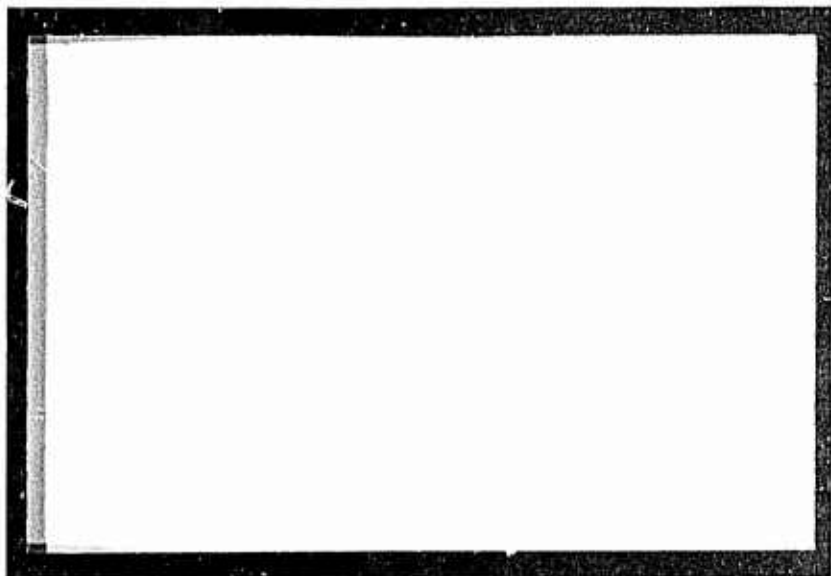
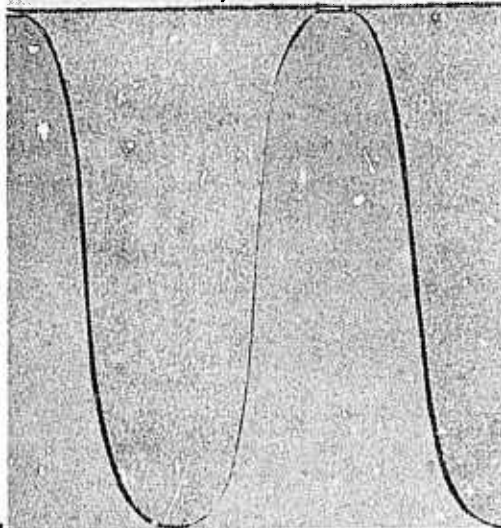


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A GENERALIZATION OF AN INEQUALITY  
OF VAN DANTZIG

B. Harris

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ABSTRACT

The inequality

$$\int_0^1 f^2(x) dx \leq \frac{1}{12}$$

was stated by van Dantzig for all functions of a certain class.

This inequality is generalized.

# A GENERALIZATION OF AN INEQUALITY OF VAN DANTZIG

B. Harris

1. Introduction. In a 1951 paper on the power of the Wilcoxon two sample test, D. van Dantzig stated without proof the following inequality.

For  $0 \leq x \leq 1$ , let  $f(x)$  satisfy

$$(a) \int_0^1 f(x) dx = 0$$

$$(b) f(1) \leq f(0)$$

(c)  $f(x) + x$  is monotone non-decreasing. Then

$$\int_0^1 f^2(x) dx \leq \frac{1}{12} .$$

In this note, we show that van Dantzig's inequality is a special case of a fairly general inequality and is in fact a consequence of convexity. We state this inequality for  $0 \leq x \leq 1$ ; however, the argument employed extends immediately to any interval  $[a, b]$ .

The inequality which is the subject of this note follows.

Theorem. Let  $\mu$  be any measure on the Borel sets of  $[0, 1]$  with  $0 < \mu[0, 1] < \infty$ , let  $h(x)$  be any  $\mu$ -integrable function with  $h(0) = 0$  and let  $\mathfrak{F}$  be the set of functions  $f(x)$  on  $[0, 1]$  with

$$(a) \int_0^1 f(x) d\mu(x) = 0$$

$$(b) f(1) \leq f(0)$$

(c)  $f(x) + h(x)$  is monotone non-decreasing in  $[0, 1]$  .

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Then, if  $g$  is any convex function defined on the range of every  $f$  satisfying

(a), (b), (c),

$$(1) \quad \sup_{f \in \mathfrak{F}} \int_0^1 g(f(x)) d\mu(x) = \max \left( \sup_{0 \leq y \leq 1} \int_0^1 g(f_{1,y}(x)) d\mu(x), \sup_{0 < y \leq 1} \int_0^1 g(f_{2,y}(x)) d\mu(x) \right)$$

where

$$f_{1,y}(x) = \begin{cases} \mu^{-1}([0,1]) \left[ \int_0^1 h(x) d\mu(x) - h(1)\mu([y,1]) \right] - h(x), & 0 \leq x < y \\ h(1) + \mu^{-1}([0,1]) \left[ \int_0^1 h(x) d\mu(x) - h(1)\mu([y,1]) \right] - h(x), & y < x \leq 1 \end{cases}$$

for  $0 \leq y \leq 1$ , and

$$f_{2,y}(x) = \begin{cases} \mu^{-1}([0,1]) \left[ \int_0^1 h(x) d\mu(x) - h(1)\mu([y,1]) \right] - h(x), & 0 \leq x < y \\ h(1) + \mu^{-1}([0,1]) \left[ \int_0^1 h(x) d\mu(x) - h(1)\mu([y,1]) \right] - h(x), & y < x \leq 1 \end{cases}$$

for  $0 < y \leq 1$ .

This reduces the problem from one of maximization over a collection of functions to maximization over a parameter ( $y$ ). With various specializations of  $\mu$ ,  $h$ , and  $g$  a variety of interesting special inequalities are obtained. In particular, if  $\mu$  is Lebesgue measure and  $h(x) = x$ , the integrals on the right hand side of (1) are independent of  $y$ . This is noted in Corollary 3.

2. Proof of Theorem. From (b) and (c) we have

$$(2) \quad f(0) + h(0) \leq f(1) + h(1) \leq f(0) + h(1)$$

and hence  $h(0) \leq h(1)$ .

It is easily seen that  $\mathfrak{F}$  is a convex set. Then for any  $\lambda \in [0, 1]$

$$(3) \quad \int_0^1 g(\lambda f_1(x) + (1-\lambda)f_2(x)) d\mu(x) \leq \int_0^1 [\lambda g(f_1(x)) + (1-\lambda)g(f_2(x))] d\mu(x)$$

and

$$(4) \quad \int_0^1 [\lambda g(f_1(x)) + (1-\lambda)g(f_2(x))] d\mu(x) \leq \max\left(\int_0^1 g(f_1(x)) d\mu(x), \int_0^1 g(f_2(x)) d\mu(x)\right).$$

Thus in determining  $\max_{f \in \mathcal{F}} \int_0^1 g(f(x)) d\mu(x)$ , it suffices to restrict attention to the extreme points of  $\mathcal{F}$ .

Let  $\Phi$  be the set of functions  $\varphi(x)$  on  $[0,1]$  with (i)  $\varphi(0) = 0$ , (ii)  $\varphi(x)$  monotone non-decreasing, and (iii)  $\varphi(1) \leq h(1)$ . In addition, let  $T: \mathcal{F} \rightarrow \Phi$  be the mapping defined by

$$T(f(x)) = \varphi_f(x) = f(x) - f(0) + h(x), \quad 0 \leq x \leq 1.$$

Then

$$T^{-1}(\varphi) = f_\varphi(x) = \varphi(x) - h(x) + \mu^{-1}([0,1]) \left( \int_0^1 h(x) d\mu(x) - \int_0^1 \varphi(x) d\mu(x) \right).$$

Thus  $T$  is one-to-one and onto. Further,  $T$  and  $T^{-1}$  preserve convex combinations so that extreme points of  $\mathcal{F}$  are the images of extreme points of  $\Phi$  and conversely. The extreme points of  $\Phi$  are clearly given by

$$(5a) \quad \varphi_{1,y}(x) = \begin{cases} 0 & 0 \leq x \leq y \\ h(1) & y < x \leq 1 \end{cases} \quad 0 \leq y \leq 1$$

and

$$(5b) \quad \varphi_{2,y}(x) = \begin{cases} 0 & 0 \leq x < y \\ h(1) & y \leq x \leq 1 \end{cases} \quad 0 < y \leq 1.$$

Note that  $\varphi_{1,1}(x) \equiv 0$ . Consequently, the extreme points of  $\mathcal{F}$  are

$$(6a) \quad f_{1,y}(x) = \begin{cases} \mu^{-1}([0,1]) \left[ \int_0^1 h(x) d\mu(x) - h(1)\mu([y,1]) \right] - h(x), & 0 \leq x \leq y \\ h(1) + \mu^{-1}([0,1]) \left[ \int_0^1 h(x) d\mu(x) - h(1)\mu([y,1]) \right] - h(x), & y < x \leq 1 \end{cases} \quad 0 \leq y \leq 1$$

and

$$(6b) \quad f_{2,y}(x) = \begin{cases} \mu^{-1}([0,1]) \left[ \int_0^1 h(x) d\mu(x) - h(1)\mu([y,1]) \right] - h(x), & 0 \leq x < y \\ h(1) + \mu^{-1}([0,1]) \left[ \int_0^1 h(x) d\mu(x) - h(1)\mu([y,1]) \right] - h(x), & y \leq x \leq 1 \end{cases} \quad 0 < y \leq 1$$

Consequently,

$$(7) \quad \sup_{f \in \mathfrak{F}} \int_0^1 g(f(x)) d\mu(x) = \max \left( \sup_{0 \leq y \leq 1} \int_0^1 g(f_{1,y}(x)) d\mu(x), \sup_{0 < y \leq 1} \int_0^1 g(f_{2,y}(x)) d\mu(x) \right).$$

Corollary 1. If  $\mu$  is absolutely continuous with respect to Lebesgue measure

$$\sup_{f \in \mathfrak{F}} \int_0^1 g(f(x)) d\mu(x) = \max_{0 \leq y \leq 1} \int_0^1 g(f_{1,y}(x)) d\mu(x).$$

The proof is trivial.

Corollary 2. If  $\mu$  is Lebesgue measure,

$$\sup_{f \in \mathfrak{F}} \int_0^1 g(f(x)) dx = \max_{0 \leq y \leq 1} \int_0^1 g(f_{1,y}(x)) dx$$

where

$$f_{1,y}(x) = \begin{cases} \int_0^1 h(x) dx - (1-y)h(1) - h(x) & 0 \leq x \leq y \\ \int_0^1 h(x) dx + yh(1) - h(x) & y < x \leq 1 \end{cases}$$

Corollary 3. If  $\mu$  is Lebesgue measure and  $h(x) = x$ ,

$$\max_{f \in \mathcal{F}} \int_0^1 g(f(x)) dx = \int_0^1 g\left(\frac{1}{2} - x\right) dx .$$

Proof: Here

$$f_{1,y}(x) = \begin{cases} y - \frac{1}{2} - x & 0 \leq x \leq y \\ \frac{1}{2} + y - x & y < x \leq 1 . \end{cases}$$

Thus,

$$\int_0^1 g(f_{1,y}(x)) dx = \int_0^y g\left(y - \frac{1}{2} - x\right) dx + \int_y^1 g\left(\frac{1}{2} + y - x\right) dx .$$

In the first integral let  $x = z + y - 1$  and in the second integral let  $x - y = z$ ,

obtaining

$$\int_0^1 g(f_{1,y}(x)) dx = \int_{1-y}^1 g\left(\frac{1}{2} - z\right) dz + \int_0^{1-y} g\left(\frac{1}{2} - z\right) dz = \int_0^1 g\left(\frac{1}{2} - x\right) dx .$$

Thus, in this case, every member of the one parameter family of functions gives the same value. The particular case  $g(x) = x^2$  is van Dantzig's Inequality.

#### REFERENCE

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Nederlandse Akad. Wetensch. Proc. Ser. A, 54(1951), 1-8.



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