19	213	28	34	326	46	49
103	10	20	217	29	34	331	47	49
108	11	20	220	29	35	333	47	50
111	11	21	223	30	35	337	48	50
114	12	21	228	30	36	340	48	51
119	12	22	230	31	36	344	49	51
120	13	22	236	32	37	348	49	52

θ_{-1} = .10 θ_1 = .20 $\alpha = \beta$ = .05 Plan 35

Comparison of A.S.N. Values

Test	θ = .10	Maximizing θ	θ = .20
Minimax	76.0	95.3	63.4
Wald	72.2	106.3	59.8
Fixed Sample Size		133.0	

n	A_n	R_n	n	A_n	R_n	n	A_n	R_n
1	--	--	117	14	20	229	32	35
6	--	6	121	14	21	235	32	36
11	--	7	123	15	21	236	33	36
19	--	8	129	16	22	242	34	36
27	--	9	135	17	22	243	34	37
32	0	9	137	17	23	249	35	37
35	0	10	142	18	23	250	35	38
38	1	10	145	18	24	255	36	38
43	1	11	148	19	24	257	36	39
44	2	11	152	19	25	262	37	39
50	3	11	154	20	25	265	37	40
51	3	12	160	21	26	268	38	40
56	4	12	166	22	26	272	38	41
59	4	13	168	22	27	275	39	41
62	5	13	173	23	27	279	39	42
67	5	14	175	23	28	281	40	42
68	6	14	179	24	28	286	40	43
74	6	15	183	24	29	288	41	43
75	7	15	185	25	29	294	42	44
81	8	15	190	25	30	301	43	45
82	8	16	191	26	30	307	44	45
87	9	16	198	27	31			
90	9	17	204	28	31			
93	10	17	205	28	32			
98	10	18	210	29	32			
99	11	18	213	29	33			
105	12	18	217	30	33			
106	12	19	220	30	34			
111	13	19	223	31	34			
114	13	20	228	31	35			

$\theta_{-1} = .10$ $\theta_1 = .20$ $\alpha = \beta = .10$ Plan 36

Comparison of A.S.N. Values

Test	$\theta = .10$	Maximizing θ	$\theta = .20$
Minimax	49.8	56.9	41.5
Wald	48.2	59.7	39.9
Fixed Sample Size		80.0	

n	A_n	R_n	n	A_n	R_n	n	A_n	R_n
1	--	--	109	14	18	225	32	34
4	--	4	115	15	19	231	33	34
7	--	5	122	16	19			
15	--	6	123	16	20			
22	0	6	128	17	20			
23	0	7	130	17	21			
29	1	7	134	18	21			
31	1	8	138	18	22			
35	2	8	141	19	22			
38	2	9	145	19	23			
41	3	9	147	20	23			
46	3	10	152	20	24			
47	4	10	153	21	24			
53	5	10	160	22	25			
54	5	11	166	23	25			
59	6	11	167	23	26			
62	6	12	173	24	26			
65	7	12	174	24	27			
69	7	13	179	25	27			
72	8	13	182	25	28			
77	8	14	186	26	28			
78	9	14	189	26	29			
84	10	14	192	27	29			
85	10	15	196	27	30			
90	11	15	199	28	30			
92	11	16	204	28	31			
96	12	16	205	29	31			
100	12	17	211	29	32			
103	13	17	212	30	32			
107	13	18	218	31	33			

$\theta_{-1} = .10$ $\theta_1 = .20$ $\alpha = \beta = .20$ Plan 37

Comparison of A.S.N. Values

Test	$\theta = .10$	Maximizing θ	$\theta = .20$
Minimax	23.1	24.0	19.4
Wald	22.8	>24.0	18.9
Fixed Sample Size		34.0	

n	A_n	R_n	n	A_n	R_n
1	--	--	103	14	16
2	--	2	107	14	17
3	--	3	109	15	17
11	--	4	114	15	18
13	0	4	116	16	18
18	0	5	122	17	19
19	1	5	129	18	20
25	2	5	135	19	20
26	2	6			
32	3	6			

33	3	7
38	4	7
41	4	8
45	5	8
48	5	9
51	6	9
56	6	10
57	7	10
63	7	11
64	8	11

70	9	11
71	9	12
77	10	12
78	10	13
83	11	13
85	11	14
90	12	14
93	12	15
96	13	15
100	13	16

θ_{-1} = .10 θ_1 = .30 $\alpha = \beta$ = .01 Plan 38

Comparison of A.S.N. Values

Test	θ = .10	Maximizing θ	θ = .30
Minimax	42.6	59.0	32.4
Wald	38.5	75.8	29.2
Fixed Sample Size		79.0	

n	A_n	R_n	n	A_n	R_n	n	A_n	R_n	n	A_n	R_n
1	--	--	42	4	11	68	10	15	97	16	20
6	--	6	44	4	12	71	10	16	99	17	20
9	--	7	46	5	12	72	11	16	103	18	21
16	--	8	50	6	12	76	12	16	108	19	21
23	--	9	51	6	13	77	12	17	109	19	22
25	0	9	55	7	13	81	13	17	113	20	22
29	1	9	57	7	14	84	13	18	116	20	23
30	1	10	59	8	14	85	14	18	117	21	23
33	2	10	63	9	14	90	15	19	122	22	24
37	3	11	64	9	15	94	16	19	127	23	24

θ_{-1} = .10 θ_1 = .30 $\alpha = \beta$ = .025 Plan 39

Comparison of A.S.N. Values

Test	θ = .10	Maximizing θ	θ = .30
Minimax	32.0	40.8	24.3
Wald	29.6	47.7	22.4
Fixed Sample Size		55.0	

n	A_n	R_n	n	A_n	R_n	n	A_n	R_n	n	A_n	R_n
1	--	--	36	4	9	67	11	14	95	17	19
5	--	5	38	4	10	71	11	15	100	18	19
11	--	6	40	5	10	72	12	15			
18	--	7	45	6	11	76	13	15			
19	0	7	49	7	11	77	13	16			
23	1	7	51	7	12	81	14	16			
24	1	8	53	8	12	83	14	17			
27	2	8	58	9	13	86	15	17			
31	2	9	62	10	13	89	15	18			
32	3	9	64	10	14	90	16	18			

θ_{-1} = .10 θ_1 = .30 $\alpha = \beta = .05$ Plan 40

Comparison of A.S.N. Values

Test	θ = .10	Maximizing θ	θ = .30
Minimax	24.0	28.8	18.5
Wald	22.8	31.6	17.3
Fixed Sample Size		39.0	

n	A_n	R_n	n	A_n	R_n	n	A_n	R_n
1	--	--	36	5	9	68	11	14
4	--	4	41	6	9	69	12	14
10	--	5	43	6	10	74	13	15
14	0	5	46	7	10	78	14	15
16	0	6	49	7	11			
19	1	6	50	8	11			
23	2	7	55	9	12			
28	3	7	59	10	12			
30	3	8	61	10	13			
32	4	8	64	11	13			

θ_{-1} = .10 θ_1 = .30 $\alpha = \beta = .10$ Plan 41

Comparison of A.S.N. Values

Test	θ = .10	Maximizing θ	θ = .30
Minimax	15.1	16.3	11.6
Wald	14.6	>16.7	11.0
Fixed Sample Size		24.0	

n	A_n	R_n	n	A_n	R_n
1	--	--	28	4	7
2	--	2	33	5	7
3	--	3	35	5	8
9	--	4	37	6	8
10	0	4	41	6	9
14	1	4	42	7	9
16	1	5	47	8	10
19	2	5	52	9	10
22	2	6			
23	3	6			

$\theta_{-1} = .10$ $\theta_1 = .30$ $\alpha = \beta = .20$ Plan 42

Comparison of A.S.N. Values

Test	$\theta = .10$	Maximizing θ	$\theta = .30$
Minimax	7.5	7.6	6.1
Wald	7.3	> 7.6	5.7
Fixed Sample Size		11.0	

n	A_n	R_n
1	--	--
2	--	2
6	0	2
8	0	3
10	1	3
14	1	4
15	2	4
20	3	5
25	4	5

$\theta_{-1} = .15$ $\theta_1 = .25$ $\alpha = \beta = .01$ Plan 43

Comparison of A.S.N. Values

Test	$\theta = .15$	Maximizing θ	$\theta = .25$
Minimax	167.9	252.7	148.0
Wald	151.3	330.2	133.1
Fixed Sample Size		330.0	

n	A_n	R_n	n	A_n	R_n	n	A_n	R_n	n	A_n	R_n
1	--	--	113	13	31	195	31	46	277	49	60
14	--	14	116	13	32	199	32	46	280	49	61
18	--	15	117	14	32	201	32	47	282	50	61
24	--	16	121	14	33	204	33	47	286	51	62
30	--	17	122	15	33	207	33	48	291	52	62
35	--	18	127	16	34	209	34	48	292	52	63
41	--	19	131	17	34	212	34	49	296	53	63
47	--	20	133	17	35	213	35	49	297	53	64
53	--	21	136	18	35	218	36	50	300	54	64
54	0	21	138	18	36	222	37	50	303	54	65
58	0	22	140	19	36	224	37	51	305	55	65
59	1	22	144	19	37	227	38	51	308	55	66
63	2	22	145	20	37	230	38	52	310	56	66
64	2	23	149	21	37	231	39	52	314	57	67
68	3	23	150	21	38	235	39	53	319	58	67
70	3	24	154	22	38	236	40	53	320	58	68
72	4	24	156	22	39	241	41	54	323	59	68
75	4	25	158	23	39	245	42	54	325	59	69
77	5	25	161	23	40	246	42	55	328	60	69
81	6	26	163	24	40	250	43	55	331	60	70
86	7	26	167	24	41	252	43	56	333	61	70
87	7	27	168	25	41	254	44	56	336	61	71
90	8	27	172	26	41	258	44	57	337	62	71
93	8	28	173	26	42	259	45	57	342	63	72
95	9	28	177	27	42	263	45	58	347	64	72
98	9	29	178	27	43	264	46	58	348	64	73
99	10	29	181	28	43	268	47	58	351	65	73
104	11	30	184	28	44	269	47	59	353	65	74
108	12	30	186	29	44	273	48	59	356	66	74
110	12	31	190	30	45	275	48	60	359	66	75

Cont. Plan 43

n	A_n	R_n	n	A_n	R_n	n	A_n	R_n
360	67	75	473	91	95	586	114	117
364	67	76	474	91	96	589	115	117
365	68	76	478	92	96	592	115	118
370	69	77	480	92	97	594	116	118
374	70	77	483	93	97	597	116	119
375	70	78	485	93	98	599	117	119
379	71	78	488	94	98	602	117	120
381	71	79	490	94	99	604	118	120
384	72	79	492	95	99	607	118	121
386	72	80	496	95	100	609	119	121
388	73	80	497	96	100	613	119	122
392	73	81	501	96	101	614	120	122
393	74	81	502	97	101	618	120	123
398	75	82	507	98	102	619	121	123
402	76	82	512	99	103	623	121	124
403	76	83	516	100	103	624	122	124
407	77	83	517	100	104	628	122	125
409	77	84	521	101	104	629	123	125
412	78	84	523	101	105	634	124	126
414	78	85	526	102	105	638	125	126
417	79	85	528	102	106			
420	79	86	531	103	106			
421	80	86	533	103	107			
425	80	87	536	104	107			
426	81	87	539	104	108			
431	82	88	541	105	108			
435	83	88	544	105	109			
436	83	89	545	106	109			
440	84	89	549	106	110			
442	84	90	550	107	110			
445	85	90	555	108	111			
447	85	91	560	109	112			
450	86	91	565	110	113			
452	86	92	570	111	113			
454	87	92	571	111	114			
458	87	93	575	112	114			
459	88	93	576	112	115			
463	88	94	579	113	115			
464	89	94	581	113	116			
469	90	95	584	114	116			

$\theta_{-1} = .15$ $\theta_1 = .25$ $\alpha = \beta = .025$ Plan 44

Comparison of A.S.N. Values

Test	$\theta = .15$	Maximizing θ	$\theta = .25$
Minimax	126.5	175.5	111.4
Wald	117.0	210.1	102.9
Fixed Sample Size		240.0	

n	A_n	R_n	n	A_n	R_n	n	A_n	R_n	n	A_n	R_n
1	--	--	108	14	28	192	33	43	280	52	58
10	--	10	109	15	28	197	34	43	281	52	59
11	--	11	113	15	29	198	34	44	285	53	59
17	--	12	114	16	29	201	35	44	287	53	60
22	--	13	119	17	30	203	35	45	290	54	60
28	--	14	123	18	30	206	36	45	292	54	61
34	--	15	125	18	31	209	36	46	295	55	61
39	--	16	128	19	31	211	37	46	298	55	62
41	0	16	130	19	32	215	38	47	299	56	62
45	1	17	132	20	32	220	39	48	303	56	63
50	2	17	136	20	33	224	40	48	304	57	63
51	2	18	137	21	33	226	40	49	309	58	64
55	3	18	141	22	33	229	41	49	313	59	64
56	3	19	142	22	34	231	41	50	314	59	65
59	4	19	146	23	34	234	42	50	318	60	65
62	4	20	147	23	35	237	42	51	320	60	66
64	5	20	151	24	35	238	43	51	323	61	66
68	6	21	153	24	36	242	43	52	325	61	67
73	7	21	155	25	36	243	44	52	328	62	67
74	7	22	158	25	37	248	45	53	331	62	68
77	8	22	160	26	37	252	46	53	332	63	68
79	8	23	164	27	38	254	46	54	336	63	69
82	9	23	169	28	38	257	47	54	337	64	69
85	9	24	170	28	39	259	47	55	341	64	70
87	10	24	174	29	39	262	48	55	342	65	70
91	11	25	175	29	40	265	48	56	347	66	71
96	12	26	178	30	40	266	49	56	351	67	71
100	13	26	181	30	41	270	49	57	352	67	72
102	13	27	183	31	41	271	50	57	356	68	72
105	14	27	187	32	42	276	51	58	358	68	73

Cont. Plan 44

n	A_n	R_n	n	A_n	R_n
361	69	73	475	92	95
363	69	74	477	93	95
366	70	74	480	93	96
369	70	75	482	94	96
370	71	75	486	94	97
374	71	76	487	95	97
375	72	76	491	95	98
379	72	77	492	96	98
380	73	77	496	96	99
385	74	78	497	97	99
390	75	79	501	97	100
394	76	79	502	98	100
395	76	80	506	98	101
399	77	80	507	99	101
401	77	81	512	100	102
404	78	81	516	101	102
406	78	82			
409	79	82			
411	79	83			
414	80	83			
417	80	84			
419	81	84			
422	81	85			
423	82	85			
427	82	86			
428	83	86			
433	84	87			
438	85	88			
443	86	89			
448	87	89			
449	87	90			
453	88	90			
454	88	91			
457	89	91			
459	89	92			
462	90	92			
465	90	93			
467	91	93			
470	91	94			
472	92	94			

θ_{-1} = .15 θ_1 = .25 $\alpha = \beta = .05$ Plan 45

Comparison of A.S.N. Values

Test	θ = .15	Maximizing θ	θ = .25
Minimax	93.9	120.8	83.0
Wald	88.9	135.5	78.4
Fixed Sample Size		168.0	

n	A_n	R_n	n	A_n	R_n	n	A_n	R_n	n	A_n	R_n
1	--	--	96	14	24	179	32	39	269	51	55
8	--	8	100	15	24	184	33	39	272	51	56
11	--	9	101	15	25	185	33	40	274	52	56
17	--	10	105	16	25	189	34	40	277	52	57
22	--	11	107	16	26	190	34	41	279	53	57
28	--	12	109	17	26	193	35	41	283	54	58
31	0	12	112	17	27	196	35	42	288	55	59
34	0	13	114	18	27	198	36	42	293	56	59
36	1	13	118	18	28	201	36	43	294	56	60
39	1	14	119	19	28	203	37	43	298	57	60
40	2	14	123	20	28	207	38	44	299	57	61
45	3	15	124	20	29	212	39	45	303	58	61
49	4	15	128	21	29	217	40	45	304	58	62
51	4	16	129	21	30	218	40	46	307	59	62
54	5	16	133	22	30	221	41	46	310	59	63
56	5	17	135	22	31	223	41	47	312	60	63
59	6	17	137	23	31	226	42	47	315	60	64
62	6	18	140	23	32	229	42	48	317	61	64
63	7	18	142	24	32	231	43	48	320	61	65
68	8	19	146	25	33	234	43	49	322	62	65
72	9	19	151	26	33	236	44	49	326	62	66
73	9	20	152	26	34	240	45	50	327	63	66
77	10	20	156	27	34	245	46	51	331	63	67
79	10	21	157	27	35	250	47	52	332	64	67
82	11	21	160	28	35	255	48	52	336	64	68
84	11	22	163	28	36	256	48	53	337	65	68
86	12	22	165	29	36	259	49	53	341	66	68
90	12	23	168	29	37	261	49	54	342	66	69
91	13	23	170	30	37	264	50	54	346	67	69
95	14	23	174	31	38	267	50	55	347	67	70

Cont. Plan 45

n	A_n	R_n
351	68	70
352	68	71
356	69	71
357	69	72
361	70	72
363	70	73
366	71	73
368	71	74
371	72	74
373	72	75
376	73	75
379	73	76
381	74	76
384	74	77
385	75	77
389	75	78
390	76	78
394	76	79
395	77	79
399	77	80
400	78	80
405	79	81
410	80	82
415	81	83
420	82	84
425	83	84

θ_{-1} = .15 θ_1 = .25 $\alpha = \beta$ = .10 Plan 46

Comparison of A.S.N. Values

Test	θ = .15	Maximizing θ	θ = .25
Minimax	60.8	71.4	53.7
Wald	58.9	75.3	51.8
Fixed Sample Size		99.0	

n	A_n	R_n	n	A_n	R_n	n	A_n	R_n	n	A_n	R_n
1	--	--	89	14	21	167	31	35	251	48	51
6	--	6	92	15	21	170	31	36	254	49	51
11	--	7	94	15	22	172	32	36	256	49	52
17	--	8	96	16	22	176	32	37	259	50	52
22	0	9	100	16	23	177	33	37	261	50	53
26	1	9	101	17	23	181	33	38	264	51	53
28	1	10	105	17	24	182	34	38	266	51	54
31	2	10	106	18	24	186	35	38	269	52	54
33	2	11	110	19	24	187	35	39	272	52	55
35	3	11	111	19	25	191	36	39	274	53	55
39	3	12	115	20	25	192	36	40	277	53	56
40	4	12	116	20	26	196	37	40	279	54	56
44	4	13	120	21	26	197	37	41	282	54	57
45	5	13	122	21	27	201	38	41	284	55	57
49	6	13	125	22	27	203	38	42	287	55	58
50	6	14	127	22	28	206	39	42	289	56	58
54	7	14	129	23	28	208	39	43	293	56	59
56	7	15	133	23	29	211	40	43	294	57	59
59	8	15	134	24	29	213	40	44	298	57	60
61	8	16	138	24	30	215	41	44	299	58	60
63	9	16	139	25	30	219	41	45	303	59	61
67	9	17	143	26	31	220	42	45	308	60	62
68	10	17	148	27	31	224	42	46	313	61	62
72	10	18	149	27	32	225	43	46			
73	11	18	153	28	32	229	43	47			
77	12	18	154	28	33	230	44	47			
78	12	19	158	29	33	235	45	48			
82	13	19	160	29	34	240	46	49			
83	13	20	163	30	34	245	47	50			
87	14	20	165	30	35	249	48	50			

$\theta_{-1} = .15$ $\theta_1 = .25$ $\alpha = \beta = .20$ Plan 47

Comparison of A.S.N. Values

Test	$\theta = .15$	Maximizing θ	$\theta = .25$
Minimax	28.4	30.1	25.1
Wald	28.0	>30.2	24.7
Fixed Sample Size			

n	A_n	R_n	n	A_n	R_n	n	A_n	R_n
1	--	--	84	15	18	167	32	34
3	--	3	86	15	19	171	32	35
5	--	4	87	15	19	172	33	35
10	--	5	89	16	19	176	33	36
12	0	5	91	16	20	177	34	36
16	0	6	94	17	20	181	34	37
17	1	6	97	17	21	182	35	37
21	1	7	99	18	21	186	35	38
22	2	7	102	18	22	187	36	38
27	3	8	103	19	22	191	36	39
31	4	8	107	19	23	192	37	39
32	4	9	108	20	23	196	38	39
36	5	9	113	21	24			
37	5	10	118	22	25			
41	6	10	123	23	26			
43	6	11	128	24	27			
46	7	11	133	25	27			
48	7	12	134	25	28			
50	8	12	137	26	28			
54	8	13	139	26	29			
55	9	13	142	27	29			
59	9	14	144	27	30			
60	10	14	147	28	30			
64	10	15	150	28	31			
65	11	15	152	29	31			
70	12	16	155	29	32			
74	13	16	157	30	32			
75	13	17	160	30	33			
79	14	17	162	31	33			
81	14	18	165	31	34			

θ_{-1} = .15 $\qquad$ θ_1 = .35 $\qquad$ $\alpha = \beta$ = .01 $\qquad$ Plan 48

Comparison of A.S.N. Values

Test	θ = .15	Maximizing θ	θ = .35
Minimax	49.5	71.7	41.1
Wald	44.6	93.3	37.0
Fixed Sample Size		94.0	

n	A_n	R_n	n	A_n	R_n	n	A_n	R_n
1	--	--	83	17	23	153	36	38
8	--	8	86	17	24	157	37	39
11	--	9	87	18	24	161	38	40
16	--	10	90	19	24	165	39	40
21	--	11	91	19	25			
24	0	11	94	20	25			
26	0	12	96	20	26			
27	1	12	98	21	26			
31	2	13	101	22	27			
34	3	13	105	23	28			

36	3	14	108	24	28
38	4	14	110	24	29
41	5	15	112	25	29
45	6	15	115	25	30
46	6	16	116	26	30
48	7	16	119	27	30
51	7	17	120	27	31
52	8	17	123	28	31
55	9	17	124	28	32
56	9	18	127	29	32

59	10	18	129	29	33
61	10	19	130	30	33
62	11	19	134	31	34
66	12	20	138	32	35
69	13	20	142	33	35
71	13	21	143	33	36
73	14	21	145	34	36
76	15	22	148	34	37
80	16	22	149	35	37
81	16	23	152	35	38

$\theta_{-1} = .15$ $\theta_1 = .35$ $\alpha = \beta = .025$ Plan 49

Comparison of A.S.N. Values

Test	$\theta = .15$	Maximizing θ	$\theta = .35$
Minimax	37.3	49.9	31.2
Wald	34.5	59.4	28.7
Fixed Sample Size		67.0	

n	A_n	R_n	n	A_n	R_n
1	--	--	78	17	21
6	--	6	82	18	22
8	--	7	86	19	22
13	--	8	87	19	23
18	0	9	89	20	23
22	1	9	92	20	24
23	1	10	93	21	24
25	2	10	96	21	25
28	2	11	97	22	25
29	3	11	101	23	26
32	4	11	104	24	26
33	4	12	106	24	27
36	5	12	108	25	27
38	5	13	110	25	28
39	6	13	112	26	28
43	7	14	115	26	29
46	8	14	116	27	29
48	8	15	119	27	30
50	9	15	120	28	30
53	10	16	123	29	30
57	11	16			
58	11	17			
60	12	17			
63	12	18			
64	13	18			
68	14	19			
71	15	19			
73	15	20			
75	16	20			
77	16	21			

θ_{-1} = .15 θ_1 = .35 $\alpha = \beta$ = .05 Plan 50

Comparison of A.S.N. Values

Test	θ = .15	Maximizing θ	θ = .35
Minimax	27.5	34.0	22.8
Wald	26.2	37.8	21.5
Fixed Sample Size		47.0	

n	A_n	R_n	n	A_n	R_n	n	A_n	R_n	n	A_n	R_n
1	--	--	33	5	10	53	11	15	76	17	20
5	--	5	34	5	11	57	12	15	79	18	20
9	--	6	35	6	11	58	12	16	81	18	21
14	0	7	38	6	12	61	13	16	83	19	21
17	1	7	39	7	12	62	13	17	85	19	22
19	1	8	42	8	12	64	14	17	87	20	22
21	2	8	43	8	13	67	14	18	90	20	23
24	3	9	46	9	13	68	15	18	91	21	23
28	4	9	48	9	14	72	16	19	94	22	24
29	4	10	50	10	14	75	17	19	98	23	24

θ_{-1} = .15 θ_1 = .35 $\alpha = \beta$ = .10 Plan 51

Comparison of A.S.N. Values

Test	θ = .15	Maximizing θ	θ = .35
Minimax	18.3	20.8	15.3
Wald	17.7	>21.7	14.6
Fixed Sample Size		29.0	

n	A_n	R_n	n	A_n	R_n	n	A_n	R_n
1	--	--	24	4	8	47	10	13
3	--	3	28	5	9	50	11	13
4	--	4	32	6	9	52	11	14
9	--	5	33	6	10	56	12	15
10	0	5	35	7	10	58	13	15
13	1	5	38	7	11	61	13	16
14	1	6	39	8	11	62	14	16
17	2	6	42	8	12	65	15	17
19	2	7	43	9	12	69	16	17
21	3	7	46	10	12			

$\theta_{-1} = .15$ $\theta_1 = .35$ $\alpha = \beta = .20$ Plan 52

Comparison of A.S.N Values

Test	$\theta = .15$	Maximizing θ	$\theta = .35$
Minimax	8.9	9.2	7 5
Wald	8.7	> 9.2	7.3
Fixed Sample Size		13.0	

n	A_n	R_n	n	A_n	R_n
1	--	--	23	4	7
2	--	2	25	5	7
4	--	3	28	6	8
6	0	3	32	7	9
9	1	4	36	8	10
13	2	4	40	9	10
14	2	5			
17	3	5			
18	3	6			
21	4	6			

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13. ABSTRACT

The author, in his masters thesis, constructed a catalog of 442 Bernoulli sampling plans which approximately minimize the maximum expected sample size among all plans which guarantee certain O.C. probability requirements. Fifty-two of these plans (which would appear to be of greatest practical interest) are presented in this report. A.S.N. curve comparisons are made with plans based on the Wald sequential probability ratio test and the fixed sample size test which guarantee the same O.C. probability requirements.

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