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HARMONIC POLYHEDRA

by

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BALLISTIC RESEARCH LABORATORIES
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ABSTRACT

A polyhedron of genus p is harmonic if the number of its faces (vertices) is the harmonic mean of its numbers of its edges and vertices (faces). The determination of all permissible combinations of numbers of vertices, edges, and faces is reduced to solution of Pell's equation. Realizations of all such polyhedra with $p = 1$ are described, as well as for all negative p with large enough numbers of edges.

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"The usefulness of considering lines, angles and figures is the greatest, because it is impossible to understand natural philosophy without these." -- Robert Grosseteste, De Lineis, Angulis et Figuris.

I. INTRODUCTION

Let V, E, and F be the number of vertices, edges, and faces respectively, of a polyhedron of genus p with simply connected faces. Then the Euler equation asserts that

(1) $$V - E + F = 2p \quad ,$$

where p is an integer less than two. If V is the harmonic mean of E and F, i.e.

(2) $$\frac{1}{V} = \frac{1}{2}\left(\frac{1}{E} + \frac{1}{F}\right) ,$$

we shall say that the polyhedron is V-harmonic. We shall also say that any triple of positive integers (V, E, F) that satisfies (1) and (2) is V-harmonic. Similarly, if.

(3) $$\frac{1}{F} = \frac{1}{2}\left(\frac{1}{E} + \frac{1}{V}\right) ,$$

we shall say that the polyhedron is F-harmonic, and that any triple of positive integers (V, E, F) that satisfies (1) and (3) is F-harmonic.

The cube, with p=1, V=8, E=12, F=6 is V-harmonic, and its dual, the regular octahedron, with p=1, V=6, E=12, F=8 is F-harmonic. We shall show that there are no toroidal (p=0) harmonic polyhedra. For other values of p the determination of harmonic triples reduces to the construction of solutions of a Pell's equation. According to well known algorithms for the construction of the complete set of solutions of Pell's equation, for each $p \neq 0$ there are infinitely many V-and F-harmonic triples. For $p = 1$ we give simple constructions for examples of harmonic polyhedra corresponding to every harmonic triple with $V \geqq 4$. For each $p \leqq -1$ we show that a natural modification of the constructions used for $p = 1$ will produce examples of all harmonic polyhedra of genus p for large enough values of E.

II. DETERMINATION OF HARMONIC TRIPLES

Let V_V and F_V be solutions of (1) and (2) for V and F as functions of E and p. Similarly, let V_F and F_F be solutions of (1) and (3). Then

(4) $$V_V(E, p) = F_F(E, p) = p + 2E - D(E, p) ,$$

and

(5) $$F_V(E, p) = V_F(E, p) = p - E + D(E, p),$$

where D satisfies a Pell's equation

(6) $$D^2(E, p) = 2E^2 + p^2 .$$

Note that by (6) D and p always have the same parity. Then by (4) to (6) E, V, <u>and</u> F <u>must be even</u>.

For $p = 0$, equation (6) has only the trivial, irrelevant solution $D = E = 0$. Hence <u>there are no toroidal harmonic polyhedra</u>. For other values of p the solutions of our Pell's equation are described at length in [1]. The discussion can be summarized as follows. First, for $p = \pm 1$, all pairs of positive integer solutions (D, E) of (6) are generated by the sequence

(7) $$D_n + E_n\sqrt{2} = (3+2\sqrt{2})^n \qquad n \geqq 1 .$$

By forming the product of (7) and

$$D_n - E_n\sqrt{2} = (3-2\sqrt{2})^n$$

we can immediately verify that $D_n^2 - 2E_n^2 = 1$. That (7) yields all desired solutions for $p = \pm 1$ will follow from our later discussion. For $p = 1$ the first five elements of the sequence and the corresponding values of V and F have been listed in Table I. The entry for $n = 1$ is irrelevant for our purpose, since $V \geqq 4$ for all polyhedra.

Table I. First Five Harmonic Triples (p = 1).

n	D_n	$V_V = F_F$	E_n	$F_V = V_F$
1	3	2	2	2
2	17	8	12	6
3	99	42	70	30
4	577	240	408	170
5	3363	1394	2378	986

The entry for $n = 2$ includes the data for the cube and regular octahedron. Since the number of faces F_V or F_F is an increasing function of n, it is apparent from the entry for $n = 3$ that none of the other regular polyhedra is harmonic. It is also clear from Table I. that none of the stellated regular polyhedra is harmonic.

From (4) and (5) we deduce that

$$V_V(E,-1) = F_F(E,-1) = V_V(E,1) - 2 = F_F(E,1) - 2$$

$$V_F(E,-1) = F_V(E,-1) = V_F(E,1) - 2 = F_V(E,1) - 2$$

which can be used to adapt Table I for $p = -1$.

Now, for any integer $p \neq 0$ let (D^*, E^*) be any pair of positive integers that satisfy (6). At least one such pair, e.g. $D^* = 3|p|$, $E^* = 2|p|$, exists. With (D^*, E^*) we can associate the sequence

$$D_n + E_n\sqrt{2} = (D^* + E^*\sqrt{2})\,(3 + 2\sqrt{2})^n \,. \tag{8}$$

If we form the product of (8) and

$$D_n - E_n\sqrt{2} = (D^* - E^*\sqrt{2})\,(3 - 2\sqrt{2})^n$$

we immediately obtain $D_n^2 - 2E_n^2 = p^2$. If we refer (6) to rectangular coordinates with horizontal D-and vertical E-axis, a point starting at (D^*, E^*) in the first quadrant and moving upward along the right-hand branch of the hyperbola (6) will pass through the successive points (D_n, E_n) for increasing $n \geqq 0$. Now we shall show that if the point moves downward along the same branch it will pass through the successive points (D_n, E_n) for decreasing $n \leqq 0$. From (8) we find

$$D_{n-1} + E_{n-1}\sqrt{2} = (D_n + E_n\sqrt{2})\,(3 - 2\sqrt{2}) \tag{9}$$

for all integers, n. If

$$D_n > 0 \,, \tag{10}$$

then

(11) $$D_n = \left(2E_n^2 + p^2\right)^{0.5}$$

(12)
$$D_{n-1} = 3D_n - 4E_n = 3\left(2E_n^2 + p^2\right)^{0.5} - 4E_n$$
$$E_{n-1} = 3E_n - 2D_n = 3E_n - 2\sqrt{2E_n^2 + p^2} .$$

From (12) we find that regardless of the sign of E_n

(13) $$D_{n-1} > 3\sqrt{2}\,|E_n| - 4E_n \geqq 0.$$

Hence, _if_ (D_n, E_n) _is on the right branch of_ (6), _so is_ (D_{n-1}, E_{n-1}), _and by induction, all points_ (D_{n-j}, E_{n-j}). Next

(14) $$E_{n-1} \geqq 0$$

if and only if

(15) $$E_n \geqq 2|p| .$$

As a corollary, if

(16) $$0 \leqq E_n < 2|p| ,$$

then

(17) $$E_j < 0 \quad , \quad j \leqq n-1 .$$

If (15) is satisfied, then by (12) and (14)

(18) $$0 \leqq E_{n-1} < (3 - 2\sqrt{2})\, E_n < 0.2\, E_n .$$

Thus, as long as $E_{n-j} > 0$ we have

(19) $$0 < E_{n-j} \leqq (0.2)^j E_n .$$

Hence for some uniquely determined $J \geqq 0$

(20) $$0 < E_{n-J} \leqq 2|p|$$

and

(21) $$E_{n-j} \leqq 0 , \quad j \geqq J + 1 .$$

By (12) and (21)

(22) $$|E_{n-j-1}| > (3 + 2\sqrt{2}) |E_{n-j}| .$$

Hence (D_n, E_n) moves downward on the right hand branch of (6) as n decreases.

In the light of what has just been proved, the complete set of solutions of (6) in the first quadrant of the DE-plane can be determined as follows. Select from the set of integers $0 < E \leqq 2|p|$ the subset Σ for which $2E^{*2} + p^2$ is a perfect square. With each E^* of Σ associate the corresponding $D^* = (2E^{*2} + p^2)^{0.5}$ and then use the pair (D^*, E^*) to form the sequence (8) for $n \geqq 0$.

For $p = \pm 1$, the subset Σ consists of the single value $E = 2$. For any $p \neq 0$, $D^* = 3|p|$, $E^* = 2|p|$ satisfies (6), so Σ is always non vacuous. The following example shows that for certain choices of p the set Σ may have two or more elements. If we rewrite (6) in the form

(23) $$D^2 - p^2 = 2E^2$$

then the choice

(24) $$E = 2^r N$$

where N is odd, allows various possibilities for the factors $D \pm |p|$ that yield

$$D = 2^{s-1} N^2 + 2^{2r-s} , \tag{25}$$
$$|p| = 2^{s-1} N^2 - 2^{2r-s} , \qquad 1 \leqq s \leqq 2r .$$

We can guarantee $p < -1$ and $E < -2p$ by taking

$$N \geqq 2^{r-s} (1 + \sqrt{3}) . \tag{26}$$

Another set of choices for $D \pm |p|$ yields

$$D = \left(2^{s-1} + 2^{2r-s}\right) N , \tag{27}$$
$$|p| = \left(2^{s-1} - 2^{2r-s}\right) N . \qquad r < s \leqq 2r .$$

III. THE AVERAGE NUMBER OF EDGES AT A VERTEX

Strong indications of the structure of harmonic polyhedra can be determined from bounds on the average number of edges at a vertex, $A(E)$, defined by

$$A(E)\, V(E) = 2E. \tag{28}$$

For V-harmonic polyhedra we must have

$$\frac{1}{A_V(E)} = \frac{V_V(E)}{2E} = 1 + \frac{p - (2E^2 + p^2)^{0.5}}{2E}$$

by (4). Then

$$\frac{d}{dE}\left(\frac{1}{A_V}\right) = -p\left[\left(2E^2 + p^2\right)^{0.5} - p\right] / 2E^2 \left(2E^2 + p^2\right)^{0.5}$$

implies that $A_V(E)$ is an increasing (decreasing) function of E for $p = 1 (p < 0)$. Furthermore

$$\lim_{E=\infty} A_V(E) = 2 + \sqrt{2} \ .$$

According to Table I, $E \geqq 12$ for $p = 1$. But then $A_V(12) = 13$. Thus

(29) $$3 \leqq A_V(E) < 2 + \sqrt{2} < 3.42 \ , \quad p = 1 \ .$$

For $p < 0$ we have

(30) $$A_V(-12p) = 4 \ .$$

Thus

(31) $$3.4 < 2 + \sqrt{2} < A_V(E) \leqq 4 \ , \quad E \geqq -12p \ , \quad p < 0 \ .$$

For F-harmonic polyhedra we have

$$1/A_F(E) = V_F/2E = \left[\left(2E^2 + p^2\right)^{0.5} + p - E\right]/2E$$

by (5). Then

$$\frac{d}{dE}\left(\frac{1}{A_F}\right) = -p\left[\left(2E^2 + p^2\right)^{0.5} + p\right]/2E^2\left(2E^2+p^2\right)^{0.5}$$

implies that $A_F(E)$ is an increasing (decreasing) function for $p = 1 (p < 0)$. Furthermore

$$\lim_{E=\infty} A_F(E) = 2(1+\sqrt{2}) .$$

Accordingt o Table I, $E \geqq 12$ for $p = 1$. But then $A_F(12) = 4$. Thus

$$(32) \qquad 4 \leqq A_F(E) \leqq 2(1+\sqrt{2}) < 4.84 , \qquad p = 1 .$$

For $p < 0$ we have

$$A_F(-12p) = 6 , \ A_F(-70p) = 5 .$$

Thus

$$(33) \qquad 4.8 < 2(1+\sqrt{2}) < A_F(E) \leqq 5 , \ E \geqq -70p , \ p < 0 .$$

IV. REALIZATIONS OF HARMONIC POLYHEDRA FOR $p = 1$.

The inequalities (29) for V-harmonic polyhedra suggest that we begin our search for examples by constructing a polyhedron with the appropriate numbers of vertices and three edges at each vertex. A prism, $P(n, 0)$, with two congruent regular n-gons as horizontal bases and n congruent squares as vertical faces has the desired property. Now let us modify $P(n, 0)$ by drawing one diagonal in each of r successive square faces. This suggests that by slightly displacing r successive vertices of the upper base to new positions in its plane we can replace r square faces of P(n, 0) by 2r triangular faces with r additional edges. In this way we can construct a set of polyhedra, P(n, r), for $3 \leqq n$ and $0 \leqq r \leqq n$ for which

(34) $$V = 2n, \quad E = 3n+r, \quad F = n+r+2 \, .$$

Note that $3 \leqq 2E/V = 3 + r/n \leqq 4$. For the values (34) let

(35) $$f(n, r) = 0.5(1/E + 1/F) - 1/V \, .$$

Then in particular,

$$f(n, 0) = (n-4)/6n(n+2).$$

According to Table I, $V = 2n \geqq 8$. Hence

$$f(n, 0) \geqq 0, \qquad n \geqq 4 \, .$$

On the other hand

$$f(n, n) = -(n+3)/8n(n+1) < 0 \, .$$

Furthermore, for fixed n, f(n, r) is a strictly decreasing function of r. Thus, if we choose for $V = 2n_o$ one of the possible values that occurs in a V-harmonic triple for p=1, then the equation

$$f(n_o, r_o) = 0$$

has a unique integer solution, r_o , such that $0 \leqq r_o \leqq n_o$. Hence $P(n_o, r_o)$ is V-harmonic.

Note that $P(4,0)$ is a cube.

Since the simplest F-harmonic polyhedron is a double pyramid, let us seek other examples by generalizing this observation. First suppose $V = 2n+2$ for an F-harmonic triple. Then let q_o be the vertex of a pyramid which has as its base a regular $2n$-gon with vertices $q_1, q_2, \ldots, q_{2n}$. For simplicity let all triangular faces of the pyramid be congruent isosceles triangles. Now construct a second pyramid with vertex q_{2n+1} and with the r-sided polygon $q_1 q_2 q_3 \ldots q_r q_1$ as a base. For convenience let q_o and q_{2n+1} be on opposite sides of the common plane of the bases of both pyramids. If $r = 2n$, let $Q(n, 2n)$ be the union of our two pyramids. If $3 \leqq r < 2n$ adjust the choice of q_{2n+1} so that the plane $q_{2n+1} q_r q_1$ will intersect each of the segments $q_o q_j$ at an interior point q_j', for $r+1 \leqq j \leqq 2n$. Then $q_o q_1 \ldots q_r q_{r+1}' \ldots q_{2n}' q_{2n+1}$ are the vertices of a polyhedron $Q(n, r)$ for $n \geqq 2$ and $3 \leqq r < 2n$. For $r \leqq 2n$, $Q(n, r)$ has

(36) $$V = 2n+2 , \quad E = 4n+r , \quad F = 2n+r .$$

Note that $q_o q_r q_{r+1}' \ldots q_{2n}' q_1 q_o$ is one of the faces of $Q(n, r)$. Also note that

$$\frac{4n+3}{n+1} \leqq \frac{2E}{V} = \frac{8n+2r}{2n+2} \leqq \frac{6n}{n+1} ,$$

which for $n \geqq 5$ covers the range required by the inequalities (32).

For the values (36) let

(37) $$g(n, r) = \tfrac{1}{2}\left(\frac{1}{V} + \frac{1}{E}\right) - \frac{1}{F} .$$

For fixed n and $r \geqq 3$, $g(n, r)$ is an increasing function of r, since

$\partial g/\partial r > 0$. Now

$$g(n, 3) = (3-n^2)/4(n+1)(2n+3)(4n+3) < 0$$

for $n \geqq 2$, and

$$g(n, 2n) = (n-2)/12n(n+1) \geqq 0$$

if $n \geqq 2$. Hence, if we choose

$$V = 2n_o + 2 = V_F$$

for any F-harmonic triple ($p = 1$), then the equation

$$g(n_o, r_o) = 0$$

will have a unique integer solution, r_o, such that $3 < r_o \leqq 2n_o$. Hence $Q(n_o, r_o)$ will be F-harmonic.

Note that $Q(2, 4)$ is a regular octahedron.

V. REALIZATIONS OF HARMONIC POLYHEDRA FOR $p < 0$.

The constructions of topologically spherical harmonic polyhedra described in Section 4 can be modified as follows to produce spheres with $1-p$ handles. For a V-harmonic triple, let $V = 2n$. Let us begin by constructing a prism $P(N, 0)$, of the type described in Section 4, with

(38) $$N = n - 4(1-p) \quad .$$

Now let d_u be an arbitrary diagonal of the upper base of $P(N, 0)$ that connects two non-consecutive vertices, and let d_l be the orthogonal projection of d_u onto the lower base. Now cut $1-p$ congruent non-intersecting prismatic holes, with square cross-sections and vertical, rectangular walls in $P(N, 0)$. If the square cross-section is small enough, we can place each of these shafts so that a diagonal of its upper (lower) square boundary lies on $d_u(d_l)$. If we delete these diagonal segments, then $2 - p$ segments of $d_u(d_l)$ remain. Now in each of r of our n vertical faces draw a diagonal. In this way we have produced a net with

(39) $$V = 2n, \quad E = 3n + 2(2-p) + r, \quad F = n + 4 + r \ .$$

If we hinge the upper and lower bases of $P(N, 0)$ along d_u and d_l slightly close the hinges, and slightly displace suitable vertices, we shall produce a polyhedron $P(n, p, r)$ for which (39) holds. Note that

$$2E/V = 3 + [r + 2(2-p)]/n$$

so for $0 \leqq r \leqq n$

(40) $$3 + 2(2-p)/n \leqq 2E/V \leqq 4 + 2(2-p)/n \ .$$

The interval (40) will include the interval (31) if

(41) $$n > 2(\sqrt{2} + 1)(2 - p) > 13.5 - 6.7p \ .$$

For the values (39) let

(42) $$f(n, p, r) = \tfrac{1}{2}(1/E + 1/F) - 1/V \ .$$

Then

$$f(n, p, 0) = \left[n^2 - 8n - 8(2-p)\right] / 2n(n+4)\left[3n + 2(2-p)\right] .$$

For $p < 0$ and $n = 2(\sqrt{2} + 1)(2-p)$

$$f(n, p, 0) > 0$$

because its numerator has the value $-4(3+2\sqrt{2})\,p\,(2-p) > 0$.

For $n \geqq 8$, $f(n,p,0)$ is an increasing function of n. Hence (41) implies

$$f(n,p,0) > 0\,, \qquad\qquad p \leqq -1\,.$$

On the other hand,

$$f(n,p,n) = -\left[n^2 + (8-p)n + 4(2-p)\right]/4n(n+2)(2n+2-p) < 0\,.$$

Furthermore, $f(n,p,r)$ is a decreasing function of r for $0 \leqq r \leqq n$. Thus, if we choose for $V = 2n_o$ one of the possible values that occurs in a V-harmonic triple for some specified $p_o < 0$, and if n_o and p_o satisfy (41), then the equation

$$f(n_o, p_o, r) = 0$$

has a unique integer solution $r = r_o$ such that $0 \leqq r_o \leqq n_o$. Hence $P(n_o, p_o, r_o)$ is V-harmonic.

For an F-harmonic triple with $p \leqq -1$ let $V = 2n+2$. Let us begin by constructing a polyhedron of the type $Q(N,r)$ described in Section 4 for

(43) $$N = n - 4(1-p).$$

If $r \geqq 4$, $Q(N,r)$ has six triangular faces $q_o q_j q_{j+1}$ and $q_{2n+1} q_j q_{j+1}$ for $1 \leqq j \leqq 3$, which we shall first modify as follows. Let q_2^*, q_3^*, q_2^+, and q_3^+ be interior points of the segments $q_o q_2$, $q_o q_3$, $q_{2n+1} q_2$, and $q_{2n+1} q_3$ chosen so that $q_2^* q_3^*$ and $q_2^+ q_3^+$ are parallel to $q_2 q_3$. Now displace q_2 slightly outward along the line $q_1 q_2$ to a new location q_2', and displace q_3 outward along $q_4 q_3$ to q_3'. Since q_1, q_2, q_3, and q_4 are coplanar, q_2' and q_3' will also be in the same plane. Let us choose q_3'

so that $q_2'q_3'$ is parallel to the original segment q_2q_3. By these changes we have determined a new polyhedron $Q^*(N, r)$ which differs from the original polyhedron $Q(N, r)$ to the extent that two vertices have been moved and four vertices have been added, with the following consequences for the set of six triangular faces mentioned above. The four "outer" triangles have been replaced by plane quadrilaterals, e.g. $q_oq_1q_2$ by $q_oq_1q_2'q_2^*$. The two "inner" triangles have been replaced by two smaller triangles and two trapezoids, e.g. $q_oq_2q_3$ by $q_oq_2^*q_3^*$ and $q_2^*q_3^*q_3'q_2'$. Thus the numbers of vertices, edges, and faces have been increased by amounts

(44) $$\Delta V = 4, \ \Delta E = 6\ , \ \Delta F = 2\ .$$

Now cut 1-p congruent non-intersecting prismatic holes with square cross-sections in $Q^*(N, r)$. If the square cross-section is small enough we can cause a pair of opposite long parallel edges of each hole to intersect both $q_2^*q_3^*$ and $q_2^+q_3^+$. In this way we break each of these segments into 2-p segments and "scallop" one edge of each of $q_oq_2^*q_3^*$, $q_2'q_2^*q_3^*q_3'$, $q_{2n+1}q_2^+q_3^+$, $q_2'q_2^+q_3^+q_3'$. We have also added 4(1-p) faces to produce a polyhedron $Q(n, p, r)$ for which by (36), (43) and (44)

(45) $$V = 2n + 6, \ E = 4n + 2p + r + 4, \ F = 2n + 4p + r - 2$$

for $4 \leqq r \leqq 2n$, where the value of E is most conveniently confirmed by use of (1). Note that

$$2E/V = (4n+r+2p)/(n+3)\ ,$$

so that for $4 \leqq r \leqq 2n$

(46) $$(4n+4+2p)/(n+3) \leqq 2E/V \leqq (6n+2p)/(n+3).$$

For $p < 0$ the lower bound in (46) is always less than 4. The upper bound will be at least 5 for

(47) $$n \geqq 15 - 2p .$$

Hence, when (47) holds, the interval (46) contains the interval (33).

For the values (45) let

(48) $$g(n, p, r) = \tfrac{1}{2}(1/V + 1/E) - 1/F .$$

Note that (47) implies

$$\partial g(n, p, r)/\partial r > 0 \quad \text{for } r \geqq 0 ,$$

i.e. g is an increasing function of r for the combinations of n, p, and r that interest us.

Now

$$g(n, p, 4) = \left[-n^2 + (5p-17)n + 2p^2 + 10p - 10\right]/4(n+3)(n+2p+2)(2n+p+4).$$

For $n > 0$ the numerator is a decreasing function of n. For

$$n \geqq -2p$$

the numerator is no greater than

$$-12p^2 + 44p - 10 < -12-44-10 = -66 .$$

Since for $n \geqq -2p$ the denominator is positive, then

$$g(n, p, 4) < 0 .$$

On the other hand

$$g(n, p, 2n) = \left[2n^2 + 8(p-2)n + 2p^2 + 3p - 17\right]/4(n+3)(2n+2p-1)(3n+p+2).$$

For $n \geqq 8 - 4p$ the numerator is an increasing function of n, and the denominator is positive. For

(49) $$n = 10 - 5p$$

the numerator has the value

$$23 - 37p + 12p^2 > 0,$$

and then

$$g(n, p, 2n) > 0 .$$

Hence if for some $p = p_o$ we consider any F-harmonic triple $V = 2n_o + 2$ with large enough n_o, the equation

$$g(n_o, p_o, r_o) = 0$$

will have a unique integer solution r_o such that $4 \leqq r_o < 2n_o$. Hence $Q(n_o, p_o, r_o)$ is F-harmonic.

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